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On the stability of stationary compressible Navier–Stokes flows in 3D

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Abstract
This paper studies the stability of the stationary solution of the compressible Navier–Stokes equation in the 3D whole space with an external force which decays at spatial infinity. We obtain the global existence result of the non-stationary problem under the smallness assumptions on the initial perturbation around the small stationary solution. We also derive the time decay rates of the perturbations under the smallness assumption on the initial perturbations, and show the optimality of the time decay rates. The proofs are based on the combination of the spectral analysis and energy method in the framework of Besov spaces. The time-space integral estimate for the linearized semigroup around the constant state in some Besov spaces plays a crucial role in the proofs.

1 Introduction

We consider the Cauchy problem for the compressible Navier–Stokes equation:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) = \mu \Delta v + (\mu + \mu')\nabla \operatorname{div} v - \nabla P(\rho) + \rho F(x), \end{cases} \tag{1}$$

with initial data $(\rho, v)|_{t=0} = (\rho_0, v_0)$ and the boundary condition $(\rho, v)(t, x) \to (\rho_\infty, 0)$ as $|x| \to \infty$. Here $t \geq 0$, $x \in \mathbb{R}^3$, $v = (v_1, v_2, v_3)$ is the fluid velocity, ρ is the fluid density, ρ_∞ is a given positive constant, P is a given pressure, μ and μ' are given viscosity coefficients and $F = (F_1, F_2, F_3)$ is a given stationary external force. We assume that μ and μ' are constants that satisfy $\mu > 0$ and $2\mu/3 + \mu' \geq 0$, and P is a smooth function of ρ in a neighborhood of ρ_∞ with $P'(\rho_\infty) > 0$. As F is a

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stationary external force, we introduce the corresponding stationary problem:

$$\begin{cases} \operatorname{div}(\rho^* v^*) = 0, \\ \operatorname{div}(\rho^* v^* \otimes v^*) = \mu \Delta v^* + (\mu + \mu')\nabla \operatorname{div} v^* - \nabla P(\rho^*) + \rho^* F(x), \end{cases} \tag{2}$$

with the boundary condition $(\rho^*, v^*)(x) \to (\rho_\infty, 0)$ as $|x| \to \infty$.

It was shown by Shibata and Tanaka [17] that if the stationary external force F has the form $F = \operatorname{div} F_1 + F_2$ where F_1 is a 3×3-matrix valued function and F_2 is a 3-vector valued function and F is small in the quantity:

$$\begin{aligned} |||F||| = \sum_{k=0}^{3} \|(1+|x|)^{k+1} \nabla^k F\|_{L^2} + \|(1+|x|)^3 F\|_{L^\infty} \\ + \|(1+|x|)^2 F_1\|_{L^\infty} + \|F_2\|_{L^1}, \end{aligned}$$

then there exists a unique stationary solution $(\rho^*, v^*) = (\sigma^* + \rho_\infty, v^*)$ which satisfies the decay properties

$$|v^*(x)| \lesssim \frac{\delta}{|x|}, \quad |\nabla v^*(x)| \lesssim \frac{\delta}{|x|^2}, \quad |\sigma^*(x)| \lesssim \frac{\delta}{|x|^2}, \tag{3}$$

as $|x| \to \infty$ with $\delta = |||F|||$, and the smoothness property

$$\|(1+|x|)^j \nabla^j \sigma^*\|_{L^2} + \|(1+|x|)^j \nabla^{j+1} v^*\|_{L^2} \lesssim \delta$$

for any $0 \le j \le 4$. In addition, Shibata and Tanaka [17] proved that if the initial perturbation $(\rho_0 - \rho^*, v_0 - v^*)$ belongs to H^3 and is small in H^3 norm, then there exists a unique solution $(\rho, v) = (\sigma + \rho^*, w + v^*)$ of (1) such that $\sigma \in C^0([0, \infty); H^3) \cap C^1([0, \infty); H^2)$, $w \in C^0([0, \infty); H^3) \cap C^1([0, \infty); H^1)$ and

$$\begin{aligned} \sup_{0 \le t < \infty} \|(\sigma, w)(t)\|_{H^3}^2 + \int_0^t \|\nabla \sigma(\tau)\|_{H^2}^2 + \|\nabla w(\tau)\|_{H^3}^2 + \|\partial_t w(\tau)\|_{H^2}^2 d\tau \\ \lesssim \|(\rho_0 - \rho^*, v_0 - v^*)\|_{H^3}^2. \end{aligned}$$

In [18], Shibata and Tanaka then derived the decay rates of the perturbations

$$\|(\rho - \rho^*, v - v^*)(t)\|_{\dot{H}^1} \lesssim_\epsilon (1+t)^{-\frac{1}{2}+\epsilon} \|(\rho_0 - \rho^*, v_0 - v^*)\|_{L^{\frac{6}{5}} \cap H^3}, \tag{4}$$

where $\epsilon > 0$ is an arbitrary constant. Here, $\dot{H}^s$ denotes the homogeneous Sobolev space whose definition is given in Sect. 2 below.

On the other hand, when the external force $F = 0$, the corresponding stationary solution is the motionless state $(\rho_\infty, 0)$, and the time-decay estimates of its perturbation are given by

$$\|(\rho - \rho_\infty, v)(t)\|_{\dot{H}^s} \lesssim (1+t)^{-\frac{s}{2} - \frac{3}{2}\left(\frac{1}{p} - \frac{1}{2}\right)} \|(\rho_0 - \rho_\infty, v_0)\|_{L^p \cap H^3} \tag{5}$$

for $0 \le s \le 2$ and $1 \le p \le 2$. (Cf. [11–13, 15].) Note that the decay rates in (4) are slower than in (5) with $s = 1$, $p = 6/5$. Since the stationary solution (ρ^*, v^*) is close to the motionless state $(\rho_\infty, 0)$, one could expect that the decay estimate (5) would also hold for the perturbation of (ρ^*, v^*). However, it is not straightforward to see this since the spatial decay of the stationary velocity field v^* is slow as is written in (3).

The aim of this paper is the following twofold. The first is to derive the optimal decay rate of the perturbations of the stationary solution (ρ^*, v^*) under the smallness assumptions on the initial perturbation of (ρ^*, v^*). We prove that there holds the decay estimate

$$\|(\rho - \rho^*, v - v^*)(t)\|_{\dot{H}^s} \lesssim_{s,p} (1+t)^{-\frac{s}{2}-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)} \|(\rho_0 - \rho^*, v_0 - v^*)\|_{L^p \cap H^3}, \tag{6}$$

where $-3/2 < s < 3/2$ and $1 \le p \le 2$ with $s/2 + 3/2(1/p - 1/2) > 0$ or $s = 0$ and $p = 2$, which especially shows that the decay estimate (5) with $0 \le s < 3/2$ and $1 \le p \le 2$ hold for the perturbation of (ρ^*, v^*). In fact, we will derive the decay estimate (6) for the stationary solutions (ρ^*, v^*) in a larger class than that studied in [17, 18]. The second is to establish the global existence result of the non-stationary problem under the smallness assumptions on the initial perturbation around the stationary solution (ρ^*, v^*), without assuming that the initial perturbation $(\rho_0 - \rho^*, v_0 - v^*)$ belongs to L^2. In fact, we shall construct the global solution when the initial perturbation $(\rho_0 - \rho^*, v_0 - v^*)$ belongs to $\dot{B}^{1/2}_{2,\infty} \cap \dot{H}^3$ with small norm. Here, $\dot{B}^s_{p,r}$ denotes the homogeneous Besov space whose definition is given in Sect. 2 below. Note that the velocity v^* of the stationary solution is not necessarily in L^2 but in $\dot{B}^{1/2}_{2,\infty} \cap \dot{H}^3$; and so our result claims the global existence of solutions of problem (1) for a class of initial data which contains not only the stationary solution obtained in [17] but also the one constructed in this paper. We shall consider the stationary solutions obtained in the following theorem.

Theorem 1.1 *There exists a constant* $\delta_0 > 0$ *such that if* $F \in \dot{B}^{-\frac{3}{2}}_{2,\infty} \cap \dot{H}^3$ *and*

$$\|F\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty} \cap \dot{H}^3} \le \delta_0,$$

then the stationary problem (2) *has a unique solution* $(\rho^*, v^*) = (\sigma^* + \rho_\infty, v^*)$ *satisfying*

$$\|\sigma^*\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty} \cap \dot{H}^4} + \|v^*\|_{\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^5} \lesssim \delta_0. \tag{7}$$

In the proof of the existence of stationary solutions, there is a difficulty in that the convection term $\mathrm{div}(\sigma^* v^*)$ causes a loss of derivative. This difficulty is overcome by rewriting the equation by using the Helmholtz decomposition as in Lemma 3.2 below and regularizing the convection term $\mathrm{div}(\sigma^* v^*)$. Theorem 1.1 shows the existence of stationary solutions for data F in a larger class than that of [17].

Let us now state our main theorem, which derives the global existence of (1) and the decay rates of the perturbations.

Theorem 1.2 *Let* (ρ^*, v^*) *be the stationary solution satisfying* (7) *with* $\|F\|_{\dot{B}^{-3/2}_{2,\infty}\cap \dot{H}^3}$ *sufficiently small. Then, there exists a constant* $\delta > 0$ *such that if the initial perturbation* $(\rho_0 - \rho^*, v_0 - v^*) \in \dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3$ *and*

$$\|(\rho_0 - \rho^*, v_0 - v^*)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}\cap \dot{H}^3} \le \delta, \tag{8}$$

then the compressible Navier–Stokes equation (1) *with initial data* (ρ_0, v_0) *has a unique global solution* (ρ, v) *satisfying* $(\rho - \rho^*, v - v^*) \in C^0([0,\infty); \dot{B}^{1/2}_{2,\infty} \cap \dot{H}^3)$ *and*

$$\sup_{0\le t<\infty} \|(\rho - \rho^*, v - v^*)(t)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}\cap \dot{H}^3} \lesssim \delta. \tag{9}$$

In addition, if $(\rho_0 - \rho^*, v_0 - v^*) \in \dot{B}^{s_0}_{2,\infty}$ *for some* $-3/2 \le s_0 \le 1/2$*, then the decay estimate*

$$\|(\rho - \rho^*, v - v^*)(t)\|_{\dot{B}^{s}_{2,\infty}} \lesssim_s (1+t)^{-\frac{s-s_0}{2}} \|(\rho_0 - \rho^*, v_0 - v^*)\|_{\dot{B}^{s_0}_{2,\infty}\cap \dot{H}^3} \tag{10}$$

holds for $-3/2 < s < 3/2$ *with* $s_0 \le s$ *and* $t \ge 0$.

Remark 1.3 The time decay estimate of the perturbation when the initial perturbation belongs to $\dot{B}^{s_0}_{2,\infty}$ with some negative s_0 has already been studied by Danchin and Xu [9], Xu [20] for the case $(\rho^*, v^*) = (\rho_\infty, 0)$. They derived the decay estimate of the solution constructed in the L^p critical regularity framework.

Remark 1.4 The time decay estimate in Theorem 1.2 can be derived without an additional smallness assumption of the initial perturbation in $\dot{B}^{s_0}_{2,\infty}$. In fact, we will show the following type estimate. (See Propositions 5.2 and 5.5 below.) For any $-3/2 \le s_0 \le 1/2$, $\epsilon > 0$ and $T > 0$, let

$$\mathcal{D}_{\epsilon,s_0}(T) \equiv \sup_{\substack{-3/2+\epsilon\le\eta\le 3/2-\epsilon,\\ s_0\le\eta}} \sup_{0\le t\le T} (1+t)^{\frac{\eta-s_0}{2}} \|(\rho - \rho^*, v - v^*)(t)\|_{\dot{B}^{\eta}_{2,\infty}}.$$

We then have

$$\begin{aligned}\mathcal{D}_{\epsilon,s_0}(T) \lesssim_\epsilon\ & \|(\rho_0 - \rho^*, v_0 - v^*)\|_{\dot{B}^{s_0}_{2,\infty}} \\ & + \Big(\sup_{0\le t<\infty} \|(\rho - \rho^*, v - v^*)(t)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}\cap \dot{H}^3} + \|F\|_{\dot{B}^{-3/2}_{2,\infty}\cap \dot{H}^3}\Big)\mathcal{D}_{\epsilon,s_0}(T),\end{aligned}$$

and hence, we arrive at

$$\sup_{T>0} \mathcal{D}_{\epsilon,s_0}(T) < \infty$$

if $\|F\|_{\dot{B}^{-3/2}_{2,\infty}\cap \dot{H}^3}$ and $\|(\rho_0-\rho^*, v_0-v^*)\|_{\dot{B}^{1/2}_{2,\infty}\cap \dot{H}^3}$ are small enough.

We also have the following estimate for the perturbation.

Theorem 1.5 *Let* (ρ^*, v^*), (ρ, v) *be as in Theorem* 1.2, *with*

$$\|(\rho_0-\rho^*, v_0-v^*)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}\cap \dot{H}^3} + \|F\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty}\cap \dot{H}^3}$$

sufficiently small. If the initial perturbation $(\rho_0-\rho^*, v_0-v^*) \in L^p$ *for some* $1 \le p \le 2$, *then the decay estimate*

$$\|(\rho-\rho^*, v-v^*)(t)\|_{\dot{H}^s} \lesssim_s (1+t)^{-\frac{s}{2}-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)} \|(\rho_0-\rho^*, v_0-v^*)\|_{L^p\cap H^3} \tag{11}$$

holds for $-3/2 < s < 3/2$ *with* $s/2 + 3/2(1/p - 1/2) > 0$ *or* $s = 0$ *and* $p = 2$.

Here, we mention that in the case where the external force $F = 0$, the time decay of the perturbation was derived by Xu [20] when the initial perturbation belongs to Besov space with some negative exponents. We also mention that the smallness of the initial perturbation in $\dot{B}^{s_0}_{2,\infty}$ or L^p are not needed in Theorems 1.2 and 1.5, since by using Lemma 5.3 below, the nonlinear estimates in the proof of decay estimate can be done under the smallness assumption in (9).

We obtain the following results regarding the optimality of the estimates in Theorem 1.5.

Theorem 1.6 *Let* (ρ^*, v^*) *be the stationary solution satisfying* (7) *with* $\|F\|_{\dot{B}^{-3/2}_{2,\infty}\cap \dot{H}^3}$ *sufficiently small. Then, the following hold*:

(i) *There exists an initial perturbation* $(\rho_0-\rho^*, v_0-v^*) \in L^1\cap H^3$ *with sufficiently small* $\|(\rho_0-\rho^*, v_0-v^*)\|_{\dot{B}^{1/2}_{2,\infty}\cap \dot{H}^3}$ *such that the corresponding global solution* (ρ, v) *satisfies*

$$\|(\rho-\rho^*, v-v^*)(t)\|_{\dot{H}^s} \sim_s (1+t)^{-\frac{s}{2}-\frac{3}{4}} \|(\rho_0-\rho^*, v_0-v^*)\|_{L^1\cap H^3},$$

where $-3/2 < s < 3/2$ *and* $t \gg 1$.

(ii) *Under the same assumption as in Theorem* 1.5, *if* $(\rho_0-\rho^*, v_0-v^*) \in L^{p,\infty}$ *for some* $1 < p < 2$, *then the following estimate*

$$\|(\rho-\rho^*, v-v^*)(t)\|_{\dot{H}^s} \lesssim_{s,p} (1+t)^{-\frac{s}{2}-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)} \|(\rho_0-\rho^*, v_0-v^*)\|_{L^{p,\infty}\cap H^3} \tag{12}$$

holds for $-3/2 < s < 3/2$ *with* $s/2 + 3/2(1/p - 1/2) > 0$ *and* $t \ge 0$. *In addition, there exists an initial perturbation* $(\rho_0-\rho^*, v_0-v^*) \in L^{p,\infty} \cap H^3$

with sufficiently small $\|(\rho_0-\rho^*, v_0-v^*)\|_{\dot{B}^{1/2}_{2,\infty}\cap H^3}$ *such that the corresponding global solution* (ρ, v) *satisfies*

$$\|(\rho-\rho^*, v-v^*)(t)\|_{\dot{H}^s} \sim_{s,p} (1+t)^{-\frac{s}{2}-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}\|(\rho_0-\rho^*, v_0-v^*)\|_{L^{p,\infty}\cap H^3},$$

where $-3/2 < s < 3/2$ *with* $s/2+3/2(1/p-1/2) > 0$ *and* $t \gg 1$.

The difficulty in deriving the time-decay estimate of the perturbation $(\rho-\rho^*, v-v^*)$ arises from the slow spatial decay of the stationary solution. The spatial decay order of ∇v^* is expected to be at most $O(1/|x|^2)$, so the linear term $w\cdot\nabla v^*$ appearing in the perturbation equation may need to be treated as the inverse-square potential term. Indeed, Davies and Simon [10] showed that the inverse-square type potential term can affect the asymptotic behavior of the solution of the heat equation. For this reason, it is not straightforward that the linear term $w\cdot\nabla v^*$ is considered as a simple perturbation of the linearized operator around the motionless state $(\rho_\infty, 0)$. The same difficulty arises in the analysis of the linear term $v^*\cdot\nabla w$, since the spatial decay order of v^* is $O(1/|x|)$.

To overcome the difficulty arising from the linear terms $v^*\cdot\nabla w$ and $w\cdot\nabla v^*$, we shall formulate the decay problem in a framework of weak-type Besov spaces and prove the estimate (10). The proof of decay estimate (10) is performed by decomposing the perturbation into low- and high-frequency parts. The analysis of the low frequency part is carried out by using the momentum formulation, while the analysis of the high-frequency part is carried out by using the velocity formulation in order to avoid the derivative loss. (Cf. [1, 19].) The low frequency part is estimated by spectral analysis around the motionless state $(\rho_\infty, 0)$. Here, a crucial role is played by the time-space integral estimate established by Danchin [6]. (See Lemma 4.1 below.) A similar analysis is found in Yamazaki [21] where the time-space integral estimate in the Lorentz spaces is effectively employed to study the incompressible Navier–Stokes equation under time-dependent external force. In this direction, we also mention that the work by Chemin [4], where the time-space integral estimate in Besov spaces was established for incompressible Navier–Stokes equation. The estimate of the high-frequency part is established by the energy method in Besov spaces developed by Danchin [6]. The proof of optimality in Theorem 1.6 is inspired by the argument in Kawashima, Matsumura and Nishida [12]. We finally note that, in the case of $F=0$, the decay rate of the perturbation of the motionless state $(\rho_\infty, 0)$ was studied in critical spaces [5, 9, 14, 20]. It is also an interesting issue to consider the decay rate of the perturbation of the stationary solution (ρ^*, v^*) in critical spaces.

Organization of the paper. In Sect. 2, we present the notation used throughout this paper and the basic facts of the homogeneous Besov spaces. Section 3 is devoted to the proof of existence of solutions to the stationary problem (2). In Sect. 4, we construct a global solution under the smallness assumptions on an initial value and an external force. In Sect. 5, we derive the decay rate of the perturbation under the smallness assumptions on the initial perturbation. We also show the result regarding the optimality of the estimates in Theorem 1.5.

2 Preliminary

The notation $A \lesssim_\alpha B$ means that there exists a constant C depending on α such that $A \leq CB$. The notation $A \sim_\alpha B$ means that $A \lesssim_\alpha B$ and $B \lesssim_\alpha A$. We denote a commutator by $[X, Y] \equiv XY - YX$. We write $\mathcal{S}$ for the set of all Schwartz functions on $\mathbb{R}^3$, and we write $\mathcal{S}'$ for the set of all tempered distributions on $\mathbb{R}^3$. The notations $\hat{\cdot}$, $\mathcal{F}$ stand for the Fourier transform

$$\hat{u}(\xi) = \mathcal{F}(u)(\xi) \equiv \int_{\mathbb{R}^3} e^{-ix\cdot\xi} u(x)dx,$$

and the notation $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. The symbol $\mathbb{P}$ denotes the Helmholtz projection: $\mathbb{P}u \equiv u - \Delta^{-1}\nabla \mathrm{div}\, u$, $u \in \mathcal{S}'$. We denote the $L^2(\mathbb{R}^3)$ inner product by $\langle u, v\rangle \equiv \int_{\mathbb{R}^3} uv dx$. Let $s \in \mathbb{R}$. The homogeneous Sobolev space $\dot{H}^s = \dot{H}^s(\mathbb{R}^3)$ is the set of tempered distributions u on $\mathbb{R}^3$ such that $\hat{u} \in L^1_{loc}$, $\|u\|_{\dot{H}^s} \equiv \||\cdot|^s \hat{u}\|_{L^2} < \infty$. The inhomogeneous Sobolev space $H^s = H^s(\mathbb{R}^3)$ is the set of tempered distributions u on $\mathbb{R}^3$ such that $\|u\|_{H^s} \equiv \|(1 + |\cdot|)^s \hat{u}\|_{L^2} < \infty$. Let $1 \leq p \leq \infty$. The weak L^p space $L^{p,\infty} = L^{p,\infty}(\mathbb{R}^3)$ is the set of measurable functions on $\mathbb{R}^3$ such that

$$\|u\|_{L^{p,\infty}} \equiv \sup_{t>0} t\, m(\{x \mid |u(x)| > t\})^{\frac{1}{p}} < \infty,$$

where m is the Lebesgue measure on $\mathbb{R}^3$. Let I be an interval in $\mathbb{R}$ and let X be a Banach space. The Bochner space $L^p(I; X)$ is the set of strongly measurable functions $u : I \to X$ such that

$$\|u\|_{L^p(I;X)} \equiv \left(\int_I \|u(t)\|_X^p dt\right)^{\frac{1}{p}} < \infty.$$

The rest of this section is devoted to introducing the homogeneous Besov spaces and presenting some basic facts. These will be applied effectively throughout this paper. To apply our analysis, we employ the squared dyadic partition of unity. Choose $\phi \in C^\infty(\mathbb{R}^3)$ supported in the annulus $\mathcal{C} = \{\xi \in \mathbb{R}^3 \mid 3/4 \leq |\xi| \leq 8/3\}$ such that

$$\sum_{j\in\mathbb{Z}} \phi^2(2^{-j}\xi) = 1 \quad \text{for } \xi \neq 0.$$

Define the dyadic blocks $(\dot{\Delta}_j)_{j\in\mathbb{Z}}$ by the Fourier multiplier

$$\dot{\Delta}_j u \equiv \mathcal{F}^{-1}[\phi^2(2^{-j}\cdot)\hat{u}],$$

and the square-rooted dyadic blocks by $\dot{\Delta}_j^{1/2} u \equiv \mathcal{F}^{-1}[\phi(2^{-j}\cdot)\hat{u}]$. The homogeneous low frequency cutoff operator is denoted by

$$\dot{S}_j u \equiv \sum_{j'<j} \dot{\Delta}_{j'} u, \quad j \in \mathbb{Z}. \tag{13}$$

At least formally, the decomposition

$$u = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u$$

can be considered, which is called a Littlewood–Paley decomposition. We fix $\phi_0 \in C_0^\infty(\mathbb{R}^3)$ satisfying $\phi_0(0) \neq 0$.

Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$. Then, the homogeneous Besov space $\dot{B}^s_{p,r} = \dot{B}^s_{p,r}(\mathbb{R}^3)$ is given by

$$\dot{B}^s_{p,r} \equiv \left\{ u \in \mathcal{S}' \;\middle|\; \lim_{j \to -\infty} \|\mathcal{F}^{-1}[\phi_0(2^{-j}\cdot)\hat{u}]\|_{L^\infty} = 0, \;\; \|u\|_{\dot{B}^s_{p,r}} < \infty \right\},$$

$$\|u\|_{\dot{B}^s_{p,r}} \equiv \left\| (2^{js} \|\dot{\Delta}_j u\|_{L^p})_{j \in \mathbb{Z}} \right\|_{\ell^r}.$$

We state some basic facts on homogeneous Besov spaces, which are frequently used in this paper.

Proposition 2.1 *Let* $s, \tilde{s} \in \mathbb{R}$, $1 \leq p, \tilde{p}, r, \tilde{r} \leq \infty$ *and* $u, v \in \mathcal{S}'$.

(i) (Derivative) *For any* $k \geq 0$, $\|\nabla^k u\|_{\dot{B}^s_{p,r}} \sim \|u\|_{\dot{B}^{s+k}_{p,r}}$.

(ii) (Duality) *Let* p' *be a conjugate exponent of* p *and let* r' *be a conjugate exponent of* r. *Let* $v \in \mathcal{S}$. *Then, we have the following duality estimates*:

$$\langle u, v \rangle \lesssim \|u\|_{\dot{B}^s_{p,r}} \|u\|_{\dot{B}^{-s}_{p',r'}} \quad \text{and} \quad \|u\|_{\dot{B}^s_{p,r}} \lesssim \sup_{\psi} \langle u, \psi \rangle,$$

where the supremum is taken over the Schwartz functions ψ *with* $\|\psi\|_{\dot{B}^{-s}_{p',r'}} \leq 1$ *and* $0 \notin \operatorname{supp} \mathcal{F}\psi$.

(iii) (Interpolation) *Let* $s_1 < s_2$ *satisfy* $s = (1-\theta)s_1 + \theta s_2$ *for some* $0 < \theta < 1$. *Then, the interpolation inequality*

$$\|u\|_{\dot{B}^s_{p,r}} \lesssim_{\theta, s_1, s_2} \|u\|^{1-\theta}_{\dot{B}^{s_1}_{p,\infty}} \|u\|^{\theta}_{\dot{B}^{s_2}_{p,\infty}} \tag{14}$$

holds.

(iv) (Fatou property) *Assume* $s < 3/p$ *or* $s = 3/p$, $r = 1$. *If* $\{u_n\}_n$ *is a bounded sequence in* $\dot{B}^s_{p,r} \cap \dot{B}^{\tilde{s}}_{\tilde{p},\tilde{r}}$, *then there exists a subsequence of* $\{u_n\}_n$ (*without relabeling*) *and* $u \in \dot{B}^s_{p,r} \cap \dot{B}^{\tilde{s}}_{\tilde{p},\tilde{r}}$ *such that*

$$\lim_{n \to \infty} u_n = u \text{ in } \mathcal{S}' \quad \text{and} \quad \|u\|_{\dot{B}^s_{p,r} \cap \dot{B}^{\tilde{s}}_{\tilde{p},\tilde{r}}} \lesssim \liminf_{n \to \infty} \|u_n\|_{\dot{B}^s_{p,r} \cap \dot{B}^{\tilde{s}}_{\tilde{p},\tilde{r}}}.$$

(v) (Paralinearization) *Let* $\Phi \in C^\infty(\mathbb{R}^3)$ *and* $u, v \in \dot{B}^s_{2,r} \cap \dot{B}^{3/2}_{2,1}$ *with* $-3/2 \le s < 3/2$ *or* $s = 3/2$, $r = 1$. *Then, we have*

$$\|\Phi(u) - \Phi(v)\|_{\dot{B}^s_{2,r}} \lesssim_\Phi (1 + \|(u, v)\|_{\dot{B}^{\frac{3}{2}}_{2,1}})\|u - v\|_{\dot{B}^s_{2,r}}.$$

(vi) (Bilinear estimate) *Let* $s_1, s_2 \in \mathbb{R}$ *satisfy* $s_1, s_2 < 3/2$ *and* $s_1 + s_2 > 0$. *Let* $1 \le r_1, r_2 \le \infty$ *satisfy* $1/r_1 + 1/r_2 = 1/r$. *Then, we have*

$$\|uv\|_{\dot{B}^{s_1+s_2-\frac{3}{2}}_{2,r}} \lesssim_{s_1,s_2} \|u\|_{\dot{B}^{s_1}_{2,r_1}} \|v\|_{\dot{B}^{s_2}_{2,r_2}}.$$

In the cases $s_1 \le 3/2$, $s_2 < 3/2$ *with* $s_1 + s_2 \ge 0$, *we have*

$$\|uv\|_{\dot{B}^{s_1+s_2-\frac{3}{2}}_{2,\infty}} \lesssim \|u\|_{\dot{B}^{s_1}_{2,1}} \|v\|_{\dot{B}^{s_2}_{2,\infty}}.$$

As for the proofs other than (v), see [2, Lemma 2.1, Proposition 2.22, 2.29, Theorem 2.25, 2.47, 2.52 and Corollary 2.91] for example. The proof of Proposition 5 (v) is same as in the proof of [6, Lemma 1.6 ii)].

Generalized Young's inequality (see [16, pp. 31-32] for example) implies that, for any $j \in \mathbb{Z}$ and any $u \in L^1 + L^\infty$,

$$\begin{aligned} 2^{-3j\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|\dot{\Delta}_j u\|_{L^{p_2}} &\lesssim \|u\|_{L^{p_1}} \quad \text{if } 1 \le p_1 \le p_2 \le \infty, \\ 2^{-3j\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|\dot{\Delta}_j u\|_{L^{p_2}} &\lesssim \|u\|_{L^{p_1,\infty}} \quad \text{if } 1 < p_1 < p_2 < \infty. \end{aligned}$$

This shows the following proposition.

Proposition 2.2 *Let* $1 \le p_1 \le p_2 \le \infty$. *Then, the space* L^{p_1} *is continuously embedded in the space* $\dot{B}^{-3(1/p_1-1/p_2)}_{p_2,\infty}$. *In addition, if* $1 < p_1 < p_2 < \infty$, *then the space* $L^{p_1,\infty}$ *is continuously embedded in the space* $\dot{B}^{-3(1/p_1-1/p_2)}_{p_2,\infty}$.

We need the following commutator estimate. The proof is the same as that in [2, Lemma 2.100].

Lemma 2.3 *Let* $-3/2 < s < 5/2$, $1 \le r \le \infty$ *and* $\phi_0 \in C^\infty_0(\mathbb{R}^3)$ *with* $\operatorname{supp}\phi_0 \subset \mathcal{C}'$ *for some annulus* $\mathcal{C}'$ *centered at the origin. Let us denote* $\chi_j v \equiv \mathcal{F}^{-1}\left[\phi_0(2^{-j}\cdot)\hat{v}\right]$ *for any* $v \in \mathcal{S}'$, $j \in \mathbb{Z}$. *Then, we have*

$$\left\|\left(2^{js}\|[\chi_j, h\partial_k]u\|_{L^2}\right)_{j\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})} \lesssim_{s,\phi_0} \|\nabla h\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \|u\|_{\dot{B}^s_{2,r}},$$

where $1 \le k \le 3$ *and* u, h *are scalar functions.*

3 Existence of stationary solutions

This section is devoted to proving the existence of the stationary solutions of (2). Let $k \in \mathbb{Z}_{\geq 1}$. Set the function spaces X, Y and Z as

$$X = L^2 \times \dot{H}^1, \quad Y = Y_0 \cap Y_1 \quad \text{and} \quad Z = \dot{B}^{-3/2}_{2,\infty} \cap \dot{H}^k,$$

where

$$Y_0 = \dot{B}^{-\frac{1}{2}}_{2,\infty} \times \dot{B}^{\frac{1}{2}}_{2,\infty}, \quad Y_1 = \dot{H}^{k+1} \times \dot{H}^{k+2}.$$

Then, we have the following results.

Theorem 3.1 *If $\|F\|_Z$ is sufficiently small, then there exists a unique stationary solution $(\rho, v) = (\sigma + \rho_\infty, v)$ of (2) such that $(\sigma, v) \in Y$ and $\|(\sigma, v)\|_Y \lesssim \|F\|_Z$.*

To prove Theorem 3.1, we first reformulate the stationary problem (2). By rewriting the stationary problem (2) using the Helmholtz decomposition, we obtain the following lemma.

Lemma 3.2 *A pair of function $(\rho, v) = (\sigma + \rho_\infty, v)$, $(\sigma, v) \in Y$ is the solution to the problem (2) if and only if $(\rho, v) = (\sigma + \rho_\infty, v)$, $(\sigma, v) \in Y$ satisfies the following equations:*

$$\begin{cases} \sigma + \alpha \mathrm{div}\,(\sigma v) = \gamma^{-2}\Delta^{-1}\mathrm{div}\, g, \\ v - \beta\Delta^{-1}\nabla\sigma = -\mu_0^{-1}\Delta^{-2}\nabla\mathrm{div}\, g - \mu^{-1}\Delta^{-1}\mathbb{P}g, \end{cases} \tag{15}$$

where $\mu_0 = 2\mu + \mu'$, $\alpha = \mu_0/(P'(\rho_\infty)\rho_\infty)$, $\beta = P'(\rho_\infty)/\mu_0$, $\gamma = P'(\rho_\infty)^{1/2}$ and

$$g(\sigma, v) = -\mathrm{div}(\rho v \otimes v) - (P'(\rho) - P'(\rho_\infty))\nabla\sigma + \rho F.$$

Proof Let $(\rho, v) = (\sigma + \rho_\infty, v)$ be a solution of the stationary problem (2). By letting the Helmholtz projection $\mathbb{P}$ and div act on the second equation of (2), respectively, we obtain the following system of equations.

$$\begin{cases} \Delta\sigma - \alpha\rho_\infty\Delta\mathrm{div}\, v = \gamma^{-2}\mathrm{div}\, g, \\ \mu\Delta\mathbb{P}v = -\mathbb{P}g, \\ \rho_\infty\mathrm{div}\, v + \mathrm{div}(\sigma v) = 0. \end{cases}$$

Therefore, (σ, v) satisfies (15). The rest can be shown in a similar way. □

We introduce the linear operator

$$\mathcal{L}_{\tilde{v}}(\sigma, v) \equiv \begin{bmatrix} \sigma + \alpha\mathrm{div}\,(\sigma\tilde{v}) \\ v - \beta\Delta^{-1}\nabla\sigma \end{bmatrix}.$$

Then, the Eq. (15) is written as

$$\mathcal{L}_v(\sigma, v) = N(\sigma, v),$$

where

$$N(\sigma, v) = \begin{bmatrix} N_1(\sigma, v) \\ N_2(\sigma, v) \end{bmatrix} = \begin{bmatrix} \gamma^{-2}\Delta^{-1}\mathrm{div}\, g(\sigma, v) \\ -\left(\mu_0^{-1}\Delta^{-2}\nabla\mathrm{div} + \mu^{-1}\Delta^{-1}\mathbb{P}\right) g(\sigma, v) \end{bmatrix}.$$

To solve (15), we first consider the following approximate problems:

$$\mathcal{L}_{\tilde{v},j}(\sigma, v) = N(\tilde{\sigma}, \tilde{v}), \;\; j \in \mathbb{Z}.$$

Here, $\mathcal{L}_{\tilde{v},j}$ is the approximation operator

$$\mathcal{L}_{\tilde{v},j}(\sigma, v) \equiv \begin{bmatrix} \sigma + \alpha \dot{S}_j \mathrm{div}\,(\sigma\tilde{v}) \\ v - \beta\Delta^{-1}\nabla\sigma \end{bmatrix}, \quad j \in \mathbb{Z},$$

where $\dot{S}_j$ is the low frequency cut-off operator defined in (13).

Lemma 3.3 *Let $\tilde{v} \in L^\infty \cap \dot{B}^{5/2}_{2,1}$. If $\|\tilde{v}\|_{\dot{B}^{5/2}_{2,1}}$ is small, then for any $j \in \mathbb{Z}$, the map*

$$\mathcal{L}_{\tilde{v},j} : X \to X$$

is bijective.

Proof Using Young's inequality, we have $\|\dot{S}_j \mathrm{div}(\sigma\tilde{v})\|_{L^2} \lesssim 2^j \|\tilde{v}\|_{L^\infty}\|\sigma\|_{L^2}$. Thus, $\mathcal{L}_{\tilde{v},j}(X) \subset X$. Fix small $d > 0$. We define the inner product of X by

$$((\sigma_1, v_1), (\sigma_2, v_2))_X \equiv \langle\sigma_1, \sigma_2\rangle + d\langle\nabla v_1, \nabla v_2\rangle,$$

where $(\sigma_1, v_1), (\sigma_2, v_2) \in X$.

By Young's inequality and $\|\mathrm{div}\tilde{v}\|_{L^\infty} \lesssim \|\tilde{v}\|_{\dot{B}^{5/2}_{2,1}}$, we have

$$|\langle\dot{S}_j((\mathrm{div}\tilde{v})\sigma), \sigma\rangle| \lesssim \|\dot{S}_j((\mathrm{div}\tilde{v})\sigma)\|_{L^2}\|\sigma\|_{L^2} \lesssim \|\tilde{v}\|_{\dot{B}^{\frac{5}{2}}_{2,1}} \|\sigma\|^2_{L^2}.$$

By using Proposition 2.1(vi), Lemma 2.3 with $s = 0$ and the identity $\langle\tilde{v}\cdot\nabla\sigma_{j'}, \sigma_{j'}\rangle = -1/2\langle\mathrm{div}\,\tilde{v}, \sigma^2_{j'}\rangle$ with $\sigma_{j'} = \dot{\Delta}^{1/2}_{j'}\sigma$, we obtain

$$\begin{aligned} |\langle\dot{S}_j(\sigma\cdot\nabla\tilde{v}), \sigma\rangle| &\lesssim \sum_{j'\le j-1} |\langle\dot{\Delta}_{j'}(v\cdot\nabla\sigma), \sigma\rangle| \\ &\lesssim \sum_{j'\in\mathbb{Z}} (|\langle\tilde{v}\cdot\nabla\sigma_{j'}, \sigma_{j'}\rangle| + |\langle[\dot{\Delta}^{\frac{1}{2}}_{j'}, \tilde{v}\cdot\nabla]\sigma, \sigma_{j'}\rangle|) \end{aligned}$$

$$
\lesssim \sum_{j'\in\mathbb{Z}} (\|\mathrm{div}\tilde{v}\|_{L^\infty}\|\sigma_{j'}\|_{L^2}^2 + \|[\dot{\Delta}_{j'}^{\frac{1}{2}}, \tilde{v}\cdot\nabla]\sigma\|_{L^2}\|\sigma_{j'}\|_{L^2})
$$

$$
\lesssim \|\tilde{v}\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\|\sigma\|_{L^2}^2. \tag{16}
$$

Thus, we have

$$
|\langle \dot{S}_j \mathrm{div}(\tilde{v}\sigma), \sigma\rangle| \lesssim \|\tilde{v}\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\|\sigma\|_{L^2}^2.
$$

Hence, if d and $\|\tilde{v}\|_{\dot{B}^{5/2}_{2,1}}$ are sufficiently small, then

$$
|(\mathcal{L}_{\tilde{v},j}(\sigma, v), (\sigma, v))_X| \gtrsim \|(\sigma, v)\|_X^2
$$

for any $(\sigma, v) \in X$, where $\|(\sigma, v)\|_X \equiv ((\sigma, v), (\sigma, v))_X^{1/2}$. The Lax–Milgram theorem completes the proof. □

We show the nonlinear estimate.

Lemma 3.4 *Let*

$$
(\tilde{\sigma}, \tilde{v}), (\tilde{\sigma}_1, \tilde{v}_1), (\tilde{\sigma}_2, \tilde{v}_2) \in Y_\delta \equiv \big\{(\sigma, v) \in Y \mid \|(\sigma, v)\|_Y \le \delta\big\}.
$$

Then, for any $0 < \delta \le 1$, *we have*

$$
\|g(\tilde{\sigma}, \tilde{v})\|_Z \lesssim \delta^2 + \|F\|_Z,
$$

$$
\|g(\tilde{\sigma}_1, \tilde{v}_1) - g(\tilde{\sigma}_2, \tilde{v}_2)\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty}} \lesssim (\delta + \|F\|_Z)\|(\tilde{\sigma}_1 - \tilde{\sigma}_2, \tilde{v}_1 - \tilde{v}_2)\|_{Y_0}.
$$

Proof By Proposition 2.1(v), (vi),

$$
\begin{aligned}
\|g(\tilde{\sigma}, \tilde{v})\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty}} &\lesssim \|\tilde{v}\|^2_{\dot{B}^{\frac{1}{2}}_{2,\infty}} + \|\tilde{\sigma}\|^2_{\dot{B}^{\frac{3}{2}}_{2,1}\cap\dot{B}^{-\frac{1}{2}}_{2,\infty}} + \|F\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty}} \\
&\lesssim \delta^2 + \|F\|_Z, \\
\|g(\tilde{\sigma}_1, \tilde{v}_1) - g(\tilde{\sigma}_2, \tilde{v}_2)\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty}} &\lesssim \delta\|\tilde{v}_1 - \tilde{v}_2\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} + \delta\|\tilde{\sigma}_1 - \tilde{\sigma}_2\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty}} \\
&\quad + \|\tilde{\sigma}_1 - \tilde{\sigma}_2\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty}}\|F\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \\
&\lesssim (\delta + \|F\|_Z)\|(\tilde{\sigma}_1 - \tilde{\sigma}_2, \tilde{v}_1 - \tilde{v}_2)\|_{Y_0}.
\end{aligned}
$$

Using Sobolev's inequality and Proposition 2.1(vi), we have

$$
\begin{aligned}
&\|\mathrm{div}\,(\tilde{\rho}\tilde{v}\otimes\tilde{v})\|_{\dot{H}^k} \lesssim \|\nabla^{k+1}(\tilde{\rho}\tilde{v}\otimes\tilde{v})\|_{L^2} \lesssim \big(1 + \|\tilde{\sigma}\|_{H^{k+1}}\big)\|\nabla\tilde{v}\|^2_{H^k}, \\
&\|(P'(\tilde{\rho}) - P'(\rho_\infty))\nabla\tilde{\sigma}\|_{\dot{H}^k} = \|Q(\tilde{\sigma})\tilde{\sigma}\nabla\tilde{\sigma}\|_{\dot{H}^k},
\end{aligned}
$$

$$Q(\tilde{\sigma}) \equiv \int_0^1 P''(\rho_\infty + t\tilde{\sigma})dt, \lesssim \left(\|Q(\tilde{\sigma})\|_{L^\infty} + \|\nabla Q(\tilde{\sigma})\|_{H^{k-1}}\right)\|\tilde{\sigma}\|^2_{H^{k+1}},$$
$$\|\tilde{\rho}F\|_{\dot{H}^k} \lesssim (1 + \|\tilde{\sigma}\|_{H^k})\|F\|_{H^k}.$$

Hence, we obtain

$$\|g(\tilde{\sigma}, \tilde{v})\|_Z \lesssim \delta^2 + \|F\|_Z.$$

□

By virtue of Lemmas 3.3 and 3.4, we can define the maps

$$\Phi_j(\tilde{\sigma}, \tilde{v}) \equiv \mathcal{L}^{-1}_{\tilde{v},j} N(\tilde{\sigma}, \tilde{v}), \quad (\tilde{\sigma}, \tilde{v}) \in Y_\delta,\ j \in \mathbb{Z}.$$

Lemma 3.5 *If $\delta > 0$ and $\|F\|_Z$ are sufficiently small, then, for any $j \geq 0$, the map $\Phi_j : Y_\delta \to Y_\delta$ satisfies $\|\Phi_j(\tilde{\sigma}, \tilde{v})\|_Y \lesssim \delta^2 + \|F\|_Z$ for any $(\tilde{\sigma}, \tilde{v}) \in Y_\delta$. Furthermore, there exists a constant $c > 0$ such that the maps Φ_j, $j \geq 0$ are contraction mappings in Y_0 norm with uniform Lipschitz constant $c\delta$, that is,*

$$\sup_{j\geq 0} \|\Phi_j(\tilde{\sigma}_1, \tilde{v}_1) - \Phi_j(\tilde{\sigma}_2, \tilde{v}_2)\|_{Y_0} \leq c\delta\|(\tilde{\sigma}_1 - \tilde{\sigma}_2, \tilde{v}_1 - \tilde{v}_2)\|_{Y_0}$$

for any $(\tilde{\sigma}_1, \tilde{v}_1), (\tilde{\sigma}_2, \tilde{v}_2) \in Y_\delta$.

