New Riemannian manifolds with L^p -unbounded Riesz transform for $p > 2$

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Received: 31 July 2017 / Accepted: 28 February 2020 / Published online: 11 September 2020
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Abstract

We construct a large class of Riemannian manifolds of arbitrary dimension with Riesz transform unbounded on $L^p(M)$ for all $p > 2$. This extends recent results for Vicsek manifolds, and in particular shows that fractal structure is not necessary for this property.

1 Introduction

Consider a Riemannian manifold M with gradient ∇ and Laplace–Beltrami operator Δ . The Riesz transform $\nabla \Delta^{-1/2}$, with $\Delta^{-1/2}$ defined via the spectral theorem, maps $L^2(M)$ boundedly to the space of square integrable vector fields $L^2(M; TM)$. Much attention has been given to the question of whether this operator extends to a bounded map from $L^p(M)$ to $L^p(M; TM)$ for $p \neq 2$, or equivalently, whether the estimate

$$(R_p) : \quad \|\nabla f\|_p \lesssim \|\Delta^{1/2} f\|_p \quad \text{for all } f \in C_c^\infty(M)$$

holds. It is conjectured that for $p \in (1, 2)$ the estimate (R_p) holds whenever M is complete, with implicit constant depending only on p ; in [1] it is shown that the failure of this uniformity among manifolds of a fixed dimension would imply the existence of a manifold for which (R_p) fails.

One is naturally led to consider also the ‘reverse’ estimate

$$(RR_p) : \quad \|\Delta^{1/2} f\|_p \lesssim \|\nabla f\|_p \quad \text{for all } f \in C_c^\infty(M).$$

A duality argument shows that for $p \in (1, \infty)$, (R_p) implies $(RR_{p'})$, where $p' = p/(p-1)$ is the Hölder conjugate of p . If (R_p) and (RR_p) both hold, then we have a norm equivalence

$$\|\nabla f\|_p \simeq \|\Delta^{1/2} f\|_p,$$

which says that the homogeneous Sobolev space $\dot{W}_1^p(M)$ may be defined either via the gradient or via the square root of the Laplace–Beltrami operator.

Generally (R_p) holds only for some interval of $p \in (1, \infty)$ including 2, and proving (R_p) presents different difficulties depending on whether $p < 2$ or $p > 2$. When $1 < p < 2$,

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(R_p) is known to follow from the volume doubling property and Gaussian or sub-Gaussian heat kernel upper estimates [9,10] (see also [15] for examples which do not satisfy such kernel estimates). The volume doubling property and an appropriately scaled L^2 -Poincaré inequality imply (R_p) for some $p > 2$ [2]. In [3] this is linked with gradient estimates for the heat kernel, and in [5] the L^2 -Poincaré inequality is replaced by a relative Faber–Krahn inequality and a reverse Hölder inequality.

Some manifolds for which (R_p) fails for some $p > 2$ are known. If M is the connected sum of two copies of $\mathbb{R}^n \setminus B(0, 1)$ with $n \geq 3$ —or more generally, an n -dimensional manifold with at least two (and finitely many) Euclidean ends— (R_p) holds if and only if $p \in (1, n)$ [7,10]. Similar results are known for conical manifolds [14] and for 2-hyperbolic, p -parabolic manifolds with at least two ends [6].

The most relevant examples to this article are Vicsek manifolds, which are ‘thickenings’ of Vicsek graphs (pictured in the 2-dimensional case in Fig. 2). The Vicsek graph, being a graphical realisation of a Vicsek fractal, is a ‘fractal at infinity’. Locally a Vicsek manifold behaves like Euclidean space (it is, of course, a manifold), but at large scale it behaves like a fractal. In [9] it is shown that for a Vicsek manifold of any dimension, (R_p) holds if and only if $p \in (1, 2]$. The result for $p < 2$ is a consequence of volume doubling and sub-Gaussian heat kernel estimates. The proof that (R_p) fails for $p > 2$ directly uses the definition of the Vicsek graph [9, Theorem 5.1].

In this article we construct a class of manifolds of arbitrary dimension for which (R_p) fails for all $p > 2$.¹ These manifolds are thickenings of what we call *spinal graphs*, satisfying generalised dimension conditions defined in terms of the spinal structure along with a polynomial volume lower bound. The Vicsek graphs satisfy these conditions, but the proof of this exploits their fractal nature. We construct a large class of non-fractal spinal graphs with the desired dimension conditions and volume lower bounds, thus yielding manifolds of arbitrary dimension with no fractal structure that fail (R_p) for all $p > 2$.

Notation

The graphs we consider are non-directed, with at most one edge per pair of vertices, and with no edges from a vertex to itself. The set of vertices of a graph G is denoted by $V(G)$, and if two vertices $x, y \in V(G)$ are neighbours we write $x \sim y$. The set of edges of G is denoted by $E(G)$. For a connected graph G we let $d_G(x, y)$ denote the combinatorial distance between x and y , given by the minimum length of a path from x to y , and for $x \in V(G)$, $r > 0$ let

$$B_G(x, r) := \{y \in V(G) : d_G(x, y) \leq r\}.$$

2 Spinal graphs

Definition 1 Let G be a connected graph, $\Sigma \subset V(G)$, and let $\pi : V(G) \rightarrow \Sigma$ be a function such that

- $\pi(x) = x$ for all $x \in \Sigma$,
- $\pi^{-1}(x)$ is finite for all $x \in \Sigma$,
- if $a, b \in V(G)$ and $\pi(a) \neq \pi(b)$, then every path from a to b contains a subpath from $\pi(a)$ to $\pi(b)$.

