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The ring of modular forms of degree two in characteristic three

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Abstract
We determine the structure of the ring of Siegel modular forms of degree 2 in characteristic 3.

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1 Introduction

Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties of dimension g. It is a Deligne-Mumford stack over $\mathbb{Z}$. It carries a natural vector bundle of rank g, the Hodge bundle $\mathbb{E}_g$. We write L for its determinant line bundle. The vector bundle $\mathbb{E}_g$ extends in a natural way over any compactification $\tilde{\mathcal{A}}_g$ of Faltings-Chai type and we will denote the extension of $\mathbb{E}_g$ and L again by the same symbols. Sections of $L^{\otimes k}$ over $\tilde{\mathcal{A}}_g$ are called modular forms of weight k. It is known that for $g \geq 2$ any section of L^k over $\mathcal{A}_g$ extends to a section of L^k over $\tilde{\mathcal{A}}_g$, a fact usually referred to as the Koecher principle, see [7, Prop. 1.5, p. 140].

If $\mathbb{F} = \mathbb{Z}$ or $\mathbb{Z}_p$ or a field one has the graded ring

$$\mathcal{R}_g(\mathbb{F}) = \oplus_k H^0(\tilde{\mathcal{A}}_g \otimes \mathbb{F}, L^k)\,.$$

It is known by [7] that it is a finitely generated $\mathbb{F}$-algebra.

In the case of $\mathbb{F} = \mathbb{C}$ the ring $\mathcal{R}_g(\mathbb{C})$ is the ring of scalar-valued Siegel modular forms of degree g. It is well-known known that $\mathcal{R}_1(\mathbb{C}) = \mathbb{C}[E_4, E_6]$ is freely generated over $\mathbb{C}$ by the Eisenstein series E_4 and E_6 of weights 4 and 6. In the 1960s Igusa [11] determined the structure of $\mathcal{R}_2(\mathbb{C})$:

$$\mathcal{R}_2(\mathbb{C}) = \mathbb{C}[\psi_4, \psi_6, \chi_{10}, \chi_{12}, \chi_{35}]/\left(\chi_{35}^2 - P\right),$$

where the indices of the generators indicate the weights and P is a polynomial in ψ_4, ψ_6, χ_{10} and χ_{12}. Moreover, the ideal of cusp forms is generated by χ_{10}, χ_{12} and χ_{35}. For $g = 3$,

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Tsuyumine showed in [20] that $\mathcal{R}_3(\mathbb{C})$ is generated by 34 elements; recently the number of generators was reduced to 19 by Lercier and Ritzenthaler [14].

For $\mathbb{F} = \mathbb{F}_p$, a finite field with p elements, the ring $R_1(\mathbb{F}_p)$ was described by Deligne [5]. Besides giving the structure of the ring over $\mathbb{Z}$

$$\mathcal{R}_1(\mathbb{Z}) = \mathbb{Z}[c_4, c_6, \Delta]/\left(c_4^3 - c_6^2 - 1728\,\Delta\right),$$

he showed that

$$\mathcal{R}_1(\mathbb{F}_2) = \mathbb{F}_2[a_1, \Delta] \quad \text{and} \quad \mathcal{R}_1(\mathbb{F}_3) = \mathbb{F}_3[b_2, \Delta],$$

where Δ is of weight 12 and a_1 (resp b_2) is of weight 1 (resp. 2). For $p \geq 5$ we have $\mathcal{R}_1(\mathbb{F}_p) = \mathbb{F}_p[c_4, c_6]$.

For $g = 2$, Igusa determined in [13] also the ring of modular forms over $\mathbb{Z}$; it is generated by elements of weight

$$4, 6, 10, 12, 12, 16, 18, 24, 28, 30, 35, 36, 40, 42, 48.$$

For finite fields the structure of $\mathcal{R}_2(\mathbb{F}_p)$ is known for $p \geq 5$. For this we refer to Ichikawa's paper [10]. For $p \geq 5$ the ring is just as in characteristic zero generated by modular forms $\psi_4, \psi_6, \chi_{10}, \chi_{12}$ and χ_{35} with χ_{35} satisfying a relation $\chi_{35}^2 = P(\psi_4, \psi_6, \chi_{10}, \chi_{12})$. Moreover for $p \geq 5$ the reduction map $\mathcal{R}_2(\mathbb{Z}_p) \to \mathcal{R}_2(\mathbb{F}_p)$ is surjective. Nagaoka studied the image of the reduction map in [17,18], see also [1].