Proof Let $(\sigma, v) = \Phi_j(\tilde{\sigma}, \tilde{v})$, $(\tilde{\sigma}, \tilde{v}) \in Y_\delta$. Then, since

$$\sigma = -\dot{S}_j \mathrm{div}(\sigma\tilde{v}) + \gamma^{-2}\Delta^{-1}\mathrm{div}\, g(\tilde{\sigma}, \tilde{v}),$$

we have the estimate

$$\begin{aligned}\|\sigma\|_{H^{k+1}} &\lesssim \|\dot{S}_j \mathrm{div}(\sigma\tilde{v})\|_{H^{k+1}} + \|g(\tilde{\sigma}, \tilde{v})\|_{H^k}\\ &\lesssim_j \|\tilde{v}\|_{L^\infty}\|\sigma\|_{L^2} + \delta^2 + \|F\|_Z,\end{aligned}$$

where the last inequality is due to the fact that $\|\nabla^n \dot{S}_j(\sigma\tilde{v})\|_{L^2} \lesssim 2^{nj}\|\sigma\tilde{v}\|_{L^2}$ for any $n \in \mathbb{Z}_{\geq 0}$ and Lemma 3.4. Thus, we have $\sigma \in H^{k+1}$. We apply the argument for (16) again, with σ replaced by $\partial_x^\alpha\sigma$, $|\alpha| = k+1$, to obtain

$$|\langle \dot{S}_j \mathrm{div}(\partial_x^\alpha\sigma\ \tilde{v}), \partial_x^\alpha\sigma\rangle| \lesssim \|\nabla\tilde{v}\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\|\partial_x^\alpha\sigma\|^2_{L^2}.$$

This gives

$$\|\sigma\|_{\dot{H}^{k+1}}^2 = \sum_{|\alpha|=k+1} \langle \partial_x^\alpha \sigma, \partial_x^\alpha \sigma \rangle \lesssim \sum_{|\alpha|=k+1,\, \beta\le\alpha} |\langle \dot{S}_j \mathrm{div}(\partial_x^\beta \sigma\, \partial_x^{\alpha-\beta} \tilde{v}), \partial_x^\alpha \sigma \rangle|$$
$$+ \|\Delta^{-1}\mathrm{div}\, g(\tilde{\sigma}, \tilde{v})\|_{\dot{H}^{k+1}} \|\sigma\|_{\dot{H}^{k+1}}$$
$$\lesssim \|\nabla \tilde{v}\|_{H^{k+1}} \|\sigma\|_{H^{k+1}}^2 + \|g(\tilde{\sigma}, \tilde{v})\|_{\dot{H}^k} \|\sigma\|_{\dot{H}^{k+1}}.$$

Thus, Lemma 3.4 shows $\|\sigma\|_{\dot{H}^{k+1}} \lesssim \delta\|(\sigma, v)\|_Y + \delta^2 + \|F\|_Z$. By directly estimating $(\sigma, v)^\mathsf{T} = (\sigma, v)^\mathsf{T} - \mathcal{L}_{\tilde{v},j}(\sigma, v) + N(\tilde{\sigma}, \tilde{v})$, we obtain

$$\|\sigma\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty}} \lesssim \|\sigma\tilde{v}\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} + \|g(\tilde{\sigma}, \tilde{v})\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty}}$$
$$\lesssim \|\sigma\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \|\tilde{v}\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} + \delta^2 + \|F\|_Z \lesssim \delta\|(\sigma, v)\|_Y + \delta^2 + \|F\|_Z,$$
$$\|v\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}\cap \dot{H}^{k+2}} \lesssim \|\sigma\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty}\cap \dot{H}^{k+1}} + \|g(\tilde{\sigma}, \tilde{v})\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty}\cap \dot{H}^{k}} \lesssim \delta\|(\sigma, v)\|_Y + \delta^2 + \|F\|_Z.$$

Then, we have $\|(\sigma, v)\|_Y \lesssim \delta^2 + \|F\|_Z$ for small $\delta > 0$. Let $(\sigma_i, v_i) = \Phi_j(\tilde{\sigma}_i, \tilde{v}_i)$, $(\tilde{\sigma}_i, \tilde{v}_i) \in Y_\delta$, $i = 1, 2$. Applying Lemma 3.4 for estimating

$$v_1 - v_2 = \beta\Delta^{-1}\nabla(\sigma_1 - \sigma_2) + N_2(\tilde{\sigma}_1, \tilde{v}_1) - N_2(\tilde{\sigma}_2, \tilde{v}_2)$$

we obtain

$$\|v_1 - v_2\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} \lesssim \|\sigma_1 - \sigma_2\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty}} + \|g(\tilde{\sigma}_1, \tilde{v}_1) - g(\tilde{\sigma}_2, \tilde{v}_2)\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty}}$$
$$\lesssim \|\sigma_1 - \sigma_2\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty}} + \delta\|(\tilde{\sigma}_1 - \tilde{\sigma}_2, \tilde{v}_1 - \tilde{v}_2)\|_{Y_0}.$$

The rest is to estimate $\|\sigma_1 - \sigma_2\|_{\dot{B}^{-1/2}_{2,\infty}}$. For any $n \in \mathbb{Z}$, there holds

$$\|\dot{\Delta}_n(\sigma_1 - \sigma_2)\|_{L^2}^2 = -\langle \dot{\Delta}_n \alpha \dot{S}_j \mathrm{div}(\sigma_1\tilde{v}_1 - \sigma_2\tilde{v}_2), \dot{\Delta}_n(\sigma_1 - \sigma_2)\rangle$$
$$+ \langle \dot{\Delta}_n \gamma^{-2}\Delta^{-1}\mathrm{div}(g(\tilde{\sigma}_1, \tilde{v}_1) - g(\tilde{\sigma}_2, \tilde{v}_2)), \dot{\Delta}_n(\sigma_1 - \sigma_2)\rangle.$$

Let $\chi_{n,j'} = \dot{\Delta}_n \dot{\Delta}_{j'}^{1/2}$, $\omega = \sigma_1 - \sigma_2$. Then, by using Proposition 2.1(vi), Lemma 2.3 and the identity

$$\langle \tilde{v}_1 \cdot \nabla\chi_{n,j'}\omega, \chi_{n,j'}\omega\rangle = -\frac{1}{2}\langle \mathrm{div}\tilde{v}_1\, \chi_{n,j'}\omega, \chi_{n,j'}\omega\rangle,$$

we have the estimate

$$|\langle \dot{\Delta}_n \dot{S}_j \mathrm{div}(\sigma_1\tilde{v}_1 - \sigma_2\tilde{v}_2), \dot{\Delta}_n\omega\rangle|$$
$$\lesssim \sum_{j'<j,\ |n-j'|\le 1} \left(\|[\chi_{n,j'}, \tilde{v}_1 \cdot \nabla]\omega\|_{L^2} \|\chi_{n,j'}\omega\|_{L^2} + |\langle \tilde{v}_1 \cdot \nabla\chi_{n,j'}\omega, \chi_{n,j'}\omega\rangle| \right)$$

$$+\left(\|\dot{\Delta}_n(\mathrm{div}\tilde{v}_1\,\omega)\|_{L^2}+\|\dot{\Delta}_n\mathrm{div}((\tilde{v}_1-\tilde{v}_2)\sigma_2)\|_{L^2}\right)\|\dot{\Delta}_n\omega\|_{L^2}$$

$$\lesssim 2^{\frac{1}{2}n}\left(\|\tilde{v}_1\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\|\omega\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty}}+\|\sigma_2\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\|\tilde{v}_1-\tilde{v}_2\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}}\right)\|\dot{\Delta}_n\omega\|_{L^2}$$

$$\lesssim 2^{\frac{1}{2}n}\delta\left(\|\omega\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty}}+\|\tilde{v}_1-\tilde{v}_2\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}}\right)\|\dot{\Delta}_n\omega\|_{L^2}.$$

Thus, if $\delta>0$ is sufficiently small, then we have

$$\|\sigma_1-\sigma_2\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty}}\lesssim\delta\|\tilde{v}_1-\tilde{v}_2\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}}+\|g(\tilde{\sigma}_1,\tilde{v}_1)-g(\tilde{\sigma}_2,\tilde{v}_2)\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty}}$$
$$\lesssim\delta\|(\tilde{\sigma}_1-\tilde{\sigma}_2,\tilde{v}_1-\tilde{v}_2)\|_{Y_0}.$$

□

Let $(\tilde{\sigma},\tilde{v})\in Y_\delta$ and let $\delta>0$ be small enough. By Proposition 2.1(iv), the bounded sequence $\{\Phi_j(\tilde{\sigma},\tilde{v})\}_{j\geq 0}$ has a convergence subsequence in $\mathcal{S}'$. Let $(\sigma,v)\in\mathcal{S}'$ be one of its convergent limits, and we write $\Phi(\tilde{\sigma},\tilde{v})=(\sigma,v)$. Then, the following proposition holds.

Proposition 3.6 *If $\delta>0$ and $\|F\|_Z$ are sufficiently small, then the map $\Phi:Y_\delta\to Y_\delta$ is well-defined and is a contraction in the Y_0 norm.*

Proof Let us show the well-definedness of Φ. Let $(\tilde{\sigma},\tilde{v})\in Y_\delta$, and let (σ_1,v_1), (σ_2,v_2) be convergent limits in $\mathcal{S}'$ of subsequences $\{\Phi_{\psi_1(j)}(\tilde{\sigma},\tilde{v})\}_{j\geq 0}$, $\{\Phi_{\psi_2(j)}(\tilde{\sigma},\tilde{v})\}_{j\geq 0}$, respectively. By Proposition 2.1(iv) and Lemma 3.5, there exist subsequences of $\{\Phi_{\psi_i(j)}(\tilde{\sigma},\tilde{v})\}_{j\geq 0}$, $i=1,2$ (without relabeling) such that

$$\|(\sigma_i,v_i)\|_Y\lesssim\liminf_{j\to\infty}\|\Phi_{\psi_i(j)}(\tilde{\sigma},\tilde{v})\|_Y\lesssim\delta^2+\|F\|_Z,\quad i=1,2.$$

Thus, we have $(\sigma_i,v_i)\in Y_\delta$, $i=1,2$ for small $\delta>0$ and $\|F\|_Z$. By subtracting $\mathcal{L}_{\tilde{v},\psi_i(j)}\Phi_{\psi_i(j)}(\tilde{\sigma},\tilde{v})=N(\tilde{\sigma},\tilde{v})$, $i=1,2$, we have

$$\begin{cases}\sigma_{1,j}-\sigma_{2,j}+\alpha\dot{S}_{\psi_1(j)}\mathrm{div}\,((\sigma_{1,j}-\sigma_{2,j})\tilde{v})-\alpha(\dot{S}_{\psi_2(j)}-\dot{S}_{\psi_1(j)})\mathrm{div}\,(\sigma_{2,j}\tilde{v})=0,\\ v_{1,j}-v_{2,j}-\beta\Delta^{-1}\nabla(\sigma_{1,j}-\sigma_{2,j})=0,\end{cases}$$

where $(\sigma_{i,j},v_{i,j})\equiv\Phi_{\psi_i(j)}(\tilde{\sigma},\tilde{v})$, $i=1,2$, $j\geq 0$. Letting $j\to\infty$ in $\mathcal{S}'$, we obtain the equalities

$$\begin{cases}\sigma_1-\sigma_2+\alpha\mathrm{div}\,((\sigma_1-\sigma_2)\tilde{v})=0,\\ v_1-v_2-\beta\Delta^{-1}\nabla(\sigma_1-\sigma_2)=0.\end{cases}$$

Then, letting $\eta=\sigma_1-\sigma_2$, we have

$$\|\eta\|_{L^2}^2=-\alpha\langle\mathrm{div}(\eta\tilde{v}),\eta\rangle=-\frac{\alpha}{2}\langle\mathrm{div}\,\tilde{v},\eta^2\rangle\lesssim\delta\|\eta\|_{L^2}^2.$$

Thus, if $\delta > 0$ is small, then we have $\sigma_1 = \sigma_2$ and $v_1 = v_2$. Hence, the map $\Phi : Y_\delta \to Y_\delta$ is well-defined.

Next, we show the map Φ is a contraction in the Y_0 norm. Let $(\sigma_i, v_i) = \Phi(\tilde{\sigma}_i, \tilde{v}_i)$, $(\tilde{\sigma}_i, \tilde{v}_i) \in Y_\delta$, $i = 1, 2$. Since the definition of Φ does not depend on taking a weakly convergent subsequence, there exits a subsequence $\{\Phi_{\psi(j)}\}_{j\geq 0}$ such that $\Phi_{\psi(j)}(\tilde{\sigma}_i, \tilde{v}_i) \to (\sigma_i, v_i)$ in $\mathcal{S}'$ as $j \to \infty$, $i = 1, 2$. By Proposition 2.1(iv) and Lemma 3.5, there exists a subsequence of $\{\Phi_{\psi(j)}(\tilde{\sigma}_i, \tilde{v}_i)\}_{j\geq 0}$, $i = 1, 2$ (without relabeling) such that

$$\|(\sigma_i, v_i)\|_Y \lesssim \liminf_{j\to\infty} \|\Phi_{\psi(j)}(\tilde{\sigma}_i, \tilde{v}_i)\|_Y \lesssim \delta^2 + \|F\|_Z, \quad i = 1, 2,$$

and

$$\begin{aligned}\|(\sigma_1 - \sigma_2, v_1 - v_2)\|_{Y_0} &\lesssim \liminf_{j\to\infty} \|\Phi_{\psi(j)}(\tilde{\sigma}_1, \tilde{v}_1) - \Phi_{\psi(j)}(\tilde{\sigma}_2, \tilde{v}_2)\|_{Y_0} \\ &\leq c\delta\|(\tilde{\sigma}_1 - \tilde{\sigma}_2, \tilde{v}_1 - \tilde{v}_2)\|_{Y_0}.\end{aligned}$$

Therefore, if $\delta > 0$ and $\|F\|_Z$ are sufficiently small, then the map $\Phi : Y_\delta \to Y_\delta$ is well-defined and is a contraction in the Y_0 norm. □

Let us now establish the proof of Theorem 3.1.

Proof of Theorem 3.1 Let δ and $\|F\|_Z$ be small. Then, by Proposition 3.6, the map $\Phi : Y_\delta \to Y_\delta$ is well-defined and is a contraction in the Y_0 norm. The contraction mapping principle and Proposition 2.1(iv) show that there exists a unique $(\sigma, v) \in Y_\delta$ such that $\Phi(\sigma, v) = (\sigma, v)$. This implies $\mathcal{L}_v(\sigma, v) = N(\sigma, v)$, since $\dot{S}_j \to 1$ in $\mathcal{S}'$ as $j \to \infty$. Hence, Lemma 3.2 shows that (σ, v) solves the stationary problem (2). □

4 Non-stationary problem

This section is devoted to proving the existence of the solutions of (1). Let us consider the equations satisfied by the perturbation of the stationary solution. Let (ρ, v) be a solution of (1) and let $(\rho^*, v^*) = (\sigma^* + \rho_\infty, v^*)$ be a stationary solution of (2). After the rescaling

$$\begin{aligned}&(\rho(t, x), v(t, x)) \to (\rho(\lambda^2 t, \lambda x), \lambda v(\lambda^2 t, \lambda x)), \\ &(\rho^*(x), v^*(x)) \to (\rho^*(\lambda x), \lambda v^*(\lambda x)),\end{aligned}$$

with $\lambda = \rho_\infty / P'(\rho_\infty)^{1/2}$, we assume without loss of generality that $\rho_\infty / P'(\rho_\infty)^{1/2} = 1$. Then, the perturbation $(\sigma, w) = (\rho - \rho^*, v - v^*)$ satisfies the following system of equations:

$$\begin{cases}\partial_t \sigma + \gamma_0 \operatorname{div} w = f(\sigma, w), \\ \partial_t w - \mathcal{A}_0 w + \gamma_0 \nabla \sigma = g(\sigma, w), \\ (\sigma, w)|_{t=0} = (\sigma_0, w_0),\end{cases} \tag{17}$$

where $\gamma_0 = P'(\rho_\infty)^{1/2}$, $\nu_0 = \mu/\rho_\infty$, $\nu_0' = \mu'/\rho_\infty$, $\mathcal{A}_0 \equiv \nu_0\Delta + (\nu_0 + \nu_0')\nabla\mathrm{div}$, $(\sigma_0, w_0) \equiv (\rho_0 - \rho^*, v_0 - v^*)$; f and g are defined by the following:

$$f(\sigma, w) = -\gamma_0\mathrm{div}\left\{(v^* + w)\sigma + \sigma^* w\right\},\ g(\sigma, w) = \sum_{i=1}^{4} g^i$$

with

$$\begin{aligned} g^1 &= -v^* \cdot \nabla w - w \cdot \nabla v^* - w \cdot \nabla w, \\ g^2 &= -(\Phi(\sigma^* + \sigma) - \Phi(\sigma^*))\nabla\sigma^* - (\Phi(\sigma^* + \sigma) - \Phi(0))\nabla\sigma, \\ g^3 &= (\Psi(\sigma^* + \sigma) - \Psi(\sigma^*))\mathcal{A}_0(v^* + w), \quad g^4 = (\Psi(\sigma^*) - \Psi(0))\mathcal{A}_0 w, \\ \Phi(\zeta) &= \frac{P'(\zeta + \rho_\infty)}{\zeta + \rho_\infty}, \quad \Psi(\zeta) = \frac{1}{\zeta + \rho_\infty}. \end{aligned}$$

Next, we present some estimates for the solution to the linearized compressible Navier–Stokes equation around the constant state $(\rho_\infty, 0)$:

$$\begin{cases} \partial_t b + \gamma\mathrm{div}\, u = 0, \\ \partial_t u - \mathcal{A}u + \gamma\nabla b = 0, \end{cases} \tag{18}$$

where $\gamma > 0$ and $\mathcal{A} = \nu\Delta + (\nu + \nu')\nabla\mathrm{div}$ with $\nu > 0$, $2\nu/3 + \nu' \geq 0$; a is a scaler function and u is a 3-vector valued function. Let e^{tA} be the semigroup associated with the linear equation (18):

$$e^{tA}U_0 = \mathcal{F}^{-1}\left[e^{t\hat{A}(\xi)}\widehat{U_0}\right], \quad U_0 = (U_{0,1}, \ldots, U_{0,4})^\mathsf{T} \in \mathcal{S}'(\mathbb{R}^3)^4, \tag{19}$$

where $\hat{A}(\xi)$ is the matrix of the form:

$$\hat{A}(\xi) = \begin{bmatrix} 0 & & -i\gamma\xi^\mathsf{T} \\ -i\gamma\xi & -\nu|\xi|^2\mathrm{I}_3 & -(\nu + \nu')\xi \otimes \xi \end{bmatrix}. \tag{20}$$

Here $\xi = (\xi_1, \xi_2, \xi_3)^\mathsf{T} \in \mathbb{R}^3$, $\xi \otimes \xi = \xi\xi^\mathsf{T}$ and I_3 is the 3×3 identity matrix. By direct calculation, the eigenvalues of $\hat{A}(\xi)$ are given by

$$\lambda_\pm(\xi) = -\frac{2\nu + \nu'}{2}|\xi|^2 \pm \frac{\sqrt{(2\nu + \nu')^2|\xi|^4 - 4\gamma^2|\xi|^2}}{2}, \quad \lambda_0(\xi) = -\nu|\xi|^2. \tag{21}$$

We set $P_\pm(\xi)$:

$$P_\pm(\xi) = \frac{V_\pm \otimes V_\pm}{V_\pm \cdot V_\pm} \quad \text{with } V_\pm = \begin{bmatrix} -i\lambda_\pm^{-1}\gamma|\xi|^2 \\ \xi \end{bmatrix}$$

for $|\xi| \neq 0, \eta_0$, where $\eta_0 = \gamma/(\nu + \nu'/2)$, $V_\pm \cdot V_\pm \equiv V_\pm^\top V_\pm$, and set the eigenprojection $P_0(\xi)$:

$$P_0(\xi) = \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{I}_3 - \frac{\xi\otimes\xi}{|\xi|^2} \end{bmatrix}.$$

Since $P_+(\xi) + P_-(\xi) + P_0(\xi) = \mathrm{I}_4$, we have the spectral resolution

$$e^{t\hat{A}(\xi)} = e^{\lambda_+ t} P_+(\xi) + e^{\lambda_- t} P_-(\xi) + e^{\lambda_0 t} P_0(\xi) \quad \text{for } |\xi| \neq 0, \eta_0. \tag{22}$$

If $|\xi| = \eta_0$, then we have

$$e^{t\hat{A}(\xi)} = e^{-\nu_0|\xi|^2 t} \begin{bmatrix} 1 - \nu_0|\xi|^2 t & -i\gamma\xi^\top t \\ -i\gamma\xi t & (1-\nu_0|\xi|^2 t)\frac{\xi\otimes\xi}{|\xi|^2} \end{bmatrix} + e^{-\nu|\xi|^2 t} P_0(\xi), \tag{23}$$

where $\nu_0 = \nu + \nu'/2$. This spectral resolution will be used in the proof of Theorem 1.6 in Sect. 5.3.