¹ In fact, we prove the stronger result that (RR_q) fails for all $q \in (1, 2)$.

Fig. 1 A spinal graph (G, Σ, π) , with a few vertices labeled. The spine $\Sigma \subset V(G)$ consists of the shaded vertices

We refer to (G, Σ, π) as a *spinal graph*, and the set of vertices Σ is called the *spine*.

Remark 1 One could formulate this definition without the finiteness condition, but it will be convenient for us to keep it.

An example of a spinal graph (G, Σ, π) is pictured in Fig. 1. There the vertices of the spine Σ are shaded black, while the other vertices are unshaded; for each vertex x , the point $\pi(x)$ is the (uniquely determined) point on Σ of minimal distance to x . The dotted lines are *not* edges of G ; if they were to be added to G , then the resulting graph would not be a spinal graph.

To help the reader familiarise themselves with the definition of a spinal graph we prove the following lemma (which will be useful later).

Lemma 1 Let (G, Σ, π) be a spinal graph, and suppose $a, b \in V(G)$ with $\pi(a) = \pi(b) =: x$. Then every minimal path from a to b is contained entirely in $\pi^{-1}(x)$.

Proof Suppose this is false. Then there exist $a, b \in V(G)$ with $\pi(a) = \pi(b) =: x$ and a path γ from a to b of minimal length which passes through a vertex c with $\pi(c) \neq x$. Since $\pi(a) \neq \pi(c)$, by the third condition in the definition of a spinal graph, there exists a subpath $\heartsuit$ of γ from $\pi(a) = x$ to $\pi^2(c) = \pi(c)$, so we can write γ as a concatenation of paths $\gamma = \alpha * \heartsuit * \beta$, where α is a path from a to x and β is a path from $\pi(c)$ to b ; viewed as a commutative diagram,

$$\begin{array}{ccc} a & \xrightarrow{\gamma} & b \\ \alpha \downarrow & & \uparrow \beta \\ x & \xrightarrow{\heartsuit} & \pi(c). \end{array}$$

Similarly, there is a subpath $\heartsuit'$ of β from $\pi(c)$ to $\pi(b) = x$, and we can write $\beta = \delta * \heartsuit' * \delta'$ as a concatenation of paths, summarised by the commutative diagram

$$\begin{array}{ccc} b & \xleftarrow{\delta'} & x \\ \beta \uparrow & & \uparrow \heartsuit' \\ \pi(c) & \xrightarrow{\delta} & \pi(c). \end{array}$$

Putting these commutative diagrams together we get

$$\begin{array}{ccccc}
 a & \xrightarrow{\gamma} & b & \xleftarrow{\delta'} & x \\
 \alpha \downarrow & & \uparrow \beta & & \uparrow \heartsuit' \\
 x & \xrightarrow{\heartsuit} & \pi(c) & \xrightarrow{\delta} & \pi(c),
 \end{array}$$

from which we can read that

$$\ell(\gamma) = \ell(\alpha * \heartsuit * \delta * \heartsuit' * \delta') = \ell(\alpha) + \ell(\heartsuit) + \ell(\delta) + \ell(\heartsuit') + \ell(\delta')$$

where $\ell(\cdot)$ denotes the length of a path. Since $x \neq \pi(c)$, the paths $\heartsuit$ and $\heartsuit'$ have positive length, so we find that

$$\ell(\alpha * \delta') = \ell(\alpha) + \ell(\delta') < \ell(\gamma).$$

Since $\alpha * \delta'$ is a path from a to b , this contradicts the assumption that γ has minimal length. $\square$

Spinal graphs may be constructed by gluing a collection of finite graphs along another graph; this is made precise in the following example. In fact, this construction yields all spinal graphs (up to isomorphism, in the usual graph-theoretical sense), as will be shown in Proposition 1.

Example 1 Let Γ be a connected graph and let $(G_x)_{x \in V(\Gamma)}$ be a collection of finite connected graphs indexed by the vertices of Γ . Suppose that for each $x \in V(\Gamma)$ a distinguished vertex $z_x \in V(G_x)$ is given. Then one can construct a graph G by gluing each G_x to Γ with the identification $z_x \sim x$. More precisely we have

$$V(G) := \bigsqcup_{x \in V(\Gamma)} V(G_x) = \{(x, z) : x \in V(\Gamma), z \in G_x\}, \quad (1)$$

and two vertices $(x_1, y_1), (x_2, y_2)$ are neighbours if and only if either

$$x_1 = x_2 = x \text{ and } y_1 \sim y_2 \text{ in } G_x,$$

or

$$y_1 = z_{x_1} \text{ and } y_2 = z_{x_2} \text{ and } x_1 \sim x_2 \text{ in } \Gamma.$$

The graph Γ , along with the graphs $(G_x)_{x \in V(\Gamma)}$, naturally embed into G . We set $\Sigma := V(\Gamma)$ in this embedding; in the disjoint union representation (1) we have

$$\Sigma := \{(x, z_x) : x \in V(\Gamma)\}.$$

Every vertex $z \in V(G)$ belongs to precisely one of the embedded graphs G_x with $x \in V(\Gamma)$, and we define $\pi : V(G) \rightarrow \Sigma$ by the relation $z \in G_{\pi(z)}$; in the disjoint union representation (1), we have $\pi(x, z) = (x, z_x)$.

