Correction to: On the L^r Hodge theory in complete non compact Riemannian manifolds

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Our aim is to correct the proofs of Lemma 5.2 and Lemma 5.3 of original article. We use the notation of original article's [Section 5].

For Lemma 5.2, the correction is straightforward: we replace the *false (in general) estimate*

$$\|Au\|_{L^r(B_x(R))} \leq \|\nabla g\|_{L^\infty(B_x(R))} \|\Delta_{\mathbb{R}^n} u\|_{L^r(B_x(R))} + \eta(\epsilon) \|\nabla u\|_{L^r(B_x(R))},$$

by the right one

$$\|Au\|_{L^r(B_x(R))} \leq \|\nabla g\|_{L^\infty(B_x(R))} \|u\|_{W^{2,r}(B_x(R))},$$

which is proved in original article.

The remainder of the computation is exactly the same and concludes the proof of Lemma 5.2.

In order to prove Lemma 5.3, we shall need the following tools.

Lemma 0.1 *Let $B_R := B(0, R)$ be the ball in $\mathbb{R}^n$ with center 0 and radius $R \leq 1$ and $B'_R = B(0, 3R/4)$. Suppose we have, with a constant C depending only on n, r , and the C^1 bound of the coefficients of Δ_φ ,*

$$\forall v \in W^{2,r}(B_1) \quad \|v\|_{W^{2,r}(B'_1)} \leq C \left(\|v\|_{L^r(B_1)} + \|\Delta_\varphi v\|_{L^r(B_1)} \right).$$

Let $u \in L^r(B_R)$ be such that $\Delta_\varphi u \in L^r(B_R)$. Then, $u \in W^{2,r}(B'_R)$ and

$$\|u\|_{W^{2,r}(B'_R)} \leq c_1 R^{-2} \|u\|_{L^r(B_R)} + c_2 \|\Delta_\varphi u\|_{L^r(B_R)},$$

where the constants c_1, c_2 depend only on n, r , and the C^1 -bound of the coefficients of Δ_φ .

The original article can be found online at <https://doi.org/10.1007/s00209-017-1844-9>.

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Proof We start with $R = 1$ and $B := B(0, 1)$. We have by assumption

$$\forall v \in W^{2,r}_p(B_1) \quad \|v\|_{W^{2,r}_p(B'_1)} \leq C \left(\|v\|_{L^r(B_1)} + \|\Delta_\varphi v\|_{L^r(B_1)} \right),$$

the constants C depending only on n, r and the C^1 bound of the coefficients of Δ_φ . It remains to make the simple change of variables $y = Rx$, $dm(y) = R^n dm(x)$, $v(x) := u(Rx)$ and to notice that $\partial_j v(x) = R \partial_j u(Rx)$, $\partial_{ij}^2 v(x) = R^2 \partial_{ij}^2 u(Rx)$ in the integrals defining the L^r -norm, to get the result. $\square$

Lemma 0.2 *Let Δ_φ be a second-order elliptic matrix operator with C^∞ coefficients operating on p -forms defined in $U \subset \mathbb{R}^n$. Let $B := B(0, R)$ be a ball in $\mathbb{R}^n$, set $B' := B(0, 3R/4)$, and suppose that $B \Subset U$. Then we have an interior estimate: there are constants c_1, c_2 depending only on $n = \dim_{\mathbb{R}} M$, r and the C^1 norm of the coefficients of Δ_φ in $\bar{B}$ such that*

$$\forall v \in W^{2,r}_p(B) \quad \|v\|_{W^{2,r}_p(B')} \leq c_1 R^{-2} \|v\|_{L^r(B)} + c_2 \|\Delta_\varphi v\|_{L^r(B)}. \quad (0.1)$$

Proof For 0-forms, this lemma is exactly [1, Theorem 9.11] plus Lemma 0.1 to get the dependency in R . For p -forms, we cannot avoid the use of deep results on elliptic systems of equations.

Let v be a p -form in $B \subset \mathbb{R}^n$. We use the interior estimates in [2, Section 6.2, Theorem 6.2.6]. In our context of a second-order elliptic system, and with our notation, with $r > 1$, we get

$$\exists C > 0 \quad \forall v \in W^{2,r}_p(B) \quad \|v\|_{W^{2,r}_p(B')} \leq c_1 R^{-2} \|v\|_{L^r(B)} + c_2 \|\Delta_\varphi v\|_{L^r(B)},$$

already including the dependency in R .