In this paper we consider the case $p = 3$ and determine the structure of $\mathcal{R}_2(\mathbb{F}_3)$. We use the close connection between the moduli space $\mathcal{A}_2$ and the moduli space $\mathcal{M}_2$ of curves of genus 2 via the Torelli map $\mathcal{M}_2 \hookrightarrow \mathcal{A}_2$ and the description of $\mathcal{M}_2$ as a quotient stack for the action of GL(2) on the space of binary sextics. In that way invariant theory can be used to construct modular forms. The relation between invariants and modular forms was already exploited by Igusa in [11], but he used theta functions and Thomae's formula to relate these to cross ratios of the zeros of a binary sextic. Here we use not only invariants but also covariants giving vector-valued modular forms as introduced in [2] to analyze the regularity of scalar-valued modular forms.

Our result is:

Theorem 1.1 *The subring $\mathcal{R}_2^{\mathrm{ev}}(\mathbb{F}_3)$ of modular forms of even weight is generated by forms of weights* 2, 10, 12, 14 *and* 36 *and has the form*

$$\mathcal{R}_2^{\mathrm{ev}}(\mathbb{F}_3) = \mathbb{F}_3[\psi_2, \chi_{10}, \psi_{12}, \chi_{14}, \chi_{36}]/J$$

with J the ideal generated by the relation

$$\psi_2^3\chi_{36} - \chi_{10}^3\psi_{12} - \psi_2^2\chi_{10}\chi_{14}^2 + \chi_{14}^3.$$

Moreover, $\mathcal{R}_2(\mathbb{F}_3) = \mathcal{R}_2^{\mathrm{ev}}(\mathbb{F}_3)[\chi_{35}]/(\chi_{35}^2 - P)$ with P a polynomial in ψ_2, χ_{10}, ψ_{12}, χ_{14} and χ_{36}. The ideal of cusp forms is generated by $\chi_{10}, \chi_{14}, \chi_{35}, \chi_{36}$.

The generator ψ_2 is the Hasse invariant that vanishes on the locus of non-ordinary abelian surfaces and χ_{10} is a form that vanishes on the locus of products of elliptic curves. The ring of modular forms of degree 2 in characteristic 2 is described in [4].

2 The proof of Theorem 1.1

Since for $g = 2$ the moduli stack $\mathcal{A}_g \otimes \mathbb{F}_3$ has a canonical compactification due to Igusa we will use this compactification $\tilde{\mathcal{A}}_2 \otimes \mathbb{F}_3$. We will denote the space of sections of L^k on

$\tilde{\mathcal{A}}_2 \otimes \mathbb{F}_3$ by $M_k(\Gamma_2)$ and we thus have $\mathcal{R}_2(\mathbb{F}_3) = \oplus_k M_k(\Gamma_2)$. We write $M_k(\Gamma_1)$ for the space $H^0(\tilde{\mathcal{A}}_1 \otimes \mathbb{F}_3, L^k)$. The Satake compactification is denoted by $\mathcal{A}_2^* \otimes \mathbb{F}_3$. We denote the first Chern class of L by λ_1.

We begin by constructing generators of weight 2 and 10. The locus V_1 of abelian surfaces with p-rank ≤ 1 is a divisor in $\mathcal{A}_2 \otimes \mathbb{F}_p$ and its closure $\overline{V}_1$ in $\tilde{\mathcal{A}}_2 \otimes \mathbb{F}_p$ has cycle class $(p-1)\lambda_1$ in the Chow group with $\mathbb{Q}$-coefficients, so $[\overline{V}_1] = 2\lambda_1$ for $p = 3$, see [6,22]. Therefore the effective divisor $\overline{V}_1$ is the divisor of a section of $L^{\otimes 2}$ and there is a modular form ψ_2 of weight 2 whose zero divisor is $\overline{V}_1$. It is determined up to multiplication by a non-zero scalar. We will normalize it later. This form is known as the Hasse invariant. Multiplication by ψ_2 implies that $\dim M_k(\Gamma_2) \leq \dim M_{k+2}(\Gamma_2)$.