It is immediate that $\pi(x) = x$ for all $x \in \Sigma$, and that each $\pi^{-1}(x)$ is finite. Now suppose $a, b \in V(G)$ with $\pi(a) \neq \pi(b)$. By construction, every path including a that does not pass through $\pi(a)$ must be entirely contained in $G_{\pi(a)}$. Since $b \notin G_{\pi(a)}$, every path from a to b must pass through $\pi(a)$, and by symmetry such a path must also pass through $\pi(b)$. That is, every path from a to b contains a subpath from $\pi(a)$ to $\pi(b)$. Therefore (G, Σ, π) is indeed a spinal graph.

Proposition 1 Suppose (G, Σ, π) is a spinal graph. Then there exists a connected graph Γ and a collection $(G_x)_{x \in V(\Gamma)}$ of finite connected graphs such that there exists an isomorphism between G and the graph constructed in Example 1, under which Σ corresponds to $V(\Gamma)$ and π corresponds to the map π' given by the relation $z \in G_{\pi'(z)}$.

Proof Let (G, Σ, π) be a spinal graph. Let Γ be the full subgraph determined by Σ , and for every $x \in \Sigma = V(\Gamma)$ let G_x be the full subgraph determined by $\pi^{-1}(x)$. Then Γ is connected, each G_x is finite and connected (by Lemma 1), and we have a bijection

$$\varphi: V(G) \rightarrow \bigsqcup_{x \in V(\Gamma)} V(G_x), \quad a \mapsto (\pi(a), a).$$

By the construction in Example 1, it suffices to show that $a, b \in V(G)$ are neighbours if and only if either $\pi(a) = \pi(b)$ and $a \sim b$ in $G_{\pi(a)}$, or $a = \pi(a)$ and $b = \pi(b)$ and $\pi(a) \sim \pi(b)$ in Γ .

By Lemma 1, if $\pi(a) = \pi(b)$ then every shortest path from a to b is entirely contained in $G_{\pi(a)}$, so in this case a and b are neighbours in G if and only if $a \sim b$ in $G_{\pi(a)}$. On the other hand, if $\pi(a) \neq \pi(b)$, then every path from a to b contains a subpath from $\pi(a)$ to $\pi(b)$, and thus a and b are neighbours if and only if either $a = \pi(a)$ and $b = \pi(b)$ or $a = \pi(b)$ and $b = \pi(a)$. In the first case, since Γ is the full subgraph determined by $\Sigma = \pi(V(G))$, we have $\pi(a) \sim \pi(b)$ in Γ , and we are done. The second case never occurs: since $\pi(b) \in \Sigma$, we would have $\pi(a) = \pi(\pi(b)) = \pi(b)$, which is a contradiction. $\square$

3 Dimensions of a spinal graph

For a spinal graph (G, Σ, π) we write d_Σ and B_Σ for the combinatorial distance and balls in the full subgraph determined by Σ .

Definition 2 Let (G, Σ, π) be a spinal graph. For all $x, y \in V(G)$ define the *spinal distance* $[x, y]$ by

$$[x, y] := d_\Sigma(\pi(x), \pi(y)),$$

and for $r > 0$ we define associated *spinal sets* by

$$D(x, r) := \{y \in G : [x, y] \leq r\} = \pi^{-1}(B_\Sigma(\pi(x), r)).$$

The spinal distance is a pseudometric on $V(G)$, and the quotient metric space is isometric to (Σ, d_Σ) , but we will not use this fact in what follows.

Definition 3 Let $\delta_\Sigma, \delta_G \geq 1$. We say that a spinal graph (G, Σ, π) has *dimensions* $(\delta_\Sigma, \delta_G)$ if there exists a point $x_0 \in \Sigma$ and an increasing sequence $(n_k)_{k \in \mathbb{N}}$ of natural numbers such that for all $k \in \mathbb{N}$,

$$|D(x_0, 2n_k)| \lesssim |D(x_0, n_k)|, \quad (2)$$

$$|B_\Sigma(x_0, 2n_k)| \lesssim n_k^{\delta_\Sigma}, \quad (3)$$

$$|D(x_0, n_k)| \simeq n_k^{\delta_G}. \quad (4)$$

Note that the dimensions of a spinal graph need not be uniquely determined, and may vary for different choices of x_0 and $(n_k)_{k \in \mathbb{N}}$.

Example 2 Let $n \in \mathbb{N}$ and consider the Vicsek graph $\mathcal{V}^n$ in $\mathbb{R}^n$, the construction of which is given in [4, Proof of Theorem 4.1], [8, Chapter 5], and [9, Sec. 5]. One can consider $\mathcal{V}^n$ as a graph with $V(\mathcal{V}^n) \subset \mathbb{Z}^n$, defined as an increasing union of subgraphs $\bigcup_{m=0}^{\infty} \mathcal{V}_m^n$. The subgraph $\mathcal{V}_0^n$ consists of $2^n + 1$ vertices: one at each corner of the unit n -cube, and a central vertex at the origin. Each corner vertex is connected to the central vertex. For $m \geq 1$, $\mathcal{V}_m^n$ is constructed inductively by connecting a copy of $\mathcal{V}_{m-1}^n$ to each ‘corner’ of $\mathcal{V}_{m-1}^n$. It follows that $|V(\mathcal{V}_m^n)| \simeq (2^n + 1)^m$ (see [4, equation (4.10)]).