The constants c_1, c_2 depend only on $r, n := \dim M$, and the bounds and moduli of continuity of the coefficients of the matrix Δ_φ .

In particular, if the coefficients of Δ_φ are close to those of $\Delta_{\mathbb{R}}$ in the C^1 norm, the constants c_1, c_2 are close to the ones obtained for $\Delta_{\mathbb{R}}$. $\square$

We are now in position to prove Lemma 5.3, which is the following.

Lemma 5.3 *Let (M, g) be a riemannian manifold. For $x \in M$, $\epsilon > 0$, we take a ϵ -admissible ball $B_x(R)$. We have a local Calderon Zygmund inequality on the manifold M : there are constants c_1, c_2 depending only on $n = \dim_{\mathbb{R}} M$, r , and ϵ such that*

$$\forall u \in W^{2,r}_p(B_x(R)) \quad \|u\|_{W^{2,r}_p(B_x(R/2))} \leq c_1 R^{-2} \|u\|_{L^r(B_x(R))} + c_2 \|\Delta u\|_{L^r(B_x(R))}.$$

Proof We transcribe the problem in $\mathbb{R}^n$ by use of a coordinates chart (V, φ) . The Hodge laplacian is the second-order elliptic matrix operator Δ_φ with C^∞ coefficients operating in $\varphi(V) \subset \mathbb{R}^n$. By the choice of a ϵ -admissible ball $B_x(R)$, and with $R' := 3R/4$, we have

$$U' := \varphi(B_x(R')) \subset B_e(0, (1 + \epsilon)R'), \quad U := \varphi(B_x(R)) \subset B_e(0, (1 + \epsilon)R) \subset \varphi(V).$$

We apply Lemma 0.2 to the euclidean balls $B' := B_e(0, (1 + \epsilon)R')$ and $B := B_e(0, (1 + \epsilon)R)$, and we get, with u_φ the p -form u read in the chart (V, φ) ,

$$\|u_\varphi\|_{W^{2,r}_p(B')} \leq c_1 R^{-2} \|u_\varphi\|_{L^r(B)} + c_2 \|\Delta_\varphi u_\varphi\|_{L^r(B)}.$$

The fact that the coefficients of Δ_φ are ϵ -close to those of $\Delta_{\mathbb{R}}$ in the C^1 norm, by condition 2) in the definition of the ϵ -admissible ball, implies that the constants c_1, c_2 depend only on n, r , and ϵ .

The Lebesgue measure on U and the canonical measure dv_g on $B_x(R)$ are equivalent; more precisely, because of condition 1), we have $(1 - \epsilon)^n \leq |\det g| \leq (1 + \epsilon)^n$, and the measure dv_g read in the chart φ is $dv_g = \sqrt{|\det g_{ij}|} d\xi$, where $d\xi$ is the Lebesgue measure in $\mathbb{R}^n$. So the Lebesgue estimates and the Sobolev estimates up to order 2 on U are valid in $B_x(R)$, up to a constant depending only on n, r and ϵ by Lemma 4.2 in original article. In particular,

$$\forall x \in M \quad \text{Vol}(B_x(R)) \leq (1 + \epsilon)^{n/2} v_n R^n, \quad (0.2)$$

where v_n is the euclidean volume of the unit ball in $\mathbb{R}^n$.

So we get, with $A := B_x((1 + 2\epsilon)R) \supset \varphi^{-1}(B)$ and $A' := \varphi^{-1}(B')$,

$$\|u\|_{W^{2,r}(A')} \leq c_1 R^{-2} \|u\|_{L^r(A)} + c_2 \|\Delta u\|_{L^r(A)}.$$

Now we notice that $A' \supset B_x(R')$, so a fortiori,

$$\|u\|_{W^{2,r}(B_x(R'))} \leq c_1 R^{-2} \|u\|_{L^r(B_x((1+2\epsilon)R))} + c_2 \|\Delta u\|_{L^r(B_x((1+2\epsilon)R))}.$$

Finally, choosing $\epsilon \leq 1/4$, we get

$$\|u\|_{W^{2,r}(B_x(R'))} \leq c_1 R^{-2} \|u\|_{L^r(B_x(3R/2))} + c_2 \|\Delta u\|_{L^r(B_x(3R/2))},$$

which is the local Calderon Zygmund inequality on $B_x(3R/4) \subset B_x(3R/2) \subset M$. So changing R for $3R/2$, this proves Lemma 5.3. $\square$

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