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On the critical regularity of nonlinearities for semilinear classical wave equations

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Abstract

In this paper, we consider the Cauchy problem for semilinear classical wave equations

$$u_{tt} - \Delta u = |u|^{p_S(n)}\mu(|u|)$$

with the Strauss exponent $p_S(n)$ and a modulus of continuity $\mu = \mu(\tau)$, which provides an additional regularity of nonlinearities in $u = 0$ comparing with the power nonlinearity $|u|^{p_S(n)}$. We obtain a sharp condition on μ as a threshold between global (in time) existence of small data radial solutions by deriving polynomial-logarithmic type weighted $L_t^\infty L_r^\infty$ estimates, and blow-up of solutions in finite time even for small data by applying iteration methods with slicing procedure. These results imply a conjecture for the critical regularity of source nonlinearities for semilinear classical wave equations. We verify this conjecture in the 3d case.

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1 Introduction

In the last 40 years, the Cauchy problem for semilinear classical wave equations with power nonlinearity, namely,

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$$\begin{cases} u_{tt} - \Delta u = |u|^p, & x \in \mathbb{R}^n,\ t > 0, \\ u(0,x) = u_0(x),\ \ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

with $p > 1$, has been deeply studied by the mathematical community. For example, the questions on global (in time) existence of solutions, blow-up of solutions in finite time and sharp lifespan estimates of solutions were of interest. In particular, the critical exponent for the semilinear Cauchy problem (1) is given by the so-called Strauss exponent $p_S(n)$, which was proposed by Strauss in [24]. Nowadays, the correctness of the Strauss exponent is well-known. The Strauss exponent $p_S(n)$ is the positive root of the quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0 \tag{2}$$

for $n \geqslant 2$, that is,

$$p_S(n) := \frac{n+1+\sqrt{n^2+10n-7}}{2(n-1)} \quad \text{when } n \geqslant 2,$$

and we put $p_S(1) := +\infty$. On one hand, for blow-up results when $1 < p \leqslant p_S(n)$, we refer interested readers to the classical papers [8, 11–13, 22, 23, 27, 28] and the new proofs proposed in [10, 26]. On the other hand, concerning global (in time) existence results when $p > p_S(n)$, we refer to [7, 9, 12, 20, 25] and references therein. Summarizing these known results, in the scale of power nonlinearities $\{|u|^p\}_{p>1}$, the critical exponent $p = p_S(n)$ for semilinear classical wave equations (1) has been found, to be the threshold condition between global (in time) existence of solutions and blow-up of local (in time) solutions with small initial data.

Nevertheless, to determine the critical nonlinearity or the critical regularity of nonlinearities, it seems too rough to restrict the consideration of semilinear wave equations (1) to the scale of power nonlinearities $\{|u|^p\}_{p>1}$. The question of the critical regularity of nonlinearities for semilinear classical wave equations is completely open as far as the authors know. For this reason, our contribution of this paper is to give an answer to this question for a class of modulus of continuity. Furthermore, we will suggest a candidate for the general critical nonlinearity via our derived results.

In this manuscript, we consider the following Cauchy problem for semilinear classical wave equations with modulus of continuity in the nonlinearity:

$$\begin{cases} u_{tt} - \Delta u = |u|^{p_S(n)}\mu(|u|), & x \in \mathbb{R}^n,\ t > 0, \\ u(0,x) = u_0(x),\ \ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{3}$$

for $n \geqslant 2$ (due to $p_S(1) = +\infty$), where $p_S(n)$ stands for the Strauss exponent, and $\mu = \mu(\tau)$ is a modulus of continuity. To be specific, a function $\mu : [0, +\infty) \to [0, +\infty)$ is called a modulus of continuity, if μ is a continuous, concave and increasing function satisfying $\mu(0) = 0$. The additional term of modulus of continuity provides an additional regularity of the nonlinear term in $u = 0$ in the Cauchy problem (3) comparing with the power nonlinearity $|u|^{p_S(n)}$. Note that the critical nonlinearity has been studied recently in semilinear classical damped wave equations [6] and the corresponding weakly coupled systems [5]. Nevertheless, due to the lack of crucial damping mechanisms, the study of the semilinear Cauchy problem (3) is not a generalization of those of [5, 6], e.g. the usual test function methods and the Matsumura type $L^p - L^q$ estimates do not work for our model (3).

The main purpose of this paper is to derive the critical regularity of nonlinearities for the semilinear Cauchy problem (3), namely, the threshold condition for modulus of continuity μ. First of all, by applying iteration methods with slicing procedure (motivated by [1, 26]) for a new weighted functional, which contains a local (in time) solution and a modulus of continuity, under some conditions of initial data, we will prove a blow-up result in Sect. 3 when

$$\lim_{\tau\to 0^+} \mu(\tau)\left(\log\frac{1}{\tau}\right)^{\frac{1}{p_S(n)}} \in [c_l, +\infty]$$

with a suitably large constant $c_l \gg 1$. Next, we will study the three dimensional Cauchy problem (3) with modulus of continuity satisfying

$$\lim_{\tau\to 0^+} \mu(\tau)\left(\log\frac{1}{\tau}\right)^{\frac{1}{p_S(3)}} = 0$$

in the radial case. By developing polynomial-logarithmic type weighted $L_t^\infty L_r^\infty$ estimates via refined analysis in the (t, r)-plane, we will demonstrate global (in time) existence of small data radial solution in Sect. 4. A typical example is that for a modulus of continuity $\mu = \mu(\tau)$ with $\mu(0) = 0$ which satisfies

$$\mu(\tau) = c_l\left(\log\frac{1}{\tau}\right)^{-\gamma} \quad \text{with } c_l \gg 1, \text{ when } \tau \in (0, \tau_0]. \tag{4}$$

Our results of this paper ensure that the critical regularity of nonlinearities $|u|^{p_S(3)}\mu(|u|)$ in semilinear classical wave equations with the modulus of continuity satisfying (4) is described by the threshold

$$\gamma = \frac{1}{p_S(3)}.$$

Namely, global (in time) existence of solutions holds when $\gamma > \frac{1}{p_S(3)}$ and blow-up of solutions holds when $0 < \gamma \leqslant \frac{1}{p_S(3)}$. Other examples will be shown in Sect. 2. To end this paper, we will give a conjecture for general conditions of the critical nonlinearity for the semilinear Cauchy problem (3) as final remarks in Sect. 5.

Notation: Firstly, c and C denote some positive constants, which may be changed from line to line. We write $f \lesssim g$ if there exists a positive constant C such that $f \leqslant Cg$. The relation $f \simeq g$ holds if and only if $g \lesssim f \lesssim g$. Moreover, $B_R(0)$ denotes the ball around the origin with radius R. We denote $\langle y\rangle := 3 + |y|$ for any $y \in \mathbb{R}$ throughout this manuscript.

2 Main results

Before stating our blow-up result, we firstly introduce the notion of energy solutions to the Cauchy problem (3) that we are going to use later.

Definition 2.1 Let $u_0 \in H^1$ and $u_1 \in L^2$. We say that $u = u(t, x)$ is an energy solution to the semilinear Cauchy problem (3) on $[0, T)$ if

$$u \in \mathcal{C}([0,T), H^1) \cap \mathcal{C}^1([0,T), L^2) \quad \text{such that } |u|^{p_S(n)}\mu(|u|) \in L^1_{\text{loc}}([0,T) \times \mathbb{R}^n)$$

fulfills the next integral relation:

$$\begin{aligned}
&\int_{\mathbb{R}^n} u_t(t,x)\phi(t,x)\mathrm{d}x + \int_0^t \int_{\mathbb{R}^n} \big(\nabla u(s,x) \cdot \nabla\phi(s,x) - u_s(s,x)\phi_s(s,x)\big)\mathrm{d}x\mathrm{d}s \\
&\quad = \int_{\mathbb{R}^n} u_1(x)\phi(0,x)\mathrm{d}x + \int_0^t \int_{\mathbb{R}^n} |u(s,x)|^{p_S(n)}\mu\big(|u(s,x)|\big)\phi(s,x)\mathrm{d}x\mathrm{d}s \qquad (5)
\end{aligned}$$

for any $\phi \in \mathcal{C}_0^\infty([0,T) \times \mathbb{R}^n)$ and any $t \in [0, T)$.

Theorem 1 *Let $u_0 \in H^1$ and $u_1 \in L^2$ be non-negative, non-trivial and compactly supported functions with supports contained in $B_R(0)$ for some $R > 0$. Moreover, let us consider a modulus of continuity $\mu = \mu(\tau)$ with $\mu(0) = 0$ satisfying*

$$\lim_{\tau \to 0^+} \mu(\tau)\left(\log\frac{1}{\tau}\right)^{\frac{1}{p_S(n)}} =: C_{\text{Str}} \in [c_l, +\infty] \qquad (6)$$

with a suitably large constant $c_l = c_l(R, n) \gg 1$. We assume that the function $g : \tau \in \mathbb{R} \to g(\tau) := \tau[\mu(|\tau|)]^{\frac{1}{p_S(n)}}$ is convex on $\mathbb{R}$. Finally, let

$$u \in \mathcal{C}([0,T), H^1) \cap \mathcal{C}^1([0,T), L^2) \quad \text{such that } |u|^{p_S(n)}\mu(|u|) \in L^1_{\text{loc}}([0,T) \times \mathbb{R}^n)$$

be an energy solution to the semilinear Cauchy problem (3) on $[0, T)$ for $n \geqslant 2$ according to Definition 2.1. Then, the energy solution u blows up in finite time.

Example 2.1 The hypothesis (6) and the supposed property for the function $g = g(\tau)$ of Theorem 1 hold for the following functions $\mu = \mu(\tau)$ on a small interval $[0, \tau_0]$ with $0 < \tau_0 \ll 1$:

- $\mu(0) = 0$ and $\mu(\tau) = (\log\frac{1}{\tau})^{-\gamma}$ with $0 < \gamma < \frac{1}{p_S(n)}$;
- $\mu(0) = 0$ and $\mu(\tau) = c_l(\log\frac{1}{\tau})^{-\frac{1}{p_S(n)}}$ with $c_l \gg 1$;
- $\mu(0) = 0$ and $\mu(\tau) = (\log\frac{1}{\tau})^{-\frac{1}{p_S(n)}}(\log^k\frac{1}{\tau})^{\gamma}$ with $\gamma > 0$ and $k \geqslant 2$, here $\log^k$ denotes the iterated logarithm (k times application).

Note that the modulus of continuity in the last cases can be continued to $\tau \in [0, +\infty)$ in such a way that $\mu = \mu(\tau)$ is a continuous, concave and increasing function, for example, a smooth and concave continuation function with $\mu(0) = 0$ such that

$$\mu(\tau) = \begin{cases} c_l(\log \frac{1}{\tau})^{-\gamma} & \text{when } \tau \in (0, \frac{1}{3}], \\ \text{strictly increasing} & \text{when } \tau \in [\frac{1}{3}, 3], \\ c_l(\log \tau)^{\gamma} & \text{when } \tau \in [3, +\infty), \end{cases}$$

with a suitably large constant $c_l \gg 1$ and $0 < \gamma \leqslant \frac{1}{p_S(n)}$. A counterexample for the condition (6) in Theorem 1 is $\mu(\tau) = \tau^{\nu}$ with $\nu > 0$. This is not surprising due to the global (in time) existence results [7, 9, 12, 20, 25] for the semilinear classical wave equations (1) with power nonlinearity $|u|^{p_S(n)+\nu}$.

Remark 1 Concerning the semilinear wave equation (1) with the critical exponent $p = p_S(n)$, by taking the additional term of modulus of continuity $\mu(|u|)$ fulfilling (6) in the nonlinearity, Theorem 1 shows that the energy solutions still blow up in finite time.

To indicate the sharpness of the condition (6), we next study the three dimensional semilinear Cauchy problem (3) with a modulus of continuity satisfying (8). Before showing our result, taking $r = |x|$, let us introduce a definition of radial solutions to our aim model in three dimensions, namely,

$$\begin{cases} u_{tt} - u_{rr} - \frac{2}{r}u_r = |u|^{p_S(3)}\mu(|u|), & r > 0,\ t > 0, \\ u(0, r) = u_0(r),\ \ u_t(0, r) = u_1(r), & r > 0. \end{cases} \tag{7}$$

Definition 2.2 The function $u = u(t, r)$ is called a global (in time) mild solution to the Cauchy problem (7) if $u \in \mathcal{C}([0, +\infty) \times \mathbb{R}_+)$ carrying its initial data, and satisfying the following integral equality:

$$\begin{aligned} u(t, r) &= \mathcal{E}_0(t, r) *_{(r)} u_0(r) + \mathcal{E}_1(t, r) *_{(r)} u_1(r) \\ &\quad + \int_0^t \mathcal{E}_1(t - s, r) *_{(r)} \left[|u(s, r)|^{p_S(3)}\mu\big(|u(s, r)|\big)\right]\mathrm{d}s. \end{aligned}$$

In the above, $\mathcal{E}_0 = \mathcal{E}_0(t, r)$ and $\mathcal{E}_1 = \mathcal{E}_1(t, r)$ are the fundamental solutions to the corresponding linear Cauchy problem to (7) with vanishing right-hand side.

We turn to the global (in time) existence of radial solutions in the subsequent theorem.

Theorem 2 *Let us consider a modulus of continuity $\mu = \mu(\tau)$ with $\mu(0) = 0$ satisfying*

$$\lim_{\tau \to 0^+} \mu(\tau) \left(\log \frac{1}{\tau}\right)^{\frac{1}{p_S(3)}} = 0. \tag{8}$$

Furthermore, with a sufficiently small τ_0 it holds

$$\mu(\tau) \left(\log \frac{1}{\tau}\right)^{\frac{1}{p_S(3)}} \lesssim \left(\log \log \frac{1}{\tau}\right)^{-1} \quad \textit{when } \tau \in (0, \tau_0]. \tag{9}$$

Let $\bar{u}_0 \in \mathcal{C}_0^2$ *and* $\bar{u}_1 \in \mathcal{C}_0^1$ *be radial. Then, there exists* $0 < \varepsilon_0 \ll 1$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$*, if* $u_0 = \varepsilon \bar{u}_0$ *and* $u_1 = \varepsilon \bar{u}_1$*, then the semilinear Cauchy problem* (3) *for* $n = 3$ *admits a uniquely determined global* (*in time*) *small data radial solution in the sense of Definition* 2.2 *such that* $u \in \mathcal{C}([0, +\infty) \times \mathbb{R}^3)$.

Remark 2 Since the assumption (9) implies (8) as $\tau \to 0^+$, one may drop the condition (8) directly. Nevertheless, to emphasize the importance of the essential condition (8) for the global (in time) existence result, we retain this condition. We conjecture that the logarithmic type decay condition (9) is a technical restriction.

