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Tautological characteristic classes II: the Witt class

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Abstract

Let K be an arbitrary infinite field. The cohomology group $H^2(SL(2, K); H_2SL(2, K))$ contains the class of the universal central extension. When studying representations of fundamental groups of surfaces in $SL(2, K)$ it is useful to have classes stable under deformations (Fenchel-Nielsen twists) of representations. We identify the maximal quotient of the universal class which is stable under twists as the Witt class of Nekovàř. The Milnor-Wood inequality asserts that an $SL(2, \mathbf{R})$-bundle over a surface of genus g admits a flat structure if and only if its Euler number is $\leq (g - 1)$. We establish an analog of this inequality, and a saturation result for the Witt class. The result is sharp for the field of rationals, but not sharp in general.

Introduction

Let $H^2(G, U)$ be the second cohomology of a discrete group G, with constant coefficients U. Elements of $H^2(G, U)$ correspond to central extensions of G by U. If G is perfect, then among all central extensions, with varying U, there is the universal one (cf. [1, Exercise IV.3.7]). The corresponding class, which we call the Moore class, lives in $H^2(G, H_2G)$.

Cohomology classes of a (discrete) group G give rise to characteristic classes of G-bundles and of representations. Let Σ be a closed oriented surface. A principal G-bundle over Σ has a monodromy representation $\rho\colon \pi_1(\Sigma) \to G$, well-defined up to G-conjugation. Given $\tau \in H^2(G, U)$, one defines the corresponding characteristic

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class of the bundle, or of ρ, by $\tau(\rho) := \rho^*\tau \in H^2(\pi_1(\Sigma), U) \simeq H^2(\Sigma, U) \simeq U$ (the last isomorphism is given by evaluation on the fundamental class of Σ). One important application of these characteristic classes is in the study of the representation variety $\mathrm{Hom}(\pi_1(\Sigma), G)/G$, i.e. of the moduli space of G-bundles over Σ (cf. [10]). This variety often has additional, topological or algebro-geometric structure, and one might be interested in its connected components. The picture one strives to achieve here is modelled on W. Goldman's description of $\mathrm{Hom}(\pi_1(\Sigma), PSL(2, \mathbf{R}))$: for a genus g surface, this space has $4g - 3$ connected components indexed by the Euler class of the representation (cf. [7]). In this classical case there are also natural continuous deformations of representations (or flat bundles) called Fenchel–Nielsen twists. For an arbitrary group G it is somewhat vague what the "connected components" of the representation variety are, but the twists do generalize (cf. [6], or Sect. 5). Thus, one wants to consider characteristic classes that are stable under twists. For a perfect G, one may ask whether the universal Moore class is stable under twists. It turns out it is not (e.g. not for $SL(2, K)$). However, a universal twist-stable cohomology class w_G does exist, and there is a beautifully simple condition that we call *equicommutativity* which is equivalent to twist-stability. The Moore class coefficient group H_2G has a largest "equicommutative quotient" $Eq(G)$, and the class $w_G \in H^2(G, Eq(G))$ can be obtained as the image of the Moore class under the coefficient map $H_2G \to Eq(G)$.

Let K be an arbitrary infinite field. The above discussion applies to $SL(2, K)$, but here we have more structure. The action of $SL(2, K)$ on its homogeneous space $\mathbf{P}^1(K)$ gives rise to another class, defined by Nekovář in [16], $\mathrm{w}^I \in H^2(SL(2, K), I^2(K))$, which we call the Witt class. Here $I^2(K)$ is the square of the fundamental ideal of $W(K)$, the Witt ring of symmetric bilinear forms over K. It follows from universality that w^I is the image of the Moore class by a certain map $H_2(SL(2, K)) \to I^2(K)$. We identify this map with $H_2(SL(2, K)) \to Eq(SL(2, K))$ and prove the following result.

Theorem A (Theorem 10.1) *Let K be an infinite field. The group $Eq(SL(2, K))$ is isomorphic to $I^2(K)$, and the Witt class $\mathrm{w}^I \in H^2(SL(2, K), I^2(K))$ is the universal equicommutative class.*

From [2] we know that the Witt class is bounded with respect to the natural seminorm on $W(K)$. This is analogous to the classical Milnor–Wood inequality for the Euler class of flat $SL(2, \mathbf{R})$-bundles. Milnor's inequality is sharp: all values allowed by it are indeed Euler classes of flat bundles. We study the corresponding saturation problem for the Witt class and prove:

Theorem B (Theorem 11.6) *Let K be an infinite field.*

(a) *The Witt class of any flat $SL(2, K)$-bundle over an oriented closed surface of genus g has norm $\leq 4(g - 1) + 2$.*
(b) *The set of Witt classes of flat $SL(2, K)$-bundles over an oriented closed surface of genus g contains the set of elements of $I^2(K)$ of norm $\leq 4(g - 1)$.*

The form of Milnor's inequality we established for general fields is not sharp. An example of non-sharpness is constructed over the field of Laurent series with rational coefficients. But for $K = \mathbf{Q}$ we have the sharp result:

Theorem C (Theorem 12.2) *The set of Witt classes of all representations of* $\pi_1(\Sigma_g)$ *in* $SL(2, \mathbf{Q})$ *is equal to the set of elements of* $I^2\mathbf{Q}$ *with norm* $\leq 4(g-1)$.

To prove Theorem B we use arithmetic properties of Markov surfaces, established in [4], to construct the required representations. The proof of Theorem C uses the classical Milnor–Wood inequality and Meyer's even more classical theorem from the theory of quadratic forms over $\mathbf{Q}$.

The Witt class can be constructed for $PSL(2, K)$, but, in general, it is not equicommutative for that group: that case requires further study.

The paper is divided into four parts, each with its own introduction.

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I. Moore and Witt classes

In this part we present the main two protagonists: the Witt and Moore classes. Both are constructed as tautological cohomology classes in the sense of [2].

1 Tautological construction of the Witt class

The Witt class was first defined by Nekovář (cf. [16]). It is a cohomology class $\mathrm{w} \in H^2(SL(2, K), W(K))$, where K is an infinite field and $W(K)$ is the Witt group of symmetric bilinear forms over K. A tautological construction of this class is given in [2, Section 7]. We briefly recall this construction now. Later, in Sect. 9, we explain how Nekovář modified the class w to $\mathrm{w}^I \in H^2(SL(2, K), I^2(K))$. In this section $G = SL(2, K)$.

The group G acts on $\mathbf{P}^1(K)$. The infinite simplex with vertex set $\mathbf{P}^1(K)$ is a contractible G-simplicial complex that we denote by X. It carries a G-invariant tautological cocycle T, defined as follows. First, we define a 2-cochain on X with values in C_2X (the chain group of X with integer coefficients) by assigning to a 2-simplex this same 2-simplex treated as an element of C_2X. This cochain is not closed—we force it to become closed by applying to its coefficient group the quotient map $C_2X \to C_2X/B_2X$. The result is closed (is a cocycle), but it is not G-invariant. We force it to become G-invariant by passing to G-coinvariants, i.e. by applying to it another quotient map $C_2X/B_2X \to (C_2X/B_2X)_G$. This finally gives T. Of course, one fears that the quotient group $(C_2X/B_2X)_G$ is trivial; however, quite miraculously, it turns out to be isomorphic to $W(K)$, the additive group of the Witt ring. (A basic discussion of the Witt ring can be found in [3, Chapter I].)

The cocycle T can be pulled back to G via (any) orbit map. In more detail, for any $x \in \mathbf{P}^1(K)$ we consider the $W(K)$-valued 2-cocycle on G defined by

$$(g_0, g_1, g_2) \mapsto T(g_0x, g_1x, g_2x). \tag{1.1}$$

(If (g_0x, g_1x, g_2x) is a degenerate simplex in X, the right hand side is interpreted as zero.) The cohomology class $\mathrm{w} \in H^2(SL(2, K), W(K))$ of this cocycle does not depend on the choice of x—this is the Witt class.

To be more explicit we recall the standard Witt group notation: for $a \in K^*(:= K \setminus \{0\})$ we denote by $[a]$ the element of $W(K)$ represented by the 1-dimensional form ax^2. The symbol [0] is interpreted as 0. The Witt class is represented by the following (homogeneous) cocycle:

$$(g_0, g_1, g_2) \mapsto [|g_0v, g_1v| \cdot |g_1v, g_2v| \cdot |g_2v, g_0v|]; \tag{1.2}$$

here v is any non-zero vector in K^2, and $|g_iv, g_jv|$ stands for the determinant of the pair of vectors (g_iv, g_jv). The cocycle depends on v, but its cohomology class does not. In the non-homogeneous setting we obtain the following cocycle representing the Witt class:

$$w(a, b) = [|v, av| \cdot |av, abv| \cdot |abv, v|] = [-|v, av| \cdot |v, abv| \cdot |v, bv|]. \tag{1.3}$$

The standard choice of v is $v = e = \binom{1}{0}$; for this v we get

$$w(a, b) = [-|e, ae| \cdot |e, abe| \cdot |e, be|] = [-a_{21} \cdot (ab)_{21} \cdot b_{21}]. \tag{1.4}$$

(In this formula a_{21} denotes the 21-entry of the matrix a.) This explicit formula will be very useful later.

For details on all the claims made above we again refer the reader to [2], especially to Sect. 7 therein.

2 Tautological construction of the Moore class

We will construct a tautological class starting from the action of a group G on the standard model of EG (cf. [8, Example 1.B.7]). This model is a Δ-complex with n-simplices given by $(n+1)$-tuples $[g_0, g_1, \dots, g_n]$ of elements of G. The G-action is $g[g_0, g_1, \dots, g_n] = [gg_0, gg_1, \dots, gg_n]$. The quotient, BG, has n-simplices given by the orbits of G on the set of n-simplices of EG. The standard notation is: $[g_1|g_2|\dots|g_n]$ for the orbit of $[1, g_1, g_1g_2, \dots, g_1g_2\dots g_n]$. In particular, in BG we have: one vertex []; a loop $[g]$ for each $g \in G$; a triangle $[g|h]$ for every pair $(g, h) \in G \times G$, with boundary glued to the edges $[g]$, $[h]$, $[gh]$.

The tautological n-cochain for the G-action on EG assigns to an n-simplex of EG this same simplex treated as an element of C_nEG. We turn this cochain into a cocycle by dividing the coefficient group by B_nEG—this produces a tautological cocycle in $Z^n(EG, C_nEG/B_nEG)$. We make this cocycle G-invariant by passing to the G-coinvariants U_n of the coefficients:

$$U_n := (C_nEG/B_nEG)_G = (C_nEG)_G/(B_nEG)_G = C_nBG/B_nBG.$$

The resulting G-invariant cocycle descends to an element $T \in Z^n(BG, U_n)$. We also get a cohomology class $\tau \in H^n(BG, U_n) \simeq H^2(G, U_n)$. The inclusion $Z_nBG \to C_nBG$ induces an inclusion $Z_nBG/B_nBG \to C_nBG/B_nBG$, thus exhibiting $H_nBG = H_nG$ as a subgroup of U_n. However, the values of the cocycle T are usually not contained in this subgroup.

Now we specialize to the case of a perfect group G and to $n = 2$. Recall that G is perfect if it has trivial abelianisation, $H_1 G = 0$. (We denote by $H_n G$ the homology group with integer coefficients, $H_n G = H_n(G, \mathbf{Z})$.) It follows that $H^1(G, A) = \mathrm{Hom}(G, A) = 0$ for all abelian groups A.

In this special case, the value $T([g|h])$ is the class of $[g|h]$ in U_2, and

$$\partial[g|h] = [g] - [gh] + [h],$$

which is never zero, so that $T([g|h]) \notin H_2 G$. We will show, however, that T is cohomologous to an $H_2 G$-valued cocycle. For each $g \in G$ the edge $[g]$ in BG is a loop; since G is perfect ($H_1 G = H_1 BG = 0$) this loop is null-homologous. Choose, for each $g \in G$, a 2-chain $n([g]) \in C_2 BG$ so that $\partial(n([g])) = [g]$ (we dub n "the Nekovář correcting chain"). Then $n \in C^1(G, C_2 BG)$, T is cohomologous to $T - \delta n$, and $T - \delta n \in Z^2(G, H_2 BG)$:

$$\partial((T - \delta n)([g|h])) = \partial(T([g|h])) - \partial(n(\partial[g|h])) = [g] - [gh] + [h] - \partial n([g]) + \partial n([gh]) - \partial n([h]) = 0.$$

The cohomology class $[T - \delta n] \in H^2(G, H_2 G)$ is mapped to the cohomology class $[T] \in H^2(G, C_2 BG/B_2 BG)$ by the map $\iota \colon H^2(G, H_2 G) \to H^2(G, C_2 BG/B_2 BG)$ (induced by the coefficients inclusion). However, there is at most one cohomology class in $H^2(G, H_2 G)$ with this property: indeed, from the Bockstein sequence

$$\cdots \longrightarrow H^1(G, C_2 BG/Z_2 BG) \longrightarrow H^2(G, H_2 G) \xrightarrow{\iota} H^2(G, C_2 BG/B_2 BG) \longrightarrow \cdots$$

we see that the map ι is injective (due to G being perfect, $H^1(G, *) = 0$). It follows that the cohomology class of $T - \delta n$ in $H^2(G, H_2 G)$ does not depend on the choice of n.

Definition 2.1 For a perfect group G, the class $u_G = [T - \delta n] \in H^2(G, H_2 G)$ is called the *Moore class* of G.

