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On semi-equivalence of generically-finite polynomial mappings

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Abstract Let $f, g : X \to Y$ be continuous mappings. We say that f is topologically equivalent to g if there exist homeomorphisms $\Phi : X \to X$ and $\Psi : Y \to Y$ such that $\Psi \circ f \circ \Phi = g$. Moreover, we say that f is topologically semi-equivalent to g if there exist open, dense subsets $U, V \subset X$ and homeomorphisms $\Phi : U \to V$ and $\Psi : Y \to Y$ such that $\Psi \circ f \circ \Phi|_U = g|_U$. Let X, Y be smooth irreducible affine complex varieties. We show that every algebraic family $F : M \times X \ni (m, x) \mapsto F(m, x) = f_m(x) \in Y$ of polynomial mappings contains only a finite number of topologically non-equivalent proper mappings and only a finite number of topologically non-semi-equivalent generically-finite mappings. In particular there are only a finite number of classes of topologically non-equivalent proper polynomial mappings $f : \mathbb{C}^n \to \mathbb{C}^m$ of a bounded (algebraic) degree. The same is true for a number of classes of topologically non-semi-equivalent generically-finite polynomial mappings $f : \mathbb{C}^n \to \mathbb{C}^m$ of a bounded (algebraic) degree.

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1 Introduction

Let $f, g : X \to Y$ be continuous mappings. We say that f is *topologically equivalent* to g if there exist homeomorphisms $\Phi : X \to X$ and $\Psi : Y \to Y$ such that $\Psi \circ f \circ \Phi = g$. Moreover, we say that f is topologically semi-equivalent to g if there exist open, dense subsets $U, V \subset X$ and homeomorphisms $\Phi : U \to V$ and $\Psi : Y \to Y$ such that $\Psi \circ f \circ \Phi|_U = g|_U$.

In the case $X = \mathbb{C}^n$ and $Y = \mathbb{C}$ René Thom stated a Conjecture that there are only finitely many topological types of polynomials $f : X \to Y$ of bounded degree. This Conjecture was confirmed by Fukuda [2]. Also a more general problem was considered: how many

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topological types are there in the family $P(n, m, k)$ of polynomial mapping $f : \mathbb{C}^n \to \mathbb{C}^m$ of degree bounded by k? Aoki and Noguchi [1] showed that there are only a finite number of topologically non-equivalent mappings in the family $P(2, 2, k)$. Finally Nakai [8] showed that each familiy $P(n, m, k)$, where $n, m, k > 3$, contains infinitely many different topological types even if we consider only generically-finite mappings. Hence the General Thom Conjecture is not true even for generically-finite mappings. However, we show in this paper that there are only a finite number of classes of topologically semi-equivalent generically-finite polynomial mappings $f : \mathbb{C}^n \to \mathbb{C}^m$ of a bounded (algebraic) degree. As a by product of our considerations we give a simple proof of the following interesting fact: for every n, m and k there are only a finite number of topological types of *proper* polynomial mappings $f : \mathbb{C}^n \to \mathbb{C}^m$ of (algebraic) degree bounded by k. Hence we can say that Thom Conjecture is true for proper polynomial mappings. We show also that if $n \leq m$ and $\Omega_n(d_1, ..., d_m)$ denotes the family of all polynomial mappings $F = (f_1, ..., f_m) : \mathbb{C}^n \to \mathbb{C}^m$ of a multi-degree bounded by $(d_1, ..., d_m)$, then any two general member of this family are topologically equivalent.

In fact we prove more: if X, Y are smooth affine irreducible varieties, then every algebraic family $\mathcal{F}$ of polynomial mappings from X to Y contains only a finite number of topologically non-semi-equivalent (non-equivalent) generically-finite (proper) mappings. Moreover, if a family $\mathcal{F}$ is irreducible, then two generic members of $\mathcal{F}$ are in the same equivalence class.

Let us recall here, that a mapping $f : X \to Y$ is generically finite, if for general $x \in X$ the set $f^{-1}(f(x))$ is finite. Our proof goes as follows. Let M be a smooth affine irreducible variety and let $\mathcal{F}$ be a family of polynomial mappings induced be a regular mapping $F : M \times X \to Y$, i.e., $\mathcal{F} := \{f_m : X \ni x \mapsto F(m, x) \in Y,\ m \in M\}$. Let us recall that if $f : X \to Z$ is a generically finite polynomial mapping of affine varieties, then the *bifurcation set* $B(f)$ of f is the set $\{z \in Z : z \in Sing(Z)\ or\ \#f^{-1}(z) \neq \mu(f)\}$, where $\mu(f)$ is the topological degree of f. The set $B(f)$ is always closed in Z. We show that there exists a Zariski open, dense subset U of M such that

(1) for every $m \in U$ we have $\mu(f_m) = \mu(\mathcal{F})$, where we treat f_m as a mapping $f_m : X \to Z_m := \overline{f_m(X)}$,
(2) for every $m_1, m_2 \in U$ the pairs $(\overline{f_{m_1}(X)}, B(f_{m_1}))$ and $(\overline{f_{m_2}(X)}, B(f_{m_2}))$ are equivalent via a homeomorphism, i.e., there is a homeomorphism $\Psi : Y \to Y$ such that $\Psi(\overline{f_{m_1}(X)}) = \overline{f_{m_2}(X)}$ and $\Psi(B(f_{m_1})) = B(f_{m_2})$.