The divisor of products of elliptic curves $H_1 := \mathcal{A}_{1,1} \otimes \mathbb{F}_3$ gives rise to a second modular form. (The notation refers to the fact that H_1 is the Humbert surface of discriminant 1.) In the Chow group of codimension 1 of $\tilde{\mathcal{A}}_2 \otimes \mathbb{F}_3$ (resp. $\mathcal{A}_2^* \otimes \mathbb{F}_3$) we have the relation (cf. e.g. [16, p. 317])

$$2[\overline{H}_1] + [D] = 10\,\lambda_1 \quad (\text{resp.} \quad 2[\overline{H}_1] = 10\,\lambda_1)\,,$$

with D the divisor at infinity, hence there exists a modular form of weight 10 vanishing with multiplicity 2 on H_1. We call this form χ_{10} (up to a normalization to be determined later). The automorphism group of a generic product of elliptic curves has an extra involution (when compared with the automorphism group of a generic principally polarized abelian surface) and it acts by -1 on L, hence every modular form of even weight vanishes with even multiplicity along H_1.

Restriction to H_1 yields for even k an exact sequence

$$0 \to H^0(\mathcal{A}_2 \otimes \mathbb{F}_3, L^k \otimes O(-2H_1)) \to H^0(\mathcal{A}_2 \otimes \mathbb{F}_3, L^k) \to H^0\left(H_1, L^k_{|H_1}\right)$$

and in view of the degree 2 morphism $\mathcal{A}_1 \times \mathcal{A}_1 \to \mathcal{A}_{1,1}$ induced by interchanging the two factors, we can identify this with

$$0 \to M_{k-10}(\Gamma_2) \to M_k(\Gamma_2) \to \mathrm{Sym}^2(M_k(\Gamma_1))\,, \tag{2.1}$$

where the second arrow is multiplication by χ_{10}. Moreover $M_{k-10}(\Gamma_2) = (0)$ for $k < 8$ since L is ample on $\mathcal{A}_2^* \otimes \mathbb{F}_3$. The exact sequence (2.1) and the fact that we know $M_k(\Gamma_1)$ implies that $\dim M_k(\Gamma_2) = 1$ for $k = 2, 4, 6, 8$ and $\dim M_{10}(\Gamma_2) = 2$ and $M_{10}(\Gamma_2)$ is generated by ψ_2^5 and χ_{10}.

We now turn to the construction of the other generators. We use the ideas of [2]. The Torelli map defines an embedding $\mathcal{M}_2 \otimes \mathbb{F}_3 \to \mathcal{A}_2 \otimes \mathbb{F}_3$. A smooth projective curve of genus 2 can be given by an equation

$$y^2 = f(x) \quad \text{with } f = \sum_{i=0}^{6} a_i\, x^{6-i}\,. \tag{2.2}$$

We let $V = \langle x_1, x_2 \rangle$ be the $\mathbb{F}_3$-vector space generated by x_1, x_2 and write f as a homogeneous polynomial $\sum_{i=0}^{6} a_i x_1^{6-i} x_2^i$. Note that a curve as in (2.2) comes with a basis of the space of regular differentials, viz. $dx/y,\ x dx/y$.

We have a description of $\mathcal{M}_2 \otimes \mathbb{F}_3$ as the stack quotient $[\mathcal{X}^0/\mathrm{GL}(V)]$ with $\mathcal{X}^0 \subset \mathcal{X} = \mathrm{Sym}^6(V) \otimes \det(V)^{-2}$ the locus given by the non-vanishing of the discriminant, see [3, Section 3, p. 3].

The pullback to $\mathcal{X}^0$ of the Hodge bundle under the composition of $\mathcal{X}^0 \to \mathcal{M}_2$ with the Torelli map $\mathcal{M}_2 \hookrightarrow \mathcal{A}_2$ is the equivariant bundle V on $\mathcal{X}^0$ as the basis $dx/y,\ x dx/y$ of the space of regular differentials on the curve $y^2 = f(x)$ shows. The pullback of L is $\det(V)$. As a consequence pulling back defines a homomorphism

