Math. Z. (2011) 267:341–366
DOI 10.1007/s00209-009-0622-8

Mathematische Zeitschrift

The divergence equation in weighted- and $L^{p(\cdot)}$-spaces

Alexander Huber

Received: 17 June 2008 / Accepted: 22 December 2008 / Published online: 30 October 2009

Abstract We study the solvability of the divergence equation in weighted spaces and Lebesgue spaces with variable exponents, where the weights are so called Muckenhoupt weights. The question of constructing divergence free test functions, which can be used for problems arising in fluid dynamics, is also addressed. The approach is based on an explicit representation formula for solutions of the divergence equation due to Bogovskiĭ and the theory of singular integral operators. The developed methods are used to prove an existence result for fluids which satisfy a $p(\cdot)$-growth condition.

Keywords Divergence equation · Muckenhoupt weights · Lebesgue spaces with variable exponents · Singular integrals · Fluids with $p(\cdot)$-growth

Mathematics Subject Classification (2000) 42B20 · 42B35 · 76D03

1 Introduction and statement of the main results

In this work we are interested in solving the divergence equation, i.e.

$$\begin{aligned} \operatorname{div} u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega\,, \end{aligned}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary. For the right hand side f, the compatibility condition $\int_\Omega f(x)\,dx = 0$ is needed since $u = 0$ at the boundary. This problem was completely solved by Bogovskiĭ [1,2] in the setting of L^p-spaces ($1 < p < \infty$) by using an explicit representation formula and the Calderón-Zygmund-theory for singular integral operators. Here we provide a generalisation to weighted- and $L^{p(\cdot)}$-spaces. To state the main results, we first fix some notations:

A. Huber (✉)
Max Planck Institute for Mathematics in the Sciences, Inselstr. 22, 04103 Leipzig, Germany
e-mail: alhuber@mis.mpg.de

A locally integrable and positive function $\omega : \mathbb{R}^d \to \mathbb{R}$ is called A_p- or Muckenhoupt weight for some $1 < p < \infty$, written $\omega \in A_p$, if

$$A_p(\omega) = \sup_{\substack{B \subset \mathbb{R}^d \\ B \text{ ball}}} \left(\fint_B \omega(x)\, dx \right) \left(k\fint_B \omega(x)^{-\frac{1}{p-1}}\, dx \right)^{p-1} < \infty.$$

The real number $A_p(\omega)$ is called the A_p-constant of ω. In the following statements, we have to deal with constants $C = C(\omega)$, which depend on the weight ω. Usually, these constants can be chosen uniformly for all ω with equibounded A_p-constant. Therefore, a mapping $C : A_p \to \mathbb{R}_+$ is called A_p-consistent if

$$\sup\{C(\omega) \mid \omega \in A_p \text{ with } A_p(\omega) \le c\} < \infty$$

for all $c \ge 1$. For an open set $\Omega \subset \mathbb{R}^d$ and a weight $\omega \in A_p$ $(1 < p < \infty)$, we write $L^p_\omega(\Omega)$ for the set of all measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\|f\|_{L^p_\omega(\Omega)} = \left(\int_\Omega |f(x)|^p \omega(x)\, dx \right)^{\frac{1}{p}} < \infty.$$

The space $(L^p_\omega(\Omega), \|\cdot\|_{L^p_\omega(\Omega)})$ is a Banach space—the so called weighted Lebesgue space. This is not surprising, since we only have replaced the Lebesgue measure by the measure μ_ω defined by

$$\mu_\omega(A) = \int_A \omega(x)\, dx$$

for measurable subsets $A \subset \Omega$. The weighted Sobolev space $W^{1,p}_\omega(\Omega)$ is defined in the usual way and $W^{1,p}_{\omega,0}(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in $W^{1,p}_\omega(\Omega)$. In view of the compatibility condition, we write $L^p_{\omega,0}(\Omega)$ for all $f \in L^p_\omega(\Omega)$ with $\int_\Omega f(x)\, dx = 0$. For more informations on weighted spaces, we refer the interested reader to the books of Journé [15], Torchinsky [20], García-Cuerva and Rubio de Francia [14].

We write $C^\infty_{0,0}(\Omega)$ for all $f \in C^\infty_0(\Omega)$ with $\int_\Omega f(x)\, dx = 0$ and show the following result in the setting of weighted spaces:

Theorem 1 *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $1 < p < \infty$ and $\omega \in A_p$. Then there exists a linear and bounded operator*

$$\mathbb{B} : L^p_{\omega,0}(\Omega) \to W^{1,p}_{\omega,0}(\Omega)^d$$

such that $\operatorname{div}(\mathbb{B} f) = f$ *for all $f \in L^p_{\omega,0}(\Omega)$. Moreover, the operator norm of $\mathbb{B}$ can be estimated by an A_p-consistent constant and we have $\mathbb{B} f \in C^\infty_0(\Omega)^d$ for $f \in C^\infty_{0,0}(\Omega)$.*

Theorem 1 generalises the result of Bogovskiĭ [1,2] to the case of weighted spaces. Durán and Muschietti [11, Theorem 3.2] proved Theorem 1 in the special case of so called power weights $\omega(x) = |x|^\alpha$ for $-d < \alpha < d\,(p-1)$, which are special examples for Muckenhoupt weights.

To define the $L^{p(\cdot)}$-spaces, we consider a measurable exponent

$$p(\cdot) : \mathbb{R}^d \to [1, \infty), \qquad 1 < p_- \le p(\cdot) \le p_+ < \infty$$

and denote the set of all measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\int_\Omega |f(x)|^{p(x)}\, dx < \infty$$

by $L^{p(\cdot)}(\Omega)$. This set is a Banach space - the so called Lebesgue space with variable exponent - when equipped with the norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 \,\middle|\, \int_\Omega \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \le 1 \right\}.$$

The $L^{p(\cdot)}$-norm of a function f can be compared with the more natural integral $\int_\Omega |f(x)|^{p(x)}\, dx$ in the following way:

$$\min \left\{ \|f\|_{p(\cdot)}^{p_-}, \|f\|_{p(\cdot)}^{p_+} \right\} \le \int_\Omega |f(x)|^{p(x)}\, dx \le \max \left\{ \|f\|_{p(\cdot)}^{p_-}, \|f\|_{p(\cdot)}^{p_+} \right\}.$$

The Hölder inequality extends in a natural way to the $L^{p(\cdot)}$-spaces, i.e.

$$\int_\Omega |f(x)g(x)|\, dx \le r_p \, \|f\|_{p(\cdot)} \, \|g\|_{p'(\cdot)} \,,$$

where $f \in L^{p(\cdot)}(\Omega)$, $g \in L^{p'(\cdot)}(\Omega)$ and $p'(x) = \frac{p(x)}{p(x)-1}$. Moreover, the dual space $\left(L^{p(\cdot)}\right)'$ of $L^{p(\cdot)}$ is isomorphic to $L^{p'(\cdot)}$. The Sobolev spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are defined in the usual way and we write $L_0^{p(\cdot)}(\Omega)$ for all $f \in L^{p(\cdot)}(\Omega)$ with $\int_\Omega f(x)\, dx = 0$. Up to here no regularity assumptions for the exponent $p(\cdot)$ have been made. By $\mathcal{P}(\mathbb{R}^d)$ we denote the set of all measurable exponents $p(\cdot)$ such that

$$1 < p_- \le p(\cdot) \le p_+ < \infty \quad \text{and} \quad M \text{ is bounded on } L^{p(\cdot)}(\mathbb{R}^d)\,,$$

where $Mf(x) = \sup_{r>0} ⨍_{B_r(x)} |f(y)|\, dy$ is the maximal operator. We refer to Kováčik and Rákosník [16] and to Diening, Hästö and Nekvinda [8] for more informations concerning these spaces.

We show the analogon to Theorem 1 in the setting of $L^{p(\cdot)}$-spaces using a different method as Diening and Růžička [10, Theorem 6.4]:

Theorem 2 *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary and let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ be a variable exponent. Then there exists a linear and bounded operator*

$$\mathbb{B} : L_0^{p(\cdot)}(\Omega) \to W_0^{1,p(\cdot)}(\Omega)^d$$

such that $\operatorname{div}(\mathbb{B} f) = f$ *for all* $f \in L_0^{p(\cdot)}(\Omega)$. *Moreover we have* $\mathbb{B} f \in C_0^\infty(\Omega)^d$ *for all* $f \in C_{0,0}^\infty(\Omega)$.

We also deal with the following question:

Given a function $f : \Omega \to \mathbb{R}^d$ such that $f = 0$ at $\partial\Omega$. Can we find another function $u : \Omega \to \mathbb{R}^d$ such that $u = 0$ at $\partial\Omega$ and $\operatorname{div}(f - u) = 0$?

The answer is simple: Just choose $u = f$. But in view of constructing divergence-free test functions, e.g. in problems arising in fluid dynamics, we are interested in small (compared to f) solutions. A suitable choice is $u = \mathbb{B} \circ \operatorname{div} f$, which automatically leads to an estimate of u in the $W^{1,p}_\omega$-norm by the L^p_ω-norm of div f thanks to Theorem 1. Moreover, it is also possible to estimate the L^p_ω-norm of u by the L^p_ω-norm of f. The idea for this construction is not new and was already discussed by Galdi [13, III.3, Theorem 3.3] in the case of classical Lebesgue spaces - known as estimates in negative norms. The next results can be seen as generalisations to weighted- and $L^{p(\cdot)}$-spaces. For given weights $w_1 \in A_{p_1}$ and $\omega_2 \in A_{p_2}$, we write $X^{p_1,p_2}_{\omega_1,\omega_2} = W^{1,p_1}_{\omega_1,0}(\Omega)^d \cap L^{p_2}_{\omega_2}(\Omega)^d$ and show the following:

Theorem 3 *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. For $1 < p_1, p_2 < \infty$ and $\omega_1 \in A_{p_1}$, $\omega_2 \in A_{p_2}$ there exists a linear operator $\mathbb{E} : X^{p_1,p_2}_{\omega_1,\omega_2} \to X^{p_1,p_2}_{\omega_1,\omega_2}$ with the following properties*:

(a) $\operatorname{div}(\mathbb{E} f) = \operatorname{div} f$ *for* $f \in X^{p_1,p_2}_{\omega_1,\omega_2}$

(b) *For all* $f \in X^{p_1,p_2}_{\omega_1,\omega_2}$ *holds*

$$\|\mathbb{E} f\|_{W^{1,p_1}_{\omega_1}} \le C_{p_1}(\omega_1)\, \|\operatorname{div} f\|_{L^{p_1}_{\omega_1}} ,$$

where $C_{p_1} > 0$ is an A_{p_1}-consistent constant and

$$\|\mathbb{E} f\|_{L^{p_2}_{\omega_2}} \le C_{p_2}(\omega_2)\, \|f\|_{L^{p_2}_{\omega_2}} ,$$

where $C_{p_2} > 0$ is an A_{p_2}-consistent constant.