Remark 2.2 For perfect G the universal coefficients (evaluation) map $H^2(G, A) \to \mathrm{Hom}(H_2 G, A)$ is an isomorphism; in particular, $H^2(G, H_2 G) \simeq \mathrm{Hom}(H_2 G, H_2 G)$. The image of the Moore class under this isomorphism is $id_{H_2 G}$. Indeed, for every homology class $[x] \in H_2 G$ we have

$$\langle u_G, [x] \rangle = \langle [T - \delta n], [x] \rangle = \langle T, x \rangle - \langle \delta n, x \rangle = \langle T, x \rangle = x.$$

This property can serve as another (more standard) definition of the Moore class—a point of view that will reappear in Sect. 6.

II. Central extensions and characteristic classes

Every cohomology class $\tau \in H^2(G, U)$ corresponds to a central extension $\overline{G}$ of G with kernel U. This extension can be used to study τ considered as a characteristic class (i.e. evaluated on G-bundles). The first (that we know of) instance of this sort of study is Milnor's paper [13]. In Sect. 3 we recall what it means to evaluate τ on a bundle P, and how Milnor expressed the result $\tau(P)$ in term of the lifts to $\overline{G}$ of the monodromies of P. In Sect. 4 we use Milnor's expression to prove several formulae computing $\tau(P)$ for a bundle P over a surface in terms of restrictions of P to subsurfaces. This will be used crucially in Sect. 11. In Sect. 5 we discuss twists—natural operations that change bundles. We compute how τ of a bundle changes under twists, and derive an algebraic condition on the corresponding central extension $\overline{G} \to G$ that is equivalent to twist–invariance of τ. We call this algebraic condition *equicommutativity*. In Sect. 6 we discuss the Moore class again, this time as the universal class; this allows us to prove that for perfect G there exists a universal twist-invariant class.

3 Characteristic class in terms of monodromies

In this section G is an arbitrary group, and U is an abelian group. We consider a class $\tau \in H^2(G, U)$. We now recall how this class can be regarded as a characteristic class. Any G-bundle P over a space B has a classifying map, i.e. a map $B \to BG$ (unique up to homotopy) such that the pull-back via this map of the universal G-bundle $EG \to BG$ is isomorphic to P. The pull-back of τ via the classifying map yields an element $\tau(P) \in H^2(B, U)$—the characteristic class (corresponding to τ) of the bundle P. We will be interested in the more specific situation when the base B is a closed surface $\Sigma = \Sigma_g$ of genus $g \geq 1$. The class $\tau(P)$ can then be evaluated on the fundamental class $[\Sigma]$ to yield an element of U. Evaluation on $[\Sigma]$ defines an isomorphism $H^2(\Sigma, U) \to U$ (e.g. by the universal coefficient theorem), so that there is no loss of information in passing from $\tau(P)$ to $\langle \tau(P), [\Sigma] \rangle$.

The following lemma is well-known (cf. [13]).

Lemma 3.1 *Let P be a G-bundle over the surface Σ, and let $\tau \in H^2(G, U)$. Choose loops $a_1, b_1, \ldots, a_g, b_g$ based at the same point that cut Σ into a $4g$-gon and generate $\pi_1(\Sigma)$ with the standard presentation ($\prod_{i=1}^{g}[a_i, b_i] = 1$). Let A_i, B_i be the monodromy of P along a_i, b_i respectively. Then*

$$\langle \tau(P), [\Sigma] \rangle = \prod_{i=1}^{g} [\overline{A}_i, \overline{B}_i], \tag{3.1}$$

where, for $g \in G$, we denote by $\overline{g}$ a lift of g to the central extension G^τ of G determined by τ.

The central extension G^τ mentioned in the lemma can be described as follows. Let τ be represented by a homogeneous cocycle $z\colon G \times G \times G \to U$. The associated non-homogeneous cocycle is given by $c(g, h) = z(1, g, gh)$. Then, on the set $G \times U$, we define the multiplication by

$$(g, u) \cdot (g', u') = (gg', uu' \cdot c(g, g')). \tag{3.2}$$

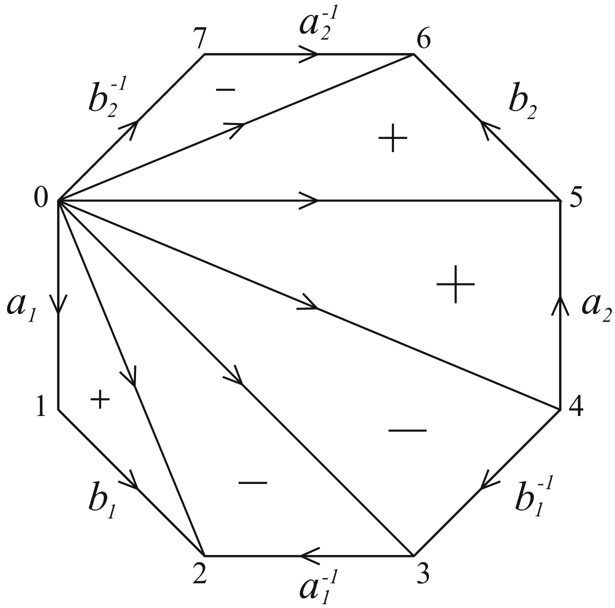

Fig. 1 The polygon Q with labels for $g = 2$. The Δ-complex structure on Σ is determined by the arrows. The fundamental cycle of Σ is the sum of the triangles with the indicated signs. The map f maps $\Delta_1 = (0, 1, 2)$ to $[g_1|B_1] = [A_1|B_1]$, $\Delta_2 = (0, 3, 2)$ to $[g_3|A_1] = [A_1 B_1 A_1^{-1}|A_1]$, etc.

The cocycle condition is equivalent to associativity. We will typically use the standard (set-theoretic) lift of G to G^τ : $\overline{g} = (g, 1)$. The abelian group U (=$\{1\} \times U$) is contained in the centre of G^τ . To prove this one checks that $c(g, 1) = c(1, g)$ (by setting $g = h$ in the cocycle condition $c(g, 1)c(g \cdot 1, h) = c(g, 1 \cdot h)c(1, h)$). We will abbreviate $(1, u)$ to u, and use multiplicative notation in U. (Eventually, for $U = W(K)$, we will switch to the additive convention.)

Conversely, for any central extension $1 \to U \to \overline{G} \to G \to 1$ we may choose a set theoretic lift $G \ni g \mapsto \overline{g} \in \overline{G}$ and define $c\colon G \times G \to U$ by $\overline{g} \cdot \overline{h} = \overline{gh} \cdot c(g, h)$. A change of the lift changes c within its cohomology class. We will always assume that the lift of the neutral element of G is the neutral element of $\overline{G}$ (more obscurely: $\overline{1} = 1$). This assumption implies that $c(1, g) = c(g, 1) = 1$ for all $g \in G$.

A more thorough discussion of central extensions can be found in [1, Chapter IV].

Proof of Lemma 3.1 (cf. [13]). Let us first describe the 2-skeleton of the model of BG that is suitable for us (cf. [8, Example 1B.7]). There is one vertex; for each element $g \in G$ there is an (oriented) edge-loop $[g]$—these form the 1-skeleton. Then, for every pair $(g, h) \in G^2$, there is a triangle $[g|h]$, with sides glued to the 1-skeleton along $[g]$, $[h]$ and $[gh]^{-1}$ (going around). A cocycle $c \in H^2(G, A) \cong H^2(BG, A)$ evaluates on the triangles as $c([g|h]) = c(g, h)$.

The surface Σ can be expressed as a (convex) polygon Q, with $4g$ sides suitably glued in pairs. We label the vertices of Q by $0, 1, \ldots, 4g-1$ (counterclockwise) and the edges (starting by $(0, 1)$ and continuing counterclockwise) by $a_1, b_1, a_1^{-1}, \ldots, b_g^{-1}$.

We also put a Δ-complex structure (cf. [8, Section 2.1]) on Σ. We divide Q into triangles, drawing line segments $(0, i)$ for $i = 2, 3 \ldots, 4g - 2$, as shown in Fig. 1.

Then we order the vertices along each edge, and in each triangle of the triangulation, in a compatible way. The orders are indicated by arrows in Fig. 1. The arrows are compatible with the boundary gluings, hence we get a Δ-complex structure on Σ. (Notice that the arrows on the boundary edges cannot all be directed counterclockwise because of the requirement of compatibility with the gluings.)

Now we describe a classifying map $f\colon \Sigma \to BG$ of the bundle P. For convenience, let $c_i^{\epsilon_i}$ be the label of the edge $(i, i+1)$, and let $C_i^{\epsilon_i}$ (equal to some $A_j^{\pm 1}$ or $B_j^{\pm 1}$) be the monodromy along that edge. The map f sends the edge $(i, i+1)$ to $[C_i]^{\epsilon_i}$ (not to $[C_i^{\epsilon_i}]$!). We map $(0, i)$ to $[g_i]$, where $g_i = C_0^{\epsilon_0} \cdot \ldots \cdot C_{i-1}^{\epsilon_{i-1}}$. For $\epsilon_i = +1$ we define the triangle $\Delta_i = (0, i, i+1)$ and map it to $[g_i|C_i]$; for $\epsilon_i = -1$ we define the triangle $\Delta_i = (0, i+1, i)$ and map it to $[g_{i+1}|C_i]$. Then the fundamental class of Σ is represented by the cycle $\sum_{i=1}^{4g-2} \epsilon_i \Delta_i$, mapped by f to the cycle

$$\sum_{i|\epsilon_i>0} [g_i|C_i] - \sum_{i|\epsilon_i<0} [g_{i+1}|C_i] \tag{3.3}$$

on which the cocycle c evaluates to

$$\langle \tau(P), [\Sigma] \rangle = \prod_{i|\epsilon_i>0} c(g_i, C_i) \prod_{i|\epsilon_i<0} c(g_{i+1}, C_i)^{-1}. \tag{3.4}$$

On the other hand, $\prod_{i=1}^{g} [\overline{A}_i, \overline{B}_i] = \prod_{i=0}^{4g-1} \overline{C}_i^{\epsilon_i}$. Using the identities

$$\overline{g} \cdot \overline{h} = \overline{gh} \cdot c(g, h), \qquad \overline{g} \cdot \overline{h}^{-1} = \overline{gh^{-1}} \cdot c(gh^{-1}, h)^{-1} \tag{3.5}$$

we see that

$$\begin{aligned} &\overline{\prod_{i=0}^{k-1} C_i^{\epsilon_i}} \cdot \overline{C}_k^{\epsilon_k} = \overline{g}_k \cdot \overline{C}_k^{\epsilon_k} \\ &= \begin{cases} \overline{g}_{k+1} \cdot c(g_k, C_k) = \overline{\prod_{i=0}^{k} C_i^{\epsilon_i}} \cdot c(g_k, C_k) & \text{if} \epsilon_k = +1, \\ \overline{g}_{k+1} \cdot c(g_{k+1}, C_k)^{-1} = \overline{\prod_{i=0}^{k} C_i^{\epsilon_i}} \cdot c(g_{k+1}, C_k)^{-1} & \text{if} \epsilon_k = -1. \end{cases} \end{aligned} \tag{3.6}$$

We apply this inductively; since $\overline{\prod_{i=0}^{4g-1} C_i^{\epsilon_i}} = \overline{1} = 1$, we end up with

$$\prod_{i=0}^{4g-1} \overline{C_i}^{\epsilon_i} = \prod_{i|\epsilon_i>0} c(g_i, C_i) \prod_{i|\epsilon_i<0} c(g_{i+1}, C_i)^{-1} = \langle \tau(P), [\Sigma] \rangle. \tag{3.7}$$

The lemma is proved. ◇

4 Gluing formulae

Let $1 \to U \to \overline{G} \to G \to 1$ be an arbitrary central group extension. We choose and fix a set-theoretic lift $G \to \overline{G}$, to be denoted $g \mapsto \overline{g}$. We call it the standard lift; we assume that it satisfies $\overline{1} = 1$. (We will sometimes use $\widetilde{g}$ to denote other, non–standard lifts of g.) We denote by c the corresponding cocycle (so that $\overline{g} \cdot \overline{h} = \overline{gh} \cdot c(g, h)$) and by τ_c its cohomology class ($\tau_c \in H^2(G, U)$).

Suppose that ξ is a flat G-bundle over an oriented compact surface S of genus g with one boundary component. A *boundary framing* of ξ will mean the following collection of data: a point $s \in \partial S$; a trivialization of the fibre ξ_s; an element $\widetilde{W} \in \overline{G}$ that lifts the monodromy $W \in G$ of ξ along ∂S (oriented compatibly with the orientation of S and based at s). We will often abusively say "boundary framing $\widetilde{W}$", because $\widetilde{W}$ is the part of the framing data that presumes the rest of it and appears explicitly in many formulas. Usually we will use $\widetilde{W} = \overline{W}$—the standard lift of W—and then call our framing *a standard framing*.

A *standard loop collection*, i.e. a collection of loops $(x_i, y_i \mid i = 1, \dots, g)$, based at s, cutting Σ into a $(4g+1)$-gon, generating $\pi_1(S)$ and satisfying $[x_1, y_1] \cdot [x_2, y_2] \cdot \dots \cdot [x_g, y_g] = \partial S$ can be chosen (in many ways). Let $X_i, Y_i \in G$ be the monodromies of ξ along these loops.

