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Solutions with moving singularities for a one-dimensional nonlinear diffusion equation

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Abstract

The aim of this paper is to study singular solutions for a one-dimensional nonlinear diffusion equation. Due to slow diffusion near singular points, there exists a solution with a singularity at a prescribed position depending on time. To study properties of such singular solutions, we define a minimal singular solution as a limit of a sequence of approximate solutions with large Dirichlet data. Applying the comparison principle and the intersection number argument, we discuss the existence and uniqueness of a singular solution for an initial-value problem, the profile near singular points and large-time behavior of solutions. We also give some results concerning the appearance of a burning core, convergence to traveling waves and the existence of an entire solution.

Mathematics Subject Classification 35K67 · 35A21 · 35K15 · 35B40

1 Introduction

In this paper, we consider positive solutions of the nonlinear diffusion equation

$$u_t = (u^m)_{xx}, \qquad 0 < m < 1, \tag{1.1}$$

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where $x \in \mathbb{R}$ is the spatial variable and $t \in \mathbb{R}$ is the time variable. Usually, this equation is called the fast diffusion equation, because the diffusion rate is large for small $u > 0$, although the diffusion rate becomes small if $u > 0$ is large.

Suppose that a solution of (1.1) is defined for $x > \xi(t)$ and t on a subinterval of $\mathbb{R}$, and is singular at $x = \xi(t)$, that is,

$$\lim_{x \downarrow \xi(t)} u(x,t) = \infty,$$

where the singular point $\xi(t)$ is a given smooth function of t. Then we are led to the study of the fast diffusion equation on a half line

$$u_t = (u^m)_{xx}, \qquad x > \xi(t). \tag{1.2}$$

When $\xi(t)$ is constant ($\xi(t) \equiv \xi(0)$), the singularity is called "standing". An example (see [2], for instance) of a solution with a standing singularity is the self-similar solution given by

$$\tilde{u}(x,t) := D_m \left\{ \frac{t}{(x-\xi(0))^2} \right\}^{\frac{1}{1-m}}, \qquad D_m := \left\{ \frac{2m(1+m)}{1-m} \right\}^{\frac{1}{1-m}}, \tag{1.3}$$

where $(x,t) \in (\xi(0), \infty) \times (0, \infty)$. When $\xi(t)$ depends on t, we say that the solution has a moving singularity. An example (see [11]) of a solution with a moving singularity is a traveling solution given by

$$u(x,t) := h(c)(x-ct)^{-\frac{1}{1-m}}, \qquad x \in (ct, \infty),\ t \in \mathbb{R}, \tag{1.4}$$

where $c > 0$ and

$$h(c) := \left\{ \frac{m}{(1-m)c} \right\}^{\frac{1}{1-m}}. \tag{1.5}$$

To study more general singular solutions, we introduce the initial-value problem

$$\begin{cases} u_t = (u^m)_{xx}, & x > \xi(t),\ 0 < t < T \in (0, \infty], \\ u(x,0) = u_0(x), & x > \xi(0), \end{cases} \tag{1.6}$$

where $u_0(x)$ is assumed to be positive and continuous but not necessarily bounded in $x \in (\xi(0), \infty)$. By a "singular" solution of (1.6), we mean a function $u(x,t)$ with the following properties:

(S1) $u(x,t)$ satisfies (1.2) in the classical sense for $t \in (0, T)$.
(S2) $u(x,t) \to \infty$ as $x \downarrow \xi(t)$ for every $t \in (0, T)$.
(S3) $u(x,t) \to u_0(x)$ as $t \downarrow 0$ uniformly in x on any closed interval in $(\xi(0), \infty)$.

Such solutions are called "extended continuous solutions" in [3] and references therein, since they are continuous with values in $[0, \infty]$.

In order to study singular solutions of (1.6), our key idea is to consider a sequence of approximate problems and its limit. For every $n \in \mathbb{N}$, let u_n be a regular solution of the following problem with a cut-off initial value and a (large) boundary value at $x = \xi(t)$:

$$\begin{cases} u_t = (u^m)_{xx}, & x > \xi(t),\ 0 < t < T, \\ u(\xi(t), t) = n, & 0 < t < T, \\ u(x, 0) = \min\left\{u_0(x), n\right\}, & x > \xi(0). \end{cases} \tag{1.7}$$

The existence and uniqueness of a solution of (1.7) follows from setting $u_\varepsilon(x, 0) = \min\left\{u_0(x) + \varepsilon, n\right\}$ and applying the standard approximation argument based on [19]. Then the comparison principle implies that $\{u_n\}$ is monotone increasing in n for every $x > \xi(t)$ and $t > 0$. We say that a solution u of (1.6) is a "minimal" singular solution if $u_n(x, t) \to u(x, t)$ as $n \to \infty$ for every $x > \xi(t)$ and $t > 0$. In fact, the minimal singular solution is smaller than any other singular solutions of (1.6) (see Remark 2.11 (iii) in Sect. 2). We note that, if a singular solution of (1.6) exists, then it is an upper bound for the approximating sequence, which implies the existence of a minimal singular solution.

Let us notice here that the minimal singular solution is a uniquely determined object which does not depend on the particular choice of an approximating sequence $\{u_n\}$ of regular solutions. It is not difficult to see that every sequence $\{u_n\}$ converging to a singular solution of (1.6) monotonically (in n) from below produces the same minimal singular solution. In other contexts, minimal solutions occur in various studies of unbounded solutions. They are sometimes called "proper" solutions (see [10, 30] for example). The advantage of minimal solutions consists in their uniqueness and in the fact that they inherit intersection-comparison properties of classical solutions via their approximating sequences.

The main aim of this paper is to discuss the existence and uniqueness of singular solutions of (1.6), their profile near $\xi(t)$, appearance of a zone where $\{u_n\}$ tends to infinity (burning core), as well as the large-time behavior of solutions. We also give some results concerning the existence and stability of solutions that travel with constant speed (traveling semi-wavefronts), and the existence of a solution which exists for all $t \in \mathbb{R}$ (entire solution). In summary, in the terminology of [3], we develop a theory of extended continuous solutions with moving strong singularities in one dimension as stated among new directions in [3, p. 178]. Both, one space-dimension and moving singularities are mentioned there separately.

As far as we know, all previous studies of moving singularities concern the higher dimensional case when $\xi(t)$ is a curve (or some other time-dependent set) in $\mathbb{R}^N$ with $N \geq 2$. See [5–8, 20] for the fast diffusion equation, [16, 17, 27] for the linear heat equation and [12, 13, 17, 18, 21–25, 28] for semilinear heat equations. Results on standing singularities for the fast diffusion equation can be found in [2, 3, 7, 14, 15, 26, 30], for example.

This paper is organized as follows. In Sect. 2, we summarize our main results. In Sect. 3, we prove the existence of a singular solution for the initial-value problem (1.6). In Sect. 4, we study the profile of singular solutions near $\xi(t)$. Sect. 5 is devoted to uniqueness for the initial-value problem (1.6) under some additional conditions. In Sect. 6, we show the appearance of a burning core in the case where $\xi(t)$ decreases. In Sect. 7, we discuss the existence of traveling solutions, and study their properties. In Sect. 8, we study the large-time behavior of singular solutions, and prove the asymptotic stability of traveling solutions. In Sect. 9, we prove the existence of an entire solution.

2 Main results

Main results of this paper are listed as follows. Our first result is concerning the existence of a singular solution of the initial-value problem (1.6) in the case where $\xi(t)$ is nondecreasing.

Theorem 2.1 *If $\xi(t)$ is nondecreasing in $t \in [0, T)$, then there exists a singular solution of (1.6).*

The next result shows that the profile of a singular solution near $\xi(t)$ is in some range for $t > 0$ and the leading term as $x \downarrow \xi(t)$ depends crucially on $\xi'(t) > 0$.

Theorem 2.2 *Assume that $\xi(t)$ is nondecreasing in $t \in [0, T)$. Let u be any singular solution of (1.6) defined on $(0, T)$.*

(i) If $c(\tau - t) > \xi(\tau) - \xi(t)$ for $t \in [0, \tau)$ with some constant $c > 0$ and $\tau \in (0, T)$, then there exists $\alpha(\tau) > 0$ such that

$$u(x, \tau) \geq \alpha(\tau)\{x - \xi(\tau)\}^{-\frac{1}{1-m}}, \qquad x \in (\xi(\tau), \xi(\tau) + 1).$$

In particular, if $\xi(t) \equiv \xi(0)$, then

$$u(x, t) \geq \tilde{u}(x, t) = D_m \left\{ \frac{t}{(x - \xi(0))^2} \right\}^{\frac{1}{1-m}}, \qquad (x, t) \in (0, \infty) \times [0, \tau],$$

where $\tilde{u}$ is the self-similar solution given by (1.3).

(ii) For any $\tau \in (0, T)$ with $\xi(\tau) > \xi(0)$, there exists $\beta(\tau) > 0$ such that

$$u(x, \tau) \leq \beta(\tau)\{x - \xi(\tau)\}^{-\frac{2}{1-m}}, \qquad x \in (\xi(\tau), \xi(\tau) + 1).$$

(iii) If $\xi(t)$ is differentiable at $\tau \in (0, T)$ and $\xi'(\tau) > 0$, then

$$u(x, \tau) = \{h(\xi'(\tau)) + o(1)\}\{x - \xi(\tau)\}^{-\frac{1}{1-m}}, \qquad x \downarrow \xi(\tau).$$

We note that $\alpha(\tau), \beta(\tau) > 0$ may become small or large depending on initial values when $x - \xi(0) > 0$ is small, and that the strength of the singularity of $u(x, t)$ for $t > 0$ may be different from that of $u_0(x)$.

Since we impose a rather mild condition (S2) at $\xi(t)$, it becomes a nontrivial question to ask about uniqueness of a singular solution for (1.6). First we give a simple sufficient condition.

Theorem 2.3 *Assume that $\xi(t)$ is nondecreasing in $t \in (0, T)$. If $u_0(x)$ is nonincreasing in $x \in (\xi(0), \infty)$, then there exists at most one singular solution of (1.6).*

In the case where $u_0(x)$ is not monotone in $x \in (\xi(0), \infty)$, we must impose the following additional conditions on the initial data to prove the uniqueness:

(U1) There exists $C_0 \geq 0$ such that $\left\{u_0(x)^m\right\}'' \geq -C_0$ for all $x \in (\xi(0), \infty)$.
(U2) $u_0(x) \to \infty$ as $x \downarrow \xi(0)$, and there exist $x_1 \in (\xi(0), \infty)$ and $\delta \in (0, x_1 - \xi(0))$ such that $u_0(x)$ is nonincreasing in $x \in (\xi(0), x_1 + \delta)$ and strictly decreasing in $x \in (x_1 - \delta, x_1 + \delta)$.
(U3) There exist $x_2 \in (x_1, \infty)$ and $\delta \in (0, x_2 - x_1)$ such that $u_0(x)$ is nonincreasing in $x \in (x_2 - \delta, \infty)$ and strictly decreasing in $x \in (x_2 - \delta, x_2 + \delta)$.

We may replace (U3) with the following alternative condition:

(U3') There exists a constant $C_1 > 0$ such that $u_0(x) \geq C_1$ for all $x \in (\xi(0), \infty)$.

Theorem 2.4 *Assume that $\xi'(t) \geq 0$ for $t \in [0, T)$. If $u_0(x)$ satisfies* (U1), (U2)*, and* (U3) *or* (U3')*, then there exists at most one singular solution of (1.6).*

Remark It seems that the condition (U1)can be relaxed somewhat. Indeed, it would be possible to extend Theorem 2.4 to more general initial values that can be well approximated by some smooth initial values satisfying (U1). However, since the class of initial values satisfying (U1), (U2), and (U3) or (U3') is rather wide, we will not pursue this possibility in this paper.