The following lemma shows some smoothing estimate for the low frequency part of the semigroup e^{tA} and its adjoint e^{tA^*}. This lemma has been proved in [2, Proposition 10.22]. (Cf. [4, 6, 21].)

Lemma 4.1 *Let $j_0 \in \mathbb{Z}$, $s \in \mathbb{R}$. Set $e_L^{tA} \equiv \dot{S}_{j_0} e^{tA}$, $e_L^{tA^*} \equiv \dot{S}_{j_0} e^{tA^*}$, where $\dot{S}_{j_0}$ is the low frequency cut-off operator defined in* (13).

(i) *For any $U_0 \in \dot{B}^s_{2,r}$ and $\alpha \geq 0$, we have*

$$\|e_L^{tA} U_0\|_{\dot{B}^{s+\alpha}_{2,r}},\ \|e_L^{tA^*} U_0\|_{\dot{B}^{s+\alpha}_{2,r}} \lesssim_{\alpha, j_0} (1+t)^{-\frac{\alpha}{2}} \|U_0\|_{\dot{B}^s_{2,r}} \tag{24}$$

for any $1 \leq r \leq \infty$.

(ii) *The following time-space integral estimate holds*:

$$\int_0^\infty \|e_L^{tA} U_0\|_{\dot{B}^{s+2}_{2,1}} dt,\ \int_0^\infty \|e_L^{tA^*} U_0\|_{\dot{B}^{s+2}_{2,1}} dt \lesssim_{j_0} \|U_0\|_{\dot{B}^s_{2,1}} \tag{25}$$

for any $U_0 \in \dot{B}^s_{2,1}$.

4.1 Existence of non-stationary solutions

Let us prove the global existence result in Theorem 1.2. We shall prove the following theorem.

Theorem 4.2 *Let (ρ^*, v^*) be a stationary solution of* (2) *satisfying* (7) *with $\|F\|_{\dot{B}^{-3/2}_{2,\infty}\cap\dot{H}^3}$ sufficiently small. Then, there exists a constant $\delta > 0$ such that if (σ_0, w_0) satisfy*

$$\|(\sigma_0, w_0)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}\cap\dot{H}^3} \leq \delta,$$

then the Eq. (17) *with initial data* (σ_0, w_0) *has a global solution* (σ, w) *satisfying* $(\sigma, w) \in C^0([0,\infty); \dot{B}^{1/2}_{2,\infty} \cap \dot{H}^3)$ *and*

$$\sup_{t>0} \|(\sigma, w)(t)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3} \lesssim \delta.$$

First, we show the a priori estimate for the perturbation.

Proposition 4.3 *Let* (ρ^*, v^*) *be a stationary solution of* (2) *satisfying* (7) *with* $\|F\|_{\dot{B}^{-3/2}_{2,\infty} \cap \dot{H}^3}$ *sufficiently small. Let* $(\sigma, w) \in C^0([0,T); \dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3)$, $w \in L^2_{loc}([0,T); \dot{H}^4)$ *be a solution of* (17) *with initial value* (σ_0, w_0) *for some* T, $0 < T \le \infty$. *Then, there exist constants* $\delta_1 > 0$ *and* $C_1 > 0$ *such that if*

$$\sup_{0\le t<T} \|(\sigma, w)(t)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3} \le \delta_1,$$

then we have

$$\sup_{0\le t<T} \|(\sigma, w)(t)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3} \le C_1 \|(\sigma_0, w_0)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3}. \tag{26}$$

Proof Let us denote $U = (\sigma, w)$ and $U_0 = (\sigma_0, w_0)$. Fix $j_0 \in \mathbb{Z}$ and decompose

$$U = U_L + U_H = (\sigma_L, w_L) + (\sigma_H, w_H), \tag{27}$$

where $U_L = (\sigma_L, w_L) \equiv (\dot{S}_{j_0}\sigma, \dot{S}_{j_0}w)$. In order to estimate the low frequency part U_L, we shall rewrite the perturbation equation (17) in the momentum formulation. Let

$$m \equiv \rho v, \quad m^* \equiv \rho^* v^* \quad \text{and} \quad n \equiv \frac{m - m^*}{\rho_\infty}.$$

Then, the pair of functions $V = (\sigma, n) = (\rho - \rho^*, n)$ satisfies the system of equations:

$$\begin{cases} \partial_t \sigma + \gamma_1 \operatorname{div} n = 0, \\ \partial_t n - \mathcal{A}_1 n + \gamma_1 \nabla \sigma = h + \gamma_1^{-1} \sigma F(x), \end{cases} \tag{28}$$

where $\gamma_1 = P'(\rho_\infty)^{1/2} = \rho_\infty$, $\mathcal{A}_1 = \mu\Delta + (\mu + \mu')\nabla \operatorname{div}$; h is defined by

$$h = \sum_{i=1}^{4} h_i \tag{29}$$

with

$$
\begin{aligned}
h_1 &= -\mathrm{div}\left(\frac{n\otimes m}{\rho}+\frac{m^*\otimes n}{\rho}+\gamma_1^{-1}\left(\Psi(\sigma^*+\sigma)-\Psi(\sigma^*)\right)m^*\otimes m\right),\\
h_2 &= -\nabla\left(\Pi(\sigma^*,\sigma)\sigma\right),\quad h_3=\mathcal{A}_1\left(\left(\Psi(\sigma^*+\sigma)-\Psi(0)\right)n\right),\\
h_4 &= \gamma_1^{-1}\mathcal{A}_1\left((\Psi(\sigma^*+\sigma)-\Psi(\sigma^*))m^*\right),\\
\Pi(\zeta_1,\zeta_2) &= \int_0^1\left(P'(\zeta_1+\theta\zeta_2+\rho_\infty)-P'(\rho_\infty)\right)d\theta,\quad \Psi(\zeta)=\frac{1}{\zeta+\rho_\infty}.
\end{aligned}
$$

Let e^{tA} be the semigroup defined in (19) with $\gamma=\gamma_1$ and $\mathcal{A}=\mathcal{A}_1$. Then, the Duhamel principle gives

$$
V_L(t)=e_L^{tA}V_0+\int_0^t e_L^{(t-\tau)A}\begin{bmatrix}0\\ h+\gamma_1^{-1}\sigma F(x)\end{bmatrix}(\tau)\,d\tau, \tag{30}
$$

where $V_L\equiv\dot{S}_{j_0}V$, $e_L^{tA}\equiv\dot{S}_{j_0}e^{tA}$ and $V_0=V(0)$. Thanks to Lemma 4.1, for any $\psi=(\psi_1,\dots,\psi_4)^{\mathsf{T}}\in\mathcal{S}^4$, we have

$$
\begin{aligned}
\langle V_L(t),\psi\rangle &= \langle e_L^{tA}V_0,\psi\rangle+\int_0^t\left\langle\begin{bmatrix}0\\ h+\gamma_1^{-1}\sigma F\end{bmatrix}(\tau),e_L^{(t-\tau)A^*}\psi\right\rangle d\tau\\
&\lesssim \|V_0\|_{\dot{B}^{\frac12}_{2,\infty}}\|\psi\|_{\dot{B}^{-\frac12}_{2,1}}+\sup_{0\le t<T}\|h+\gamma_1^{-1}\sigma F\|_{\dot{B}^{-\frac32}_{2,\infty}}\int_0^\infty\|e_L^{\tau A^*}\psi\|_{\dot{B}^{\frac32}_{2,1}}\,d\tau\\
&\lesssim\left(\|V_0\|_{\dot{B}^{\frac12}_{2,\infty}}+\sup_{0\le t<T}\|h+\gamma_1^{-1}\sigma F\|_{\dot{B}^{-\frac32}_{2,\infty}}\right)\|\psi\|_{\dot{B}^{-\frac12}_{2,1}},
\end{aligned}
$$

where $0\le t<T$. Then, Proposition 2.1(ii) implies that

$$
\sup_{0\le t<T}\|V_L(t)\|_{\dot{B}^{\frac12}_{2,\infty}}\lesssim\|V_0\|_{\dot{B}^{\frac12}_{2,\infty}}+\sup_{0\le t<T}\|h+\gamma_1^{-1}\sigma F\|_{\dot{B}^{-\frac32}_{2,\infty}}.
$$

Applying the bilinear estimate in Proposition 2.1(vi), we have

$$
\|h+\gamma_1^{-1}\sigma F\|_{\dot{B}^{-\frac32}_{2,\infty}}\lesssim\left(\|(\sigma^*,v^*,\sigma,w)\|_{\dot{B}^{\frac12}_{2,\infty}\cap\dot{B}^{\frac32}_{2,1}}+\|F\|_{\dot{B}^{-\frac12}_{2,1}}\right)\|U\|_{\dot{B}^{\frac12}_{2,\infty}}.
$$

Thus, if $\delta_1>0$ is small enough, then

$$
\sup_{0\le t<T}\|U_L(t)\|_{\dot{B}^{\frac12}_{2,\infty}}\lesssim\|U_0\|_{\dot{B}^{\frac12}_{2,\infty}}+\delta_1\sup_{0\le t<T}\|U(t)\|_{\dot{B}^{\frac12}_{2,\infty}}.
$$

Next, we estimate the high-frequency part U_H. For any $u\in\mathcal{S}'$, we denote $u_L\equiv\dot{S}_{j_0}u$ and $u_H\equiv u-\dot{S}_{j_0}u$. Since $U=(\sigma,w)$ satisfies the Eq. (17), for any multi-index α_1,α_2, we have the following identities:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_x^{\alpha_1}U_H\|_{L^2}^2+\nu\|\partial_x^{\alpha_1}\nabla w_H\|_{L^2}^2+(\nu+\nu')\|\partial_x^{\alpha_1}\mathrm{div}\, w_H\|_{L^2}^2\\
&\quad=\langle\partial_x^{\alpha_1}f_H,\partial_x^{\alpha_1}\sigma_H\rangle+\langle\partial_x^{\alpha_1}g_H,\partial_x^{\alpha_1}w_H\rangle,\\
&\frac{d}{dt}\langle\partial_x^{\alpha_2}\nabla\sigma_H,\partial_x^{\alpha_2}w_H\rangle+\gamma_0\|\partial_x^{\alpha_2}\nabla\sigma_H\|_{L^2}^2\\
&\quad=\gamma_0\|\partial_x^{\alpha_2}\mathrm{div}\, w_H\|_{L^2}^2+\langle\partial_x^{\alpha_2}\mathcal{A}w_H,\partial_x^{\alpha_2}\nabla\sigma_H\rangle\\
&\qquad+\langle\partial_x^{\alpha_2}\nabla f_H,\partial_x^{\alpha_2}w_H\rangle+\langle\partial_x^{\alpha_2}g_H,\partial_x^{\alpha_2}\nabla\sigma_H\rangle.
\end{aligned}
$$

Let $\kappa>0$ be small enough. We set

$$
\mathcal{E}(t)=\frac{1}{2}\sum_{|\alpha_1|=3}\|\partial_x^{\alpha_1}U_H(t)\|_{L^2}^2+\sum_{|\alpha_2|=2}\kappa\langle\partial_x^{\alpha_2}\nabla\sigma_H(t),\partial_x^{\alpha_2}w_H(t)\rangle.
$$

Then, we have the following inequality

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E}(t)+\widetilde{\mathcal{E}}(t)\lesssim&\sum_{|\alpha_1|=3}\langle\partial_x^{\alpha_1}f_H,\partial_x^{\alpha_1}\sigma_H\rangle+\langle\partial_x^{\alpha_1}g_H,\partial_x^{\alpha_1}w_H\rangle\\
&+\sum_{|\alpha_2|=2}\kappa\langle\partial_x^{\alpha_2}\nabla f_H,\partial_x^{\alpha_2}w_H\rangle+\langle\partial_x^{\alpha_2}g_H,\partial_x^{\alpha_2}\nabla\sigma_H\rangle,
\end{aligned}\tag{31}
$$

where $0<t<T$ and

$$
\begin{aligned}
\widetilde{\mathcal{E}}(t)\equiv&\sum_{|\alpha_1|=3}\left(\nu\|\partial_x^{\alpha_1}\nabla w_H(t)\|_{L^2}^2+(\nu+\nu')\|\partial_x^{\alpha_1}\mathrm{div}\, w_H(t)\|_{L^2}^2\right)\\
&+\sum_{|\alpha_2|=2}\kappa\left(\|\partial_x^{\alpha_2}\nabla\sigma_H(t)\|_{L^2}^2+\|\partial_x^{\alpha_2}\mathrm{div}\, w_H(t)\|_{L^2}^2\right).
\end{aligned}
$$

We also have

$$
\mathcal{E}(t)\sim\|(\sigma_H,w_H)(t)\|_{\dot H^3}^2,\quad\widetilde{\mathcal{E}}(t)\sim_{\kappa,j_0}\|\sigma_H(t)\|_{\dot H^3}^2+\|w_H(t)\|_{\dot H^4}^2
$$

for $0\le t<T$. Let us estimate the right-hand of (31). For any α_1 with $|\alpha_1|=3$, by Proposition 2.1(vi), we have the following estimate

$$
\begin{aligned}
\langle(v\cdot\nabla\partial_x^{\alpha_1}\sigma)_H,\partial_x^{\alpha_1}\sigma_H\rangle\lesssim&|\langle v\cdot\nabla\partial_x^{\alpha_1}\sigma_H,\partial_x^{\alpha_1}\sigma_H\rangle|+\big(\|v\cdot\nabla\partial_x^{\alpha_1}\sigma_L\|_{L^2}\\
&+\|(v\cdot\nabla\partial_x^{\alpha_1}\sigma)_L\|_{L^2}\big)\|\partial_x^{\alpha_1}\sigma_H\|_{L^2}\\
\lesssim_{j_0}&\|\mathrm{div}\, v\|_{L^\infty}\|\partial_x^{\alpha_1}\sigma_H\|_{L^2}^2+\|v\|_{\dot B^{\frac{3}{2}}_{2,1}}\|\partial_x^{\alpha_1}\sigma\|_{L^2}\|\partial_x^{\alpha_1}\sigma_H\|_{L^2},
\end{aligned}
$$

where $v=v^*+w$. For any α_1 with $|\alpha_1|=3$, the above estimate leads us to

$$
\begin{aligned}
&\langle \partial_x^{\alpha_1} f_H, \partial_x^{\alpha_1}\sigma_H\rangle \\
&\quad = \langle (v\cdot\nabla\partial_x^{\alpha_1}\sigma)_H, \partial_x^{\alpha_1}\sigma_H\rangle + \langle ((\operatorname{div} v)\partial_x^{\alpha_1}\sigma)_H, \partial_x^{\alpha_1}\sigma_H\rangle \\
&\qquad + \sum_{0<\beta\le\alpha_1} \langle \operatorname{div}(\partial_x^{\beta} v\, \partial_x^{\alpha_1-\beta}\sigma)_H, \partial_x^{\alpha_1}\sigma_H\rangle + \langle \partial_x^{\alpha_1}\operatorname{div}(\sigma^* w)_H, \partial_x^{\alpha_1}\sigma_H\rangle \\
&\quad \lesssim \|\operatorname{div} v\|_{L^\infty}\|\partial_x^{\alpha_1}\sigma_H\|_{L^2}^2 + \|v\|_{\dot B^{\frac{3}{2}}_{2,1}\cap \dot B^{\frac{5}{2}}_{2,1}}\|\partial_x^{\alpha_1}\sigma\|_{L^2}\|\partial_x^{\alpha_1}\sigma_H\|_{L^2} \\
&\qquad + \|\nabla v\|_{H^3}\|\nabla\sigma\|_{H^2}\|\partial_x^{\alpha_1}\sigma_H\|_{L^2} + \|\nabla\sigma^*\|_{H^3}\|\nabla w\|_{H^3}\|\partial_x^{\alpha_1}\sigma_H\|_{L^2} \\
&\quad \lesssim \delta_1\big(\|U\|_{\dot B^{\frac{1}{2}}_{2,\infty}\cap \dot H^3} + \|w\|_{\dot H^4}\big)\|\partial_x^{\alpha_1}\sigma_H\|_{L^2} \lesssim \delta\|U\|^2_{\dot B^{\frac{1}{2}}_{2,\infty}\cap \dot H^3} + \delta_1\widetilde{\mathcal{E}}.
\end{aligned}
$$

We also have

$$
\langle \partial_x^{\alpha_2}\nabla f_H, \partial_x^{\alpha_2} w_H\rangle \lesssim \delta_1\|U\|^2_{\dot B^{\frac{1}{2}}_{2,\infty}\cap \dot H^3} + \delta_1\widetilde{\mathcal{E}} \quad \text{for } |\alpha_2| = 2.
$$

By Proposition 2.1(i), (v) and (vi), we have

$$
\sum_{|\alpha_1|=3}\langle \partial_x^{\alpha_1} g_H, \partial_x^{\alpha_1} w_H\rangle + \sum_{|\alpha_2|=2}\kappa\langle \partial_x^{\alpha_2} g_H, \partial_x^{\alpha_2}\nabla\sigma_H\rangle \lesssim \delta_1\|U\|^2_{\dot B^{\frac{1}{2}}_{2,\infty}\cap \dot H^3} + \delta_1\widetilde{\mathcal{E}}.
$$

Thus, if $\delta_1 > 0$ is sufficiently small, then there exits a constant $c_0 > 0$ such that

$$
\frac{d}{dt}\mathcal{E}(t) + c_0\widetilde{\mathcal{E}}(t) \lesssim \delta_1\|U(t)\|^2_{\dot B^{\frac{1}{2}}_{2,\infty}\cap \dot H^3} \quad \text{for } 0 < t < T.
$$

Since $\mathcal{E} \lesssim_{j_0} \widetilde{\mathcal{E}}$, we have the estimate

$$
\frac{d}{dt}\mathcal{E}(t) + c_0\mathcal{E}(t) \lesssim \delta_1\|U(t)\|^2_{\dot B^{\frac{1}{2}}_{2,\infty}\cap \dot H^3} \quad \text{for } 0 < t < T.
$$

Then, Grönwall's inequality shows that

$$
\begin{aligned}
\|U_H(t)\|^2_{\dot H^3} &\lesssim e^{-c_0 t}\|U_H(0)\|^2_{\dot H^3} + \delta_1\int_0^t e^{-c_0(t-\tau)}\|U(\tau)\|^2_{\dot B^{\frac{1}{2}}_{2,\infty}\cap \dot H^3}\,d\tau \\
&\lesssim \|U_0\|^2_{\dot H^3} + \delta_1 \sup_{0\le\tau<T}\|U(\tau)\|^2_{\dot B^{\frac{1}{2}}_{2,\infty}\cap \dot H^3}
\end{aligned}
$$

for $0 \le t < T$. Thus, if $\delta_1 > 0$ is small enough, then the estimate (26) holds. □

Next, we show the following local existence result. The proof of Proposition 4.4 below is inspired by the argument in Danchin [8]. Since $(\rho^* - \rho_\infty, v^*)$ is in $\dot B^{\frac{1}{2}}_{2,\infty} \cap \dot H^3$, it suffices to show the following proposition to prove the local existence of (17).