$$\mu : \mathcal{R}_2(\mathbb{F}_3) \to I \tag{2.3}$$

with I the ring of invariants of the action of $\mathrm{GL}(V)$ on $\mathrm{Sym}^6(V)$. Here an invariant is a polynomial in $a_0, \ldots, a_6$, the coefficients of f that is invariant under $\mathrm{SL}(V)$. Since the image of $\mathcal{M}_2$ in $\mathcal{A}_2$ is a Zariski open part with complement H_1, not every invariant corresponds to a modular form; but every invariant corresponds to a rational modular form that is regular outside H_1. In particular, it becomes regular on all of $\mathcal{A}_2$ when multiplied with a sufficiently high power of χ_{10}. This provides us with homomorphisms

$$\mathcal{R}_2(\mathbb{F}_3) \xrightarrow{\mu} I \xrightarrow{\nu} \mathcal{R}_2(\mathbb{F}_3)_{\chi_{10}},$$

where $\mathcal{R}_2(\mathbb{F}_3)_{\chi_{10}}$ is obtained from $\mathcal{R}_2(\mathbb{F}_3)$ by allowing powers of χ_{10} in the denominator. We have $\nu \circ \mu = \mathrm{id}$.

This generalizes as follows to vector-valued modular forms. For each finite dimensional irreducible representation ρ of GL(2) there is a vector bundle $\mathbb{E}_2^\rho$ obtained from $\mathbb{E}_2$ by applying a Schur functor. Such a ρ is of the form $\mathrm{Sym}^j(\mathrm{St}) \otimes \det^k(\mathrm{St})$ with St the standard representation of $\mathrm{GL}(V)$. A section of $\mathrm{Sym}^j(\mathbb{E}_2) \otimes \det(\mathbb{E}_2)^k$ over $\mathcal{A}_2$ is called a modular form of degree 2 and weight (j, k). The Koecher principle also applies to these modular forms: sections of $\mathbb{E}_2^\rho$ over $\mathcal{A}_2$ extend over $\tilde{\mathcal{A}}_2$, see [7, Prop. 1.5, p. 140]. We write

$$M_{j,k}(\Gamma_2) = H^0(\tilde{\mathcal{A}}_2 \otimes \mathbb{F}_3, \mathrm{Sym}^j(\mathbb{E}_2) \otimes \det(\mathbb{E}_2)^k)$$

and we consider the $\mathcal{R}_2(\mathbb{F}_3)$-module

$$M = \oplus_{j,k} M_{j,k}(\Gamma_2).$$

It is even a ring. The map (2.3) can be extended to a map from M to the ring of covariants. Here a covariant can be described as an invariant for the action of $\mathrm{GL}(V)$ on $V \oplus \mathrm{Sym}^6(V)$. Alternatively, covariants can be obtained by taking an equivariant embedding of an irreducible $\mathrm{GL}(V)$-representation $U \to \mathrm{Sym}^d(\mathrm{Sym}^6(V))$, or equivalently, an equivariant map

$$\varphi : \mathbb{F}_3 \to \mathrm{Sym}^d(\mathrm{Sym}^6(V)) \otimes U^\vee$$

and then $\Phi = \varphi(1)$ is a covariant. If U is an irreducible representation of highest weight (w_1, w_2) then one may view Φ as a homogeneous form in $a_0, \ldots, a_6$ of degree d and in x_1, x_2 of degree $w_1 - w_2$, see [2,9,19]. For example, taking $U = \mathrm{Sym}^6(V)$ and $d = 1$ yields the covariant $\Phi = f$, the universal binary sextic. Covariants form a ring $\mathcal{C}$ that was much studied in the 19th and early 20th century. Grace and Young determined generators of this ring in [9].

The maps $\mathcal{R}_2(\mathbb{F}_3) \to I \to \mathcal{R}_2(\mathbb{F}_3)_{\chi_{10}}$ now extend to

$$M \xrightarrow{\mu} \mathcal{C} \xrightarrow{\nu} M_{\chi_{10}},$$

where $M_{\chi_{10}}$ is obtained from M by admitting powers of χ_{10} as denominators. We have $\nu \circ \mu = \mathrm{id}_M$.

The image under ν of the covariant f, the universal binary sextic, is a rational modular form $\chi_{6,-2}$, that is, a rational section of $\mathrm{Sym}^6(\mathbb{E}_2) \otimes \det(\mathbb{E}_2)^{-2}$ that is regular after multiplication by an appropriate power of χ_{10}. The power -2 comes from the twisting used in the description of the stack quotient $[\mathcal{X}^0/\mathrm{GL}(V)]$, where $\mathcal{X}^0 \subset \mathrm{Sym}^6(V) \otimes \det(V)^{-2}$, see [3, Section 3, p. 3].