Definition 4.1 The *relative class* $\overline{c}(\xi, \widetilde{W})$ of the bundle ξ with boundary framing $\widetilde{W}$ is defined as the element of U given by

$$\overline{c}(\xi, \widetilde{W}) := [\overline{X}_1, \overline{Y}_1] \cdot [\overline{X}_2, \overline{Y}_2] \cdot \dots \cdot [\overline{X}_g, \overline{Y}_g] \cdot \widetilde{W}^{-1}. \tag{4.1}$$

If the framing is standard (i.e. $\widetilde{W} = \overline{W}$), then we put $\overline{c}(\xi) = \overline{c}(\xi, \overline{W})$. (The bar over c is a reminder that the class is relative, and some framing is presumed.)

Remark 4.2 The name "relative class" is intentionally provocative. We expect that there exists a relative characteristic class $\overline{\tau}_c$ related to τ_c such that $\overline{\tau}_c(\xi)$ evaluated on the relative fundamental class of the base of ξ equals $\overline{c}(\xi)$.

Remark 4.3 Notice that since $\prod[x_i, y_i] = \partial S$, we have $\prod[X_i, Y_i] = W$; it follows that $\overline{c}(\xi) \in U$. The commutator $[\overline{X}_i, \overline{Y}_i]$ does not depend on the choice of the lifts of X_i, Y_i to $\overline{G}$: $[\widetilde{X}_i, \widetilde{Y}_i] = [\overline{X}_i, \overline{Y}_i]$. We will use different lifts to our advantage. On the other hand, changing $\widetilde{W}$ to a different lift $\widetilde{W}u$ of W results in a change:

$$\overline{c}(\xi, \widetilde{W}u) = \overline{c}(\xi, \widetilde{W}) \cdot u^{-1}. \tag{4.2}$$

It is not a priori clear that $\overline{c}(\xi)$ does not depend on the choice of the collection of loops (x_i, y_i); we will check this shortly.

If a closed surface is cut along a separating simple loop into two pieces, a G-bundle over that surface decomposes into two bundles over the pieces. The next lemma describes the relation between the (relative) classes of the three bundles.

Lemma 4.4 *Let ξ, ξ' be flat G-bundles over oriented surfaces S, S' with isomorphic boundary framings $\widetilde{W}$. The isomorphism of boundary framings allows one to glue the* *bundles ξ, ξ'; the result is a bundle $\xi \cup \xi'$ over $\Sigma = S \cup_\partial S'$. We orient Σ compatibly with S and opposite to S'. Then*

$$\langle \tau_c(\xi \cup \xi'), [\Sigma] \rangle = \overline{c}(\xi, \widetilde{W}) \overline{c}(\xi', \widetilde{W})^{-1}. \tag{4.3}$$

Proof Let (x_i, y_i) be a standard collection of loops for S, (x_i', y_i') one for S'. Then these collections together, in the order $(x_1, y_1, \dots, x_g, y_g, y'_{g'}, x'_{g'}, \dots, y'_1, x'_1)$, give a standard set of generators of $\pi_1(S \cup_\partial S')$. We have

$$\begin{aligned}\langle \tau_c(\xi \cup \xi'), [\Sigma] \rangle &= [\overline{X}_1, \overline{Y}_1] \dots [\overline{X}_g, \overline{Y}_g][\overline{Y'}_{g'}, \overline{X'}_{g'}] \dots [\overline{Y'}_1, \overline{X'}_1] \\ &= [\overline{X}_1, \overline{Y}_1] \dots [\overline{X}_g, \overline{Y}_g] \cdot \widetilde{W}^{-1} \cdot \widetilde{W} \cdot [\overline{X'}_{g'}, \overline{Y'}_{g'}]^{-1} \dots [\overline{X'}_1, \overline{Y'}_1]^{-1} \\ &= [\overline{X}_1, \overline{Y}_1] \dots [\overline{X}_g, \overline{Y}_g] \cdot \widetilde{W}^{-1} \cdot \left([\overline{X'}_1, \overline{Y'}_1] \dots [\overline{X'}_{g'}, \overline{Y'}_{g'}] \cdot \widetilde{W}^{-1} \right)^{-1} \\ &= \overline{c}(\xi, \widetilde{W}) \overline{c}(\xi', \widetilde{W})^{-1}. \end{aligned} \tag{4.4}$$

◇

Corollary 4.5 *The relative class $\overline{c}(\xi, \widetilde{W})$ does not depend on the choice of a standard loop collection (x_i, y_i).*

Proof Choose any (ξ', S') with the same boundary framing as (ξ, S) (it can be just another copy of (ξ, S)). Then compute $\tau_c(\xi \cup \xi')$ as in Lemma 4.4. We get $\tau_c(\xi \cup \xi') = \overline{c}(\xi, \widetilde{W}) \overline{c}(\xi', \widetilde{W})^{-1}$ regardless of the choices of the collections (x_i, y_i) and (x_i', y_i'). Varying one of these collections while keeping the other fixed we see the claimed independence. ◇

Now we set the notation for Lemma 4.6. Let ξ be a bundle over S with a standard boundary framing, and let $A \in G$. Then we can change (twist) the framing by A. This means that we change the trivialization $\xi_s \to G$ by A; then all monodromies M change to ${}^A M := AMA^{-1}$. In particular, W changes to ${}^A W$, and the standard framing of the A-twisted bundle is $\overline{{}^A W}$. We denote by ${}^A \xi$ the twisted bundle with this framing. Another natural choice of framing is ${}^A \overline{W}$. (Formally, we should use ${}^{\overline{A}} \overline{W} = \overline{A} \cdot \overline{W} \cdot \overline{A}^{-1}$, but this does not depend on the choice of the lift $\overline{A}$ of A and will usually be abbreviated to ${}^A \overline{W}$.) In general ${}^A \overline{W} \neq \overline{{}^A W}$, so that we have two natural A-twisted bundles with boundary framing: $(\xi, {}^A \overline{W})$ and ${}^A \xi = (\xi, \overline{{}^A W})$.

Lemma 4.6 *We have*

$$\overline{c}(\xi, {}^A \overline{W}) = \overline{c}(\xi, \overline{W}) = \overline{c}(\xi), \quad \overline{c}({}^A \xi) = \overline{c}(\xi, \overline{{}^A W}) = \overline{c}(\xi) c(A, W) c({}^A W, A)^{-1}. \tag{4.5}$$

Proof Let (x_i, y_i) be a standard collection of loops in S, and let (X_i, Y_i) be the monodromies of ξ along these loops. The monodromies for the A-twisted ξ are $({}^A X_i, {}^A Y_i)$. Since the commutator of lifts does not depend on the choice of the lifts, we have

$[\overline{{}^A X_i}, \overline{{}^A Y_i}] = [{}^A\overline{X_i}, {}^A\overline{Y_i}]$. Therefore

$$\begin{aligned}\overline{c}(\xi, {}^A\overline{W}) &= \left(\prod[\overline{{}^A X_i}, \overline{{}^A Y_i}]\right)\left({}^A\overline{W}\right)^{-1} = \left(\prod {}^A[\overline{X_i}, \overline{Y_i}]\right) {}^A\overline{W}^{-1} \\ &= {}^A\left(\prod[\overline{X_i}, \overline{Y_i}] \cdot \overline{W}^{-1}\right) = {}^A\overline{c}(\xi) = \overline{c}(\xi).\end{aligned} \tag{4.6}$$

(Conjugation does not change the central element $\overline{c}(\xi)$.) We compare this with $\overline{c}({}^A\xi)$ using the standard lift $\overline{A}$ of A:

$$\begin{aligned}{}^A\overline{W} &= \overline{A}\cdot\overline{W}\cdot\overline{A}^{-1} = \overline{AW}c(A, W)\overline{A}^{-1} = \overline{AWA^{-1}}c(AWA^{-1}, A)^{-1}c(A, W) \\ &= \overline{{}^A W}c(A, W)c({}^A W, A)^{-1}.\end{aligned} \tag{4.7}$$

Now the second formula follows from (4.2) and (4.7):

$$\begin{aligned}\overline{c}(\xi, \overline{{}^A W}) &= \overline{c}(\xi, {}^A\overline{W}c(A, W)^{-1}c({}^A W, A)) = \overline{c}(\xi, {}^A\overline{W})c(A, W)c({}^A W, A)^{-1} \\ &= \overline{c}(\xi)c(A, W)c({}^A W, A)^{-1}.\end{aligned} \tag{4.8}$$

◇

Now we set up notation for Lemma 4.7 and give another gluing construction, called "boundary connected sum". Let ξ, ξ' be bundles with standard boundary framings $\overline{W}$, $\overline{W'}$ over S, S'. We glue s to s' as well as the fibres $\xi_s, \xi'_{s'}$ (via the framing trivializations). Then we glue a triangle Δ to $\partial S \vee \partial S'$, one side along ∂S, one along $\partial S'$ (all vertices to $s = s'$), so that the path $(\partial S)(\partial S')$ is homotopic (through Δ) to the third side. This third side forms the boundary of the obtained surface Σ of genus $g + g'$. The bundle naturally extends to a bundle $\xi \vee \xi'$ over Σ (all monodromies are already visible in ξ and ξ'). The trivializations at $s = s'$ agree and trivialize the new fibre at this point. The boundary monodromy of $\xi \vee \xi'$ is WW'; we use the standard boundary framing, with lift $\overline{WW'}$.

Lemma 4.7 *In the above situation,*

$$\overline{c}(\xi \vee \xi') = \overline{c}(\xi \vee \xi', \overline{WW'}) = \overline{c}(\xi)\overline{c}(\xi')c(W, W'). \tag{4.9}$$

Proof Let (x_i, y_i) and (x'_j, y'_j) be standard loop collections in S, S'. Together they form a standard loop collection in Σ, so that

$$\begin{aligned}\overline{c}(\xi \vee \xi', \overline{WW'}) &= \prod[\overline{X}_i, \overline{Y}_i]\prod[\overline{X'}_j, \overline{Y'}_j]\cdot\overline{WW'}^{-1} = \overline{W}\overline{c}(\xi)\overline{W'}\,\overline{c}(\xi')\overline{WW'}^{-1} \\ &= \overline{W}\cdot\overline{W'}\cdot\overline{WW'}^{-1}\cdot\overline{c}(\xi)\overline{c}(\xi') = c(W, W')\overline{c}(\xi)\overline{c}(\xi').\end{aligned} \tag{4.10}$$

◇

One last piece of general calculation:

Lemma 4.8 *Let ξ be a bundle over a surface with one boundary component and genus 1 with standard boundary framing, standard loop generators (x, y) and monodromies (X, Y) (with $[X, Y] = W$). Then*

$$\overline{c}(\xi) = c(X, Y)c(Y, X)^{-1}c(W, YX)^{-1}. \tag{4.11}$$

Proof

$$\begin{aligned}\overline{c}(\xi) &= [\overline{X}, \overline{Y}]\overline{W}^{-1} = \overline{X}\cdot\overline{Y}\cdot(\overline{Y}\cdot\overline{X})^{-1}\overline{W}^{-1} = \overline{XY}c(X, Y)(\overline{YX}c(Y, X))^{-1}\overline{W}^{-1}\\ &= \overline{XY}\cdot\overline{YX}^{-1}\overline{W}^{-1}c(X, Y)c(Y, X)^{-1}\\ &= \overline{XY(YX)^{-1}}c(XY(YX)^{-1}, YX)^{-1}\overline{W}^{-1}c(X, Y)c(Y, X)^{-1}\\ &= \overline{W}\cdot\overline{W}^{-1}c(X, Y)c(Y, X)^{-1}c(W, YX)^{-1} = c(X, Y)c(Y, X)^{-1}c(W, YX)^{-1}.\end{aligned} \tag{4.12}$$

◇

5 Twists and equicommutativity

In this section we discuss the twist deformations of flat bundles over surfaces. These twists are associated to the names of Fenchel and Nielsen in the Teichmüller case (cf. [20]), and to Goldman in the Lie group case (cf. [6]).

Let P be a (flat) G-bundle over an oriented surface Σ. Choose an oriented simple loop ℓ in Σ, and a base-point $b \in \ell$. Trivialize P_b (by a right-G-equivariant isomorphism $P_b \to G$). Then, there is a well-defined element $L \in G$ representing the monodromy of P along ℓ. Choose any $V \in Z_G(L)$ (the centralizer of L in G), and trivialize the bundle P along ℓ (with ambiguity L at b). Then cut Σ and P along ℓ, and glue it back by (left) multiplication by V. (To be precise, we define the right-hand side and the left-hand side of a tubular neighbourhood of ℓ in Σ using orientations. Then, after cutting the bundle, each element p of a trivialized fibre P_x at a point $x \in \ell$ is split into a left-right pair p_L, p_R. We glue p_R to Vp_L. Since the trivialization along ℓ is L-ambiguous at b, the gluing over b is well-defined only for $V \in Z_G(L)$.) The result is a new (flat) G-bundle $P_{\ell,V}$ over Σ—the *twist* of P by V along ℓ.

It is possible to phrase the above definition in a slightly more invariant way. Suppose we refrain from choosing a trivialization of P_b. Then we still have the monodromy along ℓ. It is an element L in $\mathrm{Aut}(P_b)$, the automorphism group of the right G-space P_b. (This $\mathrm{Aut}(P_b)$ is non-canonically isomorphic to G; possible isomorphisms arise from trivializations of P_b.) Then for any $V \in Z_{\mathrm{Aut}(P_b)}(L)$ the bundle $P_{\ell,V}$ is well-defined.

Now suppose that we have a central group extension $1 \to U \to \overline{G} \to G \to 1$, as in the previous section (with a lift $g \mapsto \overline{g}$, cocycle c, cohomology class $\tau \in H^2(G, U)$).

