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An analytic proof of the stable reduction theorem

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Abstract

The stable reduction theorem says that a family of curves of genus $g \geq 2$ over a punctured curve can be uniquely completed (after possible base change) by inserting certain stable curves at the punctures. We give a new this result for curves defined over $\mathbb{C}$, using the Kähler–Einstein metrics on the fibers to obtain the limiting stable curves at the punctures.

1 Introduction

Let $X_1, X_2, \ldots$ be a sequence of compact Riemann surfaces of genus $g \geq 2$. A consequence of the Deligne–Mumford construction of moduli space is the following. There exists $N > 0$ and imbeddings $T_i : X_i \hookrightarrow \mathbb{P}^N$ such that after passing to a subsequence, $T_i(X_i) = W_i \subseteq \mathbb{P}^N$ converges to a stable algebraic curve, i.e. a curve $W_\infty \subseteq \mathbb{P}^N$ whose singular locus is either empty or consists of nodes, and whose smooth locus carries a metric of constant negative curvature. The stable reduction theorem [6, 7] (stated below) is the analogue of this result with $\{X_i : i \in \mathbb{N}\}$ replaced by an algebraic family $\{X_t : t \in \Delta^*\}$ where $\Delta^* \subseteq \mathbb{C}$ is the punctured unit disk.

The imbeddings T_i are determined by a canonical (up to a uniformly bounded automorphism) basis of $H^0(X_i, mK_{X_i})$ (here $m \geq 3$ is fixed). We are naturally led to

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ask: Can one construct the canonical basis defining T_i explicitly? In Theorem 1.1 we give an affirmative answer to this question.

The main goal of this paper is to give an independent analytic proof of these algebraic compactness results, which is the content of Theorem 1.2. We start with the Bers compactness theorem, which says that after passing to a subsequence, the X_i converge to a nodal curve in the Cheeger–Colding topology. We then use the technique of Tian [18] Donaldson–Sun [9] which uses the Kähler–Einstein metric to build a bridge between analytic convergence (in Teichmuller space) to algebraic convergence (in projective space). The main difficulty is that unlike the [9, 18] settings, the diameters of the X_i are unbounded and as a consequence, some of the pluri-canonical sections on X_∞ are not members of $L^2(X_\infty, \omega_{\mathrm{KE}})$, so one can't apply the L^2-Bergman imbedding/peak section method directly. In order to solve this problem, we introduce the "ϵ-Bergman inner product" on the vector space $H^0(X_i, mK_{X_i})$, which is defined by the L^2 norm on the thick part of the X_i (unlike the standard Bergman inner product which is the L^2 norm defined by integration on all of X_i) and we show that for fixed $m \geq 3$ the canonical basis defining T_i is an an orthonormal basis for this new inner product. This establishes Theorem 1.1 which we then use to prove Theorem 1.2 (the stable reduction theorem).

We start by reviewing the corresponding compactness results for Fano manifolds established by Tian [18] in dimension two, and Donaldson–Sun [9] in higher dimensions. Let (X_i, ω_i) be a sequence of Kähler–Einstein manifolds of dimension n with $c_1 > 0$, volume at least V and diameter at most D, normalized so that $\mathrm{Ric}(\omega_i) = \omega_i$. The first step in the proof of the Donaldson–Sun theorem is the application of Gromov's compactness theorem which implies that after passing to a subsequence, X_i converges to a compact metric space X_∞ of dimension n in the metric sense, i.e. the Cheeger–Colding (CC) sense. This first step is not not available in the $c_1 < 0$ case due to the possibility of collapsing and unbounded diameter. Nevertheless, the analogue of this Cheeger–Colding property for Riemann surfaces of genus $g \geq 2$ is available thanks to the compactness theorem of Bers [2].

For the second step, Donaldson–Sun construct explicit imbeddings $T_i : X_i \hookrightarrow \mathbb{P}^N$ with the following properties. Let $X_i \to X_\infty$ in the Cheeger–Colding sense as above. Then there is a K-stable algebraic variety $W_\infty \subseteq \mathbb{P}^N$ such that if $W_i = T_i(X_i)$ then $W_i \to W_\infty$ in the algebraic sense (i.e. as points in the Hilbert scheme). Moreover, $T_\infty : X_\infty \to W_\infty$ is a homeomorphism, biholomorphic on the smooth loci, where

$$T_\infty(x_\infty) = \lim_{i\to\infty} T_i(x_i) \quad \text{whenever} \quad x_i \to x_\infty. \tag{1.1}$$

We summarize this result with the following diagram:

$$\begin{array}{ccccc} X_i & \xrightarrow{T_i} & W_i & \hookrightarrow & \mathbb{P}^N \\ {\scriptstyle \mathrm{CC}}\downarrow & & \downarrow{\scriptstyle \mathrm{Hilb}} & & \\ X_\infty & \xrightarrow{T_\infty} & W_\infty & \hookrightarrow & \mathbb{P}^N \end{array} \tag{1.2}$$

Here the vertical arrows represent convergence in the metric (Cheeger–Colding) sense and the the algebraic (Hilbert scheme) sense respectively. The horizontal arrows isomorphisms: T_i is an algebraic isomorphism, and T_∞ is a holomorphic isomorphism. For $1 \le i \le \infty$, the maps $W_i \hookrightarrow \mathbb{P}^N$ are inclusions.

The imbeddings $T_i : X_i \to \mathbb{P}^N$ are the so called "Bergman imbeddings". This means $T_i = (s_0, \ldots, s_N)$ where the s_α form an orthonormal basis of $H^0(X_i, -mK_{X_i})$ with respect to the Bergman inner product:

$$\int_{X_i} (s_\alpha, s_\beta)\, \omega_i^n \;=\; \delta_{\alpha,\beta} \tag{1.3}$$

Here m is a fixed integer which is independent of i and the pointwise inner product is defined by $(s_\alpha, s_\beta) = s_\alpha \bar{s}_\beta \omega_i^m$. Since the definition of T_i depends on the choice of orthonormal basis $\underline{s} = (s_0, \ldots, s_N)$, we shall sometimes write $T_i = T_{i,\underline{s}}$ when we want to stress the dependence on $\underline{s}$.

Thus we assume $\mathrm{Ric}(\omega_i) = -\omega_i$ and we wish to construct imbeddings $T_i : X_i \to \mathbb{P}^N$ such that the sequence $W_i = T_i(X_i) \subseteq \mathbb{P}^N$ converges to a singular Kähler–Einstein variety W_∞ with $K_{W_\infty} > 0$.

The condition that W_∞ is a "singular Kähler–Einstein variety" can be made precise as follows. Let $W \subseteq \mathbb{P}^N$ be a projective variety with K_W ample. The work of Berman–Guenancia [1] combined with the results of Odaka [13] tell us that the following conditions are equivalent.

(1) There is a Kähler metric ω on W^{reg} such that $\mathrm{Ric}(\omega) = -\omega$ satisfying the volume condition $\int_{W^{\mathrm{reg}}} \omega^n = c_1(K_W)^n$. skip.02in
(2) W has at worst semi-log canonical singularities.
(3) W is K-stable

We wish to construct T_i in such a way that $W_\infty = \lim_{i\to\infty} T_i(X_i)$ has at worst semi-log canonical singularities. In this paper we restrict our attention to the case $n = 1$.

Our long-term goal is to generalize the above theorem of [9] to the case where the (X_i, ω_i) are smooth canonical models, of dimension n, i.e. X_i is smooth and $c_1(X_i) < 0$. The proof we present here is designed with that goal in mind. There are other approaches, but this is the one that seems to lend itself most easily to generalization. We have been able to extend the techniques to the case of dimension two, but that will be the subject a future paper.

Remark 1.1 One might guess, in parallel with the Fano setting, that the $T_i : X_i \to \mathbb{P}^N$ should be the pluricanonical Bergman imbeddings, that is $T_i = T_{i,\underline{s}}$ where $\underline{s} = (s_0, \ldots, s_N)$ and the s_α form an orthonormal basis of $H^0(X_i, mK_{X_i})$ with respect to the inner product (1.3). But as we shall see, this does not produce the correct limit, i.e. W_∞, the limiting variety, is not stable. In order to get the right imbedding into projective space, we need to replace $T_{i,\underline{s}}$ with $T^\epsilon_{i,\underline{s}}$, the so called ϵ-Bergman imbedding, defined below.