(c) $\mathbb{E} f \in C^\infty_0(\Omega)^d$ *for* $f \in C^\infty_0(\Omega)^d$.

As an application of Theorem 3, we easily obtain the following density result:

Corollary 1 *Let $1 < p < \infty, \omega \in A_p$ and $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. Set*

$$V_\omega = \left\{ f \in W^{1,p}_{\omega,0}(\Omega)^d \;\middle|\; \operatorname{div} f = 0 \right\}$$

and

$$\mathcal{V} = \left\{ f \in C^\infty_0(\Omega)^d \;\middle|\; \operatorname{div} f = 0 \right\}.$$

Then $\mathcal{V}$ is a dense subspace of V_ω.

In the setting of Lebesgue spaces with variable exponents, we show the analogon to Theorem 3 and write $X_{p(\cdot),q(\cdot)} = W^{1,p(\cdot)}_0(\Omega)^d \cap L^{q(\cdot)}(\Omega)^d$ for given exponents $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^d)$.

Theorem 4 *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary and consider exponents $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^d)$. Then there exists a linear operator*

$$\mathbb{E} : X_{p(\cdot),q(\cdot)} \to X_{p(\cdot),q(\cdot)}$$

with the following properties:

(a) $\operatorname{div}(\mathbb{E} f) = \operatorname{div} f$ *for all* $f \in X_{p(\cdot),q(\cdot)}$.

(b) *For all* $f \in X_{p(\cdot),q(\cdot)}$ *we have*

$$\|\mathbb{E} f\|_{1,p(\cdot)} \le C_{p(\cdot)}\, \|\operatorname{div} f\|_{p(\cdot)}$$

and

$$\|\mathbb{E} f\|_{q(\cdot)} \le C_{q(\cdot)}\, \|f\|_{q(\cdot)}.$$

(c) $\mathbb{E} f \in C^\infty_0(\Omega)^d$ *for all* $f \in C^\infty_0(\Omega)^d$.

We also have in the case of $L^{p(\cdot)}$-spaces:

Corollary 2 *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary and let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$. Define*

$$V_{p(\cdot)} = \left\{ f \in W_0^{1,p(\cdot)}(\Omega)^d \mid \operatorname{div} f = 0 \right\}.$$

Then $\mathcal{V}$ (defined as in Corollary 1) is a dense subspace of $V_{p(\cdot)}$.

The paper at hand is a short version of the author's diploma thesis submitted in September 2005 at the University of Freiburg, Germany supervised by Michael Růžička and is organised as follows:

In Sect. 2 we state the main technical tools for handling singular integral operators in weighted spaces. With the help of these results, we prove a special version of Theorem 1 in Sect. 3 by using the so called Bogovskiĭ-formula, valid on domains that are star like with respect to balls, and follow the work of Bogovskiĭ [1,2]. This enables us to prove Theorem 1 in Sect. 4.

We prove Theorem 3 in Sect. 5 and use recent results due to Diening [7, Theorem 8.1] and Cruz-Uribe et al. [5, Theorem 1.3] in Sect. 6 to prove Theorems 2 and 4. Instead of using [5, Theorem 1.3] for proving Theorems 2 and 4, it is also possible to work directly in the setting of $L^{p(\cdot)}$-spaces, which can be found in the work of Diening and Růžička [10].

As an application of Theorem 4, we generalise in Sect. 7 a result of Růžička [18], Frehse, et al. [12] to the case of fluids which satisfy a $p(\cdot)$-growth condition and present a simplified proof by the use of divergence free test functions.

2 Singular integral operators

In this section we state the necessary continuity results for singular integral operators in weighted spaces. We first recall a classical result due to Muckenhoupt [17] (in a weaker form) that shows why the A_p-weights are important:

Theorem 5 *Let $\omega \in A_p$ for $1 < p < \infty$. Then M is bounded on $L^p_\omega(\mathbb{R}^d)$. More precisely: For $1 < p < \infty$ exists an A_p-consistent constant $C_p > 0$ such that*

$$\|Mf\|_{L^p_\omega(\mathbb{R}^d)} \le C_p(\omega)\, \|f\|_{L^p_\omega(\mathbb{R}^d)}$$

for all $f \in L^p_\omega(\mathbb{R}^d)$ and all $\omega \in A_p$.

Next we apply this result to handle weak-singular operators:

Theorem 6 *Let $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a kernel with*

$$|k(x, y)| \le \chi_{B_R(0)}(x - y) \frac{C}{|x - y|^{d-1}}$$

for all $x \neq y \in \mathbb{R}^d$, where $R > 0, C > 0$ are given constants and $\chi_{B_R(0)}$ denotes the indicator function of $B_R(0)$. Then the mapping

$$T : L^p_\omega(\mathbb{R}^d) \to L^p_\omega(\mathbb{R}^d) : f \mapsto \left(x \mapsto \int_{\mathbb{R}^d} k(x, y) f(y)\, dy \right)$$

is well defined, linear and continuous for $1 < p < \infty$ and $\omega \in A_p$. Moreover, the operator norm of T can be estimated by a an A_p-consistent constant.

Proof Take $g(z) = \chi_{B_R(0)}(z)\frac{C}{|z|^{d-1}}$ for $z \in \mathbb{R}^d\backslash\{0\}$. Since g is radially decreasing, we find

$$\int\limits_{\mathbb{R}^d} |k(x,y)f(y)|\,dy \le \int\limits_{\mathbb{R}^d} g(x-y)|f(y)|\,dy \le A\,Mf(x)\,,$$

where $A = \int_{\mathbb{R}^d} g(z)\,dz$ (see for example Stein [19, III, Sect. 2, Theorem 2]). Now Theorem 5 shows

$$\|Tf\|_{L^p_\omega}^p \le A^p \int\limits_{\mathbb{R}^d} Mf(x)^p w(x)\,dx \le A^p C_p(\omega)^p\,\|f\|_{L^p_\omega}^p$$

and we are done. The theorem is proved.

For later use we state the following lemma:

Lemma 1 *Let* $\theta, \vartheta : \mathbb{R}^d \to \mathbb{R}$ *be bounded functions with the property that* $\operatorname{supp}\theta, \operatorname{supp}\vartheta \subset B_R(0)$ *for* $R > 0$. *Then*

$$k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} : (x,y) \mapsto \theta(x)\,\vartheta(y)$$

satisfies the assumptions of Theorem 6.

By $L_0^\infty(\mathbb{R}^d)$ we denote the set of all measurable, bounded functions f with compact support. In the rest of this section, we work with a strong singular operator $T : L_0^\infty(\mathbb{R}^d) \to L^1_{\mathrm{loc}}(\mathbb{R}^d)$ with the following properties:

(I) T is of strong type (p,p) for all $1 < p < \infty$, i.e. for every $1 < p < \infty$ exists a constant $C_p > 0$ such that

$$\|Tf\|_p \le C_p\,\|f\|_p, \quad f \in L_0^\infty(\mathbb{R}^d).$$

(II) T is associated to a kernel, i.e. there exists $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ with

(a) $k(x,\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}^d\backslash\{x\})$ for all $x \in \mathbb{R}^d$

(b)

$$Tf(x) = \int\limits_{\mathbb{R}^d} k(x,y)f(y)\,dy \quad f \in L_0^\infty(\mathbb{R}^d)\,,\ x \notin \operatorname{supp} f$$

Furthermore, the kernel k satisfies additional growth and continuity assumptions:

(III) There exists a constant $C > 0$ with

$$|k(x,y)| \le C\,|x-y|^{-d}$$

for all $x \ne y \in \mathbb{R}^d$.

(IV) There exists an open set $\emptyset \ne \Omega \subset \mathbb{R}^d$ and constants $A > 0$, $\alpha > 0$ and $c > 1$, such that

$$|k(x,y) - k(x_0,y)| \le A\,|x-x_0|^\alpha\,|x_0-y|^{-d-\alpha}$$

for all $r > 0$, $x_0 \in \mathbb{R}^d$, $x \in B_r(x_0)$ and all $y \in \Omega$ with $y \notin B_{cr}(x_0)$.

Under the above assumptions, we have the boundedness of the operator T on weighted spaces. The arguments for proving this result are in fact well known and can for example be found in the books of Journé [15] , García-Cuerva and Rubio de Francia [14]. The main difference is that assumption (IV) is valid only on a (later bounded) open set Ω, which means that we have to localise the known results. In the following we only sketch the proven methods. The first step is to prove the Fefferman-Stein inequality

$$M^\sharp(Tf) \le C_p\, M(|f|^p)^{\frac{1}{p}} \quad \text{on } \mathbb{R}^d, \qquad f \in C_0^\infty(\Omega),$$

where $M^\sharp f(x) = \sup_{r>0} ⨍_{B_r(x)} |f(y) - f_{B_r(x)}|\,dy$ is the sharp-maximal operator. With the help of the open-end-property of the A_p-weights, it easily follows

$$\|M^\sharp(Tf)\|_{L^p_\omega} \le C_p(\omega)\, \|f\|_{L^p_\omega}$$

for all $f \in C_0^\infty(\Omega)$ and all $\omega \in A_p$ with $1 < p < \infty$. Again $C_p(\omega)$ is an A_p-consistent constant. Next we have to show $Tf \in L^p_\omega(\mathbb{R}^d)$. Therefore we use a generalisation of a well known fact of Calderón and Scott [3, Proposition 4.7] to weighted spaces, which is in fact a more general version that the one used in the books of Journé [15], García-Cuerva and Rubio de Francia [14].

Theorem 7 *Let $1 < p < \infty$ and $\omega \in A_p$. Consider a function $f \in L^1_{loc}(\mathbb{R}^d)$ with the property that $M^\sharp f \in L^p_\omega(\mathbb{R}^d)$ and $\mu_\omega(\{|f| > \epsilon\}) < \infty$ for all $\epsilon > 0$. Then $f \in L^p_\omega(\mathbb{R}^d)$ and there exists an A_p-consistent constant $C_p > 0$ such that*

$$\|f\|_{L^p_\omega} \le C_p(\omega)\, \|M^\sharp f\|_{L^p_\omega}.$$

Proof The proof is exactly the same as in [3, Proposition 4.7]. We only have to use the so called "reverse doubling" property of the A_p-weights in order to compare the Lebesgue measure with the weighted measure μ_ω. We refer to Torchinsky [20, Chapter IX, Theorem 2.1 and Remark 5.6] for more informations concerning this property. □

Now the boundedness of the operator T follows from the previous estimates:

Theorem 8 *Let T be an operator with the above properties (I)–(IV). Then for all $1 < p < \infty$ there exists an A_p-consistent constant $C_p > 0$ such that*

$$\|Tf\|_{L^p_\omega} \le C_p(\omega)\, \|f\|_{L^p_\omega}$$

for all $f \in C_0^\infty(\Omega)$ and all $\omega \in A_p$.