Let $\Sigma \in V(\mathcal{V}^n)$ be the set of vertices along the 2^n diagonals: with $V(\mathcal{V}^n) \subset \mathbb{Z}^n$, we have

$$\Sigma = \{(\varepsilon_1 m, \varepsilon_2 m, \dots, \varepsilon_n m) \in \mathbb{Z}^n : m \in \mathbb{N}, \varepsilon_j \in \{1, -1\}\}.$$

For every vertex $x \in V(\mathcal{V}^n)$ there is a unique $y \in \Sigma$ such that x and y are connected by a path which intersects Σ only at y . Setting $\pi(x) := y$ makes $(\mathcal{V}^n, \Sigma)$ a spinal graph. Pictured in Fig. 2 are the first few steps of the construction of $\mathcal{V}^2$, with the spine Σ emphasised.

Let $o \in V(\mathcal{V})$ be the ‘center vertex’ of $\mathcal{V}_0^n$, and for $k \in \mathbb{N}$ let $n_k := 3^k$. Then $D(o, n_k) = \mathcal{V}_k^n$, and so

$$|D(o, n_k)| = (2^n + 1)^k = n_k^{\log_3(2^n + 1)}.$$

We also have

$$|D(o, 2n_k)| \leq |D(o, n_{k+1})| = n_{k+1}^{\log_3(2^n + 1)} \simeq n_k^{\log_3(2^n + 1)} = |D(o, n_k)|$$

and

$$|B_{\Sigma}(o, 2n_k)| = 2^n(2n_k) - 1 < 2^{n+1}3^k \simeq n_k,$$

which tells us that the spinal graph $(\mathcal{V}^n, \Sigma)$ has dimensions $(1, \log_3(2^n + 1))$. In addition, $\mathcal{V}^n$ has polynomial volume growth of dimension $\log_3(2^n + 1)$, that is

$$|B_{\mathcal{V}}(x, r)| \simeq r^{\log_3(2^n + 1)}$$

for all $x \in V(\mathcal{V}^n)$ and $r \in \mathbb{N}$. (see [4, page 632]). The lower estimate will allow us to apply Corollary 2 to $\mathcal{V}^n$.

In Section 6 we construct spinal graphs with global volume lower bounds and dimensions $(1, D)$ with $D > 1$ that do not arise from fractals.

4 Nash-type inequalities and spinal dimensional consequences

Now we assume that G is locally finite. For each vertex $x \in V(G)$ let $m(x) < \infty$ denote the number of neighbours of x , and for each $f: V(G) \rightarrow \mathbb{C}$ define the length of the gradient $|\nabla f(x)|$ by

$$|\nabla f(x)| := \left(\frac{1}{2} \sum_{\substack{y \in V(G) \\ y \sim x}} \frac{1}{m(x)} |f(y) - f(x)|^2 \right)^{1/2}.$$

For $1 < p \leq \infty$ and $\beta > 0$, we consider the Nash-type inequality

$$S(p, \beta) : \quad \|f\|_p^{1 + \frac{p'}{\beta}} \lesssim \|f\|_1^{\frac{p'}{\beta}} \|\nabla f\|_p \quad (f: V(G) \rightarrow \mathbb{C} \text{ finitely supported}),$$

which G may or may not satisfy.

Fig. 2 The first three steps of the construction of the Vicsek graph $\mathcal{V}^2$, with spine

In the presence of a spinal structure, the inequality $S(p, \beta)$ gives quantitative information connecting the ‘spinal volume growth’ of G with the volume growth of Σ . This is shown by constructing test functions, defined in terms of the spinal distance, which are constant on the fibres $\pi^{-1}(x)$. The gradients of these test functions are supported on the spine Σ , while the functions themselves are supported on spinal sets.

Lemma 2 *Let (G, Σ, π) be a spinal graph. Fix $p \in (1, \infty)$ and suppose that G satisfies $S(p, \beta)$. Then for every $x_0 \in \Sigma$ and $n \in \mathbb{N}$ we have*

$$|D(x_0, n)|^{\frac{1}{p}(1+\frac{p'}{\beta})} \lesssim n^{-1} |D(x_0, 2n)|^{\frac{p'}{\beta}} |B_{\Sigma}(x_0, 2n)|^{\frac{1}{p}}. \quad (5)$$

Proof For each $x_0 \in \Sigma$ and $n \in \mathbb{N}$ define $g_n: V(G) \rightarrow [0, 1]$ by

$$g_n(x) := \frac{\max(0, n - [x, x_0])}{n}.$$

Note that $g_n(x) = 0$ if and only if $[x, x_0] \geq n$, so that $\text{supp } g_n = D(x_0, n - 1)$. Furthermore note that g_n is constant on each $\pi^{-1}(x)$.

Since $|g_{2n}| \leq 1$ and $\text{supp } g_{2n} \subset D(x_0, 2n)$ we have

$$\|g_{2n}\|_1 \lesssim |D(x_0, 2n)|. \quad (6)$$

Next, since $g_{2n}(x) \geq 1/2$ for $x \in D(x_0, n)$, we have

$$\|g_{2n}\|_p \geq \left(\sum_{x \in D(x_0, n)} 2^{-p} \right)^{1/p} \simeq |D(x_0, n)|^{1/p}. \quad (7)$$

Finally, note that $|\nabla g_{2n}(x)| = 0$ whenever $x \in G \setminus \Sigma$ (since g_{2n} is constant on each connected component of $G \setminus \Sigma$) or $x \in G \setminus D(x_0, 2n)$ (since $\text{supp } g_{2n} = D(x_0, 2n - 1)$). When $x \in \Sigma \cap D(x_0, 2n)$ and $y \sim x$, we have

$$g_{2n}(x) - g_{2n}(y) = \begin{cases} (2n)^{-1} & \text{if } y \in \Sigma \cap D(x_0, 2n) \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\begin{aligned} \|\nabla g_{2n}\|_p &= \left(\sum_{x \in \Sigma \cap D(x_0, 2n)} \left(\frac{1}{2} \sum_{y \sim x, y \in \Sigma \cap D(x_0, 2n)} \frac{1}{m(x)} (2n)^{-2} \right)^{p/2} \right)^{1/p} \\ &\lesssim n^{-1} |\Sigma \cap D(x_0, 2n)|^{1/p} = n^{-1} |B_\Sigma(x_0, 2n)|^{1/p} \end{aligned}$$

Therefore, applying $S(p, \beta)$ to g_{2n} , we get (5). $\square$

The previous lemma can be used to show that the Nash-type inequalities $S(p, \beta)$ restrict the possible dimensions of a spinal graph.