We first need to establish some notation. Fix $g \ge 2$ and $\epsilon > 0$. If X is a compact Riemann surface of genus g, or more generally a stable analytic curve (i.e. a Riemann surface with nodes whose universal cover is the Poincaré disk) of genus g, we define the ϵ-thick part of X to be

$$X_\epsilon \;=\; \{x \in X : \mathrm{inj}_x \geq \epsilon\}$$

Here inj_x is the injectivity radius at x and the metric ω on X is the unique hyperbolic metric satisfying $\mathrm{Ric}(\omega) = -\omega$. It is well known that there exists $\epsilon(g) > 0$ such that for all X of genus g, and for all $0 < \epsilon < \epsilon(g)$, that $X \backslash X_\epsilon$ is a finite disjoint union of holomorphic annuli.

Next we define the "ϵ-Bergman imbedding" $T^\epsilon_{\underline{s}} : X \to \mathbb{P}^N$. Fix $0 < \epsilon < \epsilon(g)$ and fix $m \geq 3$. For each stable analytic curve of genus g, we choose a basis $\underline{s} = \{s_0, \ldots, s_{N_m}\}$ of $H^0(X, mK_X)$ such that

$$\int_{X_\epsilon} (s_\alpha, s_\beta)\, \omega \;=\; \delta_{\alpha,\beta}$$

Here $(s_\alpha, s_\beta) = s_\alpha \bar{s}_\beta \omega_i^{-m}$ is the usual pointwise inner product. Such a basis is uniquely determined up to the action of $U(N+1)$. Let $T^\epsilon_{\underline{s}} : X \hookrightarrow \mathbb{P}^{N_m}$ be the map $T^\epsilon_{\underline{s}} = (s_0, \ldots, s_{N_m})$. Let $W = T^\epsilon_{\underline{s}}(X)$. One easily checks that W is a stable algebraic curve and $T^\epsilon_{\underline{s}} : X \to W$ is a biholomorphic map. In particular, we have the following simple lemma.

Lemma 1.1 *If X_0 and X_0' are stable analytic curves, and $\underline{s}$, $\underline{s}'$ are orthonormal bases for $H^0(X_0, mK_{X_0})$ and $H^0(X_0', mK_{X_0'})$ respectively, then the following conditions are equivalent*

(1) $X_0 \;\approx\; X_0'$ *(i.e. X_0 and X_0' are biholomorphic).*
(2) $[T^\epsilon_{\underline{s}'}(X_0')] \;\in\; U(N+1) \cdot [T^\epsilon_{\underline{s}}(X_0)]$
(3) $[T^\epsilon_{\underline{s}'}(X_0')] \;\in\; SL(N+1, \mathbb{C}) \cdot [T^\epsilon_{\underline{s}}(X_0)]$

Here $[T^\epsilon_{\underline{s}} X_0] \in$ Hilb is the point representing $T^\epsilon_{\underline{s}} X_0 \subseteq \mathbb{P}^N$ in Hilb, *the Hilbert scheme.*

Now let X_i be a sequence of stable analytic curves of genus g (e.g Riemann surfaces of genus g). Then a basic theorem of Bers [2] (we shall outline the proof below) says there exists a stable analytic curve X_∞ (for a precise definition see Definition 2.1) such that after passing to a subsequence, $X_i \to X_\infty$. By this we mean $X_i^{\mathrm{reg}} \to X_\infty^{\mathrm{reg}}$ in the pointed Cheeger–Colding topology (see Definition 2.2). Here, for $1 \leq i \leq \infty$, $X_i^{\mathrm{reg}} \subseteq X_i$ is the smooth locus. This provides the analogue of the left vertical arrow in (1.2).

Theorem 1.1 *Let X_i be a sequence of stable analytic curves of genus g. After passing to a subsequence we have $X_i \to X_\infty$ in the Cheeger–Colding sense as above. Then there is a stable algebraic curve W_∞ and orthonormal bases $\underline{s}_i$ of $H^0(X_i, mK_{X_i})$, such that if $W_i = T_i^\epsilon(X_i)$ then $W_i \to W_\infty$ in the algebraic sense, i.e. as points in the Hilbert scheme. Moreover, $T_\infty|_{X_i^{\mathrm{reg}}}$ satisfies property* (1.1).

The idea of using Teichmuller theory to understand moduli space was advocated by Bers [2–5] in a project he initiated, and which was later completed by Hubbard–Koch [11]. They define an analytic quotient of "Augmented Teichmuller Space" whose quotient by the mapping class group is isomorphic to compactified moduli space as analytic spaces. Our approach is different and is concerned with the imbedding of the universal curve into projective space.

Remark 1.2 . As we vary ϵ, the maps T_i^ϵ differ by uniformly bounded transformations. We shall see that if $0 < \epsilon_1, \epsilon_2 < \epsilon(g)$ then $T_i^{\epsilon_1} = g_i \circ T_i^{\epsilon_2}$ where the change of basis matrices $g_i \in GL(N+1, \mathbb{C})$ converge: $g_i \to g_\infty \in GL(N+1, \mathbb{C})$. In particular, $\lim_i T_i^{\epsilon_1}(X_i)$ and $\lim_i T_i^{\epsilon_2}(X_i)$ are isomorphic.

As a corollary of our theorem we shall give a "metric" proof of the stable reduction theorem due to Deligne–Mumford [6, 7]:

Theorem 1.2 *Let C be a smooth curve and $f : \mathcal{X}^0 \to C^0$ be a flat family of stable analytic curves over a Zariski open subset $C^0 \subseteq C$. Then there exist a branched cover $\tilde{C} \to C$ and a flat family $\tilde{f} : \tilde{\mathcal{X}} \to \tilde{C}$ of stable analytic curves extending $\mathcal{X}^0 \times_{\tilde{C}} C^0$. Moreover, the extension is unique up to finite base change.*

In addition we show that the central fiber can be characterized as the Cheeger–Colding limit of the general fibers. More precisely:

Proposition 1.1 *Endow X_t with its unique Kähler–Einstein metric normalized so that* $\mathrm{Ric}(\omega_t) = -\omega_t$. *Then for every $t \in C^0$ there exist points $p_t^1, \ldots, p_t^\mu \in X_t := f^{-1}(t)$ such that the pointed Cheeger–Colding limits $Y_j = \lim_{t\to 0}(X_t, p_t^j)$ are the connected components of $\tilde{X}_0 \backslash \Sigma$ where $\tilde{X}_0 := \tilde{f}^{-1}(0)$ and $\Sigma \subseteq \tilde{X}_0$ is the set of nodes of $\tilde{X}_0$. Moreover the limiting metric on X_∞ is its unique Kähler–Einstein metric.*

Remark 1.3 A slightly modified proof also gives the log version of stable reduction, i.e for families (X_t, D_t) where D_t is an effective divisor supported on n points and $K_{X_t} + D_t$ is ample. We indicate which modifications are necessary at the end of Sect. 3.

Remark 1.4 In [16] and [17], Theorems 1.1 and Corollary 2.1 are shown to hold for smooth canonical models of dimension $n > 1$. But these papers assume the general version of Theorem 1.2, i.e. of stable reduction. In this paper we do not make these assumptions. In fact, our main purpose here is to prove these algebraic geometry results using analytic methods.

We shall first prove Theorem 1.1 under the assumption that the X_i are smooth, and Theorem 1.2 under the assumption that the generic fiber of f smooth. Afterwards we will treat the general case.

2 Background

Let X be a compact connected Hausdorff space, let $r \geq 0$ and $\Sigma = \{z_1, \ldots, z_r\} \subseteq X$. We say that X is a nodal analytic curve if $X \backslash \Sigma$ is a disjoint union $Y_1 \cup \cdots \cup Y_\mu$ of punctured compact Riemann surfaces and if for every $z \in \Sigma$, there is a small open set $z \in U \subseteq X$ and a continuous function

$$f : U \to \{(x, y) \in \mathbb{C}^2 : xy = 0\}$$

with the properties:

(1) $f(z) = (0, 0)$
(2) f is a homeomorphism onto its image
(3) $f|_{U\setminus\{z\}}$ is holomorphic

If $r = 0$ then X is a compact Riemann surface.

Definition 2.1 We say that a nodal analytic curve X is a stable analytic curve if each of the Y_j is covered by the Poincaré disk. In other words, each of the Y_j carries a unique hyperbolic metric (i.e. a metric whose curvature is -1) with finite volume.

