Infinitely divisible states on finite quantum groups

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Abstract

In this paper we study the states of Poisson type and infinitely divisible states on compact quantum groups. Each state of Poisson type is infinitely divisible, i.e., it admits n -th root for all $n \geq 1$. The main result is that on finite quantum groups infinitely divisible states must be of Poisson type. This generalizes Böge's theorem concerning infinitely divisible measures (commutative case) and Parthasarathy's result on infinitely divisible positive definite functions (cocommutative case). Two proofs are given.

Keywords Infinitely divisible states · States of Poisson type · Compact quantum group · Plancherel triple

1 Introduction

The space of bounded measures on a compact (semi)group is equipped with a natural convolution operation. The convolution of two probability measures is still a probability measure. *Infinitely divisible probability measures* are probability measures that admit n -th root for all $n \geq 1$, where the root is also a probability measure. On finite groups such probability measures have been shown to be of Poisson type, see [2] and [18].

A *positive definite function* on a compact group G is a continuous function $\phi : G \rightarrow \mathbb{C}$ such that $[\phi(g_i^{-1}g_j)]_{i,j=1}^n$ is a positive semi-definite matrix for all $g_1, \dots, g_n \in G$ and for all $n \geq 1$. It is *normalized* if $\phi(e) = 1$, where e is the unit of G . The pointwise product of two normalized positive definite functions on G is again a normalized positive definite function. From this we can define infinitely divisible normalized positive definite functions on a compact group in a natural way. This is thoroughly studied by Parthasarathy [17]. As a special case, he proved that every infinitely divisible normalized positive definite function on a finite group is of Poisson type, although the notion “Poisson type” was not explicitly defined in his paper.

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We shall consider the infinite divisibility of states on quantum groups, which provide a more general framework. Our main result is that any infinitely divisible state on a finite quantum group is of *Poisson type* (in the following also called simply a *Poisson state*). By taking the finite quantum group to be commutative or cocommutative, we recover the Böge's result [2] of infinitely divisible probability measures and Parthasarathy's result [17] on infinitely divisible normalized positive definite functions for finite groups, respectively. We will give two proofs of the main theorem. The first one is based on the ideas of [18] and the second one goes back to [17].

The plan of this paper is as follows. In Sect. 2 we recall the preliminaries on compact quantum groups and introduce the notion of Plancherel triples. In Sect. 3 we explain how to construct a Plancherel triple from an idempotent state on a finite quantum group. Section 4 is devoted to the study of Poisson states on compact quantum groups. Finally in Sect. 5 we prove the main result of this paper, namely that any infinitely divisible state on a finite quantum group is a Poisson state, in two different ways.

2 Preliminaries

2.1 Compact quantum group and its dual

Let us recall some definitions and properties of compact quantum groups. We refer to [21] and [15] for more details.

Definition 2.1 Let A be a unital C^* -algebra. If there exists a unital $*$ -homomorphism $\Delta : A \rightarrow A \otimes A$ such that

1. $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$;
2. $\{\Delta(a)(1 \otimes b) : a, b \in A\}$ and $\{\Delta(a)(b \otimes 1) : a, b \in A\}$ are linearly dense in $A \otimes A$;

then (A, Δ) is called a *compact quantum group* and Δ is called the *comultiplication* on A . Here and in the following, ι always denotes the identity map. We denote $\mathbb{G} = (A, \Delta)$ and $A = C(\mathbb{G})$. For simplicity, we write $\Delta^{(2)} = (\Delta \otimes \iota)\Delta$.

Any compact quantum group $\mathbb{G} = (A, \Delta)$ admits a unique *Haar state*, i.e. a state h on A such that

$$(h \otimes \iota)\Delta(a) = h(a)1 = (\iota \otimes h)\Delta(a), \quad a \in A.$$

Consider an element $u \in A \otimes B(H)$ with $\dim H = n$. By identifying $A \otimes B(H)$ with $\mathbb{M}_n(A)$ we can write $u = [u_{ij}]_{i,j=1}^n$, where $u_{ij} \in A$. The matrix u is called an *n -dimensional representation* of $\mathbb{G}$ if we have

$$\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}, \quad i, j = 1, \dots, n.$$

A representation u is called *unitary* if u is unitary as an element in $\mathbb{M}_n(A)$, and *irreducible* if the only matrices $T \in \mathbb{M}_n(\mathbb{C})$ such that $uT = Tu$ are multiples of identity matrix. Two representations $u, v \in \mathbb{M}_n(A)$ are said to be *equivalent* if there exists an invertible matrix $T \in \mathbb{M}_n(\mathbb{C})$ such that $Tu = vT$. Denote by $\text{Irr}(\mathbb{G})$ the set of equivalence classes of irreducible unitary representations of $\mathbb{G}$. For each $\alpha \in \text{Irr}(\mathbb{G})$, denote by $u^\alpha \in A \otimes B(H_\alpha)$ a representative of the class α , where H_α is the finite dimensional Hilbert space on which u^α acts. In the sequel we write $n_\alpha = \dim H_\alpha$.

Denote $\text{Pol}(\mathbb{G}) = \text{span} \left\{ u_{ij}^\alpha : 1 \leq i, j \leq n_\alpha, \alpha \in \text{Irr}(\mathbb{G}) \right\}$. This is a dense subalgebra of A . On $\text{Pol}(\mathbb{G})$ the Haar state h is faithful. It is well-known that $(\text{Pol}(\mathbb{G}), \Delta)$ is equipped with the Hopf*-algebra structure. That is, there exist a linear antihomomorphism S on $\text{Pol}(\mathbb{G})$, called the *antipode*, and a unital *-homomorphism $\epsilon : \text{Pol}(\mathbb{G}) \rightarrow \mathbb{C}$, called the *counit*, such that

$$\begin{aligned} (\epsilon \otimes \iota) \Delta(a) &= a = (\iota \otimes \epsilon) \Delta(a), \quad a \in \text{Pol}(\mathbb{G}), \\ m(S \otimes \iota) \Delta(a) &= \epsilon(a)1 = m(\iota \otimes S) \Delta(a), \quad a \in \text{Pol}(\mathbb{G}). \end{aligned}$$

Here m denotes the multiplication map $m : \text{Pol}(\mathbb{G}) \otimes_{\text{alg}} \text{Pol}(\mathbb{G}) \rightarrow \text{Pol}(\mathbb{G})$, $a \otimes b \mapsto ab$. Indeed, the antipode and the counit are uniquely determined by

$$\begin{aligned} S(u_{ij}^\alpha) &= (u_{ji}^\alpha)^*, \quad 1 \leq i, j \leq n_\alpha, \quad \alpha \in \text{Irr}(\mathbb{G}), \\ \epsilon(u_{ij}^\alpha) &= \delta_{ij}, \quad 1 \leq i, j \leq n_\alpha, \quad \alpha \in \text{Irr}(\mathbb{G}). \end{aligned}$$

Remark here that $* \circ S \circ * \circ S = \iota$. Also we have for all $a, b \in \text{Pol}(\mathbb{G})$

$$S((\iota \otimes h)(\Delta(b)(1 \otimes a))) = (\iota \otimes h)((1 \otimes b)\Delta(a)), \quad (2.1)$$

$$S((h \otimes \iota)((b \otimes 1)\Delta(a))) = (h \otimes \iota)(\Delta(b)(a \otimes 1)). \quad (2.2)$$

Now we add a remark on the C^* -norms on $\text{Pol}(\mathbb{G})$. We are interested in the following two C^* -norms on $\text{Pol}(\mathbb{G})$:

1. the *universal norm*:

$$\|a\|_u := \sup\{\|\pi(a)\| : \pi : \text{Pol}(\mathbb{G}) \rightarrow B(\mathcal{H}) \text{ is a unital } *- \text{homomorphism}\},$$

2. the *reduced norm*:

$$\|a\|_r := \|\pi_h(a)\|,$$

where π_h is the GNS representation associated with the Haar state h .

We shall denote by $C_u(\mathbb{G})$ and $C_r(\mathbb{G})$ the completions of $\text{Pol}(\mathbb{G})$ with respect to $\|\cdot\|_u$ and $\|\cdot\|_r$, respectively. Then the comultiplication Δ and the Haar state h on $\text{Pol}(\mathbb{G})$ admit extensions to $C_u(\mathbb{G})$ (resp. $C_r(\mathbb{G})$), denoted by Δ_u and h_u (resp. Δ_r and h_r), respectively. Both $(C_u(\mathbb{G}), \Delta_u)$ and $(C_r(\mathbb{G}), \Delta_r)$ form compact quantum groups.

Note that the counit ϵ can be always extended to $C_u(\mathbb{G})$. While, this is not always the case for $C_r(\mathbb{G})$. If ϵ can be also extended to $C_r(\mathbb{G})$, then $\mathbb{G}$ is said to be *coamenable*. An equivalent definition is, $\mathbb{G}$ is coamenable iff $\|\cdot\|_r = \|\cdot\|_u$. Note that $\|\cdot\|_r \leq \|\cdot\|_u$ always holds. We refer to [1] for more information. Throughout this paper, we shall always consider compact quantum group $\mathbb{G}$ on the universal level, so that the counit can always be extended to the $C(\mathbb{G})$.

The Peter-Weyl theory for compact groups can be extended to the quantum case. In particular, it is known that for each $\alpha \in \text{Irr}(\mathbb{G})$ there exists a positive invertible operator $Q_\alpha \in B(H_\alpha)$ such that $\text{Tr}(Q_\alpha^{-1}) := d_\alpha$, which we call *quantum dimension* of α , and the orthogonal relations hold:

$$h(u_{ij}^\alpha (u_{kl}^\beta)^*) = \frac{\delta_{\alpha\beta} \delta_{ik} (Q_\alpha)_{lj}}{d_\alpha}, \quad h((u_{ij}^\alpha)^* u_{kl}^\beta) = \frac{\delta_{\alpha\beta} \delta_{jl} (Q_\alpha^{-1})_{ki}}{d_\alpha},$$

where $\alpha, \beta \in \text{Irr}(\mathbb{G})$, $1 \leq i, j \leq n_\alpha$, $1 \leq k, l \leq n_\beta$.

We call $\mathbb{G}$ a *finite quantum group* if the underlying C^* -algebra $C(\mathbb{G})$ is finite dimensional. Note that when $\mathbb{G}$ is finite, we have $C(\mathbb{G}) = \text{Pol}(\mathbb{G})$ and then $\mathbb{G}$ is coamenable. In this case

each Q_α is identity and h is a trace, i.e. $h(ab) = h(ba)$ for any $a, b \in C(\mathbb{G})$. Then the orthogonal relation becomes

$$h(u_{ij}^\alpha (u_{kl}^\beta)^*) = h((u_{ij}^\alpha)^* u_{kl}^\beta) = \frac{\delta_{\alpha\beta} \delta_{ik} \delta_{jl}}{n_\alpha}, \quad (2.3)$$

where $\alpha, \beta \in \text{Irr}(\mathbb{G})$, $1 \leq i, j \leq n_\alpha$, $1 \leq k, l \leq n_\beta$. Moreover, the antipode S satisfies $S^2 = \iota$. Together with $* \circ S \circ * \circ S = \iota$, one obtains directly that S is $*$ -preserving.