Proposition 4.4 *There exist constants* $T_0 > 0$, $\delta_{2,0} > 0$ *and* $C_2 > 0$ *such that if an external force* $F(x)$ *and an initial value* $(\rho_0, v_0) = (b_0 + \rho_\infty, v_0)$ *satisfy*

$$\|F\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty} \cap \dot{H}^2} + \|(b_0, v_0)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3} \leq \delta_2 \tag{32}$$

with $\delta_2 \leq \delta_{2,0}$, *then there exists a unique solution* $(\rho, v) = (b + \rho_\infty, v)$ *of* (1) *on* $[0, T_0) \times \mathbb{R}^3$ *satisfying* $(b, v)|_{t=0} = (b_0, v_0)$, $(b, v) \in C^0([0, T_0); \dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3)$, $v \in L^2_t((0, T_0); \dot{H}^4)$ *and*

$$\sup_{0 \leq t < T_0} \|(b, v)(t)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3} \leq C_2 \delta_2.$$

Proof Let $0 < T_0 \leq 1$, and let $0 < \delta_2 \leq \delta_{2,0} \leq 1$ be satisfying (32). We denote the function spaces X_{T_0} and Y_{T_0} by

$$X_{T_0} \equiv \left\{ (b, v) \in C^0([0, T_0); \dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3) \,\middle|\, v \in L^2_t((0, T_0); \dot{H}^4) \right\},$$
$$Y_{T_0} \equiv \left\{ (b, v) \in C^0([0, T_0); \dot{B}^{\frac{1}{2}}_{2,\infty}) \,\middle|\, v \in C^0([0, T_0); \dot{B}^{\frac{3}{2}}_{2,\infty}) \right\}$$

with norms

$$\|(b, v)\|_{X_{T_0}} \equiv \|(b, v)\|_{C^0([0,T_0); \dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3)} + \|v\|_{L^2_t((0,T_0); \dot{H}^4)},$$
$$\|(b, v)\|_{Y_{T_0}} \equiv \|(b, v)\|_{C^0([0,T_0); \dot{B}^{\frac{1}{2}}_{2,\infty})} + \|v\|_{C^0([0,T_0); \dot{B}^{\frac{3}{2}}_{2,\infty})}.$$

We denote the norm of the Chemin–Lerner space $\tilde{L}^\infty_t((0, T_0); \dot{H}^3)$ by

$$\|u\|_{\tilde{L}^\infty_t((0,T_0); \dot{H}^3)} \equiv \left\| \{2^{3j} \|\dot{\Delta}_j u\|_{L^\infty_t((0,T_0); L^2)}\}_{j \in \mathbb{Z}} \right\|_{\ell^2(\mathbb{Z})}.$$

(Cf. [3].) The Minkowski inequality implies the estimate

$$\|u\|_{L^\infty_t((0,T_0); \dot{H}^3)} \leq \|u\|_{\tilde{L}^\infty_t((0,T_0); \dot{H}^3)}.$$

Let us construct the approximation sequence

$$\{U_n\}_{n \geq -1} \equiv \{(b_n, v_n)^\mathsf{T}\}_{n \geq -1} \subset X_{T_0}$$

as follows: The sequence $\{U_n\}_{n \geq -1}$ defined by $(b_{-1}, v_{-1}) = (0, 0)$ and

$$\begin{cases} \partial_t b_{n+1} + v_n \cdot \nabla b_{n+1} = f_0(b_n, v_n), \\ \partial_t v_{n+1} - \mathcal{A}_0 v_{n+1} = g_0(b_n, v_n), \\ (b_{n+1}, v_{n+1})|_{t=0} = (b_0, v_0), \end{cases} \tag{33}$$

where $\nu_0 = \mu/\rho_\infty$, $\nu_0' = \mu'/\rho_\infty$ and $\mathcal{A}_0 \equiv \nu_0\Delta + (\nu_0 + \nu_0')\nabla\mathrm{div}$; f_0 and g_0 are defined by the following:

$$f_0(b, v) = \rho_\infty \mathrm{div}\, v + (\mathrm{div}\, v)b, \quad g_0(b, v) = g_{0,1}(b, v) + g_{0,2}(b, v),$$
$$g_{0,1}(b, v) = -\Phi(b)\nabla b - v \cdot \nabla v, \quad g_{0,2}(b, v) = (\Psi(b) - \Psi(0))\mathcal{A}_0 v + F,$$
$$\Phi(\zeta) = \frac{P'(\zeta + \rho_\infty)}{\zeta + \rho_\infty}, \quad \Psi(\zeta) = \frac{1}{\zeta + \rho_\infty}.$$

Here, the solutions of the first equation of (33) is given by

$$b_{n+1}(t, x) = b_0(\psi_n^{-1}(t, x)) + \int_0^t f_0(b_n, v_n)(\tau, \psi_n(\tau, \psi_n^{-1}(t, x)))d\tau,$$

where ψ_n is the solution of the ordinal differential equation:

$$\begin{cases} \partial_t \psi_n(t, x) = v_n(t, \psi_n(t, x)), \\ \psi_n(0, x) = x. \end{cases}$$

We shall prove by induction that if $T_0 > 0$ and $\delta_{2,0} > 0$ are small enough, then for any $n \geq 1$,

$$\|U_n\|_{X_{T_0}} + \|U_n\|_{\tilde{L}_t^\infty((0,T_0);\dot{H}^3)} \lesssim \delta_2, \tag{34}$$
$$\|U_n - U_{n-1}\|_{Y_{T_0}} \leq \frac{1}{2}\|U_{n-1} - U_{n-2}\|_{Y_{T_0}}. \tag{35}$$

Let $n \geq 1$ and assume that the inequalities (34), (35) hold for $1 \leq k \leq n$.

Since $U_{n+1} = (b_{n+1}, v_{n+1})^\mathsf{T}$ is a solution of (33), for any $j \in \mathbb{Z}$, we have the energy estimates

$$\begin{aligned}\frac{d}{dt}\|\dot{\Delta}_j v_{n+1}\|_{L^2}^2 + c_0\|\nabla\dot{\Delta}_j v_{n+1}\|_{L^2}^2 &\lesssim \langle \dot{\Delta}_j g_0(b_n, v_n), \dot{\Delta}_j v_{n+1}\rangle \\ &\lesssim 2^{-j}\|\dot{\Delta}_j g_0(b_n, v_n)\|_{L^2}\|\nabla\dot{\Delta}_j v_{n+1}\|_{L^2},\end{aligned}$$

where $c_0 > 0$ is a constant, and

$$\begin{aligned}&\frac{d}{dt}\|\nabla\dot{\Delta}_j b_{n+1}\|_{L^2}^2 \\ &\quad\lesssim \langle -\nabla(\dot{\Delta}_j(v_n \cdot \nabla b_{n+1})) + \nabla\dot{\Delta}_j f_0(b_n, v_n), \nabla\dot{\Delta}_j b_{n+1}\rangle \\ &\quad\lesssim |\langle \nabla(\dot{\Delta}_j(v_n \cdot \nabla b_{n+1})), \nabla\dot{\Delta}_j b_{n+1}\rangle| + \|\nabla\dot{\Delta}_j f_0(b_n, v_n)\|_{L^2}\|\nabla\dot{\Delta}_j b_{n+1}\|_{L^2}.\end{aligned}$$

For any $0 \leq t < T_0$, by Proposition 2.1(v), (vi) and Lemma 2.3, there exists a sequence $\{d_j(t)\}_{j\in\mathbb{Z}}$ with $\|\{d_j(t)\}\|_{\ell^2(\mathbb{Z})} \leq 1$ such that

$$\|\dot{\Delta}_j g_{0,1}(b_n, v_n)(t)\|_{L^2} \lesssim 2^{-2j}d_j(t)\|g_{0,1}(b_n, v_n)(t)\|_{\dot{H}^2} \lesssim 2^{-2j}d_j(t)\delta_2^2,$$

$$
\begin{aligned}
\|\dot{\Delta}_j g_{0,2}(b_n, v_n)(t)\|_{L^2} &\lesssim 2^{-2j} d_j(t) \|g_{0,2}(b_n, v_n)(t)\|_{\dot{H}^2} \\
&\lesssim 2^{-2j} d_j(t) \big(\|b_n(t)\|_{\dot{H}^1 \cap \dot{H}^3} \|v_n(t)\|_{\dot{H}^2 \cap \dot{H}^4} + \|F\|_{\dot{H}^2} \big) \\
&\lesssim 2^{-2j} d_j(t) \delta_2 \|v_n(t)\|_{\dot{H}^4} + 2^{-2j} d_j(t) \delta_2^2 + 2^{-2j} d_j(t) \delta_2, \\
\|\nabla \dot{\Delta}_j f_0(b_n, v_n)(t)\|_{L^2} &\lesssim 2^{-2j} d_j(t) \|v_n(t)\|_{\dot{H}^1 \cap \dot{H}^4} \|b_n(t)\|_{\dot{H}^1 \cap \dot{H}^3} \\
&\lesssim 2^{-2j} d_j(t) \delta_2 \|v_n(t)\|_{\dot{H}^4} + 2^{-2j} d_j(t) \delta_2^2
\end{aligned}
$$

and

$$
\begin{aligned}
&|\langle \nabla(\dot{\Delta}_j (v_n \cdot \nabla b_{n+1})), \nabla \dot{\Delta}_j b_{n+1} \rangle (t)| \\
&\quad \lesssim \|[\dot{\Delta}_j, v_n \cdot \nabla] \nabla b_{n+1}(t)\|_{L^2} \|\nabla \dot{\Delta}_j b_{n+1}(t)\|_{L^2} + \|\mathrm{div} v_n(t)\|_{L^\infty} \|\nabla \dot{\Delta}_j b_{n+1}(t)\|_{L^2}^2 \\
&\qquad + 2^{-2j} d_j(t) \|U_n(t)\|_{\dot{H}^1 \cap \dot{H}^3}^2 \|\nabla \dot{\Delta}_j b_{n+1}(t)\|_{L^2} \\
&\quad \lesssim 2^{-2j} d_j(t) \delta_2 \big(\|b_{n+1}(t)\|_{\dot{H}^3} + \delta_2 \big) \|\nabla \dot{\Delta}_j b_{n+1}(t)\|_{L^2} + \delta_2 \|\nabla \dot{\Delta}_j b_{n+1}(t)\|_{L^2}^2.
\end{aligned}
$$

Here, we use the identities:

$$
\langle v_n \cdot \nabla \partial_k \dot{\Delta}_j b_{n+1}, \partial_k \dot{\Delta}_j b_{n+1} \rangle = -\frac{1}{2} \langle \mathrm{div}\, v_n\, \partial_k \dot{\Delta}_j b_{n+1}, \partial_k \dot{\Delta}_j b_{n+1} \rangle
$$

for $1 \le k \le 3$. Thus, if $T_0 > 0$ is small enough, we have the estimate

$$
\|U_{n+1}\|_{\tilde{L}_t^\infty((0,T_0);\dot{H}^3)} + \|v_{n+1}\|_{L_t^2((0,T_0);\dot{H}^4)} \lesssim \delta_2 + \delta_2^{\frac{1}{2}} \|U_{n+1}\|_{X_{T_0}}.
$$

By the Duhamel principle, we may rewrite the second equation of (33) by

$$
v_{n+1}(t) = e^{t\mathcal{A}_0} v_0 + \int_0^t e^{(t-\tau)\mathcal{A}_0} g_0(b_n, v_n)(\tau) d\tau.
$$

We use the following estimate for the semigroup $e^{t\mathcal{A}_0}$. □

Lemma 4.5 *Let $s \in \mathbb{R}$, $1 \le r \le \infty$ and $\alpha \ge 0$. For any $\psi_1 \in \dot{B}^s_{2,r}$, $\psi_2 \in \dot{B}^s_{2,1}$, we have*

$$
\|e^{t\mathcal{A}_0} \psi_1\|_{\dot{B}^{s+\alpha}_{2,r}} \lesssim t^{-\frac{\alpha}{2}} \|\psi_1\|_{\dot{B}^s_{2,r}}, \quad \int_0^t \|e^{\tau\mathcal{A}_0} \psi_2\|_{\dot{B}^{s+2}_{2,1}} d\tau \lesssim \|\psi_2\|_{\dot{B}^s_{2,1}} \quad \text{for } t > 0.
$$

As for the proof, see [4, Proposition 2.1], [7, Corollary 2.7] for example.

By Proposition 2.1(v), (vi) and Lemma 4.5, for any $0 \le t < T_0$, we have

$$
\|e^{t\mathcal{A}_0} v_0\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} \lesssim \|v_0\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}},
$$

$$\left\| \int_0^t e^{(t-\tau)\mathcal{A}_0} g_0(b_n, v_n)(\tau) d\tau \right\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} \lesssim \int_0^t \tau^{-\frac{1}{2}} \|g_0(b_n, v_n)(\tau)\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty}} d\tau$$
$$\lesssim T_0{}^{\frac{1}{2}} \left(\|U_n\|_{X_{T_0}} + \|U_n\|^2_{X_{T_0}} + \|F\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty}} \right).$$

Let us write $g_0(b_n, v_n) - g_0(b_{n-1}, v_{n-1}) = \tilde{g}_1 + \tilde{g}_2 + \tilde{g}_3$ with

$$\begin{aligned} \tilde{g}_1 &= -(\Psi(b_n) - \Psi(b_{n-1}))\nabla b_n - (v_n - v_{n-1}) \cdot \nabla v_n, \\ \tilde{g}_2 &= -\Psi(b_{n-1})\nabla(b_n - b_{n-1}) - v_{n-1} \cdot \nabla(v_n - v_{n-1}), \\ \tilde{g}_3 &= g_{0,2}(b_n, v_n) - g_{0,2}(b_{n-1}, v_{n-1}). \end{aligned}$$

By Proposition 2.1(v), (vi) and Lemma 4.5, for any $0 \le t < T_0$, we have

$$\left\| \int_0^t e^{(t-\tau)\mathcal{A}_0} \tilde{g}_1(\tau) d\tau \right\|_{\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{B}^{\frac{3}{2}}_{2,\infty}} \lesssim \int_0^t \tau^{-\frac{1}{2}} \|\tilde{g}_1(\tau)\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty} \cap \dot{B}^{\frac{1}{2}}_{2,\infty}} d\tau$$
$$\lesssim T_0^{\frac{1}{2}} (1 + \|(U_n, U_{n-1})\|_{X_{T_0}}) \sup_{0 \le \tau < T_0} \|(U_n - U_{n+1})(\tau)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}}$$

and

$$\left\| \int_0^t e^{(t-\tau)\mathcal{A}_0} \tilde{g}_2(\tau) d\tau \right\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} \lesssim \int_0^t \tau^{-\frac{1}{2}} \|\tilde{g}_2(\tau)\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty}} d\tau$$
$$\lesssim T_0^{\frac{1}{2}} (1 + \|(U_n, U_{n-1})\|_{X_{T_0}}) \sup_{0 \le \tau < T_0} \|(U_n - U_{n+1})(\tau)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}}.$$

For any $\psi \in \mathcal{S}$ and $0 \le t < T_0$, by Proposition 2.1(ii), (v), (vi) and Lemma 4.5, we have

$$\left\langle \int_0^t e^{(t-\tau)\mathcal{A}_0} \tilde{g}_2(\tau) d\tau, \psi \right\rangle \lesssim \sup_{0 \le \tau < T_0} \|\tilde{g}_2(\tau)\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty}} \int_0^\infty \|e^{\tau \mathcal{A}_0} \psi\|_{\dot{B}^{\frac{1}{2}}_{2,1}} d\tau$$
$$\lesssim \|(U_n, U_{n-1})\|_{X_{T_0}} \sup_{0 \le \tau < T_0} \|(U_n - U_{n-1})(\tau)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} \|\psi\|_{\dot{B}^{-\frac{3}{2}}_{2,1}}$$

and

$$\left\langle \int_0^t e^{(t-\tau)\mathcal{A}_0} \tilde{g}_3(\tau) d\tau, \psi \right\rangle$$
$$\lesssim \sup_{0 \le \tau < T_0} \|\tilde{g}_3(\tau)\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty} \cap \dot{B}^{-\frac{1}{2}}_{2,\infty}} \int_0^\infty \|e^{\tau \mathcal{A}_0} \psi\|_{\dot{B}^{\frac{3}{2}}_{2,1} + \dot{B}^{\frac{1}{2}}_{2,1}} d\tau$$
$$\lesssim \|(U_n, U_{n-1})\|_{X_{T_0}} \Bigg(\sup_{0 \le \tau < T_0} \|(U_n - U_{n-1})(\tau)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}}$$
$$+ \sup_{0 \le \tau < T_0} \|(v_n - v_{n+1})(\tau)\|_{\dot{B}^{\frac{3}{2}}_{2,\infty}} \Bigg) \|\psi\|_{\dot{B}^{-\frac{1}{2}}_{2,1} + \dot{B}^{-\frac{3}{2}}_{2,1}}.$$

Hence, by Proposition 2.1(ii) and the induction hypothesis,

$$\sup_{0\le t<T_0} \|v_{n+1}(t)\|_{\dot B^{\frac12}_{2,\infty}} \lesssim \delta_2,$$

$$\sup_{0\le t<T_0} \|(v_{n+1}-v_n)(t)\|_{\dot B^{\frac12}_{2,\infty}\cap \dot B^{\frac32}_{2,\infty}} \lesssim (T_0^{\frac12}+\delta_2)\|U_n-U_{n+1}\|_{Y_{T_0}}.$$

To estimate $\sup_{0\le t<T_0}\|b_{n+1}(t)\|_{\dot B^{1/2}_{2,\infty}}$ and $\sup_{0\le t<T_0}\|(b_{n+1}-b_n)(t)\|_{\dot B^{1/2}_{2,\infty}}$, we use the following lemma.

Lemma 4.6 *Let $\tilde v$, $\tilde f\in X_{T_0}$, and let $\tilde b$ be a solution of the linear transport equation*

$$\partial_t\tilde b+\tilde v\cdot\nabla\tilde b=\tilde f \quad \text{on } [0,T_0)\times\mathbb{R}^3.$$

Then, there exists a constant $C>0$ such that

$$\|\tilde b(t)\|_{\dot B^{\frac12}_{2,\infty}} \lesssim e^{CE_{\tilde v}(t)}\left(\|\tilde b(0)\|_{\dot B^{\frac12}_{2,\infty}}+\int_0^t e^{-CE_{\tilde v}(\tau)}\|\tilde f(\tau)\|_{\dot B^{\frac12}_{2,\infty}}\,d\tau\right),$$

for any $0\le t<T_0$, where $E_{\tilde v}(t)\equiv\int_0^t\|\nabla\tilde v(\tau)\|_{\dot B^{3/2}_{2,1}}d\tau$.

This lemma is a special case of [7, Theorem 2.5]. Hence, we omit the proof.

Since $E_{v_n}(T_0)\lesssim 1$, by Lemma 4.6, we have

$$\sup_{0<t\le T_0}\|b_{n+1}(t)\|_{\dot B^{\frac12}_{2,\infty}} \lesssim \|b_0\|_{\dot B^{\frac12}_{2,\infty}} + T_0\sup_{0<t\le T_0}\|f_0(b_n,v_n)(t)\|_{\dot B^{\frac12}_{2,\infty}}$$

and

$$\sup_{0<t\le T_0}\|(b_{n+1}-b_n)(t)\|_{\dot B^{\frac12}_{2,\infty}} \lesssim T_0\sup_{0<t\le T_0}\Big(\|((v_n-v_{n-1})\cdot\nabla b_n)(t)\|_{\dot B^{\frac12}_{2,\infty}} + \|(f_0(b_n,v_n)-f_0(b_{n-1},v_{n-1}))(t)\|_{\dot B^{\frac12}_{2,\infty}}\Big).$$

Then, Proposition 2.1(vi) and the induction hypothesis show that

$$\sup_{0<t\le T_0}\|b_{n+1}(t)\|_{\dot B^{\frac12}_{2,\infty}} \lesssim \delta_2,$$

$$\sup_{0<t\le T_0}\|(b_{n+1}-b_n)(t)\|_{\dot B^{\frac12}_{2,\infty}} \lesssim T_0\|U_n-U_{n-1}\|_{Y_{T_0}}.$$

If $T_0>0$ and $\delta_{2,0}>0$ are small enough, then we have

$$\|U_{n+1}\|_{X_{T_0}}+\|U_{n+1}\|_{\tilde L^\infty_t((0,T_0);\dot H^3)} \lesssim \delta_2,$$

$$\|U_{n+1}-U_n\|_{Y_{T_0}} \le \frac12\|U_n-U_{n-1}\|_{Y_{T_0}}.$$

Therefore, by induction, we have the estimates (34) and (35) for any $n \geq 1$. The estimate (35) implies that U_n converges to some $U \in Y_{T_0}$ as $n \to \infty$. Then, for any $j \in \mathbb{Z}$ and $\psi \in C_0^\infty((0, T_0) \times \mathbb{R}^3)$,

$$\langle \dot{\Delta}_j U_n, \psi \rangle \to \langle \dot{\Delta}_j U, \psi \rangle \quad \text{as } n \to \infty.$$

For any $j \in \mathbb{Z}$, there exists a sequence $\{\psi_m^{(j)}\}_{m \in \mathbb{Z}_{\geq 0}} \subset C_0^\infty((0, T_0) \times \mathbb{R}^3)$ such that $\sup_m \|\psi_m^{(j)}\|_{L_t^1((0,T_0);L^2)} \leq 1$ and

$$\langle \dot{\Delta}_j U, \psi_m^{(j)} \rangle \to \|\dot{\Delta}_j U\|_{L_t^\infty((0,T_0);L^2)} \quad \text{as } m \to \infty.$$

Thus, by Fatou's lemma, we obtain

$$\begin{aligned}
\|U\|_{\tilde{L}_t^\infty((0,T_0);\dot{H}^3)} &\leq \liminf_{m\to\infty} \liminf_{n\to\infty} \left\| \{2^{3j} \langle \dot{\Delta}_j U_n, \psi_m^{(j)} \rangle\}_{j \in \mathbb{Z}} \right\|_{\ell^2(\mathbb{Z})} \\
&\leq \liminf_{n\to\infty} \|U_n\|_{\tilde{L}_t^\infty((0,T_0);\dot{H}^3)} \lesssim \delta_2.
\end{aligned}$$

Since $U \in C^0((0, T_0); \dot{B}^{1/2}_{2,\infty})$, for any $j \in \mathbb{Z}$, the low frequency part $\dot{S}_j U$ belongs to $C^0((0, T_0); \dot{B}^{1/2}_{2,\infty} \cap \dot{H}^3)$. By $U \in \tilde{L}_t^\infty((0, T_0); \dot{H}^3)$, we have

$$\|U - \dot{S}_j U\|_{L_t^\infty((0,T_0);\dot{H}^3)} \to 0 \quad \text{as } j \to \infty.$$

Then, we have $U \in C^0([0, T_0); \dot{H}^3)$. Since U_n satisfy the relation (33), $(\rho, v) \equiv U + (\rho_\infty, 0)$ is a solution of (1) such that $U|_{t=0} = (b_0, v_0)$, $U \in X_{T_0}$, $v \in L_t^2((0, T_0); \dot{H}^1 \cap \dot{H}^4)$ and

$$\|U\|_{C^0([0,T_0);\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3)} + \|v\|_{L_t^2((0,T_0);\dot{H}^1 \cap \dot{H}^4)} \leq c_1 \delta_2,$$

where $c_1 > 0$ is a constant. Let us show the uniqueness. Let $\tilde{U} = (\tilde{b}, \tilde{v})$ be a solution of (1) satisfying

$$\|\tilde{U}\|_{C^0([0,T_0);\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3)} + \|\tilde{v}\|_{L_t^2((0,T_0);\dot{H}^1 \cap \dot{H}^4)} \leq c_1 \delta_2.$$

Then, by the proof of (34), the estimate

$$\|U - \tilde{U}\|_{Y_{T_0}} \leq \frac{1}{2} \|U - \tilde{U}\|_{Y_{T_0}}$$

holds. Hence, $U = \tilde{U}$.