In the case where $\xi(t)$ decreases, we will show that a burning core (a region where a solution becomes infinite in some sense) appears for $x \in (\xi(t), \xi(0)]$.

Theorem 2.5 *Assume that $\xi(t) < \xi(0)$ for $t \in (0, T)$. If $u_0(x) \to \infty$ as $x \downarrow \xi(0)$, then the approximating sequence $\{u_n\}$ defined by (1.7) has the following properties:*

(i) For every $t \in (0, T)$, $u_n(x, t) \to \infty$ as $n \to \infty$ uniformly in $x \in (\xi(t), \xi(0)]$.
(ii) For every $t \in (0, T)$, $u_n(x, t) \to \tilde{u}(x, t)$ as $n \to \infty$ uniformly in $x \in [\xi(0) + \rho, \infty)$, where $\rho > 0$ is an arbitrary constant, and $\tilde{u}(x, t)$ is the minimal singular solution of (1.6) with $\xi(t)$ replaced by $\tilde{\xi}(t) \equiv \xi(0)$ and $\tilde{u}(x, 0) = u_0(x)$ for $x > \xi(0)$.

When the singular solution of (1.2) exists globally in time, we are interested in its large-time behavior. The following theorem shows that any two singular solutions attract each other, and which implies that the large-time behavior of singular solutions of (1.2) is independent of initial values.

Theorem 2.6 *Assume that $\xi'(t)$ is uniformly continuous in $t \in [0, \infty)$ and satisfies $c_1 \leq \xi'(t) \leq c_2$ for $t \in [0, \infty)$ with some constants $c_1, c_2 > 0$. Let $u_1(x, t), u_2(x, t)$ be any positive singular solutions of (1.2) such that $u_1(x, 0), u_2(x, 0) \to L \geq 0$ as $x \to \infty$. Then these solutions satisfy $\left|u_1(x, t) - u_2(x, t)\right| \to 0$ as $t \to \infty$ uniformly in $x \in [\xi(t) + \rho, \infty)$, where $\rho > 0$ is an arbitrary constant.*

When $\xi(t)$ moves with a constant speed, we expect the existence of a solution of (1.2) which travels without changing its waveform. The existence and stability of traveling wave solutions for parabolic PDEs has been studied extensively in the context of reaction-diffusion equations (see [11, 29] and the references cited therein). When a traveling solution is defined on a half-line and is not extendable for all $x \in \mathbb{R}$ such as the solution given by (1.4), it is called a (strict) semi-wavefront solution (see [11, Sect. 2.1]). Although the existence of unbounded semi-wavefronts is discussed in [11, Chapter 11] including more general equations than (1.1), only little is known about their stability.

The following two theorems are concerning the existence and stability of a two-parameter family of traveling semi-wavefronts.

Theorem 2.7 *Assume $\xi(t) = ct$ for $t \in \mathbb{R}$ with some constant $c > 0$. Then for any constant $L \geq 0$, there exists a positive singular solution of (1.2) for $t \in \mathbb{R}$ of the form $u = \varphi(z)$, where $\varphi(z) = \varphi(z; c, L)$ is a positive function of $z = x - ct \in (0, \infty)$ with the following properties:*

(i) $(\varphi^m)'' + c\varphi' = 0$ *for* $z > 0$.
(ii) $\varphi'(z) < 0$ *for* $z \in (0, \infty)$, $\varphi(z) \to \infty$ *as* $z \downarrow 0$ *and* $\varphi(z) \to L$ *as* $z \to \infty$.
(iii) $\max\{h(c)z^{-\frac{1}{1-m}}, L\} \leq \varphi(z) \leq h(c)z^{-\frac{1}{1-m}} + L$ *for* $z > 0$, *where* $h(c) > 0$ *is given by (1.5).*
(iv) $\varphi(z) = \varphi(z; c, L)$ *is continuous in* $(c, L) \in (0, \infty) \times [0, \infty)$ *for every* $z > 0$, *and* $\varphi(z; c, L)$ *is decreasing in* $c \in (0, \infty)$ *and increasing in* $L \in [0, \infty)$.

Theorem 2.8 *Assume that $\xi'(t) \geq 0$ for $t \in [0, \infty)$. Let $\rho > 0$ be an arbitrary constant.*

(i) *If* $\xi'(t) \to c > 0$ *as* $t \to \infty$ *and* $u_0(x) \to L \geq 0$ *as* $x \to \infty$, *then the singular solution of (1.6) satisfies* $u(x, t) \to \varphi(x - ct; c, L)$ *as* $t \to \infty$ *uniformly in* $x \in [\xi(t) + \rho, \infty)$, *where* φ *is as in Theorem 2.7.*
(ii) *If* $\xi'(t) \to 0$ *as* $t \to \infty$, *then the singular solution of (1.6) satisfies* $u(x, t) \to \infty$ *as* $t \to \infty$ *uniformly in* $x \in (\xi(t), \xi(t) + \rho]$.
(iii) *If* $\xi'(t) \to \infty$ *as* $t \to \infty$ *and* $u_0(x) \to L \geq 0$ *as* $x \to \infty$, *then the singular solution of (1.6) satisfies* $u(x, t) \to L$ *as* $t \to \infty$ *uniformly in* $x \in [\xi(t) + \rho, \infty)$.

Any solution of (1.2) that is defined for all $t \in \mathbb{R}$ is called an entire (in time) solution. The following result gives a sufficient condition for the existence of an entire singular solution. (Some properties of this entire solution will be given in Proposition 9.1.)

Theorem 2.9 *Assume that $\xi(t)$ satisfies $c_1 \leq \xi'(t) \leq c_2$ for $t \in \mathbb{R}$, where $c_1, c_2 > 0$ are constants. Then for any $L \geq 0$, there exists a positive solution of (1.2) defined for all $t \in \mathbb{R}$ such that $u(x, t) \to \infty$ as $x \downarrow \xi(t)$ and $u(x, t) \to L$ as $x \to \infty$ for every $t \in \mathbb{R}$.*

One of the main tools to prove these theorems is a comparison method. A nonnegative function $u^+(x, t)$ (resp. $u^-(x, t)$) defined on a domain in (x, t)-space is called a supersolution (resp. subsolution) if it satisfies

$$(u^+)_t \geq \left\{(u^+)^m\right\}_{xx} \qquad (\text{resp. } (u^-)_t \leq \left\{(u^-)^m\right\}_{xx})$$

(in the sense of distribution) on a space-time domain. We note that the minimum of two supersolutions is a supersolution, and the maximum of two subsolutions is a subsolution. The standard comparison principle holds for (1.1), but we must be careful when we consider a singular solution or a singular supersolution.

We will use the following lemma frequently in the subsequent sections.

Lemma 2.10 *Let $u(x,t)$ be a (not necessarily singular) solution of (1.6). Let $u^+(x,t)$ be a supersolution of (1.1) defined for $x \in (\eta(t),\infty)$ and $t \in [0,\tau)$ with some $\tau \in (0,T]$ such that $u^+(x,t) \to \infty$ as $x \downarrow \eta(t)$ for every $t \in [0,\tau)$. Assume that $\xi(t) \le \eta(t) < \infty$ for $t \in [0,\tau)$ and $u_0(x) \le u^+(x,0)$ for $x \in (\eta(0),\infty)$. Then the inequality $u(x,t) \le u^+(x,t)$ holds for $x \in (\eta(t),\infty)$ and $t \in [0,\tau)$ if one of the following conditions is satisfied:*

(C1) There exist constants $C > 0$ and $\rho > 0$ such that $u(x,t) \le C$ for any $x \in (\xi(t), \xi(t)+\rho)$ and $t \in [0,\tau)$.
(C2) $u(x,t)$ is a minimal singular solution for $t \in (0,\tau)$.
(C3) $\xi(t) < \eta(t)$ for $t \in [0,\tau)$.

Proof First we assume (C1). We take $C > 0$ sufficiently large, and define

$$\tilde{u}^+(x,t) := \begin{cases} C, & x \in [\xi(t), \xi(t)+\tilde{\rho}(t)], \\ u^+(x,t), & x \in (\xi(t)+\tilde{\rho}(t), \infty), \end{cases} \qquad t \in [0,\tau),$$

where

$$\tilde{\rho}(t) := \inf\{\rho > 0 : u^+(\xi(t)+\rho, t) < C\} \in (0,\infty).$$

Since $u \equiv C$ satisfies (1.1), $\tilde{u}^+(x,t)$ is a supersolution of (1.1) defined for $x \in [\xi(t),\infty)$ and $t \in [0,\tau)$. Then we can apply the standard comparison principle to show that $u(x,t) \le \tilde{u}^+(x,t)$ for $x \in [\xi(t),\infty)$ and $t \in (0,\tau)$. Since $\tilde{u}^+(x,t) \le u^+(x,t)$ for $x \in (\eta(t),\infty)$ and $t \in [0,\tau)$, the proof in the case of (C1) is completed.

Next, we assume (C2). Let $\{u_n\}$ be the approximate sequence defined by (1.7). Since $u_n(x,t)$ is bounded as $x \downarrow \xi(t)$, we have $u_n(x,t) < u^+(x,t)$ for every $n \in \mathbb{N}$. Letting $n \to \infty$, we see that the minimal solution satisfies $u(x,t) \le u^+(x,t)$ for $x \in (\eta(t),\infty)$ and $t \in (0,\tau)$.

In the case of (C3), if we restrict the spatial domain of $u(x,t)$ to $[\eta(t),\infty)$, then we can apply the case of (C1) to show that $u(x,t) < u^+(x,t)$ for $x \in (\eta(t),\infty)$ and $t \in [0,\tau)$. □

Remark 2.11 (i) In the case of $\xi(t) \equiv \eta(t)$, the comparison principle may not hold for a non-minimal solution of (1.6).

(ii) If a supersolution is defined on a bounded interval $(\eta(t),\zeta(t))$ and satisfies $u(x,t) \to \infty$ as $x \downarrow \eta(t)$ and $u(x,t) \to \infty$ as $x \uparrow \zeta(t)$, then the same argument as above at $\zeta(t)$ shows that a similar result to Lemma 2.10 holds for $x \in (\eta(t),\zeta(t))$ and $t \in (0,\tau)$.

(iii) By Lemma 2.10 with (C1), the solution of (1.7) is smaller than any singular solution of (1.6). This implies that the minimal singular solution is smaller than any other singular solutions (if they exist).

The next lemma can be proved in a similar manner to Lemma 2.10. So we omit the proof.

Lemma 2.12 *Let $u(x,t)$ be any singular solution of (1.6). Let $u^-(x,t)$ be a subsolution of (1.1) defined for $x \in [\xi(t), \infty)$ and $t \in [0,\tau)$ such that $u^-(x,t)$ is bounded as $x \downarrow \xi(t)$ for every $t \in [0,\tau)$. Assume that $u_0(x) \geq u^-(x,0)$ for $x \in (\xi(0), \infty)$. Then the inequality $u(x,t) \geq u^-(x,t)$ holds for $x \in (\xi(t), \infty)$ and $t \in [0,\tau)$.*

3 Existence of singular solutions

Throughout this section, we assume that $\xi(t)$ is nondecreasing in $t \in (0,T)$. First, we give a sufficient condition for the existence of a singular solution of (1.6).

Lemma 3.1 *Suppose that the approximating sequence $\{u_n\}$ defined by (1.7) is bounded above by a continuous function $\overline{u}(x,t)$ defined for $x \in (\xi(t), \infty)$ and $t \in (0,T)$. Then there exists a singular solution of (1.6) satisfying $0 < u(x,t) \leq \overline{u}(x,t)$ for $x \in (\xi(t), \infty)$ and $t \in (0,T)$.*

Proof The standard comparison principle with respect to initial values and boundary conditions implies that the sequence $\{u_n\}$ is monotone increasing in n for every (x,t). Since the sequence is assumed to be bounded, there exists a limiting function $u(x,t)$ satisfying

$$u(x,t) := \lim_{n\to\infty} u_n(x,t) \leq \overline{u}(x,t).$$

Then the standard parabolic regularity implies that $u(x,t)$ satisfies (S1).