Example 2.2 The hypotheses (8) and (9) hold for the following functions $\mu = \mu(\tau)$ on a small interval $[0, \tau_0]$ with $0 < \tau_0 \ll 1$:

- $\mu(\tau) = \tau^\gamma$ with $\gamma \in (0, 1]$;
- $\mu(\tau) = [\log(1 + \tau)]^\gamma$ with $\gamma \in (0, 1]$;
- $\mu(0) = 0$ and $\mu(\tau) = (\log \frac{1}{\tau})^{-\gamma}$ with $\gamma > \frac{1}{p_S(3)}$;
- $\mu(0) = 0$ and $\mu(\tau) = (\log \frac{1}{\tau})^{-\frac{1}{p_S(3)}} (\log\log \frac{1}{\tau})^\gamma$ with $\gamma \leqslant -1$;
- $\mu(0) = 0$ and $\mu(\tau) = (\log \frac{1}{\tau})^{-\frac{1}{p_S(3)}} (\log\log \frac{1}{\tau})^{-1} (\log^k \frac{1}{\tau})^\gamma$ with $\gamma < 0$ and $k \geqslant 3$.

Remark 3 By assuming additionally decay properties for initial data with respect to the radial behavior, we also can derive some pointwise decay estimates for the global (in time) radial solutions. More details will be given in Corollary 4.1 and in our proof in Sect. 4.

Remark 4 The key tool to prove Theorem 2 is to derive polynomial-logarithmic type weighted $L_t^\infty L_r^\infty$ estimates. Concerning higher dimensional cases, one may recall more general representations of radial solutions to the linear wave equation associated with polynomial type weighted $L_t^\infty L_r^\infty$ estimates (see [14, 16] for odd dimensions and [17] for even dimensions). Furthermore, by setting suitable logarithmic factors to be the additional part of weighted functions, one may derive some weighted $L_t^\infty L_r^\infty$ estimates to get a global (in time) existence result for higher dimensions n, nevertheless, this purpose is beyond the scope of this manuscript.

Remark 5 Our proof of global (in time) existence of small data solutions in Theorem 2 is based on suitable weighted $L_t^\infty L_r^\infty$ estimates for the inhomogeneous wave equation in the three dimensional radial case. It is also interesting to remove the radial symmetry assumption by developing some tools in the semilinear wave equation (3), for example, some weighted Strichartz estimates with weights which are invariant under Lorentz rotations as used in [7], and some weighted $L^2 - L^2$ estimates with Morawetz multiplier as used in [18]. One has to say, that, in general, more difficulties appear for handling the modulus of continuity term in comparison with the power nonlinearity. In the moment we are satisfied for verifying our conjecture for critical nonlinearity in 3d case. To apply the above proposed tools is beyond the scope in this paper.

Remark 6 Let us summarize the given results in Theorems 1 and 2. We recall the typical modulus of continuity proposed in Examples 2.1 and 2.2. In the consideration of semilinear wave equations (3) for $n = 3$ with the modulus of continuity satisfying

(4), we may conclude that the critical regularity of nonlinearities is described by the threshold $\gamma = \frac{1}{p_S(3)}$. This is one of the main contributions of this paper and it answers the open question proposed in the introduction.

Remark 7 Motivated by the global (in time) existence condition (8) as well as the blow-up condition (6), one may introduce the following possible quantity:

$$0 \leqslant C_{\mathrm{Str}} := \lim_{\tau \to 0^+} \mu(\tau) \left(\log \frac{1}{\tau} \right)^{\frac{1}{p_S(n)}}, \tag{10}$$

to describe the critical regularity of nonlinearities for semilinear wave equations (3). The blow-up phenomenon occurs when $C_{\mathrm{Str}} \in [c_l, +\infty]$ in Theorem 1 and the global (in time) existence result holds when $C_{\mathrm{Str}} = 0$ in Theorem 2. Explanations more in detail will be provided in Sect. 5.

3 Blow-up of energy solutions

This section is organized as follows. In Sect. 3.1, we will introduce a test function, and derive sharp estimates for it in $L^1(B_{R+t}(0))$. Then, thanks to some estimates for auxiliary functions, the iteration frame and lower bound estimates for a time-dependent functional will be established in Sects. 3.2 and 3.3, respectively. Finally, in Sect. 3.4, we will demonstrate the lower bound of this functional blows up in finite time by using iteration methods with slicing procedure.

3.1 Preliminaries and auxiliary functions

Let us set a non-negative parameter

$$q := \frac{n-1}{2} - \frac{1}{p_S(n)} \quad \text{for } n \geqslant 2. \tag{11}$$

Next, we recall the following pair of auxiliary functions from [26]:

$$\xi_q(t,x) := \int_0^{\lambda_0} \mathrm{e}^{-\lambda(R+t)} \cosh(\lambda t) \Phi(\lambda x) \lambda^q \mathrm{d}\lambda, \tag{12}$$

$$\eta_q(t,s,x) := \int_0^{\lambda_0} \mathrm{e}^{-\lambda(R+t)} \frac{\sinh(\lambda(t-s))}{\lambda(t-s)} \Phi(\lambda x) \lambda^q \mathrm{d}\lambda, \tag{13}$$

where λ_0 is a fixed positive parameter and the test function $\Phi = \Phi(x)$ defined by

$$\Phi : x \in \mathbb{R}^n \to \Phi(x) := \int_{\mathbb{S}^{n-1}} \mathrm{e}^{x \cdot \omega} \mathrm{d}\sigma_\omega \quad \text{for } n \geqslant 2,$$

was introduced by [27]. The test function Φ is positive, smooth, and satisfies $\Delta\Phi = \Phi$ with

$$\Phi(x) \simeq |x|^{-\frac{n-1}{2}} \mathrm{e}^{|x|} \quad \text{as } |x| \to +\infty. \tag{14}$$

By introducing the function with separate variables

$$\Psi : (t, x) \in [0, +\infty) \times \mathbb{R}^n \to \Psi(t, x) := \mathrm{e}^{-t}\, \Phi(x),$$

it is a solution to the free wave equation $\Psi_{tt} - \Delta\Psi = 0$ and has the next property.

Lemma 3.1 *The test function Ψ fulfills the estimates*

$$K_0(R)(R+t)^{\frac{n-1}{2}} \leqslant \int_{B_{R+t}(0)} \Psi(t, x)\mathrm{d}x \leqslant K_1(R)(R+t)^{\frac{n-1}{2}}$$

for any $t \geqslant 0$ and $n \geqslant 2$ with positive constants $K_0 = K_0(R)$ and $K_1 = K_1(R)$.

Proof By using integration by parts, we arrive at

$$\begin{aligned} \mathrm{e}^{-t} \int_0^{R+t} \zeta^{\frac{n-1}{2}} \mathrm{e}^{\zeta} \mathrm{d}\zeta &= (R+t)^{\frac{n-1}{2}} \mathrm{e}^{R} - \frac{n-1}{2} \mathrm{e}^{-t} \int_0^{R+t} \zeta^{\frac{n-3}{2}} \mathrm{e}^{\zeta} \mathrm{d}\zeta \\ &\leqslant C K_1(R)(R+t)^{\frac{n-1}{2}}. \end{aligned}$$

Shrinking the domain of integration to $[t, R+t]$, one notices

$$\begin{aligned} \mathrm{e}^{-t} \int_0^{R+t} \zeta^{\frac{n-1}{2}} \mathrm{e}^{\zeta} \mathrm{d}\zeta \geqslant \int_t^{R+t} \zeta^{\frac{n-1}{2}} \mathrm{d}\zeta &= \frac{2}{n+1} \left((R+t)^{\frac{n+1}{2}} - t^{\frac{n+1}{2}} \right) \\ &\geqslant C K_0(R)(R+t)^{\frac{n-1}{2}}. \end{aligned}$$

Therefore, the previous estimates imply the desired statement because of (14) and

$$\int_{B_{R+t}(0)} \Psi(t, x)\mathrm{d}x \simeq \int_{B_{R+t}(0)} |x|^{-\frac{n-1}{2}} \mathrm{e}^{|x|-t} \mathrm{d}x \simeq \mathrm{e}^{-t} \int_0^{R+t} \zeta^{\frac{n-1}{2}} \mathrm{e}^{\zeta} \mathrm{d}\zeta.$$

The proof is completed. □

Additionally, some useful estimates of ξ_q and η_q are stated in the following lemma, whose proof can be found in [26, Lemma 3.1]. Note that our setting of q fulfills all assumptions in Lemma 3.2. Moreover, we recall the notation $\langle y \rangle = 3 + |y|$.

Lemma 3.2 *There exists $\lambda_0 > 0$ such that the following properties hold for $n \geqslant 2$:*

(i) *if $q > -1$, $|x| \leqslant R$ and $t \geqslant 0$, then*

$$\begin{aligned} \xi_q(t, x) &\geqslant A_0, \\ \eta_q(t, 0, x) &\geqslant B_0 \langle t \rangle^{-1}; \end{aligned}$$

(ii) *if* $q > -1$, $|x| \leqslant R + s$ *and* $t > s \geqslant 0$, *then*

$$\eta_q(t, s, x) \geqslant B_1 \langle t \rangle^{-1} \langle s \rangle^{-q};$$

(iii) *if* $q > \frac{n-3}{2}$, $|x| \leqslant R + t$ *and* $t > 0$, *then*

$$\eta_q(t, t, x) \leqslant B_2 \langle t \rangle^{-\frac{n-1}{2}} \langle t - |x| \rangle^{\frac{n-3}{2} - q}.$$

Here, A_0 *and* B_k, *with* $k = 0, 1, 2$, *are positive constants depending only on* λ_0, q *and* R.

Finally, we include the following generalized version of Jensen's inequality [21], whose proof also has been shown in [6, Lemma 8].

Lemma 3.3 *Let* $g = g(\tau)$ *be a convex function on* $\mathbb{R}$. *Let* $\alpha = \alpha(x)$ *be defined and non-negative almost everywhere on* Ω, *such that* α *is positive in a set of positive measure. Then, it holds*

$$g\left(\frac{\int_\Omega v(x)\alpha(x)\mathrm{d}x}{\int_\Omega \alpha(x)\mathrm{d}x}\right) \leqslant \frac{\int_\Omega g(v(x))\alpha(x)\mathrm{d}x}{\int_\Omega \alpha(x)\mathrm{d}x}$$

for all non-negative functions $v = v(x)$ *provided that all the integral terms are meaningful.*

3.2 Construction of an iteration frame

In order to prove Theorem 1, we are going to use an iteration argument to derive lower bound estimates for the weighted space average of a local (in time) solution containing modulus of continuity. For this reason, we first derive a nonlinear integral inequality to get an iteration frame.

Proposition 3.1 *Let* $u_0 \in H^1$ *and* $u_1 \in L^2$ *be non-negative, non-trivial and compactly supported functions with supports contained in* $B_R(0)$ *for some* $R > 0$. *Let* u *be an energy solution to the semilinear Cauchy problem* (3) *on* $[0, T)$ *according to Definition* 2.1. *Then, the following integral identity holds*:

$$\begin{aligned}
&\int_{\mathbb{R}^n} u(t, x)\eta_q(t, t, x)\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^n} u_0(x)\xi_q(t, x)\mathrm{d}x + t\int_{\mathbb{R}^n} u_1(x)\eta_q(t, 0, x)\mathrm{d}x \\
&\qquad + \int_0^t (t - s)\int_{\mathbb{R}^n} |u(s, x)|^{ps(n)}\mu\big(|u(s, x)|\big)\eta_q(t, s, x)\mathrm{d}x\mathrm{d}s \qquad (15)
\end{aligned}$$

for any $t \in (0, T)$, *where* ξ_q *and* η_q *are defined in* (12) *and* (13), *respectively.*

Proof From finite propagation speed for solutions of wave equations, $u(t,\cdot)$ has compact support contained in $B_{R+t}(0)$ for any $t \geqslant 0$. Therefore, we may employ (5) for a non-compactly supported test function. We now define the test function

$$\psi = \psi(s,x) := y(t,s;\lambda)\Phi(\lambda x) \quad \text{with } y(t,s;\lambda) := \frac{\sinh(\lambda(t-s))}{\lambda}.$$

As Φ is an eigenfunction of the Laplacian and $y(t,s;\lambda)$ solves $(\partial_s^2-\lambda^2)y(t,s;\lambda)=0$ with the end-points $y(t,t;\lambda)=0$ and $y_s(t,t;\lambda)=-1$, the function ψ solves the free wave equation $\psi_{ss}-\Delta\psi=0$ and satisfies

$$\begin{aligned}&\psi(t,x)=0, \quad \psi(0,x)=\lambda^{-1}\sinh(\lambda t)\Phi(\lambda x),\\ &\psi_s(t,x)=-\Phi(\lambda x), \quad \psi_s(0,x)=-\cosh(\lambda t)\Phi(\lambda x).\end{aligned}$$

Applying the test function ψ in (5) with an integration by parts once more, we may derive

$$\begin{aligned}&\int_{\mathbb{R}^n} u(t,x)\Phi(\lambda x)\mathrm{d}x\\ &\quad=\cosh(\lambda t)\int_{\mathbb{R}^n} u_0(x)\Phi(\lambda x)\mathrm{d}x + t\frac{\sinh(\lambda t)}{\lambda t}\int_{\mathbb{R}^n} u_1(x)\Phi(\lambda x)\mathrm{d}x\\ &\qquad+\int_0^t (t-s)\frac{\sinh(\lambda(t-s))}{\lambda(t-s)}\int_{\mathbb{R}^n}|u(s,x)|^{p_S(n)}\mu\big(|u(s,x)|\big)\Phi(\lambda x)\mathrm{d}x\mathrm{d}s.\end{aligned}$$

Multiplying both sides of the last equality by $\mathrm{e}^{-\lambda(R+t)}\lambda^q$, integrating the resultant with respect to λ over $[0,\lambda_0]$ and applying Tonelli's theorem, we complete the derivation of (15). $\square$

Hereafter until the end of this section, we shall assume that u_0, u_1 satisfy the assumptions from Theorem 1. Let u be an energy solution to the semilinear Cauchy problem (3) on $[0,T)$. Inspired by the modulus of continuity in its nonlinearity, let us introduce the non-negative time-dependent functional (due to $\eta_q(t,t,x)\geqslant 0$)

$$\mathcal{U}: t\in[0,T)\to\mathcal{U}(t) := \int_{\mathbb{R}^n}|u(t,x)|\big[\mu\big(|u(t,x)|\big)\big]^{\frac{1}{p_S(n)}}\eta_q(t,t,x)\mathrm{d}x \geqslant 0 \quad (16)$$

with the parameter q defined in (11).