In particular the group $G = \pi_1(\overline{f_m(X)} \setminus B(f_m))$ does not depend on $m \in U$. Using elementary facts from the theory of topological coverings, we show that the number of topological semi-types (types) of generically-finite (proper) mappings in the family $\mathcal{F}_{|U}$ is bounded by the number of subgroups of G of index $\mu(\mathcal{F})$, hence it is finite. Then we conclude the proof by induction. Finally, the case of arbitrary M can be easily reduced to the smooth, irreducible, affine case.

Remark 1.1 In this paper we use the term "polynomial mapping" for every regular mapping $f : X \to Y$ of affine varieties.

2 Bifurcation set

Let X, Z be affine irreducible varieties of the same dimension and assume that X is smooth. Let $f : X \to Z$ be a dominant polynomial mapping. It is well known that there is a Zariski open non-empty subset U of Z such that for every $x_1, x_2 \in U$ the fibers $f^{-1}(x_1)$, $f^{-1}(x_2)$

have the same number $\mu(f)$ of points. We say that $\mu(f)$ is the topological degree of f. Recall the following (see [5,6]).

Definition 2.1 Let X, Z be as above and let $f : X \to Z$ be a dominant polynomial mapping. We say that f *is finite at a point* $z \in Z$ if there exists an open neighborhood U of z such that the mapping $f_{|f^{-1}(U)} : f^{-1}(U) \to U$ is proper.

It is well-known that the set S_f of points at which the mapping f is not finite is either empty or it is a hypersurface (see [5,6]). We say that S_f is *the set of non-properness* of f.

Definition 2.2 Let X be a smooth affine n-dimensional variety and let Z be an affine variety of the same dimension. Let $f : X \to Z$ be a generically finite dominant polynomial mapping of geometric degree $\mu(f)$. The bifurcation set of f is

$$B(f) = \{z \in Z : z \in Sing(Z) \text{ or } \#f^{-1}(z) \neq \mu(f)\}.$$

Remark 2.3 The same definition makes sense for those continuous mapping $f : X \to Z$, for which we can define the topological degree $\mu(f)$ and singularities of Z. In particular if Z_1, Z_2 are affine algebraic varieties, $f : X \to Z_1$ is a dominant polynomial mapping and $\Phi : Z_1 \to Z_2$ is a homeomorphism which preserves singularities, then we can define $B(\Phi \circ f)$ as $\Phi(B(f))$. Moreover, the mapping $\Phi \circ f$ behaves topologically as an analytic covering. We will use this facts in the proof of Theorem 3.5.

We have the following theorem (see also [7]).

Theorem 2.4 *Let X, Z be affine irreducible complex varieties of the same dimension and suppose X is smooth. Let $f : X \to Z$ be a polynomial dominant mapping. Then the set $B(f)$ is closed and $B(f) = K_0 \cup S_f \cup Sing(Z)$.*

Proof Let us note that outside the set $S_f \cup Sing(Z)$ the mapping f is a (ramified) analytic covering of degree $\mu(f)$. By Lemma 2.5 below, if $z \notin Sing(Z)$ we have $\#f^{-1}(z) \leq \mu(f)$. Moreover, since f is an analytic covering outside $S_f \cup Sing(Z)$ we see that for $y \notin S_f \cup Sing(Z)$ the fiber $f^{-1}(z)$ has exactly $\mu(f)$ points counted with multiplicity. Take $X_0 := X \setminus f^{-1}(Sing(Z) \cup S_f)$. If $z \in K_0(f_{|X_0})$, the set of critical values of $f_{|X_0}$, then $\#f^{-1}(z) < \mu(f)$.

Now let $z \in S_f \setminus Sing(Z)$. There are two possibilities:

(a) $\#f^{-1}(z) = \infty$.
(b) $\#f^{-1}(z) < \infty$.

In case (b) we can assume that $f^{-1}(z) \neq \emptyset$. Let U be an affine neighborhood of z disjoint from $Sing(z)$ over which the mapping f has finite fibers. Let $V = f^{-1}(U)$. By the Zariski Main Theorem in the version given by Grothendieck (see [3]), there exists a normal variety $\overline{V}$ and a finite mapping $\overline{f} : \overline{V} \to U$ such that

(1) $V \subset \overline{V}$,
(2) $\overline{f}_{|V} = f$.