Next we show that $u(x,t)$ is singular at $\xi(t)$. By the boundary condition and continuity, for every $t \in (0,T)$, there exists $\rho_n(t) > 0$ such that $u_n(x,t) > n-1$ for $x \in (\xi(t), \xi(t)+\rho_n(t))$. Since $\{u_n\}$ is monotone increasing in n, we have $u(x,t) > n-1$ for $x \in (\xi(t), \xi(t)+\rho_n(t))$. Since $n \in \mathbb{N}$ is arbitrary, this implies that $u(x,t)$ satisfies (S2).

It remains to show that $u(x,t)$ satisfies the initial condition. Let $\xi(0) < x_0 < x_1 < x_2 < x_3 < \infty$ be arbitrarily fixed. We take (large) $n \in \mathbb{N}$ and (small) $\delta \in (0,T)$ such that

$$u(x,t) < n, \qquad (x,t) \in [x_0, x_3] \times (0,\delta].$$

Let $u^+(x,t)$ be a solution of

$$\begin{aligned}
&(u^+)_t = \left\{(u^+)^m\right\}_{xx}, && (x,t) \in (x_0,x_3) \times (0,\delta),\\
&u^+(x,0) = u_0^+(x), && x \in (x_0,x_3),\\
&u^+(x_0,t) = n, && u^+(x_3,t) = n, \qquad t \in (0,\delta),
\end{aligned}$$

where $u_0^+(x)$ is a continuous function satisfying $u_0^+(x) \equiv u_0(x)$ for $x \in [x_1, x_2]$, $u_0^+(x) \geq u_0(x)$ for $x \in [x_0, x_3]$, and $u_0^+(x_0) = u_0^+(x_3) = n$. Since

$$u_n(x_0, t) \leq u(x_0, t) \leq u^+(x_0, t), \quad u_n(x_3, t) \leq u(x_3, t) = u^+(x_3, t), \qquad t \in (0, \delta].$$

the standard comparison principle implies that

$$u_n(x, t) \leq u^+(x, t), \qquad (x, t) \in [x_0, x_3] \times (0, \delta].$$

Hence, letting $n \to \infty$, we obtain

$$u(x, t) \leq u^+(x, t), \qquad (x, t) \in [x_0, x_3] \times (0, \delta].$$

Thus we have shown

$$u_n(x, t) \leq u(x, t) \leq u^+(x, t), \qquad (x, t) \in [x_0, x_3] \times (0, \delta].$$

Since $u_n(x, t) \to u_0(x)$ and $u^+(x, t) \to u_0(x)$ as $t \downarrow 0$ uniformly in $x \in [x_1, x_2] \subset (x_0, x_3)$, we conclude that $u(x, t)$ satisfies (S3). □

We introduce a useful way to construct a supersolution of (1.1).

Lemma 3.2 *Assume that $u_a(x, t)$ and $u_b(x, t)$ satisfy (1.1), $u_a > 0$, $(u_a)_t \geq 0$, $u_b > 0$ and $(u_b)_t \geq 0$ for $(x, t) \in I \times (0, T)$, where I is an open interval. Then*

$$U^+(x; t) := \left\{u_a(x, t)^m + u_b(x, t)^m\right\}^{\frac{1}{m}}$$

satisfies

$$U_t^+ \geq \left\{(U^+)^m\right\}_{xx}, \qquad (x, t) \in I \times (0, T).$$

Proof By direct computation, we have

$$U_t^+ = (U^+)^{1-m}\left\{(u_a)^{m-1}(u_a)_t + (u_b)^{m-1}(u_b)_t\right\}.$$

Since $(u_a)_t \geq 0$ and $(u_b)_t \geq 0$ by assumption, and $U^+ > u_a$ and $U^+ > u_b$, we obtain

$$U_t^+ \geq (u_a)_t + (u_b)_t = \left\{(u_a)^m\right\}_{xx} + \left\{(u_b)^m\right\}_{xx}.$$

Then by

$$\left\{(U^+)^m\right\}_{xx} = \left\{(u_a)^m\right\}_{xx} + \left\{(u_b)^m\right\}_{xx},$$

we obtain the desired inequality. □

Let us now show the existence of a singular solution of (1.6).

Proof of Theorem 2.1 We construct a supersolution by modifying the explicit solution given by (1.3). Define

$$U^+(x,t;a,b) := \left\{u_a(x,t)^m + u_b(x,t)^m\right\}^{\frac{1}{m}}, \qquad (x,t) \in (a,b) \times (0,T), \quad (3.1)$$

where u_a and u_b are solutions of (1.1) given by

$$u_a(x,t) := D_m\left\{\frac{t+1}{(x-a)^2}\right\}^{\frac{1}{1-m}}, \qquad u_b(x,t) := D_m\left\{\frac{t+1}{(x-b)^2}\right\}^{\frac{1}{1-m}},$$

respectively, and D_m is the constant given in (1.3). Since $(u_a)_t > 0$ and $(u_b)_t > 0$ for $(x,t) \in (a,b) \times (0,T)$, we can apply Lemma 3.2 to show that U^+ satisfies

$$U_t^+ \geq \left\{(U^+)^m\right\}_{xx}, \qquad (x,t) \in (a,b) \times (0,T).$$

Note that $\min_{x\in(a,b)} U^+(x,t;a,b) \to \infty$ as $b-a \to 0$.

Now, fix $\tau \in (0,T)$. Let $\{U_k^+\}$ be a set of functions defined by $U_k^+(x,t) := U^+(x,t;a_k,b_k)$, $k = 1,2,3,\ldots$, such that

$$\xi(\tau) < a_k < b_k < \infty, \qquad \bigcup_{k=1}^{\infty}(a_k,b_k) = (\xi(\tau),\infty),$$

and $U_k^+(x,0) \geq u_0(x) \geq u_n(x,0)$ for $x \in (a_k,b_k)$. Then by Lemma 2.10 with (C2) and Remark 2.11 (ii), we have

$$u_n(x,t) \leq U_k^+(x,t), \qquad (x,t) \in (a_k,b_k) \times [0,\tau],$$

for every $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Hence we obtain

$$u_n(x,\tau) \leq u^+(x,\tau) := \inf_k U_k^+(x,\tau), \qquad x \in (\xi(\tau),\infty),$$

for every $n \in \mathbb{N}$. This implies that the approximating sequence $\{u_n(x,\tau)\}$ is bounded above by $u^+(x,\tau)$. Since $\tau \in (0,T)$ can be arbitrarily chosen, Lemma 3.1 implies the existence of a singular solution of (1.6) for $t \in (0,T)$. □

4 Asymptotic profile

Throughout this section, we assume that $\xi(t)$ is nondecreasing in $t \in (0,T)$, and study the profile of singular solutions near $\xi(t)$. Before giving a proof of Theorem 2.2, we give a useful way to construct a subsolution of (1.1).

Lemma 4.1 *Assume that a function $u(x,t)$ satisfies (1.1), $u > 0$ and $u_t \geq 0$ for $(x,t) \in I \times (0,T)$, where I is an open interval. Then for any constant $d \geq 0$,*

$$U^-(x,t) := \left\{ \max\{u(x,t)^m - d, 0\}\right\}^{\frac{1}{m}}$$

is a subsolution for $(x,t) \in I \times (0,T)$.

Proof If $u(x,t)^m \geq d$, we have

$$U_t^- - \left\{(U^-)^m\right\}_{xx} = (U^-)^{1-m}u^{m-1}u_t - (u^m)_{xx} = \left\{(U^-)^{1-m} - u^{1-m}\right\}u^{m-1}u_t.$$

Since $u_t \geq 0$ by assumption and $U^- \leq u$, U^- satisfies $U_t^- \leq \left\{(U^-)^m\right\}_{xx}$ for $(x,t) \in I \times (0,T)$ in the sense of distribution. Hence U^- is a subsolution.

We give a proof of Theorem 2.2 by using this lemma.

Proof of Theorem 2.2 First, modifying the traveling solution given by (1.4), we define

$$u^-(x,t) := \left\{ \max\{h(c)^m\left(x - c(t-\tau) - \xi(\tau)\right)^{-\frac{m}{1-m}} - d, 0\}\right\}^{\frac{1}{m}},$$

where $d > 0$ is a constant. Then by Lemma 4.1, u^- is a subsolution of (1.1). By the assumption on τ and c in (i), we have $-c(t-\tau) - \xi(\tau) > -\xi(t)$ for $t \in [0,\tau)$ so that $u^-(x,t) \to u^-(\xi(t),t) < \infty$ as $x \downarrow \xi(t)$. Moreover, since $c\tau - \xi(\tau) > -\xi(0)$, we have

$$u^-(x,0) = \left\{ \max\{h(c)^m\left(x + c\tau - \xi(\tau)\right)^{-\frac{m}{1-m}} - d, 0\}\right\}^{\frac{1}{m}} \leq u_0(x), \quad x \in (\xi(0), \infty),$$

if $d > 0$ is sufficiently large. Hence we can apply Lemma 2.12 to show that $u(x,\tau) \geq u^-(x,\tau)$ for $x \in (\xi(\tau), \infty)$. Hence there exists $\alpha(\tau) > 0$ such that

$$u(x,\tau) \geq u^-(x,\tau) \geq \alpha(\tau)\{x - \xi(\tau)\}^{-\frac{1}{1-m}}, \qquad x \in (\xi(\tau), \xi(\tau)+1).$$

In particular, if $\xi(t) \equiv \xi(0)$ for $t \in [0,\tau]$, then it follows from the comparison principle that

$$u(x,t) \geq \tilde{u}(x+\varepsilon,t) = D_m\left\{\frac{t}{(x - \xi(0) + \varepsilon)^2}\right\}^{\frac{1}{1-m}}, \qquad (x,t) \in (\xi(0),\infty) \times [0,\tau],$$

where $\tilde{u}$ is the self-similar solution given by (1.3) and $\varepsilon > 0$ is an arbitrary number. By taking the limit as $\varepsilon \downarrow 0$, the proof of (i) is completed.

Next, let $U^+(x,t;a,b)$ be the supersolution of (1.1) given by (3.1), where we take $a = \xi(\tau) > \xi(0)$ and $b \in (\xi(\tau), \xi(\tau)+1)$ sufficiently close to $\xi(\tau)$ such that $U^+(x,0;\xi(\tau),b) \geq u_0(x)$ for $x \in (\xi(\tau),b)$. Then by Lemma 3.2 and the comparison principle, we obtain

$$u(x,\tau) \leq U^+(x,\tau;\xi(\tau),b), \qquad x \in (\xi(\tau),b).$$

Since $u(x,\tau)$ is continuous in $x\in[b,\xi(\tau)+1]$, for each $\tau\in(0,T)$, there exists $\beta(\tau)$ such that

$$u(x,\tau)\le\beta(\tau)\{x-\xi(\tau)\}^{-\frac{2}{1-m}},\qquad x\in(\xi(\tau),\xi(\tau)+1).$$

This proves (ii).

Finally, we consider the case where $\xi'(\tau)>0$. Let u_a be a semi-wavefront solution given by

$$u_a(x,t):=h(a)\{x-a(t-\tau)-\xi(\tau)\}^{-\frac{1}{1-m}},$$

and let u_b be another semi-wavefront solution of (1.4), which travels leftward, given by

$$u_b(x,t):=h(b)\{-x-b(t-\tau-\delta)+\xi(\tau)\}^{-\frac{1}{1-m}},$$

where a,b are constants satisfying $0<a<\xi'(\tau)<b$. Then we take small $\delta>0$ such that $a(\tau-t)<\xi(\tau)-\xi(t)$ for $t\in[\tau-\delta,\tau)$.