A further step is to derive some estimates involving $\mathcal{U}=\mathcal{U}(t)$ both in the left- and right-hand sides, which will establish an iteration frame. According to (15) and non-negativity of initial data, we may claim

$$\int_{\mathbb{R}^n} u(t,x)\eta_q(t,t,x)\mathrm{d}x \geqslant \int_0^t (t-s)\int_{\mathbb{R}^n}|u(s,x)|^{p_S(n)}\mu\big(|u(s,x)|\big)\eta_q(t,s,x)\mathrm{d}x\mathrm{d}s. \quad (17)$$

Using Hölder's inequality, we arrive at

$$\mathcal{U}(s) \leqslant \left(\int_{\mathbb{R}^n} |u(s,x)|^{p_S(n)} \mu\big(|u(s,x)|\big)\eta_q(t,s,x)\mathrm{d}x\right)^{\frac{1}{p_S(n)}} \times \left(\int_{B_{R+s}} \frac{[\eta_q(s,s,x)]^{p'_S(n)}}{[\eta_q(t,s,x)]^{\frac{p'_S(n)}{p_S(n)}}}\mathrm{d}x\right)^{\frac{1}{p'_S(n)}}.$$

Remark that $p'_S(n)$ denotes Hölder's conjugate of $p_S(n)$. With the aid of the properties (ii) and (iii) in Lemma 3.2 (both $q > \frac{n-3}{2}$ and $q > -1$ are always fulfilled), we obtain

$$\begin{aligned}\int_{B_{R+s}} \frac{[\eta_q(s,s,x)]^{p'_S(n)}}{[\eta_q(t,s,x)]^{\frac{p'_S(n)}{p_S(n)}}}\mathrm{d}x &\lesssim \langle t\rangle^{\frac{p'_S(n)}{p_S(n)}} \langle s\rangle^{\frac{p'_S(n)}{p_S(n)}q-\frac{n-1}{2}p'_S(n)} \int_{B_{R+s}} \langle s-|x|\rangle^{(\frac{n-3}{2}-q)p'_S(n)}\mathrm{d}x\\ &\lesssim \langle t\rangle^{\frac{p'_S(n)}{p_S(n)}} \langle s\rangle^{\frac{q}{p_S(n)-1}-\frac{n-1}{2}p'_S(n)} \int_{B_{R+s}} \langle s-|x|\rangle^{-1}\mathrm{d}x\\ &\lesssim \langle t\rangle^{\frac{p'_S(n)}{p_S(n)}} \langle s\rangle^{\frac{p'_S(n)}{p_S(n)}} \log\langle s\rangle,\end{aligned}$$

due to our choice of q in (11) and

$$\begin{aligned}&\frac{q}{p_S(n)-1} - \frac{n-1}{2}p'_S(n) + n - 1\\ &\quad= \frac{p'_S(n)}{p_S(n)}\left(\frac{n-1}{2} - \frac{1}{p_S(n)} - \frac{n-1}{2}p_S(n) + (n-1)\big(p_S(n)-1\big)\right)\\ &\quad= \frac{p'_S(n)}{p_S(n)}\left[\frac{1}{p_S(n)}\left(\frac{n-1}{2}p_S^2(n) - \frac{n+1}{2}p_S(n) - 1\right) + 1\right] = \frac{p'_S(n)}{p_S(n)}.\end{aligned}$$

Note that $\log\langle s\rangle \geqslant \log 3 > 0$. Plugging the previous estimates in (17), it leads to

$$\begin{aligned}\int_{\mathbb{R}^n} u(t,x)\eta_q(t,t,x)\mathrm{d}x &\gtrsim \int_0^t (t-s)[\mathcal{U}(s)]^{p_S(n)} \left(\int_{B_{R+s}} \frac{[\eta_q(s,s,x)]^{p'_S(n)}}{[\eta_q(t,s,x)]^{\frac{p'_S(n)}{p_S(n)}}}\mathrm{d}x\right)^{-\frac{p_S(n)}{p'_S(n)}} \mathrm{d}s\\ &\gtrsim \langle t\rangle^{-1} \int_0^t (t-s)\langle s\rangle^{-1} \frac{[\mathcal{U}(s)]^{p_S(n)}}{(\log\langle s\rangle)^{p_S(n)-1}}\mathrm{d}s. \qquad (18)\end{aligned}$$

Moreover, thanks to the support condition of $u(t,\cdot)$, let us apply Lemma 3.3 with $\Omega = B_{R+t}(0)$, $\alpha = \eta_q(t,t,x)$, $v = u(t,x)$ and the convex function $g = g(\tau) = \tau[\mu(|\tau|)]^{\frac{1}{p_S(n)}}$ from our assumption in Theorem 1 to deduce

$$g\left(\frac{\int_{B_{R+t}(0)}u(t,x)\eta_q(t,t,x)\mathrm{d}x}{\int_{B_{R+t}(0)}\eta_q(t,t,x)\mathrm{d}x}\right)\leqslant\frac{\int_{B_{R+t}(0)}u(t,x)[\mu(|u(t,x)|)]^{\frac{1}{p_S(n)}}\eta_q(t,t,x)\mathrm{d}x}{\int_{B_{R+t}(0)}\eta_q(t,t,x)\mathrm{d}x}$$
$$\leqslant\frac{\mathcal{U}(t)}{\int_{B_{R+t}(0)}\eta_q(t,t,x)\mathrm{d}x}.$$

In other words,

$$\int_{B_{R+t}(0)}u(t,x)\eta_q(t,t,x)\mathrm{d}x$$
$$\leqslant\int_{B_{R+t}(0)}\eta_q(t,t,x)\mathrm{d}x\ g^{-1}\left(\frac{\mathcal{U}(t)}{\int_{B_{R+t}(0)}\eta_q(t,t,x)\mathrm{d}x}\right). \tag{19}$$

Note that the function $g=g(\tau)$ is strictly monotonic from the monotonically increasing property of $\mu=\mu(|\tau|)$. After combining (18) and (19) it follows

$$\frac{1}{\int_{B_{R+t}(0)}\eta_q(t,t,x)\mathrm{d}x}\langle t\rangle^{-1}\int_0^t(t-s)\langle s\rangle^{-1}\frac{[\mathcal{U}(s)]^{p_S(n)}}{(\log\langle s\rangle)^{p_S(n)-1}}\mathrm{d}s$$
$$\lesssim g^{-1}\left(\frac{\mathcal{U}(t)}{\int_{B_{R+t}(0)}\eta_q(t,t,x)\mathrm{d}x}\right).$$

The action of the mapping g on both sides of the last estimate yields

$$\mathcal{U}(t)\gtrsim\int_{B_{R+t}(0)}\eta_q(t,t,x)\mathrm{d}x\ g\left[\frac{1}{\int_{B_{R+t}(0)}\eta_q(t,t,x)\mathrm{d}x}\langle t\rangle^{-1}\right.$$
$$\left.\times\int_0^t(t-s)\langle s\rangle^{-1}\frac{[\mathcal{U}(s)]^{p_S(n)}}{(\log\langle s\rangle)^{p_S(n)-1}}\mathrm{d}s\right]$$
$$\gtrsim\langle t\rangle^{-1}\int_0^t(t-s)\langle s\rangle^{-1}\frac{[\mathcal{U}(s)]^{p_S(n)}}{(\log\langle s\rangle)^{p_S(n)-1}}\mathrm{d}s$$
$$\times\left[\mu\left(\left|\frac{1}{\int_{B_{R+t}(0)}\eta_q(t,t,x)\mathrm{d}x}\langle t\rangle^{-1}\int_0^t(t-s)\langle s\rangle^{-1}\frac{[\mathcal{U}(s)]^{p_S(n)}}{(\log\langle s\rangle)^{p_S(n)-1}}\mathrm{d}s\right|\right)\right]^{\frac{1}{p_S(n)}}.$$

Employing the non-negativity of $\mathcal{U}(t)$ stated in (16) as well as

$$\int_{B_{R+t}(0)}\eta_q(t,t,x)\mathrm{d}x\lesssim\langle t\rangle^{-\frac{n-1}{2}}\int_0^{R+t}\zeta^{n-1}\langle t-\zeta\rangle^{\frac{n-3}{2}-q}\mathrm{d}\zeta$$
$$\lesssim\langle t\rangle^{\frac{n-1}{2}}\int_0^{R+t}\langle t-\zeta\rangle^{-1+\frac{1}{p_S(n)}}\mathrm{d}\zeta$$
$$\leqslant C_1^{-1}\langle t\rangle^{\frac{n-1}{2}+\frac{1}{p_S(n)}} \tag{20}$$

from Lemma 3.2, in conclusion, we obtain the iteration frame

$$\mathcal{U}(t) \geqslant C_0\langle t\rangle^{-1}\int_0^t (t-s)\langle s\rangle^{-1}\frac{[\mathcal{U}(s)]^{p_S(n)}}{(\log\langle s\rangle)^{p_S(n)-1}}\mathrm{d}s \times\left[\mu\left(C_1\langle t\rangle^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-1}\int_0^t\frac{t-s}{\langle s\rangle}\frac{[\mathcal{U}(s)]^{p_S(n)}}{(\log\langle s\rangle)^{p_S(n)-1}}\mathrm{d}s\right)\right]^{\frac{1}{p_S(n)}} \tag{21}$$

for any $t\in[0,T)$, with positive constants C_0 and $C_1=C_1(R)$.

3.3 Derivation of a first lower bound estimate

By applying (17) and the property (ii) in Lemma 3.2, we may arrive at

$$\int_{\mathbb{R}^n} u(t,x)\eta_q(t,t,x)\mathrm{d}x \gtrsim \langle t\rangle^{-1}\int_0^t (t-s)\langle s\rangle^{-q}\int_{\mathbb{R}^n}|u(s,x)|^{p_S(n)}\mu\big(|u(s,x)|\big)\mathrm{d}x\mathrm{d}s.$$

An application of Hölder's inequality gives

$$\begin{aligned}&\left|\int_{\mathbb{R}^n} u(s,x)\big[\mu\big(|u(s,x)|\big)\big]^{\frac{1}{p_S(n)}}\Psi(s,x)\mathrm{d}x\right|^{p_S(n)}\\ &\leqslant \int_{\mathbb{R}^n}|u(s,x)|^{p_S(n)}\mu\big(|u(s,x)|\big)\mathrm{d}x\left(\int_{B_{R+s}(0)}|\Psi(s,x)|^{p'_S(n)}\mathrm{d}x\right)^{\frac{p_S(n)}{p'_S(n)}}\\ &\lesssim (R+s)^{(n-1)(\frac{p_S(n)}{2}-1)}\int_{\mathbb{R}^n}|u(s,x)|^{p_S(n)}\mu\big(|u(s,x)|\big)\mathrm{d}x,\end{aligned}$$

where we employed the next inequality (e.g. the proof was shown in [19, 27] by using an integration by parts):

$$\int_{B_{R+s}(0)}|\Psi(s,x)|^{p'_S(n)}\mathrm{d}x \lesssim (R+s)^{(n-1)\left(1-\frac{1}{2}p'_S(n)\right)}.$$

That is to say

$$\int_{\mathbb{R}^n} u(t,x)\eta_q(t,t,x)\mathrm{d}x \gtrsim \langle t\rangle^{-1}\int_0^t (t-s)\langle s\rangle^{-1}\left|\int_{\mathbb{R}^n} u(s,x)\big[\mu\big(|u(s,x)|\big)\big]^{\frac{1}{p_S(n)}}\Psi(s,x)\mathrm{d}x\right|^{p_S(n)}\mathrm{d}s, \tag{22}$$

due to the fact that

$$-q-(n-1)\left(\frac{p_S(n)}{2}-1\right)=-\frac{1}{p_S(n)}\left(\frac{n-1}{2}p_S^2(n)-\frac{n-1}{2}p_S(n)-1\right)=-1.$$

Let us apply Lemma 3.3 again with $g(\tau)=\tau[\mu(|\tau|)]^{\frac{1}{p_S(n)}}$ and $\alpha=\Psi(s,x)$ to arrive at

$$\begin{aligned}&\int_{\mathbb{R}^n}u(s,x)\big[\mu\big(|u(s,x)|\big)\big]^{\frac{1}{p_S(n)}}\Psi(s,x)\mathrm{d}x\\&\quad\geqslant\int_{B_{R+s}(0)}\Psi(s,x)\mathrm{d}x\;g\left(\frac{\int_{B_{R+s}(0)}u(s,x)\Psi(s,x)\mathrm{d}x}{\int_{B_{R+s}(0)}\Psi(s,x)\mathrm{d}x}\right).\end{aligned}\tag{23}$$

A further step of integration by parts to (5) shows

$$\begin{aligned}&\int_{\mathbb{R}^n}u_t(t,x)\phi(t,x)\mathrm{d}x-\int_{\mathbb{R}^n}u(t,x)\phi_t(t,x)\mathrm{d}x\\&\quad=-\int_{\mathbb{R}^n}u_0(x)\phi_t(0,x)\mathrm{d}x+\int_{\mathbb{R}^n}u_1(x)\phi(0,x)\mathrm{d}x\\&\qquad-\int_0^t\int_{\mathbb{R}^n}u(s,x)\big(\phi_{ss}(s,x)-\Delta\phi(s,x)\big)\mathrm{d}x\mathrm{d}s\\&\qquad+\int_0^t\int_{\mathbb{R}^n}|u(s,x)|^{p_S(n)}\mu\big(|u(s,x)|\big)\phi(s,x)\mathrm{d}x\mathrm{d}s.\end{aligned}$$

Again, since u is supported in a forward cone, we may apply the definition of energy solutions even though the test function is not compactly supported. Taking as test function $\phi=\phi(t,x)$ the function $\Psi=\Psi(t,x)$, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}u(t,x)\Psi(t,x)\mathrm{d}x+2\int_{\mathbb{R}^n}u(t,x)\Psi(t,x)\mathrm{d}x\geqslant\int_{\mathbb{R}^n}\big(u_0(x)+u_1(x)\big)\Phi(x)\mathrm{d}x$$

due to the non-negativity of nonlinearity and $\Psi_{tt}=\Delta\Psi$. By multiplying e^{2t} on both sides of the last inequality, we can find

$$\begin{aligned}\int_{\mathbb{R}^n}u(t,x)\Psi(t,x)\mathrm{d}x\geqslant&\frac{1}{2}\left(1+\mathrm{e}^{-2t}\right)\int_{\mathbb{R}^n}u_0(x)\Phi(x)\mathrm{d}x\\&+\frac{1}{2}\left(1-\mathrm{e}^{-2t}\right)\int_{\mathbb{R}^n}u_1(x)\Phi(x)\mathrm{d}x.\end{aligned}$$