If X is a stable analytic curve we let K_X be its canonical bundle. Thus the restriction of K_X to $X\setminus\Sigma$ is the usual canonical bundle. Moreover, in the neighborhood of a point $z \in \Sigma$, that is in a neighborhood of of $\{uv = 0\} \subseteq \mathbb{C}^2$, a section of K_X consists of a pair of meromorphic differential forms η_1 and η_2 defined on $u = 0$ and $v = 0$ respectively, with the following properties: both are holomorphic away from the origin, both have at worst simple poles at the origin, and $res(\eta_1) + res(\eta_2) = 0$.

We briefly recall the proof of the above characterization of K_X for nodal curves. A nodal singularity is Spec(B) where $B = \mathbb{C}[U, V]/(V^2 - U^2)$. Then $\mathbb{C}[U] \to \mathbb{C}[U, V]$ is generated by V which satisfies the monic equation $V^2 - U^2 = 0$. According the Lipman's characterization of the canonical sheaf [12] if $B = C[V]/(f)$ where $C = \mathbb{C}[U_1, \ldots, U_n]$ and f is a monic polynomial in V with coefficients in C, and if $X = \text{Spec}(B)$, then K_X is the sheaf of holomorphic (n, n) forms on X_{reg} which can be written as $F \cdot \frac{\pi^*(du^1 \wedge \cdots du^n)}{f'(v)}$ where $\pi : X \to \text{Spec}(C)$ and F is a regular function on X. In our case, $f(V) = V^2 - U^2$ so $f'(V) = 2V$ which means that K_X is free of rank one, generated by $\frac{du}{2v}$ or equivalently $\frac{du}{v}$. If we consider the map $\mathbb{C} \to X$ given by $t \mapsto (t, t)$ then $\frac{du}{v}$ pulls back to $\frac{dt}{t}$. On the other hand, if we consider $t \mapsto (t, -t)$ then $\frac{du}{v}$ pulls back to $-\frac{dt}{t}$.

If X is a compact Riemann surface of genus $g \geq 2$, then $\text{vol}(X) = 2g - 2$. If X is a stable analytic curve, we say that X has genus g if $\sum_j \text{vol}(Y_j) = 2g - 2$. Here the volumes are measured with respect to the hyperbolic metric and the Y_j are the irreducible components of X^{reg}.

Let X be a stable analytic curve. The following properties of K_X are proved in Harris-Morrison [10]:

(1) $h^0(X, mK_X) = (2m - 1)(g - 1) := N_m - 1$ if $m \geq 2$.
(2) mK_X is very ample if $m \geq 3$
(3) If $m \geq 3$ the m-pluricanonical imbedding of X is a stable algebraic curve in $\mathbb{P}^{N_m}$

Next we recall some basic results from Teichmuller theory. Fix $g > 0$ and fix S, a smooth surface of genus g. Teichmuller space $\mathcal{T}_g$ is the set of equivalence classes of pairs (X, f) where X is a compact Riemann surface of genus g and $f : S \to X$ is a diffeomorphism. Two pairs (X_1, f_1) and (X_2, f_2) are equivalent if there is a bi-holomorphic map $h : X_1 \to X_2$ such that $f_2^{-1} \circ h \circ f_1 : S \to S$ is in $\text{Diff}_0(S)$, diffeomorphisms isotopic to the identity. The pair (X, f) is called a "marked Riemann surface". The space $\mathcal{T}_g$ has a natural topology: A sequence $\tau_n \in \mathcal{T}_g$ converges to τ_∞

if we can find representatives $f_n : S \to X_n$, $1 \le n \le \infty$ such that the sequence of diffeomorphisms $f_\infty^{-1} \circ h \circ f_n$ converges to the identity.

The space $\mathcal{T}_g$ has a manifold structure given by Fenchel–Nielsen Coordinates whose construction we now recall. Choose a graph Γ with the following properties: Γ has $2g - 2$ vertices, each vertex is connected to three edges (which are not necessarily distinct since we allow an edge to connect a vertex to itself). For example, if $g = 2$, then there are two such graphs: Either v_1 and v_2 are connected by three edges, or they are connected by one edge, and each connected to itself by one edge.

Fix such a graph Γ. It has $3g - 3$ edges. Fix an ordering $e_1, \ldots, e_n$ on the edges where $n = 3g - 3$. Once we fix Γ and we fix an edge ordering, we can define a map $(\mathbb{R}_+ \times \mathbb{R})^n \to \mathcal{T}_g$ as follows. Given $(l_1, \theta_1, \ldots, l_n, \theta_n) \in \mathbb{R}^{2n}$ we associate to each vertex $v \in \Gamma$ the pair of pants whose geodesic boundary circles have lengths (l_i, l_j, l_k) where e_i, e_j, e_k are the three edges emanating from v. Each of those circles contains two canonically defined points, which are the endpoints of the unique geodesic segment joining it to the other geodesic boundary circles.

If all the $\theta_j = 0$, then we join the pants together, using the rules imposed by the graph Γ, in such a way that canonical points are identified. If some of the θ_j are non-zero, then we rotate an angle of $l_j\theta_j$ before joining the boundary curves together.

Thus we see that $\mathcal{T}_g$ is a manifold which is covered by a finite number of coordinate charts corresponding to different graphs Γ (each diffeomorphic to $(\mathbb{R}_+ \times \mathbb{R})^n$) If we allow some of the l_j to equal zero, then we can still glue the pants together as above, but this time we get a nodal curve. In this way, $(\mathbb{R}_{\ge 0} \times \mathbb{R})^n$ parametrizes all stable analytic curves.

Teichmuller proved that the manifold $\mathcal{T}_g$ has a natural complex structure, and that there exists a universal curve $\mathcal{C}_g \to \mathcal{T}_g$, which is a map between complex manifolds, such that the fiber above $(X, f) \in \mathcal{T}_g$ is isomorphic to X. Moreover, if $\mathcal{X} \to B$ is any family of marked Riemann surfaces, then there exists a unique holomorphic map $B \to \mathcal{T}_g$ such that $\mathcal{X}$ is the pullback of $\mathcal{C}_g$. Fenchel–Nielsen coordinates are compatible with the complex structure, i.e. they are smooth, but not holomorphic (although they are real-analytic).

Remark 2.1 One consequence of Teichmuller's theorem is the following. Let $\mathcal{X} \to B$ be a holomorphic family of marked Riemann surfaces and let $F : B \to (\mathbb{R}_+ \times \mathbb{R})^n$ be the map that sends t to the Fenchel–Nielsen coordinates of X_t. Then F is a smooth function. In particular, $X_t \to X_0$. This shows that in the stable reduction theorem, if a smooth fill-in exists then it is unique.

Now let X be a compact Riemann surface. A theorem of Bers [2], Theorem 15 (a sharp version appears in Parlier [15], Theorem 1.1) says that for $g \ge 2$ there exists a constant $C(g)$, now known as the Bers constant, with the following property. For every Riemann surface X of genus g there exists a representative $\tau = (X, f) \in \mathcal{T}_g$ and a graph Γ (i.e. a coordinate chart) such that the Fenchel–Nielsen coordinates of τ are all bounded above by $C(g)$. This is analogous to the fact that $\mathbb{P}^N$ is covered by $N + 1$ coordinate charts, each biholomorphic to $\mathbb{C}^N$, and that give a point $x \in \mathbb{P}^N$ we can choose a coordinate chart so that $x \in \mathbb{C}^N$ has the property $|x_j| \le 1$ for all j. In particular, this proves $\mathbb{P}^N$ is sequentially compact.

Bers [2] uses the existence of the Bers constant to show that the space of stable analytic curves is compact with respect to a natural topology (equivalent to the Cheeger–Colding topology). For the convenience of the reader, we recall the short argument. Let X_j be a sequence of Riemann surfaces. Then after passing to a subsequence, there is a graph Γ and representatives $\tau_j = (X_j, f_j) \in \mathcal{T}_g$ such that the Fenchel–Nielsen coordinates of τ_j with respect to Γ are all bounded above by $C(g)$ (this is due to the fact that there are only finite many allowable graphs). After passing to a further subsequence, we see $\tau_j \to \tau_\infty \in (\mathbb{R}_{\geq 0} \times \mathbb{R})^n$. If $\tau_\infty \in (\mathbb{R}_+ \times \mathbb{R})^n$ then the limit is a smooth Riemann surface. Otherwise, it is a stable analytic curve X_∞. Thus

$$X_\infty = \cup_{\alpha=1}^{\mu} X^\alpha, \quad \text{and} \quad X_\infty^{\mathrm{reg}} = \sqcup_{\alpha=1}^{\mu} Y^\alpha \tag{2.4}$$

where the second union is disjoint, and $Y^\alpha = X^\alpha \backslash F^\alpha$ where X^α is a compact Riemann surface and $F^\alpha \subseteq X^\alpha$ a finite set, consisting of the cusps.