Since $z \in \overline{f}(\overline{V} \setminus V)$, it follows from Lemma 2.5 below that $\#f^{-1}(z) < \mu(f)$. Consequently, if $z \in S_f$, we have $\#f^{-1}(z) < \mu(f)$. Finally, we have $B(f) = K_0(f_{|X_0}) \cup S_f \cup Sing(Z)$. However, the set $K_0(f_{|X_0})$ is closed in $Z \setminus (S_f \cup Sing(Z))$. Hence $B(f)$ is closed in Z. ◻

Lemma 2.5 *Let X, Z be affine normal varieties of the same dimension. Let $f : X \to Z$ be a finite mapping. Then for every $z \in Z$ we have $\#f^{-1}(z) \leq \mu(f)$.*

Proof Let $\# f^{-1}(z) = \{x_1, \dots, x_r\}$. We can choose a function $h \in \mathbb{C}[X]$ which separates all x_i (in particular we can take as h the equation of a general hyperplane section). Since f is finite, the minimal equation of h over the field $\mathbb{C}(Z)$ is of the form:

$$T^s + a_1(f)T^{s-1} + \dots + a_s(f) \in f^*\mathbb{C}[Z][T],$$

where $s \le \mu(f)$. If we substitute $f = z$ into this equation we get the desired result. □

3 Main result

We start with the following:

Lemma 3.1 *Let $f : X^k \to Y^l$ be a dominant polynomial mapping of affine irreducible varieties. There exists a Zariski open non-empty subset $U \subset Y$ such that for any $y \in U$ we have $Sing(f^{-1}(y)) = f^{-1}(y) \cap Sing(X)$.*

Proof We can assume that Y is smooth. Since there exists a mapping $\pi : Y^l \to \mathbb{C}^l$ which is generically etale, we can assume that $Y = \mathbb{C}^l$. Let us recall that if Z is an algebraic variety, then a point $z \in Z$ is smooth if and only if the local ring $\mathcal{O}_z(Z)$ is regular, or equivalently $\dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2 = \dim Z$, where $\mathfrak{m}$ denotes the maximal ideal of $\mathcal{O}_z(Z)$.

Let $y = (y_1, \dots, y_l) \in \mathbb{C}^l$ be a sufficiently generic point. Then by Sard's Theorem the fiber $Z = f^{-1}(y)$ is smooth outside $Sing(X)$ and $\dim Z = \dim X - l = k - l$. Note that the generic (scheme-theoretic) fiber F of f is reduced. Indeed, this fiber $F = Spec(\mathbb{C}(Y) \otimes_{\mathbb{C}[Y]} \mathbb{C}[X])$ is the spectrum of a localization of $\mathbb{C}[X]$ and so a domain. Since we are in characteristic zero, the reduced $\mathbb{C}(Y)$-algebra $\mathbb{C}(Y) \otimes_{\mathbb{C}[Y]} \mathbb{C}[X]$ is necessarily geometrically reduced (i.e. stays reduced after extending to an algebraic closure of $\mathbb{C}(Y)$). Since the property of fibres being geometrically reduced is open on the base, i.e. on Y, thus the fibres over an open subset of Y will be reduced. Consequently, there is a Zariski open, non-empty subset $U \subset Y$ such that for $y \in U$ the fiber $f^{-1}(y)$ is reduced. Hence we can assume that Z is reduced. It is enough to show that every point $z \in Z \cap Sing(X)$ is singular on Z.

Assume that $z \in Z \cap Sing(X)$ is smooth on Z. Let $f : X \to \mathbb{C}^l$ be given as $f = (f_1, \dots, f_l)$, where $f_i \in \mathbb{C}[X]$. Then $\mathcal{O}_z(Z) = \mathcal{O}_z(X)/(f_1 - y_1, \dots, f_l - y_l)$. In particular if $\mathfrak{m}'$ denotes the maximal ideal of $\mathcal{O}_z(Z)$ and $\mathfrak{m}$ denotes the maximal ideal of $\mathcal{O}_z(X)$ then $\mathfrak{m}' = \mathfrak{m}/(f_1 - y_1, \dots, f_l - y_l)$. Let α_i denote the class of the polynomial $f_i - y_i$ in $\mathfrak{m}/\mathfrak{m}^2$. Let us note that

$$\mathfrak{m}'/\mathfrak{m}'^2 = \mathfrak{m}/(\mathfrak{m}^2 + (\alpha_1, \dots, \alpha_l)). \tag{1}$$