Let us consider the behavior of $u(x,t)$ for $t\in[\tau-\delta,\tau]$. By Lemma 3.2, we can define a supersolution of (1.1) by

$$u^+(x,t):=\left\{u_a(x,t)^m+u_b(x,t)^m\right\}^{\frac{1}{m}}.$$

If we take $b-a>0$ and $\delta>0$ small enough, then

$$u(x,\tau-\delta)\le u^+(x,\tau-\delta),\qquad x\in(\xi(\tau)-a\delta,\xi(\tau)+2b\delta),$$

and $u^+(x,t)$ has a singularity at $\eta(t)=\xi(\tau)+a(t-\tau)>\xi(t)$ for $t\in[0,\tau)$. Hence by Lemma 2.10 with (C3) and Remark 2.11 (ii), we obtain

$$u(x,\tau)\le u^+(x,\tau),\qquad x\in(\xi(\tau),\xi(\tau)+b\delta).$$

This implies that

$$\limsup_{x\downarrow\xi(\tau)}\left\{x-\xi(\tau)\right\}^{\frac{1}{1-m}}u(x,\tau)\le h(a).$$

Letting $a\uparrow\xi'(\tau)$, we obtain

$$\limsup_{x\downarrow\xi(\tau)}\left\{x-\xi(\tau)\right\}^{\frac{1}{1-m}}u(x,\tau)\le h(\xi'(\tau)).\tag{4.1}$$

Next, we fix $c>\xi'(\tau)$ arbitrarily, and define a subsolution of (1.1) by

$$u^-(x,t):=\left\{\max\left\{h(c)^m\left(x-c(t-\tau)-\xi(\tau)\right)^{-\frac{m}{1-m}}-d,0\right\}\right\}^{\frac{1}{m}}.$$

Then we can take small $\delta > 0$ such that $c(\tau - t) > \xi(\tau) - \xi(t)$ for $t \in [\tau - \delta, \tau)$, and take large $d > 0$ such that $u(x, \tau - \delta) \geq u^-(x, \tau - \delta)$ for $x \in (\xi(\tau - \delta), \infty)$. Then, by Lemma 4.1 and the comparison principle, we obtain $u(x, \tau) \geq u^-(x, \tau)$ for $x \in (\xi(\tau), \infty)$. Hence taking the limit as $x \downarrow \xi(\tau)$ and then letting $c \downarrow \xi'(\tau)$, we deduce that

$$\liminf_{x \downarrow \xi(\tau)} \left\{x - \xi(\tau)\right\}^{\frac{1}{1-m}} u(x, \tau) \geq h(\xi'(\tau)). \tag{4.2}$$

Therefore, (4.1) and (4.2) imply that

$$\lim_{x \downarrow \xi(\tau)} u(x, \tau)\{x - \xi(\tau)\}^{-\frac{1}{1-m}} = h(\xi'(\tau)).$$

This completes the proof of (iii). □

5 Uniqueness

In this section, we prove the uniqueness of a singular solution of (1.6). We first prove the uniqueness by assuming that the initial value $u_0(x)$ is nonincreasing in $x \in (\xi(0), \infty)$.

Proof of Theorem 2.3 Let u be the minimal singular solution of (1.6). If $u_0(x)$ is nonincreasing in $x \in (\xi(0), \infty)$, then for every n, the solution $u_n(x, t)$ of (1.7) is decreasing in $x \in (\xi(t), \infty)$ for $t \in [0, T)$. Hence $u(x, t)$ is decreasing in $x \in (\xi(t), \infty)$ for every $t \in [0, T)$. Then by the comparison principle, any singular solution $\tilde{u}$ satisfies

$$u(x + \mu, t) < \tilde{u}(x, t) < u(x - \mu, t), \qquad x \in [\xi(t) + \mu, \infty),\ t \in [0, T),$$

where $\mu > 0$ is an arbitrarily small constant. Letting $\mu \downarrow 0$, we obtain $\tilde{u}(x, t) \equiv u(x, t)$ for $x \in (\xi(t), \infty)$ and $t \in [0, T)$. This proves the uniqueness. □

Next, we consider the case where the initial value $u_0(x)$ satisfies (U1), (U2), (U3) or (U3'). We need to modify the above proof when $u_0(x)$ is increasing for some $x \in (\xi(0), \infty)$. To this end, we show that the minimal singular solution of (1.6) inherits the properties (U1), (U2), (U3) and (U3').

Lemma 5.1 *Assume that $\xi(t)$ is nondecreasing in $t \in (0, T)$. Then the minimal singular solution of (1.6) has the following properties:*

(i) If $u_0(x)$ satisfies (U1)*, then $u_t(x, t) > -C_0$ for all $x \in (\xi(t), \infty)$ and $t \in (0, T)$.*
(ii) If $u_0(x)$ satisfies (U2)*, then there exist $\tau \in (0, T]$ and $\delta_0 > 0$ such that $u_x(x, t) < 0$ for $x \in (\xi(t), x_1 + \delta_0)$ and $t \in (0, \tau)$.*
(iii) If $u_0(x)$ satisfies (U3)*, then there exists $\tau \in (0, T]$ and $\delta_0 > 0$ such that $u_x(x, t) < 0$ for $x \in (x_2 - \delta_0, \infty)$ and $t \in (0, \tau)$.*
(iv) If $u_0(x)$ satisfies (U3')*, then $u(x, t) > C_1$ for all $x \in (\xi(t), \infty)$ and $t \in (0, T)$.*

Proof Let $u(x,t)$ be the minimal solution of (1.6). For each $k \in \mathbb{N}$, let $\tilde{u}_k(x,t)$ be a solution of (1.2) satisfying $\tilde{u}_k(\xi(t),t) = k$ for $t > 0$ and

$$\tilde{u}_k(x,0) := \begin{cases} k - A_k\{x - \xi(0)\}, & x \in (\xi(0), \xi_k], \\ u_0(x), & x \in (\xi_k, \infty), \end{cases}$$

where $A_k > 0$ is a constant defined by

$$A_k = \min\{A \in \mathbb{R} : k - A\{x - \xi(0)\} = u_0(x) \text{ for all } x > \xi(0)\} > 0$$

and ξ_k is defined by

$$\xi_k := \min\{\eta > 0 : k - A\{\eta - \xi(0)\} = u_0(\eta)\}.$$

Let $U^+(x,t;a,b)$ be the supersolution given by (3.1) with $\xi(T) < a < b$ such that $U^+(x,0;a,b) > u_0(x)$ for $x \in (a,b)$. If k is sufficiently large, we can define $x_k(t) > \xi(T)$ uniquely by

$$U^+(x_k(t),t;a,b) = k, \qquad (U^+)_x(x_k(t),t;a,b) < 0, \qquad t \in [0,T).$$

Then by Lemma 2.10 with (C3) and Remark 2.11 (ii), $\tilde{u}_k(x,t)$ satisfies $\tilde{u}_k(x,t) \le k$ for $x \in [\xi(t), x_k(t)]$ and $\tilde{u}_k(x,t) < U^+(x,t;a,b)$ for $x \in [x_k(t), b)$. This implies that $\tilde{u}_k(x,t)$ has a local maximum at $x = \xi(t)$. Hence we obtain $(\tilde{u}_k)_x(\xi(t),t) \le 0$ for $t \in (0,T)$.

Given any $n \in \mathbb{N}$, we take $k > n$ such that $\tilde{u}_k(x,0) \ge u_n(x,0)$ for $x > \xi(0)$ and $(\tilde{u}_k)_x(\xi(t),t) \le 0$ for $t \in (0,T)$. Then by Lemma 2.10 with (C2), $\tilde{u}_k$ satisfies $u_n(x,t) \le \tilde{u}_k(x,t) < u(x,t)$ for $x > \xi(t)$ and $t > 0$. Since $u(x,t)$ is minimal, we have $\tilde{u}_k(x,t) \to u(x,t)$ as $k \to \infty$. On the other hand, by differentiating (1.2) and $\tilde{u}_k(\xi(t),t) = k$ by t, we have $\{(\tilde{u}_k)_t\}_t = m\{(\tilde{u}_k)^{m-1}(\tilde{u}_k)_t\}_{xx}$ and

$$0 = \frac{d}{dt}\{\tilde{u}_k(\xi(t),t)\} = (\tilde{u}_k)_t(\xi(t),t) + (\tilde{u}_k)_x(\xi(t),t)\xi'(t) \le (\tilde{u}_k)_t(\xi(t),t),$$

respectively. Hence $w_k(x,t) := (\tilde{u}_k)_t(x,t)$ satisfies

$$\begin{aligned} &(w_k)_t = m\{(\tilde{u}_k)^{m-1} w_k\}_{xx}, && x \in (\xi(t),\infty),\ t \in (0,T), \\ &w_k(\xi(t),t) \ge 0, && t \in (0,T), \\ &w_k(x,0) = \{\tilde{u}_k(x,0)^m\}'' \ge -C_0, && x \in (\xi(0),\infty), \end{aligned}$$

where (U1) is used for the last inequality. Then the maximum principle implies

$$w_k(x,t) = (\tilde{u}_k)_t(x,t) \ge -C_0, \qquad x \in (\xi(t),\infty),\ t \in (0,T).$$

Taking the limit as $k \to \infty$, we obtain $u_t(x,t) \ge -C_0$. Then the maximum principle implies (i).

Next, let $U^+(x,t;a,b)$ be the supersolution given by (3.1) with $\xi(0) < a < b < x_1$ such that $U^+(x,0;a,b) > u_0(x)$ for $x \in (a,b)$. We take $\tau \in (0,T]$ such that $\xi(\tau) < a$, and take $M > 0$ and $\tilde{x}_1 \in (a,b)$ such that

$$M > \min_{x\in(a,b)} U^+(x,\tau;a,b), \quad U^+(\tilde{x}_1,\tau;a,b) = M, \quad U_x^+(\tilde{x}_1,\tau;a,b) < 0. \tag{5.1}$$

Let $u_n(x,t)$ be the solution of (1.7) with sufficiently large n. By (U2), $u_n(x,0)$ crosses the level line $u \equiv M$ exactly once. Moreover, for $t \in (0,\tau)$, $u_n(x,t)$ satisfies $u_n(x,t) > M$ if $x - \xi(t) > 0$ is sufficiently small and $u_n(\tilde{x}_1,t) < U^+(\tilde{x}_1,t;a,b) < M$. Since the intersection number of $u_n(x,t)$ with $u \equiv M$ in $(\xi(t),\tilde{x}_1)$ is nonincreasing in $t > 0$, $u_n(x,t)$ intersects with $u \equiv M$ exactly once in $(\xi(t),\tilde{x}_1)$ for every $t \in [0,\tau)$. (See [1] for the nonincreasing principle of the intersection number.) This property holds for any M satisfying (5.1). Hence $u_n(x,t)$ is decreasing in $x \in (\xi(t),\tilde{x}_1]$ for every large n and $t \in [0,\tau)$. Taking the limit as $n \to \infty$, we conclude that the minimal solution $u(x,t)$ is nonincreasing in $x \in (\xi(t),\tilde{x}_1]$ for $t \in [0,\tau)$. Then the maximum principle implies (ii). The assertion (iii) can be proved in the same manner.

Finally, since $u \equiv C_1$ satisfies (1.2), the maximum principle implies (iv). □

We say that a singular solution of (1.6) is "maximal" if it is larger than any other solutions. Let us show that the minimal singular solution of (1.6) is also maximal for small $t > 0$.

Lemma 5.2 *Assume that $\xi(t)$ is nondecreasing in $t \in (0,T)$. If $u_0(x)$ satisfies* (U1), (U2), (U3) *or* (U3'), *then the minimal singular solution of (1.6) is maximal for small* $t > 0$.