The rest of this section is devoted to proving Theorem 4.2.

Proof of Theorem 4.2 Let (ρ^*, v^*) be the stationary solution satisfying (7). By Proposition 4.4 and (7), there exist $\delta_{2,0} > 0$, $T_0 > 0$ and $C_2 \geq 1$ such that if an initial perturbation $(\sigma_0, w_0) = (\rho_0 - \rho^*, v_0 - v^*)$ and $F(x)$ satisfy

$$\|(\sigma_0, w_0)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3} + \|F\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty} \cap \dot{H}^3} \leq \delta_{2,0},$$

then there exists a unique solution $(\sigma, w) = (\rho - \rho^*, v - v^*)$ of (17) on $[0, 2T_0)$ such that $(\sigma, w)(0) = (\sigma_0, w_0)$, $(\sigma, w) \in C^0([0, 2T_0); \dot{H}^3)$ and

$$\sup_{0 \leq t < 2T_0} \|(\sigma, w)(t)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3} \leq C_2 \big(\|(\sigma_0, w_0)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3} + \|F\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty} \cap \dot{H}^3} \big).$$

Let $C_1 \geq 1$ and $\delta_1 > 0$ be constants appearing in Proposition 4.3. We take an initial perturbation (σ_0, w_0) and $F(x)$ such that

$$C_1 \|(\sigma_0, w_0)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3} + \|F\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty} \cap \dot{H}^3} \leq \min \left\{ \delta_{2,0}, \frac{\delta_1}{C_2} \right\}.$$

Let $N \geq 1$, and let $(\sigma, w) = (\rho - \rho^*, v - v^*) \in C^0([0, NT_0); \dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3)$ be a solution of (17) on $[0, NT_0)$ with initial value (σ_0, w_0) satisfying

$$\sup_{0 \leq t < NT_0} \|(\sigma, w)(t)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3} \leq C_1 \|(\sigma_0, w_0)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3}.$$

Since

$$\|(\sigma, w)((N-1)T_0)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3} + \|F\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty} \cap \dot{H}^3} \leq \delta_{2,0},$$

by Proposition 4.4, there exists a unique $(\tilde{\sigma}, \tilde{w}) \in C^0([0, 2T_0); \dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3)$ such that $(\tilde{\sigma}, \tilde{w})$ is a solution of (17) with $(\tilde{\sigma}, \tilde{w})(0) = (\sigma, w)((N-1)T_0)$ satisfying

$$\begin{aligned}
\sup_{0 \leq t < 2T_0} \|(\tilde{\sigma}, \tilde{w})(t)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3} &\leq C_2 \left(\|(\sigma, w)((N-1)T_0)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3} + \|F\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty} \cap \dot{H}^3} \right) \\
&\leq C_2 \left(C_1 \|(\sigma_0, w_0)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3} + \|F\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty} \cap \dot{H}^3} \right) \leq \delta_1.
\end{aligned}$$

Let $(\sigma, w)(t) \equiv (\tilde{\sigma}, \tilde{w})(t - NT_0)$, $t \in [NT_0, (N+1)T_0)$. Then, (σ, w) is a solution of (17) on $[0, (N+1)T_0)$ and, by Proposition 4.3, we have

$$\sup_{0 \leq t < (N+1)T_0} \|(\sigma, w)(t)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3} \leq C_1 \|(\sigma_0, w_0)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3}.$$

Hence, the proof is completed by induction on N. □

5 The proof of decay estimates

Throughout this section, we fix the stationary solution $(\rho^*, v^*) = (\sigma^* + \rho_\infty, v^*)$ obtained in Theorem 1.1 and the perturbation $(\sigma, w) = (\rho - \rho^*, v - v^*)$ obtained in Theorem 4.2. From now on, we denote the real numbers δ_1, δ_2 and δ by

$$\delta_1 \equiv \|\sigma^*\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty}\cap \dot{H}^4} + \|v^*\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}\cap \dot{H}^5}, \quad \delta_2 \equiv \sup_{t>0} \|(\sigma, w)(t)\|_{\dot{B}^{1/2}_{2,\infty}\cap \dot{H}^3}$$

and $\delta \equiv \delta_1 + \delta_2$. In order to prove the time decay estimates (10), (11) and (12), we show the following weak-type decay estimate.

Theorem 5.1 *Let $-3/2 \le s_0 \le 1/2$. If $\delta = \delta_1 + \delta_2$ is small enough, then, for any $-3/2 < s < 3/2$ with $s_0 \le s$, we have*

$$\|(\sigma, w)(t)\|_{\dot{B}^s_{2,\infty}} \lesssim_s (1+t)^{-\frac{s-s_0}{2}} \|(\sigma_0, w_0)\|_{\dot{B}^{s_0}_{2,\infty}\cap \dot{H}^3}. \tag{36}$$

The proof of Theorem 5.1 is performed by decomposing the perturbation into low- and high-frequency parts with respect to the Fourier space. We decompose the perturbation (σ, w) into low- and high-frequency components for fixed $j_0 \in \mathbb{Z}$:

$$(\sigma, w) = (\sigma_L, w_L) + (\sigma_H, w_H), \tag{37}$$

where $(\sigma_L, w_L) \equiv (\dot{S}_{j_0}\sigma, \dot{S}_{j_0} w)$.

5.1 Estimate for the low frequency part

Let us establish the time decay estimate for the low frequency part of the perturbation (σ_L, w_L).

Proposition 5.2 *Let $-3/2 \le s_0 \le 1/2$, and let $\epsilon > 0$ be a small number. If $\delta = \delta_1 + \delta_2$ is sufficiently small, then, for any $T > 0$, we have*

$$\sup_{0\le t\le T} (1+t)^{\frac{s-s_0}{2}} \|(\sigma_L, w_L)(t)\|_{\dot{B}^s_{2,\infty}} \lesssim_{\epsilon, j_0} \|(\sigma_0, w_0)\|_{\dot{B}^{s_0}_{2,\infty}} + \delta \mathcal{D}_{\epsilon, s_0}(T), \tag{38}$$

where $-3/2 + \epsilon \le s \le 3/2 - \epsilon$ with $s_0 \le s$. Here, the quantity $\mathcal{D}_{\epsilon, s_0}(T)$ is defined by

$$\mathcal{D}_{\epsilon, s_0}(T) \equiv \sup_{\substack{-3/2+\epsilon\le\eta\le 3/2-\epsilon, \\ s_0\le\eta}} \sup_{0\le t\le T} (1+t)^{\frac{\eta-s_0}{2}} \|(\sigma, w)(t)\|_{\dot{B}^\eta_{2,\infty}}. \tag{39}$$

Proof Let $n = (m - m^*)/\rho_\infty$ with $m = \rho v$, $m^* = \rho^* v^*$. Then, (σ, n) satisfies the Eq. (28). By the definition of n, we have $n = \gamma_1^{-1} w + \gamma_1(\sigma v - \sigma^* w)$. Let $n_L = \dot{S}_{j_0} n$. Then, for any $-3/2 < s < 3/2$, Proposition 2.1(vi) shows

$$\|w_L\|_{\dot{B}^s_{2,\infty}} \lesssim_s \|n_L\|_{\dot{B}^s_{2,\infty}} + \delta \|(\sigma, w)\|_{\dot{B}^s_{2,\infty}}.$$

Thus, to prove Proposition 5.2, it is sufficient to show the inequalities

$$\sup_{0\le t\le T}(1+t)^{\frac{s-s_0}{2}}\|(\sigma_L,n_L)(t)\|_{\dot B^s_{2,\infty}}\lesssim_{\epsilon,p,j_0}\|(\sigma_0,w_0)\|_{\dot B^{s_0}_{2,\infty}}+\delta\mathcal{D}_{\epsilon,s_0}(T),$$

where $-3/2+\epsilon\le s\le 3/2-\epsilon$ with $s_0\le s$. Let e^{tA} be the semigroup associated with the left-hand side of (28). Then, the Duhamel principle gives

$$\begin{bmatrix}\sigma_L\\ n_L\end{bmatrix}(t)=e^{tA}_L\begin{bmatrix}\sigma_0\\ n_0\end{bmatrix}+\int_0^t e^{(t-\tau)A}_L\begin{bmatrix}0\\ h+\gamma_1^{-1}\sigma F(x)\end{bmatrix}(\tau)\,d\tau, \tag{40}$$

where $e^{tA}_L\equiv \dot S_{j_0}e^{tA}$ and the function h is defined in (29). Let us denote $V_0=(\sigma_0,n_0)^{\mathsf T}$, $V=(\sigma,n)^{\mathsf T}$. Then, according to Proposition 2.1(vi), Lemma 4.1(i), we have the following estimate for the first term in the right-hand side of (40):

$$\left\|e^{tA}_L V_0\right\|_{\dot B^s_{2,\infty}}\lesssim_{j_0}(1+t)^{-\frac{s-s_0}{2}}\|V_0\|_{\dot B^{s_0}_{2,\infty}}\lesssim(1+t)^{-\frac{s-s_0}{2}}\|(\sigma_0,w_0)\|_{\dot B^{s_0}_{2,\infty}}. \tag{41}$$

To estimate the second term in (40), we use the following lemma. □

Lemma 5.3 *Let* $-5/2<\beta<-1/2$. *Then, we have*

$$\|h\|_{\dot B^\beta_{2,\infty}}\lesssim_{\beta,\eta}\delta_1\|V\|_{\dot B^{\beta+2}_{2,\infty}}+\|V\|_{\dot B^{\frac12+\eta}_{2,\infty}}\|V\|_{\dot B^{\beta+2-\eta}_{2,\infty}}+\|V\|_{\dot B^{\frac32-\eta}_{2,1}}\|V\|_{\dot B^{\beta+2+\eta}_{2,\infty}} \tag{42}$$

for any $0\le\eta<\min\{\beta+5/2,-\beta-1/2,1\}$.

Admitting Lemma 5.3 for a moment, we continue the proof of Proposition 5.2. Let us denote the second term in (40) by

$$\begin{aligned}N_L(t)&=\int_0^t e^{(t-\tau)A}_L\begin{bmatrix}0\\ h\end{bmatrix}(\tau)\,d\tau+\int_0^t e^{(t-\tau)A}_L\begin{bmatrix}0\\ \gamma_1^{-1}\sigma F(x)\end{bmatrix}(\tau)\,d\tau\\ &\equiv N^1_L(t)+N^2_L(t).\end{aligned} \tag{43}$$

We estimate the term $N^1_L(t)$. We first treat the case $-1/2<s\le 3/2-\epsilon$ with $s_0\le s$. In this case, we estimate $N^1_L(t)$ by using the duality argument. Let $\psi=(\psi_1,\dots,\psi_4)^{\mathsf T}\in\mathcal{S}^4$. Fix a real number α_0 satisfying $s-s_0<\alpha_0<s-s_0+2$ and $s+1/2+\epsilon\le\alpha_0\le s+5/2-\epsilon$. This α_0 can be taken if $\epsilon<1$. Proposition 2.1(ii) then yields

$$\begin{aligned}\left\langle N^1_L(t),\psi\right\rangle\lesssim&\int_{\frac t2}^t\|h(\tau)\|_{\dot B^{s-2}_{2,\infty}}\|e^{(t-\tau)A^*}_L\psi\|_{\dot B^{2-s}_{2,1}}\,d\tau\\ &+\int_0^{\frac t2}\|h(\tau)\|_{\dot B^{s-\alpha_0}_{2,\infty}}\|e^{(t-\tau)A^*}_L\psi\|_{\dot B^{\alpha_0-s}_{2,1}}\,d\tau.\end{aligned}$$

By Proposition 2.1(vi),

$$\|V(\tau)\|_{\dot{B}^{s}_{2,\infty}} \lesssim (1+\|U(\tau)\|_{\dot{B}^{\frac{3}{2}}_{2,1}})\|U(\tau)\|_{\dot{B}^{s}_{2,\infty}} \lesssim \|U(\tau)\|_{\dot{B}^{s}_{2,\infty}}, \tag{44}$$

where $U=(\sigma,w)^{\mathsf{T}}$. Then, Lemma 5.3 with $\beta=s-2$ and $\eta=0$ shows

$$\|h(\tau)\|_{\dot{B}^{s-2}_{2,\infty}} \lesssim (\delta_1+\|U(\tau)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}\cap\dot{B}^{\frac{3}{2}}_{2,1}})\|U(\tau)\|_{\dot{B}^{s}_{2,\infty}} \lesssim \delta\|U(\tau)\|_{\dot{B}^{s}_{2,\infty}}.$$

By using the time-space integral estimate

$$\int_0^\infty \|e_L^{\tau A^*}\psi\|_{\dot{B}^{2-s}_{2,1}}\,d\tau \lesssim \|\psi\|_{\dot{B}^{-s}_{2,1}}$$

which follows from Lemma 4.1(ii), we obtain

$$\begin{aligned}\int_{\frac{t}{2}}^t \|h(\tau)\|_{\dot{B}^{s-2}_{2,\infty}}\|e_L^{(t-\tau)A^*}\psi\|_{\dot{B}^{2-s}_{2,1}}\,d\tau &\lesssim_s \delta\int_{\frac{t}{2}}^t \|U(\tau)\|_{\dot{B}^{s}_{2,\infty}}\|e_L^{(t-\tau)A^*}\psi\|_{\dot{B}^{2-s}_{2,1}}\,d\tau\\ &\lesssim \delta\mathcal{D}_{\epsilon,s_0}(T)(1+t)^{-\frac{s-s_0}{2}}\int_0^\infty \|e_L^{\tau A^*}\psi\|_{\dot{B}^{2-s}_{2,1}}\,d\tau\\ &\lesssim_{j_0} \delta\mathcal{D}_{\epsilon,s_0}(T)(1+t)^{-\frac{s-s_0}{2}}\|\psi\|_{\dot{B}^{-s}_{2,1}}.\end{aligned}$$

Lemma 5.3 with $\beta=s-\alpha_0$, $\eta=0$ shows

$$\|h(\tau)\|_{\dot{B}^{s-\alpha_0}_{2,\infty}} \lesssim \left(\delta_1+\|U(\tau)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}\cap\dot{B}^{\frac{3}{2}}_{2,1}}\right)\|U(\tau)\|_{\dot{B}^{s-\alpha_0+2}_{2,\infty}} \lesssim \delta\|U(\tau)\|_{\dot{B}^{s-\alpha_0+2}_{2,\infty}}.$$

By using the time decay estimate

$$\|e_L^{(t-\tau)A}\psi\|_{\dot{B}^{\alpha_0-s}_{2,\infty}} \lesssim (1+t-\tau)^{-\frac{\alpha_0}{2}}\|\psi\|_{\dot{B}^{-s}_{2,\infty}},$$

which follows from Lemma 4.1(i), we obtain

$$\begin{aligned}&\int_0^{\frac{t}{2}} \|h(\tau)\|_{\dot{B}^{s-\alpha_0}_{2,\infty}}\|e_L^{(t-\tau)A^*}\psi\|_{\dot{B}^{\alpha_0-s}_{2,1}}\,d\tau\\ &\lesssim_s \delta\int_0^{\frac{t}{2}} \|U(\tau)\|_{\dot{B}^{s-\alpha_0+2}_{2,\infty}}\|e_L^{(t-\tau)A^*}\psi\|_{\dot{B}^{\alpha_0-s}_{2,1}}\,d\tau\\ &\lesssim \delta\mathcal{D}_{\epsilon,s_0}(T)\|\psi\|_{\dot{B}^{-s}_{2,1}}\int_0^{\frac{t}{2}}(1+\tau)^{-\frac{s-s_0+2-\alpha_0}{2}}(1+t-\tau)^{-\frac{\alpha_0}{2}}\,d\tau\\ &\lesssim_{\epsilon,j_0} \delta\mathcal{D}_{\epsilon,s_0}(T)(1+t)^{-\frac{s-s_0}{2}}\|\psi\|_{\dot{B}^{-s}_{2,1}}.\end{aligned}$$

As ψ is arbitrary, applying Proposition 2.1(ii), we obtain the inequality

$$\|N_L^1(t)\|_{\dot{B}^s_{2,\infty}} \lesssim_{\epsilon, j_0} \delta \mathcal{D}_{\epsilon, s_0}(T)\,(1+t)^{-\frac{s-s_0}{2}} \tag{45}$$

for $-1/2 < s \le 3/2 - \epsilon$, $-3/2 \le s_0 \le 1/2$ with $s_0 \le s$.

Next, we show the inequalities (45) for $-3/2 + \epsilon \le s \le -1/2$, $-3/2 \le s_0 \le 3/2$ with $s_0 \le s$. We take $\epsilon_1 > 0$ which satisfies $\epsilon_1 < 1/2$. Then, using Lemma 4.1(i) with $\alpha = 2 + s - \epsilon_1$ and Lemma 5.3 with $\beta = -2 + \epsilon_1$ and $\eta = 0$, we have

$$\begin{aligned}
\|N_L^1(t)\|_{\dot{B}^s_{2,\infty}} &\lesssim_{j_0} \int_0^t (1+t-\tau)^{-\frac{1}{2}(2+s-\epsilon_1)} \|h(\tau)\|_{\dot{B}^{-2+\epsilon_1}_{2,\infty}} d\tau \\
&\lesssim \delta \int_0^t (1+t-\tau)^{-\frac{1}{2}(2+s-\epsilon_1)} \|U(\tau)\|_{\dot{B}^{\epsilon_1}_{2,\infty}} d\tau \\
&\lesssim_{\epsilon_1, p} \delta \mathcal{D}_{\epsilon, s_0}(T)\,(1+t)^{-\frac{s-s_0}{2}}.
\end{aligned}$$

Let us estimate $N_L^2(t)$. By Proposition 2.1(vi), Lemma 4.1(i) with $\alpha = s + 3/2$, we have

$$\begin{aligned}
\|N_L^2(t)\|_{\dot{B}^s_{2,\infty}} &\lesssim_{j_0} \int_0^t (1+t-\tau)^{-\frac{s}{2}-\frac{3}{4}} \|\sigma(\tau) F\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty}} d\tau \\
&\lesssim_{\epsilon} \|F\|_{\dot{B}^{-\frac{3}{2}+\epsilon}_{2,1}} \int_0^t (1+t-\tau)^{-\frac{s}{2}-\frac{3}{4}} \|\sigma(\tau)\|_{\dot{B}^{\frac{3}{2}-\epsilon}_{2,\infty}} d\tau \\
&\lesssim_{\epsilon} \delta \mathcal{D}_{\epsilon, s_0}(T)\,(1+t)^{-\frac{s-s_0}{2}},
\end{aligned}$$

where $-3/2 + \epsilon \le s \le 3/2 - \epsilon$, $-3/2 \le s_0 \le 3/2$ with $s_0 \le s$. Hence, we obtain

$$\|N_L(t)\|_{\dot{B}^s_{2,\infty}} \lesssim_{\epsilon, j_0} \delta \mathcal{D}_{\epsilon, s_0}(T)\,(1+t)^{-\frac{s-s_0}{2}} \tag{46}$$

for $-3/2 + \epsilon \le s \le -1/2$, $-3/2 \le s_0 \le 3/2$ with $s_0 \le s$.