Lemma 3 *Let (G, Σ, π) be a spinal graph with dimensions $(\delta_\Sigma, \delta_G)$. Fix $p > 1$ and $\beta > 0$, and suppose G satisfies $S(p, \beta)$. Then*

$$\frac{\delta_G - \delta_\Sigma}{p} - \frac{\delta_G}{\beta} \leq -1. \quad (8)$$

Proof Fix $x_0 \in \Sigma$ and a sequence $(n_k)_{k \in \mathbb{N}}$ as in Definition 3. From Lemma 2 and assumptions (3) and (2), for all $k \in \mathbb{N}$ we have

$$\begin{aligned} |D(x_0, n_k)|^{\frac{1}{p} \left(1 + \frac{p'}{\beta}\right)} &\lesssim n_k^{-1} |D(x_0, 2n_k)|^{\frac{p'}{\beta}} |B_\Sigma(x_0, 2n_k)|^{\frac{1}{p}} \\ &\lesssim n_k^{-1 + \frac{\delta_\Sigma}{p}} |D(x_0, n_k)|^{\frac{p'}{\beta}}. \end{aligned}$$

Rearranging yields

$$|D(x_0, n_k)|^{\frac{1}{p} - \frac{1}{\beta}} \lesssim n_k^{-1 + \frac{\delta_\Sigma}{p}},$$

and then

$$n_k^{\delta_G \left(\frac{1}{p} - \frac{1}{\beta}\right)} \lesssim n_k^{-1 + \frac{\delta_\Sigma}{p}}.$$

follows by (4). Since n_k is increasing, taking the limit as $k \rightarrow \infty$ tells us that

$$\delta_G \left(\frac{1}{p} - \frac{1}{\beta} \right) \leq -1 + \frac{\delta_\Sigma}{p},$$

which is equivalent to (8). $\square$

Corollary 1 *Suppose the conditions of Lemma 3 are satisfied, with $\delta_G > \delta_\Sigma$. Then $\delta_G > \beta$ and*

$$p \geq \beta \frac{\delta_G - \delta_\Sigma}{\delta_G - \beta}. \quad (9)$$

Proof Rearranging (8) gives

$$\delta_G \geq \beta \left(1 + \frac{\delta_G - \delta_\Sigma}{p} \right).$$

Since $\delta_G - \delta_\Sigma > 0$, we get $\delta_G > \beta$. We can then rearrange further to get (9). $\square$

5 Riesz transform unboundedness for thickened spinal graphs

Definition 4 Let G be a uniformly locally finite graph (i.e. $\sup_{x \in V(G)} m(x) < \infty$) and $n \in \mathbb{N}$. Then an n -dimensional *thickening* of G is a smooth Riemannian manifold M constructed by replacing each vertex $x \in V(G)$ by an n -sphere, each edge $e \in E(G)$ by an n -cylinder, and welding the cylinders smoothly to the balls according to the graph structure of G , in such a way that M has bounded geometry (i.e. M has positive injectivity radius, and Ricci curvature bounded from below).

More precisely: define

$$\tilde{M} := \bigsqcup_{x \in V(G)} B_x \sqcup \bigsqcup_{e \in E(G)} C_e,$$

where B_x is isometric to a round n -sphere S^n with $m(x)$ disjoint open balls of fixed small radius ε removed, and where each C_e is isometric to a cylinder $S_\varepsilon^{n-1} \times [0, 1]$, with $S_\varepsilon^{n-1} = \partial B(0, \varepsilon) \subset \mathbb{R}^n$. A C^0 Riemannian manifold M' is constructed as a quotient of $\tilde{M}$ by gluing a cylinder C_e to two spheres B_x and B_y if and only if $x \sim y$ in G (in such a way that every 'hole' in B_x has a cylinder attached to it). A thickening M with bounded geometry may then be defined by smoothing the interface between spheres and the cylinders in M' arbitrarily (but uniformly among all the interfaces).

Remark 2 In what follows, we may replace a thickening of G (in the sense above) by any Riemannian manifold M of bounded geometry that is *isometric to G at infinity* in the sense of Coulhon–Saloff-Coste [12] (following Kanai [13]); our discretisation/thickening procedures only depend on results in [12].

The following proposition can be proven by directly following the proof of [11, Proposition 6.2] (see also the first part of [9, Theorem 5.1]). The proof involves the discretisation results of [12, Sec. 6].

Proposition 2 Let G be a locally uniformly finite graph, and let M be a thickening of G of any dimension. Fix $p \in (1, \infty)$ and suppose that M satisfies (RR_p) . Furthermore, suppose that the heat kernel h of M satisfies

$$h_t(x, x) \lesssim t^{-\alpha/2} \quad \text{for all } t > 1, x \in M.$$

Then G satisfies $S(p, \alpha)$.