Proof First we consider the case where (U1), (U2) and (U3) are satisfied. We shall construct supersolutions of (1.6) in an inner region $(\xi(t), x_1+\delta)$, an outer region $(x_2-\delta,\infty)$ and an intermediate region $(x_1-\delta, x_2+\delta)$ separately, and glue them at some points in $(x_1-\delta,x_1+\delta)$ and $(x_2-\delta,x_2+\delta)$.

Let $u(x,t)$ be the minimal solution of (1.6), and let $\tau > 0$, $C_0 \geq 0$, x_1 and x_2 be as in Lemma 5.1. We take $\delta > 0$ smaller if necessary so that Lemma 5.1 (ii) and (iii) hold for $\delta_0 = \delta$. We take a constant $C_1 > 0$ such that

$$u(x,t) > C_1, \qquad (x,t) \in (x_1-\delta, x_2+\delta) \times (0,\tau), \tag{5.2}$$

and set

$$C_2 := \max_{s\in[0,1]} \frac{(1+s)^{\frac{1}{m}-1} - 1}{s} = \max\left\{\frac{1}{m} - 1, 2^{\frac{1}{m}-1} - 1\right\} > 0. \tag{5.3}$$

Let $d := (x_1+x_2)/2$, and define

$$\psi(x,t) := \sigma e^{pt}\cosh(q(x-d)), \qquad (x,t) \in (\xi(0),\infty) \times (0,\infty),$$

where $\sigma, p, q > 0$ are constants. In the intermediate region, we define

$$u_{int}(x,t) := \left\{u(x,t)^m + \psi(x,t)\right\}^{\frac{1}{m}}, \qquad (x,t) \in (x_1-\delta, x_2+\delta) \times [0,\tau). \tag{5.4}$$

In the inner region and the outer region, we define

$$\begin{aligned} u_{in}(x,t) &: = u(x-\mu_1, t), \qquad x \in (\xi(t), x_1+\delta),\ t \in (0,\tau), \\ u_{out}(x,t) &: = u(x-\mu_2, t), \qquad x \in (x_2-\delta, \infty),\ t \in (0,\tau), \end{aligned}$$

where $\mu_1, \mu_2 > 0$ are constants. We note that $u_{in}^+(x,t)$ and $u_{out}^+(x,t)$ satisfy (1.1). Let $\varepsilon > 0$ be a small parameter. Then by (U2), $\mu_1, \mu_2 > 0$ can be determined uniquely by

$$u_0(x_1-\mu_1)^m - u_0(x_1)^m = \varepsilon = u_0(x_2-\mu_2)^m - u_0(x_2)^m, \tag{5.5}$$

and satisfy $\mu_1, \mu_2 \downarrow 0$ as $\varepsilon \downarrow 0$.

Later, it will be shown that by imposing some appropriate conditions on u_{in}, u_{int} and u_{out}, the following matching conditions are satisfied at some points $\gamma_1(t) \in (x_1-\delta, x_1+\delta)$ and $\gamma_2(t) \in (x_2-\delta, x_2+\delta)$:

$$u_{in}(\gamma_1(t), t) = u_{int}(\gamma_1(t), t), \tag{5.6}$$
$$u_{int}(\gamma_2(t), t) = u_{out}(\gamma_2(t), t), \tag{5.7}$$

respectively. Then we can define a function

$$u^+(x,t) := \begin{cases} u_{in}(x,t), & x \in (\xi(t), \gamma_1(t)), \\ u_{int}(x,t), & x \in [\gamma_1(t), \gamma_2(t)], \\ u_{out}(x,t), & x \in (\gamma_2(t), \infty), \end{cases}$$

that is continuous in $x \in (\xi(t), \infty)$ for $t \in [0,\tau)$.

We shall show that $u^+(x,t)$ becomes a supersolution if $\tau, \delta, \sigma, p, q > 0$ satisfy the following conditions:

$$\frac{pC_1^{1-m}}{m} - q^2 > C_0 C_1^{-m} C_2, \tag{5.8}$$
$$C_1^{-m}\sigma e^{p\tau}\cosh(q(x_1-d-\delta)) < 1, \tag{5.9}$$
$$\sigma\cosh(q(x_1-d)) = \varepsilon, \tag{5.10}$$
$$-q\sqrt{\varepsilon^2-\sigma^2} < \{(u_0)^m\}'(x_1-\mu_1) - \{(u_0)^m\}'(x_1), \tag{5.11}$$
$$\sigma\cosh(q(x_2-d)) = \varepsilon, \tag{5.12}$$
$$q\sqrt{\varepsilon^2-\sigma^2} > \{(u_0)^m\}'(x_2-\mu_2) - \{(u_0)^m\}'(x_2). \tag{5.13}$$

If $\varepsilon > 0$ is sufficiently small, then we can find such constants in the following order:

- We take $\sigma \in (0, \varepsilon)$ and $q > 0$ satisfying (5.10). Then (5.12) also holds by $d - x_1 = x_2 - d$.
- We take $q > 0$ larger and $\sigma > 0$ smaller to satisfy (5.11) and (5.13).
- We take $p > 0$ satisfying (5.8).
- We take $\delta > 0$ smaller and $\tau > 0$ smaller to satisfy (5.9). (Recall that C_0, C_1, C_2 can be taken independently of small $\tau > 0$ and $\delta > 0$.)

Now let us consider the intermediate region. By (5.4), we have

$$
\begin{aligned}
(u_{int})_t - \{(u_{int})^m\}_{xx} &= \frac{1}{m}(u^m + \psi)^{\frac{1}{m}-1}\{mu^{m-1}u_t + \psi_t\} - (u^m)_{xx} - \psi_{xx} \\
&= \{(u^m + \psi)^{\frac{1}{m}-1}u^{m-1} - 1\}u_t + \frac{1}{m}(u^m + \psi)^{\frac{1}{m}-1}\psi_t - \psi_{xx}.
\end{aligned}
$$

The first term in the right-hand side is estimated as follows. By (5.2) and (5.9), we have

$$
u^{-m}\psi < C_1^{-m}\sigma e^{p\tau}\cosh(q(x_1 - d - \delta)) < 1, \qquad (x, t) \in [x_1 - \delta, x_2 + \delta] \times [0, \tau),
$$

so that

$$
(u^m + \psi)^{\frac{1}{m}-1}u^{m-1} - 1 = (1 + \psi u^{-m})^{\frac{1}{m}-1} - 1 > 0.
$$

Hence by Lemma 5.1 (i), we obtain

$$
\{(u^m + \psi)^{\frac{1}{m}-1}u^{m-1} - 1\}u_t \geq -C_0\{(u^m + \psi)^{\frac{1}{m}-1}u^{m-1} - 1\}.
$$

Moreover, by $u^{-m}\psi < 1$ and (5.3), we have

$$
(u^m + \psi)^{\frac{1}{m}-1}u^{m-1} - 1 = (1 + u^{-m}\psi)^{\frac{1}{m}-1} - 1 < C_2u^{-m}\psi < C_1^{-m}C_2\psi.
$$

Hence the first term is estimated as

$$
\{(u^m + \psi)^{\frac{1}{m}-1}u^{m-1} - 1\}u_t > -C_0C_1^{-m}C_2\psi.
$$

On the other hand, the second term satisfies

$$
\frac{1}{m}(u^m + \psi)^{\frac{1}{m}-1}\psi_t = \frac{p}{m}(u^m + \psi)^{\frac{1}{m}-1}\psi > \frac{p}{m}u^{1-m}\psi > \frac{pC_1^{1-m}}{m}\psi,
$$

and the third term satisfies $-\psi_{xx} = -q^2\psi$. Thus by (5.8), we obtain

$$
(u_{int})_t - \{(u_{int})^m\}_{xx} > \left(-C_0C_1^{-m}C_2 + \frac{pC_1^{1-m}}{m} - q^2\right)\psi > 0.
$$

Next, we consider the inner region. By (5.5) and (5.10), we have

$$
\begin{aligned}
u_{int}(x_1, 0) &= \{u_0(x_1)^m + \psi(x_1, 0)\}^{\frac{1}{m}} \\
&= \{u_0(x_1)^m + \sigma \cosh(q(x_1 - d))\}^{\frac{1}{m}} = \{u_0(x_1)^m + \varepsilon\}^{\frac{1}{m}} \\
&= u_0(x_1 - \mu_1) = u_{in}(x_1, 0).
\end{aligned}
$$

Hence the matching condition (5.6) is satisfied at $t = 0$ if $\gamma_1(0) = x_1$. On the other hand, by (5.10), (5.11) and $q(x_1 - d) < 0$, we have

$$
\begin{aligned}
(u^m_{int})_x(x_1, 0) &= \{(u_0)^m\}'(x_1) + \psi_x(x_1, 0) = \{(u_0)^m\}'(x_1) + \sigma q \sinh(q(x_1 - d)) \\
&= \{(u_0)^m\}'(x_1) - q\sqrt{\sigma^2\{\cosh^2(q(x_1 - d)) - 1\}} \\
&= \{(u_0)^m\}'(x_1) - q\sqrt{\varepsilon^2 - \sigma^2} \\
&< \{(u_0)^m\}'(x_1 - \mu_1) = (u^m_{in})_x(x_1, 0).
\end{aligned}
$$

Hence we obtain

$$
(u_{in})_x(x_1, 0) > (u_{int})_x(x_1, 0).
$$

These imply that $u_{in}(x, 0)$ and $u_{int}(x, 0)$ intersect transversally at x_1. Hence $u_{in}(x, t)$ and $u_{int}(x, t)$ also intersect transversally near x_1 if $t > 0$ is small. More precisely, if $\tau > 0$ is sufficiently small, then there exists a continuous function $\gamma_1(t) \in (x_1 - \delta, x_1 + \delta)$ such that

$$
u_{in}(\gamma_1(t), t) = u_{int}(\gamma_1(t), t), \quad (u_{in})_x(\gamma_1(t), t) > (u_{int})_x(\gamma_1(t), t), \qquad t \in [0, \tau).
$$

Finally, we consider the outer region. In the same manner as the inner region, we have $u_{int}(x_2, 0) = u_{out}(x_2, 0)$ by (5.5) and (5.12), and $(u_{int})_x(x_2, 0) > (u_{out})_x(x_2, 0)$ by (5.12) and (5.13). Hence for sufficiently small $\tau > 0$, there exists a continuous function $\gamma_2(t) \in (x_2 - \delta, x_2 + \delta)$ satisfying the matching condition (5.7) together with

$$
(u_{int})_x(\gamma_2(t), t) > (u_{out})_x(\gamma_2(t), t), \qquad t \in [0, \tau).
$$

Thus it is shown that $u^+(x, t)$ is a supersolution of (1.6) for $t \in [0, \tau)$. Hence any other singular solution $\tilde{u}$ of (1.6) must satisfy

$$
u(x, t) \le \tilde{u}(x, t) \le u^+(x, t), \qquad x \in (\xi(t), \infty), \; t \in (0, \tau).
$$

Here $u^+(x, t) \downarrow u(x, t)$ as $\varepsilon \downarrow 0$ uniformly in $x \in (\xi(t) + \rho, \infty)$ and $t \in (0, \tau)$, where $\rho > 0$ is an arbitrary constant. Hence we conclude that $u(x, t)$ is maximal for small $t > 0$.

In the case where (U3') is assumed instead of (U3), we define $\psi(x,t) := \sigma e^{pt-qx}$ and

$$u^+(x,t) := \begin{cases} u_{in}(x,t), & x \in (\xi(t)+\mu_1, \gamma_1(t)],\ t \in [0,\tau). \\ u_{int}(x,t), & x \in (\gamma_1(t), \infty),\ t \in ([0,\tau). \end{cases}$$

Then the maximality can be proved in the same way as above. □

Now we are in a position to prove the uniqueness.

