Hoeffding decomposition in H^1 spaces

Maciej Rzeszut^{1,2} · Michał Wojciechowski¹

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Abstract

The well known result of Bourgain and Kwapień states that the projection $P_{\leq m}$ onto the subspace of the Hilbert space $L^2(\Omega^\infty)$ spanned by functions dependent on at most m variables is bounded in L^p with norm $\leq c_p^m$ for $1 < p < \infty$. We will be concerned with two kinds of endpoint estimates. We prove that $P_{\leq m}$ is bounded on the space $H^1(\mathbb{D}^\infty)$ of functions in $L^1(\mathbb{T}^\infty)$ analytic in each variable. We also prove that $P_{\leq 2}$ is bounded on the martingale Hardy space associated with a natural double-indexed filtration and, more generally, we exhibit a multiple indexed martingale Hardy space which contains $H^1(\mathbb{D}^\infty)$ as a subspace and $P_{\leq m}$ is bounded on it.

Keywords Hardy spaces · Hoeffding decomposition · Martingale inequalities · Decoupling

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Contents

1 Introduction	1114
2 Preliminaries	1116
3 Boundedness of P_m on $L^p(\Omega^\mathbb{N})$	1122
4 Boundedness of P_m on $H^1(\mathbb{D}^\mathbb{N})$	1126
5 Martingale Hardy spaces	1130
5.1 Double indexed martingales	1130
5.2 Multiple indexed martingales	1131
Appendix	1136
References	1140

✉ Maciej Rzeszut
maciej.rzeszut@weizmann.ac.il
Michał Wojciechowski
miwoj@impan.pl

¹ Institute of Mathematics of Polish Academy of Sciences, Śniadeckich 8, 00-656 Warsaw, Poland

² Weizmann Institute of Science, 234 Herzl St., PO Box 26, 7610001 Rehovot, Israel

1 Introduction

The Rademacher functions $(r_i)_{i \in \mathbb{N}}$ generate a well studied subspace of $L^p[0, 1]$, which we identify with $L^p(\mathbb{Z}_2^{\mathbb{N}})$. In particular by Khintchine inequality

$$\left\| \sum_i c_i r_i \right\|_{L^p} \simeq_p \left(\sum_i |c_i|^2 \right)^{\frac{1}{2}} \quad (1.1)$$

for $0 < p < \infty$, and $\overline{\text{span}}(r_i : i \in \mathbb{N})$ is complemented in L^p for $1 < p < \infty$ but not for $p = 1$. We will index the Walsh system by finite subsets of $\mathbb{N}$, i.e.

$$w_A = \prod_{i \in A} r_i. \quad (1.2)$$

The number $|A|$ is called the multiplicity of w_A . Analogous problems for Walsh functions of finite multiplicity have been resolved independently by Bonami [2] and Kiener [13]. Namely, the inequality

$$\left\| \sum_{|A| \leq m} c_A w_A \right\|_{L^p} \simeq_{p,m} \left(\sum_{|A| \leq m} |c_A|^2 \right)^{\frac{1}{2}} \quad (1.3)$$

holds true, and the orthogonal projection P_m onto $\overline{\text{span}}(w_A : |A| \leq m)$ is bounded if and only if $1 < p < \infty$. Some lower estimates for P_m are also known, see [12] or [20] for a detailed discussion on this subject.

In [4], Bourgain generalized these results to a setting in which $(\mathbb{Z}_2, \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}, \frac{1}{2}\#)$ is replaced with an arbitrary probability space $(\Omega, \mathcal{F}, \mu)$. To be more precise, let $(\Omega^\infty, \mathcal{F}^{\otimes \infty}, \mu^{\otimes \infty})$ be the infinite product space. Any $f \in L^2(\Omega^\infty, \mathcal{F}^{\otimes \infty}, \mu^{\otimes \infty})$ can be decomposed in a unique way into a series

$$f(x) = \sum_m \sum_{i_1 < \dots < i_m} f_{i_1, \dots, i_m}(x_{i_1}, \dots, x_{i_m}) \quad (1.4)$$

where $f_{i_1, \dots, i_m} \in L^2(\Omega^m)$ is mean zero in each of its m arguments. Thus, P_A and P_m defined by

$$P_{\{i_1, \dots, i_m\}} f(x) = f_{i_1, \dots, i_m}(x_{i_1}, \dots, x_{i_m}), \quad P_m = \sum_{|A|=m} P_A \quad (1.5)$$

are mutually orthogonal orthogonal projections. In the case of $\Omega = \mathbb{Z}_2$, the image of P_A is just the one-dimensional space spanned by w_A , so the above definition of P_m coincides for with the projection onto Walsh functions of multiplicity m . In [4] Bourgain proved that for $1 \leq p < \infty$,

$$\left\| \sum_{|A| \leq m} P_A f \right\|_{L^p} \simeq_{p,m} \left\| \left(\sum_{|A| \leq m} |P_A f|^2 \right)^{\frac{1}{2}} \right\|_{L^p}, \quad (1.6)$$

which is a direct generalization of (1.3). Moreover, he proved that P_m is bounded on L^p if and only if $1 < p < \infty$, with norm smaller than c_p^m where $c_p \lesssim \frac{\hat{p}^{\frac{5}{2}}}{\log \hat{p}}$ and $\hat{p} = p \vee \frac{p}{p-1}$.

It turns out that the projections P_m have a well established probabilistic interpretation. In [14], Kwapien connected them to the notion of Hoeffding decomposition, which originated

from Hoeffding's work [11]. More precisely, elements of the image of P_m are what is called generalized canonical U -statistics and the decomposition $f = \sum_m P_m f$ plays a crucial role in the proofs of many theorems concerning U -statistics. For more information, we refer the reader to [16]. Kwapień provided a shorter proof of Bourgain's result about boundedness of P_m , with a better constant $c_p \lesssim \frac{\hat{p}}{\log \hat{p}}$.

Let us describe the main results of this paper, which give certain endpoint estimates for P_m . One of them (Theorem 4.5 in the text) is obtained by restricting the domain of P_m . For exact definition of H_{all}^1 , see Sect. 2.

Theorem A P_m is bounded on the subspace $H_{\text{all}}^1(\mathbb{T}^\infty)$ of $L^1(\mathbb{T}^\infty)$ consisting of functions analytic in each variable.

We also find a norm stronger than L^1 and weaker than L^p ($p > 1$), in which P_m is bounded. The detailed construction is described in Sect. 5.2.

Theorem B For any $m \in \mathbb{N}$, there is a partition of the family of finite subsets of $\mathbb{N}$ into $\bigcup_{i \in \mathcal{I}} \mathcal{A}_i$ such that the norm

$$\|f\| := \mathbb{E} \left(\sum_i \left| \sum_{A \in \mathcal{A}_i} P_A f \right|^2 \right)^{\frac{1}{2}} \quad (1.7)$$

is between L^1 and all L^p ($p > 1$) and P_m is bounded in this norm.

It is worth noting that Theorem A translates directly to the space $\mathcal{H}^1$ of Dirichlet series, i.e. the closure of polynomials of the form $\sum_{n=1}^N b_n n^{-s}$ in the norm

$$\left\| \sum_{n=1}^N b_n n^{-s} \right\|_{\mathcal{H}^1} := \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left| \sum_{n=1}^N b_n n^{-it} \right| dt. \quad (1.8)$$

The Bohr lift, dating back to [1], is the map

$$H_{\text{all}}^1(\mathbb{T}^\infty) \ni \sum_{k \in \mathbb{N}^{\oplus \mathbb{N}}} a_k e^{i\langle k, t \rangle} \mapsto \sum_{n \in \mathbb{N}} b_n n^{-s} \in \mathcal{H}^1 \quad (1.9)$$

where $a_k = b_n$ for n having the prime number factorization $n = \prod_j p_j^{k_j}$. It is an isometry between $H_{\text{all}}^1(\mathbb{T}^\infty)$ and the space $\mathcal{H}^1$ of Dirichlet series. Thus, our result is equivalent to the fact that the projection from $\mathcal{H}^1$ onto

$$\overline{\text{span}} \{n^{-s} : n \text{ has at most } m \text{ prime factors}\} \subset \mathcal{H}^1 \quad (1.10)$$

is bounded. For a more detailed exposition of Dirichlet series and their relation to polydisc Hardy spaces, see [21].

The paper is organized as follows. In Sect. 2 we introduce necessary notation and definitions. In Sect. 3, we provide a new simple proof of the historic L^p boundedness result. The proof of the estimate $\|P_m\| \leq (e \|P_1\|)^m$ is done by means of a combinatorial identity expressing P_m in terms of tensor products of P_1 . In Sect. 4, we show that the same argument carries over with little modification showing boundedness of P_m on $H_{\text{all}}^1(\mathbb{T}^\infty)$. In Sect. 5.1, we define, purely in terms of square functions and not referring to analyticity, a multiple indexed martingale Hardy space $H^1[\mathcal{T}_m]$ of functions on Ω^∞ that admits a bounded action of P_m . It turns out that if $\Omega = \mathbb{T}$, there is a subspace $H_{m \text{ last}}^1(\mathbb{T}^\mathbb{N})$ of $L^1(\mathbb{T}^\infty)$, much bigger

than $H_{\text{all}}^1(\mathbb{T}^\infty)$, on which the L^1 norm is equivalent to $H^1[\mathcal{T}_m]$ norm. The arguments rely heavily on L^1 square function theorem for Hardy martingales and decoupling inequality of Zinn. We present two proofs of the latter in Sect. 1.

2 Preliminaries

Probability spaces and conditional expectations In all of the text, $(\Omega, \mathcal{F}, \mu)$ will be a probability space. We will equip sets of the form Ω^I , where I is an at most countable index set, with the product measure $\mu^{\otimes I}$ defined on $\mathcal{F}^{\otimes I}$. In case we are only concerned with the cardinality of I , we will write Ω^n , where n is a natural number or ∞ . By the natural filtration on $\Omega^\mathbb{N}$ we mean the filtration $(\mathcal{F}_n : n = 0, 1, \dots)$, where $\mathcal{F}_k$ is generated by the coordinate projection $\omega \mapsto (\omega_1, \dots, \omega_k)$ and denote $\mathbb{E}_k = \mathbb{E}(\cdot \mid \mathcal{F}_k)$. In general, for a subset A of the index set, $\mathcal{F}_A$ will be the sigma algebra generated by the coordinate projection $\omega \mapsto (\omega_i)_{i \in A}$ and $\mathbb{E}_A = \mathbb{E}(\cdot \mid \mathcal{F}_A)$. In more explicit terms, measurability with respect to $\mathcal{F}_A$ is equivalent to being dependent only on variables with indices belonging to A and the conditional expectation operator $\mathbb{E}_A$ integrates away the dependence on all other variables, so that the formulas

$$\mathbb{E}_k f(x) = \int_{\Omega^{[k+1, \infty)}} f(x_1, \dots, x_k, y_{k+1}, y_{k+2}, \dots) d\mu^{\otimes [k+1, \infty)}(y), \quad (2.1)$$

$$\mathbb{E}_A f(x) = \int_{\Omega^{\mathbb{N} \setminus A}} f(x_A, y_{\mathbb{N} \setminus A}) d\mu^{\otimes \mathbb{N} \setminus A}(y) \quad (2.2)$$

are satisfied (with the convention that sequences indexed by A and $\mathbb{N} \setminus A$ are merged in a natural way into a sequence indexed by $\mathbb{N}$). It will often be convenient to identify a function f defined on Ω^A with an $\mathcal{F}_A$ -measurable function $\Omega^I \ni \omega \mapsto f((\omega_i)_{i \in A})$. In order to save space, we will often write dx instead of $d\mu(x)$ whenever the measure is implied by context.

Tensor products Let $1 \leq p < \infty$. For $f_k \in L^p(\Omega_k)$, we will denote by $\bigotimes_{k=1}^n f_k$ the function on $\prod_k \Omega_k$ satisfying

$$\left(\bigotimes_k f_k \right)(x) = \prod_k f_k(x_k). \quad (2.3)$$

Because of separation of variables, we have $\|\bigotimes_k f_k\|_{L^p(\prod_k \Omega_k)} = \prod_k \|f_k\|_{L^p(\Omega_k)}$. This way we actually define an injection of the algebraic tensor product $\bigotimes_k L^p(\Omega_k)$ into $L^p(\prod_k \Omega_k)$, the image of which is dense.