From our assumption on initial data, one gets

$$\int_{B_{R+s}(0)}u(s,x)\Psi(s,x)\mathrm{d}x\gtrsim1,\tag{24}$$

where the unexpressed multiplicative constant may depend on u_0 as well as u_1. With the aid of Lemma 3.1 and (24), we are able to estimate from (23) and (22) that

$$\begin{aligned}
&\int_{\mathbb{R}^n} u(t,x)\eta_q(t,t,x)\mathrm{d}x \\
&\quad \gtrsim \langle t\rangle^{-1}\int_0^t (t-s)\langle s\rangle^{-1}\left|\int_{B_{R+s}(0)}\Psi(s,x)\mathrm{d}x\right|^{p_S(n)} \\
&\qquad \times \left|g\left(\frac{\int_{B_{R+s}(0)}u(s,x)\Psi(s,x)\mathrm{d}x}{\int_{B_{R+s}(0)}\Psi(s,x)\mathrm{d}x}\right)\right|^{p_S(n)}\mathrm{d}s \\
&\quad \gtrsim \langle t\rangle^{-1}\int_0^t (t-s)\langle s\rangle^{-1}(R+s)^{\frac{(n-1)p_S(n)}{2}}\left|g\left(C_2(R+s)^{-\frac{n-1}{2}}\right)\right|^{p_S(n)}\mathrm{d}s \\
&\quad \gtrsim \langle t\rangle^{-1}\int_0^t (t-s)\langle s\rangle^{-1}\mu\left(C_2(R+s)^{-\frac{n-1}{2}}\right)\mathrm{d}s
\end{aligned}$$

with a positive constant $C_2 = C_2(R)$. According to (19), we derive

$$\begin{aligned}
&\langle t\rangle^{-1}\int_0^t (t-s)\langle s\rangle^{-1}\mu\left(C_2(R+s)^{-\frac{n-1}{2}}\right)\mathrm{d}s \\
&\quad \lesssim \int_{B_{R+t}(0)}\eta_q(t,t,x)\mathrm{d}x\ g^{-1}\left(\frac{\mathcal{U}(t)}{\int_{B_{R+t}(0)}\eta_q(t,t,x)\mathrm{d}x}\right).
\end{aligned}$$

Furthermore, recalling the increasing property of μ and shrinking the interval of integration $[0,t]$ to $[1,t]$ for $t \geqslant 1$, one obtains

$$\begin{aligned}
\mathcal{U}(t) &\gtrsim \int_{B_{R+t}(0)}\eta_q(t,t,x)\mathrm{d}x\ g\left(\frac{1}{\int_{B_{R+t}(0)}\eta_q(t,t,x)\mathrm{d}x}\langle t\rangle^{-1}\right. \\
&\quad \left.\times\int_0^t (t-s)\langle s\rangle^{-1}\mu\left(C_2(R+s)^{-\frac{n-1}{2}}\right)\mathrm{d}s\right) \\
&\gtrsim \langle t\rangle^{-1}\mu\left(C_2(R+t)^{-\frac{n-1}{2}}\right)\int_1^t (t-s)\langle s\rangle^{-1}\mathrm{d}s \\
&\quad \times\left[\mu\left(\frac{1}{\int_{B_{R+t}(0)}\eta_q(t,t,x)\mathrm{d}x}\langle t\rangle^{-1}\mu\left(C_2(R+t)^{-\frac{n-1}{2}}\right)\int_1^t (t-s)\langle s\rangle^{-1}\mathrm{d}s\right)\right]^{\frac{1}{p_S(n)}}.
\end{aligned}$$

Taking account of

$$\begin{aligned}
\langle t\rangle^{-1}\int_1^t (t-s)\langle s\rangle^{-1}\mathrm{d}s &\gtrsim \langle t\rangle^{-1}\int_1^t \frac{t-s}{s}\mathrm{d}s = \langle t\rangle^{-1}\int_1^t \log s\,\mathrm{d}s \\
&\gtrsim \frac{1}{3t}\int_{2t/3}^t \log s\,\mathrm{d}s \gtrsim \log\left(\frac{2t}{3}\right)
\end{aligned}$$

for any $t \geqslant \frac{3}{2}$, and recalling (20), we may derive the lower bound estimate

$$\mathcal{U}(t) \gtrsim \log\left(\frac{2t}{3}\right)\mu\left(C_2(R+t)^{-\frac{n-1}{2}}\right) \\ \times\left[\mu\left(2C_3\langle t\rangle^{-\frac{n-1}{2}-\frac{1}{p_S(n)}}\log\left(\frac{2t}{3}\right)\mu\left(C_2(R+t)^{-\frac{n-1}{2}}\right)\right)\right]^{\frac{1}{p_S(n)}}$$

with a positive constant $C_3 = C_3(R)$.

Our assumption (6) shows that there is a suitably large constant $c_l \gg 1$ such that the next estimate holds:

$$\mu(\tau)\left(\log\frac{1}{\tau}\right)^{\frac{1}{p_S(n)}} > \frac{c_l}{2} \quad \text{for } 0 < \tau \leqslant \tau_0 \ll 1. \tag{25}$$

Let us choose a large constant $t_0 \geqslant 1$ such that for any $t \geqslant \frac{3}{2}t_0$, the following inequalities hold (later, we will take t_0 to be suitably large):

$$C_2(R) \leqslant \tau_0(R+t)^{\frac{n-1}{2}}, \tag{26}$$

and

$$c_l C_2^{-1} C_3\left(\frac{n-1}{2}\right)^{-\frac{1}{p_S(n)}}\log\left(\frac{2t}{3}\right) \\ \geqslant \langle t\rangle^{\frac{n-1}{2}+\frac{1}{p_S(n)}}(R+t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}}\left[\log\left(C_2^{-\frac{2}{n-1}}(R+t)\right)\right]^{\frac{1}{p_S(n)}}.$$

Note that the second inequality in the above (it will be used for reducing the argument) can be guaranteed since

$$\langle t\rangle^{\frac{n-1}{2}+\frac{1}{p_S(n)}}(R+t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}}\left[\log\left(C_2^{-\frac{2}{n-1}}(R+t)\right)\right]^{\frac{1}{p_S(n)}} \lesssim \log\left(\frac{2t}{3}\right)$$

for large time t since $p_S(n) > 1$. According to our assumption (6) for $t \geqslant \frac{3}{2}t_0$, it follows

$$\mu\left(2C_3\langle t\rangle^{-\frac{n-1}{2}-\frac{1}{p_S(n)}}\log\left(\frac{2t}{3}\right)\mu\left(C_2(R+t)^{-\frac{n-1}{2}}\right)\right) \\ \geqslant \mu\left(c_l C_3\langle t\rangle^{-\frac{n-1}{2}-\frac{1}{p_S(n)}}\left[\log\left(C_2^{-1}(R+t)^{\frac{n-1}{2}}\right)\right]^{-\frac{1}{p_S(n)}}\log\left(\frac{2t}{3}\right)\right) \\ \geqslant \mu\left(C_2(R+t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}}\right).$$

Summarizing, we deduce the following first lower bound estimate:

$$\mathcal{U}(t) \geqslant M_0 \log\left(\frac{2t}{3t_0}\right)\left[\mu\left(C_2(R+t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-\epsilon_0}\right)\right]^{1+\frac{1}{p_S(n)}} \tag{27}$$

for any $t \geqslant \frac{3}{2}t_0$ with a large parameter $t_0 \geqslant 1$, with a positive constant $M_0 = M_0(R)$ which is independent of c_l. Here, $\epsilon_0 > 0$ is a sufficiently small constant. We have to underline that such a small ϵ_0 does not bring any influence on the blow-up condition.

3.4 Iteration procedure and blow-up phenomenon: proof of Theorem 1

Up to now, we have determined among other things the iteration frame (21) for the functional $\mathcal{U} = \mathcal{U}(t)$ and the first lower bound estimate (27) containing a logarithmic factor and a factor depending on the given modulus of continuity. In this part, we are going to prove a sequence of lower bound estimates for $\mathcal{U} = \mathcal{U}(t)$ by applying the so-called slicing procedure, which has been introduced in [1].

Let us choose the sequence $\{\ell_j\}_{j\in\mathbb{N}_0}$ with $\ell_j := 2 - 2^{-(j+1)}$. Our goal is to derive the sequence of lower bound estimates for the functional $\mathcal{U} = \mathcal{U}(t)$ as follows:

$$\mathcal{U}(t) \geqslant M_j(\log\langle t\rangle)^{-b_j}\left[\log\left(\frac{t}{\ell_{2j}t_0}\right)\right]^{a_j}\left[\mu\left(C_2(R+t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-\epsilon_0}\right)\right]^{\sigma_j} \tag{28}$$

for $t \geqslant \ell_{2j}t_0$ with a suitably large constant $t_0 \gg 1$, where $\{M_j\}_{j\in\mathbb{N}_0}$, $\{a_j\}_{j\in\mathbb{N}_0}$, $\{b_j\}_{j\in\mathbb{N}_0}$ and $\{\sigma_j\}_{j\in\mathbb{N}_0}$ are sequences of non-negative real numbers that we shall determine recursively throughout the iteration procedure. From the first lower bound estimate (27), we may choose with $j = 0$ the parameters

$$a_0 := 1, \quad b_0 := 0, \quad \sigma_0 := 1 + \frac{1}{p_S(n)}. \tag{29}$$

We are going to prove the validity of (28) for any $j \in \mathbb{N}_0$ by using an inductive argument. As we have already shown the validity of the basic case (27), it remains to prove the inductive step. Let us assume that (28) holds for $j \geqslant 1$, our purpose is to demonstrate it for $j + 1$.

First of all, via the lower bound estimate (28), we know

$$\begin{aligned}
&\langle t\rangle^{-1}\int_0^t (t-s)\langle s\rangle^{-1}\frac{[\mathcal{U}(s)]^{p_S(n)}}{(\log\langle s\rangle)^{p_S(n)-1}}\mathrm{d}s \\
&\quad\geqslant M_j^{p_S(n)}\langle t\rangle^{-1}\int_0^t \frac{t-s}{\langle s\rangle}\frac{\left[\log\left(\frac{s}{\ell_{2j}t_0}\right)\right]^{a_j p_S(n)}}{(\log\langle s\rangle)^{p_S(n)-1+b_j p_S(n)}} \\
&\qquad\times\left[\mu\left(C_2(R+s)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-\epsilon_0}\right)\right]^{\sigma_j p_S(n)}\mathrm{d}s
\end{aligned}$$

$$\geqslant \frac{1}{3} M_j^{p_S(n)} \langle t\rangle^{-1} (\log\langle t\rangle)^{-(p_S(n)-1)-b_j p_S(n)} \left[\mu\left(C_2(R+t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-\epsilon_0}\right)\right]^{\sigma_j p_S(n)}$$
$$\times \int_{\ell_{2j}t_0}^{t} \frac{t-s}{s}\left[\log\left(\frac{s}{\ell_{2j}t_0}\right)\right]^{a_j p_S(n)} \mathrm{d}s$$

for $t \geqslant \ell_{2j+2}t_0$, where we shrank the interval of integration $[0, t]$ to $[\ell_{2j}t_0, t]$ so that $\langle s\rangle = 3 + |s| \leqslant 3s$ for any $s \geqslant \ell_{2j}t_0$. By employing integration by parts, we may derive

$$\begin{aligned}
&\int_{\ell_{2j}t_0}^{t} \frac{t-s}{s}\left[\log\left(\frac{s}{\ell_{2j}t_0}\right)\right]^{a_j p_S(n)} \mathrm{d}s \\
&\quad\geqslant \frac{1}{a_j p_S(n)+1}\int_{\frac{\ell_{2j}}{\ell_{2j+2}}t}^{t}\left[\log\left(\frac{s}{\ell_{2j}t_0}\right)\right]^{a_j p_S(n)+1}\mathrm{d}s \\
&\quad\geqslant \frac{1}{3(a_j p_S(n)+1)}\left(1-\frac{\ell_{2j}}{\ell_{2j+2}}\right)\langle t\rangle\left[\log\left(\frac{t}{\ell_{2j+2}t_0}\right)\right]^{a_j p_S(n)+1}
\end{aligned}$$

for $t \geqslant \ell_{2j+2}t_0$ so that $\ell_{2j}t_0 \leqslant \frac{\ell_{2j}}{\ell_{2j+2}}t$. For this reason, the last relation implies for $t \geqslant \ell_{2j+2}t_0$ immediately

$$\begin{aligned}
&\langle t\rangle^{-1}\int_0^t (t-s)\langle s\rangle^{-1}\frac{[\mathcal{U}(s)]^{p_S(n)}}{(\log\langle s\rangle)^{p_S(n)-1}}\mathrm{d}s \\
&\quad\geqslant \frac{M_j^{p_S(n)}2^{-(2j+3)}}{3\ell_{2j+2}(a_j p_S(n)+1)}(\log\langle t\rangle)^{-(p_S(n)-1)-b_j p_S(n)}\left[\log\left(\frac{t}{\ell_{2j+2}t_0}\right)\right]^{a_j p_S(n)+1} \\
&\quad\times\left[\mu\left(C_2(R+t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-\epsilon_0}\right)\right]^{\sigma_j p_S(n)}.
\end{aligned} \tag{30}$$

Plugging (28) into the iteration frame (21), we arrive, after taking into the consideration (30), at

$$\begin{aligned}
\mathcal{U}(t) \geqslant\ & \frac{C_0 2^{-(2j+3)}M_j^{p_S(n)}}{3\ell_{2j+2}(a_j p_S(n)+1)}(\log\langle t\rangle)^{-(p_S(n)-1)-b_j p_S(n)}\left[\log\left(\frac{t}{\ell_{2j+2}t_0}\right)\right]^{a_j p_S(n)+1} \\
&\times\left[\mu\left(C_2(R+t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-\epsilon_0}\right)\right]^{\sigma_j p_S(n)}[\mathcal{I}_\mu(t)]^{\frac{1}{p_S(n)}},
\end{aligned} \tag{31}$$

where we introduce

$$\mathcal{I}_\mu(t) := \mu\left(C_1\langle t\rangle^{-\frac{n-1}{2}-\frac{1}{p_S(n)}}\langle t\rangle^{-1}\int_0^t (t-s)\langle s\rangle^{-1}\frac{[\mathcal{U}(s)]^{p_S(n)}}{(\log\langle s\rangle)^{p_S(n)-1}}\mathrm{d}s\right).$$

We estimate

$$\mathcal{I}_\mu(t) \geqslant \mu\left(\frac{C_1 2^{-(2j+3)} M_j^{p_S(n)}}{3\ell_{2j+2}(a_j p_S(n)+1)} \langle t\rangle^{-\frac{n-1}{2}-\frac{1}{p_S(n)}} (\log\langle t\rangle)^{-(p_S(n)-1)-b_j p_S(n)} \right.$$
$$\left. \times \left[\mu\left(C_2(R+t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-\epsilon_0}\right)\right]^{\sigma_j p_S(n)} \left[\log\left(\frac{t}{\ell_{2j+2}t_0}\right)\right]^{a_j p_S(n)+1}\right).$$