Proof of Theorem 2.4 By Lemma 5.2, the minimal singular solution of (1.6) is maximal for $t \in (0,\tau)$, which implies the uniqueness at least for small $t > 0$. Suppose that the singular solution is unique for $t \in (0, t_m]$ but is not unique for $t > t_m$. By Lemma 5.1, the minimal solution $u(x,t)$ inherits the properties (U1), (U2), (U3) and (U3') for every $t \in (0,T)$. Hence by applying Lemma 5.2 to the initial value $u(x,t_m)$, it is shown that the singular solution of (1.6) is unique if $t - t_m > 0$ is sufficiently small. This contradiction proves the uniqueness of a singular solution of (1.6) for $t \in (0,T)$. □

6 Burning core

In this section, we show the appearance of a burning core by assuming that $\xi(t) < \xi(0)$ for $t \in (0,T)$.

Proof of Theorem 2.5 We assume $0 < T < \infty$ without loss of generality, and define

$$U(x,t) := (T-t)^{\frac{1}{1-m}} g(x),$$

where g is a positive C^2-function. Since

$$U_t = -\frac{1}{1-m}(T-t)^{\frac{m}{1-m}} g(x), \qquad (U^m)_{xx} = (T-t)^{\frac{m}{1-m}} \left\{g(x)^m\right\}'',$$

$u = U(x,t)$ satisfies (1.1) if g satisfies the equation

$$(g^m)'' + \frac{1}{1-m} g = 0. \tag{6.1}$$

For this equation, we impose the initial condition

$$g(\xi(0)) = k \in \mathbb{N}, \qquad g'(\xi(0)) = -C_k < 0,$$

where $C_k >$ is a (large) constant such that

$$u_0(x) > T^{\frac{1}{1-m}}\left\{k - C_k(x - \xi(0))\right\}, \qquad x \in (\xi(0), \xi(0) + k/C_k).$$

Since g^m is concave by (6.1) and $(g^m)'(\xi(0)) < 0$, $g(x)$ must vanish at some $x > \xi(0)$. On the other hand, integrating (6.1) on $[x, \xi(0)]$, we obtan

$$(g^m)'(x) = (g^m)'(\xi(0)) + \frac{1}{1-m}\int_x^{\xi(0)} g(s)ds.$$

Since $g(s) > g(\xi(0)) = k$ for $s \in (x, \xi(0))$ if $g'(s) < 0$ for $s \in (x, \xi(0))$, the right-hand side of this equality must vanish at some $x < \xi(0)$. Hence there exist $a_k, b_k \in \mathbb{R}$ with $a_k < \xi(0) < b_k$ such that $g(x)$ is decreasing in $x \in (a_k, b_k)$, $g'(a_k) = 0$ and $g(b_k) = 0$. Now we define a subsolution of (1.1) by

$$u_k^-(x,t) := \begin{cases} (T-t)^{\frac{1}{1-m}} g(a_k), & (x,t) \in (-\infty, a_k] \times (0,T), \\ (T-t)^{\frac{1}{1-m}} g(x), & (x,t) \in (a_k, b_k) \times (0,T), \\ 0, & (x,t) \in [b_k, \infty) \times (0,T). \end{cases}$$

Let u_n be the solution of (1.7) with $n \geq T^{\frac{1}{1-m}} g(a_k)$. Then we have $u_k^-(x,0) \leq u_n(x,0)$ for $x \in (\xi(0), \infty)$ and $u_k^-(\xi(t),t) \leq n = u_n(\xi(t),t)$ for $t \in (0,T)$. Hence by the comparison principle, we obtain

$$u_k^-(x,t) \leq u_n(x,t), \qquad x \in (\xi(t), \infty),\ t \in (0,T). \tag{6.2}$$

Here, since $u_k^-(x,t)$ is nonincreasing in $x \in \mathbb{R}$, we have

$$u_k^-(x,t) \geq (T-t)^{\frac{1}{1-m}} g(\xi(0)) = (T-t)^{\frac{1}{1-m}} k, \qquad (x,t) \in (-\infty, \xi(0)] \times [0,T).$$

Hence $u_k^-(x,t) \to \infty$ as $k \to \infty$ uniformly in $x \in (-\infty, \xi(0)]$ for every $t \in (0,T)$. Since $\{u_n\}$ is increasing in n, we see from (6.2) that $u_n(x,t) \to \infty$ as $n \to \infty$ uniformly in $x \in (\xi(t), \xi(0)]$ for every $t \in (0,T)$. This proves (i).

Let $\{\tilde{u}_n\}$ be a sequence of solutions of (1.7) with $\xi(t)$ replaced by $\tilde{\xi}(t) \equiv \xi(0)$, and let $\tilde{u}(x,t)$ be the corresponding minimal singular solution of (1.2) with a standing singularity at $\xi(0)$. Fix $\tau \in (0,T)$ arbitrarily, and take $j, k, n \in \mathbb{N}$ with $j \leq (T-\tau)^{\frac{1}{1-m}} k$ and $n \geq T^{\frac{1}{1-m}} k$. Then we have

$$\begin{aligned} \tilde{u}_j(\xi(0),t) = j &\leq (T-\tau)^{\frac{1}{1-m}} k \leq (T-t)^{\frac{1}{1-m}} k \\ &= u_k^-(\xi(0),t) \leq n = u_n(\xi(0),t), \qquad t \in [0,\tau]. \end{aligned}$$

Hence, by the comparison principle, we obtain

$$\tilde{u}_j(x,t) \leq u_n(x,t) \leq \tilde{u}(x,t), \qquad (x,t) \in [\xi(0), \infty) \times [0,\tau].$$

Here, letting $j \to \infty$, we see that $u_n(x,t) \to \tilde{u}(x,t)$ as $n \to \infty$ uniformly in $x \in [\xi(0)+\rho, \infty)$ for every $t \in [0,\tau]$, where $\rho > 0$ is an arbitrary constant. Since $\tau \in (0,T)$ is arbitrary, the proof of (ii) is complete. □

7 Traveling semi-wavefronts

In this section, we consider the existence of a two-parameter family of traveling semi-wavefronts and their properties.

Proof of Theorem 2.7 If $u = \varphi(z) > 0$, $z = x - ct$, satisfies (1.2) with $\xi(t) = ct$, then φ must satisfy

$$(\varphi^m)'' + c\varphi' = 0, \qquad z > 0. \tag{7.1}$$

Suppose that this equation has a decreasing solution such that $\varphi(z) \to \infty$ as $z \downarrow 0$ and $(\varphi(z), \varphi'(z)) \to (L, 0)$ as $z \to \infty$. Then, integrating (7.1) on (z, ∞), we have

$$(\varphi^m)' + c\varphi = cL, \qquad z > 0. \tag{7.2}$$

Rewriting this equation as

$$\frac{\varphi'}{\varphi^{1-m}(\varphi - L)} + \frac{c}{m} = 0, \tag{7.3}$$

and integrating this on $(0, z)$, we obtain

$$\int_{\varphi(z)}^{\infty} \frac{1}{s^{1-m}(s-L)} ds = \frac{cz}{m}. \tag{7.4}$$

Since the function

$$F(\varphi) := \int_{\varphi}^{\infty} \frac{1}{s^{1-m}(s-L)} ds$$

satisfies $F(\varphi) \to \infty$ as $\varphi \downarrow L$, $F(\varphi) \to 0$ as $\varphi \to \infty$ and $F'(\varphi) < 0$, it follows from the implicit function theorem that a smooth positive function $\varphi(z)$ from $(0, \infty)$ to (L, ∞) is defined by (7.4), and φ satisfies (7.1), (7.2) and $\varphi'(z) < 0$ for $z > 0$. Moreover, it is easy to see from (7.3) and (7.4) that $\varphi(z) \to \infty$ as $z \downarrow 0$ and $(\varphi(z), \varphi'(z)) \to (L, 0)$ as $z \to \infty$. Thus it is shown that there exists a unique φ satisfying (i) and (ii). The properties of φ in (iv) are easily derived from (7.4).

It remains to prove (iii). By (7.3) and $\varphi(z) > L \geq 0$, we have

$$\frac{\{\varphi(z) - L\}'}{\{\varphi(z) - L\}^{2-m}} \leq -\frac{c}{m}.$$

Integrating this on $(0, z]$, we have

$$-\frac{1}{(1-m)\{\varphi(z) - L\}^{1-m}} < -\frac{cz}{m},$$

so that

$$\varphi(z) \le \left\{ \frac{c(1-m)}{m} z \right\}^{-\frac{1}{1-m}} + L = h(c) z^{-\frac{1}{1-m}} + L, \qquad z > 0.$$

Similarly, integrating

$$\frac{\varphi'(z)}{\varphi(z)^{2-m}} \ge -\frac{c}{m}$$

on $(0, z]$, we obtain

$$\varphi(z) \ge \left\{ \frac{c(1-m)}{m} z \right\}^{-\frac{1}{1-m}} = h(c) z^{-\frac{1}{1-m}}, \qquad z > 0.$$

Since $\varphi(z) > L$, (iii) is proved. □

8 Large-time behavior

In this section, we shall show that all singular solutions of (1.2) attract each other if they have a common limit L as $x \to \infty$. This particularly means that the large-time behavior of singular solutions is independent of initial values, and is determined only by $\xi(t)$ and L. In order to prove this, we introduce a moving frame $(z, t) = (x - \xi(t), t)$, and rewrite (1.2) as

$$v_t = (v^m)_{zz} + \xi'(t) v_z, \qquad z > 0, \tag{8.1}$$

where $v(z, t) := u(z + \xi(t), t)$. In the following, we frequently use the following lemma.

Lemma 8.1 *Let $\varphi(z; c, L)$ be the function as in Theorem 2.7.*

(i) Assume that $\xi'(t) \ge c$ for $t \in [0, T)$. Then for any constants $a, b \ge 0$,

$$V^+(z) := \left\{ \varphi(z-a; c, L)^m + b \right\}^{\frac{1}{m}}, \qquad z \in (a, \infty),\ t \in [0, T),$$

is a supersolution of (8.1).

(ii) Assume that $0 < \xi'(t) \le c$ for $t \in [0, T)$. Then for any constants $a, b \ge 0$,

$$V^-(z) := \left\{ \max\{\varphi(z+a; c, L)^m - b, 0\} \right\}^{\frac{1}{m}}, \qquad z \in (0, \infty),\ t \in [0, T),$$

is a subsolution of (8.1).

Proof By $\xi'(t) \geq c$ and $\varphi' < 0$, we have

$$\begin{aligned}\left\{(V^+)^m\right\}_{zz} + \xi'(t)(V^+)_z &= (\varphi^m)'' + \xi'(t)\left\{(\varphi^m + b)^{\frac{1}{m}}\right\}_z \\ &= -c\varphi' + \xi'(t)(1 + b\varphi^{-m})^{\frac{1}{m}-1}\varphi' \\ &\leq -c(1 + b\varphi^{-m})^{\frac{1}{m}-1}\varphi' + \xi'(t)(1 + b\varphi^{-m})^{\frac{1}{m}-1}\varphi' \\ &= \{\xi'(t) - c\}(1 + b\varphi^{-m})^{\frac{1}{m}-1}\varphi' \leq 0.\end{aligned}$$

Hence V^+ is a supersolution of (8.1). Similar computations show that V^- is a subsolution of (8.1). □

Now we are in a position to prove Theorem 2.6.

