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On the gradient rearrangement of functions

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Abstract

In this paper, we introduce a symmetrization technique for the gradient of a BV function, which separates its absolutely continuous part from its singular part (sum of jump and Cantorian part). In particular, we prove a L^1 comparison between the function and the symmetrization just mentioned. Furthermore, we apply this result to obtain Saint-Venant type inequalities for some geometric functionals.

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1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^n$ with finite perimeter (see Sect. 2 for its definition) and let us denote, as in [7], by

$$\mathrm{BV}_0(\Omega) := \left\{u \in \mathrm{BV}(\mathbb{R}^n) : u \equiv 0 \text{ in } \mathbb{R}^n \setminus \Omega\right\}.$$

The aim of the present paper is to define a symmetrization of the distributional gradient of a BV function.

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The interest in this topic essentially derives from the work [17] where the authors deal with the following problems involving Hamilton-Jacobi equation

$$\begin{cases} H(\nabla u) = f & \text{a.e. in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1a}$$

$$\begin{cases} K(|\nabla v|) = f_\sharp & \text{a.e. in } \Omega^\sharp \\ v = 0 & \text{on } \partial\Omega^\sharp \end{cases} \tag{1.1b}$$

where $\Omega^\sharp$ is the ball centered at the origin with the same measure as Ω (in the sequel just centered ball), $H : \mathbb{R}^n \to \mathbb{R}$ and $K : \mathbb{R} \to \mathbb{R}$ are measurable functions, $u, v \in W_0^{1,p}$ and $f_\sharp$ is the increasing rearrangement of f (see Sect. 2 for its definition).

In particular, under suitable assumptions on H and K, it is proven ([17, Theorem 2.2]) that whenever u, v are solutions to (1.1a) and (1.1b) respectively, then

$$\|u\|_{L^1(\Omega)} \leq \|v\|_{L^1(\Omega^\sharp)}.$$

In [2] the authors study the problem of maximization of the L^q norm among functions with prescribed gradient rearrangement. Precisely, the following cases are considered

- $1 \leq q \leq \frac{np}{n-p}$ if $p < n$,
- $1 \leq q < +\infty$ if $p = n$,
- $1 \leq q \leq +\infty$ if $p > n$,

and for a fixed $\varphi = \varphi^* \in L^p(0, |\Omega|)$, they define

$$I(\Omega) := \sup \left\{ \|v\|_{L^q} \ : \ \begin{array}{c} |\nabla v| \leq f \text{ a.e. in } \Omega, \\ v \in W_0^{1,p}(\Omega) \\ f \geq 0, \ f^* = \varphi^* \end{array} \right\},$$

and they proved the following

Theorem 1.1 *[2, Theorem 3.1] Let Ω be a bounded open set in $\mathbb{R}^n$, let $\Omega^\sharp$ be the centered ball, let R be its radius and let p, q, φ be as defined above.*

Then, there exist v, g spherically symmetric on $\Omega^\sharp$ such that $g^* = \varphi$, $I(\Omega^\sharp) = \|v\|_{L^q}$, and thus

$$v \in W_0^{1,p}(\Omega), \ v \geq 0, \ |\nabla v| = g \qquad \textit{a.e. in } \Omega^\sharp.$$

Furthermore $I(\Omega^\sharp) \geq I(\Omega)$ for all open sets Ω in $\mathbb{R}^n$ with $|\Omega^\sharp| = |\Omega|$.

In [9] the author proved a representation formula for the function g, the existence of which was proved in Theorem 1.1.

Let us also mention that in [14, 15] the authors studied the optimization of the norm of a Sobolev function in the class of functions with prescribed rearrangement of the gradient.

The case of a Sobolev non-zero trace function for $q = 1$ is instead studied in [4].