The Pontryagin duality can also be extended to compact quantum groups. We only explain here for finite quantum groups. If $\mathbb{G} = (A, \Delta)$ is a finite quantum group, then we may construct its dual $\hat{\mathbb{G}} = (\hat{A}, \hat{\Delta})$ as follows. The underlying finite dimensional C^* -algebra $\hat{A}$ of $\hat{\mathbb{G}}$ is defined as A' , the set of all (bounded) linear functionals on A . For $\varphi_1, \varphi_2 \in \hat{A}$, their *convolution product* is defined as $\varphi_1 \star \varphi_2 := (\varphi_1 \otimes \varphi_2) \Delta$. We may define the involution on $\hat{A}$ as: $\varphi^* := \overline{\varphi(S(\cdot)^*)}$. Then $(\hat{A}, \star, *)$ becomes a finite dimensional C^* -algebra. For each $\varphi \in \hat{A}$, set $\hat{\Delta}(\varphi)(a \otimes b) := \varphi(ab)$, $a, b \in A$. Then $\Delta(\varphi) \in (A \otimes A)' = A' \otimes A'$ and it is easily seen that $\hat{\Delta}$ defines a comultiplication on $\hat{A}$. Hence $\hat{\mathbb{G}} = (\hat{A}, \hat{\Delta})$ becomes a finite quantum group. Moreover, it is equipped with a Hopf * -algebra structure with the antipode $\hat{S}$ and the counit $\hat{\epsilon}$ given by $\hat{S}(\varphi) := \varphi S$ and $\hat{\epsilon}(\varphi) := \varphi(1_A)$, respectively. Starting from the finite quantum group $\hat{\mathbb{G}} = (\hat{A}, \hat{\Delta})$, we may also construct its dual. Then in this way we recover the quantum group $\mathbb{G} = (A, \Delta)$.

The *Fourier transform* $\mathcal{F}$ on a finite quantum group $\mathbb{G} = (A, \Delta)$ is a map from A to $\hat{A}$ defined as $\mathcal{F}(a) = h(a \cdot) = h(\cdot a)$, where h is the Haar state on $\mathbb{G}$. We shall use $\hat{a}$ to denote the its Fourier transform $\mathcal{F}(a)$ for simplicity. Then we have the following *Parseval's identity*:

$$\hat{h}(\hat{a}_1 \hat{a}_2^*) = ch(a_1 a_2^*), \quad a_1, a_2 \in A, \quad (2.4)$$

where $\hat{h}$ is the Haar state on $\hat{\mathbb{G}}$ and $c > 0$ is a constant. If we consider the Fourier transform from $\hat{A}$ to A , then we obtain a similar equation to (2.4).

2.2 Plancherel triple

In this subsection we introduce the notion of *Plancherel triple*, which is slightly different from the same notion in [20]. Moreover, one can compare it with the so-called *D-pairs* in [13]. Let A, B be two finite dimensional C^* -algebras. Suppose that

$$\langle \cdot, \cdot \rangle : A \times B \rightarrow \mathbb{C},$$

is a non-degenerate bilinear form. Then through $\langle \cdot, \cdot \rangle$, B can be identified with A' . Indeed, the map $B \rightarrow A', b \mapsto \langle \cdot, b \rangle$ is injective, since $\langle \cdot, \cdot \rangle$ is non-degenerate. So B can be viewed as a subspace of A' and then $\dim B \leq \dim A' = \dim A$. Similarly, A can be viewed as a subspace of B' and $\dim A \leq \dim B' = \dim B$. Hence $\dim A = \dim B$ and $B = A'$ (also $A = B'$).

Definition 2.2 Let the triple $(A, B, \langle \cdot, \cdot \rangle)$ be as above. Then it is called a *Plancherel triple* if

1. the comultiplications $\Delta_A : A \rightarrow A \otimes A$ and $\Delta_B : B \rightarrow B \otimes B$ are positive, where Δ_A is the adjoint of the map given by $B \otimes B \rightarrow B, b_1 \otimes b_2 \mapsto b_1 b_2$ and Δ_B is the adjoint of the map defined through $A \otimes A \rightarrow A, a_1 \otimes a_2 \mapsto a_1 a_2$;
2. the counits $\epsilon_A := \langle \cdot, 1_B \rangle : A \rightarrow \mathbb{C}$ and $\epsilon_B := \langle 1_A, \cdot \rangle : B \rightarrow \mathbb{C}$ are $*$ -homomorphisms;

3. the Haar functionals $h_A := \langle \cdot, p_B \rangle$ and $h_B := \langle p_A, \cdot \rangle$ are faithful, positive and tracial, where $p_A \in A$ and $p_B \in B$ are support projections of ϵ_A and ϵ_B , respectively, i.e.,

$$\begin{aligned} p_A &= p_A^* = p_A^2, \quad ap_A = p_A a = \epsilon_A(a)p_A, \quad a \in A, \\ p_B &= p_B^* = p_B^2, \quad bp_B = p_B b = \epsilon_B(b)p_B, \quad b \in B. \end{aligned}$$

4. The Fourier transforms $\mathcal{F}_A : A \rightarrow B, a \mapsto \hat{a}$ and $\mathcal{F}_B : B \rightarrow A, b \mapsto \check{b}$ satisfy:

$$h_B(\hat{a}_1 \hat{a}_2^*) = ch_A(a_1 a_2^*), \quad (2.5)$$

$$h_A(\check{b}_1 \check{b}_2^*) = c'h_B(b_1 b_2^*), \quad (2.6)$$

for all $a_1, a_2 \in A, b_1, b_2 \in B$ and some $c, c' > 0$, where $\hat{a}$ and $\check{b}$ are defined through:

$$\langle x, \hat{a} \rangle := h_A(xa), \quad \langle \check{b}, y \rangle := h_B(by), \quad x \in A, y \in B.$$

Note that the above definition is self-dual, that is, $(B, A, \langle \cdot, \cdot \rangle)$ forms a Plancherel triple if $(A, B, \langle \cdot, \cdot \rangle)$ is a Plancherel triple, where $\langle b, a \rangle := \langle a, b \rangle, a \in A, b \in B$. We have the following properties of a Plancherel triple.

Proposition 2.3 *Let $(A, B, \langle \cdot, \cdot \rangle)$ be a Plancherel triple. Suppose that $A = \oplus_{\alpha=1}^m \mathbb{M}_{n_\alpha}(\mathbb{C})$ with the matrix units $\{e_{ij}^\alpha : 1 \leq i, j \leq n_\alpha, 1 \leq \alpha \leq m\}$. Let $\{\widetilde{e}_{ij}^\alpha : 1 \leq i, j \leq n_\alpha, 1 \leq \alpha \leq m\}$ be the dual basis of $\{e_{ij}^\alpha : 1 \leq i, j \leq n_\alpha, 1 \leq \alpha \leq m\}$.*

- (1) *We have $\mathcal{F}_A(e_{ij}^\alpha) = c_\alpha \widetilde{e}_{ji}^\alpha$ for some positive number c_α .*
- (2) *Suppose that $e^{\alpha_0} = p_A$. Then $\widetilde{e}^{\alpha_0} = \epsilon_A = 1_B$. Consequently, $h_B(1_B) = 1$ and then h_B is a state.*
- (3) *We have the orthogonal relation:*

$$h_B(\widetilde{e}_{ij}^\alpha (\widetilde{e}_{kl}^\beta)^*) = \frac{\delta_{\alpha\beta} \delta_{ik} \delta_{jl}}{cc_\alpha},$$

for all $\alpha, \beta, i, j, k, l$, where c is the constant appearing in (2.5).

- (4) *We have*

$$h_B(\widetilde{e}_{ij}^\alpha) = \delta_{\alpha_0\alpha}, \quad (2.7)$$

and

$$\epsilon_B(\widetilde{e}_{ij}^\alpha) = \delta_{ij}, \quad (2.8)$$

for all α, i, j .

Proof (1) For any $a = \sum_{\alpha=1}^m \sum_{i,j=1}^{n_\alpha} a_{ij}^\alpha e_{ij}^\alpha \in A$, we have by definition

$$\langle a, \mathcal{F}_A(e_{ij}^\alpha) \rangle = h_A(ae_{ij}^\alpha) = a_{ji}^\alpha h_A(e_{jj}^\alpha) = h_A(e_{jj}^\alpha) \widetilde{e}_{ji}^\alpha(a).$$

Since h_A is tracial, $h_A(e_{jj}^\alpha)$ is independent of j . So

$$\langle a, \mathcal{F}_A(e_{ij}^\alpha) \rangle = c_\alpha \widetilde{e}_{ji}^\alpha(a), \quad a \in A,$$

with $c_\alpha = h_A(e_{jj}^\alpha) > 0$. Thus $\mathcal{F}_A(e_{ij}^\alpha) = c_\alpha \widetilde{e}_{ji}^\alpha$, since $\langle \cdot, \cdot \rangle$ is non-degenerate.

(2) For any $a = \sum_{\alpha=1}^m \sum_{i,j=1}^{n_\alpha} a_{ij}^\alpha e_{ij}^\alpha \in A$, we have by definition

$$a^{\alpha_0} e^{\alpha_0} = a p_A = \epsilon_A(a) p_A = \langle a, 1_B \rangle e^{\alpha_0}.$$

Then

$$\widetilde{e^{\alpha_0}}(a) = a^{\alpha_0} = \langle a, 1_B \rangle, \quad a \in A.$$

Hence $\widetilde{e^{\alpha_0}} = \epsilon_A = 1_B$.

(3) This is a direct consequence of (2.5) and (1).

(4) By (2) and (3) we have

$$h_B(\widetilde{e_{ij}^\alpha}) = h_B(\widetilde{e_{ij}^\alpha}(\widetilde{e^{\alpha_0}})^*) = \frac{\delta_{\alpha_0\alpha}}{cc_{\alpha_0}}.$$

In particular, we have

$$1 = h_B(1_B) = h_B(\widetilde{e^{\alpha_0}}(\widetilde{e^{\alpha_0}})^*) = \frac{1}{cc_{\alpha_0}}$$

This proves (2.7). To show (2.8), recall that by definition,

$$\epsilon_B(\widetilde{e_{ij}^\alpha}) = \langle 1_A, \widetilde{e_{ij}^\alpha} \rangle = \widetilde{e_{ij}^\alpha}(1_A) = \delta_{ij}.$$

□

Example 2.4 Let (A, Δ) be a finite quantum group with dual $(\hat{A}, \hat{\Delta})$. Then $(A, \hat{A}, \langle, \rangle)$ forms a Plancherel triple. Here the bilinear form $\langle, \rangle$ is given by $\langle a, \varphi \rangle := \varphi(a)$. The comultiplications, counits, Haar states, and Fourier transforms are the usual comultiplications, counits, Haar states, and Fourier transforms on A and $\hat{A}$, respectively.

3 Plancherel triple induced from an idempotent state

In this section we will construct a Plancherel triple from an *idempotent state* on a finite quantum group $\mathbb{G} = (A, \Delta)$. An idempotent state on $\mathbb{G}$ is a state ϕ such that $\phi \star \phi = \phi$. It is well-known that (see for example [9]) considered as an element in $\hat{A}$, $p = \phi$ is a *group-like projection* in $\hat{A}$. By a group-like projection of the finite quantum group $\hat{\mathbb{G}} = (\hat{A}, \hat{\Delta})$ we mean a non-zero element $p \in \hat{A}$ such that $p = p^* = p^2$ and

$$\hat{\Delta}(p)(1 \otimes p) = p \otimes p = \hat{\Delta}(p)(p \otimes 1), \quad \hat{S}(p) = p. \quad (3.1)$$

Our aim in this section is to show that $(A_\phi, \hat{A}_p, \langle, \rangle)$ forms a Plancherel triple, where $A_\phi = (\phi \otimes \iota \otimes \phi)\Delta^{(2)}(A)$ is a C^* -subalgebra of A , $\hat{A}_p := p\hat{A}p$ is a C^* -subalgebra of $\hat{A}$ and the bilinear form is inherited from the one on the pair $(A, \hat{A})$. In the following we shall explain the constructions of A_ϕ and $\hat{A}_p$ in detail.

Before this we remark here that these constructions have already been studied by many people [5,6,8,9]. Many results are well-known and their proofs are omitted here.