Taking a suitably large t_0 such that for $t \geqslant \ell_{2j+2}t_0$, recalling the conclusion (25) from our assumption (6) and the condition (26), then the following inequality holds:

$$\begin{aligned}
&\frac{(\frac{c_l}{2})^{\sigma_j p_S(n)} C_1 C_2^{-1} 2^{-(2j+3)} M_j^{p_S(n)}}{3\ell_{2j+2}(a_j p_S(n)+1)} \left[\log\left(\frac{t}{\ell_{2j+2}t_0}\right)\right]^{a_j p_S(n)+1} \\
&\geqslant \langle t\rangle^{\frac{n-1}{2}+\frac{1}{p_S(n)}} (R+t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-\epsilon_0} (\log\langle t\rangle)^{p_S(n)-1+b_j p_S(n)} \\
&\times \left[\log\left(C_2^{-1}(R+t)^{\frac{n-1}{2}+\frac{1}{p_S(n)}+\epsilon_0}\right)\right]^{\sigma_j}
\end{aligned} \tag{32}$$

for a fixed j, because the polynomial decay factor $(R+t)^{-\epsilon_0}$ plays from the point of decay a dominant role in comparison with all logarithmic factors on the right-hand side of the last inequality. Later, we will verify the last inequality (32) uniformly for all $j \gg 1$ by choosing suitable parameters a_j, b_j, σ_j and estimating M_j. According to the last lower bounds estimates, it provides

$$\mathcal{I}_\mu(t) \geqslant \mu\left(C_2(R+t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-\epsilon_0}\right) \tag{33}$$

for $t \geqslant \ell_{2j+2}t_0$ with a suitably large $t_0 \gg 1$. Summarizing the above estimates (31) as well as (33), we claim the lower bound estimate

$$\mathcal{U}(t) \geqslant \frac{C_0 2^{-(2j+3)} M_j^{p_S(n)}}{3\ell_{2j+2}(a_j p_S(n)+1)} (\log\langle t\rangle)^{-(p_S(n)-1)-b_j p_S(n)} \left[\log\left(\frac{t}{\ell_{2j+2}t_0}\right)\right]^{a_j p_S(n)+1}$$
$$\times \left[\mu\left(C_2(R+t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-\epsilon_0}\right)\right]^{\sigma_j p_S(n)+\frac{1}{p_S(n)}}$$

for $t \geqslant \ell_{2j+2}t_0$. In other words, we have proved (28) for $j+1$ provided that

$$\begin{aligned}
M_{j+1} &:= \frac{C_0 2^{-(2j+3)}}{3\ell_{2j+2}(a_j p_S(n)+1)} M_j^{p_S(n)}, \\
a_{j+1} &:= 1 + a_j p_S(n), \\
b_{j+1} &:= p_S(n) - 1 + b_j p_S(n), \\
\sigma_{j+1} &:= \frac{1}{p_S(n)} + \sigma_j p_S(n).
\end{aligned}$$

By using recursively the relations and the initial exponents (29), we deduce

$$a_j=\frac{p_S(n)}{p_S(n)-1}p_S^j(n)-\frac{1}{p_S(n)-1},\quad b_j=p_S^j(n)-1,$$
$$\sigma_j=\frac{p_S(n)}{p_S(n)-1}p_S^j(n)-\frac{1}{(p_S(n)-1)p_S(n)}. \tag{34}$$

Due to the facts that $\ell_{2j}\leqslant 2$ and $a_j\leqslant\frac{p_S(n)}{p_S(n)-1}p_S^j(n)$, the lower bound of M_j can be estimated by

$$M_j=\frac{C_0 2^{-(2j+1)}}{3\ell_{2j}(1+a_{j-1}p_S(n))}M_{j-1}^{p_S(n)}\geqslant C_4[4p_S(n)]^{-j}M_{j-1}^{p_S(n)}$$

with the constant $C_4:=\frac{C_0(p_S(n)-1)}{12p_S(n)}>0$, which depends on n, but it is independent of j. Applying the logarithmic function to both sides of the last inequality and using iteratively the resulting inequality, we may obtain

$$\begin{aligned}\log M_j&\geqslant p_S(n)\log M_{j-1}-j\log[4p_S(n)]+\log C_4\\&\geqslant\cdots\geqslant p_S^j(n)\log M_0-\left(\sum_{k=0}^{j-1}(j-k)p_S^k(n)\right)\log[4p_S(n)]+\left(\sum_{k=0}^{j-1}p_S^k(n)\right)\log C_4\\&\geqslant p_S^j(n)\left(\log M_0-\frac{p_S(n)\log[4p_S(n)]}{[p_S(n)-1]^2}+\frac{\log C_4}{p_S(n)-1}\right)\\&\quad+\frac{j[p_S(n)-1]+p_S(n)}{[p_S(n)-1]^2}\log[4p_S(n)]-\frac{\log C_4}{p_S(n)-1},\end{aligned}$$

where we used the identities

$$\sum_{k=0}^{j-1}(j-k)p^k=\frac{1}{p-1}\left(\frac{p^{j+1}-p}{p-1}-j\right)\quad\text{and}\quad\sum_{k=0}^{j-1}p^k=\frac{p^j-1}{p-1}.$$

Let us define $j_1=j_1(p_S(n))$ as the smallest non-negative integer such that

$$j_1\geqslant\frac{\log C_4}{\log[4p_S(n)]}-\frac{p_S(n)}{p_S(n)-1}.$$

We may estimate

$$\log M_j\geqslant p_S^j(n)\left(\log M_0-\frac{p_S(n)\log[4p_S(n)]}{[p_S(n)-1]^2}+\frac{\log C_4}{p_S(n)-1}\right)=p_S^j(n)\log C_5 \tag{35}$$

carrying the positive constant

$$C_5:=M_0[4p_S(n)]^{-\frac{p_S(n)}{[p_S(n)-1]^2}}C_4^{\frac{1}{p_S(n)-1}},$$

which depends on n and R, but it is independent of j.

Let us now prove the inequality (32) uniformly for all $j \gg 1$. Due to the choices of parameters a_j, b_j, σ_j, we may estimate

$$\begin{aligned}
&\langle t\rangle^{\frac{n-1}{2}+\frac{1}{p_S(n)}}(R+t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-\epsilon_0}(\log\langle t\rangle)^{p_S(n)-1+b_jp_S(n)}\\
&\quad\times\left[\log\left(C_2^{-1}(R+t)^{\frac{n-1}{2}+\frac{1}{p_S(n)}+\epsilon_0}\right)\right]^{\sigma_j}\left[\log\left(\frac{t}{\ell_{2j+2}t_0}\right)\right]^{-a_jp_S(n)-1}\\
&\quad\leqslant C(R+t)^{-\epsilon_0}(\log\langle t\rangle)^{p_S(n)-2+b_jp_S(n)+\sigma_j-a_jp_S(n)}\\
&\quad\leqslant C(R+t)^{-\epsilon_0}(\log\langle t\rangle)^{-1+\frac{1}{p_S(n)}}.
\end{aligned}$$

Moreover, from the lower bound of M_j in (35), we know

$$\frac{(\frac{c_l}{2})^{\sigma_jp_S(n)}C_1C_2^{-1}2^{-(2j+3)}M_j^{p_S(n)}}{3\ell_{2j+2}(a_jp_S(n)+1)}\geqslant\frac{\widetilde{C}}{(p_S^{j+1}(n)-1)4^j}\left[\left(\frac{c_l}{2}\right)^{\frac{p_S(n)}{p_S(n)-1}}C_5\right]^{p_S^{j+1}(n)},$$

where $\widetilde{C}$ is a positive constant independent of j. Note that C_5 only depends on $M_0, C_0, p_S(n)$ and R, but it is independent of c_l. Taking a suitably large constant $c_l=c_l(R,n)$, in the last two inequalities, we notice

$$\frac{\widetilde{C}}{(p_S^{j+1}(n)-1)4^j}\left[\left(\frac{c_l}{2}\right)^{\frac{p_S(n)}{p_S(n)-1}}C_5\right]^{p_S^{j+1}(n)}\geqslant C(R+t)^{-\epsilon_0}(\log\langle t\rangle)^{-1+\frac{1}{p_S(n)}}$$

for any $j \gg 1$. Thus, we verified (32) uniformly for all $j \gg 1$.

Consequently, recalling the sequence of estimates (28) associated with (34), (35) and $\ell_{2j}\leqslant 2$, the lower bound estimate of $\mathcal{U}=\mathcal{U}(t)$ can be presented as follows:

$$\begin{aligned}
\mathcal{U}(t)&\geqslant\exp\left(p_S^j(n)\log C_5\right)(\log\langle t\rangle)^{-p_S^j(n)+1}\left[\log\left(\frac{t}{2t_0}\right)\right]^{\frac{p_S(n)}{p_S(n)-1}p_S^j(n)-\frac{1}{p_S(n)-1}}\\
&\quad\times\left[\mu\left(C_2(R+t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-\epsilon_0}\right)\right]^{\frac{p_S(n)}{p_S(n)-1}p_S^j(n)-\frac{1}{(p_S(n)-1)p_S(n)}}\\
&\geqslant\exp\left\{p_S^j(n)\log\left[C_5(\log\langle t\rangle)^{-1}\left[\log\left(\frac{t}{2t_0}\right)\right]^{\frac{p_S(n)}{p_S(n)-1}}\right.\right.\\
&\quad\left.\left.\times\left[\mu\left(C_2(R+t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-\epsilon_0}\right)\right]^{\frac{p_S(n)}{p_S(n)-1}}\right]\right\}\\
&\quad\times\log\langle t\rangle\left[\log\left(\frac{t}{2t_0}\right)\right]^{-\frac{1}{p_S(n)-1}}\left[\mu\left(C_2(R+t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-\epsilon_0}\right)\right]^{-\frac{1}{(p_S(n)-1)p_S(n)}}
\end{aligned}$$

for $t \geqslant 2t_0$ and any $j \geqslant j_1$. There exists a positive constant $C_6 = C_6(R)$ such that for $t \geqslant t_1$ with a suitably large constant t_1, it holds

$$C_2(R+t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-\epsilon_0} \geqslant (C_6 t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-\epsilon_0}.$$

Moreover, concerning suitably large $t \geqslant t_2$, the following estimates hold:

$$\begin{aligned}&\log\langle t\rangle = \log(3+t) \leqslant 2\log(C_6 t),\\ &\log\left(\frac{t}{2t_0}\right) \geqslant \frac{1}{2t_0}\log(C_6 t), \quad (C_6 t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-\epsilon_0} \leqslant \tau_0.\end{aligned}$$

For $t \geqslant \max\{2t_0, t_1, t_2\}$, the lower bound of the functional $\mathcal{U} = \mathcal{U}(t)$ can be controlled in the following way:

$$\begin{aligned}\mathcal{U}(t) \geqslant \exp\Bigg\{ & p_S^j(n)\log\Bigg[C_5 2^{-1}(2t_0)^{-\frac{p_S(n)}{p_S(n)-1}}[\log(C_6 t)]^{\frac{1}{p_S(n)-1}}\\ &\times\left[\mu\left((C_6 t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-\epsilon_0}\right)\right]^{\frac{p_S(n)}{p_S(n)-1}}\Bigg]\Bigg\}\log\langle t\rangle\\ &\times\left[\log\left(\frac{t}{2t_0}\right)\right]^{-\frac{1}{p_S(n)-1}}\left[\mu\left(C_2(R+t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-\epsilon_0}\right)\right]^{-\frac{1}{(p_S(n)-1)p_S(n)}}.\end{aligned} \tag{36}$$

Recalling our assumption (6), it follows that there exists a positive, continuous function $\kappa = \kappa(\tau)$ such that

$$\lim_{\tau\to 0^+}\kappa(\tau) \in [c_l, +\infty] \quad \text{and} \quad \mu(\tau) = \left(\log\frac{1}{\tau}\right)^{-\frac{1}{p_S(n)}}\kappa(\tau). \tag{37}$$

When $t \geqslant \max\{2t_0, t_1, t_2\}$, the crucial part in (36) is expressed by

$$\begin{aligned}&\log(C_6 t)\left[\mu\left((C_6 t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-\epsilon_0}\right)\right]^{p_S(n)}\\ &\quad = \left(\frac{n-1}{2}+\frac{1}{p_S(n)}+\epsilon_0\right)^{-1}\left[\kappa\left((C_6 t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-\epsilon_0}\right)\right]^{p_S(n)}.\end{aligned}$$

Hence, the lower bound estimate turns into

$$\begin{aligned}\mathcal{U}(t) \geqslant \exp\Bigg\{ & p_S^j(n)\log\Bigg[C_7\left[\kappa\left((C_6 t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-\epsilon_0}\right)\right]^{\frac{p_S(n)}{p_S(n)-1}}\Bigg]\Bigg\}\log\langle t\rangle\\ &\times\left[\log\left(\frac{t}{2t_0}\right)\right]^{-\frac{1}{p_S(n)-1}}\left[\mu\left(C_2(R+t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-\epsilon_0}\right)\right]^{-\frac{1}{(p_S(n)-1)p_S(n)}}\end{aligned}$$

with a suitable positive constant $C_7 = C_7(R)$ which is independent of j. Let us take account of the property (37) of $\kappa = \kappa(\tau)$, namely,

$$\kappa(\tau) > \frac{c_l}{2} \quad \text{when } 0 < \tau \ll 1$$

with a suitably large constant $c_l > 2(100/C_7)^{[p_S(n)-1]/p_S(n)}$, where C_7 only depends on M_0, C_0, n and R, but it is independent of j. Then, for large time $t \geqslant \max\{2t_0, t_1, t_2\}$, we find that in the last estimate

$$\log\left[C_7[\kappa(\tau)]^{\frac{p_S(n)}{p_S(n)-1}}\right] > \log\left[C_7\left(\frac{c_l}{2}\right)^{\frac{p_S(n)}{p_S(n)-1}}\right] > 1$$

for $0 < \tau = (C_6 t)^{-\frac{n-1}{2}-\frac{1}{p_S(n)}-\epsilon_0} \ll 1$. Finally, taking the limit as $j \to +\infty$ in the above estimate the lower bound for $\mathcal{U}(t)$ blows up in finite time. This completes the proof of Theorem 1. □

4 Global (in time) existence of radial solutions in three dimensions

Firstly, by introducing a polynomial-logarithmic type weighted Banach space, we will prepare uniform bounded L^∞ estimates of the radial solutions to the three dimensional free wave equation in Sect. 4.1. Then, the philosophy of the proof for Theorem 2 and its key tool will be stated in Sect. 4.2. We will demonstrate the global (in time) existence result in Sect. 4.3 by applying a refined analysis in the (t, r)-plane to estimate the nonlinear terms.

