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ERRATUM

Erratum to: Transitivity of generic semigroups of area-preserving surface diffeomorphisms

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Theorem 12 in [2] claims that for a Moser generic diffeomorphism f and an invariant residual domain U, there are no periodic points in the boundary of U. This is claimed to be a direct consequence of [3]. However, Mather's theorem is weaker than that; specifically, it says that the rotation number of the prime ends compactification on any end of U is irrational.

On the sphere, Mather's result implies what is claimed in Theorem 12, and the proof (which can be found in [1]) uses the existence of homoclinic intersections for periodic points of f guaranteed by a theorem of Pixton. This can also be done in $\mathbb{T}^2$ using [4]. However, for higher genus, the claim remains open.

We will prove a weaker version of Theorem 12 of [2]. To do this, we define for each f a C^r-residual set $\mathcal{G}_f$, and we prove that the statement of Theorem 12 of [2] holds if we additionally require that the residual domain U be g-invariant for some $g \in \mathcal{G}_f$. This is enough for our proofs. The modifications needed in the statements and proofs of [2] are as follows:

1. In Lemmas 17 and 18, "f and g are Moser generic" should be changed to "f is Moser generic and $g \in \mathcal{G}_f$", where $\mathcal{G}_f$ is the residual set from Definition 2 below. The proofs remain the same, using Proposition 5 from this article instead of Theorem 12 of [2];

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2. In the proof of Lemma 19, Proposition 4 of the present article should be used instead of Theorem 12 of [2] (because frontiers have planar neighborhoods);
3. The set $\mathcal{R}_f$ defined in the proof of Theorem 23 should be intersected with $\mathcal{G}_f$ (which still gives a residual set).

As in [2], we assume from now on that S is a compact orientable surface and $r \geq 16$, so that Moser genericity is a C^r-generic condition. From [3, Theorem 5.2], if $f \in \mathrm{Diff}^r_\omega(S)$ is Moser generic and p is a hyperbolic periodic point, then any two branches of p have the same closure (a branch is a connected component of $W^s_f(p) \setminus p$ or $W^u_f(p) \setminus p$). Thus if we define $K_{p,f} = \overline{W^s_f(p)}$, we have that $K_{p,f}$ is the closure of any stable or unstable branch of p.

Proposition 1 *For a Moser generic* $f \in \mathrm{Diff}^r_\omega(S)$*, if* p *is a hyperbolic periodic point and* $K_{p,f}$ *has a planar neighborhood, then* p *has a homoclinic intersection.*

Proof The proof of [5] assumes that the whole manifold is planar, but it works almost without modification if instead we assume that the stable and unstable manifolds of p are contained in a planar submanifold of S.

This is clear from the proof presented in [4] (the first part of the proof of Theorem 2, p. 582, dealing only with the sphere) since it is a purely topological argument taking place in a neighborhood of the union of the stable and unstable manifolds of p, and the only part where the dynamics is relevant is in a local argument in a neighborhood of p. In our setting, using the fact that $K_{p,f}$ is the closure of both the stable and unstable manifolds of p, we can restrict our attention to a planar neighborhood of p, and by embedding this neighborhood in a sphere the same proof applies. □

Definition 2 For a Moser generic f, we denote by $\mathcal{G}_f$ the set of all Moser generic $g \in \mathrm{Diff}^r_\omega(S)$ such that the following property holds: for any hyperbolic periodic point p of f and $k > 0$,

$$\text{if } K_{p,f} \cap g^k(K_{p,f}) \neq \emptyset \text{ then } W^s_f(p) \pitchfork g^k(W^s_f(p)) \neq \emptyset \neq W^u_f(p) \pitchfork g^k(W^s_f(p)) \qquad (1)$$

where $A \pitchfork B$ stands for the set of transversal intersections of A and B.