3.1 C^* -subalgebra A_ϕ

In this subsection we construct and study the C^* -subalgebra A_ϕ . Recall that an idempotent state on a compact quantum group $\mathbb{G}$ is a state ϕ such that $\phi \star \phi = \phi$. Denote by $\text{Idem}(\mathbb{G})$ the set of all idempotent states on $\mathbb{G}$. Recall also that if $\phi \in \text{Idem}(\mathbb{G})$, we have $\phi = \phi S$ on

$\text{Pol}(\mathbb{G})$, where S is the antipode on $\text{Pol}(\mathbb{G})$ [9]. We use A' to denote the set of all bounded linear functionals on A .

The first lemma is a special case of [8, Lemma 3.1], and also a variation of [15, Lemma 4.3].

Lemma 3.1 *Let ϕ be an idempotent state on a compact quantum group $\mathbb{G}$. For $b \in A$ define $\phi_b(a) := \phi(ab)$ for all $a \in A$. Then we have*

$$\phi \star \phi_b = \phi(b)\phi.$$

For $\phi \in \text{Idem}(\mathbb{G})$ set $\mathbb{E}_\phi^\ell := (\phi \otimes \iota)\Delta$ and $\mathbb{E}_\phi^r := (\iota \otimes \phi)\Delta$. The next lemma lists some useful properties of $\mathbb{E}_\phi^\ell$ and $\mathbb{E}_\phi^r$.

Lemma 3.2 *Let $\mathbb{G} = (A, \Delta)$ be a compact quantum group.*

$$(1) \mathbb{E}_\phi^\ell(a^*) = \mathbb{E}_\phi^\ell(a)^*, \mathbb{E}_\phi^r(a^*) = \mathbb{E}_\phi^r(a)^*, a \in A.$$

$$(2) \Delta \mathbb{E}_\phi^\ell = (\mathbb{E}_\phi^\ell \otimes \iota)\Delta, \Delta \mathbb{E}_\phi^r = (\iota \otimes \mathbb{E}_\phi^r)\Delta.$$

$$(3) (\iota \otimes \mathbb{E}_\phi^\ell)\Delta = (\mathbb{E}_\phi^r \otimes \iota)\Delta.$$

$$(4) \mathbb{E}_\phi^\ell \mathbb{E}_\phi^r = \mathbb{E}_\phi^r \mathbb{E}_\phi^\ell.$$

$$(5) \text{ For any } a, b \in A,$$

$$\mathbb{E}_\phi^\ell(a)\mathbb{E}_\phi^\ell(b) = \mathbb{E}_\phi^\ell(a\mathbb{E}_\phi^\ell(b)) = \mathbb{E}_\phi^\ell(\mathbb{E}_\phi^\ell(a)b), \quad (3.2)$$

$$\mathbb{E}_\phi^r(a)\mathbb{E}_\phi^r(b) = \mathbb{E}_\phi^r(a\mathbb{E}_\phi^r(b)) = \mathbb{E}_\phi^r(\mathbb{E}_\phi^r(a)b). \quad (3.3)$$

Consequently, we have $\mathbb{E}_\phi^\ell \mathbb{E}_\phi^\ell = \mathbb{E}_\phi^\ell$ and $\mathbb{E}_\phi^r \mathbb{E}_\phi^r = \mathbb{E}_\phi^r$.

Proof (1)–(4) are just straightforward computations. For (5) we prove (3.2) first. For this note that it suffices to show the first equation for any a, b the coefficients of unitary representation of $\mathbb{G}$. The case for general a, b follows from the linearity and density. Let $\{u_{ij}^\alpha, 1 \leq i, j \leq n_\alpha\}$ be the coefficients of the irreducible unitary representations $u^\alpha, \alpha \in \text{Irr}(\mathbb{G})$. On one hand, for any u_{ij}^α and u_{kl}^β we have

$$\mathbb{E}_\phi^\ell(u_{ij}^\alpha)\mathbb{E}_\phi^\ell(u_{kl}^\beta) = \sum_{s=1}^{n_\alpha} \sum_{t=1}^{n_\beta} \phi(u_{is}^\alpha)\phi(u_{kt}^\beta)u_{sj}^\alpha u_{tl}^\beta.$$

On the other hand,

$$\mathbb{E}_\phi^\ell(\mathbb{E}_\phi^\ell(u_{ij}^\alpha)u_{kl}^\beta) = \sum_{r,s=1}^{n_\alpha} \sum_{t=1}^{n_\beta} \phi(u_{ir}^\alpha)\phi(u_{rs}^\alpha u_{kt}^\beta)u_{sj}^\alpha u_{tl}^\beta.$$

Now by using Lemma 3.1 we have for any $b \in A$

$$\phi(b)\phi(u_{is}^\alpha) = \sum_{r=1}^{n_\alpha} \phi(u_{ir}^\alpha)\phi(u_{rs}^\alpha b), \quad 1 \leq i, s \leq n_\alpha, \alpha \in \text{Irr}(\mathbb{G}).$$

Choosing b as u_{kt}^β , we have

$$\begin{aligned} \mathbb{E}_\phi^\ell(\mathbb{E}_\phi^\ell(u_{ij}^\alpha)u_{kl}^\beta) &= \sum_{r=1}^{n_\alpha} \sum_{t=1}^{n_\beta} \left(\sum_{s=1}^{n_\alpha} \phi(u_{ir}^\alpha)\phi(u_{rs}^\alpha u_{kt}^\beta) \right) u_{sj}^\alpha u_{tl}^\beta \\ &= \sum_{s=1}^{n_\alpha} \sum_{t=1}^{n_\beta} \phi(u_{is}^\alpha)\phi(u_{kt}^\beta)u_{sj}^\alpha u_{tl}^\beta = \mathbb{E}_\phi^\ell(u_{ij}^\alpha)\mathbb{E}_\phi^\ell(u_{kl}^\beta). \end{aligned}$$

So $\mathbb{E}_\phi^\ell(a)\mathbb{E}_\phi^\ell(b) = \mathbb{E}_\phi^\ell(\mathbb{E}_\phi^\ell(a)b)$. In the same way one can show that $\mathbb{E}_\phi^\ell(a)\mathbb{E}_\phi^\ell(b) = \mathbb{E}_\phi^\ell(a\mathbb{E}_\phi^\ell(b))$, then (3.2) is proved. Note that by choosing $b = 1_A$ in (3.2), we obtain $\mathbb{E}_\phi^\ell\mathbb{E}_\phi^\ell = \mathbb{E}_\phi^\ell$. The proof of (3.3) and $\mathbb{E}_\phi^r\mathbb{E}_\phi^r = \mathbb{E}_\phi^r$ is similar. $\square$

Now we define $\mathbb{E}_\phi := \mathbb{E}_\phi^\ell\mathbb{E}_\phi^r = \mathbb{E}_\phi^r\mathbb{E}_\phi^\ell$. The map $\mathbb{E}_\phi$ shares the similar properties of $\mathbb{E}_\phi^\ell$ and $\mathbb{E}_\phi^r$, see the following lemma (1)–(4). Moreover, $\mathbb{E}_\phi$ commutes with the antipode S , as the following lemma (5) shows. This enables the algebra $\mathbb{E}_\phi(A)$ to possess nicer properties than $\mathbb{E}_\phi^\ell(A)$ and $\mathbb{E}_\phi^r(A)$. We shall choose $\mathbb{E}_\phi(A)$ to be our A_ϕ .

Lemma 3.3 *Let $\mathbb{G} = (A, \Delta)$ be a compact quantum group.*

- (1) $\mathbb{E}_\phi(a^*) = \mathbb{E}_\phi(a)^*, a \in A$.
- (2) $\Delta\mathbb{E}_\phi = (\mathbb{E}_\phi^\ell \otimes \mathbb{E}_\phi^r)\Delta$.
- (3) $(\mathbb{E}_\phi \otimes \mathbb{E}_\phi)\Delta = (\iota \otimes \mathbb{E}_\phi^\ell)\Delta\mathbb{E}_\phi = (\mathbb{E}_\phi^r \otimes \iota)\Delta\mathbb{E}_\phi$.
- (4) $\mathbb{E}_\phi(a)\mathbb{E}_\phi(b) = \mathbb{E}_\phi(a\mathbb{E}_\phi(b)) = \mathbb{E}_\phi(\mathbb{E}_\phi(a)b), a, b \in A$. Consequently, $\mathbb{E}_\phi\mathbb{E}_\phi = \mathbb{E}_\phi$.

Hence $A_\phi := \mathbb{E}_\phi(A)$ is a unital C^* -subalgebra of A . Moreover,

- (5) $S\mathbb{E}_\phi = \mathbb{E}_\phi S$ on $\text{Pol}(\mathbb{G})$.

Proof Again we omit the proof of (1)–(4) here. The fact that A_ϕ is a unital C^* -subalgebra follows directly from these properties. To prove (5), it suffices to check the equality for u_{ij}^α , $1 \leq i, j \leq n_\alpha, \alpha \in \text{Irr}(\mathbb{G})$, the coefficients of unitary representations of $\mathbb{G}$. And that is a consequence of

$$\begin{aligned} S\mathbb{E}_\phi(u_{ij}^\alpha) &= S(\phi \otimes \iota \otimes \phi)\Delta^{(2)}(u_{ij}^\alpha) \\ &= \sum_{k,l=1}^{n_\alpha} \phi(u_{ik}^\alpha)\phi(u_{lj}^\alpha)S(u_{kl}^\alpha) \\ &= \sum_{k,l=1}^{n_\alpha} \phi(u_{ik}^\alpha)\phi(u_{lj}^\alpha)(u_{lk}^\alpha)^*, \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}_\phi S(u_{ij}^\alpha) &= (\phi \otimes \iota \otimes \phi)\Delta^{(2)}((u_{ji}^\alpha)^*) \\ &= \sum_{k,l=1}^{n_\alpha} \phi((u_{ki}^\alpha)^*)\phi((u_{jl}^\alpha)^*)(u_{lk}^\alpha)^* \\ &= \sum_{k,l=1}^{n_\alpha} \phi(u_{ik}^\alpha)\phi(u_{lj}^\alpha)(u_{lk}^\alpha)^*, \end{aligned}$$

where the last equality follows from the facts that $\phi = \phi S$ on $\text{Pol}(\mathbb{G})$ and $S(u_{ij}^\alpha) = (u_{ji}^\alpha)^*$. $\square$

3.2 ϕ -bi-invariant functionals

Let ϕ be an idempotent state on a compact quantum group $\mathbb{G} = (A, \Delta)$. A functional $u \in A'$ is called ϕ -bi-invariant if $u\phi = \phi u = u$. In this subsection we characterize the ϕ -bi-invariant functionals. It turns out that one can transfer each ϕ -bi-invariant functional on A to its restriction to A_ϕ , preserving the norm and the $*$ -algebra structure. See [4] for related work.

For linear functionals φ_1, φ_2 on A_ϕ , one can also define their convolution product within $(A_\phi)'$: $\varphi_1 \star \varphi_2$ is defined as $(\varphi_1 \otimes \varphi_2)\Delta_\phi$. And for any linear functional φ on $\mathbb{E}_\phi(\text{Pol}(\mathbb{G}))$ one can also define its involution within $(A_\phi)'$: $\varphi^* := \overline{\varphi(S(\cdot)^*)}$. Still, we write $\varphi_1 \varphi_2$ for short to denote $\varphi_1 \star \varphi_2$. Note here that φ^* is well-defined because $S\mathbb{E}_\phi = \mathbb{E}_\phi S$.

We formulate the results of ϕ -bi-invariant functionals here without the proof. Note that in the sequel we shall use $\|u\|$ to denote the norm of $u \in A'$ as a functional on A .