We recall a classical (but strong) result of Calderón and Zygmund [4, Theorem 2] for the construction of operators with properties (I) and (II):

Theorem 9 *Let $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a kernel and set $N(x, z) = k(x, x - z)$ for $x, z \in \mathbb{R}^d$. Now assume that k is a Calderón-Zygmund-kernel, i.e. the following holds*:

1. *N is $(-d)$-homogeneous in the z-variable, i.e.*

$$N(x, \alpha z) = \alpha^{-d} N(x, z) \quad x \in \mathbb{R}^d,\ z \in \mathbb{R}^d\backslash\{0\},\ \alpha > 0.$$

2. $\sup_{x\in\mathbb{R}^d} \|N(x, \cdot)\|_{\infty, S^{d-1}} < \infty$, *where S^{d-1} is the unit sphere in $\mathbb{R}^d$.*
3. *We have*

$$\int\limits_{S^{d-1}} N(x, z)\, do(z) = 0$$

for all $x \in \mathbb{R}^d$.

For $f \in L_0^\infty(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $\epsilon > 0$ define the truncated operator T_ϵ by

$$T_\epsilon f(x) = \int\limits_{|x-y|\geq\epsilon} k(x,y) f(y)\, dy.$$

Then T_ϵ is uniformly bounded in L^p for $1 < p < \infty$ and there exists a linear operator $T_0 : L_0^\infty(\mathbb{R}^d) \to L_{loc}^1(\mathbb{R}^d)$ such that

$$\lim_{\epsilon\to 0} T_\epsilon f = T_0 f\,, \quad f \in L_0^\infty(\mathbb{R}^d)$$

in $L^p(\mathbb{R}^d)$ for $1 < p < \infty$ and a.e. on $\mathbb{R}^d$. Furthermore T_0 is associated with the kernel k and bounded on L^p for $1 < p < \infty$.

3 The Bogovskiĭ-formula

In this section we prove Theorem 1 in the case where Ω is a bounded domain that is star like with respect to a ball, i.e. there exists a ball $B \subset \Omega$ with the following property:

$$x \in \Omega,\ y \in B,\ \lambda \in (0,1) \quad \Rightarrow \quad (1-\lambda)\,x + \lambda\,y \in \Omega.$$

In this special situation, we can use an explicit representation formula due to Bogovskiĭ—the so called Bogovskiĭ-formula. Therefore we choose a function $h \in C_0^\infty(B)$ such that $\int_{\mathbb{R}^d} h(x)\,dx = 1$, where B is the above ball, and define the following kernel:

$$k(x,y) = \begin{cases} (x-y)\int_1^\infty h(y + r(x-y)) r^{d-1}\,dr & \text{for } x \neq y \\ 0 & \text{for } x = y \end{cases}.$$

Now we can state the main result of this section:

Theorem 10 *Let $\Omega \subset \mathbb{R}^d$ be bounded domain that is star like with respect to a ball B. With the above kernel set*

$$\mathbb{B} : L_\omega^p(\Omega) \to W_{\omega,0}^{1,p}(\Omega)^d : f \mapsto \left(x \mapsto \int\limits_\Omega k(x,y) f(y)\,dy \right),$$

where $1 < p < \infty$ and $\omega \in A_p$. Then $\mathbb{B}$ is well defined, bounded and the operator norm can be estimated by an A_p-consistent constant. Furthermore, we have

$$\operatorname{div}(\mathbb{B} f) = f - h \int\limits_\Omega f(y)\,dy\,, \quad f \in L_\omega^p(\Omega)$$

and $\mathbb{B} f \in C_0^\infty(\Omega)^d$ for $f \in C_0^\infty(\Omega)$.

We split the proof of this theorem into several steps and follow the proof by Bogovskiĭ [1,2]. Since k is weak-singular and Ω is bounded, we can apply Theorem 6 and find the boundedness of $\mathbb{B}$ on $L_\omega^p(\Omega)$. Moreover, it is easy to see that $\mathbb{B} f \in C^\infty(\Omega)$ for $f \in C_0^\infty(\Omega)$. With the geometric properties of Ω, we can even show:

Lemma 2 *For $f \in C_0^\infty(\Omega)$ we have $\mathbb{B} f \in C_0^\infty(\Omega)$.*

Proof Recall that Ω is star like with respect to the ball B. Define

$$M = \{ty + (1-t)z \mid y \in \operatorname{supp} f,\ z \in B,\ t \in [0,1]\}.$$

Then M is obviously compact and $M \subset \Omega$. For $x \in \mathbb{R}^d \backslash M$ and $y \in \operatorname{supp} f$, we have $y + r(x-y) \notin B$ for $r \geq 1$, since otherwise

$$x = \left(1 - \frac{1}{r}\right) y + \frac{1}{r}(y + r(x-y)) \in M.$$

This means

$$\mathbb{B} f(x) = \int\limits_{\operatorname{supp} f} f(y)(x-y) \int\limits_1^\infty h(y + r(x-y)) r^{d-1}\, dr\, dy = 0$$

for $x \in \Omega \backslash M$, hence $\operatorname{supp} \mathbb{B} f \subset M \subset \Omega$ and $\mathbb{B} f \in C_0^\infty(\Omega)$. The lemma is proved. □

Now we want to estimate the first derivatives of $\mathbb{B} f$ via f. Since k is singular on the diagonal, we work with the truncated operator $\mathbb{B}_\epsilon$, i.e.

$$\mathbb{B}_\epsilon f(x) = \int\limits_{|x-y| \geq \epsilon} k(x,y) f(y)\, dy, \quad f \in L^p_\omega(\Omega),$$

where $\epsilon > 0$. As before we find that $\mathbb{B}_\epsilon$ is bounded on $L^p_\omega(\Omega)$ and also $\mathbb{B} f \in C_0^\infty(\Omega)$ for $f \in C_0^\infty(\Omega)$. It follows from Young's inequality for convolutions that

$$\lim_{\epsilon \to 0} \mathbb{B}_\epsilon f = \mathbb{B} f \qquad \text{in } L^2(\Omega)^d$$

for all $f \in C_0^\infty(\Omega)$. By integration by parts we obtain a representation for the first derivatives of $B_\epsilon f$, where no derivative acts on f:

Lemma 3 *For $f \in C_0^\infty(\Omega)$ and $i, j = 1, \ldots, d$ we have*

$$\partial_j (\mathbb{B}_\epsilon f)^i(x) = \int\limits_{|x-y| \geq \epsilon} f(y) \frac{\partial k^i}{\partial x_j}(x,y)\, dy + \int\limits_{|x-y| = \epsilon} f(y) \frac{x_j - y_j}{|x-y|} k^i(x,y)\, do(y).$$

In order to show convergence for $\epsilon \to 0$, we handle each term separately. We start with the boundary term and set for $f \in C_0^\infty(\Omega)$, $\epsilon > 0$ and $i, j = 1, \ldots, d$

$$T_{ij}^{1\epsilon} f(x) = \int\limits_{|x-y| = \epsilon} f(y) \frac{x_j - y_j}{|x-y|} k^i(x,y)\, do(y)$$

and

$$T_{ij}^1 f(x) = f(x) \int\limits_\Omega \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^2} h(y)\, dy.$$

We remark that we can not define $T_{ij}^1 f$ for a general $f \in L^p_\omega(\Omega)$. Next lemma states the convergence result for the boundary term:

Lemma 4 *For $f \in C_0^\infty(\Omega)$ and $i, j = 1, \dots, d$ it holds*

$$\lim_{\epsilon \to 0} T_{ij}^{1\epsilon} f = T_{ij}^1 f$$

uniformly on Ω and in $L^2(\Omega)$. Furthermore, we have

$$\|T_{ij}^1 f\|_{L_\omega^p} \le \|f\|_{L_\omega^p}$$

for all $f \in C_0^\infty(\Omega)$, $\omega \in A_p$ with $1 < p < \infty$ and $i, j = 1, \dots, d$.

Proof Let $f \in C_0^\infty(\Omega)$. For $x \in \Omega$ we find

$$T_{ij}^1 f(x) = \int\limits_{|y|=1} f(x) y_i y_j \int\limits_0^\infty h(x + ry) r^{d-1}\, dr\, do(y).$$

In a similar way we can write

$$T_{ij}^{1\epsilon} f(x) = \int\limits_{|y|=1} y_j f(x - \epsilon y) k^i(x, x - \epsilon y) \epsilon^{d-1}\, do(y).$$

By the substitution $r \to \frac{1}{\epsilon} r + 1$ and the definition of $k^i(x, y)$, we get

$$T_{ij}^{1\epsilon} f(x) = \int\limits_{|y|=1} f(x - \epsilon y) y_i y_j \int\limits_0^\infty h(x + ry)(r + \epsilon)^{d-1}\, dr\, do(y).$$

With these expressions we can estimate:

$$\begin{aligned}
&|T_{ij}^1 f(x) - T_{ij}^{1\epsilon} f(x)| \\
&\quad \le \left| \int\limits_{|y|=1} (f(x) - f(x - \epsilon y))\, y_i y_j \int\limits_0^\infty h(x + ry) r^{d-1}\, dr\, do(y) \right| \\
&\quad\quad + \left| \int\limits_{|y|=1} f(x - \epsilon y) y_i y_j \int\limits_0^\infty h(x + ry) \sum_{s=0}^{d-2} \binom{d-1}{s} r^s \epsilon^{d-s-1}\, dr\, do(y) \right|.
\end{aligned}$$

With the mean value theorem, it follows

$$|T_{ij}^1 f(x) - T_{ij}^{1\epsilon} f(x)| \le \epsilon\, c(f, h, d, R),$$

hence $T_{ij}^{1\epsilon} f \to T_{ij}^1 f$ uniformly on Ω and in $L^2(\Omega)$ for $\epsilon \to 0$. Having in mind $\int_\Omega h(y)\, dy = 1$, we find

$$\|T_{ij}^1 f\|_{L_\omega^p}^p \le \int\limits_\Omega |f(x)|^p \left(\int\limits_\Omega h(y)\, dy \right)^p w(x)\, dx = \|f\|_{L_\omega^p}^p.$$

The lemma is proved.