Proof of Theorem 2.6 Let $L \geq 0$ and $c \in [c_1, c_2]$ be fixed, and let v^c be the minimal singular solution of (8.1) with the initial condition $v^c(z, 0) = \varphi(z; c, L)$. Since $c_1 \leq \xi'(t) \leq c_2$ by assumption, $\varphi(z - \varepsilon; c_1, L)$ and $\varphi(z + \varepsilon; c_2, L)$ with $\varepsilon > 0$ are a supersolution and a subsolution of (8.1), respectively. Moreover, by Theorem 2.7 (ii) and (iv), we have

$$\begin{aligned}&v^c(z, 0) = \varphi(z; c, L) < \varphi(z; c_1, L) < \varphi(z - \varepsilon; c_1, L), \qquad z \in (\varepsilon, \infty), \\ &v^c(z, 0) = \varphi(z; c, L) > \varphi(z; c_2, L) > \varphi(z + \varepsilon; c_2, L), \qquad z \in (0, \infty).\end{aligned}$$

Hence it follows from Lemma 2.10 with (C3) and Lemma 2.12 that

$$\begin{aligned}&v^c(z, t) < \varphi(z - \varepsilon; c_1, L), \qquad (z, t) \in (\varepsilon, \infty) \times [0, \infty), \\ &v^c(z, t) > \varphi(z + \varepsilon; c_2, L), \qquad (z, t) \in (0, \infty) \times [0, \infty).\end{aligned}$$

Letting $\varepsilon \downarrow 0$, we obtain

$$\varphi(z; c_2, L) \leq v^c(z, t) \leq \varphi(z; c_1, L), \qquad (z, t) \in (0, \infty) \times [0, \infty). \tag{8.2}$$

We define

$$v^+(z, t; \theta) := \left\{v^c(z - \theta, t)^m + \theta\right\}^{\frac{1}{m}}, \qquad z \in (\theta, \infty),\ t \in [0, \infty),$$

where $\theta > 0$ is a parameter. Since $\{v^c(z, 0)\}'' = \varphi''(z; c, L) > 0$ for all $z > 0$, it follows from Lemma 5.1 (i) with $C_0 = 0$ that $u^c(x, t) := v^c(x - \xi(t), t)$ satisfies

$$\begin{aligned}0 < (u^c)_t(x, t) &= (v^c)_t(x - \xi(t), t) - \xi'(t)(v^c)_z(x - \xi(t), t) \\ &= (v^c)_t(z, t) - \xi'(t)(v^c)_z(z, t), \qquad (z, t) \in (0, \infty) \times (0, \infty).\end{aligned}$$

Hence $v^+ = \left\{(v^c)^m + \theta\right\}^{\frac{1}{m}}$ satisfies

$$\begin{aligned}
&(v^+)_t - \{(v^+)^m\}_{zz} - \xi'(t)(v^+)_z \\
&\quad = \left\{(v^c)^m + \theta\right\}^{\frac{1}{m}-1}(v^c)^{m-1}(v^c)_t - \left\{(v^c)^m + \theta\right\}_{zz} \\
&\qquad - \xi'(t)\left\{(v^c)^m + \theta\right\}^{\frac{1}{m}-1}(v^c)^{m-1}(v^c)_z. \\
&\quad = \left[\left\{1 + \theta(v^c)^{-m}\right\}^{\frac{1-m}{m}} - 1\right]\left\{(v^c)_t - \xi'(t)(v^c)_z\right\} > 0.
\end{aligned}$$

Namely, $v^+(z, t; \theta)$ is a supersolution of (8.1).

Let v be any singular solution of (8.1) satisfying $v(z, 0) \to L$ as $z \to \infty$. If $\theta > 0$ is sufficiently large, then $0 < v(z, 0) < v^+(z, 0; \theta)$ for $z \in (\theta, \infty)$. Then by Lemma 2.10 with (C3), we obtain

$$0 < v(z, t) < v^+(z, t; \theta), \qquad (z, t) \in (\theta, \infty) \times [0, \infty).$$

Therefore, by (8.2), v satisfies

$$0 < v(z, t) < \left\{\varphi(z - \theta; c_1, L)^m + \theta\right\}^{\frac{1}{m}}, \qquad (z, t) \in (\theta, \infty) \times [0, \infty). \tag{8.3}$$

Here we define

$$\theta^+(t) := \inf \left\{\theta > 0 : v(z, t) \le v^+(z, t; \theta) \text{ for } z \in (\theta, \infty)\right\} \ge 0.$$

Then for each $t > 0$, we have

$$0 < v(z, t) \le v^+(z, t; \theta^+(t)), \qquad z \in (\theta^+(t), \infty). \tag{8.4}$$

Again by Lemma 2.10 with (C3), we have

$$v(z, t + \tau) \le v^+(z, t + \tau; \theta^+(t)), \qquad z \in (\theta^+(t), \infty),\ \ \tau \in [0, \infty),$$

which implies $\theta^+(t + \tau) \le \theta^+(t)$ for $\tau \ge 0$. Hence we conclude that $\theta(t)$ is nonincreasing in $t \in [0, \infty)$.

We shall show that $\theta^+(t) \downarrow 0$ as $t \to \infty$. Suppose on the contrary that $\theta^+(t) \downarrow \theta_\infty > 0$ as $t \to \infty$. For any $d > \theta_\infty$,

$$V^-(z) := \left\{\max\{\varphi(z - \theta_\infty; c, L)^m - d, 0\}\right\}^{\frac{1}{m}}, \qquad (z, t) \in (\theta_\infty, \infty) \times (0, \infty),$$

is a subsolution of (8.1) and satisfies $V^-(z) \le v^+(z, 0; \theta_\infty)$ for $z \in (\theta_\infty, \infty)$. Hence by Lemma 2.12, we have

$$V^-(z) \le v^+(z, t; \theta_\infty), \qquad (z, t) \in (\theta_\infty, \infty) \times [0, \infty).$$

On the other hand, by Theorem 2.7 (iii), we can take $x_1, x_3 > 0$ such that $0 < x_1 < \theta_\infty < x_3$ and

$$v(z,0) \le V_1^+(z) := \left\{\varphi(z-x_1;c,L)^m + x_3\right\}^{\frac{1}{m}}, \qquad z \in (x_1,\infty).$$

Then there exists $\rho_1 \in (x_1,\infty)$ such that

$$v(z,t) \le V_1^+(z) < V^-(z) \le v^+(z,t;\theta_\infty), \qquad (z,t) \in (\theta_\infty,\rho_1) \times [0,\infty). \tag{8.5}$$

Similarly, we can take $x_2, x_4 > 0$ such that $0 < x_4 < \theta_\infty < x_2$ and

$$v(z,0) < V_2^+(z) := \left\{\varphi(z-x_2;c,L)^m + x_4\right\}^{\frac{1}{m}}, \qquad z \in (x_2,\infty).$$

Then there exists $\rho_2 \in (x_2,\infty)$ such that

$$v(z,t) \le V_2^+(z) < V^-(z) \le v^+(z,t;\theta_\infty), \quad (z,t) \in (\rho_2,\infty) \times [0,\infty). \tag{8.6}$$

Since $\theta_\infty \le \theta^+(t)$, by (8.5) and (8.6), there exists $z^+(t) \in [\rho_1,\rho_2]$ such that

$$v(z^+(t),t) = v^+(z^+(t),t;\theta^+(t)) \tag{8.7}$$

at some $z^+(t) \in [\rho_1,\rho_2]$. Moreover, we see from (8.4) that $v(z,t)$ and $v^+(z,t;\theta^+(t))$ must be tangent at $z = z^+(t) \in [\rho_1,\rho_2]$.

Now, by the uniform continuity of $\xi'(t)$, we can apply the Ascoli–Arzelà theorem to show that there exists an increasing sequence $\{t_i\}$ such that $t_i \to \infty$ and $\xi'(t+t_i) \to \xi'_\infty(t)$ as $i \to \infty$ uniformly in $t \in [0,1]$, where $\xi_\infty(t)$ is a differentiable function of $t \in [0,1]$. Let $[\sigma_1,\sigma_2]$ be an interval with $0 < \sigma_1 < \rho_1 < \rho_2 < \sigma_2$. By (8.2) and (8.3), $v^c(z,t)$, $v^c_z(z,t)$, $v(z,t)$ and $v_z(z,t)$ are bounded for $(z,t) \in [\sigma_1,\sigma_2] \times [0,\infty)$. Hence there exists a subsequence of $\{t_i\}$ (still denoted by $\{t_i\}$) such that $v^+(z,t+t_i;\theta^+(t))$ and $v(z,t+t_i)$ converge to their limiting functions $v^+_\infty(z,t)$ and $v_\infty(z,t)$, respectively, as $i \to \infty$ uniformly in $(z,t) \in [\sigma_1,\sigma_2] \times [0,1]$. Then the standard parabolic regularity implies that $v_\infty(z,t)$ and $v^+_\infty(z,t)$ satisfy

$$(v_\infty)_t = \{(v_\infty)^m\}_{zz} + \xi'_\infty(t)(v_\infty)_z, \qquad (z,t) \in [\sigma_1,\sigma_2] \times [0,1],$$

and

$$(v^+_\infty)_t \ge \{(v^+_\infty)^m\}_{zz} + \xi'_\infty(t)(v^+_\infty)_z, \qquad (z,t) \in [\sigma_1,\sigma_2] \times [0,1],$$

respectively. Furthermore, by (8.4) and (8.7), $v_\infty(z,0)$ and $v^+_\infty(z,0)$ satisfy $v_\infty(z,t) \le v^+_\infty(z,t)$ for $z \in [\sigma_1,\sigma_2]$ and are tangent at some $z \in [\rho_1,\rho_2]$ for every $t \in [0,1]$. Since $v_\infty(z,t) < v^+_\infty(z,t)$ for $z \in [\sigma_1,\rho_1) \cup (\rho_2,\sigma_2]$ by (8.5) and (8.6), the maximum principle implies

$$v_\infty(z,t) < v^+(z,t;\theta_\infty), \qquad (z,t) \in [\rho_1,\rho_2] \times (0,1].$$

Then by continuity, if i is sufficiently large, then

$$v(z, t + t_i) < v^+(z, t + t_i; \theta^+(t + t_i)), \qquad (z, t) \in [\rho_1, \rho_2] \times [0, 1].$$

However, this contradicts the fact that $v(z, t_i + t)$ and $v^+(z, t_i + t; \theta^+(t_i + t))$ are tangent at some $z \in [\rho_1, \rho_2]$. Thus it is shown that $\theta^+(t) \downarrow 0$ and $v^+(z, t; \theta^+(t)) \to v^c(z, t)$ as $t \to \infty$ uniformly in $[\rho, \infty)$, where $\rho > 0$ is an arbitrary constant.

Similarly, if we define

$$v^-(z, t; \theta) := \left\{ \max\{v(z + \theta, t)^m - \theta, 0\} \right\}^{\frac{1}{m}}, \qquad (z, t) \in (0, \infty) \times (0, \infty),$$

and

$$\theta^-(t) := \inf \left\{ \theta > 0 : v(z, t) > v^-(z, t) \text{ for } z \in (0, \infty) \right\} \in [0, \infty),$$

then we can show that $\theta^-(t) \downarrow 0$ as $t \to \infty$. Hence $v^-(z, t; \theta^-(t)) \to v^c(z, t)$ as $t \to \infty$ uniformly in $[\rho, \infty)$. Since

$$v^-(z, t; \theta^-(t)) \le v(z, t) \le v^+(z, t; \theta^+(t)), \qquad z \in (\theta^+(t), \infty),\ t \in [0, \infty),$$

we conclude that $v(z, t) \to v^c(z, t)$ as $t \to \infty$ uniformly in $z \in [\rho, \infty)$.

Now, let $u_1(x, t), u_2(x, t) > 0$ be singular solutions of (1.2) such that $u_1(x, 0)$, $u_2(x, 0) \to L \ge 0$ as $x \to \infty$, and set $v_1(z, t) := u_1(z + \xi(t), t)$ and $v_2(z, t) := u_2(z + \xi(t), t)$, respectively. Then we have

$$\begin{aligned} \left|u_1(x, t) - u_2(x, t)\right| = \left|v_1(z, t) - v_2(z, t)\right| &\le \left|v_1(z, t) - v^c(z, t)\right| \\ &+ \left|v_2(z, t) - v^c(z, t)\right| \to 0 \end{aligned}$$

as $t \to \infty$ uniformly in $x = z + \xi(t) \in [\xi(t) + \rho, \infty)$. This completes the proof of Theorem 2.6. □

Next, we consider the case where $\xi'(t) \to c$ as $t \to \infty$. Applying Theorem 2.6, we obtain the following proposition.