Corollary 2.1 *Let $p_\infty^\alpha \in Y^\alpha$. Then there exist $p_i^1, \ldots, p_i^\mu \in X_i$ such that in the pointed Cheeger–Colding topology, $(Y^\alpha, p_\infty^\alpha) = \lim_{j\to\infty}(X_j, p_j^\alpha)$. Moreover, for every open set $p_\infty^\alpha \in U_\infty^\alpha \subseteq\subseteq Y^\alpha$ there exist open sets $p_i^\alpha \subseteq U_i^\alpha \subseteq X_i$ and diffeomorphisms $f_j^\alpha : U_\infty^\alpha \to U_j^\alpha$ so that $(f_j^\alpha)^*\omega_j^\alpha \to \omega_\infty^\alpha$ and $(f_j^\alpha)^* J_j^\alpha \to J_\infty^\alpha$ where ω_j^α and ω_∞^α are the hyperbolic metrics on U_j^α and U_∞^α, and J_j^α and J_∞^α are the complex structures on U_j^α and U_∞^α*

Definition 2.2 In the notation of Corollary 2.1, we shall say $\omega_j \to \omega_\infty$ in the pointed Cheeger–Colding sense and we shall write $X_i \to X_\infty$.

Remark: Odaka [14] uses pants decompositions to construct a "tropical compactification" of moduli space which attaches metrized graphs (of one real dimension) to the boundary of moduli space. These interesting compactifications are compact Hausdorff topological spaces but are no longer algebraic varieties.

3 Limits of Bergman imbeddings

Now let $\mathcal{X}$ be as in the theorem, and let $t_i \in C^0$ with $t_i \to 0$. Let $X_i = X_{t_i}$ and fix a pants decomposition of X_i. Then Bers' theorem implies that after passing to a subsequence we can find a nodal curve X_∞ as above so that $X_j \to X_\infty$.

In order to prove the theorem, we must show:

(1) X_∞ is independent of the choice of subsequence.
(2) After making a finite base change, we can insert X_∞ as the central fiber in such a way that the completed family is algebraic.

We begin with (2). Let X be a hyperbolic Riemann surface with finite area (i.e. possibly not compact, but only cusps). The Margulis "thin-thick decomposition" says that there exists $\epsilon(g) > 0$ with the following property. There exists at most $3g - 3$ closed geodesics of length less that $\epsilon(g)$. Moreover, for every $\epsilon \leq \epsilon(g)$ the set

$$X \backslash X_\epsilon \ = \ \{x \in X : \mathrm{inj}_x < \epsilon\}$$

is a finite union of of holomorphic annuli (which are open neighborhoods of short geodesics) if X is compact, and a finite union of annuli as well as punctured disks, which correspond to cusp neighborhoods if X is has singularities. We call these annuli "Margulis annuli". Moreover, $V(\epsilon)$, the volume of $X\backslash X_\epsilon$, has the property $\lim_{\epsilon\to 0} V(\epsilon) = 0$. An elementary proof is given in Proposition 52, Chapter 14 of Donaldson [8].

Now we define a modified Bergman kernel as follows: For convenience we write $\epsilon = \epsilon(g)$. This is a positive constant, depending only on the genus g. Let X be a stable analytic curve. For $\eta_1, \eta_2 \in H^0(X, mK_X)$ let

$$\langle \eta_1, \eta_2\rangle_\epsilon = \int_{X_\epsilon} \eta_1\bar{\eta}_2 h^m_{KE}\omega_{KE} \tag{3.5}$$

and $\|\eta\|^2_\epsilon = \langle \eta, \eta_\epsilon$. If we replace X_ϵ by X, we get the standard Bergman inner product.

Now fix $m \geq 3$. Choosing orthonormal bases with respect to the inner product (3.5) defines imbeddings $T_i^\epsilon : X_i \to \mathbb{P}^{N_m}$ and $T_\infty^\epsilon : X_\infty \to \mathbb{P}^{N_m}$, which we call ϵ-Bergman imbeddings. Our goal is to show

Theorem 3.1 *Let $X_1, X_2, \ldots$ be a sequence of stable analytic curves of genus g. Then there exists a stable analytic curve X_∞ such that after passing to a subsequence if necessary, $X_i \to X_\infty$ in the Cheeger–Colding topology. For $1 \leq i < \infty$, we fix an orthonormal basis $\underline{s}_i$ of $H^0(X_i, mK_X)$. Then there exists a choice of orthonormal basis $\underline{s}_\infty$ for X_∞ such that after passing to a subsequence,*

$$\lim_{i\to\infty} T^\epsilon_{i,\underline{s}_i} = T^\epsilon_{\infty,\underline{s}_\infty} \tag{3.6}$$

In other words, if $x_i \in X_i$ and $x_\infty \in X_\infty$ with $x_i \to x_\infty$, then

$$T_i^\epsilon(x_i) \to T_\infty^\epsilon(x_\infty)$$

We assume first that the X_i are smooth and then later explain how to remove this assumption. The proof of Theorem 3.1 rests upon the following.

Theorem 3.2 *Fix $g \geq 2$ and $m, \epsilon > 0$. Then there exist $C(g, m, \epsilon)$ with the following property.*

$$\|s\|_\epsilon \leq \|s\|_{\epsilon/2} \leq C(g, m, \epsilon)\|s\|_\epsilon$$

for all Riemann surfaces X of genus g and all $s \in H^0(X, mK_X)$.

To prove the theorem, we need the following adapted version of a result of Donaldson–Sun. We omit the proof which is very similar to [9] (actually easier since the only singularities of X_∞ are nodes so the pointed limit of the X_i in the Cheeger–Colding topology is smooth).

Proposition 3.1 *Let $X_i \to X_\infty$ be a sequence of Riemann surfaces of genus g converging in the pointed Cheeger–Colding sense to a stable curve X_∞. Fix $\{s_0^\infty, \ldots, s_M^\infty\} \subseteq$*

$H^0(X_\infty, mK_{X_\infty})$ an ϵ-orthonormal basis of the bounded sections (i.e. the $L^2(X_\infty)$ sections, i.e. the sections which vanish at all nodes). Then there exists an ϵ-orthonormal subset

$$\{s_0^i, \ldots, s_M^i\} \subseteq H^0(X_i, mK_{X_i})$$

such that for $0 \le \alpha \le M$, we have

$$s_\alpha^i \to s_\alpha^\infty$$

in L^2 and uniformly on compact subsets of X_∞^{reg}. In particular, if $x_i \in X_i^{\text{reg}}$

$$x_i \to x_\infty \iff s_\alpha^i(x_i) \to s_\alpha^\infty(x_\infty) \text{ for all } 0 \le \alpha \le M \iff T_i^{\nu,\epsilon}(x_i) \to T_\infty^{\nu,\epsilon}(x_\infty). \tag{3.7}$$

where $T_i^{\nu,\epsilon} : X_i^{\text{reg}} \hookrightarrow \mathbb{P}^M$ is the map $x_i \mapsto (s_0^i, \ldots, s_M^i)(x_i)$ for $1 \le i \le \infty$.

Proof of Theorem 3.2 Let $X_i \to X_\infty$ as in Proposition 3.1. Choose $(s_0^\infty, \ldots, s_M^\infty)$ and $(t_0^\infty, \ldots, t_M^\infty)$ which are ϵ and $\epsilon/2$ orthonormal bases of the subspace of bounded sections in $H^0(X_\infty, mK_{X_\infty})$ in such a way that $t_\alpha^\infty = \lambda_\alpha^\infty s_\alpha^\infty$ for real numbers $0 < \lambda_\alpha^\infty < 1$. Choose $s_\alpha^i \to s_\alpha^\infty$ and $t_\alpha^i \to t_\alpha^\infty$ as in Proposition 3.1 in such a way that $t_\alpha^i = \lambda_\alpha^i s_\alpha^i$ with $0 < \lambda_\alpha^i < 1$. Clearly

$$\lambda_\alpha^i \to \lambda_\alpha^\infty > 0 \quad \text{for} \quad 0 \le \alpha \le M \tag{3.8}$$

Choose additional sections s_α^i and t_α^i for $M+1 \le \alpha \le N$ so that $\{s_0^i, \ldots, s_N^i\}$ and $\{t_0^i, \ldots, t_N^i\}$ are ϵ and $\epsilon/2$ bases of $H^0(X_i, mK_{X_i})$ and $t_\alpha^i = \lambda_\alpha^i s_\alpha^i$ with $0 < \lambda_\alpha^i < 1$ for $0 \le \alpha \le N$.