For the remaining part of $\partial_j (\mathbb{B}_\epsilon f)^i$, we set

$$T_{ij}^{2\epsilon} f(x) = \int\limits_{|x-y| \ge \epsilon} f(y) \frac{\partial k^i}{\partial x_j}(x, y)\, dy$$

for $f \in C_0^\infty(\Omega), x \in \Omega, \epsilon > 0$ and $i, j = 1, \ldots, d$. The operator $T_{ij}^{2\epsilon}$ is the sum of a weak- and strong-singular operator:

Lemma 5 *There exist measurable functions $l_{ij}, N_{ij} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ $(i, j = 1, \ldots, d)$ such that the following holds*:

1. *There are constants $c_{ij} > 0$ and $R > 0$ such that*

$$|l_{ij}(x, y)| \leq \chi_{B_{2R}(0)}(x - y) \frac{c_{ij}}{|x - y|^{d-1}}$$

for all $x \neq y$.

2. (a) *N_{ij} is $(-d)$-homogeneous in the second variable, i.e.*

$$N_{ij}(x, \alpha z) = \alpha^{-d} N_{ij}(x, z)$$

for all $x \in \mathbb{R}^d, z \in \mathbb{R}^d \backslash \{0\}$ and all $\alpha > 0$.

   (b) *We have* $\sup_{x \in \mathbb{R}^d} \|N_{ij}(x, \cdot)\|_{\infty, S^{d-1}} < \infty$.

   (c) *For all $x \in \mathbb{R}^d$ we have*

$$\int_{S^{d-1}} N_{ij}(x, z)\, do(z) = 0.$$

   (d) *For $k_{ij}(x, y) = N_{ij}(x, x - y)$ holds*

$$|k_{ij}(x, y)| \leq \frac{C_{ij}}{|x - y|^d},$$

where $C_{ij} > 0$ is a suitable constant.

   (e) *For $k_{ij}(x, y) = N_{ij}(x, x - y)$ we have*

$$|k_{ij}(x, y) - k_{ij}(x_0, y)| \leq C_{ij}\, |x - x_0|\, |x_0 - y|^{-d-1}$$

for all $x_0, x \in \mathbb{R}^d$ and all $y \in \Omega$ such that $|x - x_0| < \frac{1}{2}|x_0 - y|$.

3. *We have*

$$T_{ij}^{2\epsilon} f(x) = \int_{|x-y| \geq \epsilon} l_{ij}(x, y) f(y)\, dy + \int_{|x-y| \geq \epsilon} N_{ij}(x, x - y) f(y)\, dy.$$

for all $f \in C_0^\infty(\Omega), x \in \Omega$ and all $\epsilon > 0$.

Proof We compute $\frac{\partial k^i}{\partial x_j}(x, y)$ for $x \neq y$ and $i, j = 1, \ldots, d$:

$$\frac{\partial k^i}{\partial x_j}(x, y) = \delta_{ij} \int_1^\infty h(y + r(x - y)) r^{d-1}\, dr$$
$$+ (x_i - y_i) \int_1^\infty \left(\partial_j h\right)(y + r(x - y)) r^d\, dr$$

By substituting $r \mapsto |x-y|^{-1} r + 1$ and applying the binomial formula, we can write

$$\frac{\partial k^i}{\partial x_j}(x,y) = \frac{\delta_{ij}}{|x-y|^d} \int_0^\infty h\left(x + r\frac{x-y}{|x-y|}\right) \sum_{s=0}^{d-2} \binom{d-1}{s} r^s |x-y|^{d-1-s}\, dr$$
$$+ \frac{x_i - y_i}{|x-y|^{d+1}} \int_0^\infty (\partial_j h)\left(x + r\frac{x-y}{|x-y|}\right) \sum_{s=0}^{d-1} \binom{d}{s} r^s |x-y|^{d-s}\, dr$$
$$+ \frac{\delta_{ij}}{|x-y|^d} \int_0^\infty h\left(x + r\frac{x-y}{|x-y|}\right) r^{d-1}\, dr$$
$$+ \frac{x_i - y_i}{|x-y|^{d+1}} \int_0^\infty (\partial_j h)\left(x + r\frac{x-y}{|x-y|}\right) r^d\, dr.$$

The first two terms on the right hand side define $l_{ij}(x,y)$, i.e

$$l_{ij}(x,y) = m(x,y) \left(\sum_{s=0}^{d-2} \binom{d-1}{s} \frac{\delta_{ij}}{|x-y|^{s+1}} \int_0^\infty h\left(x + r\frac{x-y}{|x-y|}\right) r^s\, dr \right.$$
$$\left. + \sum_{s=0}^{d-1} \binom{d}{s} \frac{x_i - y_i}{|x-y|^{s+1}} \int_0^\infty (\partial_j h)\left(x + r\frac{x-y}{|x-y|}\right) r^s\, dr \right),$$

where $m(x,y) = \chi_\Omega(x)\chi_{B_{2R}(0)}(x-y)$. The last two terms define $N_{ij}(x,z)$, i.e.

$$N_{ij}(x,z) = \varphi(x) \left(\frac{\delta_{ij}}{|z|^d} \int_0^\infty h\left(x + r\frac{z}{|z|}\right) r^{d-1}\, dr \right.$$
$$\left. + \frac{z_i}{|z|^{d+1}} \int_0^\infty (\partial_j h)\left(x + r\frac{z}{|z|}\right) r^d\, dr \right),$$

where φ is a smooth function such that $\varphi = 1$ on Ω and $\operatorname{supp}\varphi \subset B_R(0)$. It immediately follows

$$T_{ij}^{2\epsilon} f(x) = \int_{|x-y|\geq\epsilon} l_{ij}(x,y) f(y)\, dy + \int_{|x-y|\geq\epsilon} N_{ij}(x, x-y) f(y)\, dy.$$

It is easy to see that we have

$$|l_{ij}(x,y)| \leq \chi_{B_{2R}(0)}(x-y) \frac{c_{ij}}{|x-y|^{d-1}} \quad \text{and} \quad |k_{ij}(x,y)| \leq \frac{C_{ij}}{|x-y|^d},$$

where $k_{ij}(x,y) = N_{ij}(x, x-y)$ and $c_{ij}, C_{ij} > 0$ are suitable constants. Furthermore, $N_{ij}(x,z)$ is $(-d)$-homogeneous in the z-variable. Since $\Omega \subset B_R(0)$, we have the estimate

$$|N_{ij}(x,z)| \leq \delta_{ij} \int_0^{2R} h(x+rz) r^{d-1}\, dr + \int_0^{2R} \left(\partial_j h\right)(x+rz) r^d\, dr$$
$$\leq \frac{(2R)^d}{d} \|h\|_\infty + \frac{(2R)^{d+1}}{d+1} \|\partial_j h\|_\infty$$

for $x \in \mathbb{R}^d$ and $z \in S^{d-1}$, hence $\sup_{x\in\mathbb{R}^d} \|N_{ij}(x,\cdot)\|_{\infty,S^{d-1}} < \infty$. We can also write

$$\int\limits_{S^{d-1}} N_{ij}(x,z)\,do(z) = \varphi(x)\delta_{ij} \int\limits_{\mathbb{R}^d} h(x+z)\,dz + \varphi(x) \int\limits_{\mathbb{R}^d} z_i \frac{\partial}{\partial z_j} h(x+z)\,dz\,,$$

thus $\int_{S^{d-1}} N_{ij}(x,z)\,do(z) = 0$ by integration by parts. The remaining estimate

$$|k_{ij}(x,y) - k_{ij}(x_0,y)| \le C_{ij}\,|x-x_0|\,|x_0-y|^{-d-1}$$

for all $x_0, x \in \mathbb{R}^d$ and all $y \in \Omega$ such that $|x-x_0| < \frac{1}{2}|x_0-y|$ can be found in [10, p. 215, Eq. (31)]. The lemma is proved. □

Applying Theorem 6 to the weak singular-, Theorems 9 and 8 to the singular part of $T^2_{ij} f$ implies:

Lemma 6 *There exists a linear mapping* $T^2_{ij} : L^\infty_0(\mathbb{R}^d) \to L^1_{loc}(\mathbb{R}^d)$ *such that*:

(a) T^2_{ij} *is bounded on* $L^p_\omega(\Omega)$, *where* $1 < p < \infty$ *and* $\omega \in A_p$. *More precisely*: *For all* $1 < p < \infty$ *there exists an* A_p*-consistent constant* $C_p > 0$ *such that*

$$\|T^2_{ij} f\|_{L^p_\omega} \le C_p(\omega)\,\|f\|_{L^p_\omega}$$

for $f \in C^\infty_0(\Omega)$, $\omega \in A_p$ *and* $i, j = 1, \ldots, d$.

(b) *For all* $f \in C^\infty_0(\Omega)$ *and* $i, j = 1, \ldots, d$ *we have*

$$\lim_{\epsilon\to 0} T^{2\epsilon}_{ij} f = T^2_{ij} f$$

in $L^2(\Omega)$.

Collecting all facts we know about the derivatives of $\mathbb{B}_\epsilon f$ yields:

Lemma 7 *For* $f \in C^\infty_0(\Omega)$ *and* $i, j = 1, \ldots, d$ *we have*:

$$\partial_j(\mathbb{B} f)^i = T^1_{ij} f + T^2_{ij} f$$

a.e. on Ω. *In particular, for all* $1 < p < \infty$ *there exists an* A_p*-consistent constant* $C_p > 0$ *such that*

$$\|\partial_j(\mathbb{B} f)^i\|_{L^p_\omega} \le C_p(\omega)\,\|f\|_{L^p_\omega}$$

for all $f \in C^\infty_0(\Omega)$, $\omega \in A_p$ *and* $i, j = 1, \ldots, d$. *Furthermore,*

$$\lim_{\epsilon\to 0} \partial_j(\mathbb{B}_\epsilon f)^i = \partial_j(\mathbb{B} f)^i \qquad \text{in } L^2(\Omega)$$

for all $f \in C^\infty_0(\Omega)$ *and* $i, j = 1, \ldots, d$.

Proof We know

$$\partial_j(\mathbb{B}_\epsilon f)^i(x) = T^{1\epsilon}_{ij} f(x)$$

for all $f \in C^\infty_0(\Omega)$, $x \in \mathbb{R}^d$, $\epsilon > 0$ and $i, j = 1, \ldots, d$. From Lemmas 4 and 6 we get

$$\lim_{\epsilon\to 0} \partial_j(\mathbb{B}_\epsilon f)^i = T^1_{ij} f + T^2_{ij} f$$

in $L^2(\Omega)$ for all $f \in C^\infty_0(\Omega)$ and $i, j = 1, \ldots, d$, thus

$$\partial_j(\mathbb{B} f)^i = T^1_{ij} f + T^2_{ij} f.$$

The estimate follows again from Lemmas 4 and 6. The lemma is proved.