Lemma 3.4 *Let $\phi \in \text{Idem}(\mathbb{G})$ and $u \in A'$. Then u is ϕ -bi-invariant if and only if $u = u|_{A_\phi} \mathbb{E}_\phi$. In this case, the following hold:*

- (1) $\|u\| = \|u|_{A_\phi}\|$;
- (2) u is a positive linear functional (resp. a state) on A if and only if $u|_{A_\phi}$ is a positive linear functional (resp. a state) on A_ϕ ;
- (3) $u^*|_{A_\phi} = (u|_{A_\phi})^*$ on $\mathbb{E}_\phi(\text{Pol}(\mathbb{G}))$;
- (4) if u and v in A' are both ϕ -bi-invariant, then $(uv)|_{A_\phi} = u|_{A_\phi} v|_{A_\phi}$;
- (5) $\phi = \epsilon \mathbb{E}_\phi = \epsilon|_{A_\phi} \mathbb{E}_\phi$.

3.3 The Plancherel triple $(A_\phi, \hat{A}_p, \langle, \rangle)$

Let $\mathbb{G} = (A, \Delta)$ be a finite quantum group. Let ϕ be an idempotent state on $\mathbb{G}$, then $p = \phi$ is a group-like projection in $\hat{A}$. By Lemma 3.3, $A_\phi = \mathbb{E}_\phi(A)$ is a finite dimensional C^* -algebra. Clearly $\hat{A}_p = p\hat{A}p$ is also a finite dimensional C^* -algebra. Moreover, Lemma 3.4 implies that $(A_\phi)' = \hat{A}_p$. The main result of this section is the following proposition.

Proposition 3.5 *Let A_ϕ and $\hat{A}_p$ be as above. Then $(A_\phi, \hat{A}_p, \langle, \rangle)$ forms a Plancherel triple, where the bilinear form is inherited from the one on the pair $(A, \hat{A})$.*

Proof Since $(A_\phi)' = \hat{A}_p$, the bilinear form $\langle, \rangle$ on the pair $(A_\phi, \hat{A}_p)$ is non-degenerate. Now we check that the triple $(A_\phi, \hat{A}_p, \langle, \rangle)$ verifies the conditions 1–4 of Definition 2.2. For simplicity, in the sequel we shall use $\langle, \rangle$ to denote the bilinear forms on the different pairs $(A, \hat{A})$, $(A_\phi, \hat{A}_p)$ and their corresponding tensor products $(A \otimes A, \hat{A} \otimes \hat{A})$ and $(A_\phi \otimes A_\phi, \hat{A}_p \otimes \hat{A}_p)$. The readers can distinguish them easily. We shall use $h := h_A$ and $\hat{h} := h_B$ to denote the Haar states on $\mathbb{G}$ and $\hat{\mathbb{G}}$, respectively.

1. In this case, we claim that the comultiplications on A_ϕ and $\hat{A}_p$ are respectively $\Delta_\phi := (\mathbb{E}_\phi \otimes \mathbb{E}_\phi)\Delta|_{A_\phi}$ and $\hat{\Delta}_p := (p \otimes p)\hat{\Delta}|_{\hat{A}_p}(\cdot)(p \otimes p)$. Then automatically they are positive. Indeed, by definition,

$$\langle \Delta_{A_\phi}(a), x \otimes y \rangle = \langle a, xy \rangle = (x \otimes y)\Delta(a), \quad a \in A_\phi, x, y \in \hat{A}_p.$$

By Lemma 3.4, we have $x = x\mathbb{E}_\phi$ and $y = y\mathbb{E}_\phi$. Thus $(x \otimes y)\Delta(a) = (x \otimes y)\Delta_\phi(a)$ and

$$\langle \Delta_{A_\phi}(a), x \otimes y \rangle = \langle \Delta_\phi(a), x \otimes y \rangle, \quad a \in A_\phi, x, y \in \hat{A}_p.$$

Since $\langle, \rangle$ is non-degenerate, we have $\Delta_{A_\phi} = \Delta_\phi$. This proves the claim. Similarly, one can show $\Delta_{\hat{A}_p} = \hat{\Delta}_p$.

2. The counits on A_ϕ and $\hat{A}_p$ are respectively $\epsilon_\phi := \epsilon|_{A_\phi}$ and $\hat{\epsilon}_p := \hat{\epsilon}|_{\hat{A}_p}$. Here ϵ and $\hat{\epsilon}$ are respectively the counits on A and $\hat{A}$. In fact, by definition,

$$\epsilon_{A_\phi} = \langle \cdot, 1_{\hat{A}_p} \rangle = \langle \cdot, 1_{\hat{A}} \rangle|_{A_\phi} = \epsilon_\phi, \quad \epsilon_{\hat{A}_p} = \langle 1_{A_\phi}, \cdot \rangle = \langle 1_A, \cdot \rangle|_{\hat{A}_p} = \hat{\epsilon}_p.$$

Then it is easy to check that $\epsilon_\phi : A_\phi \rightarrow \mathbb{C}$ and $\hat{\epsilon}_p : \hat{A}_p \rightarrow \mathbb{C}$ are both $*$ -isomorphisms.

3. On one hand, the support projection $p_{\hat{A}}$ of $\hat{\epsilon} = \epsilon_{\hat{A}}$ verifies

$$p_{\hat{A}}p = pp_{\hat{A}} = \hat{\epsilon}(p)p_{\hat{A}} = \phi(1_A)p_{\hat{A}} = p_{\hat{A}}.$$

So $p_{\hat{A}} \in \hat{A}_p$. Moreover,

$$bp_{\hat{A}} = p_{\hat{A}}b = \hat{\epsilon}(b)p_{\hat{A}} = \hat{\epsilon}_p(b)p_{\hat{A}}, \quad b \in \hat{A}_p.$$

Therefore the support projection of $\hat{\epsilon}_p$ is $p_{\hat{A}_p} = p_{\hat{A}}$. Thus the Haar functional on A_ϕ is

$$h_\phi := h_{A_\phi} = \langle \cdot, p_{\hat{A}_p} \rangle = \langle \cdot, p_{\hat{A}} \rangle|_{A_\phi} = h|_{A_\phi}.$$

On the other hand, by Lemma 3.3,

$$\mathbb{E}_\phi(p_A)\mathbb{E}_\phi(a) = \mathbb{E}_\phi(p_A\mathbb{E}_\phi(a)) = \epsilon(\mathbb{E}_\phi(a))\mathbb{E}_\phi(p_A) = \epsilon_\phi(\mathbb{E}_\phi(a))\mathbb{E}_\phi(p_A), \quad a \in A.$$

Similarly, $\mathbb{E}_\phi(p_A)\mathbb{E}_\phi(a) = \epsilon_\phi(\mathbb{E}_\phi(a))\mathbb{E}_\phi(p_A)$, $a \in A$. By choosing $a = p_A$, we obtain that $\mathbb{E}_\phi(p_A)^2 = \phi(p_A)\mathbb{E}_\phi(p_A)$. So $q := \frac{1}{\phi(p_A)}\mathbb{E}_\phi(p_A)$ is a self-adjoint projection in A_ϕ such that

$$qa = aq = \epsilon_\phi(a)q, \quad a \in A_\phi.$$

That is, $p_{A_\phi} = q$ is the support projection of ϵ_ϕ . So the Haar functional on $\hat{A}_p$ is

$$\hat{h}_p := h_{\hat{A}_p} = \langle p_{A_\phi}, \cdot \rangle = \frac{1}{\phi(p_A)} \langle \mathbb{E}_\phi(p_A), \cdot \rangle = \frac{1}{\phi(p_A)} \langle p_A, \cdot \rangle|_{\hat{A}_p} = \frac{1}{\phi(p_A)} \hat{h}|_{\hat{A}_p}.$$

Hence the Haar functionals h_ϕ and $\hat{h}_p$ are faithful, positive and tracial.

4. For any $a \in A_\phi$, we claim that $\mathcal{F}_A(a) = h(\cdot a) \in \hat{A}_p$, i.e. $\mathcal{F}_A(a)$ is ϕ -bi-invariant. Indeed, by (2.1),

$$\begin{aligned} (\phi \otimes h(\cdot a))\Delta(x) &= (\phi \otimes h)(\Delta(x)(1 \otimes a)) \\ &= (\phi \circ S \otimes h)((1 \otimes x)\Delta(a)), \quad a \in A_\phi, x \in A. \end{aligned}$$

Since $\phi = \phi S$ and $\mathbb{E}_\phi(a) = a$, we have

$$(\phi \otimes h(\cdot a))\Delta(x) = (\phi \otimes h)((1 \otimes x)\Delta(a)) = h(x\mathbb{E}_\phi^\ell(a)) = h(xa), \quad a \in A_\phi, x \in A.$$

Hence $\phi \star h(\cdot a) = h(\cdot a)$, for all $a \in A_\phi$. From (2.2) and $\mathbb{E}_\phi^r(a) = a$ one can deduce in a similar way that $h(\cdot a) \star \phi = h(\cdot a)$. Thus $\mathcal{F}_A(a) = h(\cdot a)$ is ϕ -bi-invariant for all $a \in A_\phi$. This proves our claim. Now for any $a \in A_\phi$ we have

$$\langle x, \mathcal{F}_{A_\phi}(a) \rangle = h_\phi(xa) = h(xa) = \langle x, \mathcal{F}_A(a) \rangle, \quad x \in A_\phi.$$

Since $\mathcal{F}_A(a) \in \hat{A}_p$, $\mathcal{F}_A(a)$ is the image of $\mathcal{F}_{A_\phi}(a)$ under the natural inclusion $\hat{A}_p \rightarrow \hat{A}$. Hence the Parseval's identity (2.4) on the pair $(A, \hat{A})$ yields

$$\hat{h}_p(\mathcal{F}_{A_\phi}(a_1)\mathcal{F}_{A_\phi}(a_2)^*) = \hat{h}(\mathcal{F}_A(a_1)\mathcal{F}_A(a_2)^*) = ch(a_1a_2^*) = ch_\phi(a_1a_2^*), \quad a_1, a_2 \in A_\phi.$$

Similarly, one can show that

$$h_\phi(\mathcal{F}_{\hat{A}_p}(b_1)\mathcal{F}_{\hat{A}_p}(b_2)^*) = c'\hat{h}_p(b_1b_2^*), \quad b_1, b_2 \in \hat{A}_p,$$

for some constant $c' > 0$. □

4 Poisson states on compact quantum groups

Let $\mathbb{G} = (A, \Delta)$ be a compact quantum group. Denote by $\mathcal{S}(A)$ the set of all states on A . For each $\phi \in \text{Idem}(\mathbb{G})$, we say that $\{\omega_t\}_{t \geq 0}$ is a *convolution semigroup of functionals* on A starting from ϕ if

- (1) $\omega_t \in A'$ for each $t \geq 0$.
- (2) $\omega_{s+t} = \omega_s \omega_t$ for all $s, t \geq 0$.
- (3) $\omega_0 = \phi$.

If moreover, each $\omega_t \in \mathcal{S}(A)$, we call $\{\omega_t\}_{t \geq 0}$ a *convolution semigroup of states* starting from ϕ . We say that the convolution semigroup of states $\{\omega_t\}_{t \geq 0}$ is *norm continuous* if

$$\lim_{t \rightarrow 0^+} \|\omega_t - \phi\| = 0.$$

Recall here that $\|u\|$ denotes the norm of $u \in A'$ as a functional on A . Moreover, it is a Banach norm, since

$$\|uv\| = \|(u \otimes v)\Delta\| \leq \|u\| \|v\|, \quad u, v \in A'.$$

For a ϕ -bi-invariant functional $u \in A'$ define

$$\exp_\phi(u) := \phi + \sum_{n \geq 1} \frac{u^n}{n!}.$$

Then it is easy to check that $\{\exp_\phi(tu)\}_{t \geq 0}$ form a norm continuous convolution semigroup of functionals. We aim to find sufficient and necessary conditions on u such that $\{\exp_\phi(tu)\}_{t \geq 0}$ is a convolution semigroup of states. For this we make some notations. A functional $u \in A'$ is called *Hermitian* if $u(x^*) = \overline{u(x)}$ for all x ; it is further called *conditionally positive definite with respect to ϕ* if $u(x^*x) \geq 0$ for all x such that $\phi(x^*x) = 0$. The main theorem in this section is as follows.