Let X_k be subspaces (by a subspace we always mean a closed linear subspace) of $L^p(\Omega_k)$. By $\bigotimes_k X_k$ we will denote the subspace of $L^p(\prod_k \Omega_k)$ spanned by functions of the form $\bigotimes_k f_k$, where $f_k \in X_k$, and the norm is inherited from $L^p(\prod_k \Omega_k)$ (care has to be taken, as $\bigotimes_k X_k$ is not determined solely by X_k as Banach spaces, but rather by the particular way they are embedded in $L^p(\Omega_k)$). If $T_k : X_k \rightarrow L^p(\Omega_k)$ are bounded operators, then we can define an operator $\bigotimes_k T_k : \bigotimes_k X_k \rightarrow L^p(\prod_k \Omega_k)$ by the formula

$$\left(\bigotimes_k T_k \right) \left(\bigotimes_k f_k \right) = \bigotimes_k T_k f_k, \quad (2.4)$$

and easily check that the property

$$\left\| \bigotimes_k T_k : \bigotimes_k X_k \rightarrow L^p \left(\prod_k \Omega_k \right) \right\| \leq \prod_k \|T_k : X_k \rightarrow L^p(\Omega_k)\| \quad (2.5)$$

is satisfied. Indeed, $\bigotimes_k T_k = \prod_k \text{id}_{L^p(\prod_{j \neq k} \Omega_j)} \otimes T_k$, and any operator of the form $\text{id} \otimes T$ has norm bounded by $\|T\|$, because $(\text{id} \otimes T) f(\omega_1, \omega_2) = T(f(\omega_1, \cdot))(\omega_2)$.

Fourier transform Let $\mathbb{T}$ be the interval $[0, 2\pi)$ equipped with addition modulo 2π and normalized Lebesgue measure $d\mu = \frac{dx}{2\pi}$. We will be exclusively dealing with Fourier transforms of functions on $\mathbb{T}$ or some power of $\mathbb{T}$. Since the group dual to $\mathbb{T}$ is $\mathbb{Z}$, the dual group to the product $\mathbb{T}^{\mathbb{N}}$ is the direct sum $\mathbb{Z}^{\oplus \mathbb{N}}$ (i.e., integer-valued sequences that are eventually 0), on which we define the Fourier transform by

$$\widehat{f}(n) = \int_{\mathbb{T}^{\mathbb{N}}} f(x) e^{-i \sum_{k \in \mathbb{N}} n_k x_k} d\mu^{\otimes \mathbb{N}}(x). \quad (2.6)$$

Hardy spaces of martingales and analytic functions By $\mathbb{D}$ we denote the unit disk in the complex plane. We can identify $\mathbb{T}$ with the unit circle by the map $t \mapsto e^{it}$. For $N \in \mathbb{N}$, the space $H^1(\mathbb{D}^N)$ is defined as the space of functions analytic in the polydisc $\mathbb{D}^N$ such that the norm

$$\|F\|_{H^1(\mathbb{D}^N)} = \sup_{0 < r_1, \dots, r_N < 1} \int_{\mathbb{T}^N} \left| F(r_1 e^{it_1}, \dots, r_N e^{it_N}) \right| \frac{dt}{(2\pi)^N} \quad (2.7)$$

is finite. It is well-known [10] that such a function has an a.e. radial limit $f(t_1, \dots, t_N) = \lim_{r \rightarrow 1} F(re^{it_1}, \dots, re^{it_N})$ on the distinguished boundary $\mathbb{T}^N$ and F can be recovered from f by convolution with a Poisson kernel. This sets a one-to-one correspondence between $H^1(\mathbb{D}^N)$ and the space

$$H_{\text{all}}^1(\mathbb{T}^N) = \overline{\text{span}} \left\{ e^{i \langle n, t \rangle} : n_1, \dots, n_N \geq 0 \right\} \subset L^1(\mathbb{T}^N). \quad (2.8)$$

We also can define $H_{\text{all}}^1(\mathbb{T}^{\mathbb{N}})$ in the same manner as in (2.8), but care has to be taken, since these functions are can no longer be extended analytically to $\mathbb{D}^{\mathbb{N}}$ in general (hence the shorthand $H^1(\mathbb{D}^{\mathbb{N}})$, which we will sometimes use, is an abuse of notation). Later we will use two more H^1 spaces, namely $H_{\text{last}}^1(\mathbb{T}^{\mathbb{N}})$ (also called Hardy martingales) and $H_{m \text{ last}}^1(\mathbb{T}^{\mathbb{N}})$, which we will define as follows.

$$H_{\text{last}}^1(\mathbb{T}^{\mathbb{N}}) = H_{1 \text{ last}}^1(\mathbb{T}^{\mathbb{N}}) = \overline{\text{span}} \left\{ e^{i \langle n, t \rangle} : n_{i_0} > 0 \text{ for } i_0 = \max \{i : n_i \neq 0\} \right\} \subset L^1(\mathbb{T}^{\mathbb{N}}), \quad (2.9)$$

$$H_{m \text{ last}}^1(\mathbb{T}^{\mathbb{N}}) = \overline{\text{span}} \left\{ e^{i \langle n, t \rangle} : m \text{ last of nonzero } n_i \text{'s are } > 0 \right\} \subset L^1(\mathbb{T}^{\mathbb{N}}). \quad (2.10)$$

In the space $H_{m \text{ last}}^1(\mathbb{T}^{\mathbb{N}})$ we allow characters of the form $e^{i \langle n, t \rangle}$, where $|\text{supp } n| < m$ and $n_j \geq 0$ for all j .

Now we recall the definition of a martingale Hardy space and some related inequalities. A standard reference in this matter is [9]. Let $(\mathcal{F}_n)_{n=0}^{\infty}$ be an arbitrary filtration on a probability space $(\Omega, \mathcal{F}, \mu)$, where $\mathcal{F}$ is generated by $\bigcup \mathcal{F}_n$. We denote $\mathbb{E}_k = \mathbb{E}(\cdot | \mathcal{F}_k)$, $\Delta_0 = \mathbb{E}_0$, $\Delta_k = \mathbb{E}_k - \mathbb{E}_{k-1}$ for $k \geq 1$, and define the square function and maximal function of f respectively by

$$Sf = \left(\sum_{n=0}^{\infty} |\Delta_n f|^2 \right)^{\frac{1}{2}}, \quad f^* = \sup_n |\mathbb{E}_n f|. \quad (2.11)$$

This allows us to define the martingale Hardy space.

Definition 2.1 The space $H^1[(\mathcal{F}_n)_{n=1}^\infty]$ is a function space on Ω with the norm

$$\|f\|_{H^1[(\mathcal{F}_n)_{n=1}^\infty]} = \mathbb{E}Sf. \quad (2.12)$$

We will make use of three following classical martingale inequalities.

Theorem 2.2 (Burkholder, Gundy [7] for $1 < p < \infty$; Davis [8] for $p = 1$) For $1 \leq p < \infty$,

$$\|Sf\|_{L^p} \simeq_p \|f^*\|_{L^p}. \quad (2.13)$$

Theorem 2.3 (Burkholder [6]) For $1 < p < \infty$,

$$\|f\|_{L^p} \simeq_p \|Sf\|_{L^p}. \quad (2.14)$$

Theorem 2.4 (Stein [24]) For $1 < p < \infty$ and an arbitrary sequence $(f_n)_{n=0}^\infty$,

$$\left\| \left(\sum_{n=0}^{\infty} |\mathbb{E}_n f_n|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim_p \left\| \left(\sum_{n=0}^{\infty} |f_n|^2 \right)^{\frac{1}{2}} \right\|_{L^p}. \quad (2.15)$$

Definition 2.5 A martingale atom is a function of the form

$$a = u - \mathbb{E}_{j-1}u, \quad (2.16)$$

where

$$A \in \mathcal{F}_j, \quad \text{supp } u \subset A, \quad \|u\|_{L^2} \leq |A|^{-\frac{1}{2}}. \quad (2.17)$$

Theorem 2.6 Let $f \in H^1[(\mathcal{F}_n)_{n=1}^\infty]$ be of mean 0. Then there are atoms $a_1, a_2, \dots$ and scalars $c_1, c_2, \dots$ such that

$$f = \sum_{n=1}^{\infty} c_n a_n \quad (2.18)$$

and

$$\sum_{n=1}^{\infty} |c_n| \lesssim \|f\|_{H^1[(\mathcal{F}_n)_{n=1}^\infty]}. \quad (2.19)$$

Theorem 2.7 [Fefferman] The dual space to $H^1[(\mathcal{F}_n)_{n=1}^\infty]$ is $BMO[(\mathcal{F}_n)_{n=1}^\infty]$, where

$$\|g\|_{BMO[(\mathcal{F}_n)_{n=1}^\infty]} \simeq \sup_k \left\| \left(\mathbb{E}_k \sum_{n \geq k} |\Delta_n g|^2 \right)^{\frac{1}{2}} \right\|_{L^\infty}, \quad (2.20)$$

where the duality is given by $\langle f, g \rangle = \lim_{n \rightarrow \infty} \mathbb{E}(\mathbb{E}_n f \mathbb{E}_n g)$.

Vector-valued inequalities For a Banach space B , by $L^p(S, B)$ we denote the Bochner space of strongly measurable B -valued random variables equipped with the norm

$$\|f\|_{L^p(S, B)} = \left(\int_S \|f(x)\|_B^p d\mu(s) \right)^{\frac{1}{p}} \quad (2.21)$$

(or, equivalently, the closed span of functions of the form $(f \otimes v)(x) = f(x)v$, where $f \in L^p(S)$ and $v \in B$, in the $L^p(S, B)$ norm). For an operator T between subspaces of

$L^p(S_1)$ and $L^p(S_2)$ and a linear operator $F : B_1 \rightarrow B_2$ we can define $T \otimes F$ and the algebraic tensor product by $(T \otimes F)(f \otimes v) = T(f) \otimes F(v)$, but this construction does not necessarily produce a bounded operator on the closure. The main tool for obtaining vector-valued extensions of inequalities will be the following lemma, which for I_1, I_2 being singletons is due to Marcinkiewicz and Zygmund [18] (in this case $\lesssim \|T\|$ can be replaced with $\leq \|T\|$).

Lemma 2.8 *Let $X_i \subset L^1(S_i, \ell^2(I_i))$ for $i = 1, 2$, B be a Hilbert space and $T : X_1 \rightarrow X_2$ be bounded. Then $T \otimes \text{id}_B : X_1 \otimes B \rightarrow X_2 \otimes B$, where $X_i \otimes B$ is treated as a subspace of $L^1(\Omega_i, \ell^2(I_i, B))$, is bounded with norm $\lesssim \|T\|$.*

Proof Without loss of generality, B is finite-dimensional, say $B = \ell^2(J)$ for some finite J . Let $X_1 \otimes \ell^2(J) \ni f = (f_j)_{j \in J}$, so that $f_j \in X_1$. Let also r_j for $j \in J$ be Rademacher variables. Then, applying $\ell^2(I_2)$ -valued Khintchine inequality,

$$\begin{aligned} & \| (T \otimes \text{id}) f \|_{L^1(S_2, \ell^2(I_2 \times J))} \\ &= \int_{S_2} \left(\sum_{j \in J} \| T f_j(s) \|_{\ell^2(I_2)}^2 \right)^{\frac{1}{2}} d\mu_2(s) \end{aligned} \quad (2.22)$$

$$\simeq \int_{S_2} \mathbb{E} \left\| \sum_j r_j T f_j(s) \right\|_{\ell^2(I_2)} d\mu_2(s) \quad (2.23)$$

$$= \mathbb{E} \int_{S_2} \left\| T \left(\sum_j r_j f_j \right)(s) \right\|_{\ell^2(I_2)} d\mu_2(s) \quad (2.24)$$

$$\leq \|T\| \mathbb{E} \int_{S_1} \left\| \sum_j r_j f_j(s) \right\|_{\ell^2(I_1)} d\mu_1(s) \quad (2.25)$$

$$\leq \|T\| \int_{S_1} \left(\sum_j \|f_j(s)\|_{\ell^2(I_1)}^2 \right)^{\frac{1}{2}} d\mu_1(s) \quad (2.26)$$

$$= \|T\| \|f\|_{L^1(S_1, \ell^2(I_1 \times J))}. \quad (2.27)$$

□

Hoeffding decomposition Now we define the main object of our interest. In order to avoid technicalities with convergence in strong operator topology, we will work in a finite product of Ω (all the results extend automatically to Ω^∞ by density). We will see in a moment that any function $f \in L^1(\Omega^n)$ can be decomposed in a unique way as

$$f = \sum_{m=0}^n \sum_{1 \leq i_1 < \dots < i_m \leq n} P_{i_1, \dots, i_m} f,$$

where $P_{i_1, \dots, i_m} f(x_1, \dots, x_n)$ depends only on $x_{i_1}, \dots, x_{i_m}$ and is of mean 0 with respect to each of $x_{i_1}, \dots, x_{i_m}$ (equivalently, $P_A f$ is $\mathcal{F}_A$ -measurable and is orthogonal to all $\mathcal{F}_B$ -measurable functions for $B \subsetneq A$). This decomposition has been studied in [4, 14]. In

particular, $P_{i_1, \dots, i_m}$ are pairwise orthogonal orthogonal projections. Let

$$P_m = \sum_{1 \leq i_1 < \dots < i_m \leq n} P_{i_1, \dots, i_m}$$

and U_m be the range of P_m . It is known [4, 14] that P_m is bounded on $L^p(\Omega^n)$, $1 < p < \infty$, with norm independent on n , but this is not true for $L^1(\Omega^n)$.