With the usual density argument, it easily follows that the mapping

$$\mathbb{B} : L^p_\omega(\Omega) \to W^{1,p}_{\omega,0}(\Omega)^d : f \mapsto \left(x \mapsto \int_\Omega k(x,y) f(y)\, dy \right)$$

is well defined, linear and bounded. Furthermore, we know that the operator norm of $\mathbb{B}$ can be estimated by an A_p-consistent constant. The conclusion now follows from the next lemma:

Lemma 8 *We have*

$$\operatorname{div}(\mathbb{B} f) = f - h \int_\Omega f(y)\, dy$$

for all $f \in L^p_\omega(\Omega)$, where $1 < p < \infty$ and $\omega \in A_p$.

Proof By density it is enough to show the statement for $f \in C_0^\infty(\Omega)$. We know from Lemma 3 for $\epsilon > 0$ and $x \in \Omega$:

$$\operatorname{div}(\mathbb{B}_\epsilon f)(x) = \sum_{j=1}^d \partial_j (\mathbb{B}_\epsilon f)^j(x) = \sum_{j=1}^d T^{1\epsilon}_{jj} f(x) + \sum_{j=1}^d T^{2\epsilon}_{jj} f(x).$$

For the second sum we calculate:

$$\begin{aligned} \sum_{j=1}^d T^{2\epsilon}_{jj} f(x) &= \sum_{j=1}^d \int_{|x-y|\geq\epsilon} f(y) \frac{\partial k^j}{\partial x_j}(x,y)\, dy \\ &= d \int_{|x-y|\geq\epsilon} f(y) \int_1^\infty h(y + r(x-y)) r^{d-1}\, dr\, dy \\ &\quad + \int_{|x-y|\geq\epsilon} f(y) \int_1^\infty r^d \frac{d}{dr} h(y + r(x-y))\, dr\, dy. \end{aligned}$$

By integration by parts we find:

$$\sum_{j=1}^d T^{2\epsilon}_{jj} f(x) = -h(x) \int_{|x-y|\geq\epsilon} f(y)\, dy.$$

Lemma 7 implies

$$\lim_{\epsilon\to 0} \operatorname{div}(\mathbb{B}_\epsilon f) = \operatorname{div}(\mathbb{B} f)$$

in $L^2(\Omega)$, thus

$$\operatorname{div}(\mathbb{B} f) = \sum_{j=1}^d T^1_{jj} f - h \int_\Omega f(y)\, dy$$

in view of Lemma 4. We remark that

$$\sum_{j=1}^d T^1_{jj} f(x) = \sum_{j=1}^d f(x) \int_\Omega \frac{(x_j - y_j)^2}{|x-y|^2} h(y)\, dy = f(x),$$

since $\int_\Omega h(y)\,dy = 1$. Putting all together yields

$$\operatorname{div}(\mathbb{B} f) = f - h \int_\Omega f(y)\,dy.$$

The lemma is proved.

The proof of Theorem 10 is complete.

4 Proof of Theorem 1

From Theorem 10 we know how to solve the divergence equation on bounded domains that are star like with respect to a ball. In view of this we have to find a suitable decomposition of domains with Lipschitz boundary. It is also necessary to split the right hand side with respect to this decomposition in order to apply the result of the previous section. In the next lemma we state the required result:

Lemma 9 *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. Then there exist open and bounded sets $G_1, \dots, G_N \subset \mathbb{R}^d$ with the following properties*:

(a) $\overline{\Omega} \subset G_1 \cup \dots \cup G_N$
(b) $\Omega_i = \Omega \cap G_i$ *is star like with respect to a ball* $B_i \subset\subset \Omega_i$ *for all* $i = 1, \dots, N$

For this decomposition exists a linear mapping

$$H = (H_1, \dots, H_N) : C_{0,0}^\infty(\Omega) \to C_{0,0}^\infty(\Omega_1) \times \dots \times C_{0,0}^\infty(\Omega_N)$$

such that:

(c) $\sum_{i=1}^N H_i f = f$ *for all* $f \in C_{0,0}^\infty(\Omega)$
(d) *For all* $f \in C_{0,0}^\infty(\Omega)$ *we have*

$$H_i f = \eta_i f + \sum_{j=1}^{m_i} \theta_{ij} \int_{\mathbb{R}^d} \vartheta_{ij} f\,dx\,,$$

where $m_i \in \mathbb{N}$, $\eta_i \in C_0^\infty(G_i)$, $\theta_{ij} \in C_0^\infty(\Omega_i)$ *and* $\vartheta_{ij} \in C_0^\infty(\mathbb{R}^d)$ *are independent of* f *(*$j = 1, \dots, m_i$ *and* $i = 1, \dots, N$*).*

Moreover there exists an A_p*-consistent constant* $C_p > 0$ *such that*

$$\|H_i f\|_{L^p_\omega} \le C_p(\omega)\,\|f\|_{L^p_\omega}$$

for all $f \in C_{0,0}^\infty(\Omega)$, $\omega \in A_p$ *and* $i = 1, \dots, N$.

Proof The construction for the decomposition and for the mapping H can be found in the book of Galdi [13, III.3, Lemma 3.4]. From Lemma 1 and Theorem 6 we get the remaining estimate for H_i. The lemma is proved.

We are now in a position to solve the divergence equation on bounded domains with Lipschitz boundary:

Proof of Theorem 1 We use Lemma 9 to split Ω into the sets $\Omega_i = \Omega \cap G_i$, which are star like with respect to balls B_i, and moreover, we choose functions $h_i \in C_0^\infty(\Omega)$ such that $\int_{B_i} h_i\, dx = 1$. We set

$$\mathbb{B} : C_{0,0}^\infty(\Omega) \to C_0^\infty(\Omega)^d \subset W_{\omega,0}^{1,p}(\Omega)^d : f \mapsto \sum_{i=1}^N \mathbb{B}_i \circ H_i f\,,$$

where $\mathbb{B}_i : L_{\omega,0}^p(\Omega_i) \to W_{\omega,0}^{1,p}(\Omega_i)^d$ is defined as in Theorem 10 with help of the function h_i. Since $H_i f \in C_{0,0}^\infty(\Omega_i)$ and $\mathbb{B}_i \circ H_i f \in C_0^\infty(\Omega_i)^d \subset C_0^\infty(\Omega)^d$ for $f \in C_{0,0}^\infty(\Omega)$, we find that $\mathbb{B}$ is well defined and linear. Now we compute

$$\operatorname{div}(\mathbb{B} f) = \operatorname{div} \sum_{i=1}^N \mathbb{B}_i \circ H_i f = \sum_{i=1}^N \operatorname{div}(\mathbb{B}_i \circ H_i f) = \sum_{i=1}^N H_i f = f$$

for all $f \in C_{0,0}^\infty(\Omega)$. With the help of Theorem 10, we find

$$\|\mathbb{B} f\|_{W_\omega^{1,p}} \le \sum_{i=1}^N \|\mathbb{B}_i(H_i f)\|_{W_\omega^{1,p}} \le C_p(\omega) \sum_{i=1}^N \|H_i f\|_{L_\omega^p}\,,$$

where $C_p > 0$ is an A_p-consistent constant. The estimate of Lemma 9 implies

$$\|\mathbb{B} f\|_{W_\omega^{1,p}} \le C_p(\omega)\, \|f\|_{L_\omega^p}$$

for all $f \in C_{0,0}^\infty(\Omega)$ with an A_p-consistent constant $C_p > 0$. A density argument with the help of Lemma 10 leads to the desired result. The theorem is proved.

It remains to prove the following lemma:

Lemma 10 *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Then the subspace $C_{0,0}^\infty(\Omega)$ is dense in $L_{\omega,0}^p(\Omega)$ for $1 < p < \infty$ and $\omega \in A_p$.*

Proof Given a function $f \in L_\omega^p(\Omega)$, we can find a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_0^\infty(\Omega)$ such that $\lim_{n\to\infty} f_n = f$ in $L_\omega^p(\Omega)$. The embedding $L_\omega^p(\Omega) \hookrightarrow L^1(\Omega)$ also implies $\lim_{n\to\infty} f_n = f$ in $L^1(\Omega)$. Now we choose a function $\psi \in C_0^\infty(\Omega)$ such that $\int_\Omega \psi\, dx = 1$ and define

$$g_n = f_n - \psi \int_\Omega f_n\, dx$$

for $n \in \mathbb{N}$. We then have $g_n \in C_{0,0}^\infty(\Omega)$ and since $\int_\Omega f\, dx = 0$ it follows

$$\|f - g_n\|_{L_\omega^p} \le \|f - f_n\|_{L_\omega^p} + \|\psi\|_{L_\omega^p}\, \|f - f_n\|_1$$

for all $n \in \mathbb{N}$, hence $\lim_{n\to\infty} g_n = f$ in $L_\omega^p(\Omega)$. In particular, $C_{0,0}^\infty(\Omega)$ is a dense subspace of $L_{\omega,0}^p(\Omega)$. The lemma is proved.

5 Proof of Theorem 3

In this section we prove Theorem 3:

Proof of Theorem 3 We make again use of the decomposition lemma (Lemma 9) and find open and bounded sets $G_1, \ldots, G_N \subset \mathbb{R}^d$ and a linear mapping

$$H = (H_1, \ldots, H_N) : C_{0,0}^\infty(\Omega) \to C_{0,0}^\infty(\Omega_1) \times \cdots \times C_{0,0}^\infty(\Omega_N),$$

where $\Omega_i = \Omega \cap G_i$ for $i = 1, \ldots, N$. The sets $\Omega_1, \ldots, \Omega_N$ are star like with respect to balls $B_1, \ldots, B_N$, and we choose functions $h_i \in C_0^\infty(B_i)$ with the property that $\int_{B_i} h_i \, dx = 1$ in order to define the linear operator

$$\mathbb{B}_i : C_0^\infty(\Omega_i) \to C_0^\infty(\Omega_i)^d \subset C_0^\infty(\Omega)^d$$

as in Theorem 10. From the Gauss theorem it follows $\operatorname{div} f \in C_{0,0}^\infty(\Omega)$ for every $f \in C_0^\infty(\Omega)^d$ and so we can define

$$\mathbb{E} : C_0^\infty(\Omega)^d \to C_0^\infty(\Omega)^d \subset X_{\omega_1,\omega_2}^{p_1,p_2} : f \mapsto \sum_{i=1}^N \mathbb{B}_i \circ H_i(\operatorname{div} f).$$