Since the point z is smooth on Z we have $\dim_{\mathbb{C}} \mathfrak{m}'/\mathfrak{m}'^2 = \dim Z = \dim X - l$. Take a basis $\beta_1, \dots, \beta_{k-l}$ of the space $\mathfrak{m}'/\mathfrak{m}'^2$ and let $\overline{\beta_i} \in \mathfrak{m}/\mathfrak{m}^2$ correspond to β_i under the correspondence (1). Note that the vectors $\overline{\beta_1}, \dots, \overline{\beta_{k-l}}, \alpha_1, \dots, \alpha_l$ generate the space $\mathfrak{m}/\mathfrak{m}^2$. This means that $\dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2 \le k - l + l = k = \dim X$. Hence the point z is smooth on X, a contradiction. □

We have:

Lemma 3.2 *Let X, Y be smooth complex irreducible algebraic varieties and $f : X \to Y$ a regular dominant mapping. Let $N \subset W \subset X$ be closed subvarieties of X. Then there exists a non-empty Zariski open subset $U \subset Y$ such that for every $y_1, y_2 \in U$ the triples $(f^{-1}(y_1), W \cap f^{-1}(y_1), N \cap f^{-1}(y_1))$ and $(f^{-1}(y_2), W \cap f^{-1}(y_2), N \cap f^{-1}(y_2))$ are homeomorphic.*

Proof Let X_1 be an algebraic completion of X and let $\overline{Y}$ be a smooth algebraic completion of Y. Take $X_1' := \overline{graph(f)} \subset X_1 \times \overline{Y}$ and let X_2 be a desingularization of X_1'.

We can assume that $X \subset X_2$. We have an induced mapping $\overline{f} : X_2 \to \overline{Y}$ such that $\overline{f}_{|X} = f$. Let $Z = X_2 \backslash X$. Denote by $\overline{N}, \overline{W}$ the closures of N and W in X_2. Let $\mathcal{R} = \{\overline{N} \cap Z, \overline{W} \cap Z, \overline{N}, \overline{W}, Z\}$, a collection of algebraic subvarieties of X_2. There is a Whitney stratification $\mathcal{S}$ of X_2 which is compatible with $\mathcal{R}$.

For any smooth strata $S_i \in \mathcal{S}$ let B_i be the set of critical values of the mapping $\overline{f}_{|S_i}$ and denote $B = \overline{\bigcup B_i}$. Take $X_3 = X_2 \backslash \overline{f}^{-1}(B)$. The restriction of the stratification $\mathcal{S}$ to X_3 gives a Whitney stratification which is compatible with the family $\mathcal{R}' := \mathcal{R} \cap X_3$. We have a proper mapping $f' := \overline{f}_{|X_3} : X_3 \to \overline{Y} \backslash B$ which is a submersion on each stratum. By the Thom first isotopy theorem there is a trivialization of f' which preserves the strata. It is an easy observation that this trivialization gives a trivialization of the mapping $f : X \backslash f^{-1}(B) \to Y \backslash B := U$. In particular the fibers $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are homeomorphic via a stratum preserving homeomorphism. This means that the triples $(f^{-1}(y_1), W \cap f^{-1}(y_1), N \cap f^{-1}(y_1))$ and $(f^{-1}(y_2), W \cap f^{-1}(y_2), N \cap f^{-1}(y_2))$ are homeomorphic. □

We also need the following:

Definition 3.3 Let X, Y be smooth affine varieties. By a family of regular mappings $\mathcal{F}_M(X, Y, F) := \mathcal{F}$ we mean a regular mapping $F : M \times X \to Y$, where M is an algebraic variety. The members of a family $\mathcal{F}$ are the mappings $f_m : X \ni x \to F(m, x) \in Y$. Let

$$G : M \times X \ni (m, x) \mapsto (m, F(m, x)) \in Z = \overline{G(M \times X)} \subset M \times Y.$$

If G is generically finite, then by the topological degree $\mu(\mathcal{F})$ we mean the number $\mu(G)$. Otherwise we put $\mu(\mathcal{F}) = 0$.

Later we will sometimes identify the mapping f_m with the mapping $G(m, \cdot) = (m, f_m) : X \to m \times Y$. The following lemma is important:

Lemma 3.4 *Let X, Y be smooth affine complex varieties. Let M be a smooth affine irreducible variety and let $\mathcal{F}$ be the family induced by a mapping $F : M \times X \to Y$, i.e., $\mathcal{F} = \{f_m : X \ni x \mapsto F(m, x) \in Y,\ m \in M\}$. Assume that $\mu(\mathcal{F}) > 0$. Take $Z = \overline{G(M \times X)}$ and put $Z_m = (m \times Y) \cap Z$.*