This construction was given in [2] in characteristic zero and yields a meromorphic modular form, here denoted $\varphi_{6,-2}$, that becomes holomorphic after multiplication by χ_{10}. The reduction of the characteristic zero rational modular form $\varphi_{6,-2}$ yields a rational modular form in characteristic 3. This implies that $\chi_{6,-2}$ becomes regular after multiplication by χ_{10}. We can write the form $\chi_{6,-2}$ locally on $\mathcal{A}_2 \otimes \mathbb{F}_3$ symbolically as

$$\chi_{6,-2} = \sum_{i=0}^{6} \alpha_i X_1^{6-i} X_2^i \,, \tag{2.4}$$

where the monomials $X_1^{6-i}X_2^i$ are dummies to indicate the coordinates in the fibres of $\mathrm{Sym}^6(\mathbb{E}_2) \otimes \det(\mathbb{E}_2)^{-2}$. Here we view α_i locally as a rational function on $\mathcal{A}_2 \otimes \mathbb{F}_3$. Using the local expression (2.4) one can give the image $\nu(T)$ of an invariant $T = T(a_0, \ldots, a_6)$ locally by $T(\alpha_0, \ldots, \alpha_6)$.

We note that interchanging X_1 and X_2 induces an involution replacing α_i by α_{6-i}.

Comparing with the characteristic 0 case and using semi-continuity we see that the orders of the rational functions α_i along the divisor H_1 are at least equal to the orders of their complex analogues along H_1. The Fourier expansion in characteristic 0 given in [2, page 1658] implies the following inequalities for the orders of α_i along H_1 in characteristic 3:

$$\mathrm{ord}_{H_1}(\alpha_0, \ldots, \alpha_6) = (\geq 2, \geq 1, \geq 0, \geq -1, \geq 0, \geq 1, \geq 2)\,. \tag{2.5}$$

Moreover, the symmetry that interchanges x_1 and x_2 implies that the orders of α_i and α_{6-i} along H_1 are equal. Another way to see the estimates for the orders is by developing $\chi_{6,8} = \chi_{6,-2}\chi_{10}$ along the locus $\mathcal{A}_{1,1} \otimes \mathbb{F}_3 \subset \mathcal{A}_2 \otimes \mathbb{F}_3$. Since the pullback of the Hodge bundle $\mathbb{E}_2$ to $\mathcal{A}_1 \times \mathcal{A}_1$ via $\mathcal{A}_1^2 \to \mathcal{A}_{1,1} \subset \mathcal{A}_2$ is $\oplus_{i=0}^{6} p_1^*(\mathbb{E}_1)^i \otimes p_2^*(\mathbb{E}_1)^{6-i}$ the restriction of $\alpha_i \chi_{10}$ lies in $S_{14-i}(\Gamma_1) \otimes S_{8+i}(\Gamma_1)$ and this is zero. The next Taylor term in the Taylor development along $\mathcal{A}_{1,1}$ lies in $S_{15-i}(\Gamma_1) \otimes S_{9+i}(\Gamma_1)$ and this is zero for $i \neq 3$.

The ring of invariants I for the action of $\mathrm{GL}(V)$ on $\mathrm{Sym}^6(V)$ in characteristic 3 is generated by invariants A, B, C, D and E of degree 2, 4, 6, 10 and 15, see e.g. [11] or [8]. The invariants A, B, C, D that we use here can be expressed in the reductions modulo 3 of the invariants J_2, J_4, J_6 et J_{10} given in [15]: $A = -J_2(\mathrm{mod} 3)$, $B = -J_4(\mathrm{mod} 3)$, $C = -J_6 - A^3(\mathrm{mod} 3)$, $D = J_{10}(\mathrm{mod} 3)$. The invariant E can be found in [12, p. 848].

The invariant A has the form $A = a_1a_5 - a_2a_4$. We know of the existence of a modular form ψ_2 of weight 2. Under the map μ it must map to a non-zero multiple of A. We fix ψ_2 by requiring $\mu(\psi_2) = A$. The restriction to H_1 of the Hasse invariant ψ_2 is a non-zero multiple of $\mathrm{Sym}^2(b_2)$, with b_2 the Hasse invariant for $g = 1$, hence ψ_2 does not vanish identically on H_1.