Obviously, $\mathbb{E}$ is linear and we have

$$\operatorname{div}(\mathbb{E} f) = \sum_{i=1}^N \operatorname{div}(\mathbb{B}_i \circ H_i(\operatorname{div} f)) = \sum_{i=1}^N H_i(\operatorname{div} f) = \operatorname{div} f$$

for all $f \in C_0^\infty(\Omega)^d$. From Theorem 10 and Lemma 9, we find

$$\|\mathbb{E} f\|_{W_{\omega_1}^{1,p_1}} \le C_{p_1}(\omega_1) \sum_{i=1}^N \|H_i(\operatorname{div} f)\|_{L_{\omega_1}^{p_1}} \le C_{p_1}(\omega_1) \, \|\operatorname{div} f\|_{L_{\omega_1}^{p_1}},$$

where $C_{p_1}(\omega_1) > 0$ is an A_{p_1}-consistent constant. In the following we show the remaining estimate

$$\|\mathbb{E} f\|_{L_{\omega_2}^{p_2}} \le C_{p_2}(\omega_2) \, \|f\|_{L_{\omega_2}^{p_2}}$$

for all $f \in C_0^\infty(\Omega)^d$, where $C_{p_2} > 0$ is an A_{p_2}-consistent constant. Therefore we consider $\mathbb{B}_{i_0} \circ H_{i_0}(\operatorname{div} f)$ for a $i_0 \in \{1, \ldots, N\}$. For the sake of simplicity, we neglect the index i_0. With help of Lemma 9 and integration by parts, we obtain

$$\mathbb{B} \circ H(\operatorname{div} f) = \mathbb{B}(\eta \operatorname{div} f) - \sum_{i=1}^d \sum_{j=1}^m \mathbb{B}\left(\theta_j \int_{\mathbb{R}^d} \partial_i \vartheta_j f_i \, dy\right).$$

Now we define

$$Tf = \mathbb{B}(\eta \operatorname{div} f) \quad \text{and} \quad Sf = -\sum_{i=1}^d \sum_{j=1}^m \mathbb{B}\left(\theta_j \int_{\mathbb{R}^d} \partial_i \vartheta_j f_i \, dy\right)$$

for $f \in C_0^\infty(\Omega)^d$. Lemma 1, Theorems 6 and 10 imply

$$\|Sf\|_{L_{\omega_2}^{p_2}} \le C_{p_2}(\omega_2) \, \|f\|_{L_{\omega_2}^{p_2}}$$

for all $f \in C_0^\infty(\Omega)^d$, where $C_{p_2} > 0$ is an A_{p_2}-consistent constant. Next we write

$$T^\epsilon f = \mathbb{B}_\epsilon(\eta \operatorname{div} f)$$

for $\epsilon > 0$, $f \in C_0^\infty(\Omega)^d$ and remark

$$\lim_{\epsilon \to 0} T^\epsilon f = \lim_{\epsilon \to 0} \mathbb{B}_\epsilon(\eta \operatorname{div} f) = \mathbb{B}(\eta \operatorname{div} f) = Tf$$

in $L^2(\Omega)^d$ by Young's inequality for convolutions. By integration by parts, we have

$$\begin{aligned} T_i^\epsilon f(x) &= \sum_{j=1}^d \int_{|x-y|=\epsilon} \eta(y) f_j(y) \frac{x_j - y_j}{|x-y|} k^i(x, y)\, do(y) \\ &\quad - \sum_{j=1}^d \int_{|x-y|\geq\epsilon} k^i(x, y) \frac{\partial \eta}{\partial y_j}(y) f_j(y)\, dy \\ &\quad - \sum_{j=1}^d \int_{|x-y|\geq\epsilon} \frac{\partial k^i}{\partial y_j}(x, y) \eta(y) f_j(y)\, dy \\ &= \sum_{j=1}^d T_{ij}^{1\epsilon} f(x) + \sum_{j=1}^d T_{ij}^{2\epsilon} f(x) + \sum_{j=1}^d T_{ij}^{3\epsilon} f(x). \end{aligned}$$

We pass to the limit in each term separately. Lemma 4 implies

$$\lim_{\epsilon \to 0} T_{ij}^{1,\epsilon} f = T_{ij}^1 f$$

uniformly on Ω and in $L^2(\Omega)$, where

$$T_{ij}^1 f(x) = \eta(x) f_j(x) \int_\Omega \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^2} h(y)\, dy.$$

Obviously, we have

$$\|T_{ij}^1 f\|_{L_{\omega_2}^{p_2}} \leq \|\eta\|_\infty \, \|f\|_{L_{\omega_2}^{p_2}}.$$

Again by standard properties of convolutions we find

$$\lim_{\epsilon \to 0} T_{ij}^{2\epsilon} f = \lim_{\epsilon \to 0} \mathbb{B}_\epsilon^i(\partial_j \eta f_j) = \mathbb{B}^i(\partial_j \eta f_j) = T_{ij}^2 f$$

in $L^2(\Omega)$ and

$$\|T_{ij}^2 f\|_{L_{\omega_2}^{p_2}} = \|\mathbb{B}^i(\partial_j \eta f_j)\|_{L_{\omega_2}^{p_2}} \leq C_{p_2}(\omega_2) \, \|f\|_{L_{\omega_2}^{p_2}}$$

by Theorem 6, where $C_{p_2}(\omega_2) > 0$ is an A_{p_2}-consistent constant. By a similar computation as in Lemma 5, we have that

$$T_{ij}^{3\epsilon} f(x) = \int_{|x-y|\geq\epsilon} l_{ij}(x, y) \eta(y) f_j(y)\, dy + \int_{|x-y|\geq\epsilon} N_{ij}(x, x-y) \eta(y) f_j(y)\, dy\,,$$

where the kernels l_{ij} and N_{ij} satisfy the properties stated in Lemma 5. Again by Theorems 6, 9 and 8 there exists a linear mapping $T_{ij}^3 : L_0^\infty(\mathbb{R}^d)^d \to L_{\text{loc}}^1(\mathbb{R}^d)$ such that

$$\lim_{\epsilon \to 0} T_{ij}^{3\epsilon} f = T_{ij}^3 f$$

in $L^2(\Omega)$ and

$$\|T^3_{ij} f\|_{L^{p_2}_{\omega_2}} \le C_{p_2}(\omega_2)\, \|f\|_{L^{p_2}_{\omega_2}}$$

for all $f \in C_0^\infty(\Omega)^d$. As usual $C_{p_2}(\omega_2) > 0$ is an A_{p_2}-consistent constant. This shows

$$\|\mathbb{E} f\|_{L^{p_2}_{\omega_2}} \le C_{p_2}(\omega_2)\, \|f\|_{L^{p_2}_{\omega_2}}$$

for all $f \in C_0^\infty(\Omega)^d$. Summarising $\mathbb{E} : C_0^\infty(\Omega)^d \to C_0^\infty(\Omega)^d \subset X^{p_1,p_2}_{\omega_1,\omega_2}$ is a linear mapping with the following properties:

(i) $\operatorname{div}(\mathbb{E} f) = \operatorname{div} f$ for all $f \in C_0^\infty(\Omega)^d$
(ii) $\|\mathbb{E} f\|_{W^{1,p_1}_{\omega_1}} \le C_{p_1}(\omega_1)\, \|f\|_{L^{p_1}_{\omega_1}}$ for all $f \in C_0^\infty(\Omega)^d$, where $C_{p_1} > 0$ is an A_{p_1}-consistent constant.
(iii) $\|\mathbb{E} f\|_{L^{p_2}_{\omega_2}} \le C_{p_2}(\omega_2)\, \|f\|_{L^{p_2}_{\omega_2}}$ for all $f \in C_0^\infty(\Omega)^d$, where $C_{p_2} > 0$ is an A_{p_2}-consistent constant.

By the usual density argument, we are done. The theorem is proved.

6 Proof of Theorems 2 and 4

Recall that $\mathcal{P}(\mathbb{R}^d)$ is the set of all measurable exponents $p(\cdot)$ such that

$$1 < p_- \le p(\cdot) \le p_+ < \infty \quad \text{and} \quad M \text{ is bounded on } L^{p(\cdot)}(\mathbb{R}^d).$$

For $\mathcal{P}(\mathbb{R}^d)$ we have the following characterisation due to Diening [7, Theorem 8.1], which uses the pointwise defined conjugate exponent $p'(x) = \frac{p(x)}{p(x)-1}$:

Theorem 11 *Let $p : \mathbb{R}^d \to [1,\infty)$, $1 < p_- \le p(\cdot) \le p_+ < \infty$ be a measurable exponent. Then the following statements are equivalent*:

(a) $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$.
(b) $p'(\cdot) \in \mathcal{P}(\mathbb{R}^d)$.
(c) $p(\cdot)/q \in \mathcal{P}(\mathbb{R}^d)$ *for some* $1 < q < p_-$.
(d) $(p(\cdot)/q)' \in \mathcal{P}(\mathbb{R}^d)$ *for some* $1 < q < p_-$.

Next we state a recent result of Cruz-Uribe et al. [5, Theorem 1.3] that shows that it is possible to transform continuity results for operators on weighted spaces to the case of Lebesgue spaces with variable exponents.

Theorem 12 *Given a set $\mathcal{F} = \{(f,g)\}$ of tuples consisting of nonnegative and measurable functions $f, g : \mathbb{R}^d \to \mathbb{R}^+$. Let $1 < q < \infty$ and $C_q > 0$ be an A_q-consistent constant such that*

$$\int_{\mathbb{R}^d} f(x)^q \omega(x)\, dx \le C_q(\omega) \int_{\mathbb{R}^d} g(x)^q \omega(x)\, dx$$

for all $(f,g) \in \mathcal{F}$ and all weights $\omega \in A_q$. Here the left hand side of the inequality is assumed to be finite. Now let $p : \mathbb{R}^d \to [1,\infty)$ be a variable exponent such that $q < p_- \le p(\cdot)$ and $(p(\cdot)/q)' \in \mathcal{P}(\mathbb{R}^d)$. Then there exists a constant $C > 0$ such that

$$\|f\|_{p(\cdot)} \le C\, \|g\|_{p(\cdot)}$$

for all $(f,g) \in \mathcal{F}$, where $f \in L^{p(\cdot)}(\mathbb{R}^d)$.

Combining Theorems 1 and 3 with Theorems 11 and 12, we obtain Theorems 2 and 4. Theorems 2 and 4 are proved.