Theorem 5.1 *Let P be a G-bundle over a closed oriented surface Σ. Let ℓ be an oriented simple loop in Σ, based at b. Let $L \in G$ be the monodromy of P along ℓ*

(with respect to some trivialization of P_b), let $V \in Z_G(L)$, and let $P_{\ell,V}$ be the twist of P. Let $\tau \in H^2(G,U)$ be a cohomology class represented by a cocycle c. Then

$$\langle \tau(P_{\ell,V}), \Sigma\rangle = \langle \tau(P), [\Sigma]\rangle c(V,L)c(L,V)^{-1}. \tag{5.1}$$

Proof We use the commutator product expression from Lemma 3.1. The basic calculation (valid for any two commuting elements $V, L \in G$) is

$$\begin{aligned}\overline{V}\,\overline{L} &= \overline{VL}c(V,L) = \overline{LV}c(V,L) = \overline{L}\,\overline{V}c(L,V)^{-1}c(V,L)\\ &= \overline{L}\,\overline{V}c(V,L)c(L,V)^{-1}.\end{aligned} \tag{5.2}$$

Case 1. The loop ℓ does not separate Σ. Then the standard presentation loops $a_1,\ldots,b_g$ in Σ can be chosen so that $b_1 = \ell$. If $A_1,\ldots,B_g$ are the elements of G representing the monodromies of the bundle P along $a_1,\ldots,b_g$ (with $B_1 = L$), then the monodromies of $P_{\ell,V}$ along these loops are represented by the same elements except for one change: A_1 gets replaced by A_1V. In the commutator product expression the first term $[\overline{A}_1, \overline{B}_1]$ $(= [\overline{A}_1, \overline{L}])$ changes to $[\overline{A_1V}, \overline{B}_1]$ $(= [\overline{A_1V}, \overline{L}])$. Using (5.2) we compute:

$$\begin{aligned}[\overline{A_1V}, \overline{L}] &= [\overline{A}_1\overline{V}c(A_1,V)^{-1}, \overline{L}] = [\overline{A}_1\overline{V}, \overline{L}]\\ &= \overline{A}_1\overline{V}\,\overline{L}\,\overline{V}^{-1}\overline{A}_1^{-1}\overline{L}^{-1} = \overline{A}_1\overline{L}\,\overline{V}c(V,L)c(L,V)^{-1}\overline{V}^{-1}\overline{A}_1^{-1}\overline{L}^{-1}\\ &= [\overline{A}_1, \overline{L}]c(V,L)c(L,V)^{-1}.\end{aligned} \tag{5.3}$$

The claim follows.

Case 2. The loop ℓ separates Σ. Then we cut Σ and P along ℓ into two components, say P_0 over Σ_0 and P_1 over Σ_1. The assumptions of the theorem induce (isomorphic) boundary framings for P_0 and P_1 (except for lifts of the boundary monodromy L—we take standard lifts). We have $P = P_0 \cup P_1$, $P_{\ell,V} = {}^VP_0 \cup P_1$. Lemmas 4.4, 4.6 give

$$\begin{aligned}\langle \tau(P_{\ell,V}), [\Sigma]\rangle &= \overline{c}({}^VP_0)\overline{c}(P_1)^{-1} = \overline{c}(P_0)c(V,L)c({}^VL,V)^{-1}\overline{c}(P_1)^{-1}\\ &= \langle \tau(P), [\Sigma]\rangle c(V,L)c(L,V)^{-1}.\end{aligned} \tag{5.4}$$

The last equality uses the fact that ${}^VL = L$, a consequence of $V \in Z_G(L)$. ◇

This theorem leads to the following definition.

Definition 5.2 A cocycle $c\colon G \times G \to U$ is called *equicommutative*, if it satisfies $c(g,h) = c(h,g)$ whenever $gh = hg$. A cohomology class is *equicommutative* if it is represented by an equicommutative cocycle.

Proposition 5.3 *Let c be a cocycle with cohomology class $\tau_c \in H^2(G,U)$, let $U \to \overline{G} \to G$ be the corresponding central extension, and let $g \mapsto \overline{g}$ be the lift corresponding to c. The following conditions are equivalent*:

(a) *for every commuting pair $g, h \in G$, the lifts $\overline{g}, \overline{h} \in \overline{G}$ commute*;
(b) *the cocycle c is equicommutative*;

(c) *the cohomology class* τ_c *is equicommutative*;
(d) *for every commuting pair* $g, h \in G$, *every lift of* g *commutes with every lift of* h;
(e) *every cocycle representing* τ_c *is equicommutative*;
(f) *for every commutative subgroup* $H < G$ *the pre-image of* H *in* $\overline{G}$ *is commutative.*

Proof It is straightforward to see that:

- the weak conditions (a), (b) are equivalent;
- the strong conditions (d), (e), (f) are equivalent;
- the strong conditions imply the weak conditions;
- (b) implies (c).

To finish the proof we show that (c) implies (e). It is enough to check that all 2-coboundaries are equicommutative. Let $g, h \in G$ be commuting elements, and let $n \in C^1(G, U)$. Then

$$(\delta n)(g,h) = n(h)n(gh)^{-1}n(g) = n(g)n(hg)^{-1}n(h) = (\delta n)(h,g). \tag{5.5}$$

◇

Definition 5.4 A cohomology class $\tau \in H^2(G, U)$ is *twist–invariant*, if for every G-bundle P over a closed oriented surface Σ, and for every twist $P_{\ell,V}$ of that bundle, we have $\tau(P_{\ell,V}) = \tau(P)$.

In Definition 5.4 one could, equivalently, use the condition $\langle \tau(P_{\ell,V}), [\Sigma]\rangle = \langle \tau(P), [\Sigma]\rangle$. This is because evaluation on $[\Sigma]$ is an isomorphism $H^2(\Sigma, U) \to U$.

Corollary 5.5 *A cohomology class is twist-invariant if and only if it is equicommutative.*

Proof It follows from Theorem 5.1 that equicommutativity implies twist-invariance. For the converse, let c be a cocycle representing a twist-invariant class $\tau \in H^2(G, U)$. For any pair of commuting elements $g, h \in G$ there exists a G-bundle $\xi_{(g,h)}$ over $\Sigma_1 = T^2$ with monodromies (along standard generating loops a_1, b_2) equal to g, h. This bundle is a twist of $\xi_{(1,h)}$, and Theorem 5.1 gives:

$$\tau(\xi_{(g,h)}) = \tau(\xi_{(1,h)})c(g,h)c(h,g)^{-1}. \tag{5.6}$$

Now the assumption of twist-invariance implies that $c(g,h) = c(h,g)$. ◇

Corollary 5.6 *The Witt class is twist-invariant.*

Proof We check that the cocycle (1.4) is equicommutative. Let $a, b \in SL(2, K)$, and suppose that $ab = ba$. Then

$$w(a,b) = [-a_{21}(ab)_{21}b_{21}] = [-a_{21}(ba)_{21}b_{21}] = w(b,a). \tag{5.7}$$

◇

Remark 5.7 Unlike the Witt class, the Moore class, in general, is not twist–invariant. This is more fully explained in Sect. 8.

6 Universal classes

In this section we specialize our discussion of characteristic classes to perfect groups. Recall that a group G is perfect if it is equal to its commutator subgroup $[G, G]$. In homological terms this means that $H_1G = 0$ (we denote $H_i(G, \mathbf{Z})$ by H_iG). Consequently, the universal coefficients map $H^2(G, A) \to \mathrm{Hom}(H_2G, A)$ is an isomorphism for every abelian group A. In particular, for $A = H_2G$ we have a well-defined class $u_G \in H^2(G, H_2G)$ that corresponds to id_{H_2G} under this isomorphism. (Thus, by Remark 2.2, u_G coincides with the class constructed in Sect. 2.) The class u_G is universal in the following sense: for any abelian group A and any class $v \in H^2(G, A)$ there exists a unique homomorphism $f : H_2G \to A$ such that $f_*u_G = v$ (cf. [1, Exercise IV.3.7]). Also the central extension $1 \to H_2G \to \overline{G} \to G \to 1$ defined by u_G is universal. (This extension was one of the early reasons for considering the second homology of a group; whence the name "Schur multiplier" for H_2G.) It is known that all universal central extensions of a perfect group G are canonically isomorphic. Quite often the construction of such an extension and the study of its kernel is the way to calculate H_2G and to describe u_G. We call the class u_G the Moore class, because it was investigated by Moore for $G = SL(2, K)$.

Definition 6.1 Let G be a perfect group, and let u be a cocycle representing the universal class $u_G \in H^2(G, H_2G)$. Let $G^{[,]}$ be the subgroup of H_2G generated by the set

$$\{u(g, h)u(h, g)^{-1} \mid g, h \in G, gh = hg\}. \tag{6.1}$$

We put $Eq(G) := H_2G/G^{[,]}$. Let $w_G \in H^2(G, Eq(G))$ be the image of the universal class u_G by the quotient map $q : H_2G \to Eq(G)$.

Remark 6.2 The set (6.1) does not depend on the choice of the cocycle u. Indeed, it can be described as the set of commutators, in $\overline{G}$, of lifts of pairs of commuting elements $g, h \in G$, as the following calculation shows:

$$\begin{aligned} u(g, h)u(h, g)^{-1} &= \overline{gh}^{-1}\overline{g}\,\overline{h}\left(\overline{hg}^{-1}\overline{h}\,\overline{g}\right)^{-1} = \overline{gh}^{-1}\overline{g}\,\overline{h}\,\overline{g}^{-1}\overline{h}^{-1}\overline{hg} \\ &= \overline{gh}^{-1}[\overline{g}, \overline{h}]\overline{hg} = [\overline{g}, \overline{h}]. \end{aligned} \tag{6.2}$$

The last equality follows from the fact that for commuting g, h the commutator $[\overline{g}, \overline{h}]$ is central in $\overline{G}$. This commutator does not depend on the choice of lifts of g, h, because all possible lifts are of the form $\overline{g}u$, $\overline{h}v$ with u, v central in $\overline{G}$.

Remark 6.3 Another, more topological description of the set (6.1): it consists of "genus 1 classes", i.e. the classes in H_2G that are images of the fundamental class of the 2-dimensional torus T^2 under some map $T^2 \to BG$. Indeed, such a map associates to the generators of $\pi_1(T^2)$ an (arbitrary) commuting pair $g, h \in G$; the image of the fundamental class of T^2 is then $u(g, h)u(h, g)^{-1}$ by the computation in the proof of Lemma 3.1.

Theorem 6.4 *Let G be a perfect group. The class $w_G \in H^2(G, Eq(G))$ is a universal equicommutative class in the following sense: for every equicommutative cohomology class $v \in H^2(G, A)$ there exists a unique homomorphism $g\colon Eq(G) \to A$ such that $g_* w_G = v$.*

Proof Let $f\colon H_2G \to A$ be the homomorphism that maps u_G to v. Choose a cocycle u representing u_G. Then f_*u is a cocycle representing v, hence it is equicommutative. It follows that, for every commuting pair $g, h \in G$, we have

$$f\left(u(g,h)u(h,g)^{-1}\right) = f_*u(g,h) \cdot f_*u(h,g)^{-1} = 1; \tag{6.3}$$

therefore f factors through the quotient map $q\colon H_2G \to Eq(G)$, i.e. $f = g \circ q$ for some $g\colon Eq(G) \to A$. We get

$$g_* w_G = g_* q_* u_G = f_* u_G = v. \tag{6.4}$$

The uniqueness statement is proved by contradiction. Suppose two different homomorphisms $g, g'\colon Eq(G) \to A$ map w_G to v; then $g \circ q, g' \circ q\colon H_2G \to A$ are different, and both map u_G to v—contradiction. $\diamond$

Remark 6.5 The class $w_G \in H^2(G, Eq(G))$ is, up to a unique isomorphism, the unique universal equicommutative class; this is a standard consequence of universality.

We finish this section by indicating a more general point of view on universality. It will not be used later in this paper.

