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ERRATUM

Erratum to: Bloch–Wigner theorem over rings with many units

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1 Introduction

In this erratum, we correct a mistake that I made in [1]. The formulation of our main theorem [1, Theorem 5.1] is not correct. The correct form of the theorem, which is sufficient for our applications, is as follows:

Let R be a commutative ring with many units and define

$$\tilde{H}_3(\mathrm{SL}_2(R), \mathbb{Z}) := H_3(\mathrm{GL}_2(R), \mathbb{Z})/(H_3(\mathrm{GL}_1(R), \mathbb{Z}) + R^* \cup H_2(\mathrm{GL}_1(R), \mathbb{Z}),$$

which is induced by the diagonal inclusion of $R^* \times \mathrm{GL}_1(R)$ in $\mathrm{GL}_2(R)$. Then, there is a quotient M of $H_1(\Sigma_2, R^* \otimes R^*)$ which fits into exact sequences

$$0 \longrightarrow T_R \longrightarrow \tilde{H}_3(\mathrm{SL}_2(R), \mathbb{Z}) \longrightarrow \mathfrak{p}(R) \longrightarrow (R^* \otimes_{\mathbb{Z}} R^*)_\sigma \longrightarrow K_2(R) \longrightarrow 0,$$
$$\mathrm{Tor}_1^{\mathbb{Z}}(\mu(R), \mu(R)) \longrightarrow T_R \longrightarrow M \longrightarrow 0.$$

When R is an integral domain, the left-hand side map in the second exact sequence is injective.

2 The main theorem

Lemma 3.1 in [1] is not correct. In fact, in the proof of the lemma, the map

$$F_q \otimes_{\mathrm{Stab}_{\mathrm{GL}_2}(\infty)} \mathbb{Z} \to F_q \otimes_{\mathrm{GL}_2} C_1(R^2), \quad s \otimes 1 \mapsto s \otimes (\infty, 0)$$

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is not well defined. Because of this mistake, we made few uncorrect claims in [1]. Here, we give a correct formulations of the lemma and these claims that are sufficient for our applications. We follow the notations in [1].

Lemma 3.1 *The groups* $E^2_{1,0}$ *and* $E^2_{1,1}$ *are trivial. Also, there is a surjective map* $H_1(\Sigma_2, R^*\otimes R^*) \twoheadrightarrow E^2_{1,2}$*, where the action of* $\Sigma_2 = \{1, \sigma\}$ *on* $R^*\otimes R^*$ *is defined by* $\sigma(a\otimes b) = -b\otimes a$*. In particular,* $E^2_{1,2}$ *is a* 2*-torsion group.*

Proof In [1] on page 335, we have shown that $H_0(\mathrm{GL}_2, H_1(X)) \simeq \mathbb{Z}$ and $d^1_{2,0} = \mathrm{id}_{\mathbb{Z}}$. Consider the differential $d^1_{2,1} : E^1_{2,1} = H_1(\mathrm{GL}_2, H_1(X)) \to \ker(d^1_{1,1}) \simeq R^*$. Let $\varphi : R^* \simeq H_1(\mathrm{GL}_2, C_2(R^2)) \to H_1(\mathrm{GL}_2, H_1(X))$. It is easy to see that $d^1_{2,1}\circ\varphi = \mathrm{id}_{R^*}$. These facts immediately imply the triviality of $E^2_{1,0}$ and $E^2_{1,1}$. To compute $E^2_{1,2}$, first note that $\ker(d^1_{1,2}) \simeq H_2(R^*) \oplus (R^*\otimes R^*)^\sigma$. Again, one can easily see that the composition

$$H_2(R^*) \simeq H_2\left(\mathrm{GL}_2, C_1\left(R^2\right)\right) \to H_2(\mathrm{GL}_2, H_1(X)) \to H_2(R^*)\oplus(R^*\otimes R^*)^\sigma$$

is given by $x \mapsto (x, 0)$. Thus, $E^2_{1,2} \simeq (R^*\otimes R^*)^\sigma/A$. By an easy analysis of our main spectral sequence, we have

$$E^2_{1,2} = (R^*\otimes R^*)^\sigma/A \hookrightarrow H_3(\mathrm{GL}_2)/H_3(R^*\times R^*).$$