Then

(1) *There is an open non-empty subset $U_1 \subset M$ such that for every $m \in U_1$ we have $\mu(f_m) = \mu(\mathcal{F})$;*
(2) *There is a non-empty open subset $U_2 \subset U_1$ such that for every $m \in U_2$ we have $\overline{f_m(X)} = Z_m := (m \times Y) \cap Z$ and $B(f_m) = B(G)_m := (m \times Y) \cap B(G)$;*
(3) *There is a non-empty open subset $U_3 \subset U_2$ such that for every $m_1, m_2 \in U_3$ the pairs $(\overline{f_{m_1}(X)}, B(f_{m_1}))$ and $(\overline{f_{m_2}(X)}, B(f_{m_2}))$ are equivalent by means of a homeomorphism, i.e., there is a homeomorphism $\Psi : Y \to Y$ such that $\Psi(\overline{f_{m_1}(X)}) = \overline{f_{m_2}(X)}$ and $\Psi(B(f_{m_1})) = B(f_{m_2})$.*

Proof (1) Take $G : M \times X \ni (m, x) \mapsto (m, F(m, x)) \in Z$. The mapping $G : M \times X \ni (m, x) \mapsto (m, F(m, x)) \in Z$ has a constant number of points in the fibers outside the bifurcation set $B(G) \subset Z$. Take $U = Z \backslash B(G)$. By Theorem 2.4 the set U is open. Let $\pi : Z \ni (m, y) \mapsto m \in M$ be the projection. We show that the constructible set $\pi(U)$ is dense in M. Indeed, assume that $\overline{\pi(U)} = N$ is a proper subset of M. Since U is dense in Z, we have $\pi(Z) \subset N$, i.e., $Z \subset N \times Y$. This is a contradiction. In particular the set $\pi(U)$ is dense in M and it contains a Zariski open, non-empty subset $U_1 \subset M$. Of course $\mu(f_m) = \mu(\mathcal{F})$ for $m \in U_1$.

(2) Consider the projection $\pi : Z \ni (m, y) \mapsto m \in M$. As we know from (1), the mapping π is dominant. By a well known result, after shrinking U_1 we can assume that every fiber Z_m of π ($m \in U_2 \subset U_1$) is of pure dimension $d = \dim Z - \dim M = \dim X$. However, $Z_m = \overline{f_m(X)} \cup B(G)_m$. Generically the dimension of $B(G)_m$ is less than d. Hence if we possibly shrink U_2, we get $Z_m = \overline{f_m(X)}$ for $m \in U_2$. Moreover, by Lemma 3.1 (after shrinking U_2 if necessary), we can assume that $Sing(Z_m) = Sing(Z)_m := (m \times Y) \cap Sing(Z)$. Now it is easy to see that $B(f_m) = B(G)_m$.
(3) We have $\overline{f_m(X)} = Z_m$ and $B(f_m) = B(G)_m$ for $m \in U_2$. Now apply Lemma 3.2 with $X = U_2 \times Y$, $W = (U_2 \times Y) \cap Z$, $N = (U_2 \times Y) \cap B(G)$ and $f : U_1 \times Y \ni (m, y) \mapsto m \in U_1$.

□

Now we are ready to prove our main result:

Theorem 3.5 *Let X, Y be smooth affine irreducible varieties. Every algebraic family $\mathcal{F}$ of polynomial mappings from X to Y contains only a finite number of topologically non-semi-equivalent (non-equivalent) generically-finite (proper) mappings.*

Proof The proof is by induction on $\dim M$. We can assume that M is affine, irreducible and smooth. Indeed, M can be covered by a finite number of affine subsets M_i, and we can consider the families $\mathcal{F}_{|M_i}$ separately. For the same reason we can assume that M is irreducible. Finally $\dim M \setminus Reg(M) < \dim M$ and we can use induction to reduce the general case to the smooth one.

Assume that M is smooth and affine. If $\mu(\mathcal{F}) = 0$, then $\mathcal{F}$ does not contain any generically-finite mapping. Hence we can assume that $\mu(\mathcal{F}) = k > 0$. By Lemma 3.4 there is a non-empty open subset $U \subset M$ such that for every $m_1, m_2 \in U$ we have

(1) $\mu(f_{m_1}) = \mu(f_{m_2}) = k$,
(2) The pairs $(\overline{f_{m_1}(X)}, B(f_{m_1}))$ and $(\overline{f_{m_2}(X)}, B(f_{m_2}))$ are equivalent by means of a homeomorphism, i.e., there is a homeomorphism $\Psi : Y \to Y$ such that $\Psi(\overline{f_{m_1}(X)}) = \overline{f_{m_2}(X)}$ and $\Psi(B(f_{m_1})) = B(f_{m_2})$.