One of the possible ways to prove the existence of the above decomposition in $L^2(\Omega^n)$ is as follows. First we define the subspace

$$U_{\leq m} = \overline{\text{span}} \bigcup_{|A| \leq m} \{f \in L^2(\Omega^n) : f \text{ is } \mathcal{F}_A\text{-measurable}\} \subset L^2(\Omega^n) \quad (2.28)$$

for each $m \geq 0$. The sequence of subspaces $U_{\leq 0}, U_{\leq 1}, \dots, U_{\leq n}$ is increasing, so by putting

$$U_0 = U_{\leq 0}, \quad U_m = U_{\leq m} \cap U_{\leq m-1}^\perp \quad (2.29)$$

we obtain a decomposition

$$L^2(\Omega^n) = \bigoplus_{m=0}^n U_m \quad (2.30)$$

into an orthogonal direct sum of U_m . We will denote the orthogonal projection onto U_m by P_m .

A more explicit formula for P_m can be obtained. For $A \subset [1, n]$, let

$$P_A = (\text{id} - \mathbb{E})^{\otimes A} \otimes \mathbb{E}^{\otimes [1, n] \setminus A}, \quad (2.31)$$

where id and $\mathbb{E}$ are understood to act on $L^2(\Omega)$, and let U_A be the range of the projection P_A . It is easy to see that

$$\mathbb{E}_A = (\text{id} - \mathbb{E} + \mathbb{E})^{\otimes A} \otimes \mathbb{E}^{\otimes [1, n] \setminus A} = \sum_{B \subset A} (\text{id} - \mathbb{E})^{\otimes B} \otimes \mathbb{E}^{\otimes [1, n] \setminus B} \quad (2.32)$$

and, since the subspaces U_B are mutually orthogonal,

$$L^2(\Omega^n, \mathcal{F}_A) = \bigoplus_{B \subset A} U_B. \quad (2.33)$$

Moreover

$$U_{\leq m} = \overline{\text{span}} \bigcup_{|A| \leq m} L^2(\Omega^n, \mathcal{F}_A) \quad (2.34)$$

$$= \overline{\text{span}} \bigcup_{|A| \leq m} \bigoplus_{B \subset A} U_B \quad (2.35)$$

$$= \bigoplus_{|B| \leq m} U_B \quad (2.36)$$

and consequently

$$U_m = \bigoplus_{|B|=m} U_B, \quad P_m = \sum_{|B|=m} P_B. \quad (2.37)$$

Decoupling inequalities We are going to present a special case of a theorem of J. Zinn [26], which will be one of the most important tools.

Theorem 2.9 [Zinn] For $k = 1, \dots, N$, let f_k be a function on Ω^k . Then

$$\int_{\Omega^N} \left(\sum_{k=1}^N |f_k(x_1, \dots, x_k)|^2 \right)^{\frac{1}{2}} dx \simeq \int_{\Omega^N} \int_{\Omega^N} \left(\sum_{k=1}^N |f_k(x_1, \dots, x_{k-1}, y_k)|^2 \right)^{\frac{1}{2}} dx dy. \quad (2.38)$$

We will provide two new proofs of the above in Sect. 1. Below, we state two corollaries obtained by iterating Zinn's inequality.

Corollary 2.10 For $1 \leq a < b \leq N$, let $f_{a,b} \in L^1(\Omega^N, \mathcal{F}_{[a,b]})$. Denote $(x_a, \dots, x_b)$ by $x_{[a,b]}$. Then

$$\int_{\Omega^N} \left(\sum_{a < b} |f_{a,b}(x_{[a,b]})|^2 \right)^{\frac{1}{2}} dx \simeq \int_{(\Omega^N)^3} \left(\sum_{a < b} |f_{a,b}(z_a, x_{[a+1,b-1]}, y_b)|^2 \right)^{\frac{1}{2}} dx dy dz \quad (2.39)$$

Proof Let $F_b \in L^1(\Omega^N, \mathcal{F}_b, \ell^2)$ be defined by $(F_b)_a = f_{a,b}$ for $a < b$ and 0 otherwise. Then, by Theorem 2.9 applied for functions $\|F_b\|_{\ell^2}$,

$$\begin{aligned} & \int_{\Omega^N} \left(\sum_{a < b} |f_{a,b}(x_{[a,b]})|^2 \right)^{\frac{1}{2}} dx \\ &= \int_{\Omega^N} \left(\sum_b \|F_b(x_{\leq b})\|_{\ell^2}^2 \right)^{\frac{1}{2}} dx \end{aligned} \quad (2.40)$$

$$\simeq \int_{(\Omega^N)^2} \left(\sum_b \|F_b(x_{\leq b-1}, y_b)\|_{\ell^2}^2 \right)^{\frac{1}{2}} dx dy \quad (2.41)$$

$$= \int_{(\Omega^N)^2} \left(\sum_{a < b} |f_{a,b}(x_{[a,b-1]}, y_b)|^2 \right)^{\frac{1}{2}} dx dy. \quad (2.42)$$

Analogously, by setting y as fixed, and applying Theorem 2.9 with reversed order of variables (which we can do, because we are dealing with finite sums),

$$\int_{(\Omega^N)^2} \left(\sum_{a \leq b} |f_{a,b}(x_{[a,b-1]}, y_b)|^2 \right)^{\frac{1}{2}} dx dy \simeq \int_{(\Omega^N)^3} \left(\sum_{a \leq b} |f_{a,b}(z_a, x_{[a+1,b-1]}, y_b)|^2 \right)^{\frac{1}{2}} dx dy dz \quad (2.43)$$

as desired. $\square$

Corollary 2.11 For all $i = (i_1, \dots, i_m)$ such that $i_1 < \dots < i_m$, let f_i be an $\mathcal{F}_{[1, i_1-1] \cup \{i_1, \dots, i_m\}}$ -measurable function on $\Omega^{\mathbb{N}}$. Then, treating each f_i as a function on $\Omega^{[1, i_1-1]} \times \Omega^{\{i_1, \dots, i_m\}}$,

$$\int_{\Omega^{\mathbb{N}}} \left(\sum_i |f_i(x_{< i_1}, x_{i_1}, \dots, x_{i_m})|^2 \right)^{\frac{1}{2}} dx$$

$$\simeq_m \int_{\Omega^{\mathbb{N}}} \int_{(\Omega^{\mathbb{N}})^m} \left(\sum_i \left| f_i \left(x_{<i_1}, y_{i_1}^{(1)}, \dots, y_{i_m}^{(m)} \right) \right|^2 \right)^{\frac{1}{2}} dy^{(1, \dots, m)} dx, \quad (2.44)$$

where $y^{(1)}, \dots, y^{(m)}$ are variables in $\Omega^{\mathbb{N}}$.

Proof Let us fix $k \in \{1, \dots, m\}$ and for each $j \in \mathbb{N}$ define a function φ_j on $\Omega^{[1, j]} \times (\Omega^{\mathbb{N}})^{m-k}$ by the formula

$$\begin{aligned} & \varphi_j \left(x_{\leq j}, y^{(k+1)}, \dots, y^{(m)} \right) \\ &= \left(\sum_{i_1 < \dots < i_{k-1} < j < i_{k+1} < \dots < i_m} \left| f_{i_1, \dots, i_{k-1}, j, i_{k+1}, \dots, i_m} \left(x_{<i_1}, x_{i_1}, \dots, x_{i_{k-1}}, x_j, y_{i_{k+1}}^{(k+1)}, \dots, y_{i_m}^{(m)} \right) \right|^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (2.45)$$

Then, for fixed $y^{(>k)} = (y^{(k+1)}, \dots, y^{(m)}) \in (\Omega^{\mathbb{N}})^{m-k}$,

$$\int_{\Omega^{\mathbb{N}}} \left(\sum_{i_1 < \dots < i_m} \left| f_i \left(x_{<i_1}, x_{i_1}, \dots, x_{i_k}, y_{i_{k+1}}^{(k+1)}, \dots, y_{i_m}^{(m)} \right) \right|^2 \right)^{\frac{1}{2}} dx \quad (2.46)$$

$$= \int_{\Omega^{\mathbb{N}}} \left(\sum_{j \in \mathbb{N}} \left| \varphi_j \left(x_{\leq j}, y^{(k+1)}, \dots, y^{(m)} \right) \right|^2 \right)^{\frac{1}{2}} dx \quad (2.47)$$

$$\simeq \int_{\Omega^{\mathbb{N}}} \int_{\Omega^{\mathbb{N}}} \left(\sum_{j \in \mathbb{N}} \left| \varphi_j \left(x_{<j}, y_j^{(k)}, y^{(k+1)}, \dots, y^{(m)} \right) \right|^2 \right)^{\frac{1}{2}} dx dy^{(k)} \quad (2.48)$$

$$= \int_{\Omega^{\mathbb{N}}} \int_{\Omega^{\mathbb{N}}} \left(\sum_{i_1 < \dots < i_m} \left| f_i \left(x_{<i_1}, x_{i_1}, \dots, x_{i_{k-1}}, y_{i_k}^{(k)}, y_{i_{k+1}}^{(k+1)}, \dots, y_{i_m}^{(m)} \right) \right|^2 \right)^{\frac{1}{2}} dx dy^{(k)}. \quad (2.49)$$

Here, i_k plays the role of j and (2.48) is an application of Theorem 2.9 to functions $|\varphi_j|^2$. Integrating the resulting inequality with respect to $y^{(>k)}$, we get

$$\begin{aligned} & \int_{(\Omega^{\mathbb{N}})^{m-k}} \int_{\Omega^{\mathbb{N}}} \left(\sum_{i_1 < \dots < i_m} \left| f_i \left(x_{<i_1}, x_{i_1}, \dots, x_{i_k}, y_{i_{k+1}}^{(k+1)}, \dots, y_{i_m}^{(m)} \right) \right|^2 \right)^{\frac{1}{2}} dx dy^{(\geq k+1)} \\ & \simeq \int_{(\Omega^{\mathbb{N}})^{m-k+1}} \int_{\Omega^{\mathbb{N}}} \left(\sum_{i_1 < \dots < i_m} \left| f_i \left(x_{<i_1}, x_{i_1}, \dots, x_{i_{k-1}}, y_{i_k}^{(k)}, y_{i_{k+1}}^{(k+1)}, \dots, y_{i_m}^{(m)} \right) \right|^2 \right)^{\frac{1}{2}} dx dy^{(\geq k)}, \end{aligned} \quad (2.50)$$

which by induction from $k = m$ to $k = 1$ proves (2.44). $\square$

3 Boundedness of P_m on $L^p(\Omega^{\mathbb{N}})$

The main motivation for this part is the following theorem, proved by Bourgain with $c_p \lesssim \frac{\hat{p}^{\frac{5}{2}}}{\ln \hat{p}}$ and by Kwapien with $c_p \lesssim \frac{\hat{p}}{\ln \hat{p}}$, where $\hat{p} = \max \left(p, \frac{p}{p-1} \right)$.

Theorem 3.1 [4, 14] P_m is bounded on $L^p(\Omega^{\mathbb{N}})$ for $1 < p < \infty$, with norm $\lesssim c_p^m$.

We will present a proof that yields $\|P_1 : L^p \odot\| < \infty$ and $c_p = e\|P_1 : L^p \odot\|$.