Theorem 4.1 *Suppose that $\mathbb{G} = (A, \Delta)$ is a compact quantum group. Let $\phi \in \text{Idem}(\mathbb{G})$. Then for $u \in A'$, the following statements are equivalent.*

- (1) $u(1_A) = 0$, u is ϕ -bi-invariant and conditionally positive definite w.r.t. ϕ .
- (2) $u = r(v - \phi)$, where $r \geq 0$ and v is a ϕ -bi-invariant state.

The following proposition proves Theorem 4.1 in the level of unital C^* -algebras. It can be considered as a special case of Theorem 4.1 when $\phi = \epsilon$ is the counit. In this case each $u \in A'$ is ϵ -bi-invariant and $\phi = \epsilon$ is a character.

Proposition 4.2 *Let A be a unital C^* -algebra with ϵ a character. Then for any non-zero bounded linear functional u on A such that $u(1_A) = 0$ and $u(x^*x) \geq 0$ for all $\epsilon(x^*x) = 0$, we have $u = r(v - \epsilon)$, where $r > 0$ and v is a state.*

Proof Note first that $\epsilon(x^*x) = |\epsilon(x)|^2$. So $\epsilon(x^*x) = 0$ if and only if $x \in \ker \epsilon = \{x : \epsilon(x) = 0\}$. Let $u_0 := u|_{\ker \epsilon}$ be the restriction of u to $\ker \epsilon$. By assumption, u_0 is a bounded linear positive functional on the ideal $\ker \epsilon$. So it admits a unique positive linear extension $\tilde{u}_0$ to A such that $\tilde{u}_0|_{\ker \epsilon} = u_0$ and $\|\tilde{u}_0\| = \|u_0\|$. Hence for any $x \in A$, we have $x - \epsilon(x)1_A \in \ker \epsilon$ and thus

$$u(x) = u(x - \epsilon(x)1_A) = u_0(x - \epsilon(x)1_A) = \tilde{u}_0(x - \epsilon(x)1_A) = r(v - \epsilon)(x),$$

where $r := \|\tilde{u}_0\| = \|u_0\| > 0$ and $v := \frac{1}{r}\tilde{u}_0$ is a state. □

Now we are ready to prove Theorem 4.1. The idea is to restrict the problem to A_ϕ , and then transfer the decomposition from A_ϕ to A .

Proof of Theorem 4.1 The direction (2) $\Rightarrow$ (1) is clear. To prove (1) $\Rightarrow$ (2), suppose $u \neq 0$ and write $u = u|_{A_\phi} \mathbb{E}_\phi$ by Lemma 3.4. Note that $\epsilon|_{A_\phi}$ is a character on the unital C^* -algebra A_ϕ . From the definition of u , we have $u|_{A_\phi}(1_{A_\phi}) = 0$. Moreover, for any $x \in A$ such that $\epsilon|_{A_\phi}(\mathbb{E}_\phi(x)^* \mathbb{E}_\phi(x)) = 0$, we have by Lemma 3.2 and Lemma 3.4 that

$$0 = \epsilon|_{A_\phi}(\mathbb{E}_\phi(x)^* \mathbb{E}_\phi(x)) = \epsilon|_{A_\phi} \mathbb{E}_\phi(\mathbb{E}_\phi(x)^* \mathbb{E}_\phi(x)) = \phi(\mathbb{E}_\phi(x)^* \mathbb{E}_\phi(x)).$$

Again, by Lemma 3.2 and Lemma 3.4, the conditionally positive definiteness of u with respect to ϕ implies

$$u|_{A_\phi}(\mathbb{E}_\phi(x)^* \mathbb{E}_\phi(x)) = u|_{A_\phi} \mathbb{E}_\phi(\mathbb{E}_\phi(x)^* \mathbb{E}_\phi(x)) = u(\mathbb{E}_\phi(x)^* \mathbb{E}_\phi(x)) \geq 0.$$

So we have by Proposition 4.2 that $u|_{A_\phi} = r(w - \epsilon|_{A_\phi})$ with $r > 0$ and w a state on A_ϕ . Set $v := w\mathbb{E}_\phi$, then v is, by Lemma 3.4, a ϕ -bi-invariant state on A such that

$$u = u|_{A_\phi} \mathbb{E}_\phi = r(w\mathbb{E}_\phi - \epsilon|_{A_\phi} \mathbb{E}_\phi) = r(v - \phi),$$

as desired. $\square$

Definition 4.3 Let $\mathbb{G}$ be a compact quantum group. Let $\phi \in \text{Idem}(\mathbb{G})$. We denote by $\mathcal{N}_\phi(\mathbb{G})$ the class of functionals $u \in A'$ that satisfy the conditions (2) in Theorem 4.1. Then for each $u \in \mathcal{N}_\phi(\mathbb{G})$, $\omega := \exp_\phi(u)$ is a state. Denote by $\mathcal{P}_\phi(\mathbb{G})$ the family of all such states. Set $\mathcal{P}(\mathbb{G}) := \bigcup_{\phi \in \text{Idem}(\mathbb{G})} \mathcal{P}_\phi(\mathbb{G})$. Then any $\omega \in \mathcal{P}(\mathbb{G})$ is said to be of *Poisson type*, or a *Poisson state* on $\mathbb{G}$.

Recall that any norm continuous convolution semigroup of states $\{\omega_t\}_{t \geq 0}$ on a compact quantum group $\mathbb{G} = (A, \Delta)$ can be recovered by exponentiation with a bounded generator $u := \lim_{t \rightarrow 0^+} \frac{1}{t}(\omega_t - \omega_0)$. It is not difficult to see that $u(1_A) = 0$ and u is conditionally positive definite with respect to ω_0 , since these hold for each $\frac{1}{t}(\omega_t - \omega_0)$, $t > 0$. Then together with Theorem 4.1 we have the following result.

Theorem 4.4 Let ϕ be an idempotent state on a compact quantum group $\mathbb{G} = (A, \Delta)$. For any non-zero bounded linear functional ω on A such that $\omega\phi = \phi\omega = \omega$, the following are equivalent

1. $\omega = \omega_1$ with $\{\omega_t\}_{t \geq 0}$ a norm continuous convolution semigroup of states such that $\omega_0 = \phi$;
2. $\omega = \exp_\phi(u)$, where $u \in A'$ is ϕ -bi-invariant, $u(1_A) = 0$, and $u(x^*x) \geq 0$ for all $x \in A$ such that $\phi(x^*x) = 0$;
3. $\omega = \exp_\phi(u)$, where $u = r(v - \phi)$, with $r > 0$ and v a ϕ -bi-invariant state on A .

Remark 4.5 The convolution semigroup of states $\{\omega_t\}_{t \geq 0}$ on a compact quantum group $\mathbb{G}$ is said to be *weakly continuous* if $\omega_t(a) \rightarrow \omega_0(a)$, $t \rightarrow 0^+$ for any $a \in C(\mathbb{G})$. Clearly norm continuous convolution semigroup of states is weakly continuous. When $\mathbb{G}$ is a finite quantum group, the converse also holds. But for general compact quantum group $\mathbb{G}$, there exists weakly continuous convolution semigroup of states which is not norm continuous. In this case, the generator is unbounded.

5 Infinitely divisible states on finite quantum groups

In this section we prove the main result of the paper.

Definition 5.1 Let $\mathbb{G} = (A, \Delta)$ be a compact quantum group. A state $\omega \in \mathcal{S}(A)$ is said to be *infinitely divisible* if $\omega = \omega_n^n$ for some $\omega_n \in \mathcal{S}(A)$ and for all $n \geq 1$. We use $\mathcal{S}(\mathbb{G})$ to denote the set of all infinitely divisible states on $\mathbb{G}$.

Clearly Poisson states are infinitely divisible. Our main result in this section is that any infinitely divisible state on a finite quantum group is a Poisson state. From now on, unless stated otherwise, $\mathbb{G} = (A, \Delta)$ always denotes a finite quantum group.

The following lemma is well-known, and the proof follows from standard arguments.

Lemma 5.2 Let $A = \oplus_{k=1}^m \mathbb{M}_{n_k}(\mathbb{C})$ with matrix units $\{e_{ij}^k : 1 \leq i, j \leq n_k, 1 \leq k \leq m\}$. Denote the dual basis by $\{\omega_{ij}^k\}$. Then for any $\omega = \sum_{k=1}^m \sum_{i,j=1}^{n_k} a_{ij}^{(k)} \omega_{ij}^k$, ω is a positive linear functional if and only if $[a_{ij}^{(k)}]_{i,j=1}^{n_k}$ is positive semi-definite for each k . In this case, $\|\omega\| = \sum_{k=1}^m \sum_{i=1}^{n_k} a_{ii}^{(k)}$.

As a direct consequence, we have the following decomposition, which is quite easy but very helpful.

Corollary 5.3 Let $A = \oplus_{k=1}^m \mathbb{M}_{n_k}(\mathbb{C})$ with matrix units $\{e_{ij}^k : 1 \leq i, j \leq n_k, 1 \leq k \leq m\}$. Denote the dual basis by $\{\omega_{ij}^k\}$. Let $\omega = \sum_{k=1}^m \sum_{i,j=1}^{n_k} a_{ij}^{(k)} \omega_{ij}^k$ such that for each k either $[a_{ij}^{(k)}]_{i,j=1}^{n_k} \geq 0$ or $[a_{ij}^{(k)}]_{i,j=1}^{n_k} \leq 0$. Then $\omega_+ := \sum_{k \in \Lambda} \sum_{i,j=1}^{n_k} a_{ij}^{(k)} \omega_{ij}^k$ and $\omega_- := \sum_{k \notin \Lambda} \sum_{i,j=1}^{n_k} a_{ij}^{(k)} \omega_{ij}^k$ are positive functionals on A such that

$$\omega = \omega_+ - \omega_- \text{ and } \|\omega\| = \|\omega_+\| + \|\omega_-\|,$$

where Λ is the set of all the k 's such that $[a_{ij}^{(k)}]_{i,j=1}^{n_k} \geq 0$.

Another important corollary is as follows.

Corollary 5.4 Let $(A, B, \langle, \rangle)$ be a Plancherel triple. Suppose that $A = \oplus_{\alpha=1}^m \mathbb{M}_{n_\alpha}(\mathbb{C})$ with the matrix units $\{e_{ij}^\alpha : 1 \leq i, j \leq n_\alpha, 1 \leq \alpha \leq m\}$. Then for any positive linear functional $u \in A'$, we have

$$h_B(u) \leq \epsilon_B(u).$$

Proof By Lemma 5.2 we may write u as $u = \sum_{\alpha=1}^m \sum_{i,j=1}^{n_\alpha} a_{ij}^\alpha \widetilde{e}_{ij}^\alpha$, where each $[a_{ij}^\alpha]_{i,j=1}^{n_\alpha}$ is positive semi-definite and $\{\widetilde{e}_{ij}^\alpha\}$ is the dual basis of $\{e_{ij}^\alpha\}$. Suppose that $\widetilde{e}^{\alpha_0} = 1_B$. Then by (2.7) and (2.8), we have

$$h_B(u) = a^{\alpha_0} \leq \sum_{\alpha=1}^m \sum_{i=1}^{n_\alpha} a_{ii}^\alpha = \epsilon_B(u).$$

□

Let $\mathbb{G} = (A, \Delta)$ be a finite quantum group. Then for any $u \in A' = \hat{A}$ we denote by $\|u\|_{\hat{A}}$ the C^* -norm of u as an element in C^* -algebra $\hat{A}$. Recall that $\|u\|$ is the norm of u as a functional. Moreover, we have $\|u\|_{\hat{A}} \leq \|u\|$.