Proposition 6.6 *Let G be a perfect group, $u_G \in H^2(G, H_2G)$ its Moore class, and let $\varphi\colon H_2G \to Q$ be a group epimorphism (coefficient reduction map). We set $u_{G,Q} = \varphi_* u_G$. Suppose that a cohomology class $v \in H^2(G, A)$ satisfies the following condition: $v(x) = 0$ for all $x \in \ker\varphi$. Then there exists a unique group homomorphism $\psi\colon Q \to A$ such that $\psi_* u_{G,Q} = v$.*

Proof Let $\Psi\colon H_2G \to A$ be the unique map giving $\Psi_* u_G = v$. For each $x \in \ker\varphi$ we have:

$$\Psi(x) = \Psi(u_G(x)) = \Psi_* u_G(x) = v(x) = 0.$$

It follows that there exists a $\psi\colon Q \to A$ such that $\psi \circ \varphi = \Psi$. Then

$$\psi_* u_{G,Q} = \psi_* \varphi_* u_G = \Psi_* u_G = v.$$

Now we show that the homomorphism ψ is unique. Suppose that ψ and ψ' satisfy the conditions of the theorem. Then $\psi \circ \varphi, \psi' \circ \varphi\colon H_2G \to A$ are coefficient maps that map u_G to v; thus, these maps are equal, by the universality property of u_G. Since φ is epimorphic, we deduce $\psi = \psi'$. $\diamond$

III. ***SL(2,K)***

In this part we specialize our considerations to the discrete group $SL(2, K)$, where K is an infinite field. This group is perfect, so that the results of Sect. 6 apply. Our main result is that the (reduced) Witt class is the universal equicommutative class for this group (Theorem 10.1). The proof relies on several known results which we review carefully. The Schur multiplier of $SL(2, K)$, denoted $\pi_1(SL(2, K))$ henceforth to honour Calvin Moore, is classically described by generators and relations; we recall this description in Sect. 7. The generators are "symbols" $\{a, b\}$, $a, b \in K^*$. (We use K^* to denote $K\setminus\{0\}$.) In the quotient $Eq(SL(2, K))$ of $\pi_1(SL(2, K))$ the symbols become symmetric: $\{a, b\} = \{b, a\}$. We are thus led to consider the group $\pi_1(SL(2, K))/\text{sym}$, defined by adjoining to the classical presentation of $\pi_1(SL(2, K))$ all the symbol symmetry relations $\{a, b\} = \{b, a\}$, $a, b \in K^*$. This group is a natural mid-step in the quotient sequence

$$\pi_1(SL(2, K)) \to \pi_1(SL(2, K))/\text{sym} \to Eq(SL(2, K)).$$

In Sect. 8 we show that in fact $\pi_1(SL(2, K))/\text{sym} \simeq I^2(K)$ (here $I^2(K)$ is the square of the fundamental ideal $I(K)$ of the Witt ring $W(K)$, cf. [3, Chapter I]). A cocycle b representing the universal class $u_{SL(2,K)}$ was given explicitly (though slightly erroneously) by Moore; we recall the correct description in Sect. 9. There we also present the results of Nekovář, and Kramer and Tent, proving that the image of the Moore cocycle b in $I^2(K)$ is cohomologous to the (reduced) Witt cocycle. In Sect. 10 we use this compatibility to show that $I^2(K) \simeq Eq(SL(2, K))$ and that the reduced Witt class is (equivalent to) the universal equicommutative class.

7 Schur multiplier of $SL(2, K)$

In this section we recall the standard description of the universal central extension and of the Schur multiplier of $SL(2, K)$. (As always, we assume that K is an infinite field.) The classical references are [15, Sections 8,9], [12], [17, §7].

The universal central extension of $SL(2, K)$ is called the Steinberg group and is denoted $St(2, K)$. It is generated by two families of symbols: $x_{12}(t)$, $t \in K$; $x_{21}(t)$, $t \in K$. For $t \in K^*$ one defines an auxiliary element $w_{ij}(t) = x_{ij}(t)x_{ji}(-t^{-1})x_{ij}(t)$; then the relations defining $St(2, K)$ are:

$$\begin{aligned} x_{ij}(t)x_{ij}(s) &= x_{ij}(t+s) \qquad (t, s \in K), \\ w_{ij}(t)x_{ij}(r)w_{ij}(t)^{-1} &= x_{ji}(-t^{-2}r) \qquad (t \in K^*, r \in K), \end{aligned} \tag{7.1}$$

where $\{i, j\} = \{1, 2\}$. Other noteworthy elements of $St(2, K)$ are $h_{ij}(t) = w_{ij}(t)w_{ij}(-1)$.

The projection $\pi : St(2, K) \to SL(2, K)$ is defined by

$$\pi : \quad x_{12}(t) \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad x_{21}(t) \mapsto \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}. \tag{7.2}$$

Then one easily checks that

$$\pi: \quad w_{12}(t) \mapsto \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}, \quad h_{12}(t) \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}. \tag{7.3}$$

The kernel of π, i.e. the Schur multiplier of $SL(2, K)$ (denoted $H_2(SL(2, K))$, $\pi_1(SL(2, K))$ or $KSp_2(K)$ in various sources) is generated by elements

$$\{s, t\} := h_{12}(s)h_{12}(t)h_{12}(st)^{-1} \quad (s, t \in K^*). \tag{7.4}$$

With this generating set it is described by an explicit family of relations:

$$\begin{aligned} &\{st, r\}\{s, t\} = \{s, tr\}\{t, r\}, \quad \{1, s\} = \{s, 1\} = 1; \\ &\{s, t\} = \{t^{-1}, s\}; \\ &\{s, t\} = \{s, -st\}; \\ &\{s, t\} = \{s, (1 - s)t\} \quad \text{if } s \neq 1. \end{aligned} \tag{7.5}$$

(For this presentation see [15, Theorem 9.2] or [17, §7, Theorem 12]. Even though the group is abelian, the convention is multiplicative.)

Notice that $h_{12}(s)$ and $h_{12}(t)$ are lifts to $St(2, K)$ of commuting elements $\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}, \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$. Therefore, the commutator $[h_{12}(s), h_{12}(t)] = \{s, t\}\{t, s\}^{-1}$ belongs to $SL(2, K)^{[,]}$—the kernel of the quotient map $\pi_1(SL(2, K)) \to Eq(SL(2, K))$. Imposing in $\pi_1(SL(2, K))$ the extra "symbol symmetry" relations $\{s, t\} = \{t, s\}$ we obtain the group $\pi_1(SL(2, K))/\text{sym}$, "the symmetrized Schur multiplier"—an intermediate step in passing from $\pi_1(SL(2, K))$ to $Eq(SL(2, K))$. We will prove that this group is in fact equal to $Eq(SL(2, K))$. For this we use quadratic form theory.

8 Symmetrized Schur multiplier and quadratic forms

The fundamental ideal $I(K)$ of $W(K)$ is generated by non-degenerate symmetric bilinear forms on even-dimensional spaces. Another suitable collection of generators consists of the forms $\langle\langle a\rangle\rangle = \langle 1, -a\rangle$, $a \in K^*$. The ideal $I^2(K)$ is the square of $I(K)$; it is generated by Pfister forms $\langle\langle a, b\rangle\rangle$ (for $a, b \in K^*$), where $\langle\langle a, b\rangle\rangle = \langle\langle a\rangle\rangle \otimes \langle\langle b\rangle\rangle = \langle 1, -a\rangle \otimes \langle 1, -b\rangle = [1] - [a] - [b] + [ab]$ (the last equality valid in $W(K)$). More on these generating sets (in particular, the relations) can be found in [3, I.4].

Let us state (in our current notation) [18, Corollary 6.4]: there exists a natural homomorphism $\Phi\colon \pi_1(SL(2, K)) \to I^2(K)$, sending $\{a, b\}$ to $\langle\langle a, b\rangle\rangle$; the kernel of Φ is generated by the elements $\{a^2, b\}$. We will also need [15, Lemma 3.2]: in $\pi_1(SL(2, K))$

$$\{a, b\}\{b, a\}^{-1} = \{a^2, b\}. \tag{8.1}$$

Putting these two facts together we get the following.

Proposition 8.1 *The homomorphism*

$$\Phi\colon \pi_1(SL(2,K)) \ni \{a,b\} \mapsto \langle\langle a,b\rangle\rangle \in I^2(K)$$

induces an isomorphism

$$\phi\colon \pi_1(SL(2,K))/\mathrm{sym} \to I^2(K).$$

Remark 8.2 It is known that, in general, the map Φ is not an isomorphism. This means that for some field K and some $a, b \in K^*$ we have $\{a,b\} \neq \{b,a\}$ in $\pi_1(SL(2,K))$. It follows that for that field K the Moore class is not equicommutative. In topological terms, we may consider the $SL(2,K)$-bundle $\xi_{a,b}$ over the torus T^2 with monodromies $\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}$, $\begin{pmatrix} b & 0 \\ 0 & b^{-1}\end{pmatrix}$. Then the inequality $\{a,b\} \neq \{b,a\}$ implies that the Moore class of $\xi_{a,b}$ is non-trivial.

Remark 8.3 [18, Corollary 6.4], crucial for the proof of Proposition 8.1, follows from [18, Lemma 6.3]. In the statement of that lemma the second of the defining relations (2) is misprinted; the correct version is

$$\langle\langle a\rangle\rangle + \langle\langle b\rangle\rangle = \langle\langle a+b\rangle\rangle + \langle\langle (a+b)ab\rangle\rangle, \quad a+b \neq 0. \tag{8.2}$$

In the proof of part (3) of the lemma Suslin writes: "It is trivial to check [...] that the elements [...] satisfy relations (2)". To check Relation (8.2) we needed the following calculation in $\pi_1(SL(2,K))/\mathrm{sym}$ (inspired by a calculation of Rost, cf. [5, Lemma 7.6.8]):

$$\begin{aligned}\{a,b\} &= \{b,a\} = \{b,-ba\} = \{-ab,b\} = \{-ab^{-1},b\}\\ &= \{-ab^{-1},(1+ab^{-1})b\} = \{-ab,a+b\} = \{a+b,-ab\}\\ &= \{a+b,ab(a+b)\}.\end{aligned} \tag{8.3}$$

Each step uses symbol symmetry, applies one of the defining relations (7.5), or multiplies a symbol entry by a square. The latter operation is equivalent to multiplication by a symbol of the form $\{z^2,y\}$, as asserted in [15, Appendix, (7)]; by (8.1), the symbol $\{z^2,y\}$ is trivial in $\pi_1(SL(2,K))/\mathrm{sym}$.

9 Comparison of the Moore and Witt cocycles

In this section we recall the explicit form of a cocycle b representing the universal class $u_{SL(2,K)} \in H^2(SL(2,K), \pi_1(SL(2,K)))$ as given in [15, 9.1-4], with later corrections (cf. [9, 9.1]). We also describe the image of b under the map $\Phi\colon \pi_1(SL(2,K)) \to I^2(K)$. Kramer and Tent show that this image, an $I^2(K)$-valued cocycle on $SL(2,K)$, is cohomologous to the Witt cocycle. In the next section we will use this fact to show that the $I^2(K)$-valued Witt class is the universal equicommutative class for $SL(2, K)$.

Kramer and Tent do their calculation in the generic case, and argue that this is enough to claim cocycle equality. We present the details in all cases as this allows us to give an explicit formula for a universal equicommutative cocycle.

Every element of $SL(2, K)$ is uniquely represented in one of the forms:

$$g_1(u,t) = x(u)h(t) = \begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}; \tag{9.1}$$

$$g_2(u,t,v) = x(u)w(t)x(v) = \begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & t\\ -t^{-1} & 0\end{pmatrix}\begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix}. \tag{9.2}$$

This leads to the following definition of a lift $SL(2, K) \to St(2, K)$:

$$\overline{g_1(u,t)} = \overline{x(u)h(t)} := x_{12}(u)h_{12}(t); \tag{9.3}$$

$$\overline{g_2(u,t,v)} = \overline{x(u)w(t)x(v)} := x_{12}(u)w_{12}(t)x_{12}(v) \tag{9.4}$$

The corresponding cocycle b was calculated by Moore, with later correction by Schwarze (cf. [9, 9.1]). We present the formulae for the cocycle b, and for its image under Φ in $W(K)$.

(1) $b(g_2(u,t,v), g_2(u',t',v')) =$

$$\begin{cases}\{-w't^{-1}t'^{-1}, -tt'^{-1}\}\{-t,-t'\}^{-1} & \text{if } w' := -(v+u') \neq 0,\\ \{-t,-t'\}^{-1} & \text{if } w' = 0.\end{cases}$$

These are mapped by Φ to

$$\begin{aligned}\langle\langle -w't^{-1}t'^{-1}, -tt'^{-1}\rangle\rangle - \langle\langle -t,-t'\rangle\rangle &= \langle\langle -w'tt', -tt'\rangle\rangle - \langle\langle -t,-t'\rangle\rangle\\ &= [w'] - [t] - [t'] + [tt'w'].\end{aligned}$$

and to

$$-\langle\langle -t,-t'\rangle\rangle = -[1] - [t] - [t'] - [tt'].$$

(2) $b(g_2(u,t,v), g_1(u',t')) = \{t, t'^{-1}\}$. This is mapped by Φ to $\langle\langle t, t'^{-1}\rangle\rangle = \langle\langle t, t'\rangle\rangle$.

(3) $b(g_1(u,t), g_2(u',t',v')) = \{t,t'\}$, mapped to $\langle\langle t,t'\rangle\rangle$.

(4) $b(g_1(u,t), g_1(u',t')) = \{t,t'\}$, mapped to $\langle\langle t,t'\rangle\rangle$.

We summarize:

$(\Phi_* b)(g,h)$

$$= \begin{cases} [w'] - [t] - [t'] + [tt'w'] & \text{if } g = g_2(u,t,v), h = g_2(u',t',v'), w' = -(v+u') \neq 0, \\ -[1] - [t] - [t'] - [tt'] & \text{if } g = g_2(u,t,v), h = g_2(u',t',v'), w' = -(v+u') = 0, \\ \langle\langle t, t'\rangle\rangle & \text{if } g = g_1(u,t), h = g_2(u',t',v'), \\ \langle\langle t, t'\rangle\rangle & \text{if } g = g_2(u,t,v), h = g_1(u',t'), \\ \langle\langle t, t'\rangle\rangle & \text{if } g = g_1(u,t), h = g_1(u',t'). \end{cases} \tag{9.5}$$

On the other hand, we have the Witt class w $\in H^2(SL(2,K), W(K))$, given by the cocycle w defined by (1.4):

$$w(g,h) = [-|e, ge| \cdot |e, ghe| \cdot |e, he|], \tag{9.6}$$

where $e = \binom{1}{0}$. We now express this cocycle in the parametrization of $SL(2,K)$ used by Moore.