Proof Without loss of generality, we may assume that we are working in $L^p(\Omega^{[1, N]}, \mathcal{F}^{\otimes N})$. Indeed, by (2.33) and (2.37), P_m preserves $L^2(\Omega^{\mathbb{N}}, \mathcal{F}_{[1, N]})$, which can be canonically identified with $L^2(\Omega^{[1, N]}, \mathcal{F}^{\otimes N})$. Since the sequence $(L^2(\Omega^{\mathbb{N}}, \mathcal{F}_{[1, N]})) : N \in \mathbb{N}$ is increasing and its sum is dense in $L^p(\Omega^{\mathbb{N}}, \mathcal{F}^{\otimes \mathbb{N}})$, all we need to prove is

$$\lim_{N \rightarrow \infty} \|P_m : L^p(\Omega^{[1, N]}, \mathcal{F}^{\otimes N}) \odot\| \leq \frac{1}{e} c_p^m. \quad (3.1)$$

$P_0 = \mathbb{E}$ is bounded. The L^p boundedness of P_1 is essentially a known result [5], but we provide a proof for the sake of completeness. Let $(\mathcal{F}_k : k \in [0, N])$ be the natural filtration and $(\mathcal{F}_k^*)_{k=0}^N$ be the natural reversed filtration, i.e. $\mathcal{F}_k^* = \mathcal{F}_{[k, N]}$. By (2.32) and (2.37) we see that

$$P_1 = \sum_{k=1}^N P_{\{k\}}, \quad \Delta_k = \sum_{\max A=k} P_A, \quad \mathbb{E}_k^* = \sum_{A \subset [k, N]} P_A. \quad (3.2)$$

By mutual orthogonality of P_A 's

$$\Delta_k P_1 = P_{\{k\}} = \mathbb{E}_k^* \Delta_k. \quad (3.3)$$

Applying Theorem 2.3, (3.3) and Theorem 2.4, we obtain

$$\|P_1 f\|_{L^p} \simeq_p \left\| \left(\sum_{k=0}^N |\Delta_k P_1 f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \quad (3.4)$$

$$= \left\| \left(\sum_{k=0}^N |\mathbb{E}_k^* \Delta_k f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \quad (3.5)$$

$$\lesssim_p \left\| \left(\sum_{k=0}^N |\Delta_k f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \quad (3.6)$$

$$\simeq_p \|f\|_{L^p}. \quad (3.7)$$

We will now proceed by induction. Suppose that (3.1) is satisfied with $m-1$ in the place of m . Let $N = mn$ and define an operator Q_m acting on $L^p(\Omega^{[1, N]})$ by

$$Q_m = \frac{1}{\binom{N}{n}} \sum_{\substack{A \subset [1, N] \\ |A|=n}} \left((P_1 : L^p(\Omega^A) \odot) \otimes (P_{m-1} : L^p(\Omega^{[1, N] \setminus A}) \odot) \right). \quad (3.8)$$

Utilising (2.37) we get

$$(P_1 : L^p(\Omega^A) \odot) = \sum_{\substack{B_1 \subset A \\ |B_1|=1}} (P_{B_1} : L^p(\Omega^A) \odot), \quad (3.9)$$

$$(P_{m-1} : L^p(\Omega^{[1, N] \setminus A}) \odot) = \sum_{\substack{B_2 \subset [1, N] \setminus A \\ |B_2|=m-1}} (P_{B_2} : L^p(\Omega^{[1, N] \setminus A}) \odot). \quad (3.10)$$

By (2.31),

$$\left(P_{B_1} : L^p\left(\Omega^A\right) \odot\right) \otimes\left(P_{B_2} : L^p\left(\Omega^{[1, N] \backslash A}\right) \odot\right)=\left(P_{B_1 \cup B_2} : L^p\left(\Omega^{[1, N]}\right) \odot\right) . \quad (3.11)$$

Putting the last four equations together, we get

$$Q_m=\frac{1}{\binom{N}{n}} \sum_{\substack{A \subset[1, N] \\ |A|=n}} \sum_{\substack{B_1 \subset A, B_2 \subset[1, N] \backslash A \\ \left|B_1\right|=1,\left|B_2\right|=m-1}}\left(P_{B_1 \cup B_2} : L^p\left(\Omega^{[1, N]}\right) \odot\right) \quad (3.12)$$

$$=\frac{1}{\binom{N}{n}} \sum_{\substack{A \subset[1, N] \\ |A|=n}} \sum_{\substack{B \subset[1, N] \\ \left|B \cap A\right|=1,\left|B \backslash A\right|=m-1}}\left(P_B : L^p\left(\Omega^{[1, N]}\right) \odot\right) \quad (3.13)$$

$$=\frac{1}{\binom{N}{n}} \sum_{\substack{B \subset[1, N] \\ \left|B\right|=m}}|\{A \subset[1, N]:|A|=n,\left|B \cap A\right|=1\}| P_B=\frac{1}{\binom{N}{n}} \sum_{\substack{B \subset[1, N] \\ \left|B\right|=m}} m \binom{N-m}{n-1} P_B \quad (3.14)$$

$$=m \frac{\binom{N-m}{n-1}}{\binom{N}{n}} P_m . \quad (3.15)$$

However, by (3.8) and the induction hypothesis,

$$\left\|Q_m\right\| \leq \frac{1}{\binom{N}{n}} \sum_{\substack{A \subset[1, N] \\ |A|=n}}\left\|P_1: L^p\left(\Omega^A\right) \odot\right\| \cdot\left\|P_{m-1}: L^p\left(\Omega^{[1, N] \backslash A}\right) \odot\right\| \quad (3.16)$$

$$\leq\left\|P_1: L^p\left(\Omega^{[1, N]}\right) \odot\right\| \frac{c_p^{m-1}}{e} \quad (3.17)$$

$$=\frac{c_p^m}{e^2} . \quad (3.18)$$

Let $a_n \approx b_n$ denote $\lim _{n \rightarrow \infty} \frac{a_n}{b_n}=1$. By the Stirling formula,

$$\binom{n m}{n} \approx \frac{(2 \pi n m)^{\frac{1}{2}}\left(\frac{n m}{e}\right)^{n m}}{(2 \pi n)^{\frac{1}{2}}\left(\frac{n}{e}\right)^n(2 \pi n(m-1))^{\frac{1}{2}}\left(\frac{n(m-1)}{e}\right)^{n(m-1)}} \quad (3.19)$$

$$=\left(\frac{m}{2 \pi n(m-1)}\right)^{\frac{1}{2}} \frac{n^{n m} m^{n m}}{n^n n^{n(m-1)}(m-1)^{n(m-1)}} \quad (3.20)$$

$$=\left(\frac{m}{2 \pi n(m-1)}\right)^{\frac{1}{2}}\left(\frac{m^m}{(m-1)^{m-1}}\right)^n . \quad (3.21)$$

Thus

$$\frac{\binom{nm}{n}}{m \binom{(n-1)m}{n-1}} \approx \frac{1}{m} \left(\frac{n+1}{n} \right)^{\frac{1}{2}} \frac{m^m}{(m-1)^{(m-1)}} \quad (3.22)$$

$$\approx \left(\frac{m}{m-1} \right)^{m-1}. \quad (3.23)$$

Finally, by (3.15), (3.18) and (3.23),

$$\lim_{N \rightarrow \infty} \left\| P_m : L^p \left(\Omega^{[1,N]} \right) \circlearrowleft \right\| = \lim_{n \rightarrow \infty} \left\| P_m : L^p \left(\Omega^{[1,nm]} \right) \circlearrowleft \right\| \quad (3.24)$$

$$= \lim_{n \rightarrow \infty} \frac{\binom{nm}{n}}{m \binom{nm-m}{n-1}} \|Q_m\| \quad (3.25)$$

$$= \left(1 + \frac{1}{m-1} \right)^{m-1} \|Q_m\| \quad (3.26)$$

$$\leq \frac{c_p^m}{e}. \quad (3.27)$$

□

We provide a short proof of a fact taken from [5] that Theorem 3.1 can not be, extended to $p = 1$ or ∞ , which motivates the next section.

Proposition 3.2 *If Ω is not a single atom, then P_m for $m \geq 1$ is not bounded on $L^1(\Omega^\infty)$ or $L^\infty(\Omega^\infty)$.*

Proof It is enough to consider $L^1(\Omega^\infty)$, because P_m 's are self-adjoint. Let $f \in L^2(\Omega)$ be such that $\mathbb{E}f = 1$, $f \geq 0$ and $\mu(\text{supp } f) < 1$. Then $\mathbb{E}|f-1|^2 > 0$. For $F_n = f^{\otimes n} \in L^2(\Omega^n)$ we have

$$\|P_1 F_n\|_{L^1(\Omega^n)} = \int_{\Omega^n} \left| \sum_i (f(x_i) - 1) \right| dx \quad (3.28)$$

$$\simeq \int_{\Omega^n} \left(\sum_i |f(x_i) - 1|^2 \right)^{\frac{1}{2}} dx \quad (3.29)$$

$$\geq (n \mathbb{E}|f-1|^2)^{\frac{1}{2}} \quad (3.30)$$

which is not dominated by $\|F_n\|_{L^1(\Omega^n)} = \|f\|_{L^1(\Omega)}^n = 1$. To prove the unboundedness of P_m for $m > 1$, we simply notice that

$$P_m \left((f-1)^{\otimes(m-1)} \otimes F_n \right) = (f-1)^{\otimes(m-1)} \otimes P_1 F_n. \quad (3.31)$$

□

4 Boundedness of P_m on $H^1(\mathbb{D}^{\mathbb{N}})$

The projection P_m can be described even more explicitly in the case $\Omega = \mathbb{T}$. Indeed, if $n \in \mathbb{Z}^{\oplus \mathbb{N}}$ is supported on the set A , then

$$P_A e^{i\langle n, t \rangle} = \bigotimes_{j \in A} (\text{id} - \mathbb{E}) e^{in_j t_j} = \prod_{j \in A} e^{in_j t_j} = e^{i\langle n, t \rangle}. \quad (4.1)$$

Thus

$$e^{i\langle n, t \rangle} \in U_{\text{supp } n} \quad (4.2)$$

and

$$U_m = \overline{\text{span}} \left\{ e^{i\langle n, t \rangle} : |\text{supp } n| = m \right\}. \quad (4.3)$$

In particular, P_m preserves the space $H^2(\mathbb{D}^{\mathbb{N}}) = L^2(\mathbb{T}^{\mathbb{N}}) \cap H^1(\mathbb{D}^{\mathbb{N}})$.

In order to adapt the proof of Theorem 3.1 to the $H^1(\mathbb{D}^{\mathbb{N}})$ case, we will need a replacement for the argument proving that P_1 is bounded. The role of the combination of Burkholder–Gundy and Doob inequalities will be played by the following theorem, which can be found in [3].

Theorem 4.1 (Bourgain) *For $f \in H^1_{\text{last}}(\mathbb{T}^{\mathbb{N}})$, there is an equivalence of norms*

$$\|f\|_{L^1(\mathbb{T}^{\mathbb{N}})} \simeq \|f\|_{H^1[(\mathcal{F}_n)_{n=0}^{\infty}]}, \quad (4.4)$$

where $(\mathcal{F}_n)_{n=0}^{\infty}$ is the natural filtration on $\mathbb{T}^{\mathbb{N}}$.

For later use, we note the Hilbert space valued extension.

Corollary 4.2 *Let B be a Hilbert space. For $f \in H^1_{\text{last}}(\mathbb{T}^{\mathbb{N}}, B)$, there is an equivalence of norms*

$$\|f\|_{L^1(\mathbb{T}^{\mathbb{N}}, B)} \simeq \|f\|_{H^1[(\mathcal{F}_n)_{n=0}^{\infty}, B]} = \int_{\mathbb{T}^{\mathbb{N}}} \left(\sum_{k=0}^{\infty} \|\Delta_k f(t)\|_B^2 \right)^{\frac{1}{2}} dt, \quad (4.5)$$

where $(\mathcal{F}_n)_{n=0}^{\infty}$ is the natural filtration on $\mathbb{T}^{\mathbb{N}}$.

Proof Theorem 4.1 gives a map

$$T : H^1_{\text{last}}(\mathbb{T}^{\mathbb{N}}) \rightarrow L^1(\mathbb{T}^{\mathbb{N}}, \ell^2), \quad (4.6)$$

which is an isomorphism onto the subspace of $L^1(\mathbb{T}^{\mathbb{N}}, \ell^2)$ consisting of functions f such that f_k is a k -th martingale difference and is analytic in the k -th variable, defined by

$$Tf = (\Delta_k f)_{k=0}^{\infty}. \quad (4.7)$$

Thus, applying Lemma 2.8 with I_1 being a singleton, $I_2 = \mathbb{N}$, T as above (and then the same for T^{-1}) we get

$$\|f\|_{H^1_{\text{last}}(\mathbb{T}^{\mathbb{N}}, B)} \simeq \|(T \otimes \text{id}_B) f\|_{L^1(\mathbb{T}^{\mathbb{N}}, B)} = \int_{\mathbb{T}^{\mathbb{N}}} \left(\sum_k \|\Delta_k f(t)\|_B^2 \right)^{\frac{1}{2}} dt. \quad (4.8)$$

□

The role of the Stein martingale inequality will be played by the following simple observation.