Proposition 3 *If* f *is Moser generic, then* $\mathcal{G}_f$ *is* C^r*-residual.*

Proof Fix a Moser generic $g \in \mathrm{Diff}^r_\omega(S)$, $k \in \mathbb{N}$, and a hyperbolic periodic point p of f. Since the set $\mathrm{Per}_k(g)$ of periodic points of g with period at most k is finite, using a perturbation of g of the form hgh^{-1} with h close to the identity we may assume that $\mathrm{Per}_k(g) \cap \partial K_{p,f} = \emptyset$ (because $\partial K_{p,f}$ has empty interior). Suppose $K_{p,f}$ intersects $g^k(K_{p,f})$. Since g preserves area and $K_{p,f}$ is connected, this implies that $g^k(K_{p,f}) \cap \partial K_{p,f} \neq \emptyset$. Choose $x \in g^k(K_{p,f}) \cap \partial K_{p,f}$, and note that $x \notin \mathrm{Per}_k(g)$. Let $x_n \in W^s_f(p)$ and $y_n \in W^s_f(p)$ be such that $x_n \to x$ and $g^k(y_n) \to x$. Let U be a neighborhood of x such that $g^i(U)$ is disjoint from U for $-k \leq i \leq 0$. By standard arguments we may choose, for any sufficiently large n, a map $\tilde{h} \in \mathrm{Diff}^r_\omega(S)$ supported in $g^{-1}(U)$ and C^r-close to the identity such that $\tilde{h}(g^{k-1}(y_n)) = g^{-1}(x_n)$. Letting $\tilde{g} = g\tilde{h}$ we obtain a map C^r-close to g such that $\tilde{g}^k(y_n) = x_n \in W^s_f(p)$. Thus $\tilde{g}^k(W^s_f(p)) \cap W^s_f(p) \neq \emptyset$, and this intersection can be made transverse with a perturbation of $\tilde{g}$. Note that once the intersection is transverse, it persists under new perturbations. Since $K_{p,f}$ is the closure of $W^u_f(p)$ as well, and $\tilde{g}^k(K_{p,f}) \cap K_{p,f} \neq \emptyset$, we may use the same argument to obtain a perturbation $\hat{g}$ of $\tilde{g}$ which also has a point of transversal intersection between $W^u_f(p)$ and $\hat{g}^k(W^s_f(p))$. This shows that, for p and k fixed, the set of maps g satisfying (1) is dense in $\mathrm{Diff}^r_\omega(S)$. Since condition (1) is also open, the set $\mathcal{U}_{p,k} \subset \mathrm{Diff}^r_\omega(S)$ where property (1) holds for this choice of p and k is open and dense. Since $\mathcal{G}_f$ is the intersection of the sets $\mathcal{U}_{p,k}$ over all $k \in \mathbb{N}$ and all hyperbolic periodic points p of f (which are countably many), this completes the proof. □

Recall that a residual domain is a connected component of the complement of a nontrivial continuum. By [3, Lemma 2.3], a residual domain has finitely many boundary components.

Proposition 4 *Let f be Moser generic, U a periodic connected open set and $K \subset \partial U$ a periodic nontrivial continuum. If K has a planar neighborhood, then K contains no periodic points.*

Proof Let n be such that $f^n(K) = K$ and $f^n(U) = U$. Suppose that there is a periodic point p of f in K. We assume that $f^n(p) = p$, by increasing n if necessary. Since f (and so f^n) is Moser generic, p is either elliptic (Moser stable) or hyperbolic. By an argument of Mather, we can show that p is hyperbolic: suppose on the contrary that p is Moser stable. Then there is a sequence of invariant disks converging to p such that the dynamics of f^n on the boundary of each disk is minimal. If D is such a disk and K intersects ∂D, then $\partial D \subset K \subset \partial U$, which implies by connectedness that U is either contained in D or disjoint form D. The first case is not possible if we choose D small enough, while the second case contradicts the fact that $p \in \partial U$. Thus p is hyperbolic.