Fix a pair $Q = \overline{f_{m_0}(X)}$, $B = B(f_{m_0})$ for some $m_0 \in U_3$. For $m \in U_3$ the mapping $f_m : X \to Y$ is topologically equivalent to the continuous mapping $f'_m = \Psi_m \circ f_m$ with $\overline{f'_m(X)} = Q$ and $B(f'_m) = B$ (Lemma 3.4). Every mapping f'_m induces a topological covering $f'_m : X \setminus {f'_m}^{-1}(B) = P_{f'_m} \to R = Q \setminus B$. Take a point $a \in R$ and let $a_{f'_m} \in {f'_m}^{-1}(a)$. We have an induced homomorphism

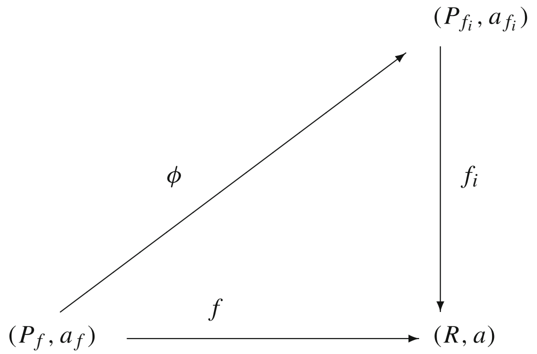
$$f_* : \pi_1(P_{f'_m}, a_{f'_m}) \to \pi_1(R, a).$$

Denote $H_f = f_*(\pi_1(P_f, a_f))$ and $G = \pi_1(R, a)$. Hence $[G : H_f] = k$. It is well known that the fundamental group of a smooth algebraic variety is finitely generated. In particular the group $G := \pi_1(Q \setminus B, a)$ is finitely generated. Let us recall the following result of M. Hall (see [4]):

Lemma 3.6 *Let G be a finitely generated group and let k be a natural number. Then there are only a finite number of subgroups $H \subset G$ such that $[G : H] = k$.*

By Lemma 3.6 there are only a finite number of subgroups $H_1, ..., H_r \subset G$ with index k. Choose generically-finite (proper) mappings $f_i = f'_{m_i} = \Psi_i \circ f_{m_i} : X \to Y$ such that $H_{f_i} = H_i$ (of course only if such a mapping f_i does exist). We show that every generically-finite (proper) mapping f'_m ($m \in U$) is semi-equivalent (equivalent) to one of mappings f_i.

Indeed, let $H_{f'_m} = H_{f_i}$ (here $f'_m = \Psi_m \circ f_m$). We show that $f'_m := f$ is equivalent to f_i. Let us consider two coverings $f : (P_f, a_f) \to (R, a)$ and $f_i : (P_{f_i}, a_{f_i}) \to (R, a)$. Since $f_*(\pi_1(P_f, a_f)) = f_{i*}(\pi_1(P_{f_i}, a_{f_i}))$ we can lift the covering f to a homeomorphism $\phi : P_f \to P_{f_i}$ such that following diagram commutes:



Hence for generically-finite mappings we have

$$(\Psi_i)^{-1} \circ \Psi_m \circ f_m \circ \phi^{-1}|_U = f_{m_i}|_U,$$

where $V = X \setminus f_m^{-1}(B(f_m))$ and $U = X \setminus f_{m_i}^{-1}(B(f_{m_i}))$. Hence f_m is semi-equivalent to f_{m_i}.

In the case of proper mappings we show additionally that the mapping ϕ can be extended to a continuous mapping Φ on the whole of X. Indeed, take a point $x \in f^{-1}(B)$ and let $y = f(x)$. The set $f_i^{-1}(y) = \{b_1, ..., b_s\}$ is finite. Take small open disjoint neighborhoods $W_i(r)$ of b_i, such that $W_i(r)$ shrinks to b_i as r tends to 0. We can choose an open neighborhood $V(r)$ of y so small that $f_i^{-1}(V(r)) \subset \bigcup_{j=1}^{s} W_i(r)$. Now take a small connected neighborhood $P_x(r)$ of x such that $f(P_x(r)) \subset V(r)$. The set $P_x(r) \setminus f^{-1}(B)$ is still connected and it is transformed by ϕ into one particular set $W_{i_0}(r)$. We take $\Phi(x) = b_{i_0}$. It is easy to see that the mapping Φ so defined is a continuous extension of ϕ. In fact $\phi(P_x(r) \setminus f^{-1}(B))$ shrinks to b_{i_0} if r goes to 0. Moreover, we still have $f = f_i \circ \Phi$.

In a similar way the mapping Λ determined by ϕ^{-1} is continuous. It is easy to see that $\Lambda \circ \Phi = \Phi \circ \Lambda = identity$, hence Φ is a homeomorphism. Consequently, the mapping $f_i \circ \Phi = \Psi_i \circ f_{m_i} \circ \Phi$ is equal to $f = \Psi_m \circ f_m$. Finally, we get

$$(\Psi_i)^{-1} \circ \Psi_m \circ f_m \circ \Phi^{-1} = f_{m_i}.$$

This means that the family $\mathcal{F}_{|U}$ contains only a finite number of topologically non-semi-equivalent (non-equivalent) generically-finite (proper) mappings. In fact, the number of topological semi-types (types) of generically-finite (proper) mappings in $\mathcal{F}_{|U}$ is bounded by the number of subgroups of G of index $\mu(\mathcal{F})$.