Now assume the theorem is false. Then there exists $X_i \to X_\infty$ as above such that $\lambda_\alpha^i \to 0$ for some α. We must have $\alpha \ge M+1$ by (3.8). Choose $M+1 \le A < N$ such that $\lambda_\alpha^i \to 0$ if and only if $A \le \alpha \le N$. Since $\|s_\alpha^i\|_{L^2(X_\epsilon)} = 1$ we may choose $s_\alpha^\infty(\epsilon) \in H^0(X_i^\epsilon, K_{X_\infty}|_{X_\infty^\epsilon})$ such that

$$s_\alpha^i|_{X_i^\epsilon} \to s_\alpha^\infty(\epsilon) \text{ for } M+1 \le \alpha \le N \text{uniformly on compact subsets of } X_\epsilon \tag{3.9}$$

Let $T_i^\epsilon : X_i \to W_i^\epsilon \subseteq \mathbb{P}^N$ be the Kodaira map given by the sections $s_0^i, \ldots, s_N^i$ and let $W_\infty^\epsilon = \lim_{i\to\infty} W_i^\epsilon$. Let

$$T_\infty^\epsilon : X_\infty^\epsilon \hookrightarrow W_\infty^\epsilon \quad \text{and} \quad T_\infty^{\nu,\epsilon} : X_\infty^{\text{reg}} \hookrightarrow \mathbb{P}^M$$

be the Kodaira maps given by $(s_0^\infty, \ldots, s_M^\infty, s_{M+1}^\infty(\epsilon), \ldots, s_N^\infty(\epsilon))$ and $(s_0^\infty, \ldots, s_M^\infty)$. Thus

$$\pi \circ T_\infty^\epsilon = T_\infty^{\nu,\epsilon}|_{X_\infty^\epsilon} \tag{3.10}$$

where $\pi : \mathbb{P}^N_M := \mathbb{P}^N \backslash \{z_0 = \cdots = z_M = 0\} \to \mathbb{P}^M$ is defined by $(z_0, \ldots, z_N) \mapsto (z_0, \ldots, z_M)$. Moreover

$$\pi(W^\epsilon_\infty \cap \mathbb{P}^N_M) \subseteq T^{\nu,\epsilon}_\infty(X^{\text{reg}}_\infty)$$

Now the definition of A implies

$$T^{\epsilon/2}_\infty(X^\epsilon_\infty) \subseteq Z^{\epsilon/2}_\infty = \{z \in W^{\epsilon/2}_\infty : z_A = z_{A+1} = \cdots = z_N = 0\}$$

Thus (3.10) implies

$$T^{\nu,\epsilon/2}_\infty(X^{\text{reg}}_\infty) \supset \pi(Z^{\epsilon/2}_\infty \cap \mathbb{P}^N_M) \supset \pi(T^{\epsilon/2}_\infty(X^\epsilon_\infty)) = T^{\nu,\epsilon/2}_\infty(X^\epsilon_\infty)$$

Since the second set is constructible,

$$\pi(Z^{\epsilon/2}_\infty \cap \mathbb{P}^N_M) = T^{\nu,\epsilon/2}_\infty(X^{\text{reg}}_\infty \backslash \Sigma_\epsilon)$$

where $\Sigma_\epsilon \subseteq X^{\text{reg}}_\infty \backslash X^\epsilon_\infty$ is a finite set.

Let $x_\infty \in X^{\text{reg}}_\infty \backslash \Sigma_\epsilon$. Then $T^{\nu,\epsilon/2}_\infty(x_\infty) = \pi(w_\infty)$ for some $w_\infty \in Z^{\epsilon/2}_\infty \cap \mathbb{P}^N_M$. Choose $w_i \in W^{\epsilon/2}_i$ such that $w_i \to w_\infty$ and choose $x_i \in X_i$ such that $T^{\epsilon/2}_i(x_i) = w_i$. Then (3.7) implies

$$\begin{aligned} T^{\epsilon/2}_i(x_i) \to w_\infty &\implies \pi(T^{\epsilon/2}_i(x_i)) \to \pi(w_\infty) \\ &\implies T^{\nu,\epsilon/2}_i(x_i) \to T^{\nu,\epsilon/2}_\infty(x_\infty) \implies x_i \to x_\infty \end{aligned}$$

Thus we see that if $x_\infty \in X^{\text{reg}}_\infty \backslash \Sigma_\epsilon$ there exists $x_i \to x_\infty$ such that

$$\lim_{i\to\infty} T^{\epsilon/2}_i(x_i) \in Z^{\epsilon/2}_\infty$$

Let $x_\infty \in X^{\text{reg}}_\infty$. We say that x_∞ is an ϵ-*good* point if for every $x_i \to x_\infty$, $\lim_{i\to\infty} s^i_\alpha(x_i) = 0$ for all $A \le \alpha \le N$. The set of ϵ-bad points is finite (otherwise W^ϵ_∞ would have infinitely many components by the intermediate value theorem). Also, every point in $X^{2\epsilon}_\infty$ is ϵ-good.

Lemma 3.1 *Let $x_\infty \in X^{\text{reg}}_\infty$ and $A + 1 \le \alpha \le N$. Then for every $R > 0$ we have*

$$\lim_{i\to 0} ⨍_{B_R(x_i)} |t^i_\alpha|^2 = 0 \tag{3.11}$$

Proof Assume first that $B_{2R}(x_i)$ contains only good points. If (3.11) fails, there exists $c > 0$ such that for infinitely many i we have $⨍_{B_R(x_i)} |\tau_i|^2 \ge c$. Since $|\tau_i(x_\infty)| \to 0$ we see that $|\tau_i|^2(x'_i) = c$ for some $x'_i \in B_R(x_\infty)$. After passing to a subsequence $x'_i \to x'_\infty \in B_{2R}(x_\infty)$ and $|\tau_i|^2(x'_i) \to c$. This contradicts the assumption that $B_{2R}(x_i)$ contains only good points. To prove the lemma, it suffices to show that all points are good. Suppose not and assume x_∞ is a bad point and choose R so that x_∞ is the only bad point in $B_{2R}(x_\infty)$. Assume that (3.11) fails and that $\int_{B_R(x_i)} |t_i^\alpha|^2 > c\mathrm{vol}(B_R(x_\infty))$ for infinitely many i. Since all points in $B_R \backslash B_r$ are good, the previous step implies for every $0 < r < R$ and for i sufficiently large,

$$\fint_{B_r(x_i)} |t_i^\alpha|^2 \ \geq \ c \cdot \frac{\mathrm{vol}(B_R(x_\infty))}{\mathrm{vol}(B_r(x_\infty))} \text{and} \lim_{j\to\infty} \int_{B_R(x_j)\backslash B_r(x_j)} |t_j^\alpha|^2 \ = \ 0$$

But t_α^i is a holomorphic section so this contradicts the maximum principle if r is sufficiently small. $\square$

Now we can complete the proof of Theorem 3.2. Assume $A < N$ and fix $A + 1 \leq \alpha \leq N$. Choose $x_\infty \in X_\infty^{\mathrm{reg}}$ and $x_i \to x_\infty$. Choose $R > 0$ so that $X_i^{\epsilon/2} \subseteq B_R(x_i)$ for all i. By Lemma 3.1 we see that $1 = \int_{X^{\epsilon/2}} |t_\alpha^i|^2 \to 0$, a contradiction.

We conclude that if $\eta_j \in H^0(X_j, mK_{X_j})$ is a sequence such that the norms $\|\eta_j\|_\epsilon^2 = \langle \eta_j, \eta_j\ _\epsilon = 1$, then after passing to a subsequence, we have $(f_j^\alpha)^*\eta_j \to \eta_\infty$ for some $\eta_\infty \in H^0(X_\infty^{reg}, mK_{X_\infty}|_{X_\infty^{\mathrm{reg}}})$ with $\|\eta\|_\epsilon = 1$. Here the $f_j^\alpha : U_j^\alpha \to U^\alpha$ are as in the statement of Corollary 1 and this is true for all U^α and all α. Moreover, an orthornormal basis of $H^0(X_j, mK_{X_j})$, which is a vector space of dimension $(2m-1)(g-1)$, will converge to an orthonormal set of $(2m-1)(g-1)$ elements in $H^0(X_\infty^{reg}, mK_{X_\infty})$. The main problem is to now show that these $(2m-1)(g-1)$ elements extend to elements of $H^0(X_\infty, mK_{X_\infty})$. If they extend, then they automatically form a basis since $H^0(X_\infty, mK_{X_\infty})$ has dimension $(2m-1)(g-1)$ and this would prove Theorem 3.1.