4.1 Preliminary and weighted $L^\infty_t L^\infty_r$ estimates for the linear model

As preparations for studying nonlinear models, we will state some polynomial-logarithmic type weighted $L^\infty_t L^\infty_r$ estimates for the linear wave equation in the radial case. Let us first extend initial data $u_0 = u_0(r)$ and $u_1 = u_1(r)$ by even reflections, namely,

$$u_0(-r) = u_0(r) \quad \text{and} \quad u_1(-r) = u_1(r) \qquad \text{for } r < 0.$$

Note that our assumptions $u_0 \in \mathcal{C}_0^2$ and u_0 radially symmetric ensure $u_0'(0) = 0$. Due to our interest of radial solutions and the application of even reflections, we may rewrite the semilinear Cauchy problem (3) in three dimensions as

$$\begin{cases} u_{tt} - u_{rr} - \frac{2}{r}u_r = |u|^{p_S(3)}\mu(|u|), & r \in \mathbb{R},\ t > 0, \\ u(0, r) = u_0(r),\ \ u_t(0, r) = u_1(r), & r \in \mathbb{R}. \end{cases} \tag{38}$$

Now we turn our focus to the linear model with vanishing right-hand side and the same initial data as those of (38), namely,

$$\begin{cases} v_{tt} - v_{rr} - \frac{2}{r} v_r = 0, & r \in \mathbb{R},\ t > 0, \\ v(0,r) = u_0(r), \quad v_t(0,r) = u_1(r), & r \in \mathbb{R}. \end{cases} \tag{39}$$

Let us recall $u_0 \in \mathcal{C}_0^2$ as well as $u_1 \in \mathcal{C}_0^1$. According to the well-known d'Alembert's formula, the solution to the linear Cauchy problem (39) can be represented as follows:

$$\begin{aligned} v(t,r) &= \frac{\partial}{\partial t}\left(\int_{-1}^{1} H_{u_0}(t+r\sigma)\mathrm{d}\sigma\right) + \int_{-1}^{1} H_{u_1}(t+r\sigma)\mathrm{d}\sigma \\ &= \frac{1}{2r}\big((t+r)u_0(t+r) - (t-r)u_0(t-r)\big) + \frac{1}{r}\int_{t-r}^{t+r} H_{u_1}(\rho)\mathrm{d}\rho, \end{aligned} \tag{40}$$

where we denoted

$$H_{u_j}(\rho) := \frac{\rho}{2} u_j(\rho) \quad \text{with } j = 0, 1.$$

Its proof is standard by taking the new variable $rv(t,r)$ and the representation of solution for the one dimensional free wave equation.

Motivated by the papers [2, 4, 15], we are able to derive some decay estimates in some weighted $L_t^\infty L_r^\infty$ spaces for radial solutions of the linear Cauchy problem (39). In order to overcome some difficulties from the influence of modulus of continuity when we consider the nonlinear model (38), we will include an additional logarithmic type weighted function in the solution space. To be specific, we introduce the Banach space

$$X_\kappa := \left\{ v \in \mathcal{C}([0,+\infty) \times \mathbb{R}) :\ v \text{ is even in } r \text{ and } \|v\|_{X_\kappa} < +\infty \right\}$$

with a polynomial-logarithmic type weighted norm

$$\|v\|_{X_\kappa} := \sup_{t \geqslant 0,\ r \in \mathbb{R}} \big(\omega(\langle t - |r|\rangle)\langle t + |r|\rangle\langle t - |r|\rangle^{\kappa - 1}|v(t,r)|\big) \quad \text{with } \kappa := 1 + \frac{1}{p_S(3)}.$$

Here, the new weighted factor is defined by

$$\omega(\tau) := (\log \tau)^{\frac{1}{p_S(3)}} \quad \text{for any } \tau \geqslant 3. \tag{41}$$

Then, we have the next result for bounded estimates in the family of Banach spaces $\{X_\kappa\}_{\kappa > 1}$.

Proposition 4.1 *Let $u_0 \in \mathcal{A}_\kappa \cap \mathcal{C}_0^2$ and $u_1 \in \mathcal{B}_{\kappa+1} \cap \mathcal{C}_0^1$ with $\kappa > 1$. Then, the following estimate holds*:

$$\|v\|_{X_\kappa} \lesssim \|u_0\|_{\mathcal{A}_\kappa} + \|u_1\|_{\mathcal{B}_{\kappa+1}},$$

where the Banach spaces for initial data are defined as follows:

$$\mathcal{A}_\kappa := \left\{h \in \mathcal{C}^1 : \ h \text{ is an even function and } \|h\|_{\mathcal{A}_\kappa} < +\infty\right\},$$
$$\mathcal{B}_\kappa := \left\{h \in \mathcal{C} : \ h \text{ is an even function and } \|h\|_{\mathcal{B}_\kappa} < +\infty\right\},$$

carrying the corresponding norms

$$\|h\|_{\mathcal{A}_\kappa} := \sup_{r\geqslant 0}\left(\omega(\langle r\rangle)\langle r\rangle^{\kappa}|h(r)|\right) + \sup_{r\geqslant 0}\left(\omega(\langle r\rangle)\langle r\rangle^{\kappa+1}|h'(r)|\right),$$
$$\|h\|_{\mathcal{B}_\kappa} := \sup_{r\geqslant 0}\left(\omega(\langle r\rangle)\langle r\rangle^{\kappa}|h(r)|\right).$$

In the above, we used the property to be even for the functions $\langle r\rangle$ and $|h(r)|$, $|h'(r)|$ so that we just need to consider $r \geqslant 0$ in these norms.

Proof By using the definitions of $\|u_0\|_{\mathcal{A}_\kappa}$ and $\|u_1\|_{\mathcal{B}_{\kappa+1}}$, respectively, we may estimate

$$|H_{u_0}(\rho)| \lesssim [\omega(\langle\rho\rangle)]^{-1}\langle\rho\rangle^{1-\kappa}\|u_0\|_{\mathcal{A}_\kappa},$$
$$|H'_{u_0}(\rho)| \lesssim |u_0(\rho)| + \langle\rho\rangle|u_0'(\rho)| \lesssim [\omega(\langle\rho\rangle)]^{-1}\langle\rho\rangle^{-\kappa}\|u_0\|_{\mathcal{A}_\kappa},$$
$$|H_{u_1}(\rho)| \lesssim [\omega(\langle\rho\rangle)]^{-1}\langle\rho\rangle^{-\kappa}\|u_1\|_{\mathcal{B}_{\kappa+1}}.$$

Let us employ the triangle inequality to observe

$$\langle t-|r|\rangle = 3 + \big|t-|r|\big| \leqslant 3 + |t \pm r| = \langle t\pm r\rangle. \tag{42}$$

According to the solution formula (40), one may derive

$$\begin{aligned}
|v(t,r)| &\leqslant \frac{1}{|r|}\big(|H_{u_0}(t+r)| + |H_{u_0}(t-r)|\big) + \frac{1}{|r|}\int_{t-|r|}^{t+|r|}|H_{u_1}(\rho)|\mathrm{d}\rho \\
&\lesssim \frac{1}{|r|}\Big([\omega(\langle t+r\rangle)]^{-1}\langle t+r\rangle^{1-\kappa} + [\omega(\langle t-r\rangle)]^{-1}\langle t-r\rangle^{1-\kappa}\Big) \\
&\quad \times\left(\|u_0\|_{\mathcal{A}_\kappa} + \|u_1\|_{\mathcal{B}_{\kappa+1}}\right) + \frac{1}{|r|}\int_{t-|r|}^{t+|r|}[\omega(\langle\rho\rangle)]^{-1}\langle\rho\rangle^{-\kappa}\mathrm{d}\rho\left(\|u_0\|_{\mathcal{A}_\kappa} + \|u_1\|_{\mathcal{B}_{\kappa+1}}\right) \\
&\lesssim \left(\frac{1}{|r|}[\omega(\langle t-|r|\rangle)]^{-1}\langle t-|r|\rangle^{1-\kappa} + \frac{1}{|r|}\int_{t-|r|}^{t+|r|}[\omega(\langle\rho\rangle)]^{-1}\langle\rho\rangle^{-\kappa}\mathrm{d}\rho\right) \\
&\quad \times\left(\|u_0\|_{\mathcal{A}_\kappa} + \|u_1\|_{\mathcal{B}_{\kappa+1}}\right),
\end{aligned}$$

because of $\kappa > 1$, where we used the relation (42). Let us divide our next considerations into two cases with respect to the interplay between t and $|r|$.

- When $t \geqslant 2|r|$, since the integrand takes its maximum for $\rho = t-|r|$ and $\langle t+|r|\rangle \approx \langle t-|r|\rangle$, thanks to the representation (40), we may estimate

$$
\begin{aligned}
&\omega(\langle t-|r|\rangle)\langle t+|r|\rangle\langle t-|r|\rangle^{\kappa-1}|v(t,r)|\\
&\quad\lesssim \frac{\omega(\langle t-|r|\rangle)\langle t+|r|\rangle\langle t-|r|\rangle^{\kappa-1}}{|r|}|H_{u_0}(t+r)-H_{u_0}(t-r)|+\frac{\langle t+|r|\rangle}{\langle t-|r|\rangle}\|u_1\|_{\mathcal{B}_{\kappa+1}}\\
&\quad\lesssim \omega(\langle t-|r|\rangle)\langle t+|r|\rangle\langle t-|r|\rangle^{\kappa-1}|H'_{u_0}(\zeta)|+\|u_1\|_{\mathcal{B}_{\kappa+1}}\\
&\quad\lesssim \|u_0\|_{\mathcal{A}_\kappa}+\|u_1\|_{\mathcal{B}_{\kappa+1}},
\end{aligned}
$$

where we employed the mean value theorem with $\zeta\in(t-r,t+r)$, (42) and

$$
\begin{aligned}
|H'_{u_0}(\zeta)|&\lesssim[\omega(\langle\zeta\rangle)]^{-1}\langle\zeta\rangle^{-\kappa}\|u_0\|_{\mathcal{A}_\kappa}\\
&\lesssim[\omega(\langle t-|r|\rangle)]^{-1}\langle t-|r|\rangle^{-\kappa}\|u_0\|_{\mathcal{A}_\kappa}.
\end{aligned}
$$

- When $t\leqslant 2|r|$, we have some further discussions.
  - If $|r|\leqslant 1$, since $\langle t-|r|\rangle\approx\langle t+|r|\rangle\approx 3$, this set is compact, then we get

$$
\omega(\langle t-|r|\rangle)\langle t+|r|\rangle\langle t-|r|\rangle^{\kappa-1}|v(t,r)|\lesssim\|u_0\|_{\mathcal{A}_\kappa}+\|u_1\|_{\mathcal{B}_{\kappa+1}},
$$

    whose approach is the same as the one for $t\geqslant 2|r|$.
  - If $|r|\geqslant 1$, since $\langle t+|r|\rangle\leqslant 3\langle r\rangle$ and $|r|\approx\langle r\rangle$, then we get

$$
\begin{aligned}
&\omega(\langle t-|r|\rangle)\langle t+|r|\rangle\langle t-|r|\rangle^{\kappa-1}|v(t,r)|\\
&\quad\lesssim\left(\frac{\langle t+|r|\rangle}{|r|}+\frac{\langle t+|r|\rangle}{|r|}\langle t-|r|\rangle^{\kappa-1}\int_{t-|r|}^{t+|r|}\langle\rho\rangle^{-\kappa}\mathrm{d}\rho\right)\left(\|u_0\|_{\mathcal{A}_\kappa}+\|u_1\|_{\mathcal{B}_{\kappa+1}}\right)\\
&\quad\lesssim\left(1+\frac{\langle t-|r|\rangle^{\kappa-1}}{\kappa-1}\left(\langle t-|r|\rangle^{1-\kappa}-\langle t+|r|\rangle^{1-\kappa}\right)\right)\left(\|u_0\|_{\mathcal{A}_\kappa}+\|u_1\|_{\mathcal{B}_{\kappa+1}}\right)\\
&\quad\lesssim\|u_0\|_{\mathcal{A}_\kappa}+\|u_1\|_{\mathcal{B}_{\kappa+1}},
\end{aligned}
$$

    because of $\kappa>1$.

In other words, we arrive at

$$
\left\|\omega(\langle t-|r|\rangle)\langle t+|r|\rangle\langle t-|r|\rangle^{\kappa-1}v(t,r)\right\|_{L^\infty([0,+\infty)\times\mathbb{R})}\lesssim\|u_0\|_{\mathcal{A}_\kappa}+\|u_1\|_{\mathcal{B}_{\kappa+1}},
$$

which completes our proof. □

4.2 Philosophy of our approach

By Duhamel's principle, the solution to the inhomogeneous linear Cauchy problem

$$
\begin{cases}
v_{tt}-v_{rr}-\frac{2}{r}v_r=F(t,r), & r\in\mathbb{R},\ t>0,\\
v(0,r)=0,\ \ v_t(0,r)=0, & r\in\mathbb{R}
\end{cases}
\tag{43}
$$

is given by

$$v^{\text{inh}}(t,r)=\int_0^t\int_{-1}^1 H_F[s](t-s+r\sigma)\mathrm{d}\sigma\mathrm{d}s=\frac{1}{r}\int_0^t\int_{t-s-r}^{t-s+r}H_F[s](\rho)\mathrm{d}\rho\mathrm{d}s$$

with $H_F[s](\rho):=\frac{\rho}{2}F(s,\rho)$. Concerning the semilinear Cauchy problem (38), inspired by the last representation (43), we introduce

$$Lu(t,r):=\int_0^t\int_{-1}^1 H_u[s](t-s+r\sigma)\mathrm{d}\sigma\mathrm{d}s=\frac{1}{r}\int_0^t\int_{t-s-r}^{t-s+r}H_u[s](\rho)\mathrm{d}\rho\mathrm{d}s$$

with the nonlinear term

$$H_u[s](\rho):=\frac{\rho}{2}|u(s,\rho)|^{p_S(3)}\mu\big(|u(s,\rho)|\big).$$

In view of Duhamel's principle as well as the above setting, we expect that if we find $u\in X_\kappa$ with $\kappa>1$ such that

$$\begin{aligned}u(t,r)&=v(t,r)+Lu(t,r)\\&=\frac{\partial}{\partial t}\int_{-1}^1 H_{u_0}(t+r\sigma)\mathrm{d}\sigma+\int_{-1}^1 H_{u_1}(t+r\sigma)\mathrm{d}\sigma\\&\quad+\frac{1}{r}\int_0^t\int_{t-s-r}^{t-s+r}H_u[s](\rho)\mathrm{d}\rho\mathrm{d}s,\end{aligned}$$

then $u=u(t,r)$ is a solution of the Cauchy problem (38). Note that $v\in X_\kappa$ when $\kappa>1$ for the linear Cauchy problem (39) via Proposition 4.1 and initial data u_0, u_1 with compact support. In order to prove Theorem 2, in the subsequent part, we will verify the following two crucial inequalities:

$$\|Lu\|_{X_\kappa}\lesssim\|u\|_{X_\kappa}^{p_S(3)},\tag{44}$$

$$\|Lu-L\tilde{u}\|_{X_\kappa}\lesssim\|u-\tilde{u}\|_{X_\kappa}\left(\|u\|_{X_\kappa}^{p_S(3)-1}+\|\tilde{u}\|_{X_\kappa}^{p_S(3)-1}\right),\tag{45}$$

for any $u,\tilde{u}\in X_\kappa$, under some conditions for the modulus of continuity. Let us recall $u_0=\varepsilon\bar{u}_0$ and $u_1=\varepsilon\bar{u}_1$. Combining (44) with Proposition 4.1, we immediately claim

$$\|v+Lu\|_{X_\kappa}\lesssim\varepsilon\left(\|\bar{u}_0\|_{\mathcal{A}_\kappa}+\|\bar{u}_1\|_{\mathcal{B}_{\kappa+1}}\right)+\|u\|_{X_\kappa}^{p_S(3)}.\tag{46}$$

Providing that we take a small parameter $0<\varepsilon<\varepsilon_0\ll 1$ and compactly supported initial data, we combine (46) and (45) to claim that there exists a global (in time) small data radial solution $u\in X_\kappa$ by using Banach's fixed point theorem.