It remains to prove Lemma 5.3.

Proof of Lemma 5.3 Let $\Psi(\zeta) = 1/(\zeta + \rho_\infty)$. By using Proposition 2.1(i), (vi), we obtain

$$\begin{aligned}
\|h_1\|_{\dot{B}^\beta_{2,\infty}} &\lesssim \left\| \frac{n \otimes m}{\rho} + \frac{m^* \otimes n}{\rho} + \gamma_1^{-1} \left(\Psi(\sigma^* + \sigma) - \Psi(\sigma^*) \right) m^* \otimes m \right\|_{\dot{B}^{\beta+1}_{2,\infty}} \\
&\lesssim_{\beta, \eta} \|(\sigma^*, v^*)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} \|U\|_{\dot{B}^{\beta+2}_{2,\infty}} + \|U\|_{\dot{B}^{\frac{1}{2}+\eta}_{2,\infty}} \|U\|_{\dot{B}^{\beta+2-\eta}_{2,\infty}},
\end{aligned}$$

since $0 < \eta < 1$. We also have bounds for h_2 as

$$\|h_2\|_{\dot{B}^\beta_{2,\infty}} \lesssim \|\Pi(\sigma^*, \sigma)\sigma\|_{\dot{B}^{\beta+1}_{2,\infty}} \lesssim \|\sigma^*\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} \|\sigma\|_{\dot{B}^{\beta+2}_{2,\infty}} + \|\sigma\|_{\dot{B}^{\frac{1}{2}+\eta}_{2,\infty}} \|\sigma\|_{\dot{B}^{\beta+2-\eta}_{2,\infty}}.$$

By Proposition 2.1(i), (v), (vi), we obtain bounds for g^3, g^4 as

$$
\begin{aligned}
\|h_3\|_{\dot{B}^{\beta}_{2,\infty}} &\lesssim \left\| \left(\Psi(\sigma^* + \sigma) - \Psi(0) \right) \nabla n \right\|_{\dot{B}^{\beta+1}_{2,\infty}} \\
&\quad + \left\| \nabla \left(\Psi(\sigma^* + \sigma) - \Psi(0) \right) n \right\|_{\dot{B}^{\beta+1}_{2,\infty}} \\
&\lesssim_{\beta,\eta} \|\sigma^*\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \|n\|_{\dot{B}^{\beta+2}_{2,\infty}} + \|U\|_{\dot{B}^{\frac{3}{2}-\eta}_{2,\infty}} \|U\|_{\dot{B}^{\beta+2+\eta}_{2,\infty}}, \\
\|h_4\|_{\dot{B}^{\beta}_{2,\infty}} &\lesssim_{\beta,\eta} \left\| \left(\Psi(\sigma^* + \sigma) - \Psi(\sigma^*) \right) \nabla m^* \right\|_{\dot{B}^{\beta+1}_{2,\infty}} \\
&\quad + \left\| \nabla \left(\Psi(\sigma^* + \sigma) - \Psi(\sigma^*) \right) m^* \right\|_{\dot{B}^{\beta+1}_{2,\infty}} \\
&\lesssim_{\beta,\eta} \|m^*\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \|\sigma\|_{\dot{B}^{\beta+2}_{2,\infty}},
\end{aligned}
$$

since $-3/2 < \beta,\ \beta \pm \eta < -1/2$. Therefore, we have the desired estimate (42). □

Proposition 2.2 implies the continuous inclusions

$$
L^{p,\infty} \hookrightarrow \dot{B}^{-3\left(\frac{1}{p}-\frac{1}{2}\right)}_{2,\infty} \quad \text{for } 1 < p < 2, \quad L^1 \hookrightarrow \dot{B}^{-\frac{3}{2}}_{2,\infty} \quad \text{and} \quad L^2 \hookrightarrow \dot{B}^{0}_{2,\infty}.
$$

Then, by Lemma 4.1(i), we have the estimate for the first term in the right-hand side of (40),

$$
\left\| e^{tA}_L V_0 \right\|_{\dot{B}^{s}_{2,\infty}} \lesssim_{j_0} (1+t)^{-\frac{s}{2}-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)} \|V_0\|_{L^{p,\infty}} \quad \text{if } 1 < p < 2, \tag{47}
$$

$$
\left\| e^{tA}_L V_0 \right\|_{\dot{B}^{s}_{2,\infty}} \lesssim_{j_0} (1+t)^{-\frac{s}{2}-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)} \|V_0\|_{L^{p}} \quad \text{if } p = 1, 2, \tag{48}
$$

where $s \in \mathbb{R}$ and $1 \le p \le 2$ with $s/2 + 3/2(1/p - 1/2) \ge 0$. Thus, by the estimate (46), we obtain the following proposition.

Proposition 5.4 *Let $\epsilon > 0$ be a small number. If $\delta = \delta_1 + \delta_2$ is sufficiently small, then, for any $T > 0$, we have*

$$
\sup_{0 \le t \le T} (1+t)^{\frac{s}{2}+\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)} \|(\sigma_L, w_L)(t)\|_{\dot{B}^{s}_{2,\infty}} \lesssim_{\epsilon,p,j_0} \|(\sigma_0, w_0)\|_{\Lambda_p} + \delta \widetilde{\mathcal{D}}_{\epsilon,p}(T), \tag{49}
$$

where $-3/2 + \epsilon \le s \le 3/2 - \epsilon$, $1 \le p \le 2$ with $s/2 + 3/2(1/p - 1/2) \ge 0$. Here, the quantity $\widetilde{\mathcal{D}}_{\epsilon,p}(T)$ is defined by

$$
\widetilde{\mathcal{D}}_{\epsilon,p}(T) \equiv \sup_{\substack{-3/2+\epsilon \le \eta \le 3/2-\epsilon, \\ \eta/2+3/2(1/p-1/2) \ge 0}} \ \sup_{0 \le t \le T} (1+t)^{\frac{\eta}{2}+\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)} \|(\sigma, w)(t)\|_{\dot{B}^{\eta}_{2,\infty}},
$$

and the function space Λ_p *is defined by*

$$\Lambda_p = \begin{cases} L^{p,\infty}(\mathbb{R}^3) & \text{if } 1 < p < 2, \\ L^p(\mathbb{R}^3) & \text{if } p = 1, 2. \end{cases}$$

5.2 Estimate for the high frequency part

The estimate of the high frequency part of the perturbation (σ_H, w_H) is proved as follows.

Proposition 5.5 *If* $\delta = \delta_1 + \delta_2$ *is small enough, then we have the following estimate for all* $T > 0$ *and small* $\epsilon > 0$,

$$\sup_{0\le t\le T} (1+t)^{\frac{s-s_0}{2}} \|(\sigma_H, w_H)(t)\|_{\dot{B}^s_{2,\infty}} \lesssim_{j_0} \|(\sigma_0, w_0)\|_{\dot{B}^s_{2,\infty}} + \delta \mathcal{D}_{\epsilon,s_0}(T), \tag{50}$$

where $\delta = \delta_1 + \delta_2$, $-3/2+\epsilon \le s \le 3/2-\epsilon$, $-3/2 \le s_0 \le 1/2$ *with* $s_0 \le s$. *Here,* $\mathcal{D}_{\epsilon,s_0}(T)$ *is the quantity defined in* (39).

Proof Since $\|(\sigma_H, w_H)(t)\|_{\dot{B}^{s_1}_{2,\infty}} \lesssim_{j_0} \|(\sigma_H, w_H)(t)\|_{\dot{B}^{s_2}_{2,\infty}}$ for $s_1 \le s_2$, it is sufficient to show that the estimate (50) with the highest order case $s = 3/2-\epsilon$. Since (σ, w) satisfies Eq. (17), we have the following identities for $(\sigma_j, w_j, f_j, g_j) \equiv \dot{\Delta}_j(\sigma, w, f, g)$, where $\dot{\Delta}_j$ are the dyadic blocks,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla(\sigma_j, w_j)\|_{L^2}^2 + \nu\|\nabla^2 w_j\|_{L^2}^2 + (\nu+\nu')\|\nabla \mathrm{div}\, w_j\|_{L^2}^2 \\
&\quad = \langle \nabla f_j, \nabla\sigma_j\rangle + \langle \nabla g_j, \nabla w_j\rangle, \\
&\frac{d}{dt}\langle \nabla\sigma_j, w_j\rangle + \gamma\|\nabla\sigma_j\|_{L^2}^2 = \gamma\|\mathrm{div}\, w_j\|_{L^2}^2 + \langle \mathcal{A}w_j, \nabla\sigma_j\rangle + \langle \nabla f_j, w_j\rangle + \langle g_j, \nabla\sigma_j\rangle.
\end{aligned}$$

Hence, there exists a constant $c_0 > 0$ such that for sufficiently small $\kappa > 0$,

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla(\sigma_j, w_j)\|_{L^2}^2 + \kappa\langle \nabla\sigma_j, w_j\rangle) + c_0\left(\kappa\|\nabla\sigma_j\|_{L^2}^2 + \|\nabla^2 w_j\|_{L^2}^2\right) \\
&\quad \lesssim \langle \nabla f_j, \nabla\sigma_j\rangle + \kappa\langle \nabla f_j, w_j\rangle + \sum_{i=1}^{4}(\kappa\langle g_j^i, \nabla\sigma_j\rangle + \langle \nabla g_j^i, \nabla w_j\rangle),
\end{aligned}$$

where $g_j^i = \dot{\Delta}_j g^i$. By using Proposition 2.1(i), (vi) and Lemma 2.3, for any $j \ge j_0-1$, we have

$$\begin{aligned}
\langle \nabla f_j, \nabla\sigma_j\rangle &= -\langle \nabla\dot{\Delta}_j \mathrm{div}\,((v^*+w)\sigma + \sigma^* w), \nabla\sigma_j\rangle \\
&\lesssim 2^j\|\dot{\Delta}_j(\mathrm{div}(v^*+w)\,\sigma + \nabla\sigma^*\cdot w)\|_{L^2}\|\nabla\sigma_j\|_{L^2} \\
&\quad + 2^j\|[\dot{\Delta}_j, (v^*+w)\cdot\nabla]\sigma + [\dot{\Delta}_j, \sigma^*\mathrm{div}\,]w\|_{L^2}\|\nabla\sigma_j\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&- \langle \nabla((v^* + w) \cdot \nabla\sigma_j), \nabla\sigma_j\rangle - \langle \nabla(\sigma^* \mathrm{div}\, w_j), \nabla\sigma_j\rangle \\
&\lesssim_{\epsilon, j_0} 2^{-j(s-1)} \|(\sigma^*, v^*, w)\|_{\dot{B}^{\frac{5}{2}}_{2,1}} \|(\sigma, w)\|_{\dot{B}^s_{2,\infty}} \|\nabla\sigma_j\|_{L^2} \\
&\quad + \|\nabla v^*\|_{L^\infty} \|\nabla\sigma_j\|_{L^2}^2 + \|(\sigma^*, \nabla\sigma^*, \sigma, \nabla\sigma)\|_{L^\infty} \|\nabla^2 w_j\|_{L^2} \|\nabla\sigma_j\|_{L^2} \\
&\lesssim \delta 2^{-j(s-1)} \|(\sigma, w)\|_{\dot{B}^s_{2,\infty}} \|\nabla\sigma_j\|_{L^2} + \delta \|(\nabla\sigma_j, \nabla^2 w_j)\|_{L^2}^2,
\end{aligned}
$$

where we use the identities:

$$
\langle v^* \cdot \nabla\partial_k\sigma_j, \partial_k\sigma_j\rangle = -\frac{1}{2}\langle \mathrm{div}\, v^* \partial_k\sigma_j, \partial_k\sigma_j\rangle \quad \text{for } 1 \le k \le 3.
$$

Using Proposition 2.1(i), (v) and (vi), we have

$$
\begin{aligned}
\langle \nabla f_j, w_j\rangle &\lesssim 2^{-j(s-1)} \|\mathrm{div}\,((v^* + w)\sigma + \sigma^* w)\|_{\dot{B}^{s-1}_{2,\infty}} \|\nabla w_j\|_{L^2} \\
&\lesssim 2^{-j(s-1)} \|(\sigma^*, \sigma, v^*)\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \|(\sigma, w)\|_{\dot{B}^s_{2,\infty}} \|\nabla w_j\|_{L^2} \\
&\lesssim \delta 2^{-j(s-1)} \|(\sigma, w)\|_{\dot{B}^s_{2,\infty}} \|\nabla w_j\|_{L^2}.
\end{aligned}
$$

Similarly to the estimate of $\langle \nabla f_j, w_j\rangle$, by applying Proposition 2.1(v), we have, for any $j \ge j_0 - 1$,

$$
\begin{aligned}
&\sum_{i=1}^{2} (\kappa\langle g^i_j, \nabla\sigma_j\rangle + \langle \nabla g^i_j, \nabla w_j\rangle) \\
&\quad \lesssim_\kappa \delta 2^{-j(s-1)} \|(\sigma, w)\|_{\dot{B}^s_{2,\infty}} \|(\nabla\sigma_j, \nabla^2 w_j)\|_{L^2} + \delta \|(\nabla\sigma_j, \nabla^2 w_j)\|_{L^2}^2.
\end{aligned}
$$

We next consider bounds for $\kappa\langle g^3_j, \nabla\sigma_j\rangle + \langle \nabla g^3_j, \nabla w_j\rangle$. Let $\Psi(\zeta) = 1/(\zeta + \rho_\infty)$. If $\epsilon > 0$ is sufficiently small, then, by Proposition 2.1(v), (vi), we have

$$
\begin{aligned}
&\kappa\langle g^3_j, \nabla\sigma_j\rangle + \langle \nabla g^3_j, \nabla w_j\rangle \\
&\quad \lesssim_\kappa 2^{-j(s-1)} \left\|(\Psi(\sigma^* + \sigma) - \Psi(\sigma^*))\mathcal{A}(v^* + w)\right\|_{\dot{B}^{s-1}_{2,\infty}} \|(\nabla\sigma_j, \nabla^2 w_j)\|_{L^2} \\
&\quad \lesssim \delta 2^{-j(s-1)} \|\sigma\|_{\dot{B}^s_{2,\infty}} \|(\nabla\sigma_j, \nabla^2 w_j)\|_{L^2}.
\end{aligned}
$$

To estimate $\kappa\langle g^4_j, \nabla\sigma_j\rangle + \langle \nabla g^4_j, \nabla w_j\rangle$, we introduce the function $h = \Psi(\sigma^*) - \Psi(0)$. By Proposition 2.1(iii), (v), (vi) and Lemma 2.3, we have

$$
\begin{aligned}
&\kappa\langle g^4_j, \nabla\sigma_j\rangle + \langle \nabla g^4_j, \nabla w_j\rangle \\
&\quad = \langle \dot{\Delta}_j(h\mathcal{A}w), \kappa\nabla\sigma_j - \Delta w_j\rangle \lesssim \Big(2^j \|[\dot{\Delta}_j, h\nabla]w\|_{L^2} + \|\nabla\left(h\nabla w_j\right)\|_{L^2} \\
&\qquad + \sum_{1\le k,l\le 3} \|\dot{\Delta}_j\left(\partial_k h\, \partial_l w\right)\|_{L^2}\Big) \|(\nabla\sigma_j, \nabla^2 w_j)\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim_{\epsilon.j_0} 2^{-j(s-1)} \left(\|\sigma^*\|_{\dot{B}^{\frac{5}{2}}_{2,1}} \|w\|_{\dot{B}^s_{2,\infty}} + \|(\sigma^*, \nabla\sigma^*)\|_{L^\infty} \|\nabla^2 w_j\|_{L^2} \right) \\
&\quad \times \|(\nabla\sigma_j, \nabla^2 w_j)\|_{L^2} + 2^{-j(s-1)} \sum_{1\le k,l\le 3} \|\partial_k h\, \partial_l w\|_{\dot{B}^{s-1}_{2,\infty}} \|(\nabla\sigma_j, \nabla^2 w_j)\|_{L^2} \\
&\lesssim \delta 2^{-j(s-1)} \|w\|_{\dot{B}^s_{2,\infty}} \|(\nabla\sigma_j, \nabla^2 w_j)\|_{L^2} + \delta \|(\nabla\sigma_j, \nabla^2 w_j)\|^2_{L^2}.
\end{aligned}
$$

Let $\kappa = \kappa(j_0) > 0$ be small enough. Then, for any $j \ge j_0 - 1$, we have

$$
\mathcal{E}_j(t) \equiv \|\nabla(\sigma_j, w_j)\|^2_{L^2} + \kappa\langle \nabla\sigma_j, w_j\rangle \sim_{j_0} \|\nabla(\sigma_j, w_j)\|^2_{L^2}. \tag{51}
$$

It then follows that there exists a constant $c_1 = c_1(j_0) > 0$ such that

$$
\frac{d}{dt}\mathcal{E}_j + 2c_1\widetilde{\mathcal{E}_j} \lesssim d_j \widetilde{\mathcal{E}_j}^{\frac{1}{2}},
$$

where $\widetilde{\mathcal{E}_j} \equiv \|(\nabla\sigma_j, \nabla^2 w_j)\|^2_{L^2}$, $d_j \equiv \delta 2^{-j(s-1)}\|(\sigma, w)\|_{\dot{B}^s_{2,\infty}}$. From Young's inequality and the fact that $\mathcal{E}_j \lesssim_{j_0} \widetilde{\mathcal{E}_j}$ for any $j \ge j_0 - 1$, we deduce that

$$
\frac{d}{dt}\mathcal{E}_j + c_1\mathcal{E}_j \lesssim_{j_0} d_j^2.
$$

Therefore, Grönwall's inequality and the relation (51) ensure that

$$
\begin{aligned}
&2^{2js}\|(\sigma_j, w_j)(t)\|^2_{L^2} \\
&\quad\lesssim_{j_0} e^{-c_1 t}\|(\sigma_0, w_0)\|^2_{\dot{B}^s_{2,\infty}} + \delta^2 \int_0^t e^{-c_1(t-\tau)}\|(\sigma, w)(\tau)\|^2_{\dot{B}^s_{2,\infty}} d\tau \\
&\quad\lesssim e^{-c_1 t}\|(\sigma_0, w_0)\|^2_{\dot{B}^s_{2,\infty}} + \delta^2 \mathcal{D}_{\epsilon,s_0}(T)^2 \int_0^t e^{-c_1(t-\tau)}(1+\tau)^{-(s-s_0)} d\tau \\
&\quad\lesssim (1+t)^{-(s-s_0)}\big(\|(\sigma_0, w_0)\|_{\dot{B}^s_{2,\infty}} + \delta\mathcal{D}_{\epsilon,s_0}(T)\big)^2,
\end{aligned}
$$

where $j \ge j_0$. Hence, we obtain

$$
(1+t)^{\frac{s-s_0}{2}}\|(\sigma_H, w_H)(t)\|_{\dot{B}^s_{2,\infty}} \lesssim_{j_0} \|(\sigma_0, w_0)\|_{\dot{B}^s_{2,\infty}} + \delta\mathcal{D}_{\epsilon,s_0}(T).
$$

□

Propositions 5.2 and 5.5 derive the weak-type estimate (36). This completes the proof of Theorem 5.1.

Next, we show Theorem 1.5 and the decay estimate (12) in Theorem 1.6(ii). Since we have the interpolation inequality (see Proposition 2.1(iii))

$$
\|u\|_{\dot{H}^s} \lesssim_{s_1,s_2,\theta} \|u\|^{1-\theta}_{\dot{B}^{s_1}_{2,\infty}} \|u\|^{\theta}_{\dot{B}^{s_2}_{2,\infty}} \quad \text{with } s = (1-\theta)s_1 + \theta s_2,\ \theta \in (0,1), \tag{52}
$$

Propositions 5.4 and 5.5 allow us to complete the proof of the estimate (11) when $(p, s) \neq (2, 0)$ and the estimate (12) in Theorem 1.6(ii). The rest of the proof of Theorem 1.5 is established by the following proposition.

Proposition 5.6 *If $\delta = \delta_1 + \delta_2$ is small enough and $(\sigma_0, w_0) \in L^2$, then the energy estimate*

$$\|(\sigma, w)(t)\|_{H^3}^2 + \int_0^t \left(\|\nabla w(\tau)\|_{H^3}^2 + \|\nabla\sigma(\tau)\|_{H^2}^2\right) d\tau \lesssim \|(\sigma_0, v_0)\|_{H^3}^2$$

holds for any $t \geq 0$.