Lemma 9.1

$$w(g,h) = \begin{cases} [w'] & \textit{if } g = g_2(u,t,v), h = g_2(u',t',v'), w' = -(v+u') \neq 0, \\ 0 & \textit{in all other cases}. \end{cases} \tag{9.7}$$

Proof Notice that $g_1(u,t)e = x(u)h(t)e = \binom{t}{0}$, so that $|e, g_1(u,t)e| = 0$. Therefore, the value of $w(g,h)$ is zero if any of the arguments g, h is of the form $g_1(u,t)$. Even so, for later use we need the following easily checked formulae:

$$g_2(u,t,v)g_1(u',t')e = \begin{pmatrix} * \\ -t^{-1}t' \end{pmatrix}, \quad g_1(u,t)g_2(u',t',v')e = \begin{pmatrix} * \\ -t^{-1}t'^{-1} \end{pmatrix},$$
$$g_1(u,t)g_1(u',t')e = \begin{pmatrix} tt' \\ 0 \end{pmatrix}. \tag{9.8}$$

Let us turn to case 1:

$$g_2(u,t,v)e = x(u)w(t)x(v)e = x(u)w(t)e = x(u)\begin{pmatrix} 0 \\ -t^{-1} \end{pmatrix} = \begin{pmatrix} -t^{-1}u \\ -t^{-1} \end{pmatrix}, \tag{9.9}$$

so that $|e, g_2(u,t,v)e| = -t^{-1}$. Furthermore (setting $w' = -v - u'$),

$$g_2(u,t,v)g_2(u',t',v')e = x(u)w(t)x(v)\begin{pmatrix} -t'^{-1}u' \\ -t'^{-1} \end{pmatrix} = x(u)\begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}\begin{pmatrix} t'^{-1}w' \\ -t'^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -tt'^{-1} \\ -t^{-1}t'^{-1}w' \end{pmatrix} = \begin{pmatrix} -tt'^{-1} - t^{-1}t'^{-1}uw' \\ -t^{-1}t'^{-1}w' \end{pmatrix},$$

so that

$$|e, g_2(u,t,v)g_2(u',t',v')e| = -t^{-1}t'^{-1}w'. \tag{9.10}$$

Therefore

$$w(g_2(u,t,v), g_2(u',t',v')) = \begin{cases} [-(-t^{-1})(-t'^{-1})(-t^{-1}t'^{-1}w')] = [w'] & \text{if } w' \neq 0, \\ 0 & \text{if } w' = 0. \end{cases} \tag{9.11}$$

◇

Nekovář [16, §2] noticed that w is cohomologous to an $I^2(K)$-valued cocycle; it is enough to add the coboundary of the following cochain:

$$n(g) = \begin{cases} [|e, ge|] & \text{if } |e, ge| \neq 0, \\ [1] - [t] & \text{if } ge = te \text{ for some } t \in K^*. \end{cases} \tag{9.12}$$

In the parametrization used by Moore: $n(g_1(u,t)) = [1] - [t]$, $n(g_2(u,t,v)) = [-t^{-1}] = -[t]$,

$$\begin{aligned} n(g_2(u,t,v)g_2(u',t',v')) &= \begin{cases} [-t^{-1}t'^{-1}w'] = -[tt'w'] & \text{if } w' \neq 0, \\ [1] - [-tt'^{-1}] = [1] + [tt'] & \text{if } w' = 0; \end{cases} \\ n(g_2(u,t,v)g_1(u',t')) &= [-t^{-1}t'] = -[tt'], \\ n(g_1(u,t)g_2(u',t',v')) &= [-t^{-1}t'^{-1}] = -[tt'], \\ n(g_1(u,t)g_1(u',t')) &= [1] - [tt']. \end{aligned} \tag{9.13}$$

We are ready to calculate δn and see that $\Phi_* b = w + \delta n$. Using the formula $(\delta n)(g,h) = n(g) - n(gh) + n(h)$ we get:

(1) $(\delta n)(g_2(u,t,v), g_2(u',t',v'))$
$$= \begin{cases} -[t] + [tt'w'] - [t'] & \text{if } w' = -(v+u') \neq 0, \\ -[t] - ([1] + [tt']) - [t'] & \text{if } w' = 0. \end{cases}$$
(2) $(\delta n)(g_2(u,t,v), g_1(u',t')) = -[t] - (-[tt']) + [1] - [t'] = \langle\langle t, t'\rangle\rangle$

(3) $(\delta n)(g_1(u,t), g_2(u',t',v')) = [1] - [t] - (-[tt']) - [t'] = \langle\langle t, t'\rangle\rangle$

(4) $(\delta n)(g_1(u,t), g_1(u',t')) = [1] - [t] - ([1] - [tt']) + [1] - [t'] = \langle\langle t, t'\rangle\rangle$

Comparing these four formulae, (9.5) and (9.1) we obtain the following proposition.

Proposition 9.2 [9, 9.2] *The Witt cocycle w and the cocycle $\Phi_* b$ are cohomologous (in the complex of $SL(2, K)$-cochains with coefficients in $W(K)$).*

The phrasing of Proposition 9.2 is slightly awkward, because $\Phi_* b$ has coefficients in $I^2(K)$, while w has coefficients in $W(K)$. Fortunately, it is not hard to check that the relevant cohomology groups embed:

Proposition 9.3 *The inclusion* $\iota\colon I^2(K) \to W(K)$ *induces a monomorphism*

$$\iota_*\colon H^2(SL(2, K), I^2(K)) \to H^2(SL(2, K), W(K)). \tag{9.14}$$

Proof Let $Q = W(K)/I^2(K)$. Consider the short exact sequence of coefficient groups:

$$0 \to I^2(K) \to W(K) \to Q \to 0 \tag{9.15}$$

and the associated long exact sequence

$$\cdots \to H^1(SL(2, K), Q) \to H^2(SL(2, K), I^2(K)) \to H^2(SL(2, K), W(K)) \to \cdots \tag{9.16}$$

We have

$$H^1(SL(2, K), Q) = \mathrm{Hom}(SL(2, K), Q) = 0, \tag{9.17}$$

because $SL(2, K)$ is perfect, hence has no nontrivial homomorphisms to abelian groups. ◇

To summarize: $\iota_*([\Phi_* b]) = [w]$—or, in terms of cohomology classes, $\iota_*(\Phi_*(u_{SL(2,K)})) = \mathrm{w}$.

Definition 9.4 The *reduced* (or $I^2(K)$-*valued*) Witt class $\mathrm{w}^I \in H^2(SL(2, K), I^2(K))$ is defined by $\mathrm{w}^I = \iota_*^{-1}(\mathrm{w})$; it is equal to $\Phi_*(u_{SL(2,K)})$ and represented by the cocycle $\Phi_* b$, explicitly given by (9.5).

For practical purposes, one can ignore the difference between the classes w and w^I, mainly because of the following corollary of Proposition 9.2.

Corollary 9.5 *For any* $SL(2, K)$*-bundle* P *over a closed oriented surface* Σ *we have*

$$\langle \mathrm{w}(P), [\Sigma] \rangle = \langle \mathrm{w}^I(P), [\Sigma] \rangle \in I^2(K). \tag{9.18}$$

Proposition 9.6 *The class* w^I *is equicommutative.*

Proof A cocycle c representing w^I treated as a $W(K)$-valued cocycle (via the embedding $I^2(K) \to W(K)$) is cohomologous to the (standard) Witt cocycle w. The latter is equicommutative, hence, by Proposition 5.3, so is c. ◇

10 The Witt class is universal equicommutative

We prove what is in the title of this section.

Theorem 10.1 (Theorem A) *Let K be an infinite field. The group $Eq(SL(2, K))$ is isomorphic to $I^2(K)$, and the Witt class* $\mathrm{w}^I \in H^2(SL(2, K), I^2(K))$ *is the universal equicommutative class.*

Proof Consider the following diagram of coefficient groups.

$$\begin{array}{ccccc} \pi_1(SL(2,K)) & \xrightarrow{q_1} & \pi_1(SL(2,K))/\mathrm{sym} & \xrightarrow{q_2} & Eq(SL(2,K)) \\ & d_1 \searrow & \Big\downarrow \phi & \swarrow d_2 & \\ & & I^2(K) & & \end{array} \tag{10.2}$$

The diagonal arrows are the unique maps deduced from the universal properties of $u_{SL(2,K)}$ and $w_{SL(2,K)}$, applied to the Witt class w^I. Uniqueness of the universal map implies commutativity of the diagram. Namely, the left triangle commutes because $\phi q_1 = \Phi$ maps $u_{SL(2,K)}$ to w^I (Proposition 9.2), hence is equal to the diagonal map d_1. Similarly, we have $\mathrm{w}^I = (d_2)_* w_{SL(2,K)} = (d_2)_*(q_2 q_1)_* u_{SL(2,K)}$, hence $d_2 q_2 q_1 = d_1 = \phi q_1$; but q_1 is surjective, therefore we deduce $d_2 q_2 = \phi$, i.e. the commutativity of the right triangle. Now d_2 is an isomorphism, because ϕ is an isomorphism (Proposition 8.1) and q_2 is surjective. The isomorphism d_2 maps the universal equicommutative class $w_{SL(2,K)}$ to w^I. ◇

Another corollary of the proof is that q_2 is an isomorphism: the quotient of $\pi_1(SL(2, K))$ by the symbol symmetry relations is already equal to $Eq(SL(2, K))$.

IV. Witt range

What are the possible values of the Witt class for $SL(2, \mathbf{Q})$-bundles over surfaces? We know that these values reside in $I^2\mathbf{Q}$, which is a direct sum of $4\mathbf{Z}$ (the real, signature part) and an infinite direct sum of $\mathbf{Z}/2$ (p-adic parts, one per odd prime). More details of this description are given in Sect. 12. While the real part of the Witt class is related to the Euler class, hence non-trivial (cf. [2, Section 13]), the p-adic parts are more mysterious—perhaps trivial? Using (1.4) we did some computer calculations in FriCAS that indicated non-triviality of the p-adic parts. In Sect. 12 we give a complete description of the range of the Witt class over $\mathbf{Q}$, proving that the Milnor–Wood inequality (restricting the real part in terms of the genus of the base surface) is the only restriction—the p-adic parts can be arbitrary. The challenging part of this result is the construction of sufficiently many non-trivial bundles. This is done in Sect. 11 in much greater generality, for arbitrary infinite fields K. We analyse $SL(2, K)$-bundles over simple surfaces (pair of pants; genus 1 surface with one boundary component) using some results on the Markov equation (quoted from [4]). Then we use the gluing results from Sect. 4. The formula (1.4) is used throughout to control the Witt class. Our final result, Theorem 11.6, gives a large subset of Witt classes in $I^2(K)$. This subset is quite close to the one defined by the boundedness restriction for the Witt class; the difference is discussed in Sect. 13.

An early paper where many $SL(2, \mathbf{Q})$-bundles were constructed by gluing is [19].

11 Markov surfaces and representations

In this section K will be an arbitrary infinite field. Let w be the Witt class. We will use the representing cocycle

$$w(X, Y) = [-|e, Xe| \cdot |e, XYe| \cdot |e, Ye|] = [-X_{21} \cdot (XY)_{21} \cdot Y_{21}], \qquad (11.1)$$

where $e = \binom{1}{0}$, X_{21} denotes the 21-entry of the matrix X, and $[0]$ is interpreted as 0. Additive convention will be used for the cocycle.

Remark 11.1 If either X, Y or XY is diagonal (or even upper-triangular), then $w(X, Y) = 0$.

We will use notions and notation discussed at the beginning of Sect. 4, in particular the notion of standard framing, and the notion of "relative class" $\overline{w}$ (cf. Definition 4.1).