From the condition (8), there exists a positive and continuous function $\bar{\kappa} = \bar{\kappa}(\tau)$ such that

$$\mu(\tau) = \left(\log \frac{1}{\tau}\right)^{-\frac{1}{p_S(3)}} \bar{\kappa}(\tau) \quad \text{and} \quad \lim_{\tau \to 0^+} \bar{\kappa}(\tau) = 0. \tag{47}$$

Furthermore, the condition (9) means that the above function $\bar{\kappa}$ satisfies the next additional condition:

$$\bar{\kappa}(\tau) \log\log \frac{1}{\tau} \lesssim 1 \quad \text{when } \tau \in (0, \tau_0] \tag{48}$$

with a sufficiently small τ_0. Before starting our proof, let us introduce a crucial statement.

Proposition 4.2 *Let us consider a modulus of continuity $\mu = \mu(\tau)$ with $\mu(0) = 0$ satisfying the conditions (8) and (9). Let us recall the weighted factor via (41). Then, the integral*

$$I(\xi) := \int_{-|\xi|}^{|\xi|} [\omega(\langle\eta\rangle)]^{-p_S(3)} \langle\xi+\eta\rangle\langle\eta\rangle^{-1} \mu\left(\varepsilon_0 [\omega(\langle\eta\rangle)]^{-1} \langle\xi\rangle^{-1} \langle\eta\rangle^{-\frac{1}{p_S(3)}}\right) \mathrm{d}\eta$$

fulfills the following estimate:

$$I(\xi) \lesssim \langle\xi\rangle [\log(\langle\xi\rangle)]^{-\frac{1}{p_S(3)}} (\log\log\langle\xi\rangle)\, \bar{\kappa}\left(\varepsilon_0\langle\xi\rangle^{-1}\right).$$

Proof To begin with, let us split $I(\xi)$ into two parts

$$I_1(\xi) := \int_{-\frac{|\xi|}{2}}^{\frac{|\xi|}{2}} [\omega(\langle\eta\rangle)]^{-p_S(3)} \langle\xi+\eta\rangle\langle\eta\rangle^{-1} \mu\left(\varepsilon_0 [\omega(\langle\eta\rangle)]^{-1} \langle\xi\rangle^{-1} \langle\eta\rangle^{-\frac{1}{p_S(3)}}\right) \mathrm{d}\eta,$$

$$I_2(\xi) := \left(\int_{\frac{|\xi|}{2}}^{|\xi|} + \int_{-|\xi|}^{-\frac{|\xi|}{2}}\right) [\omega(\langle\eta\rangle)]^{-p_S(3)} \langle\xi+\eta\rangle\langle\eta\rangle^{-1} \mu\left(\varepsilon_0 [\omega(\langle\eta\rangle)]^{-1} \langle\xi\rangle^{-1} \langle\eta\rangle^{-\frac{1}{p_S(3)}}\right) \mathrm{d}\eta.$$

For the first integral, we use the asymptotic behaviors $\langle\xi\rangle \approx \langle\eta+\xi\rangle \approx \langle\eta-\xi\rangle$ when $\eta \in [-\frac{|\xi|}{2}, \frac{|\xi|}{2}]$ to deduce

$$\begin{aligned}
I_1(\xi) &\lesssim \langle\xi\rangle \int_{-\frac{|\xi|}{2}}^{\frac{|\xi|}{2}} \langle\eta\rangle^{-1} [\log(\langle\eta\rangle)]^{-1} \mu\left(\varepsilon_0 \langle\xi\rangle^{-1} \langle\eta\rangle^{-\frac{1}{p_S(3)}} [\log(\langle\eta\rangle)]^{-\frac{1}{p_S(3)}}\right) \mathrm{d}\eta \\
&\lesssim \langle\xi\rangle \mu\left(\varepsilon_0\langle\xi\rangle^{-1}\right) \int_0^{\frac{|\xi|}{2}} \langle\eta\rangle^{-1} [\log(\langle\eta\rangle)]^{-1} \mathrm{d}\eta \\
&\lesssim \langle\xi\rangle [\log(\langle\xi\rangle)]^{-\frac{1}{p_S(3)}} (\log\log\langle\xi\rangle)\, \bar{\kappa}\left(\varepsilon_0\langle\xi\rangle^{-1}\right),
\end{aligned} \tag{49}$$

in which we applied the condition (47), and the increasing property of $\mu = \mu(\tau)$ as well as $[\langle\eta\rangle \log(\langle\eta\rangle)]^{-\frac{1}{p_S(3)}} < 1$.

Let us turn to the first part of $I_2(\xi)$, which is denoted by $I_{2,1}(\xi)$. When $\xi \geqslant 0$, due to $\eta \in [\frac{\xi}{2}, \xi]$, the equivalences $\langle \eta+\xi\rangle \approx \langle \eta\rangle \approx \langle \xi\rangle$ hold. It leads to

$$
\begin{aligned}
I_{2,1}(\xi) &\lesssim \langle \xi\rangle[\log(\langle \xi\rangle)]^{-1}\mu\left(\varepsilon_0\langle \xi\rangle^{-1-\frac{1}{p_S(3)}}[\log(\langle \xi\rangle)]^{-\frac{1}{p_S(3)}}\right)\\
&\lesssim \langle \xi\rangle[\log(\langle \xi\rangle)]^{-1-\frac{1}{p_S(3)}}\,\bar{\kappa}\left(\varepsilon_0\langle \xi\rangle^{-1}\right),
\end{aligned}
$$

because $[\langle \xi\rangle\log(\langle \xi\rangle)]^{-\frac{1}{p_S(3)}} < 1$. When $\xi \leqslant 0$, the equivalences $\langle \eta-\xi\rangle \approx \langle \xi\rangle \approx \langle \eta\rangle$ are valid for $\eta \in [-\xi, -\frac{\xi}{2}]$. Then, one deduces

$$
\begin{aligned}
I_{2,1}(\xi) &\lesssim [\log(\langle \xi\rangle)]^{-1}\langle \xi\rangle^{-1}\mu\left(\varepsilon_0\langle \xi\rangle^{-1-\frac{1}{p_S(3)}}[\log(\langle \xi\rangle)]^{-\frac{1}{p_S(3)}}\right)\int_{-\xi}^{-\frac{\xi}{2}}\langle -\xi-\eta\rangle \mathrm{d}\eta\\
&\lesssim \langle \xi\rangle[\log(\langle \xi\rangle)]^{-1-\frac{1}{p_S(3)}}\,\bar{\kappa}\left(\varepsilon_0\langle \xi\rangle^{-1}\right).
\end{aligned}
$$

The above two estimates are stronger than the estimate (49). Repeating the same procedure as those for $I_{2,1}(\xi)$, by symmetry we are able to get

$$
I_{2,2}(\xi) \lesssim \langle \xi\rangle[\log(\langle \xi\rangle)]^{-1-\frac{1}{p_S(3)}}\,\bar{\kappa}\left(\varepsilon_0\langle \xi\rangle^{-1}\right).
$$

Summarizing the above derived estimates, we complete the proof of this proposition. □

4.3 Some estimates for solutions to nonlinear models: proof of Theorem 2

Let us choose $u \in X_\kappa$. From the definition of the Banach space X_κ with $\kappa > 1$, we obtain

$$
\begin{aligned}
&|u(s,\rho)|^{p_S(3)}\mu\big(|u(s,\rho)|\big)\\
&\quad\lesssim [\omega(\langle s-|\rho|\rangle)]^{-p_S(3)}\langle s+|\rho|\rangle^{-p_S(3)}\langle s-|\rho|\rangle^{-(\kappa-1)p_S(3)}\|u\|_{X_\kappa}^{p_S(3)}\\
&\qquad\times\mu\left([\omega(\langle s-|\rho|\rangle)]^{-1}\langle s+|\rho|\rangle^{-1}\langle s-|\rho|\rangle^{-(\kappa-1)}\|u\|_{X_\kappa}\right)\\
&\quad\lesssim [\omega(\langle s-|\rho|\rangle)]^{-p_S(3)}\langle s+|\rho|\rangle^{-p_S(3)}\langle s-|\rho|\rangle^{-(\kappa-1)p_S(3)}\|u\|_{X_\kappa}^{p_S(3)}\\
&\qquad\times\mu\left(\varepsilon_0[\omega(\langle s-|\rho|\rangle)]^{-1}\langle s+|\rho|\rangle^{-1}\langle s-|\rho|\rangle^{-(\kappa-1)}\right),
\end{aligned}
$$

where we assumed $\|u\|_{X_\kappa} \leqslant \varepsilon_0$ for some $\varepsilon_0 > 0$ sufficiently small.

With the aim of proving (44), we just need to show

$$
\omega(\langle t-|r|\rangle)\langle t+|r|\rangle\langle t-|r|\rangle^{\kappa-1}|Lu(t,r)| \lesssim \|u\|_{X_\kappa}^{p_S(3)}.
$$

Because Lu is even in r, we may restrict ourselves to non-negative values of r. Concerning $r \geqslant 0$, applying the definition of Lu, we get

$$|Lu(t,r)| \leqslant \frac{1}{r}\int_0^t \int_{t-s-r}^{t-s+r} |H_u[s](\rho)|\mathrm{d}\rho\mathrm{d}s \lesssim \frac{1}{r} I_0(t,r)\|u\|_{X_\kappa}^{p_S(3)},$$

where

$$I_0(t,r) := \int_0^t \int_{t-s-r}^{t-s+r} \langle\rho\rangle[\omega(\langle s-|\rho|\rangle)]^{-p_S(3)}\langle s+|\rho|\rangle^{-p_S(3)}\langle s-|\rho|\rangle^{-(\kappa-1)p_S(3)} \\ \times \mu\left(\varepsilon_0[\omega(\langle s-|\rho|\rangle)]^{-1}\langle s+|\rho|\rangle^{-1}\langle s-|\rho|\rangle^{-(\kappa-1)}\right)\mathrm{d}\rho\mathrm{d}s$$

for the case $t \geqslant r$. In the case $t \leqslant r$, we can slightly modify the representation formulate for Lu. Precisely, being $H_u[s](\rho)$ an odd function with respect to ρ, one notices

$$\int_{(t-s)-r}^{r-(t-s)} H_u[s](\rho)\mathrm{d}\rho = 0.$$

The additivity of the integral regions shows

$$Lu(t,r) = \frac{1}{r}\int_0^t \int_{r-(t-s)}^{(t-s)+r} H_u[s](\rho)\mathrm{d}\rho\mathrm{d}s.$$

As a consequence, when $t \leqslant r$, we may replace $I_0(t,r)$ by

$$\widetilde{I}_0(t,r) := \int_0^t \int_{r-(t-s)}^{r+(t-s)} \langle\rho\rangle[\omega(\langle s-|\rho|\rangle)]^{-p_S(3)}\langle s+|\rho|\rangle^{-p_S(3)}\langle s-|\rho|\rangle^{-(\kappa-1)p_S(3)} \\ \times \mu\left(\varepsilon_0[\omega(\langle s-|\rho|\rangle)]^{-1}\langle s+|\rho|\rangle^{-1}\langle s-|\rho|\rangle^{-(\kappa-1)}\right)\mathrm{d}\rho\mathrm{d}s.$$

All in all, we already derived

$$|Lu(t,r)| \lesssim \begin{cases} r^{-1}I_0(t,r)\|u\|_{X_\kappa}^{p_S(3)} & \text{when } t \geqslant r, \\ r^{-1}\widetilde{I}_0(t,r)\|u\|_{X_\kappa}^{p_S(3)} & \text{when } t \leqslant r, \end{cases}$$

for any $r \geqslant 0$.

Estimates for $I_0(t,r)$ **with** $t \geqslant r$. Since $|t-s-r| \leqslant t-s+r$ (we are working with $r \geqslant 0$) and the even function with respect to ρ of the integrand in $I_0(t,r)$ so that

$$I_0(t,r) \lesssim \int_0^t \int_{\max\{0,t-s-r\}}^{t-s+r} \langle s+\rho\rangle^{-p_S(3)}\langle s-\rho\rangle^{-(\kappa-1)p_S(3)}\langle\rho\rangle[\omega(\langle s-\rho\rangle)]^{-p_S(3)} \\ \times \mu\left(\varepsilon_0[\omega(\langle s-\rho\rangle)]^{-1}\langle s+\rho\rangle^{-1}\langle s-\rho\rangle^{-(\kappa-1)}\right)\mathrm{d}\rho\mathrm{d}s.$$

We now perform the change of two variables

$$\xi = s + \rho, \quad \eta = \rho - s, \quad \text{namely,} \quad \rho = \frac{\xi + \eta}{2}, \quad s = \frac{\xi - \eta}{2}. \tag{50}$$

Since $\rho, s \geqslant 0$, we get $|\eta| \leqslant \xi$. Furthermore, $\rho \in [\max\{0, t-s-r\}, t-s+r]$ leads to

$$t - r \leqslant s + \max\{0, t-s-r\} \leqslant s + \rho = \xi \leqslant t + r.$$

Hence, recalling $\kappa := 1 + \frac{1}{ps(3)}$ in the definition of the Banach space X_κ, we may employ Proposition 4.2 to derive

$$\begin{aligned} I_0(t,r) &\lesssim \int_{t-r}^{t+r} \langle \xi \rangle^{-ps(3)} \int_{-\xi}^{\xi} [\omega(\langle \eta \rangle)]^{-ps(3)} \langle \xi + \eta \rangle \langle \eta \rangle^{-1} \\ &\quad \times \mu\left(\varepsilon_0 [\omega(\langle \eta \rangle)]^{-1} \langle \xi \rangle^{-1} \langle \eta \rangle^{-\frac{1}{ps(3)}}\right) \mathrm{d}\eta \mathrm{d}\xi \\ &\lesssim \int_{t-r}^{t+r} \langle \xi \rangle^{-ps(3)} I(\xi) \mathrm{d}\xi \\ &\lesssim \int_{t-r}^{t+r} \langle \xi \rangle^{1-ps(3)} [\log(\langle \xi \rangle)]^{-\frac{1}{ps(3)}} (\log\log\langle \xi \rangle)\, \bar{\kappa}\left(\varepsilon_0 \langle \xi \rangle^{-1}\right) \mathrm{d}\xi. \end{aligned}$$