By the inequalities (2.5) and the expression for A we see that $\mathrm{ord}_{H_1}(\alpha_2) = 0 = \mathrm{ord}_{H_1}(\alpha_4)$ and

$$\mathrm{ord}_{H_1}(\alpha_0, \ldots, \alpha_6) = (\geq 2, \geq 1, 0, \geq -1, 0, \geq 1, \geq 2)\,.$$

In degree 4 we find another invariant B, not a multiple of A^2:

$$\begin{aligned} B = {} & 2\,a_0a_1a_5a_6 + a_0a_2a_4a_6 + 2\,a_0a_2a_5^2 + 2\,a_0a_4^3 + 2\,a_1^2a_4a_6 + 2\,a_1a_2a_4a_5 \\ & + a_1a_3^2a_5 + a_1a_3a_4^2 + 2\,a_2^3a_6 + a_2^2a_3a_5 + a_2^2\,a_4^2 + 2\,a_2a_3^2a_4. \end{aligned}$$

Since we know $\dim M_4(\Gamma_2) = 1$ there cannot be a regular modular form in weight 4 that is not a multiple of ψ_2^2. This implies that $\mathrm{ord}_{H_1}(\alpha_3) < 0$ and hence $\mathrm{ord}_{H_1}(\alpha_3) = -1$. Thus $B = (a_1a_5 - a_2a_4)a_3^2 + (a_1a_4^2 + a_2^2a_5)a_3 + \cdots$ defines a rational modular form $\chi_B = \nu(B)$

of weight 4 with order -2 along H_1. Since χ_{10} vanishes with multiplicity 2 along H_1 we thus find that

$$\chi_{14} := \chi_B \chi_{10}$$

is a regular modular form of weight 14.

The vector space of invariants of degree 6 is generated by A^3, AB and an invariant C

$$C = 2\,a_3^6 + A\,a_3^4 + 2(a_1a_4^2 + a_2^2a_5)a_3^3 + \cdots$$

and we see that $\chi_C = \nu(C)$ has order -6 along H_1. In degree 10 there is a new invariant

$$D = (a_1a_5)^3a_3^4 + (a_0a_2^3a_5^3 + a_1^3a_4^3a_6 + 2a_1^3a_4^2a_5^2 + 2a_1^2a_2^2a_5^3)a_3^3 + \cdots$$

yielding a modular form that vanishes with multiplicity ≥ 2 on H_1. Indeed, since $\alpha_1\alpha_5$ vanishes with multiplicity ≥ 2 the first term $(\alpha_1\alpha_5)^3\alpha_3^4$ vanishes with order ≥ 2; the next terms also vanish with order ≥ 2 as one easily checks. Therefore χ_D is regular and vanishes with multiplicity ≥ 2. Since χ_D is not zero, it must be a multiple of χ_{10} and then vanishes on H_1 with multiplicity 2. This implies that the order of vanishing of α_1 and α_5 along H_1 is 1.

Corollary 1 *We have* $\mathrm{ord}_{H_1}(\alpha_0, \ldots, \alpha_6) = (\geq 2, 1, 0, -1, 0, 1, \geq 2)$.

We fix χ_{10} by setting it equal to $\chi_D = \nu(D)$. This fixes χ_{14} too.

In a similar manner one checks that the rational modular form $\psi_S = \nu(S)$ with S equal to

$$S = B^3 + A^3C - A^2B^2 = (a_1a_4^2 + a_2^2a_5)^3a_3^3 + \cdots$$

is regular too. We put $\psi_{12} = \psi_S$. We thus find a 3-dimensional subspace of $M_{12}(\Gamma_2)$ generated by ψ_2^6, $\psi_2\chi_{10}$ and ψ_{12}. From the fact that B and D are not divisible by A we see that χ_{14} does not lie in $\psi_2 M_{12}(\Gamma_2)$. Therefore $\dim M_{12}(\Gamma_2) < \dim M_{14}(\Gamma_2)$. Since we know by (2.1) that $\dim M_{14}(\Gamma_2) \leq 4$ we conclude that $\dim M_{12}(\Gamma_2) = 3$.