Let $T = M \setminus U$. Hence dim $T <$ dim M. By the induction the family $\mathcal{F}_{|T}$ also contains only a finite number of topologically non-semi-equivalent (non-equivalent) generically-finite (proper) mappings. Consequently so does $\mathcal{F}$. □

Corollary 3.7 *There is only a finite number of topologically non-semi-equivalent (non-equivalent) generically-finite (proper) polynomial mappings $f : \mathbb{C}^n \to \mathbb{C}^m$ of a bounded algebraic degree.* □

4 Families of proper mappings

In this section we slightly extend our previous result in the case of irreducible families of proper (or generically-finite) mappings. First we prove a following lemma:

Lemma 4.1 *Let $Y = \mathbb{R}^n$ and let $Z \subset Y$ be a linear subspace of Y. Fix $\epsilon > 0$ and take $\eta < \epsilon$. Let $B(0, \eta)$ be a ball of radius η. Let $\gamma : I \ni t \mapsto \gamma(t) \in B(0, \eta) \cap Z$ be a smooth path. Then there exists a continuous family of diffeomorphisms $\Phi_t : Y \to Y$, $t \in [0, 1]$ such that*

(1) $\Phi_t(\gamma(t)) = \gamma(0)$ *and* $\Phi_t(z) = z$ *for* $\|z\| \geq \epsilon$.
(2) $\Phi_0 =$ identity.
(3) $\Phi_t(Z) = Z$.

Proof Let $v_t = \gamma(0) - \gamma(t) \in T\mathbb{R}^n$. We construct a family of diffeomorphisms Φ_t, which are interpolation between translation $x \to x + v_t$ and identity.

Let $\sigma : Y \to [0, 1]$ be a differentiable function such that $\sigma = 1$ on $B(0, \eta)$ and $\sigma = 0$ outside $B(0, \epsilon)$. Define a vector field $V(x) = \sigma(x)v_t$. Integrating this vector field we get desired diffeomeorphisms Φ_t, for any t. □

Corollary 4.2 *Let Y be a smooth manifold and Z be a smooth submanifold. For every point $a \in Z$ and every open neighborhood V_a of the point a, there is an open connected neighborhood U_a of the point a, such that:*

(a) $\overline{U_a} \subset V_a$,
(b) *if $\gamma : I \ni t \mapsto \gamma(t) \in U_a \cap Z$ is a smooth path, then there is a continuous family of diffeomorphism $\psi_t : Y \to Y$, $t \in [0, 1]$ such that*

(1) $\psi_t(\gamma(t)) = \gamma(0)$,
(2) $\psi_t(x) = x$ *for* $x \notin V_a$ *and* $\psi_0 =$ identity,
(3) $\psi_t(Z) = Z$.

Now we are in a position to prove:

Theorem 4.3 *Let X, Y be smooth affine irreducible varieties. Let $\mathcal{F} : M \times X \to Y$ be an algebraic family of proper polynomial mappings from X to Y. Assume that M is an irreducible variety. Then there exists a Zariski open dense subset $U \subset M$ such that for every $m, m' \in U$ mappings f_m and $f_{m'}$ are topologically equivalent.*

Proof We follow the proof of Theorem 3.5 and we use here the same notation. By Lemma 3.4 there is a non-empty open subset $U \subset M$ such that for every $m_1, m_2 \in U$ we have

(1) $\mu(f_{m_1}) = \mu(f_{m_2}) = k$,
(2) The pairs $(\overline{f_{m_1}(X)}, B(f_{m_1}))$ and $(\overline{f_{m_2}(X)}, B(f_{m_2}))$ are equivalent by means of a homeomorphism, i.e., there is a homeomorphism $\Psi : Y \to Y$ such that $\Psi(\overline{f_{m_1}(X)}) = \overline{f_{m_2}(X)}$ and $\Psi(B(f_{m_1})) = B(f_{m_2})$.

Fix a pair $(Q = \overline{f_{m_0}(X)}, B = B(f_{m_0}))$ for some $m_0 \in U$. For $m \in U$ the mappings f_m and f_{m_0} can be connected by a continuous path f_t, $f_0 = f_{m_0}$, $f_1 = f_m$. Moreover we have also a continuous family of homeomorphisms $\Psi_t : Y \to Y$ such that $\Psi_t(\overline{f_t(X)}) = \overline{f_0(X)}$ and $\Psi(B(f_t)) = B(f_0)$. It is enough to prove that mappings $F_t = \Psi_t \circ f_t$ are locally (in the sense of parameter t) equivalent.