We call this map β. We know that

$$H_1(\Sigma_2, R^*\otimes R^*) = \frac{(R^*\otimes R^*)^\sigma}{(1+\sigma)(R^*\otimes R^*)} = \frac{(R^*\otimes R^*)^\sigma}{\langle a\otimes b - b\otimes a : a, b\in R^*\rangle}.$$

Let $x_{a,b} = a\otimes b - b\otimes a$. Then,

$$h = [(a,1)|(1,b)] - [(1,b)|(a,1)] - [(b,1)|(1,a)] + [(1,a)|(b,1)]$$

is the cycle that represents the element $x_{a,b} \in (R^*\otimes R^*)^\sigma \subseteq H_2(R^*\times R^*)^\sigma$. Let τ be the automorphism of transposition of terms. Then, $\tau(h) - h = 0$. Now, by [2, Lemma 2.5], the image of the class $\overline{h}$ under β is given by $-\overline{\rho_s(h)}$, where $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$\begin{aligned}\rho_s(h) := &+[s|(1,a)|(b,1)] - [(a,1)|s|(b,1)] + [(a,1)|(1,b)|s]\\ &-[s|(b,1)|(1,a)] + [(1,b)|s|(1,a)] - [(1,b)|(1,1)|s]\\ &-[s|(1,b)|(a,1)] + [(b,1)|s|(a,1)] - [(b,1)|(1,a)|s]\\ &+[s|(a,1)|(1,b)] - [(1,a)|s|(1,b)] + [(1,a)|(b,1)|s].\end{aligned}$$

Now, by a direct computation, one can see that $\overline{\rho_s(h)} = 0$. Thus, $x_{a,b} \in A$ and therefore $(1+\sigma)(R^*\otimes R^*) \subseteq A$. This implies the surjectivity that we are looking for. □

Now, we are ready to correct Theorem 5.1 in [1].

Theorem 5.1 *Let* R *be a commutative ring with many units and define* $\tilde{H}_3(\mathrm{SL}_2(R)) := H_3(\mathrm{GL}_2)/(H_3(\mathrm{GL}_1) + R^*\cup H_2(\mathrm{GL}_1))$*. There is a quotient* M *of* $H_1(\Sigma_2, R^*\otimes R^*)$ *which fits into exact sequences*

$$0 \longrightarrow T_R \longrightarrow \tilde{H}_3(\mathrm{SL}_2(R)) \longrightarrow B(R) \longrightarrow 0,$$
$$\mathrm{Tor}_1^{\mathbb{Z}}(\mu(R), \mu(R)) \longrightarrow T_R \longrightarrow M \longrightarrow 0.$$

When R *is an integral domain, the left-hand side map in the second exact sequence is injective.*

Proof The proof is very similar to the proof of Theorem 5.1 in [1]. □

Remark 0.1 Here, we further make some minor corrections.

(i) In Proposition 2.1 and Remark 5.2 in [1], we have to assume that either the coefficient group is $\mathbb{Z}[1/2]$ or the ring R has the property $R^* = {R^*}^2$ ($K_1(R) = K_1(R)^2$ for Remark 5.2).
(ii) The claim made in Remark 2.2 in [1] is not correct, and Suslin's claim in [2, Remark 2.2] remains true.

The rest of our claims in [1] remains true.

References

1. Mirzaii, B.: Bloch–Wigner theorem over rings with many units. Math. Z. **268**, 329–346 (2011)
2. Suslin, A.A.: K_3 of a field and the Bloch group. Proc. Steklov Inst. Math. **183**(4), 217–239 (1991)