Proposition 8.2 *Assume that $\xi'(t)$ is nonnegative, bounded and uniformly continuous in $t \in [0, \infty)$, and that $u_0(x) \to L \ge 0$ as $x \to \infty$. Let $\varphi(z; c, L)$ be the function as in Theorem 2.7, and $\rho > 0$ be an arbitrary constant.*

(i) If $\limsup_{t\to\infty} \xi'(t) \le c$, then there exists $\alpha(t) > 0$ such that $\alpha(t) \to 0$ as $t \to \infty$ and the singular solution of (1.6) satisfies

$$u(x, t) \ge \varphi(x - \xi(t); c, L) - \alpha(t), \qquad x \in [\xi(t) + \rho, \infty),\ t \in (0, \infty).$$

(ii) If $\liminf_{t\to\infty} \xi'(t) \ge c$, then there exists $\beta(t) > 0$ such that $\beta(t) \to 0$ as $t \to \infty$ and the singular solution of (1.6) satisfies

$$u(x, t) \le \varphi(x - \xi(t); c, L) + \beta(t), \qquad x \in [\xi(t) + \rho, \infty),\ t \in (0, \infty).$$

Proof Given $\varepsilon > 0$ and $\rho > 0$, we take $z_0 > \rho$ such that $\varphi(z_0; c, L) < L + \varepsilon$. Then we have

$$\varphi(z; c+\delta, L) > L > \varphi(z_0; c, L) - \varepsilon > \varphi(z; c, L) - \varepsilon, \qquad z \in [z_0, \infty).$$

On the other hand, by Theorem 2.7 (iv), if $\delta > 0$ is sufficiently small, we have

$$\varphi(z; c+\delta, L) > \varphi(z; c, L) - \varepsilon, \qquad z \in [\rho, z_0].$$

Hence there exists $\delta > 0$ such that

$$\varphi(z; c+\delta, L) > \varphi(z; c, L) - \varepsilon, \qquad z \in [\rho, \infty). \tag{8.8}$$

By the assumption in (i), we can take $t_1 > 0$ such that $\xi'(t) \le c+\delta$ for $t \in [t_1, \infty)$, and denote by $u^c(x,t)$ a solution of (1.2) with $u^c(x, t_1) = \varphi(x - \xi(t_1); c, L)$. Since

$$u^-(x,t) := \varphi(x - \xi(t); c+\delta, L), \qquad x \in (\xi(t), \infty),\ t \in [t_1, \infty),$$

is a subsolution of (1.2), we have

$$u^c(x,t) \ge \varphi(x - \xi(t); c+\delta, L), \qquad x \in (\xi(t), \infty),\ t \in [t_1, \infty). \tag{8.9}$$

On the other hand, by Theorem 2.6, there exists $t_2 > t_1$ such that

$$|u(x,t) - u^c(x,t)| < \varepsilon, \qquad x \in (\xi(t)+\rho, \infty),\ t \in [t_2, \infty).$$

Hence by (8.8) and (8.9), we obtain

$$\begin{aligned} u(x,t) &\ge u^c(x,t) - \varepsilon \ge \varphi(x-\xi(t); c+\delta, L) - \varepsilon \\ &> \varphi(x - \xi(t); c, L) - 2\varepsilon, \qquad x \in (\xi(t)+\rho, \infty),\ t \in [t_2, \infty). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, (i) is proved.

Similarly, for any $\varepsilon > 0$ and $\rho > 0$, there exists a small $\delta > 0$ such that $\varphi(z; c - \delta, L) < \varphi(z; c, L) + \varepsilon$ for $z \in [\rho, \infty)$. By the assumption in (ii), we can take $t_1 > 0$ such that $\xi'(t) \ge c - \delta$ for $t \in [t_1, \infty)$. Then there exists $t_2 > t_1$ such that

$$\begin{aligned} u(x,t) &\le u^c(x,t) + \varepsilon \le \varphi(x-\xi(t); c-\delta, L) + \varepsilon \\ &< \varphi(x - \xi(t); c, L) + 2\varepsilon, \qquad x \in (\xi(t)+\rho, \infty),\ t \in [t_2, \infty). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, (ii) is proved. □

Proof of Theorem 2.8 First, we consider the case where $\xi'(t) \to c \in (0, \infty)$ as $t \to \infty$. By Proposition 8.2, the singular solution of (1.6) satisfies

$$\begin{aligned} &\varphi(x - \xi(t); c, L) - \alpha(t) \le u(x,t) \le \varphi(x - \xi(t); c, L) + \beta(t), \\ &\quad x \in [\xi(t)+\rho, \infty),\ t \in (0, \infty). \end{aligned}$$

where $\alpha(t), \beta(t) \to 0$ as $t \to \infty$ and $\rho > 0$ is an arbitrary constant. This proves Theorem 2.8 (i).

Next, we consider the case where $\xi'(t) \to 0$ as $t \to \infty$. Fix $c > 0$ arbitrarily. Let u be the singular solution of (1.6). For a function $\eta(t)$, let $\hat{u}$ be a solution of

$$\begin{cases} \hat{u}_t = (\hat{u}^m)_{xx}, & x > \eta(t),\ 0 < t < T, \\ u(x,0) = \hat{u}_0(x), & x > \eta(0) = \xi(0), \end{cases}$$

where $\hat{u}_0(x) \geq u_0(x)$ and $\hat{u}_0'(x) \leq 0$ for $x > \eta(0)$. Then $\hat{u}_x(x,t) < 0$ for all $x \in (\eta(t), \infty)$ and $t \in (0, \infty)$. If $\eta(0) = \xi(0)$ and $\eta'(t) \geq \xi'(t)$ for $t > 0$, then

$$u^-(x,t) := \hat{u}(x + \eta(t) - \xi(t), t)$$

is a a subsolution of (1.6), because

$$\begin{aligned} (u^-)_t - \{(u^-)^m\}_{xx} &= \{\eta'(t) - \xi'(t)\}\hat{u}_x + \hat{u}_t - (\hat{u}^m)_{xx} \\ &= \{\eta'(t) - \xi'(t)\}\hat{u}_x \leq 0, \qquad x \in (\xi(t), \infty),\ t \in (0, \infty). \end{aligned}$$

Moreover, if $\eta(t) \to c$ as $t \to \infty$, then by Proposition 8.2 (i), $\hat{u}$ satisfies

$$\hat{u}(x,t) \geq \varphi(x - \eta(t); c, L) - \alpha(t), \qquad x \in (\eta(t), \infty),\ t \in (0, \infty).$$

Thus we obtain

$$\begin{aligned} u(x,t) &\geq u^-(x,t) = \hat{u}(x + \eta(t) - \xi(t), t) \geq \varphi(x - \xi(t); c, L) \\ &\quad -\alpha(t), \quad x \in (\xi(t) + \rho, \infty),\ t \in (0, \infty). \end{aligned}$$

Here, by Theorem 2.7 (iii), $\varphi(z; c, L) \to \infty$ as $c \downarrow 0$ uniformly in $z \in [\rho, \infty)$, and $\alpha(t) \to 0$ by Proposition 8.2 (i). Since $c > 0$ is arbitrary and $\hat{u}(x,t)$ is decreasing in $x \in (\xi(t), \infty)$, the proof of Theorem 2.8 (ii) is complete.

Similarly, if $0 \leq \eta'(t) \leq \xi'(t)$ and $\eta'(t) \to c$ as $t \to \infty$, then

$$u^+(x,t) := \hat{u}(x + \eta(t) - \xi(t), t)$$

is a supersolution of (1.6) so that

$$u(x,t) \leq u^+(x,t) \leq \varphi(x - \xi(t); c, L) + \beta(t).$$

Since $c > 0$ is arbitrary, Theorem 2.7 (iii) and Proposition 8.2 (ii) imply Theorem 2.8 (iii). □

9 Entire solution

In this section, we prove the following proposition concerning the existence of an entire solution and its properties. (We note that Theorem 2.9 follows immediately from this proposition.)

Proposition 9.1 *Assume that $\xi(t)$ satisfies $c_1 \le \xi'(t) \le c_2$ for all $t \in \mathbb{R}$, where $c_1, c_2 > 0$ are constants. Then for each $L \ge 0$, there exists a positive solution of (1.2) defined for all $t \in \mathbb{R}$ with the following properties:*

(i) $u(x,t) \to \infty$ as $x \downarrow \xi(t)$ and $u(x,t) \to L$ as $x \to \infty$ for every $t \in \mathbb{R}$.
(ii) $u_t(x,t) > 0$ and $u_x(x,t) < 0$ for all $x > \xi(t)$.
(iii) $h(c_2)\{x-\xi(t)\}^{-\frac{1}{1-m}} \le u(x,t) \le h(c_1)\{x-\xi(t)\}^{-\frac{1}{1-m}}$ for all $x > \xi(t)$.

Proof Let $\varphi(z; c, L)$ be as in Theorem 2.7. We define

$$\begin{aligned} u^+(x,t) &:= \varphi(x-\xi(t); c_1), \\ u^-(x,t) &:= \varphi(x-\xi(t); c_2), \end{aligned} \qquad x \in (\xi(t), \infty),\ t \in \mathbb{R}.$$

Then we have $u^-(x,t) \le u^+(x,t)$ for $x \in (\xi(t), \infty)$ and $t \in (0, \infty)$ by Theorem 2.7 (iv). We compute

$$\begin{aligned} u_t^+(x,t) - \{(u^+)^m\}_{xx} &= -\xi'(t)\varphi'(x-\xi(t); c_1) - \left\{\varphi(x-\xi(t); c_1)^m\right\}_{xx} \\ &= \{c_1 - \xi'(t)\}\varphi'(x-\xi(t); c_1) \ge 0, \\ u_t^-(x,t) - \{(u^-)^m\}_{xx} &= -\xi'(t)\varphi'(x-\xi(t); c_2) - \left\{\varphi(x-\xi(t); c_2)^m\right\}_{xx} \\ &= \{c_2 - \xi'(t)\}\varphi'(x-\xi(t); c_2) \le 0. \end{aligned}$$

Hence u^+ and u^- are a supersolution and a subsolution of (1.1), respectively, defined for all $t \in \mathbb{R}$.

Let $\{t_i\}$ be a decreasing sequence such that $t_i \to -\infty$ as $i \to \infty$, and let u_i denote the unique singular solution of the initial-value problem

$$\begin{cases} (u_i)_t = \left\{(u_i)^m\right\}_{xx}, & x > \xi(t),\ t > t_i, \\ u_i(x, t_i) = u^-(x, t_i), & x > \xi(t_i). \end{cases}$$

Then by the comparison principle, we have

$$u^-(x,t) \le u_i(x,t) \le u^+(x,t), \qquad x \in (\xi(t), \infty),\ t \in (t_i, \infty).$$

These inequalities imply that the sequence $\{u_i(x,t)\}$ is increasing in i and is bounded above by $u^+(x,t)$. Then by the same argument as in [4, 9] (see also [5, Sect. 4]), we can show that $\{u_i(x,t)\}$ converges to an entire solution of (1.2) as $i \to \infty$, and the entire solution lies between $u^-(x,t)$ and $u^+(x,t)$ for all $t \in \mathbb{R}$. Moreover, since $(u_i)_x(x,t_i) < 0$ and $\left\{(u_i)^m\right\}_{xx}(x,t_i) > 0$ for $x \in (\xi(t_i), \infty)$, the entire solution satisfies $u_x(x,t) < 0$ and $u_t(x,t) > 0$ for all $x \in (\xi(t), \infty)$ and $t \in \mathbb{R}$. This completes the proof. □

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper, and that the paper is not currently submitted to other journals and that it will not be submitted to other journals during the reviewing process.

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