Since $S(p, \alpha)$ restricts the possible dimensions of a spinal graph, we may argue by contraposition to show that dimension and volume information on a spinal graph implies unboundedness of the Riesz transform on $L^p(M)$ for sufficiently large $p > 2$ (in fact, we prove that $(RR_{p'})$ does not hold for sufficiently large $p > 2$, which is strictly stronger).

Theorem 1 Let (G, Σ, π) be a locally uniformly finite spinal graph with dimensions $(\delta_\Sigma, \delta_G)$, with $\delta_G > \delta_\Sigma$. Furthermore, suppose that $B_G(x, r) \gtrsim r^v$ for all $r \geq 1$. Let M be a thickening of G of any dimension. Then for all

$$p > 2 \frac{\delta_G - \delta_\Sigma}{\frac{\delta_G}{v} - 2\delta_\Sigma + 2} =: p_c(\delta_\Sigma, \delta_G, v),$$

M does not satisfy (R_p) .

Proof The volume assumption on G implies a corresponding large-scale volume estimate

$$V(x, r) \gtrsim r^\nu \quad (\text{for all } r > 1, x \in M)$$

on M (this may be derived from the results of [12, Sec. 6]). Since M has bounded geometry, [4, Theorem 1.1] implies the heat kernel estimate

$$h_t(x, x) \lesssim t^{-\nu/(v+1)} \quad \text{for all } t > 1$$

on M . Fix $q > 1$ and suppose that M satisfies (R_q) , hence also $(RR_{q'})$. Proposition 2 then implies that G satisfies $S(q', 2\nu/(v+1))$, and Corollary 1 then yields

$$q' \geq \frac{2\nu}{\nu+1} \left(\frac{\delta_G - \delta_\Sigma}{\delta_G - \frac{2\nu}{\nu+1}} \right) = p'_c$$

or equivalently that $q \leq p_c$. Therefore M does not satisfy (R_p) for any $p > p_c$. $\square$

Taking $\delta_\Sigma = 1$ and $\nu = \delta_G$ gives the following corollary.

Corollary 2 *Let (G, Σ, π) be a locally uniformly finite spinal graph with dimensions $(1, \delta_G)$, with $\delta_G > 1$, and suppose that $B_G(x, r) \gtrsim r^{\delta_G}$ for all $r \in \mathbb{N}$. Let M be a thickening of G of any dimension. Then the Riesz transform bound (R_p) for M fails for all $p > 2$.*

As remarked in Example 2, these assumptions are satisfied by the Vicsek graphs $\mathcal{V}^n$, reproving [9, Theorem 5.1].

6 Non-fractal spinal graphs with volume lower bounds

Fix $D > 1$. In this section we show how to construct locally uniformly finite spinal graphs (G, Σ, π) with dimensions $(1, D)$ and the volume lower bound

$$|B_G(x, r)| \gtrsim r^D \quad (x \in V(G), r \in \mathbb{N}), \quad (10)$$

but which need not possess any ‘fractal’ structure (in contrast with the Vicsek graph example). Corollary 2 applies to such spinal graphs, thus yielding many manifolds M for which (R_p) fails for all $p > 2$.

First we need a technical lemma on volumes of intersections of balls in doubling graphs. We defer the proof to Sec. 7. Recall that a graph G is *doubling* if there exist constants $C_d, \nu > 0$ such that for all $0 < r < R < \infty$ and $x \in V(G)$,

$$|B_G(x, R)| \leq C_d(R/r)^\nu |B_G(x, r)|.$$

Taking $R = 1$ and $r = 1 - \varepsilon$ for ε arbitrarily small shows that a doubling graph is locally uniformly finite, with $m(x) \leq C_d$ for all $x \in V(G)$.

Lemma 4 *Let G be a doubling graph. Then there exists $C > 0$, depending only on the doubling constants of G , such that for all $y \in V(G)$ and $R > 0$, and for all $x \in B(y, R)$ and $r \leq 2R$, we have*

$$|B_G(x, r) \cap B_G(y, R)| \geq C |B_G(x, r)|.$$

Now we move on to our construction. This is inspired by the ‘plate’ construction in [4, Theorem 5.1].

Example 3 Fix $\delta > D$, and let $(P_n)_{n \in \mathbb{N}}$ be a family of graphs satisfying

$$|B_{P_n}(x, r)| \simeq r^\delta \quad (x \in V(P_n), r \in \mathbb{N})$$

with implicit constants independent of n . For simplicity one can take each P_n to be equal to a fixed graph P ; one could even take $\delta \in \mathbb{N}$ and $P = \mathbb{Z}^\delta$. We allow for arbitrary choices to emphasise that self-similarity is not necessary. Let $\alpha = (D - 1)/\delta$ (so that $\alpha\delta + 1 = D$ and $\alpha < 1$) and for each $n \in \mathbb{N}$ choose an arbitrary vertex $o_n \in V(P_n)$. Construct a spinal graph (G, Σ, π) with $\Sigma = \mathbb{N}$ as in Example 1 by taking G_n to be the full subgraph of P_n determined by $B_{P_n}(o_n, n^\alpha)$, and choosing as distinguished points $z_n = o_n$.

To show that this spinal graph has dimensions $(1, D)$, take the sequence $n_k = k$ and observe that

$$|D(1, k)| = \sum_{n=1}^k |B_{P_n}(o_n, n^\alpha)| \simeq \sum_{n=1}^k n^{\alpha\delta} \simeq k^{\alpha\delta+1} = k^D$$

(the second sum may be estimated by comparing with integrals of the function $t \mapsto t^{\alpha\delta}$). In particular we have $|D(1, 2k)| \simeq (2k)^D \simeq |D(1, k)|$, and furthermore it is clear that $|B_{\mathbb{N}}(x, r)| \simeq r$. Therefore the spinal graph has dimensions $(1, D)$.