Corollary 4.3 *For any sequence $(f_n : n \in \mathbb{N})$ adapted to the natural filtration on $\Omega^{\mathbb{N}}$,*

$$\mathbb{E} \left(\sum_{n=1}^{\infty} |f_n|^2 \right)^{\frac{1}{2}} \gtrsim \mathbb{E} \left(\sum_{n=1}^{\infty} |\mathbb{E}_{\{n\}} f_n|^2 \right)^{\frac{1}{2}}. \quad (4.9)$$

Proof Let $\tilde{f}_n$ be a sequence of functions on $\Omega^{\mathbb{N}} \times \Omega^{\mathbb{N}}$ defined by

$$\tilde{f}_n(x, y) = f_n(x_1, \dots, x_{n-1}, y_n). \quad (4.10)$$

Applying Theorem 2.9 and conditional expectation with respect to the second of two sets of variables,

$$\mathbb{E} \left(\sum_{n=1}^{\infty} |f_n|^2 \right)^{\frac{1}{2}} \gtrsim \mathbb{E} \left(\sum_{n=1}^{\infty} |\tilde{f}_n|^2 \right)^{\frac{1}{2}} \quad (4.11)$$

$$\geq \mathbb{E} \left(\sum_{n=1}^{\infty} |(\mathbb{E} \otimes \text{id}) \tilde{f}_n|^2 \right)^{\frac{1}{2}} \quad (4.12)$$

$$= \mathbb{E} \left(\sum_{n=1}^{\infty} |1 \otimes \mathbb{E}_{\{n\}} f_n|^2 \right)^{\frac{1}{2}} \quad (4.13)$$

$$= \mathbb{E} \left(\sum_{n=1}^{\infty} |\mathbb{E}_{\{n\}} f_n|^2 \right)^{\frac{1}{2}}. \quad (4.14)$$

□

By conditioning with respect to the first set of variables, we obtain the inequality

$$\mathbb{E} \left(\sum |f_n|^2 \right)^{\frac{1}{2}} \gtrsim \mathbb{E} \left(\sum |\mathbb{E}_{n-1} f_n|^2 \right)^{\frac{1}{2}} \quad (4.15)$$

due to Lepingle [17].

Theorem 4.4 *For any Ω , P_1 is bounded on $H^1[(\mathcal{F}_n)_{n=0}^{\infty}]$.*

Proof We proceed as in the proof of Theorem 3.1. First, we reduce the problem to the $\Omega^{[1, N]}$ realm. Then we notice that

$$\Delta_k P_1 = P_{\{k\}} = \mathbb{E}_{\{k\}} \Delta_k, \quad (4.16)$$

which by Corollary 4.3 yields

$$\|P_1 f\|_{H^1} = \left\| \left(\sum_{k=0}^N |\Delta_k P_1 f|^2 \right)^{\frac{1}{2}} \right\|_{L^1} \quad (4.17)$$

$$= \left\| \left(\sum_{k=0}^N |\mathbb{E}_{\{k\}} \Delta_k f|^2 \right)^{\frac{1}{2}} \right\|_{L^1} \quad (4.18)$$

$$\lesssim \left\| \left(\sum_{k=0}^N |\Delta_k f|^2 \right)^{\frac{1}{2}} \right\|_{L^1} \quad (4.19)$$

$$= \|f\|_{H^1}. \quad (4.20)$$

□

Theorem 4.5 P_m is bounded on $H^1(\mathbb{D}^{\mathbb{N}})$ with norm $\leq \frac{1}{e} c_1^m$, where

$$c_1 = e \left\| P_1 : H^1(\mathbb{D}^{\mathbb{N}}) \circlearrowright \right\|. \quad (4.21)$$

Proof The case $m = 0$ is trivial, $m = 1$ follows directly from Theorems 4.1 and 4.4. The induction step is identical to the proof of Theorem 3.1, up to changing $L^p(\Omega^I)$ to $H^1(\mathbb{D}^I)$. Alternatively, we can prove the same in a single step. Set

$$Q_{m,n} = \frac{1}{\binom{nm}{n,\dots,n}} \sum_{\substack{A_1 \cup \dots \cup A_m = [1, nm] \\ A_i \text{'s disjoint} \\ |A_1| = \dots = |A_m| = n}} \bigotimes_{i=1}^m \left(P_1 : H^1(\mathbb{D}^{A_i}) \circlearrowright \right). \quad (4.22)$$

It is easily seen that for each set B of cardinality m , P_B appears $m! \binom{(n-1)m}{n-1, \dots, n-1}$ times in the sum. Therefore

$$\|P_1 : H^1(\mathbb{D}^{\infty})\|^m \geq \|Q_{m,n}\| \quad (4.23)$$

$$= \frac{m! \binom{(n-1)m}{n-1, \dots, n-1}}{\binom{nm}{n, \dots, n}} \|P_m : H^1(\mathbb{D}^{nm}) \circlearrowright\| \quad (4.24)$$

and since

$$\lim_{n \rightarrow \infty} \frac{\binom{nm}{n, \dots, n}}{m! \binom{(n-1)m}{n-1, \dots, n-1}} = \lim_{n \rightarrow \infty} \frac{\frac{(nm)!}{n!^m}}{m! \frac{(nm-m)!}{(n-1)!^m}} \quad (4.25)$$

$$= \lim_{n \rightarrow \infty} \frac{nm}{m} n^{-m} \quad (4.26)$$

$$= \frac{m^m}{m!} \quad (4.27)$$

we get

$$\|P_m : H^1(\mathbb{D}^{\infty}) \circlearrowright\| \leq \frac{m^m}{m!} \|P_1 : H^1(\mathbb{D}^{\infty})\|^m \leq \frac{1}{(2\pi m)^{\frac{1}{2}}} (e \|P_1 : H^1(\mathbb{D}^{\infty})\|)^m. \quad (4.28)$$

□

It has to be noted that our proofs of Theorems 3.1 and 4.5 extend naturally to a vector valued case, respectively UMD and AUMD valued. Indeed, Bourgain's proof of Theorem 2.4, as presented in [20], extends to the UMD valued version, while Theorem 4.1 is just the statement that a one-dimensional space has the AUMD property. In both cases, the induction follows without change. There is also a second direction in which we can generalize. Namely, by looking carefully at the proof of Theorem 4.1, one can see that the only place in which

analyticity plays a role is the $H^1 = H^2 \cdot H^2$ theorem, which is true for H^1 on any compact and connected group with ordered dual [22], which means that we can replace $\mathbb{T}$ with any such group.

Given that Kwapien's constant c_p in Theorem 3.1 has the best known asymptotics as a function of p for $m = 1$, one can ask about the dependence of $\|P_m : L^p(\Omega^\infty) \odot\|$ and $\|P_m : H^1(\mathbb{D}^\mathbb{N}) \odot\|$ on m .

Proposition 4.6 *The inequalities*

$$\|P_m : L^p(\Omega^\infty) \odot\| \geq \|P_1 : L_0^p(\Omega^\infty) \odot\|^m \quad (4.29)$$

for nontrivial Ω and

$$\|P_m : H^1(\mathbb{D}^\mathbb{N}) \odot\| \geq \|P_1 : H_0^1(\mathbb{D}^\mathbb{N}) \odot\|^m, \quad (4.30)$$

where L_0^p and H_0^1 stand for functions of mean 0, are true. Also,

$$\|P_1 : H_0^1(\mathbb{D}^\mathbb{N}) \odot\| > 1. \quad (4.31)$$

Proof Let $f \in L^p(\Omega^n)$ be of mean 0. Then $f^{\otimes m} \in L^p(\Omega^{mn})$ and

$$(P_m : L^p(\Omega^{mn}) \odot)(f^{\otimes m}) = ((P_1 : L^p(\Omega^n) \odot)(f))^{\otimes m}. \quad (4.32)$$

Indeed, we have $f = \sum_{|A| \geq 1} P_A f$ because $P_\emptyset f = \mathbb{E}f = 0$, hence

$$f^{\otimes m} = \sum_{\substack{A_i \subset [n(i-1)+1, ni] \\ |A_i| \geq 1 \text{ for } i=1, \dots, m}} \bigotimes_{i=1}^m P_{A_i} f \quad (4.33)$$

The only way to get a summand in U_m is to have $|A_i| = 1$ for all i and the sum of such summands is the right hand side of (4.32). Taking an f which is close to attaining the norm of P_1 on a respective space proves (4.29) and (4.30).

In order to see (4.31), assume for the sake of contradiction that P_1 is a contraction on $H_0^1(\mathbb{D}^2)$. We will test it on functions of the form $\mathbb{T}^2 \ni (w, z) \mapsto F(z) + w + azw \in \mathbb{C}$, where $F \in H_0^1(\mathbb{D})$ and a is a scalar. It is easy to see that

$$\mathbb{E}|\alpha + \beta w| = \mathbb{E}|\alpha| + |\beta|w| \quad (4.34)$$

for $\alpha, \beta \in \mathbb{C}$. Hence, from the inequality

$$\mathbb{E}|F(z) + w(1 + az)| \geq \mathbb{E}|F(z) + w| \quad (4.35)$$

we get

$$\mathbb{E}|F(z)| + w(1 + az)| \geq \mathbb{E}|F(z)| + w|. \quad (4.36)$$

Since any nonnegative function can be approximated by the modulus of an $H_0^1(\mathbb{D})$ function, (4.35) is true for any nonnegative F . In particular, the left hand side attains a local minimum at $a = 0$, so by $|u + v| = |u| + \operatorname{Re} \frac{u\bar{v}}{|u|} + o(v)$ we infer that

$$\operatorname{Re} \mathbb{E} \frac{(F(z) + w)\bar{w}z}{|F(z) + w|} = 0. \quad (4.37)$$

Now let

$$\phi(r) = \mathbb{E} \frac{1 + r\bar{w}}{|1 + r\bar{w}|} \quad (4.38)$$

for $r \geq 0$. This is a continuous function, whose values lie on some curve γ connecting 0 and 1 (because $\phi(0) = 1$ and $\lim_{r \rightarrow \infty} \phi(r) = 0$). The condition (4.37) can be rewritten as

$$\operatorname{Re} \mathbb{E} \bar{z} \phi(F(z)) = 0. \quad (4.39)$$

Since F was allowed to be any positive function, $\phi(F)$ can be any function with values in γ , making (4.39) obviously false. $\square$

5 Martingale Hardy spaces

5.1 Double indexed martingales

Above we noticed that the boundedness of P_1 on $H^1(\mathbb{D}^{\mathbb{N}})$ follows from the boundedness of P_1 on a bigger space $H^1[(\mathcal{F}_n)]$. It is tempting to find an abstract martingale inequality responsible for the boundedness of P_m on $H^1(\mathbb{D}^{\mathbb{N}})$. We can do this for $m = 2$.

By the natural double-indexed filtration on $\Omega^{\mathbb{N}}$ we will mean the family $(\mathcal{F}_{[a,b]} : a \leq b)$ (note that the inclusion order in the first index is reversed). Let $\Delta_n = \mathbb{E}_n - \mathbb{E}_{n-1}$ be the martingale differences with respect to $(\mathcal{F}_n)$ and $\Delta_n^* = \mathbb{E}_n^* - \mathbb{E}_{n+1}^*$ be the martingale differences with respect to $(\mathcal{F}_n^*)$, where $\mathcal{F}_n^* = \mathcal{F}_{[n,\infty)}$. We define the martingale differences with respect to $(\mathcal{F}_{[a,b]})$ by

$$\Delta_{[a,b]} = \Delta_a^* \Delta_b = \mathbb{E}_{[a+1,b-1]} + \mathbb{E}_{[a,b]} - \mathbb{E}_{[a+1,b]} - \mathbb{E}_{[a,b-1]} \quad (5.1)$$

and an H^1 norm for this filtration by

$$\|f\|_{H^1[(\mathcal{F}_{[a,b]})]} = \mathbb{E} \left(|\mathbb{E} f|^2 + \sum_{1 \leq a \leq b} |\Delta_{[a,b]} f|^2 \right)^{\frac{1}{2}}. \quad (5.2)$$

The definition of double martingale differences coincides with what is considered in [25].