Let ϕ be an idempotent state on finite quantum group $\mathbb{G} = (A, \Delta)$. For any $u \in A'$ such that $u = u\phi = \phi u$ and $\|u - \phi\| < 1$, define the *logarithm of u with respect to ϕ* as

$$\log_{\phi}(u) := - \sum_{k \geq 1} \frac{(\phi - u)^k}{k}.$$

Then we have the following properties of logarithm and exponential.

Lemma 5.5 Suppose that $\mathbb{G} = (A, \Delta)$ is a finite quantum group. Let ϕ be an idempotent state on A , then for any bounded linear functionals u, v on A such that $u = u\phi = \phi u$ and $v = v\phi = \phi v$, we have

- (1) $\exp_{\phi}(\log_{\phi}(u)) = u$, if $\|u - \phi\| < 1$.
- (2) $\log_{\phi}(\exp_{\phi}(u)) = u$, if $\|u\| < \log 2$.
- (3) $\exp_{\phi}(u + v) = \exp_{\phi}(u) \exp_{\phi}(v)$ if $uv = vu$.
- (4) $\log_{\phi}(uv) = \log_{\phi}(u) + \log_{\phi}(v)$, if $uv = vu$ and the following holds:

$$\|u - \phi\| < 1, \quad \|v - \phi\| < 1, \quad \text{and} \quad \|uv - \phi\| < 1.$$

- (5) If moreover, u is a state such that

$$\|u - \phi\| < \frac{1}{2} \quad \text{and} \quad \|u^n - \phi\| < \frac{1}{2}$$

for some $n \geq 1$, then

$$\|u^k - \phi\| < \frac{1}{2} \quad \text{for all } 1 \leq k \leq n.$$

Consequently, in such a case we have

$$\log_{\phi}(u^k) = k \log_{\phi}(u) \quad \text{for all } 1 \leq k \leq n.$$

Proof (1)-(4) are direct and hold for all Banach algebras. To show (5), let u_{ϕ} be the restriction of u to A_{ϕ} . Then by Lemma 3.4, u_{ϕ} is a state on a finite-dimensional C^* -algebra A_{ϕ} . Moreover,

$$\|u - \phi\| = \|(u_{\phi} - \epsilon_{\phi})\mathbb{E}_{\phi}\| = \|u_{\phi} - \epsilon_{\phi}\|,$$

where ϵ_{ϕ} denotes the restriction of counit ϵ of A to A_{ϕ} . Write $A_{\phi} = \bigoplus_{k=1}^m \mathbb{M}_{n_k}(\mathbb{C})$ with matrix units $\{e_{ij}^k : 1 \leq i, j \leq n_k, 1 \leq k \leq m\}$. Let $\{\omega_{ij}^k\}$ be the dual basis. By Lemma 5.2, $u_{\phi} = \sum_{k=1}^m \sum_{i,j=1}^{n_k} b_{ij}^{(k)} \omega_{ij}^k$ with $[b_{ij}^{(k)}]_{i,j=1}^{n_k} \geq 0$ and $\|u_{\phi}\| = \sum_{k=1}^m \sum_{i=1}^{n_k} b_{ii}^{(k)} = 1$. Since ϵ_{ϕ} is a character on A_{ϕ} , there exists k_0 such that $n_{k_0} = 1$ and $\omega^{(k_0)} = \epsilon_{\phi}$. Thus $u_{\phi} - \epsilon_{\phi} = (b^{(k_0)} - 1)\epsilon_{\phi} + \sum_{k \neq k_0} \sum_{i,j=1}^{n_k} b_{ij}^{(k)} \omega_{ij}^k$ verifies the condition of Corollary 5.3 and it follows that

$$\|u_{\phi} - \epsilon_{\phi}\| = 1 - b^{(k_0)} + \sum_{k \neq k_0} \sum_{i=1}^{n_k} b_{ii}^{(k)} = 2 - 2b^{(k_0)} = -2(u_{\phi} - \epsilon_{\phi})(e^{k_0}).$$

So for $v_1 := u - \phi$ we have $\|v_1\| = \|w_1\| = -2w_1(e^{k_0})$, where $w_1 = v_1|_{A_{\phi}}$. Similarly for $v_j := u^j - \phi$ and $w_j := v_j|_{A_{\phi}}$ we have

$$\|v_j\| = \|w_j\| = -2w_j(e^{k_0}), \quad j \geq 1. \quad (5.1)$$

Now we show (5) by the induction argument. Clearly it holds for $n = 1$. Suppose for now that it holds for n . Set $r := \|v_1\|$, $s := \|v_n\|$ and $t := \|v_{n+1}\|$. From $(v_1 + \phi)(v_n + \phi) = v_{n+1} + \phi$

it follows that $v_{n+1} = v_1 + v_n + v_1 v_n$ and thus by Lemma 3.4 (4) $w_{n+1} = w_1 + w_n + w_1 w_n$. This, together with (5.1) and Lemma 3.4 (1), yields

$$t = r + s - 2(w_1 w_n)(e^{k_0}) \geq r + s - 2\|w_1 w_n\| = r + s - 2\|v_1 v_n\| \geq r + s - 2rs.$$

Then

$$(1 - 2r)(1 - 2s) = 4rs - 2r - 2s + 1 \geq 1 - 2t.$$

By assumption, $1 - 2r, 1 - 2t > 0$, so $1 - 2s > 0$. Hence u and u^n verify the conditions in (4), and we obtain

$$\log_\phi(u^{n+1}) = \log_\phi(u) + \log_\phi(u^n) = \log_\phi(u) + n \log_\phi(u) = (n + 1) \log_\phi(u),$$

where in the second equality we have used the induction for n . This finishes the proof for $n + 1$ and then shows (5). $\square$

Remark 5.6 In fact, to prove (5) we have used the fact that $\|u + v\| = \|u\| + \|v\|$ for all $u, v \in \mathcal{N}_\phi(\mathbb{G})$.

Proposition 5.7 Let ω be an infinitely divisible state on a finite quantum group $\mathbb{G} = (A, \Delta)$. Let ϕ be an idempotent state on A . Assume that there exists a sequence $\{\omega_{m_j}\}_{j \geq 0}$ of roots of states of ω , with $\omega = \omega_{m_j}^{m_j}$, for all j , such that

- (1) $\{m_j\}_{j \geq 0}$ is strictly increasing;
- (2) $\omega_{m_j} = \omega_{m_j} \phi = \phi \omega_{m_j}$ for all j ;
- (3) $\omega_{m_j} = \omega_{m_{j+1}}^{n_j}$ for some positive integer $n_j, j \geq 0$;
- (4) $\omega_{m_j} \rightarrow \phi$, as $j \rightarrow \infty$.

Then $\omega \in \mathcal{P}_\phi(\mathbb{G})$.

Proof Assume that $\{\omega_{m_j}\}$ contains infinitely many different elements, otherwise $\omega = \phi \in \mathcal{P}_\phi(\mathbb{G})$. By (4), we can choose $j_0 > 0$ such that $\|\omega_{m_j} - \phi\| < 1/2$ for all $j \geq j_0$. This inequality, together with (2), allows us to define

$$v_0 := \log_\phi(\omega_{m_{j_0}}), \text{ and } v := m_{j_0} v_0.$$

Then by the definition of ω_{j_0} and Lemma 5.5 (1)(2),

$$\omega = \omega_{m_{j_0}}^{m_{j_0}} = \left(\exp_\phi(\log_\phi(\omega_{m_{j_0}})) \right)^{m_{j_0}} = \exp_\phi(m_{j_0} v_0) = \exp_\phi(v).$$

To prove $\omega \in \mathcal{P}_\phi(\mathbb{G})$, it suffices to show that $v \in \mathcal{N}_\phi(\mathbb{G})$. For this we check that v verifies Theorem 4.1 (1). Clearly, $v(1_A) = 0$, since ω_{j_0} is a state. By the definition of logarithm, $v = v\phi = \phi v$. It remains to show that for any $x \in A$ such that $\phi(x^*x) = 0$, we have $v_0(x^*x) \geq 0$. By (3) we have

$$\omega_{m_{j_0}} = \omega_{m_j}^{N_j}, \quad j \geq j_0,$$

where $N_j := n_{j_0} \cdots n_{j-1}$. Recall that for all $j \geq j_0$, $\|\omega_{m_j} - \phi\| < 1/2$. Thus by Lemma 5.5 (5) we have

$$\log_\phi(\omega_{m_j}) = \frac{1}{N_j} \log_\phi(\omega_{m_{j_0}}) = \frac{v_0}{N_j},$$

and by Lemma 5.5 (1)

$$\omega_{m_j} = \exp_\phi\left(\frac{v_0}{N_j}\right), \quad j \geq j_0.$$

The condition (1) implies that $N_j \rightarrow \infty$ as $j \rightarrow \infty$. Now for any $x \in A$ such that $\phi(x^*x) = 0$, we have

$$0 \leq \omega_{m_j}(x^*x) = \phi(x^*x) + \frac{v_0(x^*x)}{N_j} + \sum_{m \geq 2} \frac{v_0^m(x^*x)}{N_j^m \cdot m!} = \frac{v_0(x^*x)}{N_j} + O\left(\frac{1}{N_j^2}\right),$$

for all $j \geq j_0$. Hence

$$v_0(x^*x) + O\left(\frac{1}{N_j}\right) \geq 0, \quad j \geq j_0.$$

Letting $j \rightarrow \infty$, we have $v_0(x^*x) \geq 0$, which ends the proof. $\square$

As this proposition suggests, to show that an infinitely divisible state is of Poisson type, it is important to capture the idempotent state where the infinitely divisible state is “supported on”. For this we need two lemmas. The first one is an easy fact in matrix theory.

Lemma 5.8 *Let $P \in \mathbb{M}_n(\mathbb{C})$ be a self-adjoint projection. Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $A = AP = PA$, $AB = P$ and $\|A\| \leq 1$, $\|B\| \leq 1$. Here $\|\cdot\|$ denote the C^* -norm in matrix algebra. Then $A^*A = AA^* = P$. Consequently, if u and v are states on a finite quantum group $\mathbb{G} = (A, \Delta)$ such that $u = u\phi = \phi u$ and $uv = \phi$, where ϕ is an idempotent state on $\mathbb{G}$, then $u^*u = uu^* = \phi$.*

Proof Since P is a self-adjoint projection, we may assume without loss of generality that

$$P = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

where I_r is the identity in $\mathbb{M}_r(\mathbb{C})$ with $r = \text{rank}(P)$. From $A = AP = PA$ and $AB = P$ it follows

$$A = \begin{pmatrix} A_r & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} B_r & * \\ * & * \end{pmatrix},$$

with $A_r B_r = I_r$. Note that

$$1 = \|I_r\| = \|A_r B_r\| \leq \|A_r\| \|B_r\| \leq \|A\| \|B\| \leq 1.$$

So $\|A_r\| = \|B_r\| = 1$. This is to say,

$$\|A_r^* A_r\| = 1 = \|B_r B_r^*\| = \|(A_r^* A_r)^{-1}\|.$$

Then all the eigenvalues of $A_r^* A_r$ must be 1 and thus $A_r^* A_r = I_r$. Hence $B_r = A_r^*$ and thus $A^* A = AA^* = P$.