A further generator is

$$\chi_{36} = \nu(CD^3) = \chi_C\chi_{10}^3\,.$$

Since the orders of χ_C and χ_{10} along H_1 are -6 and 2 the modular form χ_{36} is regular and does not vanish identically on H_1. The modular form χ_{36} is not contained in the subring generated by ψ_2,χ_{10}, ψ_{12} and χ_{14} as one sees by looking at the invariants. We have the identity

$$(B^3 + A^3C - A^2B^2)D^3 = B^3D^3 + A^3CD^3 - A^2DB^2D^2$$

by which we can express $\psi_{12}\chi_{10}^3$ in the other generators:

$$\psi_{12}\chi_{10}^3 = \chi_{14}^3 + \psi_2^3\chi_{36} - \psi_2^2\chi_{10}\chi_{14}^2\,. \tag{2.6}$$

Since A, B, C, D are generators of the ring of invariants and are algebraically independent the forms ψ_2, χ_{10}, ψ_{12}, χ_{14} are algebraically independent. The form χ_{36} then satisfies the algebraic relation (2.6) and since there is no non-trivial relation of lower weight involving χ_{36} it implies that this relation generates the ideal of relations between the generators ψ_2, χ_{10}, ψ_{12}, χ_{14} and χ_{36}.

The forms ψ_2, χ_{10}, ψ_{12}, χ_{14} and χ_{36} generate a subring R^{ev} of the ring $\mathcal{R}_2^{\mathrm{ev}}(\mathbb{F}_3)$ with generating function

$$G = \frac{(1-t^{42})}{(1-t^2)(1-t^{10})(1-t^{12})(1-t^{14})(1-t^{36})}\,.$$

and by the Riemann-Roch theorem we have $\dim M_k(\Gamma_2) = k^3/1080 + O(k^2)$ for even k. Note that

$$\frac{42}{2 \cdot 10 \cdot 12 \cdot 14 \cdot 36} = \frac{1}{2880}\,.$$

On the other hand we have $c_1(L)^3 = 1/2880$, see [22, p. 74]. We can use the degree of $\mathrm{Proj}(\mathcal{R}_2^{\mathrm{ev}}(\mathbb{F}_3))$ to show that there cannot be more generators of $\mathcal{R}_2^{\mathrm{ev}}(\mathbb{F}_3)$, but one can see this also in a more elementary way as follows.

Let $d(k) = \dim M_k(\Gamma_2)$ and $r(k) = \dim R_k$ where $R_k = R^{\mathrm{ev}} \cap M_k(\Gamma_2)$.

Proposition 1 *We have $d(k) = r(k)$ for even $k \geq 0$.*

Proof We know that $d(k) \geq r(k)$ for even k and $d(k) = r(k)$ for even $0 \leq k \leq 14$. Suppose by induction that $d(k) = r(k)$ for even $k \leq m$. The exact sequence (2.1) gives the upper bound $d(k) \leq r(k-10) + c(k)(c(k)+1)/2$ for $k \leq m+10$, where $c(k) = \dim M_k(\Gamma_1) = \lfloor k/12 \rfloor + 1$. Using the generating function G one sees that $r(k) - r(k-10) = c(k)(c(k)+1)/2$ for $k \not\equiv 0 \,(\mathrm{mod}\,12)$ and $k \not\equiv 2\,(\mathrm{mod}\,12)$. Hence $d(k) = r(k)$ for even $k \leq m+10$ with $k \not\equiv 0, 2 (\mathrm{mod}\,12)$. But we have

$$d(k+2) - d(k) \geq r(k+2) - r(k)\,,$$

as we show in the next lemma. This proves $d(k) = r(k)$ for even $k \leq m+10$. Therefore we conclude the proof by induction. □