7 An application of Theorem 4

In this section we show how Theorem 4 can be used to handle problems arising in fluid dynamics. More precisely, we study the existence of weak solutions of the system

$$\begin{aligned} -\operatorname{div} T(\cdot, Du) + [\nabla u]u + \nabla \pi &= f && \text{in } \Omega \\ \operatorname{div} u &= 0 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary. For a given force $f : \Omega \to \mathbb{R}^d$, we want to find the velocity field u and the pressure π of the fluid. By $Du = \frac{1}{2}\nabla u + \frac{1}{2}(\nabla u)^T$ we denote the symmetric part of the gradient of u, and we use the abbreviation $[\nabla u]u = (\sum_{j=1}^d u^j \partial_j u^i)_{i=1,\dots,d}$. Moreover, we assume that $T : \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is a Carathéodory function satisfying

$$\begin{aligned} |T(x,\eta)| &\le c\,|\eta|^{p(x)-1} + \varphi_1(x), \\ T(x,\eta) : \eta &\ge c|\eta|^{p(x)} - \varphi_2(x), \\ (T(x,\eta_1) - T(x,\eta_2)) : (\eta_1 - \eta_2) &> 0, \end{aligned}$$

for all $x \in \Omega$ and all $\eta_1 \neq \eta_2 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, where $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $\varphi_1 \in L^{p'(\cdot)}(\Omega)$ and $\varphi_2 \in L^1(\Omega)$. We define the spaces

$$\begin{aligned} V_{p(\cdot)} &= \left\{ f \in W_0^{1,p(\cdot)}(\Omega)^d \mid \operatorname{div} f = 0 \right\}, \\ X^\sigma_{p(\cdot),q} &= \left\{ \varphi \in X_{p(\cdot),q}(\Omega) \mid \operatorname{div} \varphi = 0 \right\} \end{aligned}$$

and show the following result:

Theorem 13 *Let $p_- > \frac{2d}{d+1}$. Then for every right hand side $f \in W^{-1,p'(\cdot)}(\Omega)^d$ there exists a weak solution $u \in V_{p(\cdot)}$, i.e. we have*

$$\int_\Omega T(\cdot, Du) : D\varphi\, dx + \int_\Omega [\nabla u]u \cdot \varphi\, dx = \langle f, \varphi \rangle$$

for all $\varphi \in C_0^\infty(\Omega)^d$ with $\operatorname{div} \varphi = 0$.

This theorem generalises a result of Růžička [18], Frehse et al. [12] to the case of fluids with $p(\cdot)$-growth, and moreover, we present a simplified proof by the use of divergence free test functions. With a more refined (and more technical) method - the Lipschitz truncation method - it is also possible to prove the existence of solutions for $p_- > \frac{2d}{d+2}$. We refer to Diening, Málek and Steinhauer [9] for a presentation of this method.

We start with an approximation procedure as in [12, 18] and introduce the following system:

$$\begin{aligned} -\operatorname{div} T(\cdot, Du_n) + [\nabla u_n]u_n + \frac{1}{n}|u_n|^{q-2}u_n + \nabla \pi_n &= f && \text{in } \Omega \\ \operatorname{div} u &= 0 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

The idea is to choose $q \in (0, \infty)$ large enough, such that we can use the theory of pseudo-monotone operators (see for example Zeidler [21]) even for small values of p_- to obtain the existence of approximate solutions u_n. For the coercivity of the corresponding operator equation, we need a $L^{p(\cdot)}$-version of Korn's inequality:

Theorem 14 *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary and consider an exponent $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$. Then there exists a constant $C > 0$ such that*

$$\|f\|_{1,p(\cdot)} \leq C \, \|Df\|_{p(\cdot)}$$

for all $f \in W_0^{1,p(\cdot)}(\Omega)^d$.

Proof The conclusion follows from [10, Corollary 5.6] after combining [7, Theorem 8.1] and [6, Lemma 5.5].

Now we can show the existence of approximate solutions:

Lemma 11 *Let $q > \max\{2(p_-)', p_-\}$ and $f \in W^{-1,p'(\cdot)}(\Omega)^d$. Then there exists a weak solution $u_n \in X^{\sigma}_{p(\cdot),q}$ of the approximated system, i.e. we have*

$$\int_{\Omega} T(\cdot, Du_n) : D\varphi \, dx + \int_{\Omega} [\nabla u_n] u_n \cdot \varphi \, dx + \frac{1}{n} \int_{\Omega} |u_n|^{q-2} u_n \cdot \varphi \, dx = \langle f, \varphi \rangle$$

for all $\varphi \in X^{\sigma}_{p(\cdot),q}$. Furthermore, there exists a constant $C > 0$ independent of n, such that we have the a priori estimate

$$\|u_n\|_{1,p(\cdot)} + \frac{1}{n} \|u_n\|_q^q \leq C.$$

Proof The proof is standard in the case of fluids with p-growth (see [12,18]) and can be adapted to our situation by minor changes. To be more precise, we only have to remind that we can compare the $L^{p(\cdot)}$-norm of a function f with the integral $\int_{\Omega} |f(x)|^{p(x)} \, dx$ and that we have a $L^{p(\cdot)}$-version of Korn's inequality (Theorem 14). □

Thanks to the a priori estimate, we can find a subsequence of u_n, still denoted by u_n, such that

$$u_n \rightharpoonup u_0 \quad \text{in } W_0^{1,p(\cdot)}(\Omega)^d.$$

In order to prove that u_0 is a weak solution of our system, we show in the sequel that $Du_n \to Du_0$ pointwise in Ω:

First of all, we use as in [18] a cut-off function $\psi \in C^{\infty}(\mathbb{R}^d, \mathbb{R}^d)$ which satisfies

$$\psi(y) = y \text{ for } |y| \leq 1, \quad \psi(y) = 0 \text{ for } |y| \geq 2$$
$$|\psi(y)| \leq 2 \text{ for } y \in \mathbb{R}^d, \quad |\nabla \psi(y)| \leq C_0 \text{ for } y \in \mathbb{R}^d.$$

For a bounded sequence $\delta_n > 0$, which will be chosen later, we define

$$\psi_{\delta_n}(y) = \delta_n \, \psi(y/\delta_n)$$

and deduce $\psi_{\delta_n}(u_n - u_0) \in W_0^{1,p(\cdot)}(\Omega)^d \cap L^{\infty}(\Omega)^d$ as well as

$$\psi_{\delta_n}(u_n - u_0) \to 0 \qquad \text{in } L^r(\Omega)^d, \; 1 \leq r < \infty,$$
$$\psi_{\delta_n}(u_n - u_0) \rightharpoonup 0 \qquad \text{in } W_0^{1,p(\cdot)}(\Omega)^d.$$

For choosing the sequence δ_n, we define

$$\begin{aligned} E(n,\kappa) &= \{x \in \Omega \mid |u_n(x) - u_0(x)| < \kappa\} \\ F(n,\kappa) &= \{x \in \Omega \mid \kappa \le |u_n(x) - u_0(x)| < 2\kappa\} \\ G(n,\kappa) &= \{x \in \Omega \mid 2\kappa \le |u_n(x) - u_0(x)|\} \end{aligned}$$

for $\kappa > 0$ and

$$h_n = (C_0\,|\nabla u_n - \nabla u_0|)^{p(\cdot)} + C_0\,|T(\cdot, Du_n) - T(\cdot, Du_0)|\,|\nabla u_n - \nabla u_0|.$$

We have the following statement:

Lemma 12 *Let $\epsilon > 0$ be given. Then there exists a bounded sequence δ_n with $\delta_n \ge 1$ and*

$$\int_{F(n,\delta_n)} h_n\,dx \le \epsilon$$

for all $n \in \mathbb{N}$.

Proof From the a priori estimate, we conclude $\int_\Omega h_n\,dx \le C$ for all $n \in \mathbb{N}$. We choose $N \in \mathbb{N}$ such that $\frac{C}{N} \le \epsilon$ and find

$$\sum_{j=1}^{N} \int_{F(n,2^{j-1})} h_n\,dx \le \int_\Omega h_n\,dx \le C.$$

This especially means that for all $n \in \mathbb{N}$ there exists a $j_n \in \{1,\ldots,N\}$ such that

$$\int_{F(n,2^{j-1})} h_n\,dx \le \frac{C}{N} \le \epsilon.$$

Now we define $\delta_n = 2^{j_n-1}$, hence the lemma.

Up to here we mainly followed [12,18]. A significant difference to the mentioned articles is that at this point we use the theory developed in the previous sections to construct completely divergence free test functions, i.e. for a given $0 < \epsilon < 1$ we define

$$\varphi_n = \psi_{\delta_n}(u_n - u_0) - \phi_n = \psi_{\delta_n}(u_n - u_0) - \mathbb{E}\left(\psi_{\delta_n}(u_n - u_0)\right)$$

for all $n \in \mathbb{N}$, where δ_n is the sequence found in Lemma 12. We then have

$$\begin{aligned} \phi_n,\, \varphi_n &\rightharpoonup 0 \quad \text{in } W_0^{1,p(\cdot)}(\Omega)^d\,, \\ \phi_n,\, \varphi_n &\to 0 \quad \text{in } L^r(\Omega)^d,\ 1 \le r < \infty\,, \end{aligned}$$

for $n \to \infty$ and $\operatorname{div}\varphi_n = 0$. Now we show that ϕ_n is small in the $W^{1,p(\cdot)}$-norm: First of all, we have

$$\|\phi_n\|_{1,p(\cdot)} = \left\|\mathbb{E}\left(\psi_{\delta_n}(u_n - u_0)\right)\right\|_{1,p(\cdot)} \le C_{p(\cdot)}\left\|\operatorname{div}\left(\psi_{\delta_n}(u_n - u_0)\right)\right\|_{p(\cdot)}$$

thanks to Theorem 4. Furthermore,

$$\begin{aligned}\int_\Omega \left|\operatorname{div}\left(\psi_{\delta_n}(u_n-u_0)\right)\right|^{p(\cdot)}dx &= \int_{E(n,\delta_n)} \left|\operatorname{div}\left(\psi_{\delta_n}(u_n-u_0)\right)\right|^{p(\cdot)}dx\\ &\quad+\int_{F(n,\delta_n)} \left|\operatorname{div}\left(\psi_{\delta_n}(u_n-u_0)\right)\right|^{p(\cdot)}dx\\ &\quad+\int_{G(n,\delta_n)} \left|\operatorname{div}\left(\psi_{\delta_n}(u_n-u_0)\right)\right|^{p(\cdot)}dx\end{aligned}$$

and since

$$\begin{aligned}\left|\operatorname{div}\left(\psi_{\delta_n}(u_n-u_0)\right)\right| &\le C_0|\nabla u_n-\nabla u_0|,\\ \operatorname{div}\left(\psi_{\delta_n}(u_n-u_0)\right)&=0 \quad \text{on } E(n,\delta_n)\cup G(n,\delta_n),\end{aligned}$$

we end up with

$$\int_\Omega \left|\operatorname{div}\left(\psi_{\delta_n}(u_n-u_0)\right)\right|^{p(\cdot)}dx \le \int_{F(n,\delta_n)} h_n\,dx \le \epsilon.$$

We conclude $\left\|\operatorname{div}\left(\psi_{\delta_n}(u_n-u_0)\right)\right\|_{p(\cdot)}^{p_+} \le \epsilon$ and $\|\phi_n\|_{1,p(\cdot)} \le C_{p(\cdot)}\epsilon^{\frac{1}{p_+}}$. With the above constructed test functions, we now show the pointwise convergence of the symmetric part of the gradient.