The remaining part follows from the facts that $p = \phi$ is a self-adjoint projection in $\hat{A}$ and

$$\|u\|_{\hat{A}} \leq \|u\| = 1, \quad \|v\|_{\hat{A}} \leq \|v\| = 1.$$

$\square$

Lemma 5.9 *Let $(A, B, \langle, \rangle)$ be a Plancherel triple. Suppose that $u \in B$ is a state on A such that $uu^* = u^*u = \epsilon_A$, where ϵ_A is the counit of A . Then u is an n -th root of ϵ_A for some $n \leq \dim B$.*

Proof Suppose that $A = \bigoplus_{\alpha=1}^m \mathbb{M}_{n_\alpha}(\mathbb{C})$ with the matrix units $\{e_{ij}^\alpha : 1 \leq i, j \leq n_\alpha, 1 \leq \alpha \leq m\}$. Let $\{\widetilde{e}_{ij}^\alpha\}$ be the dual basis. Then by Lemma 5.2 we can write u as

$$u = \sum_{\alpha=1}^m p_\alpha \sum_{i,j=1}^{n_\alpha} a_{ij}^\alpha \widetilde{e}_{ij}^\alpha,$$

where $p_\alpha \geq 0$, $\sum_{\alpha=1}^m p_\alpha = 1$ and $[a_{ij}^\alpha]_{i,j=1}^{n_\alpha}$ positive semi-definite with trace one. Let α_0 be such that $\widetilde{e}^{\alpha_0} = 1_B$. Then we can write $u = p_{\alpha_0} \widetilde{e}^{\alpha_0} + v$ with $v = \sum_{\alpha \neq \alpha_0} p_\alpha \sum_{i,j=1}^{n_\alpha} a_{ij}^\alpha \widetilde{e}_{ij}^\alpha$ such that $\|v\| = \sum_{\alpha \neq \alpha_0} p_\alpha = 1 - p_{\alpha_0}$. Note that $uu^* = p_{\alpha_0}^2 \widetilde{e}^{\alpha_0} + p_{\alpha_0}(v + v^*) + vv^*$. Then by (2.7), $h_B(v) = h_B(v^*) = 0$. Since v and v^* are both positive functionals, $vv^*(\cdot) = \langle \Delta_A(\cdot), v \otimes v^* \rangle$ is also a positive functional on A . Hence we have by Corollary 5.4 that

$$h_B(vv^*) \leq \epsilon_B(vv^*) = |\epsilon_B(v)|^2 = (1 - p_{\alpha_0})^2.$$

So we have

$$1 = h_B(1_B) = h_B(uu^*) = p_{\alpha_0}^2 + h_B(vv^*) \leq p_{\alpha_0}^2 + (1 - p_{\alpha_0})^2 = 1 - 2p_{\alpha_0} + 2p_{\alpha_0}^2,$$

which yields $p_{\alpha_0}(p_{\alpha_0} - 1) \geq 0$. Recall that $0 \leq p_{\alpha_0} \leq 1$, hence either $p_{\alpha_0} = 0$ or $p_{\alpha_0} = 1$. That is to say, either $h_B(u) = 0$ or $u = \epsilon_A$. Since for any $n \geq 1$, u^n is again a state such that $u^n u^{*n} = \epsilon_A$, we obtain, by a similar argument, that either $h_B(u^n) = 0$ or $u^n = \epsilon_A$.

If u is not a n -th root of ϵ_A for all $1 \leq n \leq \dim B$, then we have

$$h_B(u^n) = 0, \quad n = 1, 2, \dots, \dim B. \quad (5.2)$$

Note that we may choose $m \leq \dim B$ such that u is unitary in $\mathbb{M}_m(\mathbb{C})$. Set $P(\lambda) := \det(\lambda I_m - u) = \sum_{i=0}^m a_i \lambda^i$, then Cayley-Hamilton Theorem implies that $P(u) = 0$, where I_m denotes the identity matrix in $\mathbb{M}_m(\mathbb{C})$. Since u is unitary in $\mathbb{M}_m(\mathbb{C})$, we have $a_0 = (-1)^m \det(u) \neq 0$. But by (5.2)

$$a_0 h_B(I_m) = a_0 h_B(I_m) + \sum_{i=1}^m a_i h_B(u^i) = h_B(P(u)) = 0.$$

Thus $a_0 = 0$, which leads to a contradiction. So we must have $u^m = \epsilon_A$ for some $1 \leq m \leq \dim B$. $\square$

The following proposition, gathering the main ingredients of preceding lemmas, will be used to prove Theorem 5.11.

Proposition 5.10 *Let $\mathbb{G} = (A, \Delta)$ be a finite quantum group with dual $\widehat{\mathbb{G}} = (\widehat{A}, \widehat{\Delta})$. Suppose that $u, v \in \mathcal{S}(A)$ and $\phi \in \text{Idem}(\mathbb{G})$ are such that $u = u\phi = \phi u$ and $uv = \phi$. Then there exists a positive integer $m \leq \dim \widehat{A}$ such that $u^m = \phi$.*

Proof From Lemma 5.8 it follows $u^*u = uu^* = \phi$. Let u_ϕ and ϵ_ϕ be the restrictions of u and the counit ϵ of A to A_ϕ , respectively. From Proposition 3.5 $(A_\phi, \widehat{A}_\phi, \langle, \rangle)$ forms a Plancherel triple. By Lemma 3.4, u_ϕ is a state on A_ϕ such that $u_\phi u_\phi^* = \epsilon_\phi$. So Lemma 5.9 implies $u_\phi^m = \epsilon_\phi$ for some $m \leq \dim \widehat{A}_\phi \leq \dim \widehat{A}$. Hence Lemma 3.4 gives $u^m = u_\phi^m \mathbb{E}_\phi = \epsilon_\phi \mathbb{E}_\phi = \phi$. $\square$

Now we are ready to prove the main result of this paper.

Theorem 5.11 *Let $\mathbb{G} = (A, \Delta)$ be a finite quantum group. Then $\mathcal{P}(\mathbb{G}) = \mathcal{S}(\mathbb{G})$.*

The first proof $\mathcal{P}(\mathbb{G}) \subset \mathcal{J}(\mathbb{G})$ is clear. Let $\omega \in \mathcal{J}(\mathbb{G})$. We claim that for any positive integer $N \geq 2$, there exists a sequence $\{b_n\}_{n \geq 0}$ of roots of ω such that $b_0 = \omega$, $b_{n-1} = b_n^N$, $n \geq 1$. Indeed, since A is finite dimensional, the set of states $Z = \mathcal{S}(A)$ is compact with respect to the norm topology. Thus $\prod_{j \geq 0} Z_j$, where $Z_j = Z$ for all j , is compact with respect to the product topology. Let $a_n \in Z$ be any n -th root of ω for all $n \geq 0$. Then the sequence of non-empty closed sets

$$W_k := \overline{\prod_{j \geq k} \{a_{Nj}^{Nj}\} \times \{a_{Nj}^{Nj-1}\} \times \cdots \times \{a_{Nj}\}} \times \prod_{i \geq j} Z_i, \quad k \geq 1,$$

is decreasing: $W_1 \supset W_2 \supset \cdots$, and thus any finite intersection of $\{W_k\}_{k \geq 1}$ is non-empty. By compactness of $\prod_{j \geq 0} Z_j$, $\bigcap_{k \geq 1} W_k \neq \emptyset$. Hence one can choose $(b_0, b_1, \dots) \in \bigcap_{k \geq 1} W_k$ such that

$$b_0 = \omega, \quad b_{n-1} = b_n^N, \quad n \geq 1.$$

This proves the claim.

Choose $N = (\dim \hat{A})! \geq 2$ and let $\{b_n\}_{n \geq 0}$ be as above. Since Z is compact, there exists a subsequence $\{c_j\}_{j \geq 0}$ of $\{b_i\}_{i \geq 0}$ such that c_j converges to some $c \in Z$. If we fix a non-negative integer i , we have $b_i = c_j^{r_j}$ for sufficient large j and some integer $r_j \geq N \geq 2$. That is,

$$b_i = c_j c_j^{r_j-1} = c_j^{r_j-1} c_j. \quad (5.3)$$

We can assume that $c_j^{r_j-1}$ converges to some $d_i \in Z$, otherwise consider some subsequence, since $\{c_j^{r_j-1}\}_{j \geq 0} \subset Z$. Thus letting $j \rightarrow \infty$ in (5.3), we have

$$b_i = c d_i = d_i c, \quad i \geq 0. \quad (5.4)$$

This implies $b_i \in cZ \cap Zc$ for all $i \geq 0$. From the choice of c_j we have $c_j \in cZ \cap Zc$ for all $j \geq 0$. Then for any i the corresponding $c_j^{r_j-1} \in cZ \cap Zc$ for all j , which implies that $d_i \in cZ \cap Zc$ by the compactness of $cZ \cap Zc$. Now consider (5.3) for $\{c_j\}_{j \geq 0}$, instead of $\{b_i\}_{i \geq 0}$, we obtain an updated version of (5.4):

$$c_j = c d'_j = d'_j c, \quad j \geq 0, \quad (5.5)$$

where $d'_j \in cZ \cap Zc$. Letting $j \rightarrow \infty$, consider the subsequence of $\{d'_j\}_{j \geq 0}$ if necessary, one obtains

$$c = c d = d c, \quad (5.6)$$

where $d \in cZ \cap Zc$ by the compactness of $cZ \cap Zc$. Suppose $d = c e$ for some $e \in Z$, then $d^2 = d c e = c e = d$, i.e. d is an idempotent state. By Proposition 5.10, we obtain $c^m = d$ for some $m \leq \dim \hat{A}$. Since our choice of N satisfies $m|N$, we have

$$c^N \rightarrow c^N = (c^m)^{\frac{N}{m}} = d, \quad \text{as } j \rightarrow \infty.$$

Denote by ϕ the idempotent state d . Set $\omega_0 := \omega$ and $\omega_n := c_n^N$ for all $n \geq 1$. Then $\omega_n \rightarrow \phi$ as n tends to ∞ . By definition, $\{\omega_n\}_{n \geq 0}$ is a subsequence of $\{b_j\}_{j \geq 0}$, thus $\omega_{n-1} = \omega_n^{s_n}$ with $N|s_n$ for all $n \geq 1$. Moreover, from (5.5) we have

$$\omega_n = c_n^N = (c d'_n)^N = c^N d_n'^N = \phi d_n'^N = \phi(\phi d_n'^N) = \phi \omega_n, \quad n \geq 0.$$

Similarly, $\omega_n = \omega_n \phi$, $n \geq 0$. Hence $\{\omega_n\}_{n \geq 0}$ verifies the conditions of Proposition 5.7, and consequently $\omega \in \mathcal{P}_\phi(\mathbb{G})$. $\square$

Before giving the second proof, we introduce the following proposition, which could be formulated and proved for a general Banach algebra.

Proposition 5.12 *Let $\mathbb{G} = (A, \Delta)$ be a compact quantum group, with A separable. Let ω be an infinitely divisible state on $\mathbb{G}$. Suppose that there exist an idempotent state ϕ and a sequence of ϕ -bi-invariant roots $\{\omega_{n_k}\}_{k \geq 1}$ of ω , where $\{n_k\}_{k \geq 0}$ is an increasing sequence of positive integers, such that $\omega_{n_k}^{n_k} = \omega$ for all $k \geq 1$, and*

$$\sup_{k \geq 1} n_k \|\omega_{n_k} - \phi\| = M < \infty, \quad (5.7)$$

then $\omega \in \mathcal{P}_\phi(\mathbb{G})$.