By [3, Proposition 11.1] it follows that $W^s_f(p)$ and $W^u_f(p)$ are contained in K. Since K has a planar neighborhood, it follows from Proposition 1 that p has a homoclinic intersection. By the λ-lemma, this implies that there is a decreasing sequence of rectangles R_i with boundary in $W^s_f(p) \cup W^u_f(p) \subset K$ accumulating on p. This contradicts the connectedness of U: in fact U cannot be contained in all of the rectangles R_i, but by connectedness if U is not contained in R_i then $\overline{U}$ is disjoint from the interior of R_{i+1}, which contains points of ∂U. This is a contradiction. □

Proposition 5 *Let f be Moser generic, and let U be an f-periodic residual domain. If U is also g-periodic for $g \in \mathcal{G}_f$, then ∂U has no periodic points of f.*

Proof If K is a boundary component of U, then there is $n > 0$ such that $f^n(K) = K = g^n(K)$. Suppose that there is a periodic point p of f in K. We assume that $f^n(p) = p$ and $f^n(U) = U = g^n(U)$ by increasing n if necessary. Since f is Moser generic, the first paragraph of the proof of Proposition 4 shows that p must be hyperbolic.

By [3, Proposition 11.1], it follows that the stable and unstable manifolds of p are contained in K, so that $K_p \doteq K_{p,f}$ is contained in K. We will now show that K_p has a planar neighborhood. Assume by contradiction that K_p has no planar neighborhood, and consider two cases.

Suppose first that $K_p, g^n(K_p), g^{2n}(K_p), \dots$ are pairwise disjoint. Then, for any $m > 0$ we can find m disjoint nonplanar manifolds with boundary $N_1, \dots, N_m$ in S, by taking disjoint neighborhoods of $K_p, g^n(K_p), \dots, g^{n(m-1)}(K_p)$. Moreover, we may assume that no component of $\overline{S \setminus N_i}$ is a disk, by adding disks to N_i if necessary. This implies that each component of $\overline{S \setminus \cup_i N_i}$ has Euler characteristic at most 0 (for it has at least two boundary components). On the other hand, the Euler characteristic of each N_i is strictly negative (because by our assumption they are nonplanar and have nonempty boundary), so we have

$$2-2g=\chi(S)=\chi(N_1)+\cdots+\chi(N_m)+\chi\left(\overline{S\setminus\cup_i N_i}\right)\leq -m,$$

where g is the genus of S. Since this can be done for all m, we get a contradiction.

Now suppose that $g^k(K_p)\cap K_p\neq\emptyset$ for some $k>0$ multiple of n. Then by Proposition 3, we can find arcs $\gamma_s,\gamma_u\subset g^k(W^s_f(p))\subset K$ such that γ_s has a nonempty transversal intersection with $W^s_f(p)$, and γ_u has a nonempty transversal intersection with $W^u_f(p)$. By a standard λ-lemma argument, this implies that, arbitrarily close to p, we can find a decreasing sequence of arbitrarily small rectangles R_i bounded by arcs in $W^s_f(p)$, $W^u_f(p)$, $f^{n_ik}(\gamma_u)$ and $f^{-n_ik}(\gamma_s)$ (for some $n_i\in\mathbb{Z}$) accumulating on p. Since all of these arcs are contained in $K\subset\partial U$, the same argument used in the proof of Proposition 4 produces a contradiction from the connectedness of U. This completes the proof that K_p has a planar neighborhood.

But if K_p has a planar neighborhood, we know from Proposition 4 that f cannot have periodic points on K_p, contradicting the definition of K_p. This completes the proof. □

References

1. Franks, J., Le Calvez, P.: Regions of instability for non-twist maps. Ergodic Theory Dynam. Syst. **23**, 111–141 (2003)
2. Koropecki, A., Nassiri, M.: Transitivity of generic semigroups of area-preserving surface diffeomorphisms. Math. Z. **266**, 707–718 (2010)
3. Mather, J.: Invariant subsets of area-preserving homeomorphisms of surfaces. Adv. Math. Suppl. Stud. **7B**, 531–561 (1981)
4. Oliveira, F.: On the generic existence of homoclinic points. Ergodic Theory Dynam. Syst. **7**(4), 567–595 (1987)
5. Pixton, D.: Planar homoclinic points. J. Differ. Equ. **44**(3), 365–382 (1982)