To proceed, we make use of the discussion of the Margulis collar in section 14.4.1 of [8]. Let $\lambda > 0$ be the length of C a collapsing geodesic in X_j which forms a node in the limit in X_∞. We fix j and we write $X = X_j$. Let

$$A_\lambda \ = \ \{z \in \mathbb{C} : 1 \leq |z| \leq e^{2\pi\lambda},\ \lambda \leq \arg(z) \leq \pi - \lambda\}/\sim$$

where the equivalence relation identifies the circles $|z| = 1$ and $|z| = e^{2\pi\lambda}$. Then [8] shows A injects holomorphically into X in such a way that $1 \leq y \leq e^{2\pi\lambda}$ maps to C. The point is that the segment $1 \leq y \leq e^{2\pi\lambda}$ is very short - it has size λ. But the segments $A \cap \{\arg(z) = \lambda\}$ and $A \cap \{\arg(z) = \pi - \lambda\}$ have size 1. So for λ small, A is a topologically a cylinder, but metrically very long and narrow in the middle but not narrow at the ends. In other words, the middle of A is in the thin part, but the boundary curves are in the thick part.

The transformation

$$\tau = \exp\left(i\frac{\ln z}{\lambda}\right)$$

maps A_λ to the annulus

$$A'_\lambda \ = \ \{\exp(-(\pi - \lambda)/\lambda) \leq |\tau| \leq \exp(-1)\}$$

To summarize: We are given a sequence X_j, and a geodesic C_j in X_j that collapses to a node ν in Y^α for some α. We are also given a sequence of orthonormal bases $\{\eta_{j,1},\ldots,\eta_{j,N}\}$ of $H^0(X_j, kK_{X_j})$ where $N=(2k-1)(g-1)$ and $\eta_{j,\mu}\to\eta_{\infty,\mu}$. Here $\eta_{\infty,\mu}$ is a section of $kK_{X\infty}$ on X_∞^{reg}. Fix μ and write $\eta_j=\eta_{j,\mu}$ and $\eta_\infty=\eta_{\infty,\mu}$. We need to show that η_∞ extends to all of X_∞.

We may view η_j as a k form on A_{λ_j} or on A'_{λ_j} and η_∞ as a k form on the punctured disk A'_0. Write $\eta_j=f_j(z)dz^k=h_j(\tau)d\tau^k$ and $\eta_\infty=h_\infty(\tau)d\tau^k$. The discussion in [8] shows that if we fix a relatively compact open subset $U\subseteq A'_0$, then $h_j\to h_\infty$ uniformly on U (this makes sense since $U\subseteq A'_{\lambda_j}$ for j sufficiently large).

Since $\|\eta_j\|_{L^2}=1$ we have uniform sup norm bounds on the thick part of X_j. Thus

$$\|\eta_j\|_{L^\infty((X_i)_\epsilon}\ \le\ C(\epsilon) \tag{3.12}$$

We want to use (3.12) to get a bound on the thin part. In z coordinates, (3.12) implies

$$|\eta|_\omega\ =\ |\mathrm{Im}(z)|^k\cdot|f(z)|\ \le\ C(\epsilon)\ \text{ if}\arg(z)=\lambda\text{or}\arg(z)=2\pi-\lambda \tag{3.13}$$

since the boundary curves $\arg(z)=\lambda$ and $\arg(z)=2\pi-\lambda$ are in the thick part. Here we write η for η_j and f for f_j to lighten the notation.

Now

$$\mathrm{Im}(z)\ =\ -\exp(\lambda\arg\tau)(\sin(\lambda\ln|\tau|) \tag{3.14}$$

if we write $f(z)=g(\tau)$, then (3.13) implies

$$|g(\tau)|\le\frac{C(\epsilon)}{\lambda^k}\ \text{ for }\tau\in\partial A' \tag{3.15}$$

Since $f(z)dz^k=h(\tau)d\tau^k=g(\tau)(\frac{dz}{d\tau})^k\,d\tau^k$ and $\frac{dz}{d\tau}=\frac{z\lambda}{i\tau}$ we see for λ small

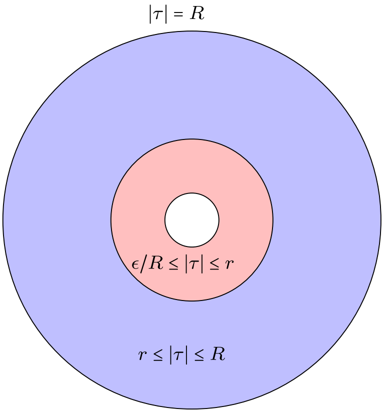
$$|h_j(\tau)|\ \le\frac{1}{\lambda^k}\left|\frac{dz}{d\tau}\right|^k\ =\ \frac{1}{\lambda^k}\frac{|z|^k\lambda^k}{|\tau|^k}\ \le\ \frac{2}{|\tau|^k}$$

where the last inequality follows from the fact $1\le|z|\le 2$. Writing

$$u_j(\tau)\ =\ h_j(\tau)\tau^k$$

Thus we see $|u_j(\tau)|\le 2$ for $\tau\in\partial A'$. The maximum principle now implies that $|u_j(\tau)|\le 2$ for $\tau\in A'$. Since this is true for all X_i, we see that any limit u_∞ must satisfy the same inequality in the limit of the annuli, which is a punctured disk: $|h_\infty(\tau)|\cdot|\tau|^k\le C$. This shows h_∞ has at most a pole of order k.

Fig. 1 A'_λ



Moreover $u(0)$ is the residue

$$u(0) \;=\; \lim_{j\to\infty} \frac{1}{2\pi\sqrt{-1}} \int_{|\tau|=r} u_j(\tau)\,\frac{d\tau}{\tau} \tag{3.16}$$

Here $0 < r \le \exp(-1)$ is any fixed number (independent of j).

To summarize, we have now seen that a collar degenerates to a union of two punctured disks and so the limit of the η_j is a pair of k forms, $\eta_\infty = u_\infty(\tau)\left(\frac{d\tau}{\tau}\right)^k$ and $\tilde\eta_\infty = \tilde u_\infty(\tau')\left(\frac{d\tau'}{\tau'}\right)^k$ where u and $\tilde u$ are holomorphic in a neighborhood of the origin in $\mathbb{C}$. There is one final condition that we need to check in order to verify that the limit is in $H^0(X_\infty, kK_{X_\infty})$: Let $R = \exp(-1)$, $r = \exp(-\pi/2\lambda_j)$ and $\epsilon = \exp(-\pi/\lambda_j)$ (so $\epsilon_{\lambda_j}/r = r$). We must show $\tilde u(0) = (-1)^k u(0)$.

To check this, let $\tilde\tau = \frac{\epsilon_j}{\tau}$. Then Fig. 1 remains the same, with τ replaced by $\tilde\tau$ and

$$f(z)dz^k \;=\; u_j(\tau)\left(\frac{d\tau}{\tau}\right)^k \;=\; u_j(\epsilon_j/\tilde\tau)(-1)^k\left(\frac{d\tilde\tau}{\tilde\tau}\right)^k \;=\; \tilde u_j(\tilde\tau)\left(\frac{d\tilde\tau}{\tilde\tau}\right)^k$$

Now we see

$$\int_{|\tau|=r} u_j(\tau)\,\frac{d\tau}{\tau} \;=\; (-1)\int_{|\tilde\tau|=\epsilon_j/r} u_j\left(\frac{\epsilon_j}{\tilde\tau}\right)(-1)\frac{d\tilde\tau}{\tilde\tau} \;=\; (-1)^k\int_{|\tilde\tau|=r} \tilde u_j(\tilde\tau)\,\frac{d\tilde\tau}{\tilde\tau}$$

In the second integral, the factor of (-1) outside the integral is due to the fact that the orientation of the circle has been reversed and the (-1) inside the integral comes from the change of variables. The second identity is a result of the fact that $u(\tilde\tau)$ is holomorphic on the annulus $\{\tilde\tau \in \mathbb{C} : \epsilon_j/r < \tilde\tau < r\}$. Taking limits as $j \to \infty$ we obtain $\tilde u(0) = (-1)^k u(0)$. This establishes Theorem 3.1 when the X_i are smooth.