Lemma 11.2 *Let K be an infinite field. Then, for any $\alpha, \beta \in K^*$ there exists a flat $SL(2, K)$-bundle ξ with a standard framing, over the oriented genus 1 compact surface with one boundary component, such that $\overline{w}(\xi) = [\alpha] + [\beta] \in W(K)$. Moreover, for every $z \in -\alpha\beta K^{*2}$ (with finitely many exceptions) the bundle ξ may be chosen so that its boundary monodromy is diagonal with eigenvalues z, z^{-1}.*

Proof To construct the bundle as in the lemma, with boundary monodromy Z, we need to find $X, Y \in SL(2, K)$ such that $[X, Y] = Z$. We heavily rely on the classical description of the space of solutions of the commutator equation $[X, Y] = Z$; we use the version from [4], though some results go back as early as to Fricke. To start, if $[X, Y] = Z$, then the Fricke identity says that the scalars $x_1 = \operatorname{tr} X$, $x_2 = \operatorname{tr} Y$, $x_3 = \operatorname{tr} XY$ and $m = \operatorname{tr} Z + 2$ satisfy the Markov equation

$$x_1^2 + x_2^2 + x_3^2 - x_1 x_2 x_3 = m. \qquad (M_m)$$

Conversely, if

(a) $\operatorname{tr} Z \neq \pm 2$;
(b) (x_1, x_2, x_3) is a solution of (M_m);
(c) $m - x_2^2 \neq 0$;
(d) $Y \in SL(2, K)$ satisfies $\operatorname{tr} Y = \operatorname{tr} ZY = x_2$;

then there exists a unique $X \in SL(2, K)$ that satisfies $\operatorname{tr} X = x_1$, $\operatorname{tr} XY = x_3$ and $[X, Y] = Z$ (cf. [4, Lemma 3.5]). Explicitly, this X is given by

$$(m - x_2^2) X = -x_3 ZY + x_1 Z + (x_3 - x_1 x_2) Y^{-1} + x_1 I. \qquad (11.2)$$

We will work out the case $Z = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$, $z \neq \pm 1$ (then (a) is fulfilled). The matrices Y satisfying (d) can be found explicitly. If $Y = \begin{pmatrix} a & * \\ * & d \end{pmatrix}$, then $\operatorname{tr} Y = \operatorname{tr} ZY = x_2$ are equivalent to a linear system of equations on a, d with unique solution

$$a = \frac{x_2}{1+z}, \quad d = \frac{z x_2}{1+z}. \qquad (11.3)$$

The condition $\det Y = 1$ gives the following form of Y:

$$Y = \frac{1}{1+z}\begin{pmatrix} x_2 & c^{-1}(zx_2^2 - (1+z)^2) \\ c & zx_2 \end{pmatrix}. \tag{11.4}$$

The solution depends on a unique parameter $c \in K^*$. (We have $c \neq 0$, since otherwise $1 = \det Y$ would imply $1 = zx_2^2(1+z)^{-2} = x_2^2 m^{-1}$, contradicting (c).) Condition (c) ensures that $m - x_2^2 \neq 0$; therefore we may plug (11.4) into (11.2) and determine X:

$$(m - x_2^2)X = \frac{1}{1+z}\begin{pmatrix} x_1(1+z)^2 - x_1x_2^2z & * \\ (x_1x_2 - (1+z^{-1})x_3)c & x_1(1+z)(1+z^{-1}) - x_1x_2^2 \end{pmatrix}. \tag{11.5}$$

Further direct computations show that

$$\begin{aligned} (m - x_2^2)(XY)_{21} &= \frac{x_1(1+z) - x_2x_3}{z(1+z)}c, \\ (m - x_2^2)(YX)_{21} &= \frac{x_1(1+z) - x_2x_3}{1+z}c. \end{aligned} \tag{11.6}$$

Let ξ_c be the bundle defined by these (X, Y). Finally, using Lemma 4.8 and Remark 11.1we get

$$\begin{aligned} \overline{w}(\xi_c) &= w(X, Y) - w(Y, X) - w(Z, YX) \\ &= [-X_{21} \cdot (XY)_{21} \cdot Y_{21}] - [-X_{21} \cdot (YX)_{21} \cdot Y_{21}] \\ &= [-\frac{c}{z(1+z)}(x_1x_2 - (1+z^{-1})x_3)(x_1(1+z) - x_2x_3)] \\ &\quad -[-\frac{c}{1+z}(x_1x_2 - (1+z^{-1})x_3)(x_1(1+z) - x_2x_3)]. \end{aligned} \tag{11.7}$$

If our solution (x_1, x_2, x_3) satisfies two further conditions

(e) $x_1x_2 - (1+z^{-1})x_3 \neq 0$, $x_1(1+z) - x_2x_3 \neq 0$;

then we modify c to a new parameter

$$C := -\frac{c}{z(1+z)}(x_1x_2 - (1+z^{-1})x_3)(x_1(1+z) - x_2x_3) \tag{11.8}$$

(C runs through K^* as c does), and then we get

$$\overline{w}(\xi_c) = [C] - [Cz] = [C] + [-Cz]. \tag{11.9}$$

To summarize, to realize $[\alpha] + [\beta] \in W(K)$ (for given $\alpha, \beta \in K^*$) as $\overline{w}(\xi_c)$ we may choose $z = -\alpha\beta\lambda^2$, $C = \alpha$ (for some $\lambda \in K^*$) and apply the above construction—provided that we find a solution (x_1, x_2, x_3) of M_m that satisfies (c) and (e). In [4, proof of Proposition 3.6] the following solution of M_{t+2} is given:

$$x_2 = \zeta + \zeta^{-1}, \qquad x_3 = \frac{-x_2^2 + t + 1}{\zeta - \zeta^{-1}}, \qquad x_1 = 1 + \zeta x_3. \tag{S}$$

(We have corrected a sign mistake in x_3; $\zeta \in K^*\backslash\{\pm 1\}$ is arbitrary.) Putting $t = z+z^{-1}$ we get:

$$x_1 = \frac{\zeta z - \zeta(\zeta^2 + 2\zeta^{-2}) + \zeta z^{-1}}{\zeta - \zeta^{-1}}, \qquad x_2 = \frac{\zeta^2 - \zeta^{-2}}{\zeta - \zeta^{-1}}, \tag{S'}$$
$$x_3 = \frac{z - (\zeta^2 + 1 + \zeta^{-2}) + z^{-1}}{\zeta - \zeta^{-1}}$$

Let us now discuss the conditions (c) and (e) for the solution (S'). For a fixed ζ, these are three non-trivial polynomial inequalities on z, hence they hold except for some finite set E of values of z. (Since the general procedure described above requires $z \neq \pm 1$, we include these two values in E as well.) Clearly, for all but finitely many values of λ we have $z \notin E$; this proves the lemma. $\diamond$

Lemma 11.3 *Let* $\lambda_1, \lambda_2, \lambda_3 \in K^* \setminus \{\pm 1\}$. *Let*

$$\Lambda_i = \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i^{-1} \end{pmatrix}, \quad t_i = \mathrm{tr}\Lambda_i = \lambda_i + \lambda_i^{-1}. \tag{11.10}$$

Suppose that the triple (t_1, t_2, t_3) *does not satisfy Markov's equation* M_4. *Then for every* $c \in K^*$ *there exist a unique pair of matrices* (L, M) *in* $SL(2, K)$ *that satisfies the following conditions*:

(a) $L = {}^A\Lambda_1$, $M = {}^B\Lambda_2$ *for some* $A, B \in SL(2, K)$;
(b) $LM = \Lambda_3$;
(c) $L_{21} = c$.

Moreover, regardless of the choice of A, B in condition (a), we then have

$$w({}^A\Lambda_1, A) + w({}^B\Lambda_2, B) = [-c\lambda_1] + [c\lambda_2\lambda_3]. \tag{11.11}$$

(Recall that ${}^A\Lambda$ is our notation for the conjugate $A\Lambda A^{-1}$.)

Corollary 11.4 *Let* ξ, ξ' *be flat* $SL(2, K)$*-bundles with standard boundary framings and diagonal boundary monodromies* $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix} \neq \pm I$. *Let* $\alpha, \beta \in K^*$. *Then, for every* $z \in -\alpha\beta\lambda_1\lambda_2 K^{*2}$ (*with finitely many exceptions*) *there exist matrices* $A, B \in SL(2, K)$ *such that* ${}^A\xi \vee {}^B\xi'$ *has diagonal boundary monodromy* $\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$ $\neq \pm I$ *and with standard boundary framing satisfies*

$$\overline{w}({}^A\xi \vee {}^B\xi') = [\alpha] + [\beta] + \overline{w}(\xi) + \overline{w}(\xi'). \tag{11.12}$$

Proof of the Corollary We use Lemma 11.3 with Λ_1, Λ_2 equal to the boundary monodromies of ξ, ξ'. We choose $\lambda_3 = z = -\alpha\beta\lambda_1\lambda_2\lambda^2$ and $c = \alpha\lambda_1$ for some $\lambda \in K^*$. (There is a finite set of values that λ_3 has to avoid: ± 1 and the values for which (t_1, t_2, t_3) would satisfy M_4; we avoid them for all but finitely many choices of $\lambda \in K^*$.) To prove (11.12) we use Lemmas 4.6 and 4.7 (that describe the change of $\overline{w}$ under twists and $\vee$), the remark about vanishing of w for diagonal matrices, and (11.11):

$$\begin{aligned}
\overline{w}({}^A\xi \vee {}^B\xi') &= \overline{w}({}^A\xi) + \overline{w}({}^B\xi') + w({}^A\Lambda_1, {}^B\Lambda_2)\\
&= \overline{w}(\xi) + w(A, \Lambda_1) - w({}^A\Lambda_1, A) + \overline{w}(\xi') + w(B, \Lambda_2) - w({}^B\Lambda_2, B)\\
&= \overline{w}(\xi) + \overline{w}(\xi') - (w({}^A\Lambda_1, A) + w({}^B\Lambda_2, B))\\
&= \overline{w}(\xi) + \overline{w}(\xi') - ([-c\lambda_1] + [c\lambda_2\lambda_3])\\
&= \overline{w}(\xi) + \overline{w}(\xi') + [\alpha] + [\beta].
\end{aligned} \tag{11.13}$$

◇

Proof of Lemma 11.3 Suppose that (L, M) is a pair satisfying (a), (b) and (c). Let

$$L = \begin{pmatrix} a & d \\ c & t_1 - a \end{pmatrix}.$$

We claim that $c \neq 0$. Indeed, if c were 0 then L would be upper-triangular (with eigenvalues $\lambda_1^{\pm 1}$), and $M = L^{-1}\Lambda_3$ would also be upper-triangular (with eigenvalues $\lambda_2^{\pm 1}$). Direct calculation shows that then $[L, M]$ would be upper-triangular and of trace 2 (with both diagonal entries equal to 1). The Fricke identity for the traces ($t_1 = \mathrm{tr} L$, $t_2 = \mathrm{tr} M$, $t_3 = \mathrm{tr}\Lambda_3$) would show that this triple satisfies the Markov equation (M_4)—contradiction.

Thus, we have $L = \begin{pmatrix} a & d \\ c & t_1 - a \end{pmatrix}$ for some $c \in K^*$. It follows that $d = -(a^2 - t_1 a + 1)/c$. Next:

$$M = L^{-1}\Lambda_3 = \begin{pmatrix} t_1 - a & -d \\ -c & a \end{pmatrix}\begin{pmatrix} \lambda_3 & 0 \\ 0 & \lambda_3^{-1} \end{pmatrix} = \begin{pmatrix} \lambda_3(t_1 - a) & -\lambda_3^{-1}d \\ -\lambda_3 c & \lambda_3^{-1}a \end{pmatrix}. \tag{11.14}$$

The trace of M is t_2: $\lambda_3(t_1 - a) + \lambda_3^{-1}a = t_2$, which holds for

$$a = \frac{\lambda_3 t_1 - t_2}{\lambda_3 - \lambda_3^{-1}}. \tag{11.15}$$

To finish the discussion of the existence and uniqueness question we remark that any matrix with determinant 1 and trace t_i is conjugate (in $SL(2, K)$) to Λ_i.

We now pass to the calculation of w. The equation ${}^A\Lambda_1 = L$ implies that A_1, the left column of $A = (A_1, *)$, is a λ_1-eigenvector of L. Therefore

$$(LA)_{21} = (\lambda_1 A_1, *)_{21} = \lambda_1 A_{21},$$

so that

$$w(L, A) = [-L_{21}(LA)_{21}A_{21}] = [-c \cdot \lambda_1 A_{21} \cdot A_{21}] = [-c\lambda_1]. \tag{11.16}$$

(We know that $A_{21} \neq 0$, for otherwise A and ${}^A\Lambda_1 = L$ would be upper-triangular, contradicting $L_{21} = c \neq 0$.) Similarly,

$$w(M, B) = [-M_{21}(MB)_{21}B_{21}] = [-(-\lambda_3 c) \cdot (\lambda_2 B_{21}) \cdot B_{21}] = [c\lambda_2\lambda_3]. \tag{11.17}$$

◇

One of the ways to look at the Witt ring $W(K)$ is the following. Any element of $W(K)$ is represented by a unique (up to isomorphism) anisotropic form over K. The dimension of this anisotropic representative defines a norm $\|\cdot\|$ on $W(K)$ (cf. [14, 3.1.7, 3.1.8]). On the other hand, any element $x \in W(K)$ can be represented—in many ways—as a finite sum

$$x = \sum_{i \in I} n_i[a_i], \tag{11.18}$$

where $n_i \in \mathbf{Z}$, $a_i \in K^*$. The symbolic norm $\|x\|_s$ is the minimum—over all such representations—of the expression $\sum_{i \in I} |n_i|$.

Lemma 11.5 *For each $x \in W(K)$ we have $\|x\| = \|x\|_s$.*

Proof If $x = \sum_{i \in I} n_i[a_i]$, then the form $x = \bigoplus_{i \in I} |n_i|\langle(\mathrm{sgn} n_i)a_i\rangle$ represents x. This form has a largest anisotropic direct summand—the unique (up to isomorphism) representative of x, of dimension $\|x\|$. It follows that

$$\|x\| \leq \dim\left(\bigoplus_{i \in I} |n_i|\langle(\mathrm{sgn} n_i)a_i\rangle\right) = \sum_{i \in I} |n_i|. \tag{11.19}$$

Therefore $\|x\| \leq \|x\|_s$. On the other hand, a diagonalization of the anisotropic representative of x expresses x as a sum of $\|x\|$ symbols $[a_i]$, so that $\|x\|_s \leq \|x\|$. ◇

The diameter of $W(K)$ with respect to the above norm is sometimes called the u-invariant of K (cf. [11, XI.6]). However, it is too often infinite (e.g. for $K = \mathbf{Q}$), hence a refinement is widely used: $u(K)$ is the diameter of the set $W_t(K)$ of torsion elements in $W(K)$ (cf. [3, Chapter VI] or [11, Definition XI.6.24]). It is classically known that $u(K) = 4$ for local (non–archimedean) and for global fields (cf. [11, Examples XI.6.2, XI.6.29] or [3, Example 36.2]).

We use the following facts from [11, II.2]. The determinant of a quadratic form $q = \langle a_1, \ldots, a_n\rangle$ is defined as $d(q) := \prod_{i=1}^n a_i \in K^*/K^{*2}$, and the discriminant as $d_\pm(q) = (-1)^{n(n-1)/2} d(q)$. An even–dimensional form q is in $I^2(K)$ if and only if $d_\pm(q) = 1$.