Corollary 5.1 *For $f \in H^1(\mathbb{D}^{\mathbb{N}})$, there is an equivalence of norms*

$$\|f\|_{H^1(\mathbb{D}^{\mathbb{N}})} \simeq \|f\|_{H^1[(\mathcal{F}_{[a,b]})_{a \leq b}]}, \quad (5.3)$$

where $(\mathcal{F}_{[a,b]})_{a \leq b}$ is the natural double-indexed filtration on $\mathbb{T}^{\mathbb{N}}$.

Proof For any ± 1 -valued sequence $(\varepsilon_n : n \in \mathbb{N})$, we define operators S_ε and S_ε^* by

$$S_\varepsilon f = \mathbb{E} f + \sum_{n=1}^{\infty} \varepsilon_n \Delta_n f, \quad S_\varepsilon^* f = \mathbb{E} f + \sum_{n=1}^{\infty} \varepsilon_n \Delta_n^* f. \quad (5.4)$$

By Theorem 4.1, S_ε is an isomorphism from $H^1(\mathbb{D}^{\mathbb{N}})$ to itself, uniformly in ε . By reversing the order of variables, the same can be said about S_ε^* . Thus for any $\varepsilon, \varepsilon'$,

$$\|f\|_{H^1(\mathbb{D}^{\mathbb{N}})} \simeq \|S_\varepsilon S_{\varepsilon'}^* f\|_{H^1(\mathbb{D}^{\mathbb{N}})} \quad (5.5)$$

$$= \left\| \mathbb{E} f + \sum_{a \leq b} \varepsilon_a \varepsilon'_b \Delta_a^* \Delta_b f \right\|_{H^1(\mathbb{D}^{\mathbb{N}})} \quad (5.6)$$

$$= \mathbb{E} \left| \mathbb{E} f + \sum_{a \leq b} \varepsilon_a \varepsilon'_b \Delta_{[a,b]} f \right|. \quad (5.7)$$

By averaging the last quantity over all choices of $\varepsilon, \varepsilon'$ and applying the Khintchine-Kahane inequality twice, we get the desired inequalities. $\square$

Theorem 5.2 P_2 is bounded on $H^1 \left[(\mathcal{F}_{a,b})_{a \leq b} \right]$, for any Ω .

Proof As usual, we reduce the problem to the $\Omega^{[1,N]}$ version. By (3.2),

$$\Delta_{[a,b]} = \sum_{\substack{\min A=a \\ \max A=b}} P_A. \quad (5.8)$$

Thus

$$\Delta_{[a,b]} P_2 = P_{[a,b]} = \mathbb{E}_{[a,b]} \Delta_{[a,b]} \quad (5.9)$$

for $a < b$. We can assume that $P_0 f = P_1 f = 0$ (i.e. $\mathbb{E} f = 0$ and $\Delta_{[a,a]} f = 0$ for all a), because $U_{\leq 1}$, being the image of $\mathbb{E} + \sum_a \Delta_{[a,a]}$ is trivially complemented in the underlying norm and P_2 is 0 on $U_{\leq 1}$. By applying Corollary 2.10,

$$\|f\|_{H^1} = \mathbb{E} \left(\sum_{a < b} |\Delta_{[a,b]} f|^2 \right)^{\frac{1}{2}} \quad (5.10)$$

$$= \int_{\Omega^N} \left(\sum_{a < b} |\Delta_{[a,b]} f(x_{[a,b]})|^2 \right)^{\frac{1}{2}} dx \quad (5.11)$$

$$\simeq \int_{(\Omega^N)^3} \left(\sum_{a < b} |\Delta_{[a,b]} f(z_a, x_{[a+1, b-1]}, y_b)|^2 \right)^{\frac{1}{2}} dx dy dz \quad (5.12)$$

$$\geq \int_{(\Omega^N)^2} \left(\sum_{a < b} \left| \int_{\Omega^N} \Delta_{[a,b]} f(z_a, x_{[a+1, b-1]}, y_b) dx \right|^2 \right)^{\frac{1}{2}} dy dz \quad (5.13)$$

$$\simeq \mathbb{E} \left(\sum_{a < b} |\mathbb{E}_{[a,b]} \Delta_{[a,b]} f|^2 \right)^{\frac{1}{2}} \quad (5.14)$$

$$= \mathbb{E} \left(\sum_{a < b} |\Delta_{[a,b]} P_2 f|^2 \right)^{\frac{1}{2}} \quad (5.15)$$

$$= \|P_2 f\|_{H^1} \quad (5.16)$$

as desired. $\square$

5.2 Multiple indexed martingales

We will make an attempt at generalizing the above for multiple indexed martingales. Suppose there is a family $(T_i, \partial T_i)_{i \in \mathcal{I}}$ of pairs of finite subsets of some set X (finite or not) indexed by some set $\mathcal{I}$, such that $\partial T_i \subseteq T_i$ (∂T_i is not a boundary in a topological sense - we use this

notation for resemblance with the case where T_i are intervals and ∂T_i are their endpoints). We would like to define operators Δ_i on $L^2(\Omega^X)$ by the formula

$$\Delta_i = (\text{id} - \mathbb{E})^{\otimes \partial T_i} \otimes \text{id}^{\otimes T_i \setminus \partial T_i} \otimes \mathbb{E}^{\otimes T'_i}, \quad (5.17)$$

where T'_i stands for the complement of T_i in X . This is supposed to mimic the standard martingale differences when $X = \mathbb{N}$, $\mathcal{I} = \mathbb{N}$, $T_i = [0, i]$, $\partial T_i = \{i\}$ and double martingale differences when $\mathcal{I} = \{(a, b) : a \leq b\}$, $T_{a,b} = [a, b]$, $\partial T_{a,b} = \{a, b\}$. The natural condition

$$\sum_i \Delta_i f = f \quad (5.18)$$

is guaranteed by

$$\text{for any finite } A \subset X, \text{ there exists unique } i \in \mathcal{I} \text{ such that } \partial T_i \subseteq A \subseteq T_i. \quad (5.19)$$

Indeed,

$$\Delta_i = (\text{id} - \mathbb{E})^{\otimes \partial T_i} \otimes \text{id}^{\otimes T_i \setminus \partial T_i} \otimes \mathbb{E}^{\otimes T'_i} \quad (5.20)$$

$$= (\text{id} - \mathbb{E})^{\otimes \partial T_i} \otimes \mathbb{E}^{\otimes T'_i} \otimes \sum_{B \subseteq T_i \setminus \partial T_i} \left(P_B : L^2(\Omega^{T_i \setminus \partial T_i}) \right) \odot \quad (5.21)$$

$$= (\text{id} - \mathbb{E})^{\otimes \partial T_i} \otimes \mathbb{E}^{\otimes T'_i} \otimes \sum_{B \subseteq T_i \setminus \partial T_i} (\text{id} - \mathbb{E})^{\otimes B} \otimes \mathbb{E}^{\otimes T_i \setminus (\partial T_i \cup B)} \quad (5.22)$$

$$= \sum_{B \subseteq T_i \setminus \partial T_i} P_{\partial T_i \cup B}. \quad (5.23)$$

Hence

$$\sum_{i \in \mathcal{I}} \Delta_i = \sum_{i \in \mathcal{I}} \sum_{B \subseteq T_i \setminus \partial T_i} P_{\partial T_i \cup B} \quad (5.24)$$

and each P_A appears in the above sum exactly once if and only if the condition (5.19) is satisfied. For a family $(T_i, \partial T_i)_{i \in \mathcal{I}}$ we may define a norm by the formula

$$\|f\|_{H^1[(T_i, \partial T_i)_{i \in \mathcal{I}}]} = \mathbb{E} \left(\sum_{i \in \mathcal{I}} |\Delta_i f|^2 \right)^{\frac{1}{2}} \quad (5.25)$$

and ask the following:

- Is it true that

$$\|f\|_{H^1[(T_i, \partial T_i)_{i \in \mathcal{I}}]} \simeq \|f\|_{H^1(\mathbb{D}^{\mathbb{N}})} \quad (5.26)$$

for $f \in \|f\|_{H^1(\mathbb{D}^{\mathbb{N}})}$?

- If yes, is there any interesting example of a set $\mathbb{N}^{\oplus \infty} \subset \Gamma \subset \mathbb{Z}^{\oplus \infty}$ such that (5.26) is true for $f \in L^1(\mathbb{T}^{\infty})$ with $\text{supp } \hat{f} \subset \Gamma$?
- For which, if any, m is P_m bounded on $H^1[(T_i, \partial T_i)_{i \in \mathcal{I}}]$?

We are able to answer them in the case when

$$\mathcal{I} = \{A \subset \mathbb{N} : |A| \leq m\} \quad (5.27)$$

$$\partial T_A = A, \quad T_A = \begin{cases} A & \text{if } |A| < m, \\ (0, \min A) \cup A & \text{if } |A| = m. \end{cases} \quad (5.28)$$

For a finite set $B \subset \mathbb{N}$, the unique $A \in \mathcal{I}$ such that $\partial T_A \subseteq B \subseteq T_A$, which we will denote by ∂B , is

$$\partial B = \begin{cases} B & \text{if } |B| < m, \\ m \text{ last elements of } B & \text{if } |B| \geq m. \end{cases} \quad (5.29)$$

Theorem 5.3 *Let $m \geq 1$ be fixed and $(T_i, \partial T_i)_{i \in \mathcal{I}}$ be defined by (5.27), (5.28). Then*

$$\|f\|_{H_{m \text{ last}}^1(\mathbb{T}^{\mathbb{N}})} \simeq \|f\|_{H^1[(T_i, \partial T_i)_{i \in \mathcal{I}}]} \quad (5.30)$$

for $f \in H_{m \text{ last}}^1(\mathbb{T}^{\mathbb{N}})$, where $\mathbb{T}$ is used as Ω . Moreover, for $m' \in \mathbb{N}$ and nontrivial Ω , the following are equivalent.

- (i) $m' \leq m$
- (ii) $P_{m'}$ is bounded on $H^1[(T_i, \partial T_i)_{i \in \mathcal{I}}]$
- (iii) $P_{m'}$ is bounded on $H_{m \text{ last}}^1(\mathbb{T}^{\mathbb{N}})$.

Proof For $A, B \subset \mathbb{N}$, we will write $T_A^{(m)}, \partial T_A^{(m)}, \Delta_A^{(m)}, \partial^{(m)} B$ to indicate the value of m we are currently using. For brevity we will denote $(T_A^{(m)}, \partial T_A^{(m)})_{|A| \leq m}$ by $\mathcal{T}_m$. For $|A| < m$, we have $\Delta_A^{(m)} = (\text{id} - \mathbb{E})^{\otimes A} \otimes \mathbb{E}^{\otimes \mathbb{N} \setminus A} = P_A$. In particular, $\Delta_\emptyset^{(m)} = \mathbb{E}$. Therefore, by definition of the $H^1[\mathcal{T}_m]$ norm and Corollary 2.11,

$$\begin{aligned} \|f\|_{H^1[\mathcal{T}_m]} &\simeq \int_{\Omega^{\mathbb{N}}} \int_{(\Omega^{\mathbb{N}})^m} \left(\sum_{i_1 < \dots < i_m} \left| \Delta_i^{(m)} f(x_{<i_1}, y_{i_1}^{(1)}, \dots, y_{i_m}^{(m)}) \right|^2 \right)^{\frac{1}{2}} dy^{(1, \dots, m)} dx \\ &\quad + \sum_{0 < s < m} \int_{(\Omega^{\mathbb{N}})^s} \left(\sum_{i_1 < \dots < i_s} \left| P_i f(y_{i_1}^{(1)}, \dots, y_{i_s}^{(s)}) \right|^2 \right)^{\frac{1}{2}} dy^{(1, \dots, s)} + |\mathbb{E}f|. \end{aligned} \quad (5.31)$$