It is more difficult to show the global volume lower bound (10), but luckily the proof of [4, Theorem 5.1] already does this for a similar problem. Note that it suffices to assume $r \geq 2$.

First we show that $|B_G(n, r)| \gtrsim r^D$ for all $n \in \mathbb{N}$. To see this, write

$$\begin{aligned} |B_G(n, r)| &\geq \sum_{k=0}^r |B_{P_{n+k}}(o_{n+k}, \min(r - k, (n + k)^\alpha))| \\ &\geq \sum_{k=0}^{\lfloor r/2 \rfloor} |B_{P_{n+k}}(o_{n+k}, \min(r - k, k^\alpha))| \\ &\simeq \sum_{k=0}^{\lfloor r/2 \rfloor} k^{\alpha\delta} \simeq \lfloor r/2 \rfloor^{\alpha\delta+1} \simeq r^D, \end{aligned}$$

using that $k^\alpha < r - k$ for $k \leq r/2$ in the third line.

Now suppose $x \in V(G)$ with $\pi(x) = n$. After identifying x with the appropriate vertex $z \in B_{P_n}(o_n, n^\alpha)$ (which, recall, is identified with $\pi(x)$), we get an identification of $B_G(x, r) \cap \pi^{-1}(n)$ with $B_{P_n}(z, r) \cap B_{P_n}(o_n, n^\alpha)$. Thus for $r \leq 2n^\alpha$ we have

$$\begin{aligned} |B_G(x, r)| &\geq |B_G(x, r) \cap \pi^{-1}(n)| \\ &= |B_{P_n}(z, r) \cap B_{P_n}(o_n, n^\alpha)| \\ &\gtrsim |B_{P_n}(z, r)| \simeq r^\delta > r^D \end{aligned}$$

using Lemma 4 in the third line. On the other hand, if $r > 2n^\alpha$, then $B_G(x, r)$ contains both $\pi^{-1}(n)$ and $B_G(n + 1, r - 1 - n^\alpha)$, so

$$\begin{aligned} |B_G(x, r)| &\geq \max(|\pi^{-1}(n)|, |B_G(n + 1, r - 1 - n^\alpha)|) \\ &\gtrsim |\pi^{-1}(n)| + |B_G(n + 1, r - 1 - n^\alpha)| \\ &\gtrsim n^{\alpha D} + (r - 1 - n^\alpha)^D \\ &\simeq (n^\alpha + r - 1 - n^\alpha)^D \\ &\simeq r^D. \end{aligned}$$

This completes the proof of (10).

The following corollary is then an immediate consequence of Corollary 2.

Corollary 3 *Suppose M is a thickening of a spinal graph (G, Σ, π) as constructed as in Example 3. Then the Riesz transform bound (R_p) for M fails for all $p > 2$.*

Remark 3 It is probably possible to construct spinal graphs with dimensions $(\delta_\Sigma, \delta_G)$ with $1 < \delta_\Sigma < \delta_G$ and with a polynomial volume lower bound of dimension δ_G , thus yielding manifolds M for which (R_p) fails for all $p > p_c > 2$. Since our construction exploits taking $\Sigma = \mathbb{N}$, this is beyond the scope of this article. It may even be possible to show that (R_p) holds on such manifolds for $p \in (2, p_c)$, but this is very much beyond the scope of this article.

7 Proof of Lemma 4

Here we prove the technical lemma needed in the construction of the previous section. We write $B(x, r) := B_G(x, r)$ and $d(x, y) := d_G(x, y)$.

Proof First note that if the result is true for $r \leq R$, then it holds for $r \leq 2R$, because in this case for $r > R$ we have

$$\begin{aligned} |B(x, r) \cap B(y, R)| &\geq |B(x, R) \cap B(y, R)| \geq C|B(x, R)| \\ &\geq CC_d^{-1}(R/r)^\nu |B(x, r)| \\ &\geq CC_d^{-1}2^{-\nu} |B(x, r)| \end{aligned}$$

using the doubling condition and $R \geq r/2$ in the last step. Thus we assume that $r \leq R$, and split the proof into two cases.

Case 1: $r > 2d(x, y)$. By definition of the combinatorial distance, there exists a vertex z such that $d(y, z) + d(z, x) = d(y, x)$ and $d(y, z) = \lceil d(y, x)/2 \rceil$. Suppose that $z' \in B(z, r - d(y, z))$. Then

$$\begin{aligned} d(z', x) &\leq r - d(y, z) + d(z, x) \\ &= r - 2\lceil d(y, x)/2 \rceil + d(y, x) \\ &\leq r - 2d(y, x) \leq r \end{aligned}$$

and

$$d(z', y) \leq r - d(y, z) + d(z, y) = r \leq R,$$

so $B(z, r - d(y, z)) \subset B(x, r) \cap B(y, R)$. Therefore

$$\begin{aligned} |B(x, r) \cap B(y, R)| &\geq |B(z, r - d(y, z))| \\ &\gtrsim \left(\frac{r - d(y, z)}{r + d(x, z)} \right)^\nu |B(z, r + d(x, z))| \\ &\geq \left(\frac{r - d(y, z)}{r + d(x, z)} \right)^\nu |B(x, r)| \end{aligned}$$

for some $\nu > 0$ determined by the doubling constant of G . To see that the bracketed expression above is uniformly bounded below, estimate its reciprocal from above by

$$\begin{aligned} \frac{r + d(x, z)}{r - d(y, z)} &= \frac{r - d(y, z) + d(x, y)}{r - d(y, z)} \\ &= 1 + \frac{d(x, y)}{r - d(y, z)} \\ &< 1 + \frac{d(x, y)}{2d(x, y) - d(y, z)} \\ &= 1 + \frac{d(x, y)}{d(x, y) + d(z, x)} \\ &\leq 2. \end{aligned}$$

using $r > 2d(x, y)$ in the third line and $d(z, x) = d(x, y) - d(y, z)$ in the fourth line.