The literature concerning rearrangements in the spaces $W^{1,p}$ is exhaustive, whereas, to our knowledge, results on BV functions are rarer. One of the most relevant papers in this framework is [10] where the authors extend the validity of Polya-Szegö inequality to BV functions. More specifically, they proved that if $u \in \mathrm{BV}(\mathbb{R}^n)$, then its Schwarz rearrangement $u^\sharp$ (see Sect. 2 for its definition) belongs to $\mathrm{BV}(\mathbb{R}^n)$ and it holds [10, Theorem 1.3]

$$\begin{aligned} |Du^\sharp|(\mathbb{R}^n) &\le |Du|(\mathbb{R}^n) \\ |D^{\mathrm{s}}u^\sharp|(\mathbb{R}^n) &\le |D^{\mathrm{s}}u|(\mathbb{R}^n) \\ |D^{\mathrm{j}}u^\sharp|(\mathbb{R}^n) &\le |D^{\mathrm{j}}u|(\mathbb{R}^n) \end{aligned} \tag{1.2}$$

where D^s and D^j denote respectively the singular and the jump part of the gradient (see [10] for their definitions). There is no analogue of (1.2) for the absolutely continuous and the Cantorian part of the gradient, i.e. in the symmetrization procedure the total variation of D^a and D^c can be increased, as shown in the example given in [10].

In this paper, we want to introduce a symmetrization that keeps the absolutely continuous part separate from the singular part (sum of jump and Cantorian part) of the gradient. To be more precise, we define the radial function $u^\star \in W^{1,1}(\Omega^\sharp) \cap \mathrm{BV}_0(\Omega^\sharp) \cap L^\infty(\Omega^\sharp)$ such that

$$\begin{cases} |\nabla u^\star|(x) = |\nabla^a u|_\sharp(x) & \text{a.e. in } \Omega^\sharp \\ u^\star(x) = \dfrac{1}{\mathrm{Per}(\Omega^\sharp)}|D^s u|(\mathbb{R}^n) & \text{on } \partial\Omega^\sharp \end{cases}, \tag{1.3}$$

where $\nabla^a u$ and $D^s u$ will be defined in Sect. 2.

The main theorem can be stated as follows.

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with finite perimeter and let $\Omega^\sharp$ be the centered ball. Assume that u is a non-negative function belonging to $BV_0(\Omega)$ and assume that $u^\star$ is defined as in (1.3), then*

$$\|u\|_{L^1(\Omega)} \le \left\|u^\star\right\|_{L^1(\Omega^\sharp)}.$$

We will also deal with some applications, in particular we will consider

- a penalized torsional rigidity problem

$$T_{\mathcal{F}}(\Omega, \Lambda) := -\inf_{\psi \in H_0^1(\Omega)} \left(\frac{1}{2} \int_\Omega |\nabla\psi|^2 \, dx - \int_\Omega |\psi| \, dx + \Lambda |\{|\nabla\psi| \neq 0\}| \right);$$

- a modified torsional rigidity

$$\frac{1}{T_{\mathcal{G}}(\Omega, m)} := \inf_{\psi \in H^1(\Omega)} \frac{\displaystyle\int_\Omega |\nabla\psi|^2 \, dx + \frac{1}{m}\left(\int_{\partial\Omega} |\psi| \, d\mathcal{H}^{n-1}\right)^2}{\left(\displaystyle\int_\Omega |\psi| \, dx\right)^2}.$$

In both cases, we will prove a Saint-Venant type inequality:

$$T_{\mathcal{F}}(\Omega, \Lambda) \leq T_{\mathcal{F}}(\Omega^{\sharp}, \Lambda), \qquad T_{\mathcal{G}}(\Omega, m) \leq T_{\mathcal{G}}(\Omega^{\sharp}, m).$$

The paper is organized as follows: in Sect. 2 we recall some preliminary results and useful tools for our aim, in Sect. 3 we prove our main result on the symmetrization of the gradient for a BV function, while in Sect. 4 we present some applications of this kind of symmetrization.

2 Notations and preliminaries

2.1 Functions of bounded variation

Let us summarize some basic notions concerning BV functions, for all the details we refer for instance to [6, 10, 13].

In the following, Ω will be an open set of $\mathbb{R}^n$.