Here, we identify an increasing sequence with the set of its elements, write $dy^{(1, \dots, m)}$ to denote $dy^{(1)} \dots dy^{(m)}$ and treat $\Delta_i^{(m)} f$ as a function on $\Omega^{[1, i_1-1]} \times \Omega^{i_1, \dots, i_m}$. From this expression, we immediately see the implication (i) $\implies$ (ii). Indeed, for $m' < m$,

$$\Delta_A^{(m)} P_{m'} = \begin{cases} \Delta_A^{(m)} & \text{if } |A| = m' \\ 0 & \text{if } |A| \neq m', \end{cases} \quad (5.32)$$

which trivializes the inequality $\|f\|_{H^1[\mathcal{T}_m]} \gtrsim \|P_{m'} f\|_{H^1[\mathcal{T}_m]}$. For $m' = m$, we notice that

$$\Delta_A^{(m)} P_m = \begin{cases} \mathbb{E}^{\otimes [1, \min A-1]} \Delta_A^{(m)} & \text{if } |A| = m \\ 0 & \text{if } |A| \neq m, \end{cases} \quad (5.33)$$

and the desired inequality follows from

$$\|f\|_{H^1[\mathcal{T}_m]} \geq \int_{\Omega^{\mathbb{N}}} \int_{(\Omega^{\mathbb{N}})^m} \left(\sum_{i_1 < \dots < i_m} \left| \Delta_i^{(m)} f(x_{<i_1}, y_{i_1}^{(1)}, \dots, y_{i_m}^{(m)}) \right|^2 \right)^{\frac{1}{2}} dy^{(1, \dots, m)} dx \quad (5.34)$$

$$\geq \int_{(\Omega^{\mathbb{N}})^m} \left(\sum_{i_1 < \dots < i_m} \left| \int_{\Omega^{\mathbb{N}}} \Delta_i^{(m)} f(x_{<i_1}, y_{i_1}^{(1)}, \dots, y_{i_m}^{(m)}) dx \right|^2 \right)^{\frac{1}{2}} dy^{(1, \dots, m)} \quad (5.35)$$

$$= \int_{(\Omega^{\mathbb{N}})^m} \left(\sum_{i_1 < \dots < i_m} \left| \Delta_i^{(m)} P_m f(x_{<i_1}, y_{i_1}^{(1)}, \dots, y_{i_m}^{(m)}) \right|^2 \right)^{\frac{1}{2}} dy^{(1, \dots, m)} \quad (5.36)$$

$$= \|P_m f\|_{H^1[\mathcal{T}_m]}. \quad (5.37)$$

The implication (ii) $\implies$ (iii) follows from (5.30), which we will prove by induction with respect to m . For $m = 1$ this is just Theorem 4.1. Suppose it is true for some m and let $f \in H_{m+1 \text{ last}}^1(\mathbb{T}^{\mathbb{N}})$. In particular, $f \in H_{m \text{ last}}^1(\mathbb{T}^{\mathbb{N}})$. By (5.30), which is now the induction hypothesis, and (5.31),

$$\begin{aligned} \|f\|_{L^1(\mathbb{T}^{\mathbb{N}})} &\simeq \int_{\mathbb{T}^{\mathbb{N}}} \int_{(\mathbb{T}^{\mathbb{N}})^m} \left(\sum_{i_1 < \dots < i_m} \left| \Delta_i^{(m)} f(x_{<i_1}, y_{i_1}^{(1)}, \dots, y_{i_m}^{(m)}) \right|^2 \right)^{\frac{1}{2}} dy^{(1, \dots, m)} dx \\ &\quad + \sum_{0 < s < m} \int_{(\mathbb{T}^{\mathbb{N}})^s} \left(\sum_{i_1 < \dots < i_s} \left| P_i f(y_{i_1}^{(1)}, \dots, y_{i_s}^{(s)}) \right|^2 \right)^{\frac{1}{2}} dy^{(1, \dots, s)} + |\mathbb{E}f|. \end{aligned} \quad (5.38)$$

The last two summands are as they are in the desired expression for $m + 1$ instead of m and we only have to deal with the first. For any $i_1 < \dots < i_m$ and $t \in \mathbb{T}^{\mathbb{N}}$ we have

$$\Delta_{i_1, \dots, i_m}^{(m)} f(t) = \sum_{\partial^{(m)} \text{supp } n = i} \widehat{f}(n) e^{i \langle n, t \rangle}. \quad (5.39)$$

Thus, treating $\Delta_i^{(m)} f$ as a function on $\mathbb{T}^{[1, i_1-1]} \times \mathbb{T}^{\{i_1, \dots, i_m\}}$,

$$\Delta_{i_1, \dots, i_m}^{(m)} f(x_{<i_1}, y_1, \dots, y_m) = \sum_{\partial^{(m)} \text{supp } n = i} \widehat{f}(n) e^{i \sum_{j < i_1} n_j x_j + i \sum_{1 \leq j \leq m} n_{i_j} y_j}. \quad (5.40)$$

Let y be fixed and $\Delta_k^{(1)}$, where $k \in \mathbb{N} \cup \{\emptyset\}$, act with respect to the variable $x \in \mathbb{T}^{\mathbb{N}}$ (so, technically, $\Delta_k^{(1)}$ stands for $\Delta_k^{(1)} \otimes \text{id}$). Then

$$\Delta_{\emptyset}^{(1)} \Delta_{i_1, \dots, i_m}^{(m)} f(x_{<i_1}, y_1, \dots, y_m) = P_i f(y_1, \dots, y_m) \quad (5.41)$$

and

$$\Delta_k^{(1)} \Delta_{i_1, \dots, i_m}^{(m)} f(x_{<i_1}, y_1, \dots, y_m) = 0 \quad \text{for } k \geq i_1. \quad (5.42)$$

For $k < i_1$,

$$\Delta_k^{(1)} \Delta_{i_1, \dots, i_m}^{(m)} f(x_{<i_1}, y_1, \dots, y_m) \quad (5.43)$$

$$= \sum_{\partial^{(m)} \text{supp } n=i} \widehat{f}(n) e^{i \sum_{1 \leq j \leq m} n_{ij} y_j} \Delta_k^{(1)} e^{i \sum_{j < i_1} n_{j \cdot j} x_j} \quad (5.44)$$

$$= \sum_{\substack{\partial^{(m)} \text{supp } n=i \\ \max(\text{supp } n \setminus i)=k}} \widehat{f}(n) e^{i \sum_{1 \leq j \leq m} n_{ij} y_j + i \sum_{j \leq k} n_{j \cdot j} x_j} \quad (5.45)$$

$$= \sum_{\partial^{(m+1)} \text{supp } n=\{k, i_1, \dots, i_m\}} \widehat{f}(n) e^{i \sum_{j < k} n_{j \cdot j} x_j + i n_{kk} x_k + i \sum_{1 \leq j \leq m} n_{ij} y_j} \quad (5.46)$$

$$= \Delta_{k, i_1, \dots, i_m}^{(m+1)} f(x_{<k}, x_k, y_1, \dots, y_m). \quad (5.47)$$

By (5.40), $\Delta_{i_1, \dots, i_m}^{(m)} f(x_{<i_1}, y_1, \dots, y_m)$ is in $H_{\text{last}}^1(\mathbb{T}^{\mathbb{N}})$ with respect to x . Therefore, applying Corollary 4.2 to the vector valued function $x \mapsto \left(\Delta_i^{(m)} f(x_{<i_1}, y_{i_1}^{(1)}, \dots, y_{i_m}^{(m)}) \right)_{i_1 < \dots < i_m}$ with fixed $y^{(1, \dots, m)}$, plugging in (5.41), (5.42), (5.47) and using Corollary 2.11, we get

$$\int_{\mathbb{T}^{\mathbb{N}}} \left(\sum_{i_1 < \dots < i_m} \left| \Delta_i^{(m)} f(x_{<i_1}, y_{i_1}^{(1)}, \dots, y_{i_m}^{(m)}) \right|^2 \right)^{\frac{1}{2}} dx \quad (5.48)$$

$$= \int_{\mathbb{T}^{\mathbb{N}}} \left\| \left(\Delta_i^{(m)} f(x_{<i_1}, y_{i_1}^{(1)}, \dots, y_{i_m}^{(m)}) \right)_{i_1 < \dots < i_m} \right\|_{\ell^2} dx \quad (5.49)$$

$$\simeq \left\| \left(\Delta_{\emptyset}^{(1)} \Delta_i^{(m)} f(x_{<i_1}, y_{i_1}^{(1)}, \dots, y_{i_m}^{(m)}) \right)_{i_1 < \dots < i_m} \right\|_{\ell^2} \quad (5.50)$$

$$+ \int_{\mathbb{T}^{\mathbb{N}}} \left(\sum_k \left\| \left(\Delta_k^{(1)} \Delta_i^{(m)} f(x_{<i_1}, y_{i_1}^{(1)}, \dots, y_{i_m}^{(m)}) \right)_{i_1 < \dots < i_m} \right\|_{\ell^2}^2 \right)^{\frac{1}{2}} dx \quad (5.51)$$

$$= \left\| \left(P_{i_1, \dots, i_m} f(y_{i_1}^{(1)}, \dots, y_{i_m}^{(m)}) \right)_{i_1 < \dots < i_m} \right\|_{\ell^2} \quad (5.52)$$

$$+ \int_{\mathbb{T}^{\mathbb{N}}} \left(\sum_{k < i_1 < \dots < i_m} \left| \Delta_{k, i_1, \dots, i_m}^{(m+1)} f(x_{<k}, x_k, y_{i_1}^{(1)}, \dots, y_{i_m}^{(m)}) \right|^2 \right)^{\frac{1}{2}} dx \quad (5.53)$$

$$= \left(\sum_{i_1 < \dots < i_m} \left| P_{i_1, \dots, i_m} f(y_{i_1}^{(1)}, \dots, y_{i_m}^{(m)}) \right|^2 \right)^{\frac{1}{2}} \quad (5.54)$$

$$+ \int_{\mathbb{T}^{\mathbb{N}}} \int_{\mathbb{T}^{\mathbb{N}}} \left(\sum_{k < i_1 < \dots < i_m} \left| \Delta_{k, i_1, \dots, i_m}^{(m+1)} f(x_{<k}, y_k^{(0)}, y_{i_1}^{(1)}, \dots, y_{i_m}^{(m)}) \right|^2 \right)^{\frac{1}{2}} dx dy^{(0)}. \quad (5.55)$$

Integrating the resulting equivalence with respect to $y^{(1, \dots, m)}$ and plugging into (5.38), we verify that $\|f\|_{L^1(\mathbb{T}^{\mathbb{N}})} \simeq H^1[\mathcal{T}_{m+1}]$, which finishes the proof of (5.30).

In order to see that (iii) $\implies$ (i), let us take $m' > m$. For any $g \in L^1(\mathbb{T}^{\mathbb{N}})$, the function $G \in L^1(\mathbb{T}^{\mathbb{N}})$ defined by

$$G(t) = g(t_1, \dots, t_n) e^{i \sum_{j=n+1}^{n+m} t_j} \quad (5.56)$$

is in $H_{m \text{ last}}^1(\mathbb{T}^{\mathbb{N}})$. But

$$(P_{m'}G)(t) = (P_{m'-m}g)(t_1, \dots, t_n) e^{i \sum_{j=n+1}^{n+m} t_j}, \quad (5.57)$$

so

$$\|P_{m'} : H_{m \text{ last}}^1(\mathbb{T}^{\mathbb{N}}) \circlearrowleft\| \geq \frac{\int_{\mathbb{T}^{\mathbb{N}}} |(P_{m'}G)(t)| dt}{\int_{\mathbb{T}^{\mathbb{N}}} |G(t)| dt} \quad (5.58)$$

$$= \frac{\int_{\mathbb{T}^{\mathbb{N}}} |(P_{m'-m}g)(t_1, \dots, t_n)| dt}{\int_{\mathbb{T}^{\mathbb{N}}} |g(t_1, \dots, t_n)| dt}, \quad (5.59)$$

which by Proposition 3.2 can be arbitrarily big. $\square$

It is worth noting that by repeating the above proof of the equivalence between $H_{m \text{ last}}^1$ norm and $H^1[\mathcal{T}_m]$, one can obtain

$$\|f\|_{H^p[\mathcal{T}_m]} \simeq_p \|f\|_{L^p} \quad (5.60)$$

where $H^p[\mathcal{T}_m]$ is defined in a natural way. Moreover, by iterating the $\|f\|_{H^1} \geq \|f\|_{L^1}$ inequality for linearly ordered martingales,

$$\|f\|_{H^1[\mathcal{T}_m]} \gtrsim \|f\|_{L^1}. \quad (5.61)$$

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Appendix

We present two proofs of Theorem 2.9 different from the original one by Zinn.

Let us recall the non-linear telescoping lemma due to Bourgain and Müller.