Proof Let $\kappa > 0$ be sufficiently small. We set

$$\mathcal{E}_1(t) = \sum_{|\alpha_1| \leq 3} \|\partial_x^{\alpha_1}(\sigma, w)(t)\|_{L^2}^2 + \sum_{|\alpha_2| \leq 2} \kappa \langle \partial_x^{\alpha_2} \nabla\sigma(t), \partial_x^{\alpha_2} w(t)\rangle.$$

Since (σ, w) satisfies Eq. (17), we have the following inequality

$$\begin{aligned} \mathcal{E}_1(t) - \mathcal{E}_1(0) + \int_0^t \widetilde{\mathcal{E}}_1(\tau)\, d\tau \lesssim \int_0^t & \sum_{|\alpha_1| \leq 3} \left(\langle \partial_x^{\alpha_1} f, \partial_x^{\alpha_1}\sigma\rangle + \langle \partial_x^{\alpha_1} g, \partial_x^{\alpha_1} w\rangle\right) \\ &+ \sum_{|\alpha_2| \leq 2} \kappa \left(\langle \partial_x^{\alpha_2}\nabla f, \partial_x^{\alpha_2} w\rangle + \langle \partial_x^{\alpha_2} g, \partial_x^{\alpha_2}\nabla\sigma\rangle\right) d\tau, \end{aligned} \tag{53}$$

where $t \geq 0$ and

$$\begin{aligned} \widetilde{\mathcal{E}}_1(t) = &\sum_{|\alpha_1| \leq 3} \left(\nu \|\partial_x^{\alpha_1}\nabla w(t)\|_{L^2}^2 + (\nu + \nu')\|\partial_x^{\alpha_1} \mathrm{div}\, w(t)\|_{L^2}^2\right) \\ &+ \sum_{|\alpha_2| \leq 2} \kappa \left(\|\partial_x^{\alpha_2}\nabla\sigma(t)\|_{L^2}^2 + \|\partial_x^{\alpha_2}\mathrm{div}\, w(t)\|_{L^2}^2\right). \end{aligned}$$

We also have

$$\mathcal{E}_1(t) \sim \|(\sigma, w)(t)\|_{H^3}^2, \quad \widetilde{\mathcal{E}}_1(t) \sim \|\nabla\sigma(t)\|_{H^2}^2 + \|\nabla w(t)\|_{H^3}^2$$

for $t \geq 0$. Next, we establish the estimate of the right-hand of (53). By Proposition 2.1(ii), (vi), we have

$$\langle f, \sigma\rangle + \kappa\langle \nabla f, w\rangle \lesssim \|(\sigma^*, v^*, w)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} \|(\nabla\sigma, \nabla w, \nabla^2 w)\|_{L^2}^2 \lesssim \delta\widetilde{\mathcal{E}}_1.$$

Using the identities

$$\langle \mathrm{div}((v^* + w)\partial_x^{\alpha_1}\sigma), \partial_x^{\alpha_1}\sigma\rangle = \frac{1}{2}\langle \mathrm{div}(v^* + w)\partial_x^{\alpha_1}\sigma, \partial_x^{\alpha_1}\sigma\rangle,$$

for any $1 \le |\alpha_1| \le 3$, we have

$$\begin{aligned}
&\langle \partial_x^{\alpha_1} f, \partial_x^{\alpha_1}\sigma\rangle \\
&\quad = \frac{1}{2}\langle \mathrm{div}(v^* + w)\, \partial_x^{\alpha_1}\sigma, \partial_x^{\alpha_1}\sigma\rangle + \langle \partial_x^{\alpha_1}\mathrm{div}(\sigma^* w), \partial_x^{\alpha_1}\sigma\rangle \\
&\qquad + \sum_{0<\beta\le\alpha_1} \langle \mathrm{div}(\partial_x^{\beta}(v^* + w)\, \partial_x^{\alpha_1-\beta}\sigma), \partial_x^{\alpha_1}\sigma\rangle \\
&\quad \lesssim \|\mathrm{div}(v^* + w)\|_{L^\infty}\|\nabla\sigma\|_{H^2}^2 + \|\sigma^*\|_{H^4}\|\nabla w\|_{H^3}\|\nabla\sigma\|_{H^2} \\
&\qquad + \|\nabla v^*\|_{H^4}\|\nabla\sigma\|_{H^2}^2 + \|w\|_{H^3}\|\nabla\sigma\|_{H^2}^2 + \|\nabla w\|_{H^3}\|\sigma\|_{H^3}\|\nabla\sigma\|_{H^2} \\
&\quad \lesssim \delta\widetilde{\mathcal{E}}_1. \qquad (54)
\end{aligned}$$

We also have

$$\langle \partial_x^{\alpha_2}\nabla f, \partial_x^{\alpha_2} w\rangle \lesssim \delta\widetilde{\mathcal{E}}_1 \quad \text{for } 0 \le |\alpha_2| \le 2.$$

Similarly to the estimate of (54) by applying Proposition 2.1(v), we have

$$\sum_{|\alpha_1|\le 3} \langle \partial_x^{\alpha_1} g, \partial_x^{\alpha_1} w\rangle + \sum_{|\alpha_2|\le 2} \kappa\langle \partial_x^{\alpha_2} g, \partial_x^{\alpha_2}\nabla\sigma\rangle \lesssim \delta\widetilde{\mathcal{E}}_1.$$

If δ is sufficiently small, then

$$\mathcal{E}_1(t) - \mathcal{E}_1(0) + c_0\int_0^t \widetilde{\mathcal{E}}_1(\tau)\, d\tau \le 0,$$

where $t \ge 0$ and $c_0 > 0$ is a constant. Hence, we obtain

$$\|(\sigma, w)(t)\|_{H^3}^2 + \int_0^t \|\nabla w(\tau)\|_{H^3}^2 + \|\nabla\sigma(\tau)\|_{H^2}^2 d\tau \lesssim \|(\sigma_0, v_0)\|_{H^3}^2$$

for any $t \ge 0$. □

5.3 The proof of optimality

This section is devoted to the proof of Theorem 1.6. We first show the following lemma which derives the estimate of N_L defined in (43):

$$\begin{aligned}
N_L(t) &= \int_0^t e_L^{(t-\tau)A}\begin{bmatrix} 0 \\ h \end{bmatrix}(\tau)\, d\tau + \int_0^t e_L^{(t-\tau)A}\begin{bmatrix} 0 \\ \gamma_1^{-1}\sigma F(x) \end{bmatrix}(\tau)\, d\tau \\
&= N_L^1(t) + N_L^2(t).
\end{aligned}$$

Recall that δ_1 is given by

$$\delta_1 = \|\sigma^*\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty}\cap\dot{H}^4} + \|v^*\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}\cap\dot{H}^5}.$$

Lemma 5.7 *Let an initial perturbation $U_0 = (\sigma_0, w_0)$ be sufficiently small in $\dot{B}^{1/2}_{2,\infty} \cap H^3$ norm. If $U_0 \in L^1$ and $-3/2 < s < 3/2$, then there exists a small $d > 0$ such that*

$$\|N_L(t)\|_{\dot{H}^s} \lesssim_{s,d,U_0} \delta_1(1+t)^{-\frac{s}{2}-\frac{3}{4}} + (1+t)^{-\frac{s}{2}-\frac{3}{4}-d}.$$

In the case $1 < p < 2$, $-3/2 < s < 3/2$ with $s/2 + 3/2\,(1/p - 1/2) > 0$, if $U_0 \in L^{p,\infty}$, then there exists a small $d > 0$ such that

$$\|N_L(t)\|_{\dot{H}^s} \lesssim_{s,d,U_0} \delta_1(1+t)^{-\frac{s}{2}-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)} + (1+t)^{-\frac{s}{2}-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-d}.$$

Proof Theorem 1.5 and the decay estimate (12) in Theorem 1.6(ii) imply the decay estimate of the perturbation $U = (\sigma, w)$:

$$\|U(t)\|_{\dot{H}^s} \lesssim_{s,p} (1+t)^{-\frac{s}{2}-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)} \|U_0\|_{\Lambda_p \cap H^3},$$

where $-3/2 < s < 3/2$, $1 \le p \le 2$ with $s/2 + 3/2(1/p - 1/2) > 0$, and Λ_p is the function space defined in Theorem 5.4. Let us denote $s_0(p) = -3(1/p - 1/2)$. Let $\psi = (\psi_1, \ldots, \psi_4)^\mathsf{T} \in \mathcal{S}^4$. Fix a real number α_0 that satisfies $s - s_0(p) < \alpha_0 < s - s_0(p) + 2$, $s + 1/2 < \alpha_0 < s + 5/2$. Proposition 2.1(ii) then yields

$$\begin{aligned}\left\langle N_L^1(t), \psi \right\rangle \lesssim \int_{\frac{t}{2}}^{t} \|h(\tau)\|_{\dot{B}^{s-2}_{2,\infty}} \|e_L^{(t-\tau)A^*}\psi\|_{\dot{B}^{2-s}_{2,1}}\, d\tau \\ + \int_0^{\frac{t}{2}} \|h(\tau)\|_{\dot{B}^{s-\alpha_0}_{2,\infty}} \|e_L^{(t-\tau)A^*}\psi\|_{\dot{B}^{\alpha_0-s}_{2,1}}\, d\tau.\end{aligned}$$

Lemmas 4.1(ii) and 5.3 imply that if $-1/2 < s < 3/2$ and $s/2+3/2(1/p-1/2) > 0$, then for any small $d, \eta > 0$ with $d \le \eta$, we have

$$\begin{aligned}&\int_{\frac{t}{2}}^{t} \|h(\tau)\|_{\dot{B}^{s-2}_{2,\infty}} \|e_L^{(t-\tau)A^*}\psi\|_{\dot{B}^{2-s}_{2,1}}\, d\tau \\ &\lesssim_s \delta_1 \int_{\frac{t}{2}}^{t} \|U(\tau)\|_{\dot{B}^{s}_{2,\infty}} \|e_L^{(t-\tau)A^*}\psi\|_{\dot{B}^{2-s}_{2,1}}\, d\tau \\ &\quad + \int_{\frac{t}{2}}^{t} \|U(\tau)\|_{\dot{B}^{\frac{1}{2}+\eta}_{2,\infty} \cap \dot{B}^{\frac{3}{2}-\eta}_{2,\infty}} \|U(\tau)\|_{\dot{B}^{s-\eta}_{2,\infty} \cap \dot{B}^{s+\eta}_{2,\infty}} \|e_L^{(t-\tau)A^*}\psi\|_{\dot{B}^{2-s}_{2,1}}\, d\tau \\ &\lesssim_{s,\eta,U_0} \left(\delta_1(1+t)^{-\frac{s-s_0}{2}} + (1+t)^{-\frac{s-s_0}{2}-d}\right) \int_0^\infty \|e_L^{\tau A^*}\psi\|_{\dot{B}^{2-s}_{2,1}}\, d\tau \\ &\lesssim_{s,d} \left(\delta_1(1+t)^{-\frac{s-s_0}{2}} + (1+t)^{-\frac{s-s_0}{2}-d}\right)\|\psi\|_{\dot{B}^{-s}_{2,1}}.\end{aligned}$$

By Proposition 2.1(vi), Lemmas 4.1(i) and 5.3, for any small $d, \eta > 0$ with $d \leq \eta$, we have

$$
\begin{aligned}
&\int_0^{\frac{t}{2}} \|h(\tau)\|_{\dot{B}^{s-\alpha_0}_{2,\infty}} \|e_L^{(t-\tau)A^*}\psi\|_{\dot{B}^{\alpha_0-s}_{2,1}}\,d\tau \\
&\lesssim_s \delta_1 \int_0^{\frac{t}{2}} \|U(\tau)\|_{\dot{B}^{s-\alpha_0+2}_{2,\infty}} \|e_L^{(t-\tau)A^*}\psi\|_{\dot{B}^{\alpha_0-s}_{2,1}}\,d\tau \\
&\quad + \int_{\frac{t}{2}}^{t} \|U(\tau)\|_{\dot{B}^{\frac{1}{2}+\eta}_{2,\infty}\cap \dot{B}^{\frac{3}{2}-\eta}_{2,\infty}} \|U(\tau)\|_{\dot{B}^{s-\alpha_0+2-\eta}_{2,\infty}\cap \dot{B}^{s-\alpha_0+2+\eta}_{2,\infty}} \|e_L^{(t-\tau)A^*}\psi\|_{\dot{B}^{\alpha_0-s}_{2,1}}\,d\tau \\
&\lesssim_{s,d} \left(\delta_1(1+t)^{-\frac{s-s_0}{2}} + (1+t)^{-\frac{s-s_0}{2}-d}\right)\|\psi\|_{\dot{B}^{-s}_{2,1}}.
\end{aligned}
$$

By Proposition 2.1(vi), Lemma 4.1(i) with $\alpha = s + 3/2$, we have

$$
\begin{aligned}
\|N_L^2(t)\|_{\dot{B}^s_{2,\infty}} &\lesssim_{j_0} \int_0^t (1+t-\tau)^{-\frac{s}{2}-\frac{3}{4}} \|\sigma(\tau)F\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty}}\,d\tau \\
&\lesssim_{\epsilon,p} \|F\|_{\dot{B}^{-s}_{2,1}} \int_0^t (1+t-\tau)^{-\frac{s}{2}-\frac{3}{4}} \|\sigma(\tau)\|_{\dot{B}^s_{2,\infty}}\,d\tau \\
&\lesssim_{\epsilon,p} \delta_1(1+t)^{-\frac{s-s_0}{2}}.
\end{aligned}
$$

Since $\psi \in \mathcal{S}$ is arbitrary, we have

$$
\|N_L(t)\|_{\dot{B}^s_{2,\infty}} \lesssim_{s,d} \delta_1(1+t)^{-\frac{s-s_0}{2}} + (1+t)^{-\frac{s-s_0}{2}-d}.
$$

By the interpolation inequality in Proposition 2.1(iii), we have

$$
\|N_L(t)\|_{\dot{H}^s} \lesssim_{s,d} \delta_1(1+t)^{-\frac{s-s_0}{2}} + (1+t)^{-\frac{s-s_0}{2}-d} \tag{55}
$$

for $-1/2 < s < 3/2$, $1 \leq p \leq 2$ with $s/2 + 3/2(1/p - 1/2) > 0$. As in the proof of the inequality (46), we obtain the inequality (55) for $-3/2 < s \leq -1/2$, $1 \leq p \leq 2$ with $s/2 + 3/2(1/p - 1/2) > 0$. □

Next, we prove Theorem 1.6. The proof of optimality in Theorem 1.6 below is inspired by the argument in Kawashima et al. [12].

Proof of Theorem 1.6 Let an initial value $U_0 = (\sigma_0, w_0)^\top$ satisfy $U_0 \in L^1$ and $\|U_0\|_{H^3} \ll \delta_2$. Then, the decay estimate (11) with $p = 1$ holds. Lemma 5.7 ensures that there exists a small $d > 0$ such that

$$
\begin{aligned}
\|U\|_{\dot{H}^s} &\geq \|e_L^{tA}U_0\|_{\dot{H}^s} - \|N_L(t)\|_{\dot{H}^s} \\
&\gtrsim_{s,U_0} \|e_L^{tA}U_0\|_{\dot{H}^s} - \delta_1(1+t)^{-\frac{s}{2}-\frac{3}{4}} - (1+t)^{-\frac{s}{2}-\frac{3}{4}-d},
\end{aligned}
$$

where $-3/2 < s < 3/2$. We recall the spectral resolution in (22), (23):

$$e^{t\hat{A}(\xi)} = e^{\lambda_+ t} P_+(\xi) + e^{\lambda_- t} P_-(\xi) + e^{\lambda_0 t} P_0(\xi).$$

Since $\lambda_\pm(\xi) \sim i|\xi|$ as $\xi \to 0$, we have the asymptotic behavior of $P_\pm(\xi)$:

$$P_\pm(\xi) \sim Q_\pm(\xi) + O(|\xi|) \quad \text{as } |\xi| \to 0, \quad Q_\pm(\xi) = \frac{1}{2}\begin{bmatrix} 1 & \mp\frac{\xi}{|\xi|} \\ \mp\frac{\xi^{\mathsf{T}}}{|\xi|} & \frac{\xi\otimes\xi}{|\xi|^2} \end{bmatrix}. \tag{56}$$

Let

$$e_L^{tA_0} U_0 \equiv \dot{S}_{j_0}\mathcal{F}^{-1}\left[\left(e^{\lambda_+ t} Q_+(\xi) + e^{\lambda_- t} Q_-(\xi) + e^{\lambda_0 t} P_0(\xi)\right)\widehat{U_0}\right],$$

where $\dot{S}_j$ is the low frequency cut-off operator defined in (13). Then, there exist constants $c, c_0 > 0$ such that

$$\|(e_L^{tA} - e_L^{tA_0})U_0\|_{\dot{H}^s}^2 \lesssim \int_{|\xi|\le c_0} |\xi|^{2(s+1)} e^{-ct|\xi|^2} d\xi \; \|U_0\|_{L^1} \lesssim_{j_0} (1+t)^{-s-\frac{5}{2}} \|U_0\|_{L^1}.$$

As real orthogonal projections $Q_\pm$, P_0 satisfy

$$Q_+ + Q_- + P_0 = \mathrm{I}_4, \quad Q_\pm Q_\mp = Q_\pm P_0 = 0$$

and $\widehat{U_0}(\xi)$ is continuous at $\xi = 0$, we obtain the following estimate for sufficiently small $j_0 \in \mathbb{Z}$,

$$\begin{aligned} &\|e_L^{tA_0} U_0\|_{\dot{H}^s}^2 \\ &\quad = \|\dot{S}_{j_0}\mathcal{F}^{-1}[e^{\lambda_+ t} Q_+ \widehat{U_0}]\|_{\dot{H}^s}^2 + \|\dot{S}_{j_0}\mathcal{F}^{-1}[e^{\lambda_- t} Q_- \widehat{U_0}]\|_{\dot{H}^s}^2 + \|\dot{S}_{j_0}\mathcal{F}^{-1}[e^{\lambda_0 t} P_0 \widehat{U_0}]\|_{\dot{H}^s}^2 \\ &\quad \gtrsim \left\|\dot{S}_{j_0} e^{ct\Delta} U_0\right\|_{\dot{H}^s}^2 \gtrsim_{j_0, U_0} \left|\widehat{U_0}(0)\right|^2 \left\|\dot{S}_{j_0} e^{ct\Delta}\right\|_{\dot{H}^s}^2 \gtrsim M(U_0)^2 (1+t)^{-s-\frac{3}{2}}, \end{aligned}$$

where $M(U_0) \equiv |\int U_0 dx|$, $-3/2 < s < 3/2$. Here, we use the identity

$$|V|^2 = |Q_+ V|^2 + |Q_- V|^2 + |P_0 V|^2 \quad \text{for any } V \in \mathbb{C}^4.$$

Therefore, if $\delta_1 = \delta_1(U_0) > 0$ is sufficiently small, then we have

$$\|U(t)\|_{\dot{H}^s} \gtrsim_{s, U_0} (1+t)^{-\frac{s}{2}-\frac{3}{4}} \quad \text{for } t \gg 1,$$

where $-3/2 < s < 3/2$. We next treat the case $1 < p < 2$, $-3/2 < s < 3/2$ with $s/2 + 3/2\,(1/p - 1/2) > 0$. Let $U_0 = \delta_0 \dot{S}_{j_0}(|\cdot|^{-3/p})\boldsymbol{e}_1$ with $\boldsymbol{e}_1 = (1, 0, 0, 0)^{\mathsf{T}}$ and small $\delta_0 > 0$. Then, we have $U_0 \in L^{p,\infty}$ and $\|U_0\|_{H^3} \lesssim_{j_0} \delta_0$, since $\mathcal{F}(|\cdot|^{-3/p}) \sim$

$|\cdot|^{3/p-3}$. By (56), there exist constants $c, c_0 > 0$ such that

$$\|(e_L^{tA} - e_L^{tA_0})U_0\|_{\dot{H}^s}^2 \lesssim \int_{|\xi|\leq c_0} |\xi|^{2\left(s+\frac{3}{p}-2\right)} e^{-ct|\xi|^2} d\xi \lesssim_{j_0} (1+t)^{-s-3\left(\frac{1}{p}-\frac{1}{2}\right)-1}. \tag{57}$$

Let j_0 be sufficiently small. Then, there exists a constant $c_1 > 0$ such that

$$\begin{aligned}
&\|e_L^{tA_0} U_0\|_{\dot{H}^s}^2 \\
&\quad = \|\dot{S}_{j_0}\mathcal{F}^{-1}[e^{\lambda_+ t} Q_+ \widehat{U_0}]\|_{\dot{H}^s}^2 + \|\dot{S}_{j_0}\mathcal{F}^{-1}[e^{\lambda_- t} Q_- \widehat{U_0}]\|_{\dot{H}^s}^2 + \|\dot{S}_{j_0}\mathcal{F}^{-1}[e^{\lambda_0 t} P_0 \widehat{U_0}]\|_{\dot{H}^s}^2 \\
&\quad \gtrsim \int_{|\xi|\leq c_1} |\xi|^{2\left(s+\frac{3}{p}-3\right)} e^{-ct|\xi|^2} d\xi \gtrsim (1+t)^{-s-3\left(\frac{1}{p}-\frac{1}{2}\right)}.
\end{aligned} \tag{58}$$

Hence, the corresponding global solution $U = (\sigma, w)^{\mathsf{T}}$ satisfies

$$\|U(t)\|_{\dot{H}^s} \gtrsim (1+t)^{-\frac{s}{2}-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)} \quad \text{for } t \gg 1.$$

□

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