(1) *First step of the proof.* Let $C_t \subset X$ denotes the preimage by F_t of the set B (in fact $C_t = f_t^{-1}(B(f_t))$ and put $X_t = X \backslash C_t$. Put $Q' := Q \backslash B$.Assume that for all mappings F_t there is a point $a \in (X \backslash \bigcup_{t \in I} C_t)$ such that for all $t \in I$ we have $F_t(a) = b$.

We have an induced homomorphism $G_{t*} : \pi_1(X_t, a) \to \pi_1(Q', b)$. We show that the subgroup $F_{t*}(\pi_1(X_t, a)) \subset \pi_1(Q', b)$ does not depend on t.

Indeed let $\gamma_1, ..., \gamma_s$ be generators of the group $\pi_1(X_{t_0}, a)$. Let U_i be an open relatively compact neighborhoods of γ_i such that $\overline{U_i} \cap C_{t_0} = \emptyset$. For sufficiently small number $\epsilon > 0$ and $t \in (t_0 - \epsilon, t_0 + \epsilon)$ we have $\overline{U_i} \cap C_t = \emptyset$. Let $t \in (t_0 - \epsilon, t_0 + \epsilon)$. Note that the loop $F_t(\gamma_i)$ is homotopic with the loop $F_{t_0}(\gamma_i)$. In particular the group $F_{t_0 *}(\pi_1(X_{t_0}, a))$ is contained in the group $F_{t*}(\pi_1(X_t, a))$. Since they have the same (finite!) index in $\pi_1(Y', b)$ they are equal. This means that the subgroup $G_{t*}(\pi_1(X_t, a)) \subset \pi_1(Y', b)$ is locally constant, hence it is constant.

Let us consider two coverings $F_t : (X_t, a) \to (Q', b)$ and $F_0 : (X_0, a) \to (Q', b)$. Since $F_{t*}\pi_1(X_t, a) = F_{0*}\pi_1(X_0, a)$ we can lift the covering F_t to a homeomorphism $\phi_t : X_t \to X_0$. As before we can extend the homeomorphism ϕ_t to the homeomorphism $\Phi_t : X \to X$, such that $F_0 \circ \Phi_t = F_t$.

(2) *The general case.* Now we can prove Theorem 4.3. Since in general there is no a point $a \in (X \setminus \bigcup_{t \in I} C_t)$ such that for all $t \in I$ we have $F_t(a) = b$, we have to modify our construction.

First we prove that for every $t_0 \in I$ there exists $\epsilon > 0$ and a family of homeomorphisms $\Phi_t : X \to X$, $t \in (t_0 - \epsilon, t_0 + \epsilon)$ such that $F_t = F_{t_0} \circ \Phi_t$ for $t \in (t_0 - \epsilon, t_0 + \epsilon)$. Take a point $a \in X_{t_0}$ and choose $\epsilon > 0$ so small that $a \in X_t$ for $t \in (t_0 - \epsilon, t_0 + \epsilon)$. Put $\gamma(t) \ni t \mapsto F_t(a) \in Y'$. We can take ϵ so small that the hypothesis of Corollary 4.2 is satisfied. Applying Corollary 4.2 with $Y' = Y \setminus B$ and $Z = Q \setminus B$ we have a continuous family of diffeomeorphisms $\psi_t : Y \to Y$ which preserves Q and B, $t \in (t_0 - \epsilon, t_0 + \epsilon)$ such that $\psi_t(F_t(a)) = F_0(a)$. Take $G_t = \psi_t \circ F_t$. Arguing as in the first part of our proof all G_t are topologically equivalent for $t \in (t_0 - \epsilon, t_0 + \epsilon)$. Hence also all F_t are topologically equivalent for $t \in (t_0 - \epsilon, t_0 + \epsilon)$. Since F_t are locally topologically equivalent, they are topologically equivalent for every $t \in I$. □

Corollary 4.4 *Let $n \leq m$ and let $\Omega_n(d_1, ..., d_m)$ denotes the family of all polynomial mappings $F = (f_1, ..., f_m) : \mathbb{C}^n \to \mathbb{C}^m$ of a multi-degree bounded by $(d_1, ..., d_m)$. Then any two general members of this family are topologically equivalent.*

Proof Indeed, it is enough to note that a generic mapping $f \in \Omega_n(d_1, ..., d_m)$ is proper. □

Using the same method we can prove:

Theorem 4.5 *Let X, Y be smooth affine irreducible varieties. Let $\mathcal{F} : M \times X \to Y$ be an algebraic family of generically-finite polynomial mappings from X to Y. Assume that M is an irreducible variety. Then there exists a Zariski open dense subset $U \subset M$ such that for every $m, m' \in U$ the mappings f_m and $f_{m'}$ are topologically semi-equivalent.*

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