Lemma 6.1 [3, 19] *Let $\lambda_1, \dots, \lambda_n, \varphi_1, \dots, \varphi_n$ be nonnegative random variables such that*

$$\mathbb{E}\lambda_k \geq \mathbb{E}(\varphi_k^2 + \lambda_{k-1}^2)^{\frac{1}{2}}. \quad (6.1)$$

Then

$$\mathbb{E}\left(\sum_{k=1}^n \varphi_k^2\right)^{\frac{1}{2}} \leq 2\left(\mathbb{E}\lambda_n \mathbb{E} \max_{1 \leq k \leq n} \lambda_k\right)^{\frac{1}{2}}. \quad (6.2)$$

Corollary 6.2 *Let $X_1, \dots, X_n$ be independent and set*

$$\lambda_0 = 0, \quad \lambda_k = \mathbb{E}(X_k^2 + \lambda_{k-1}^2)^{\frac{1}{2}}. \quad (6.3)$$

Then

$$\lambda_n \leq \mathbb{E} \left(\sum_{k=1}^n X_k^2 \right)^{\frac{1}{2}} \leq 2\lambda_n. \quad (6.4)$$

Proof The right inequality of (6.4) follows directly from Lemma 6.1, since λ_k is an increasing sequence of constants. To prove the other inequality, we see that conditioning with respect to $\sigma(X_{k+1}, \dots, X_n)$ gives

$$\mathbb{E} \left(\sum_{j=k}^n X_j^2 + \lambda_{k-1}^2 \right)^{\frac{1}{2}} = \mathbb{E} \left(\sum_{j=k+1}^n X_j^2 + \left((X_k^2 + \lambda_{k-1}^2)^{\frac{1}{2}} \right)^2 \right)^{\frac{1}{2}} \quad (6.5)$$

$$\geq \mathbb{E} \left(\sum_{j=k+1}^n X_j^2 + \lambda_k^2 \right)^{\frac{1}{2}}, \quad (6.6)$$

thus by induction

$$\mathbb{E} \left(\sum_{k=1}^n X_k^2 \right)^{\frac{1}{2}} \geq \mathbb{E} \left(\sum_{j=k+1}^n X_j^2 + \lambda_k^2 \right)^{\frac{1}{2}}, \quad (6.7)$$

which for $k = n$ is the desired inequality. $\square$

Proof of Theorem 2.9 In order to prove the $\gtrsim$ inequality in (2.38), we merely perform a slight modification of the proof of Lepingle inequality presented in [3]. Let us denote $\Omega^N \times \Omega^N \ni (x, y) \mapsto f_n(x_1, \dots, x_{n-1}, y_n)$ by $\tilde{f}_n$. By tensoring (f_n) against the Rademacher sequence, we may assume that it is a martingale difference sequence. Then the left hand side equals $\|F\|_{H^1}$, where $F = \sum_{n=1}^{\infty} f_n$ and $f_n = \Delta_n F$. By Theorem 2.6 it is enough to check the boundedness of the right hand side in the case when F is an atom, because we have an a priori bound for finite sums. Let $F = u - \mathbb{E}_{j-1} u$, where u satisfies (2.17). Then

$$f_k = \begin{cases} 0 & \text{if } k < j \\ \Delta_k u & \text{if } k \geq j. \end{cases} \quad (6.8)$$

By $A \in \mathcal{F}_j$, the support of $\mathbb{E}_k u$ for $k \geq j$ is contained in A as well, because

$$\mathbb{E}(|\mathbb{E}_k u| \cdot \mathbb{1}_{\Omega^\infty \setminus A}) \leq \mathbb{E}(\mathbb{E}_k |u| \cdot \mathbb{1}_{\Omega^\infty \setminus A}) \quad (6.9)$$

$$= \mathbb{E}(|u| \cdot \mathbb{E}_k \mathbb{1}_{\Omega^\infty \setminus A}) \quad (6.10)$$

$$= 0. \quad (6.11)$$

Thus for $k > j$ we have $\text{supp } f_k \subset \text{supp } \mathbb{E}_k u \cup \text{supp } \mathbb{E}_{k-1} u \subset A$. Consequently

$$\text{supp } \tilde{f}_k \subset A \times \Omega^\infty \text{ for } k > j, \quad (6.12)$$

because if $(x, y) \in \text{supp } \tilde{f}_k$, then $\{(x_1, \dots, x_{k-1}, y_k)\} \times \Omega^{\{k+1, \dots\}} \subset \text{supp } f_k \subset A$, which by $A \in \mathcal{F}_j$ implies $x \in A$. By and (6.8) we have

$$\mathbb{E} |\tilde{f}_j| = \mathbb{E} |f_j| = \mathbb{E} |\Delta_j u| \leq 2\mathbb{E}|u|. \quad (6.13)$$

Combining (2.17), (6.8), (6.12), (6.13) with the inequality

$$\|X\|_{L^1} \leq |\text{supp } X|^{\frac{1}{2}} \|X\|_{L^2} \quad (6.14)$$

and the fact that the projections Δ_k are mutually orthogonal, we obtain

$$\mathbb{E} \left(\sum_{k=1}^{\infty} \tilde{f}_k^2 \right)^{\frac{1}{2}} \leq \mathbb{E} |\tilde{f}_j| + \mathbb{E} \left(\sum_{k=j+1}^{\infty} \tilde{f}_k^2 \right)^{\frac{1}{2}} \quad (6.15)$$

$$\leq 2\mathbb{E} |u| + \mathbb{E} \left(\sum_{k=j+1}^{\infty} \tilde{f}_k^2 \right)^{\frac{1}{2}} \quad (6.16)$$

$$\leq 2|A|^{\frac{1}{2}} (\mathbb{E} u^2)^{\frac{1}{2}} + |A|^{\frac{1}{2}} \left(\sum_{k=j+1}^{\infty} \mathbb{E} \tilde{f}_k^2 \right)^{\frac{1}{2}} \quad (6.17)$$

$$= 2|A|^{\frac{1}{2}} (\mathbb{E} u^2)^{\frac{1}{2}} + |A|^{\frac{1}{2}} \left(\sum_{k=j+1}^{\infty} \mathbb{E} (\Delta_k u)^2 \right)^{\frac{1}{2}} \quad (6.18)$$

$$\leq 3|A|^{\frac{1}{2}} (\mathbb{E} u^2)^{\frac{1}{2}} \quad (6.19)$$

$$\leq 3. \quad (6.20)$$

We will prove the $\lesssim$ inequality in (2.38) now. It is clear that it is enough to prove it with only finitely many of f_k nonzero. We define the sequence $(\lambda_k(x))_{k=1}^n$ of functions in $L^1(\Omega^\infty)$ inductively by

$$\lambda_0(x) = 0, \quad \lambda_k(x) = \int_{\Omega} (f_k^2(x_1, \dots, x_{k-1}, y_k) + \lambda_{k-1}^2(x))^{\frac{1}{2}} dy_k. \quad (6.21)$$

For any fixed $x \in \Omega^\infty$, this sequence coincides with the sequence defined by (6.3) applied to the independent random variables $f_k(x_1, \dots, x_{k-1}, \cdot) \in L^1(\Omega)$, so the left part of (6.4) yields the pointwise inequality

$$\int_{\Omega^\infty} \left(\sum_{k=1}^n f_k^2(x_1, \dots, x_{k-1}, y_k) \right)^{\frac{1}{2}} dy \geq \lambda_n(x). \quad (6.22)$$

By induction it is obvious that λ_k is $\mathcal{F}_{k-1}$ -measurable. Thus

$$\lambda_k(x) = \int_{\Omega} (f_k^2(x_1, \dots, x_{k-1}, y_k) + \lambda_{k-1}^2(x_1, \dots, x_{k-2}))^{\frac{1}{2}} dy_k \quad (6.23)$$

$$= \left(\mathbb{E}_{k-1} (f_k^2 + \lambda_{k-1}^2)^{\frac{1}{2}} \right) (x_1, \dots, x_{k-1}). \quad (6.24)$$

In particular, λ_k verify the condition (6.1) with respect to f_k and are pointwise increasing, so Lemma 6.1 gives

$$\mathbb{E} \left(\sum_{k=1}^n f_k^2 \right)^{\frac{1}{2}} \leq 2 \left(\mathbb{E} \lambda_n \mathbb{E} \max_{k \leq n} \lambda_k \right)^{\frac{1}{2}} = 2\mathbb{E} \lambda_n. \quad (6.25)$$

Integrating (6.22) with respect to x and applying (6.25) we obtain

$$\mathbb{E} \left(\sum_{k=1}^n \tilde{f}_k^2 \right)^{\frac{1}{2}} = \int_{\Omega^\infty} \int_{\Omega^\infty} \left(\sum_{k=1}^n f_k^2(x_1, \dots, x_{k-1}, y_k) \right)^{\frac{1}{2}} dy dx \quad (6.26)$$

$$\geq \int_{\Omega^\infty} \lambda_n(x) dx \quad (6.27)$$

$$\geq \frac{1}{2} \mathbb{E} \left(\sum_{k=1}^n f_k^2 \right)^{\frac{1}{2}}. \quad (6.28)$$

□

Proof (Yet another proof of Theorem 2.9) Without loss of generality, we may assume that Ω has a structure of a compact abelian group with Haar measure, e.g. by embedding Ω in $\mathbb{T}$. Just like previously, we also may assume that f_k is a k -th martingale difference and notice that the left hand side is just $\|\sum_k f_k\|_{H^1}$. For $\xi \in \Omega^N$ we define an operator T_ξ by

$$T_\xi f(x) = \sum_k \Delta_k f(x_1, \dots, x_{k-1}, x_k + \xi_k). \quad (6.29)$$

Since $\Delta_k T_\xi f$ is just a translation of $\Delta_k f$,

$$\|\Delta_k T_\xi f\|_{L^\infty} = \|\Delta_k f\|_{L^\infty}. \quad (6.30)$$

For $k > n$, by translation in the variable y_k ,

$$\mathbb{E}_n |\Delta_k T_\xi f|^2(x) = \int_{\Omega^{>n}} |\Delta_k f(x_1, \dots, x_n, y_{n+1}, \dots, y_{k-1}, y_k + \xi_k)|^2 dy \quad (6.31)$$

$$= \int_{\Omega^{>n}} |\Delta_k f(x_1, \dots, x_n, y_{n+1}, \dots, y_{k-1}, y_k)|^2 dy \quad (6.32)$$

$$= \mathbb{E}_n |\Delta_k f|^2(x). \quad (6.33)$$

Therefore

$$\|T_\xi f\|_{BMO} \lesssim \|f\|_{BMO}. \quad (6.34)$$

We have $T_\xi^* = T_{-\xi}$, because

$$\langle T_\xi f, g \rangle = \mathbb{E} \sum_k \Delta_k T_\xi f \Delta_k g \quad (6.35)$$

$$= \sum_k \int_{\Omega^k} \Delta_k f(x_1, \dots, x_{k-1}, x_k + \xi_k) \Delta_k g(x_1, \dots, x_k) dx \quad (6.36)$$

$$= \sum_k \int_{\Omega^k} \Delta_k f(x_1, \dots, x_{k-1}, x_k) \Delta_k g(x_1, \dots, x_{k-1} - \xi_k) dx \quad (6.37)$$

$$= \langle f, T_{-\xi} g \rangle \quad (6.38)$$

By Theorem 2.7, T_ξ are uniformly bounded on martingale H^1 , so

$$\|T_\xi f\|_{H^1} \lesssim \|f\|_{H^1} = \|T_{-\xi} T_\xi f\|_{H^1} \lesssim \|T_\xi f\|_{H^1} \quad (6.39)$$

and thus

$$\|f\|_{H^1} \simeq \int_{\Omega^N} \|T_\xi f\|_{H^1} d\xi. \quad (6.40)$$

Ultimately, by translating ξ for fixed x ,

$$\int_{\Omega^N} \left(\sum_k |f_k(x_1, \dots, x_k)|^2 \right)^{\frac{1}{2}} dx$$

$$= \left\| \sum_k f_k \right\|_{H^1} \quad (6.41)$$

$$\simeq \int_{\Omega^N} \left\| T_\xi \sum_k f_k \right\|_{H^1} d\xi \quad (6.42)$$

$$= \int_{\Omega^N} \int_{\Omega^N} \left(\sum_k |f_k(x_1, \dots, x_{k-1}, x_k + \xi_k)|^2 \right)^{\frac{1}{2}} d\xi dx \quad (6.43)$$

$$= \int_{\Omega^N} \int_{\Omega^N} \left(\sum_k |f_k(x_1, \dots, x_{k-1}, \xi_k)|^2 \right)^{\frac{1}{2}} d\xi dx. \quad (6.44)$$

□

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