Now assume the X_i are stable analytic curves, but not necessarily smooth. The Fenchel–Nielsen coordinates of X_i determine a point $[X_i] \in (\mathbb{R}_{\ge 0}\times\mathbb{R})^n$. The simple observation we need is that $(\mathbb{R}_{>0}\times\mathbb{R})^n \subseteq (\mathbb{R}_{\ge 0}\times\mathbb{R})^n$ is dense so we may choose a smooth Riemann surface $\tilde{X}_i$ such that $[X_i] \in (\mathbb{R}_{\geq 0} \times \mathbb{R})^n$ is ϵ_i close to $[X_i]$ where $\epsilon_i \to 0$ (i.e. X_i is smoothable). Now Corollary 2.1 implies that after passing to a subsequence, $\tilde{X}_i \to X_\infty$ in the pointed Cheeger–Colding topology. We conclude that $X_i \to X_\infty$ as well. Moreover, one easily sees that T_i^ϵ and $\tilde{T}_i^\epsilon$ have the same limit. This proves (3.6) and completes the proof of Theorem 3.1 □

Remark 3.1 The proof of the log version Theorem 3.1 is almost the same. The only observation we need is the following. If X is a compact Riemann surface and $D = p_1 + \cdots + p_n$ is a divisor supported on n points such that $K_X + D$ is ample, then $X \backslash D$ has a unique metric ω such that $\mathrm{Ric}(\omega) = -\omega$ and ω has cusp singularities at the points p_j. Moreover, just as in the case $n = 0$, X has a pants decomposition. The only difference is that we allow some of the length parameters to vanish, but this does not affect the arguments. In particular, we can use the Fenchel–Nielson coordinates to find a limit of the (X_j, D_j) (after passing to a subsequence) and the T_j^ϵ are defined exactly as before.

Now suppose X_i is a sequence of compact Riemann surfaces of genus g converging analytically to a nodal curve X_∞ and let η_i be a Kähler metric on X_i is the same class as the Kähler–Einstein metric ω_i. We have seen that $\omega_i \to \omega_\infty$, the Kähler–Einstein metric on X_∞, in the pointed Cheeger–Colding sense. Let $\tilde{\omega}_\infty$ be a Kähler metric on X_∞^{reg} and assume $\tilde{\omega}_i \to \tilde{\omega}_\infty$ in the pointed Cheeger–Colding sense. Let $T_i(\tilde{\omega}_i) : X_i \to \mathbb{P}^N$ be the embedding defined by an orthonormal basis of $H^0(X_i, 3K_{X_i})$ using the metric $\tilde{\omega}_i$ on the thick part of X_i and define $T_\infty(\tilde{\omega}_\infty) : X_\infty \to \mathbb{P}^N$ similarly. Thus the T_i and T_∞ of Theorem 2 can be written as $T_i(\omega_i)$ and $T_\infty(\omega_\infty)$ and in this notation, Theorem 2 says $T_i(\omega_i) \to T_\infty(\omega_\infty)$

Corollary 3.1 *After passing to a subsequence*

$$T_i(\tilde{\omega}_i) \;\to\; T_\infty(\tilde{\omega}_\infty)$$

Proof. Since $\tilde{\omega}_\infty$ and ω_∞ are equivalent on the thick part of X_∞, we see that

$$T_i(\tilde{\omega}_i) = \gamma_i \circ T_i(\omega_i)$$

where $\gamma_i \in GL(N+1, \mathbb{C})$ has uniformly bounded entries as does γ_i^{-1}. Thus after passing to a subsequence, $\gamma_i \to \gamma_\infty \in GL(N+1, \mathbb{C})$ and

$$\lim_{i\to\infty} T_i(\tilde{\omega}_i) \;=\; \lim_{i\to\infty} \gamma_i \circ T_i(\omega_\infty) \;=\; \gamma_\infty \circ T_\infty(\omega_\infty) \;=\; T_\infty(\tilde{\omega}_\infty)$$

Remark: The proof shows we only need to assume $\tilde{\omega}_i \to \tilde{\omega}_\infty$ on the thick part of X_∞.

4 Existence of stable fill-in

Proof of Theorem 1.2 Let $f : \mathcal{X}^0 \to C^0 = C\backslash\{p_1, \ldots, p_m\}$ be a flat family of stable analytic curves of genus $g \geq 2$. We first observe that we can find some completion (not necessarily nodal) $\mathcal{Y} \to C$ of the family $\mathcal{X}^0 \to C^0$. To see this let $\Omega_{\mathcal{X}^0/C^0}$ be the sheaf of relative differential forms (i.e. the relative canonical line bundle when $\mathcal{X}^0$ is smooth). Then the Hodge bundle $f_*K_{\mathcal{X}^0/C^0}$ is a vector bundle over C^0 of rank $3g-3$ (see page 694 of Vakil [V]) and $f_*K_{\mathcal{X}^0/C^0}^{\otimes m}$ is a vector bundle $\mathcal{E}_m^0$ of rank $N_m - 1 := (2m-1)(g-1)$ for $m \geq 2$. Choose $\mathcal{E}_m \to C$ an extension of the vector bundle $\mathcal{E}_m^0 \to C^0$ to the curve C.

For example, let $U \subseteq C^0$ be any affine open subset over which $\mathcal{E}_m^0$ is trivial and let $s_0, \ldots, s_{N_m}$ be a fixed $\mathcal{O}(U)$ basis. Then if $p_j \in V \subseteq C^0$ is an affine open set such that $V \backslash \{p_j\} \subseteq U$, then define $\mathcal{E}(V)$ to be the $\mathcal{O}(V)$ submodule of $\mathcal{E}^0(V\backslash\{p_j\})$ spanned by the s_α.

Once $\mathcal{E}$ is fixed, we choose $m \geq 3$ and let $\mathcal{X}^0 \hookrightarrow \mathbb{P}(\mathcal{E}^0) \subseteq \mathbb{P}(\mathcal{E})$ be the canonical imbedding. Then we define

$$\mathcal{Y} \subseteq \mathbb{P}(\mathcal{E}) \tag{4.1}$$

to be the flat limit of $\mathcal{X}^0 \to C^0$ inside $\mathbb{P}(\mathcal{E}) \to C$.

Now we complete the proof of Theorem 1.2. To lighten the notation, we shall assume $m = 1$ and write $C^0 = C\backslash\{0\}$ where $0 := p_1$. Suppose $t_i \in C^0$ with $t_i \to 0$ and such that we have analytic convergence $X_{t_i} \to X_\infty$ where X_∞ is an stable analytic curve. We wish to show that there exists a smooth curve $\tilde{C}$ and a finite cover $\mu : \tilde{C} \to C$ with the following property. If we let $\Sigma = \mu^{-1}(0)$ (a finite set) there exists a unique completion $\tilde{f} : \tilde{\mathcal{X}} \to \tilde{C}$ of $\mu^*\mathcal{X}^0 \to \tilde{C}\backslash\Sigma$ with $X_\infty = p^{-1}(\tilde{0})$ for all $\tilde{0} \in \Sigma$.

Define

$$Z^0 \ = \ \{(t, z) \in C^0 \times \mathrm{Hilb}(\mathbb{P}^{N_m}) : \ z \in \mathcal{T}_t\}$$

where $\mathcal{T}_t$ is the set of all Hilbert points $[T(X_t)]$. Here $T : X_t \to \mathbb{P}^{N_m}$ ranges over the set of all Bergman imbeddings. In particular, $\mathcal{T}_t \subseteq \mathrm{Hilb}(\mathbb{P}^{N_m})$ is a single $G = SL(N_m+1)$ orbit.