Thus, from the even behavior of Lu with respect to r, taking account of $t \geqslant r \geqslant 0$, we may arrive at

$$\begin{aligned} &\omega(\langle t-r \rangle) \langle t+r \rangle \langle t-r \rangle^{\frac{1}{ps(3)}} |Lu(t,r)| \\ &\quad \lesssim \omega(\langle t-r \rangle) \langle t+r \rangle \langle t-r \rangle^{\frac{1}{ps(3)}} r^{-1} |I_0(t,r)|\, \|u\|_{X_\kappa}^{ps(3)} \\ &\quad \lesssim J_0(t,r) \|u\|_{X_\kappa}^{ps(3)}, \end{aligned}$$

where

$$\begin{aligned} J_0(t,r) &:= [\log(\langle t-r \rangle)]^{\frac{1}{ps(3)}} \frac{\langle t+r \rangle \langle t-r \rangle^{\frac{1}{ps(3)}}}{r} \\ &\quad \times \int_{t-r}^{t+r} \langle \xi \rangle^{1-ps(3)} [\log(\langle \xi \rangle)]^{-\frac{1}{ps(3)}} (\log\log\langle \xi \rangle)\, \bar{\kappa}\left(\varepsilon_0 \langle \xi \rangle^{-1}\right) \mathrm{d}\xi. \end{aligned}$$

Noticing that $\varepsilon_0 \langle \xi \rangle^{-1} \leqslant \varepsilon_0 3^{-1} \leqslant \tau_0$ with $0 < \varepsilon_0 \ll 1$ sufficiently small, we directly apply (48) to derive

$$\bar{\kappa}\left(\varepsilon_0 \langle \xi \rangle^{-1}\right) \left[\log\log\left(\varepsilon_0^{-1} \langle \xi \rangle\right)\right] \lesssim 1. \tag{51}$$

Due to $\varepsilon_0^{-1} \gg 1$, we simplify our aim as

$$J_0(t,r) \lesssim [\log(\langle t-r\rangle)]^{\frac{1}{p_S(3)}} \frac{\langle t+r\rangle\langle t-r\rangle^{\frac{1}{p_S(3)}}}{r} \int_{t-r}^{t+r} \langle\xi\rangle^{1-p_S(3)}[\log(\langle\xi\rangle)]^{-\frac{1}{p_S(3)}}\mathrm{d}\xi.$$

Next, we will estimate $J_0(t,r)$ more precisely by estimating the last right-hand side in different zones of the (t,r)-plane.

Zone I: $t \geqslant 2r \geqslant 0$ For $\xi \in [t-r, t+r]$, we know the equivalence $\langle\xi\rangle \approx \langle t+r\rangle$. It follows

$$\begin{aligned} J_0(t,r) &\lesssim \langle t+r\rangle^{2-p_S(3)}\langle t-r\rangle^{\frac{1}{p_S(3)}}[\log(\langle t+r\rangle)]^{-\frac{1}{p_S(3)}}[\log(\langle t-r\rangle)]^{\frac{1}{p_S(3)}} \\ &\lesssim \langle t+r\rangle^{2-p_S(3)+\frac{1}{p_S(3)}} \lesssim 1, \end{aligned}$$

since $-p_S^2(3)+2p_S(3)+1=0$ for three dimensions (see (2) with $n=3$).

Zone II: $0 \leqslant r \leqslant 1$ and $t \leqslant 2r$ We own the equivalence $\langle t+r\rangle \approx 3$ so that

$$J_0(t,r) \lesssim \langle t+r\rangle^{1+\frac{1}{p_S(3)}}\langle t-r\rangle^{1-p_S(3)} \lesssim 1$$

by using the bounds $2 \leqslant \langle t-r\rangle \leqslant \langle t+r\rangle$.

Zone III: $r \geqslant 1$ and $r \leqslant t \leqslant 2r$ Via the equivalences $r \approx \langle r\rangle \approx \langle t+r\rangle$, we are able to conclude

$$J_0(t,r) \lesssim \frac{\langle t+r\rangle}{r}\langle t-r\rangle^{\frac{1}{p_S(3)}} \int_{t-r}^{t+r}\langle\xi\rangle^{1-p_S(3)}\mathrm{d}\xi \lesssim \langle t-r\rangle^{2-p_S(3)+\frac{1}{p_S(3)}} \lesssim 1,$$

where we applied $p_S(3) = 1+\sqrt{2} > 2$.

Summarizing, we have derived the key estimate

$$\omega(\langle t-r\rangle)\langle t+r\rangle\langle t-r\rangle^{\frac{1}{p_S(3)}}|Lu(t,r)| \lesssim J_0(t,r)\|u\|_{X_\kappa}^{p_S(3)} \lesssim \|u\|_{X_\kappa}^{p_S(3)}$$

for any $t \geqslant r$.

Estimates for $\widetilde{I}_0(t,r)$ with $t \leqslant r$. This part just discusses the case for $r \geqslant 1$ since the influence coming from the compact set $\{(t,r): 0 \leqslant t \leqslant r \leqslant 1\}$ is negligible. As before, we take the change of variables (50). According to $s \geqslant 0$ and $\rho \geqslant r-(t-s)$, we deduce $r-t \leqslant \eta \leqslant \xi$, while from $s \geqslant 0$ and $|\rho - r| \leqslant t-s$ it follows $r-t \leqslant \xi \leqslant r+t$. For this reason, we find

$$\begin{aligned} \widetilde{I}_0(t,r) \lesssim &\int_{r-t}^{r+t}\langle\xi\rangle^{-p_S(3)}\int_{r-t}^{\xi}[\omega(\langle\eta\rangle)]^{-p_S(3)}\langle\xi+\eta\rangle\langle\eta\rangle^{-1} \\ &\times \mu\left(\varepsilon_0[\omega(\langle\eta\rangle)]^{-1}\langle\xi\rangle^{-1}\langle\eta\rangle^{-\frac{1}{p_S(3)}}\right)\mathrm{d}\eta\mathrm{d}\xi. \end{aligned}$$

The relation $[r-t,\xi]\subset[-\xi,\xi]$ and the application of Proposition 4.2 associated with (51) yield

$$\begin{aligned}\widetilde{I}_0(t,r)&\lesssim\int_{r-t}^{r+t}\langle\xi\rangle^{1-p_S(3)}[\log(\langle\xi\rangle)]^{-\frac{1}{p_S(3)}}(\log\log\langle\xi\rangle)\,\bar{\kappa}\left(\varepsilon_0\langle\xi\rangle^{-1}\right)\mathrm{d}\xi\\&\lesssim[\log(\langle r-t\rangle)]^{-\frac{1}{p_S(3)}}\int_{r-t}^{r+t}\langle\xi\rangle^{1-p_S(3)}\mathrm{d}\xi\\&\lesssim\langle t-r\rangle^{2-p_S(3)}[\log(\langle t-r\rangle)]^{-\frac{1}{p_S(3)}},\end{aligned}$$

due to $2-p_S(3)=1-\sqrt{2}<0$. Hence,

$$\begin{aligned}&\omega(\langle t-r\rangle)\langle t+r\rangle\langle t-r\rangle^{\frac{1}{p_S(3)}}|Lu(t,r)|\\&\quad\lesssim\omega(\langle t-r\rangle)\langle t+r\rangle\langle t-r\rangle^{\frac{1}{p_S(3)}}r^{-1}|\widetilde{I}_0(t,r)|\,\|u\|_{X_\kappa}^{p_S(3)}\\&\quad\lesssim\frac{\langle t+r\rangle}{r}\langle t-r\rangle^{2-p_S(3)+\frac{1}{p_S(3)}}\|u\|_{X_\kappa}^{p_S(3)}\\&\quad\lesssim\|u\|_{X_\kappa}^{p_S(3)}\end{aligned}$$

by the same reason as the one in *Zone III* of estimates for $I_0(t,r)$.

Thus, we may claim

$$\left\|\omega(\langle t-|r|\rangle)\langle t+|r|\rangle\langle t-|r|\rangle^{\frac{1}{p_S(3)}}Lu(t,r)\right\|_{L^\infty([0,+\infty)\times\mathbb{R})}\lesssim\|u\|_{X_\kappa}^{p_S(3)}\tag{52}$$

with the weighted factor defined in (41) and under the conditions (8) as well as (9). As a by-product, due to

$$\begin{aligned}|Lu(t,r)-L\tilde{u}(t,r)|\lesssim\frac{1}{r}\|u-\tilde{u}\|_{X_\kappa}&\left(\|u\|_{X_\kappa}^{p_S(3)-1}+\|\tilde{u}\|_{X_\kappa}^{p_S(3)-1}\right)\\&\times\begin{cases}r^{-1}I_0(t,r)&\text{when }t\geqslant r,\\r^{-1}\widetilde{I}_0(t,r)&\text{when }t\leqslant r,\end{cases}\end{aligned}$$

one can prove

$$\begin{aligned}&\left\|\omega(\langle t-|r|\rangle)\langle t+|r|\rangle\langle t-|r|\rangle^{\frac{1}{p_S(3)}}\left(Lu(t,r)-L\tilde{u}(t,r)\right)\right\|_{L^\infty([0,+\infty)\times\mathbb{R})}\\&\lesssim\|u-\tilde{u}\|_{X_\kappa}\left(\|u\|_{X_\kappa}^{p_S(3)-1}+\|\tilde{u}\|_{X_\kappa}^{p_S(3)-1}\right),\end{aligned}$$

by the same way as those for (52).

Summarizing the last statements, we have completed the proof of the desired inequalities (44) as well as (45), namely, the proof of Theorem 2 is completed. □

Furthermore, by employing Banach's fixed point theorem, with the weighted data, we arrive at the next global (in time) well-posedness result with some pointwise decay estimates.

Corollary 4.1 *Let us consider a modulus of continuity* $\mu = \mu(\tau)$ *with* $\mu(0) = 0$ *fulfilling* (8) *and* (9). *Let* $u_0 \in (\mathcal{A}_\kappa \cap \mathcal{C}_0^2)$ *and* $u_1 \in (\mathcal{B}_{\kappa+1} \cap \mathcal{C}_0^1)$ *be radial with* $\kappa = 1 + \frac{1}{p_S(3)}$. *Then, there exists* $0 < \varepsilon_0 \ll 1$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *fulfilling* $\|u_0\|_{\mathcal{A}_\kappa} + \|u_1\|_{\mathcal{B}_{\kappa+1}} < \varepsilon_0$, *the semilinear Cauchy problem* (3) *for* $n = 3$ *admits a uniquely determined global* (*in time*) *small data radial solution* $u \in \mathcal{C}([0, +\infty) \times \mathbb{R}^3)$. *Furthermore, the solution fulfills the following pointwise decay estimates*:

$$|u(t, r)| \lesssim [\log(\langle t - |r|\rangle)]^{-\frac{1}{p_S(3)}} \langle t + |r|\rangle^{-1} \langle t - |r|\rangle^{-\frac{1}{p_S(3)}} \left(\|u_0\|_{\mathcal{A}_\kappa} + \|u_1\|_{\mathcal{B}_{\kappa+1}}\right).$$

The data spaces $\mathcal{A}_\kappa$ and $\mathcal{B}_\kappa$ were defined in Proposition 4.1.

Remark 8 One of our new tools is the introduction of new weighted data spaces with logarithmic factors. With these spaces we are able to derive polynomial-logarithmic type decay estimates for the global (in time) radial solutions to the Cauchy problem (38) with modulus of continuity from the previous Corollary 4.1.

5 Final remarks

In this paper, we derived a blow-up result and a global (in time) existence result for semilinear classical wave equations with a modulus of continuity in the nonlinearity $|u|^{p_S(n)}\mu(|u|)$. Specially, considering a modulus of continuity $\mu = \mu(\tau)$ with $\mu(0) = 0$ carrying the form (4) with $0 < \tau_0 \ll 1$, we described the critical regularity of nonlinearities $|u|^{p_S(3)}\mu(|u|)$ by the threshold $\gamma = \frac{1}{p_S(3)}$.

According to the general blow-up condition (6) and the global (in time) existence result in Theorem 2, we expect that the general threshold is described by the quantity (10), i.e.

$$C_{\text{Str}} = \lim_{\tau \to 0^+} \mu(\tau) \left(\log \frac{1}{\tau}\right)^{\frac{1}{p_S(n)}}. \tag{53}$$

Let us explain it via three situations concerning the value of $C_{\text{Str}} \geqslant 0$.

- **Blow-up of solutions when** $C_{\text{Str}} = +\infty$: Due to the proposed condition (6) in Theorem 1, the validity of the last conjecture from the blow-up viewpoint is already known when $C_{\text{Str}} = +\infty$.
- **Blow-up of solutions when** $C_{\text{Str}} \in (0, +\infty)$: Due to the proposed condition (6) in Theorem 1, we already know the blow-up phenomenon occurs when $C_{\text{Str}} \in [c_l, +\infty)$ with a suitably large constant $c_l \gg 1$. Recently, the blow-up result for semilinear classical wave equations with modulus of continuity in derivative type nonlinearity has been studied by [3]. Considering the model $u_{tt} - \Delta u = |u_t|^{p_G(n)}\mu(|u_t|)$ with the Glassey exponent $p_G(n) := \frac{n+1}{n-1}$ for $n \geqslant 2$, the author of [3] proposes a new blow-up condition even for the completed intermediate case as follows:

$$\lim_{\tau \to 0^+} \mu(\tau) \left(\log \frac{1}{\tau}\right) = C_{\text{Gla}} \in (0, +\infty].$$

Motivated by the above explanation, we believe that the blow-up result still holds in the intermediate case $C_{\mathrm{Str}} \in (0, c_l)$. However, due to some technical difficulties, the rigorous justification is still challenging.

- **Global (in time) existence of solutions when** $C_{\mathrm{Str}} = 0$: Due to the proposed condition (8), i.e. $C_{\mathrm{Str}} = 0$, and the decay assumption (9) in Theorem 2, our conjecture is partially verified from the global (in time) existence perspective.

Lastly, we underline that the validity of our conjecture (53) has been verified in the present manuscript for the semilinear three dimensional Cauchy problem (3) with a modulus of continuity fulfilling $\mu(0) = 0$ and $\mu(\tau) = c_l(\log\frac{1}{\tau})^{-\gamma}$ with $c_l \gg 1$ when $\tau \in (0, \tau_0]$, because the global (in time) existence result for $\gamma > \frac{1}{p_S(3)}$ and the blow-up result for $0 < \gamma \leqslant \frac{1}{p_S(3)}$ have been rigorously demonstrated in Theorems 2 and 1, respectively.

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Data availability we have no any data.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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