Proof Recall that $\{\varphi \in A' : \|\varphi\| \leq M\}$ is compact with respect to weak* topology for each $M > 0$. Then from (5.7) we have for some subsequence of $\{n_k\}_{k \geq 1}$, still denoted by $\{n_k\}_{k \geq 1}$, that $n_k(\omega_{n_k} - \phi)$ converges to an element $u \in A'$ with respect to the weak* topology. Then $u = \lim_{k \rightarrow \infty} n_k(\omega_{n_k} - \phi) \in \mathcal{N}_\phi(\mathbb{G})$ and $\exp_\phi(u)$ is a Poisson state. It suffices to show that $\omega = \exp_\phi(u)$. Set

$$u_{n_k} := \sum_{m \geq 2} \frac{(\omega_{n_k} - \phi)^m}{m!}, \quad k \geq 1.$$

It is well-defined, since $\sup_{k \geq 1} \|\omega_{n_k} - \phi\| < \infty$. Moreover, from (5.7) it follows

$$\|u_{n_k}\| \leq \sum_{m \geq 2} \frac{1}{m!} \left(\frac{M}{n_k}\right)^m \leq \frac{1}{n_k^2} \sum_{m \geq 2} \frac{M^m}{m!},$$

whence

$$\lim_{k \rightarrow \infty} n_k \|u_{n_k}\| = 0,$$

and

$$\lim_{k \rightarrow \infty} (1 + \|u_{n_k}\|)^{n_k} = \lim_{k \rightarrow \infty} (1 + \|u_{n_k}\|)^{\frac{1}{\|u_{n_k}\|} \cdot n_k \|u_{n_k}\|} = 1.$$

Hence

$$\begin{aligned} \|\omega - \exp_\phi(n_k(\omega_{n_k} - \phi))\| &= \|\omega_{n_k}^{n_k} - (\omega_{n_k} + u_{n_k})^{n_k}\| \\ &\leq \sum_{m=1}^{n_k} \binom{n_k}{m} \|\omega_{n_k}^{n_k-m} u_{n_k}^m\| \\ &\leq \sum_{m=1}^{n_k} \binom{n_k}{m} \|u_{n_k}\|^m = (1 + \|u_{n_k}\|)^{n_k} - 1, \end{aligned}$$

which tends to 0 as $k \rightarrow \infty$. This shows $\omega = \exp_\phi(u)$ and finishes the proof. $\square$

The second proof of Theorem 5.11 Again, $\mathcal{P}(\mathbb{G}) \subset \mathcal{I}(\mathbb{G})$ is clear. Let $\omega \in \mathcal{I}(\mathbb{G})$. From the first proof we know that there exist an idempotent state $\phi \in \text{Idem}(\mathbb{G})$ and a sequence of roots $\{\omega_{n_k}\}_{k \geq 0} \subset \mathcal{I}(A)$ with $\{n_k\}_{k \geq 0}$ an increasing sequence of positive integers such that

$$\omega_{n_k}^{n_k} = \omega, \quad \omega_{n_k} = \omega_{n_k} \phi = \phi \omega_{n_k}, \quad k \geq 0,$$

and $\omega_{n_k} \rightarrow \phi$ as $k \rightarrow \infty$. Let u and u_{n_k} be the restrictions of ω and ω_{n_k} to A_ϕ for all $k \geq 0$, respectively. Then from Lemma 3.4 u is a state on A_ϕ such that $\{u_{n_k}\}_{k \geq 0}$ is a sequence of roots of u in $\mathcal{S}(A_\phi)$ verifying

$$u_{n_k}^{n_k} = u \text{ and } u_{n_k} \rightarrow \epsilon_\phi, \quad k \rightarrow \infty,$$

where ϵ_ϕ is the counit of A_ϕ . Now we repeat a calculation in Lemma 5.9. Suppose that $A = \oplus_{\alpha=1}^m \mathbb{M}_{n_\alpha}(\mathbb{C})$ with the matrix units $\{e_{ij}^\alpha : 1 \leq i, j \leq n_\alpha, 1 \leq \alpha \leq m\}$. Let $\{\widetilde{e}_{ij}^\alpha\}$ be the dual basis. Then by Lemma 5.2 we can write u as

$$u_{n_k} = \sum_{\alpha=1}^m p_{\alpha,k} \sum_{i,j=1}^{n_\alpha} a_{ij}^{\alpha,k} \widetilde{e}_{ij}^\alpha$$

with $p_{\alpha,k} \geq 0$, $\sum_{\alpha=1}^m p_{\alpha,k} = 1$ and $[a_{ij}^{\alpha,k}]_{i,j=1}^{n_\alpha}$ is positive semi-definite with trace one for each k . Let α_0 be such that $\widetilde{e}^{\alpha_0} = \epsilon_\phi = 1_{\hat{A}_p}$. By Corollary 5.3 and the assumption,

$$\|u_{n_k} - \epsilon_\phi\| = 2(1 - p_{\alpha_0,k}) \rightarrow 0, \quad k \rightarrow \infty. \quad (5.8)$$

So $p_{\alpha_0,k} \rightarrow 1$ as $k \rightarrow \infty$.

Now for each $q > 0$ we introduce the L_q -norm induced by $\hat{h}_p$. Recall that $\hat{h}_p = \frac{1}{\phi(p_A)} \hat{h}|_{\hat{A}_p}$ is a tracial state on $\hat{A}_p$. Namely, we define

$$\|x\|_q := [\hat{h}_p(|x|^q)]^{\frac{1}{q}}, \quad x \in \hat{A}_p,$$

where $|x| = (x^*x)^{\frac{1}{2}}$. Then Hölder's inequality still holds and we have

$$\|u\|_{\frac{2}{n_k}} = \|u_{n_k}^{n_k}\|_{\frac{2}{n_k}} \leq \|u_{n_k}\|_2^{n_k}, \quad k \geq 0. \quad (5.9)$$

Note that u is invertible in $\hat{A}_p$. Indeed, for large k there holds

$$\|u_{n_k} - \epsilon_\phi\|_{\hat{A}_p} \leq \|u_{n_k} - \epsilon_\phi\| < 1, \quad k \geq 0,$$

then we have that u_{n_k} is invertible for large k . So is $u = u_{n_k}^{n_k}$ is also invertible in $\hat{A}_p$.

Following a similar calculation to that in Lemma 5.9, we obtain

$$\hat{h}_p(u_{n_k} u_{n_k}^*) \leq p_{\alpha_0,k}^2 + (1 - p_{\alpha_0,k})^2.$$

This, together with (5.9), yields

$$\hat{h}_p(|u|^{\frac{1}{n_k}}) = \|u\|_{\frac{2}{n_k}}^{\frac{2}{n_k}} \leq \|u_{n_k}\|_2^{\frac{2}{n_k}} = \hat{h}_p(u_{n_k} u_{n_k}^*) \leq p_{\alpha_0,k}^2 + (1 - p_{\alpha_0,k})^2,$$

for all k . Since $p_{\alpha_0,k} \rightarrow 1$ as $k \rightarrow \infty$, there exists $K > 0$ such that for all $k \geq K$, $\frac{1}{2} \leq p_{\alpha_0,k} \leq 1$. Thus for all $k \geq K$, $1 - p_{\alpha_0,k} \leq p_{\alpha_0,k}$ and then

$$\hat{h}_p(|u|^{\frac{1}{n_k}}) \leq p_{\alpha_0,k}^2 + (1 - p_{\alpha_0,k})^2 \leq p_{\alpha_0,k}^2 + p_{\alpha_0,k}(1 - p_{\alpha_0,k}) = p_{\alpha_0,k}. \quad (5.10)$$

Combining this with (5.8), we have

$$n_k \|u_{n_k} - \epsilon_\phi\| = 2n_k(1 - p_{\alpha_0,k}) \leq 2n_k(1 - \hat{h}_p(|u|^{\frac{1}{n_k}})) = 2n_k \hat{h}_p(1_{\hat{A}_p} - |u|^{\frac{1}{n_k}})$$

for all $k \geq K$. Since u is invertible in $\hat{A}_p$, $n(1_{\hat{A}_p} - |u|^{\frac{1}{n}})$ converges to $-\log |u|$ in norm as $n \rightarrow \infty$. Then there exists a constant $M < \infty$ such that

$$\sup_{n \geq 1} n \hat{h}_p(1_{\hat{A}_p} - |u|^{\frac{1}{n}}) \leq M.$$

Thus from Lemma 3.4 it follows that

$$\sup_{k \geq 0} n_k \|\omega_{n_k} - \phi\| = \sup_{k \geq 0} n_k \|u_{n_k} - \epsilon_\phi\| \leq \sup_{k \geq 0} 2n_k \hat{h}_p(1_{\hat{A}_p} - |u|^{\frac{1}{n_k}}) < \infty.$$

Hence $\omega \in \mathcal{P}_\phi(\mathbb{G})$ by Proposition 5.12. $\square$

Remark 5.13 Both proofs rely on the capture of idempotent state where the infinitely divisible state is “supported on” and the sequence of roots converging to this idempotent state. After this the first proof aims to show that this sequence of roots can be chosen to form a *submonogeneous convolution semigroup* (Proposition 5.7 (3)), while the idea of the second proof is derived from a general result Proposition 5.12, concerning the decay property (5.7) of this sequence of roots. The inequality (5.10) also allows us to simplify the proof of the main theorem in [17] for the finite group case.

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References

- Bédos, E., Murphy, G.J., Tuset, L.: Co-amenability of compact quantum groups. *J. Geom. Phys.* **40**, 130–153 (2001)
- Böge, W.: Über die Charakterisierung unendlich teilbarer Wahrscheinlichkeitsverteilungen. *J. Reine Angew. Math.* **201**, 150–156 (1959)
- Chapovsky, Y.A., Vainerman, L.I.: Compact quantum hypergroups. *J. Oper. Theory.* **41**(2), 261–289 (1999)
- Das, B., Franz, U., Wang, X.: Invariant Markov semigroups on quantum homogeneous spaces. [arXiv:1901.00791](https://arxiv.org/abs/1901.00791) (2019)
- Delvaux, L., Van Daele, A.: Algebraic quantum hypergroups. *Adv. Math.* **226**(2), 1134–1167 (2011)
- Delvaux, L., Van Daele, A.: Algebraic quantum hypergroups II. Constructions and examples. *Int. J. Math.* **22**(3), 407–434 (2011)
- Faal, R., Kasprzak, P.: Group-like projections for locally compact quantum groups. *J. Oper. Theory.* **80**(1), 153–166 (2018)
- Franz, U., Skalski, A.: A new characterisation of idempotent states on finite and compact quantum groups. *C. R. Math. Acad. Sci. Paris.* **347**(17–18), 991–996 (2009)
- Franz, U., Skalski, A.: On idempotent states on quantum groups. *J. Algebra* **322**(5), 1774–1802 (2009)
- Kalyuzhnyi, A.A.: Conditional expectations on compact quantum groups and new examples of quantum hypergroups. *Methods Funct. Anal. Topol.* **7**(4), 49–68 (2001)
- Kawada, Y., Itô, K.: On the probability distribution on a compact group. I. *Proc. Phys.-Math. Soc. Jpn.* (3) **22**, 977–998 (1940)
- Kasprzak, P., Sołtan, P.M.: The Lattice of Idempotent States on a Locally Compact Quantum Group. [arXiv:1802.03953](https://arxiv.org/abs/1802.03953) (2018)
- Kerov, S.V.: Duality of finite-dimensional *-algebras. *Vestnik Leningrad. Univ.* **7**, 23–239 (1974)

14. Landstad, M.B., Van Daele, A.: Groups with compact open subgroups and multiplier Hopf $\ast$ -algebras. *Expo. Math.* **26**(3), 197–217 (2008)
15. Maes, A., Van Daele, A.: Notes on compact quantum groups. *Nieuw Arch. Wisk. (4)* **16**(1–2), 73–112 (1998)
16. Pal, A.: A counterexample on idempotent states on a compact quantum group. *Lett. Math. Phys.* **37**(1), 75–77 (1996)
17. Parthasarathy, K.R.: Infinitely divisible representations and positive definite functions on a compact group. *Commun. Math. Phys.* **16**, 148–156 (1970)
18. Schmetterer, L.: On Poisson laws and related questions. In: *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability* (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory pages 169–185, (1972)
19. Van Daele, A.: Discrete quantum groups. *J. Algebra* **180**(2), 431–444 (1996)
20. Evdokimov, S.A., Ponomarenko, I.N., Vershik, A.M.: Algebras in Plancherel duality and algebraic combinatorics. *Funct. Anal. Appl.* **31**(4), 252–261 (1997)
21. Woronowicz, S.L.: *Compact quantum groups. Symétries quantiques* (Les Houches, 1995), North-Holland, Amsterdam, 845–884 (1998)

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