Theorem 11.6 (Theorem B) *Let K be an infinite field.*

(a) *The Witt class of any flat $SL(2, K)$-bundle over an oriented closed surface of genus g has norm $\leq 4(g-1)+2$.*
(b) *The set of Witt classes of flat $SL(2, K)$-bundles over an oriented closed surface of genus g contains the set of elements of $I^2(K)$ of norm $\leq 4(g-1)$.*

Proof (a) This part is straightforward. The closed orientable genus g surface Σ_g has a Δ-complex structure with $4g-2$ triangles (cf. the proof of Lemma 3.1). The value of the Witt cocycle $w(\xi)$ (of any flat $SL(2, K)$-bundle ξ over Σ_g) on each of these triangles has norm ≤ 1. This implies claim (a).

(b) Let $q \in I^2(K)$ be an element of norm $\leq 4g-4$. Then we can find $\alpha_i, \beta_i, \gamma_j, \delta_j \in K^*$ $(1 \leq i \leq g,\ 2 \leq j \leq g-1)$ such that

$$q = \sum_i([\alpha_i] + [\beta_i]) + \sum_j([\gamma_j] + [\delta_j]). \tag{11.20}$$

If the norm of q is smaller than $4g-4$, we add some extra trivial terms $[1] + [-1]$ to obtain the above form; since $q \in I^2(K)$ we know that $\dim q$ is even, and that the following product formula holds:

$$1 = d_{\pm}(q) = \prod_i(\alpha_i\beta_i)\prod_j(\gamma_j\delta_j). \tag{11.21}$$

Now we find, using Lemma 11.2, bundles ξ_i (for $i < g$), (over genus 1 oriented surfaces with one boundary component), with standard boundary framing, with $\overline{w}(\xi_i) = [\alpha_i] + [\beta_i]$, with diagonal boundary monodromy with eigenvalues $z_i^{\pm 1}$ that satisfy $[z_i] = [-\alpha_i\beta_i]$. We put $\zeta_1 = \xi_1$, $u_1 = z_1$, and then, using Corollary 11.4, we recursively define

$$\zeta_j := {}^{A_j}\zeta_{j-1} \vee {}^{B_j}\xi_j, \tag{11.22}$$

with standard boundary framing, diagonal boundary monodromy with eigenvalues $u_j^{\pm 1}$ that satisfy

$$[u_j] = [-\gamma_j\delta_j u_{j-1} z_j] \tag{11.23}$$

and

$$\overline{w}(\zeta_j) = [\gamma_j] + [\delta_j] + \overline{w}(\zeta_{j-1}) + \overline{w}(\xi_j). \tag{11.24}$$

Induction then gives:

$$[u_j] = [-\prod_{i=1}^{j}(\alpha_i\beta_i)\prod_{\ell=2}^{j}(\gamma_\ell\delta_\ell)], \tag{11.25}$$

$$\overline{w}(\zeta_j) = \sum_{i=1}^{j}([\alpha_i] + [\beta_i]) + \sum_{\ell=2}^{j}([\gamma_j] + [\delta_j]). \tag{11.26}$$

Now we construct ξ_g just as the other ξ_i, but with $\overline{w}(\xi_g) = [-\alpha_g] + [-\beta_g]$. For $j = g - 1$ we obtain, by the product formula, $[u_{g-1}] = [z_g]$. It follows that ζ_{g-1} and ξ_g can be constructed with the same boundary monodromy. Then $\zeta_{g-1} \cup \xi_g$ is the desired bundle:

$$\mathrm{w}(\zeta_{g-1} \cup \xi_g) = \overline{w}(\zeta_{g-1}) - \overline{w}(\xi_g) = q. \tag{11.27}$$

◇

12 The range of the Witt class over Q

The goal of this section is to describe the range of the Witt class for all representations of $\pi_1(\Sigma_g)$ (the fundamental group of the genus g orientable surface) in $SL(2, \mathbf{Q})$.

We now recall a description of $I^2\mathbf{Q}$ from [11, VI.5.8], slightly modified using [14, 4.2.5]: the sequence

$$0 \to I^2\mathbf{Q} \to I^2\mathbf{R} \oplus \bigoplus_{p\neq\infty} I^2\mathbf{Q}_p \xrightarrow{f} \mathbf{Z}/2 \to 0 \tag{12.1}$$

is exact. Here the middle terms are: $I^2\mathbf{R} \simeq 4\mathbf{Z}$, $I^2\mathbf{Q}_p \simeq \mathbf{Z}/2$. The first embedding is defined by functorial maps associated to the completion embeddings $\mathbf{Q} \to \mathbf{R}$, $\mathbf{Q} \to \mathbf{Q}_p$, while f is the "reciprocity law" map:

$$f(4n, a_2, a_3, a_5, \ldots) = \left(n + \sum_{p\neq\infty} a_p\right) \bmod 2. \tag{12.2}$$

(Each $I^2\mathbf{Q}_p$ is isomorphic to $\mathbf{Z}/2$; each non-trivial a_p is interpreted as 1.) The torsion part of $I^2\mathbf{Q}$ is isomorphically mapped onto the subgroup

$$\left\{(a_p)_{p\neq\infty} \in \bigoplus_{p\neq\infty} I^2\mathbf{Q}_p \mid \sum_{p\neq\infty} a_p \equiv 0 \bmod 2\right\}. \tag{12.3}$$

We denote by $\sigma : W(\mathbf{Q}) \to W(\mathbf{R}) \simeq \mathbf{Z}$ the signature map, normalized by $\sigma([1]) = 1$. Then the torsion elements of $W(\mathbf{Q})$ are the ones of signature zero.

Lemma 12.1 (a) *Elements of $I^2\mathbf{Q}$ of signature $\pm 4h$ (where $h \geq 1$) have norm $4h$. The set of such elements can be described as*

$$\left\{\sum_{i=1}^{4h}[a_i] \in W(\mathbf{Q}) \mid \pm a_i > 0, \prod_i a_i \in \mathbf{Q}^{*2}\right\}. \tag{12.4}$$

(b) *Non-trivial elements of* $I^2\mathbf{Q}$ *of signature* 0 *have norm* 4. *The set of such elements can be described as*

$$\{[a]+[b]+[c]+[d] \in W(\mathbf{Q}) \mid abcd \in \mathbf{Q}^{*2},\ \textit{and exactly two of}\ a,b,c,d\ \textit{are positive}\}. \tag{12.5}$$

Proof Let us start with the proof of part (b). Let $x \in I^2\mathbf{Q}$, $x \neq 0$, $\sigma(x) = 0$. Since $u(\mathbf{Q}) = 4$ we know that $\|x\| \leq 4$. As the dimension function takes even values on $I\mathbf{Q}$ (by definition), hence also on $I^2\mathbf{Q}$, an anisotropic representative of x is 4- or 2-dimensional. If $x = \langle a, b\rangle$, however, we get $1 = d_\pm(\langle a, b\rangle) = -ab$; then $ab = -1$ in $\mathbf{Q}^*/\mathbf{Q}^{*2}$, and $x = \langle a, -a\rangle = 0$ in $W(\mathbf{Q})$. Therefore, an anisotropic representative of x is 4-dimensional. Thus, we have $x = [a]+[b]+[c]+[d]$ for some $a, b, c, d \in \mathbf{Q}^*$ that satisfy $abcd \in \mathbf{Q}^{*2}$ (this is equivalent to $x \in I^2\mathbf{Q}$), and exactly two of a, b, c, d are positive (equivalent to $\sigma(x) = 0$).

For part (a) we use Meyer's theorem (cf. [14, Corollary II.3.2]): a quadratic $\mathbf{Q}$-form of dimension greater than 4 is $\mathbf{Q}$-isotropic if it is $\mathbf{R}$-isotropic. Let $x \in I^2\mathbf{Q}$, $\sigma(x) = 4h > 0$. Let q be an anisotropic representative of x. If $\dim q > 4h$, then q is $\mathbf{R}$-isotropic, hence, by Meyer's theorem, also $\mathbf{Q}$-isotropic—contradiction. The rest of the statement is seen as in part (b): the condition $\prod_i a_i \in \mathbf{Q}^{*2}$, i.e. $d_\pm = 1$, characterizes elements in $I^2\mathbf{Q}$, while positivity of a_i is equivalent to $\sigma = 4h$. The claim for signature $-4h$ can be deduced by switching from x to $-x$. ◇

Using Lemma 12.1 we can now give a complete description of possible Witt classes of $SL(2,\mathbf{Q})$-bundles.

Theorem 12.2 (Theorem C) *The set of Witt classes of all representations of* $\pi_1(\Sigma_g)$ *in* $SL(2,\mathbf{Q})$ *is equal to the set of elements of* $I^2\mathbf{Q}$ *with norm* $\leq 4(g-1)$.

Proof The classical Milnor–Wood inequality states that the Euler class of a flat $SL(2,\mathbf{R})$-bundle over Σ_g has absolute value $\leq g-1$. For an $SL(2,\mathbf{Q})$-bundle (treated as a flat $SL(2,\mathbf{R})$-bundle) this Euler class is equal to $\frac{1}{4}$ of the signature of the Witt class (cf. [2, Theorem 13.4]); therefore, the Witt class of an $SL(2,\mathbf{Q})$-bundle over Σ_g has signature of absolute value $\leq 4(g-1)$. Then it has also norm $\leq 4(g-1)$, by Lemma 12.1—except, possibly, for $g = 1$. But for $g = 1$ we know that the Witt class is 0 (by equicommutativity and Lemma 3.1), so that the norm bound holds also in this case.

Conversely, by Theorem 11.6, every element of $I^2\mathbf{Q}$ of norm $\leq 4(g-1)$ is realizable as the Witt class of some $SL(2,\mathbf{Q})$-bundle over Σ_g. ◇

Remark 12.3 Assume $g \geq 2$. Then in the statement of Theorem 12.2 one can replace "with norm $\leq 4(g-1)$" by "with signature of absolutes value $\leq 4(g-1)$" (as is evident from the proof).

13 The easy norm bound is not sharp

The Witt class $\mathrm{w}(\xi)$ of a flat $SL(2,K)$-bundle ξ over Σ_g has norm $\leq 4g-2$. For $K = \mathbf{R}$ and the Euler class this bound can be improved to $4g-4$ (the already mentioned

Milnor–Wood). It is unclear whether this stronger estimate ($\|\mathrm{w}(\xi)\| \leq 4g - 4$) holds for the general Witt class; we have not found any counterexamples. The question is meaningful for fields with $u > 4$. We now present an example of an element of $I^2(K)$ with norm 6 that is not realizable as $\mathrm{w}(\xi)$ over Σ_2.

Proposition 13.1 *Let $K = \mathbf{Q}((x))$, the field of Laurent series with rational coefficients. Let*

$$q = \langle 1, 1, 1, 7, x, -7x \rangle. \tag{13.1}$$

Then $q \in I^2(K)$, $\|q\| = 6$ and q is not realizable as the Witt class of an $SL(2, K)$-bundle over Σ_2.

Proof We have $\dim(q) = 6$ and $d_{\pm}(q) = 1$, hence $q \in I^2(K)$. Suppose that $q = \mathrm{w}(\xi)$. Let $\pi : \Sigma_3 \to \Sigma_2$ be a degree-2 covering map. Then $\mathrm{w}(\pi^*\xi) = 2\mathrm{w}(\xi) = 2q$. Since Σ_3 has a Δ-complex structure with 10 triangles, we would have $\|2q\| = \|\mathrm{w}(\pi^*\xi)\| \leq 10$. We will get a contradiction by showing that $\|2q\| = 12$. (This equality also implies $\|q\| = 6$.)

We will show that $q \oplus q$ is an anisotropic form. To prove it we use (iteratively) the following fact (cf. [11, Proposition VI.1.9]). Let F be a nondyadic complete discretely valued field with residue field $\overline{F}$ and uniformizer t. Then the form $q_1 \oplus t q_2$ is anisotropic over F if the forms $\overline{q}_1, \overline{q}_2$ over $\overline{F}$ are anisotropic. We will use this proposition for $F = \mathbf{Q}((x)), \mathbf{Q}_7$.

First, we split

$$2q = \langle 1, 1, 1, 1, 1, 1, 7, 7 \rangle \oplus x \langle 1, 1, -7, -7 \rangle. \tag{13.2}$$

Thus, we want to show that $\langle 1, 1, 1, 1, 1, 1, 7, 7 \rangle$ and $\langle 1, 1, -7, -7 \rangle$ are anisotropic over $\mathbf{Q}$. For the first form this is clear—it is positive–definite. We extend $\langle 1, 1, -7, -7 \rangle$ to $\mathbf{Q}_7$ and split again:

$$\langle 1, 1, -7, -7 \rangle = \langle 1, 1 \rangle \oplus (-7)\langle 1, 1 \rangle. \tag{13.3}$$

Now $\langle 1, 1 \rangle$ is anisotropic over $\mathbf{F}_7$, which finishes the proof. ◇

Not all forms of norm 6 are susceptible to this argument, however. For example, consider

$$q' = \langle 1, 1, 1, 5, x, -5x \rangle \tag{13.4}$$

over the same field $\mathbf{Q}((x))$. This form is in $I^2\mathbf{Q}((x))$, has norm 6, but $2q'$ is isotropic, hence has norm ≤ 10. We do not know whether q' is realizable as the Witt class over Σ_2.

Data availability There is no data associated to this article.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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