Lemma 1 *We have $d(k+2) - d(k) \geq r(k+2) - r(k)$ for even $k \geq 0$.*

Proof We can write $R_{k+2} = \psi_2 R_k \oplus N_{k+2}$ with N_{k+2} the subspace with basis the forms $\chi_{10}^a \psi_{12}^b \chi_{14}^c \chi_{36}^d$ with $a, b, c, d \geq 0$ and $c \leq 2$ in view of the relation (2.6). Then we have $\dim N_{k+2} = r(k+2) - r(k)$. The inequality $d(k+2) - d(k) \geq \dim N_{k+2}$ follows from the fact that $N_{k+2} \cap \psi_2 M_k(\Gamma_2) = (0)$. To see this fact, suppose that $f \in M_k(\Gamma_2)$ such that $f \notin R_k$ and $\psi_2 f \in R_{k+2}$. Then $\psi_2 f = P$ with P a sum of monomials $\chi_{10}^a \psi_{12}^b \chi_{14}^c \chi_{36}^d$ with $c \leq 2$. Then $P = \nu(Q)$ with Q a polynomial in

$$D,\, B^3 + A^3C - A^2B^2,\, BD,\, CD^3\,.$$

Since $P = \psi_2 f$ this polynomial must be divisible by A. But this implies that if $Q \neq 0$ then it must have at least one monomial with $c \geq 3$, but we excluded this. □

The invariant E of degree 15 is of the form

$$E = (a_1 a_4^2 - a_2^2 a_5)^3 a_3^6 + \cdots$$

and $\nu(E)$ has order -3 along H_1. Therefore

$$\chi_{35} := \nu(ED^2)$$

is a regular modular form. It vanishes on H_1 and on the Humbert surface H_4 of discriminant 4, both with multiplicity 1. The surfaces H_1 and H_4 parametrize abelian surfaces that possess an extra involution. Locally near H_4 the extra automorphism corresponds to the symmetry that interchanges x_1 and x_2.

We know that the cycle class of $2\,H_4$ on $\mathcal{A}_2^* \otimes \mathbb{F}_3$ is $60\lambda_1$, see [21, Prop. 3.3, p. 217]. Therefore the divisor of χ_{35} is $H_1 + H_4$ and since the closure of H_1 contains the 1-dimensional cusp χ_{35} is a cusp form. Then χ_{35}^2 is of even weight, hence can be expressed as a polynomial in $\psi_2, \chi_{10}, \psi_{12}, \chi_{14}$ and χ_{36}. If ψ is an odd weight modular form then it must vanish on H_1 and H_4, hence it will be divisible by χ_{35}.

The relation between the space of binary sextics and the moduli space $\overline{\mathcal{M}}_2$ (see for example [3, Section 4]) implies that a modular form χ is a cusp form if and only if the invariant $\mu(\chi)$ is divisible by the discriminant D in I. From the form of the generators one easily sees that $\chi_{10}, \chi_{14}, \chi_{36}$ and χ_{35} generate the ideal of cusp forms. This completes the proof.

Remark 1 One can use the knowledge of the dimensions of $M_k(\Gamma_2)$ to deduce non-vanishing of $H^1(\tilde{\mathcal{A}}_2 \otimes \mathbb{F}_3, L^k)$ for certain values of k. The short exact sequence of sheaves on $\tilde{\mathcal{A}}_2 \otimes \mathbb{F}_3$

$$0 \to L^k \otimes \mathcal{O}(-\overline{V}_1) \to L^k \to L^k_{|\overline{V}_1} \to 0$$

gives rise to a long exact sequence which can be identified with

$$0 \to M_{k-2}(\Gamma_2) \to M_k(\Gamma_2) \to H^0(\overline{V}_1, L^k) \to H^1(\tilde{\mathcal{A}}_2 \otimes \mathbb{F}_3, L^{k-2}) \to \cdots$$

For example, if $\dim M_{k-2}(\Gamma_2) = \dim M_k(\Gamma_2)$ we get an injection $H^0(\overline{V}_1, L^k) \to H^1(\tilde{\mathcal{A}}_2 \otimes \mathbb{F}_3, L^{k-2})$ and if $k \equiv 0 (\mathrm{mod}\, 4)$ and $k \geq 0$ one can show that $H^0(\overline{V}_1, L^k) \neq 0$ by showing that $H^0(\overline{V}_1[2], L^k)^{\mathfrak{S}_6} \neq (0)$, the space of invariants under the symmetric group $\mathfrak{S}_6$ acting on $H^0(\overline{V}_1[2], L^k)$ with $V_1[2]$ the 3-rank ≤ 1 locus in the level 2 moduli space $\tilde{\mathcal{A}}_2[2]$. Thus for example, $H^1(\tilde{\mathcal{A}}_2 \otimes \mathbb{F}_3, L^{14}) \neq (0)$.

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