Lemma 13 *Let $p_- > \frac{2d}{d+1}$. Then*

$$Du_n \to Du_0 \quad \text{a.e. in } \Omega$$

for a subsequence.

Proof Define

$$g_n = (T(\cdot, Du_n) - T(\cdot, Du_0)) : (Du_n - Du_0)$$

for $n \in \mathbb{N}$. With help of the Hölder inequality and the a priori estimate, we find

$$\int_\Omega g_n\,dx \le r_p \|T(\cdot, Du_n) - T(\cdot, Du_0)\|_{p'(\cdot)} \|Du_n - Du_0\|_{p(\cdot)} \le C < \infty,$$

where $C > 0$ is independent of $n \in \mathbb{N}$. We now choose $\theta \in (0, 1)$ and show in the following $g_n^\theta \to 0$ in $L^1(\Omega)$ for $n \to \infty$. With the Hölder inequality, we find once more

$$\begin{aligned}\int_\Omega g_n^\theta\,dx &= \int_{E(n,1)} g_n^\theta\,dx + \int_{|u_n-u_0|\ge 1} g_n^\theta\,dx\\ &\le |\Omega|^{1-\theta}\left(\int_{E(n,1)} g_n\,dx\right)^\theta + C\left|\{|u_n-u_0|\ge 1\}\right|^{1-\theta}\end{aligned}$$

and since $u_n \to u_0$ in $L^{p_-}(\Omega)^d$, we conclude

$$\limsup_{n\to\infty} \int_\Omega g_n^\theta \, dx \le |\Omega|^{1-\theta} \limsup_{n\to\infty} \left(\int_{E(n,1)} g_n \, dx \right)^\theta .$$

We insert our divergence free test function $\varphi_n \in X^\sigma_{p(\cdot),q}$ in the approximated system, subtract $\int_\Omega T(\cdot, Du_0) : D\left(\psi_{\delta_n}(u_n - u_0)\right) dx$ on both sides and after rearranging we find

$$\begin{aligned} &\int_\Omega (T(\cdot, Du_n) - T(\cdot, Du_0)) : D\left(\psi_{\delta_n}(u_n - u_0)\right) dx \\ &= \langle f, \varphi_n \rangle - \int_\Omega [\nabla u_n] u_n \cdot \varphi_n \, dx - \frac{1}{n} \int_\Omega |u_n|^{q-2} u_n \cdot \varphi_n \, dx \\ &\quad + \int_\Omega T(\cdot, Du_n) : D\phi_n \, dx - \int_\Omega T(\cdot, Du_0) : D\left(\psi_{\delta_n}(u_n - u_0)\right) dx \\ &= I_1^n + \cdots + I_5^n . \end{aligned}$$

We analyse each term $I_1^n, \ldots, I_5^n$ separately. Since $\varphi_n \rightharpoonup 0$ in $W_0^{1,p(\cdot)}(\Omega)^d$, we get $I_1^n \to 0$. Sobolev's embedding theorem and the assumption $p_- > \frac{2d}{d+1}$ guarantee the existence of a number s such that

$$(p_-)' < s \qquad \text{and} \qquad W_0^{1,p(\cdot)}(\Omega)^d \hookrightarrow W_0^{1,p_-}(\Omega)^d \hookrightarrow L^s(\Omega)^d$$

and because of $\frac{1}{p_-} + \frac{1}{s} < 1$, we find $r \in (1, \infty)$ such that $\frac{1}{p_-} + \frac{1}{s} + \frac{1}{r} = 1$, hence

$$\left| \int_\Omega [\nabla u_n] u_n \cdot \varphi_n \, dx \right| \le C \, \|\nabla u_n\|_{p(\cdot)} \, \|u_n\|_{1,p(\cdot)} \, \|\varphi_n\|_r .$$

Together with the a priori estimate and $\varphi_n \to 0$ in $L^r(\Omega)^d$, we find $I_2^n \to 0$. For I_3^n we have

$$|I_3^n| \le \left(\frac{1}{n} \right)^{\frac{1}{q}} \left(\frac{1}{n} \|u_n\|_q^q \right)^{\frac{1}{q'}} \|\varphi_n\|_q ,$$

hence $I_3^n \to 0$. Moreover, the duality of $L^p(\cdot)$ and $L^{p'(\cdot)}$ shows that $I_5^n \to 0$. The Hölder inequality implies

$$|I_4^n| \le r_p \, \|T(\cdot, Du_n)\|_{p'(\cdot)} \, \|D\phi_n\|_{p(\cdot)} \le C \, \epsilon^{\frac{1}{p_+}} ,$$

where $C > 0$ is a constant independent of n and ϵ. Collecting all facts which we know for $I_1^n, \ldots, I_5^n$ leads to

$$\limsup_{n\to\infty} \left| \int_\Omega (T(\cdot, Du_n) - T(\cdot, Du_0)) : D\left(\psi_{\delta_n}(u_n - u_0)\right) dx \right| \le C \, \epsilon^{\frac{1}{p_+}} .$$

Since

$$\begin{aligned} D\left(\psi_{\delta_n}(u_n - u_0)\right) &= D(u_n - u_0) \qquad \text{on } E(n, \delta_n) , \\ D\left(\psi_{\delta_n}(u_n - u_0)\right) &= 0 \qquad \text{on } G(n, \delta_n) , \\ \left| D\left(\psi_{\delta_n}(u_n - u_0)\right) \right| &\le C_0 \, |\nabla u_n - \nabla u_0| \qquad \text{on } \Omega , \end{aligned}$$

we get the estimate

$$\int_{E(n,\delta_n)} g_n\,dx \leq \left|\int_\Omega (T(\cdot, Du_n) - T(\cdot, Du_0)) : D\left(\psi_{\delta_n}(u_n - u_0)\right)\,dx\right| + \int_{F(n,\delta_n)} h_n\,dx.$$

This and the fact that $\delta_n \geq 1$ implies

$$\limsup_{n\to\infty} \int_{E(n,1)} g_n\,dx \leq C\,\epsilon^{\frac{1}{p_+}} + \epsilon\,,$$

hence

$$\limsup_{n\to\infty} \int_\Omega g_n^\theta\,dx \leq |\Omega|^{1-\theta}\left(C\,\epsilon^{\frac{1}{p_+}} + \epsilon\right)^\theta\,,$$

where $C > 0$ is a constant independent of ϵ. Since $\epsilon > 0$ was arbitrary, we conclude $g_n^\theta \to 0$ in $L^1(\Omega)$, and therefore, we find a subsequence such that $g_n \to 0$ a.e. in Ω. Thanks to the strict monotonicity of T, we get

$$Du_n \to Du_0 \quad \text{a.e. in } \Omega$$

for this subsequence. The lemma is proved.

Having the pointwise convergence of the symmetric part of the gradient, we can continue as in [12, 18] to show that u_0 is actually a weak solution of our system. The proof of Theorem 13 is complete.

Acknowledgments This work is a short version of the author's diploma thesis submitted in September 2005 at the University of Freiburg, Germany. The author would like to thank his supervisor Michael Růžička for suggesting the topic and continuous support. In addition the author would like to thank Lars Diening for stimulating discussions.

References

1. Bogovskii, M.: Solution of the first boundary value problem for the equation of continuity of an incompressible medium. Dokl. Akad. Nauk SSSR **248**, 1037–1040 (1979). English transl. in Soviet Math. Dokl. **20**, 1094–1098 (1979)
2. Bogovskii, M.: Solution of some vector analysis problems connected with operators div and grad. Trudy Seminar S.L. Sobolev, Akademia Nauk SSSR **80**, 5–40 (1980)
3. Calderón, A.P., Scott, R.: Sobolev type inequalities for $p > 0$. Studia Math. **62**(1), 75–92 (1978)
4. Calderón, A.P., Zygmund, A.: On singular integrals. Am. J. Math. **78**, 289–309 (1956)
5. Cruz-Uribe, D., Fiorenza, A., Martell, J.M., Pérez, C.: The boundedness of classical operators on variable L^p spaces. Ann. Acad. Sci. Fenn. Math. **31**, 239–264 (2006)
6. Diening, L.: Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$. Math. Nachr. **268**, 31–43 (2004)
7. Diening, L.: Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces. Bull. Sci. Math. **129**(8), 657–700 (2005)
8. Diening, L., Hästö, P., Nekvinda, A.: Open problems in variable exponent Lebesgue and Sobolev spaces. In: Drábek, P., Rákosník, J. (eds.) Function Spaces, Differential Operators and Nonlinear Analysis (2004), pp. 38–58. Czech Academy of Sciences, Prague (2005)

9. Diening, L., Málek, J., Steinhauer, M.: On Lipschitz truncations of Sobolev functions (with variable exponent) and their selected applications. ESAIM Control Optim. Calc. Var. **14**(2), 211–232 (2008)
10. Diening, L., Růžička, M.: Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics. J. Reine Angew. Math. **563**, 197–220 (2003)
11. Durán, R.G., Muschietti, M.A.: An explicit right inverse of the divergence operator which is continuous in weighted norms. Studia Math. **148**(3), 207–219 (2001)
12. Frehse, J., Málek, J., Steinhauer, M.: An existence result for fluids with shear dependent viscosity - steady flows. Non. Anal. Theory Meth. Appl. **30**, 3041–3049 (1997)
13. Galdi, G.: An introduction to the mathematical theory of the Navier-Stokes equations, linearized steady problems. In: Tracts in Natural Philosophy, vol. 38. Springer, New York (1994)
14. García-Cuerva, J., Rubio de Francia, J.L.: Weighted norm inequalities and related topics. In: North-Holland Mathematics Studies, vol. 116. North-Holland Publishing Co., Amsterdam (1985)
15. Journé, J.L.: Calderón-Zygmund operators, pseudodifferential operators and the Cauchy integral of Calderón. Lecture Notes in Mathematics, vol. 994. Springer, Berlin (1983)
16. Kováčik, O., Rákosník, J.: On spaces $L^{p(x)}$ and $W^{k,p(x)}$. Czechoslovak Math. J. **41**(4), 592–618 (1991)
17. Muckenhoupt, B.: Weighted norm inequalities for the Hardy maximal function. Trans. Amer. Math. Soc. **165**, 207–226 (1972)
18. Růžička, M.: A note on steady flow of fluids with shear dependent viscosity. In: Proceedings of the Second World Congress of Nonlinear Analysts, Part 5 (Athens, 1996), vol. 30. pp. 3029–3039 (1997)
19. Stein, E.: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton (1970)
20. Torchinsky, A.: Real-variable methods in harmonic analysis. In: Pure and Applied Mathematics, vol. 123. Academic Press Inc., Orlando (1986)
21. Zeidler, E.: Nonlinear functional analysis and its applications. II/B. Springer, New York (1990)