Case 2: $r \leq 2d(x, y)$. Note that the estimate for $r < 2$ follows from the fact that G is locally uniformly finite, so it suffices to consider $r \geq 2$. As in the previous case, there exists a vertex z such that $d(y, z) + d(z, x) = d(y, x)$ and $d(x, z) = \lfloor r/2 \rfloor$ (here we use that $r/2 \leq d(x, y)$). If $z' \in B(z, \lfloor r/2 \rfloor)$, then

$$d(z', x) < 2 \left\lfloor \frac{r}{2} \right\rfloor \leq r$$

and

$$d(z', y) < \left\lfloor \frac{r}{2} \right\rfloor + d(z, y) = \left\lfloor \frac{r}{2} \right\rfloor + d(x, y) - \left\lfloor \frac{r}{2} \right\rfloor = d(x, y) \leq R,$$

using that $x \in B(y, R)$ by assumption, and so $B(z, \lfloor r/2 \rfloor) \subset B(x, r) \cap B(y, R)$. Therefore

$$\begin{aligned} |B(x, r) \cap B(y, R)| &\geq |B(z, \lfloor r/2 \rfloor)| \\ &\geq |B(z, r/4)| \\ &\geq (8^\nu C_d)^{-1} |B(z, 2r)| \\ &\geq (8^\nu C_d)^{-1} |B(z, r + d(z, x))| \\ &\geq (8^\nu C_d)^{-1} |B(x, r)| \end{aligned}$$

since $\lfloor r/2 \rfloor \geq r/4$ holds whenever $r \geq 2$, and using the doubling condition. $\square$

Acknowledgements I thank Li Chen and Thierry Coulhon for many interesting discussions on this material. I also thank the anonymous referee for their careful reading and suggestions. Portions of this work were carried out while the author was a postdoc at Université Grenoble-Alpes and at TU Delft (supported by the VIDI subsidy 639.032.427 of the Netherlands Organisation for Scientific Research (NWO)), and also while the author held an Australian Mathematical Society Lift-off Fellowship.

Funding Open Access funding enabled and organized by Projekt DEAL.

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References

1. Amenta, A., Tolomeo, L.: A dichotomy concerning uniform boundedness of Riesz transforms on Riemannian manifolds. *Proc. Am. Math. Soc.* **147**(11), 4797–4803 (2019)
2. Auscher, P., Coulhon, T.: Riesz transform on manifolds and Poincaré inequalities. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **4**(3), 531–555 (2005)
3. Auscher, P., Coulhon, T., Duong, X.T., Hofmann, S.: Riesz transform on manifolds and heat kernel regularity. *Ann. Sci. École Norm. Sup.* **37**(6), 911–957 (2004)
4. Barlow, M., Coulhon, T., Grigor'yan, A.: Manifolds and graphs with slow heat kernel decay. *Invent. Math.* **144**(3), 609–649 (2001)
5. Bernicot, F., Frey, D.: Riesz transforms through reverse Hölder and Poincaré inequalities. *Math. Z.* **284**(3–4), 791–826 (2016)
6. Carron, G.: Riesz transform on manifolds with quadratic curvature decay. *Rev. Mat. Iberoam.* **33**(3), 749–788 (2017)
7. Carron, G., Coulhon, T., Hassell, A.: Riesz transform and L^p -cohomology for manifolds with Euclidean ends. *Duke Math. J.* **133**(1), 59–93 (2006)
8. Chen, L.: Quasi Riesz transforms, Hardy spaces and generalized sub-Gaussian heat kernel estimates, Ph.D. thesis, Université Paris-Sud and Australian National University (2014)
9. Chen, L., Coulhon, T., Feneuil, J., Russ, E.: Riesz transform for $1 \leq p \leq 2$ without Gaussian heat kernel bound. *J. Geom. Anal.* **27**(2), 1489–1514 (2017)
10. Coulhon, T., Duong, X.T.: Riesz transforms for $1 \leq p \leq 2$. *Trans. Amer. Math. Soc.* **351**(3), 1151–1169 (1999)
11. Coulhon, T., Duong, X.T.: Riesz transform and related inequalities on noncompact Riemannian manifolds. *Comm. Pure Appl. Math.* **56**(12), 1728–1751 (2003)
12. Coulhon, T., Saloff-Coste, L.: Variétés riemanniennes isométriques à l'infini. *Rev. Mat. Iberoamericana* **11**(3), 687–726 (1995)
13. Kanai, M.: Analytic inequalities, and rough isometries between noncompact Riemannian manifolds, Curvature and topology of Riemannian manifolds (Katata, : Lecture Notes in Math., vol. 1201. Springer, Berlin **1986**, 122–137 (1985)
14. Li, H.-Q.: La transformation de Riesz sur les variétés coniques. *J. Funct. Anal.* **168**(1), 145–238 (1999)
15. Li, H.-Q., Zhu, J.-X.: A note on “Riesz transform for $1 \leq p \leq 2$ without Gaussian heat kernel bound”. *J. Geom. Anal.* **28**(2), 1597–1609 (2018)

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