We claim that $Z^0 \subseteq C^0 \times \mathrm{Hilb}(\mathbb{P}^{N_m})$ is a constructible subset. To see this, let $U \subseteq C^0$ be an affine open subset and let $\sigma_0, \ldots, \sigma_{N_m}$ be a fixed $\mathcal{O}(U)$ basis of $\mathcal{E}_m(U)$. This basis defines an imbedding

$$S : \pi^{-1}(U) \ \to \ U \times \mathbb{P}^{N_m} \tag{4.2}$$

given by $x \mapsto (\pi(x), \sigma_0(x), \ldots, \sigma_{N_m}(x))$. Define $H : U \to \mathrm{Hilb}(\mathbb{P}^{N_m})$ by $H(t) = \mathrm{Hilb}(S(X_t))$ and define the map

$$f_U : G \times U \ \to \ U \times \mathrm{Hilb}(\mathbb{P}^{N_m}) \ \text{ given by} (g, t) \mapsto (t, g \cdot H(t))$$

Then f_U is an algebraic map so its image is constructible. This shows $Z^0|_U$ is constructible for every affine subset $U \subseteq C^0$ and hence Z^0 is constructible.

Now we fix $0 < \epsilon < \epsilon(g)$ and let $W_j = T_j(X_{t_j})$ where T_j is the ϵ-Bergman imbedding. Then (3.6) implies $T_j(X_j) = W_j \to T_\infty(X_\infty) = Y_\infty$, a stable algebraic curve in $\mathbb{P}^{N_m}$. Let $Z \to C$ be the closure of Z^0 in $C \times \mathrm{Hilb}(\mathbb{P}^{N_m}) \subseteq C \times \mathbb{P}^M$. Here $\mathbb{P}^M \supset \mathrm{Hilb}(\mathbb{P}^{N_m})$ is chosen so that there is a G action on $\mathbb{P}^M$ which restricts to the G

action on $\mathrm{Hilb}(\mathbb{P}^{N_m})$. Then Z is a subvariety of $C \times \mathrm{Hilb}(\mathbb{P}^{N_m})$ whose dimension we denote by d. Let Z_t the fiber of Z above $t \in C$. Then $[Y_\infty] \in Z_0$.

To construct $\tilde{C}$ we use the Luna Slice Theorem: There exists $W \subseteq \mathbb{C}^{M+1}$ a $G_{[Y_0]}$ invariant subspace such that $[Y_\infty] \in \mathbb{P}(W) \subseteq \mathbb{P}^M$ and such that the map

$$\mathbb{P}(W) \times \mathrm{Lie}(G) \;\to\; \mathbb{P}^M \quad \text{given by } (x, \xi) \mapsto \exp(\xi)x$$

is a diffeomorphism of some small neighborhood $U_W \times V \subseteq \mathbb{P}(W) \times \mathrm{Lie}(G)$ onto an open set $\Omega \subseteq \mathbb{P}^M$, with $U_W \subseteq \mathbb{P}(W)$ invariant under the finite group $G_{[Y_0]}$. After shrinking U_W if necessary, the intersection of a G orbit with $U_W \backslash [Y_0]$ is a finite set of order $m_1 | m$ where $m = |G_{[Y_0]}|$. In other words, the quotient $G_{[Y_0]} \backslash U_W$ parametrizes the G-orbits in $\mathbb{P}^M$ that intersect U_W.

Note that Ω contains $(t_i, [Y_i])$ for infinitely many i so $(C \times \mathbb{P}(W)) \cap Z$ is a projective variety C_1 of dimension at least one. Moreover, if we let C_2 be the union of the components of C_1 containing $\{0\} \times [Y_\infty]$, then C_2 contains infinitely many of $(t_i, [Y_i])$ so the image of $C_2 \to C$ contains infinitely many t_i and thus $C_2 \to C$ is surjective. On the other hand, $C_2 \to C$ is finite of degree m_1 (this follows from the construction of $U(W)$).

Let $\tilde{C} \subseteq C_1$ be an irreducible component of C_1 containing $(t_i, [Y_i])$ for infinitely many i. Let $H \subseteq G_{[Y_\infty]}$ be the set of all $\sigma \in G_{[Y_\infty]}$ such that $\sigma(\tilde{C}) = \tilde{C}$. Then H has order d for some $d | m_1$ and $\tilde{C} \to C$ is finite of degree d.

Finally, we have $\tilde{C} \subseteq Z \subseteq C \times \mathrm{Hilb}(\mathbb{P}^{N_m})$. This gives us a map $\tilde{C} \to \mathrm{Hilb}(\mathbb{P}^{N_m})$. If we pull back the universal family we get a flat family $\tilde{\mathcal{X}} \to \tilde{C}$ which extends $\mathcal{X}^0 \times_{\tilde{C}} C^0$. This completes the proof of Theorem 1.2. ◻

5 Uniqueness of the stable fill-in

Let $\pi : X^* \to \Delta^* \subset \Delta$ be an algebraic family of stable curves genus g. We claim that there exists a unique stable analytic curve X_0 such that $X_t \to X_0$ in the Cheeger–Colding sense as $t \to 0$. This will establish the uniqueness statement of Theorem 1.2, and since existence was demonstrated in the previous section, it completes the proof.

Let $S : \mathcal{X}^* \to \Delta^* \times \mathbb{P}^{N_m}$ as in (4.2). For each $t \in \Delta^*$, the set $\underline{\sigma}_t = (\sigma_0(t), \ldots, \sigma_{N_m}(t))$ is a basis of $H^0(X_t, mK_{X_t})$. Let $\underline{s}_t = (s_0(t), \ldots, s_{N_m}(t))$ be the orthonormal basis of $H^0(X_t, mK_{X_t})$ obtained by applying the Gram-Schmidt process to the basis $\underline{\sigma}_t$ and let $T_t^\epsilon : X_t \to \mathbb{P}^N$ be the map $T_t^\epsilon = T_{\underline{s}_t}^\epsilon$. Here $0 < \epsilon < \epsilon(g)$ is fixed once and for all. Remark 2.1 implies that $t \mapsto [T_t^\epsilon(X_t)]$ defines a continuous function $\Delta^* \to \mathrm{Hilb}$. Let

$$z : \Delta^* \times SL(N+1, \mathbb{C}) \;\to\; \Delta^* \times \mathrm{Hilb}$$

and

$$f : \Delta^* \to \Delta^* \times \mathrm{Hilb}$$

be the maps

$$z(t,g) = (t, g\cdot[T_t(X_t)]) \quad \text{and} \quad f(t) = z(t,[T_t(X_t)]).$$

Let $F = \overline{\mathrm{Im}(f)} \subseteq \Delta \times \mathrm{Hilb}$ and $Z = \overline{\mathrm{Im}z} \subseteq \Delta \times \mathrm{Hilb}$. Let $\pi_F : F \to \Delta$ and $\pi_Z : Z \to \Delta$ be the projection maps and $F_0 = \pi_F^{-1}(0)$, $Z_0 = \pi_Z^{-1}(0)$. Observe that $F_0 \subseteq \mathrm{Hilb}$ is closed and connected (this easily follows from the fact that Δ^* is connected and Hilb is compact and connected). Moreover, Theorem 3.1 implies that every element of F_0 is of the form $T^{\epsilon}_{\underline{s}}(X_0)$ for some stable analytic curve X_0 and some basis $\underline{s}$.
Claim: F_0 is contained in the $U(N+1)$ orbit of $[X_0]$.

Assume the claim for the moment, and let us show that it implies uniqueness. Suppose there exist subsequences $t_i, t_i' \in \Delta^*$ such that $X_{t_i} \to X_0$ and $X_{t_i'} \to X_0'$. We must show that $X_0 \approx X_0'$, i.e. X_0 and X_0' are isomorphic stable analytic curves. Theorem 3.1 implies there are bases $\underline{s}$ and $\underline{s}'$ such that $[T^{\epsilon}_{\underline{s}}(X_0)], [T^{\epsilon}_{\underline{s}'}(X_0')] \in F_0$ so $T^{\epsilon}_{\underline{s}'u}(X_0') \in U(N+1)\cdot T^{\epsilon}_{\underline{s}}(X_0)$. Now Lemma 1.1 implies $X_0 \approx X_0'$. This gives uniqueness.

The set $U = SL(N+1,\mathbb{C})\cdot[T^{\epsilon}_{\underline{s}}(X_0)] \subseteq Z_0$ is open since $\dim Z_0 = \dim SL(N+1,\mathbb{C})$ and the stabilizer of $[T^{\epsilon}_{\underline{s}}(X_0)]$ is finite. Lemma 1.1 implies

$$F_0 \cap U \;\subseteq\; U(N+1)[T^{\epsilon}_{\underline{s}}(X_0)] \;\subseteq\; U \tag{5.1}$$

Now $U(N+1)[T^{\epsilon}_{\underline{s}}(X_0)]$ is compact and F_0 is connected, so $F_0 \cap U = F_0$. Thus the claim follows from (5.1). □

Data availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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