Definition 2.1 A function $u \in L^1(\Omega)$ is said to be a **function of bounded variation** in Ω if its distributional derivative is a Radon measure, i.e.

$$\int_{\Omega} u \frac{\partial \varphi}{\partial x_i}\, dx = \int_{\Omega} \varphi \, dD^i u \qquad \forall \varphi \in C_C^{\infty}(\Omega),$$

with Du a $\mathbb{R}^n$-valued measure in Ω. The total variation of Du will be denoted with $|Du|$.

The set of functions of bounded variation in Ω is denoted by $\mathrm{BV}(\Omega)$ and it is a Banach space with respect to the norm $\|u\|_{\mathrm{BV}(\Omega)} := \|u\|_{L^1(\Omega)} + |Du|(\Omega)$.

Definition 2.2 Let E be a $\mathcal{L}^n$-measurable set. The **perimeter** of E inside Ω is defined as

$$\mathrm{Per}(E, \Omega) := |D\chi_E|(\Omega),$$

and we say that E is a **set of finite perimeter** in Ω if $\chi_E \in \mathrm{BV}(\Omega)$. If $\Omega = \mathbb{R}^n$, we denote $\mathrm{Per}(E) := \mathrm{Per}(E, \mathbb{R}^n)$.

It is also worth mentioning the isoperimetric inequality for sets of finite perimeter.

Theorem 2.1 (Isoperimetric inequality) *Let $E \subset \mathbb{R}^n$ be a bounded set of finite measure. Then it holds*

$$|E| \leq n^{-\frac{n}{n-1}} \omega_n^{-\frac{1}{n-1}} \left[Per(E)\right]^{\frac{n}{n-1}},$$

where ω_n is the measure of n-dimensional ball of radius 1.

By the Lebesgue decomposition Theorem, each component of Du can be decomposed with respect to the Lebesgue measure, namely

$$D_i u = D_i^{\mathrm{a}} u + D_i^{\mathrm{s}} u \qquad \text{with } D_i^{\mathrm{a}} u \ll \mathcal{L}^n, \quad D_i^{\mathrm{s}} u \perp \mathcal{L}^n.$$

and

$$D_i^{\mathrm{a}} u = f_i \llcorner \mathcal{L}^n,$$

for some $f_i \in L^1(\Omega)$. So, defining

$$\frac{\partial u}{\partial x_i} := f_i, \qquad \nabla^{\mathrm{a}} u = \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n} \right) \qquad \text{and } D^{\mathrm{s}} u = \left(D_1^{\mathrm{s}} u, \ldots, D_n^{\mathrm{s}} u \right),$$

we can write

$$dDu = \nabla^{\mathrm{a}} u \llcorner \mathcal{L}^n + dD^{\mathrm{s}} u.$$

Clearly it holds

$$|Du|(A) = |D^{\mathrm{a}} u|(A) + |D^{\mathrm{s}} u|(A) = \int_A \left| \nabla^{\mathrm{a}} u \right| dx + |D^{\mathrm{s}} u|(A),$$

for every Borel set $A \subseteq \Omega$.

Let us recall the following **Fleming-Rishel formula** (see [16] or [13]):

Theorem 2.2 (Fleming-Rishel formula) *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in BV(\Omega)$, then for almost every $t \in (-\infty, +\infty)$ the set $\{u > t\}$ has finite perimeter in Ω and it holds*

$$|Du|(\Omega) = \int_{-\infty}^{+\infty} Per(\{u > t\}, \Omega)\, dt. \tag{2.1}$$

Moreover if $u \in L^1(\Omega)$ and

$$\int_{-\infty}^{+\infty} Per(\{u > t\}, \Omega)\, dt < +\infty,$$

then $u \in BV(\Omega)$ and consequently (2.1) holds.

2.2 Rearrangements of functions

We now briefly recall some notions about rearrangements. We refer for instance to [18, 19, 23] for all the details.

Definition 2.3 Let Ω be a measurable set and let $u : \Omega \to \mathbb{R}$ be a measurable function, the **distribution function** of u is defined as

$$\mu : [0, +\infty) \to [0, +\infty) \qquad \mu(t) = \left| \left(\{x \in \Omega \,:\, |u(x)| > t\} \right) \right|$$

where, here and throughout the paper, $|E|$ denotes the n-dimensional Lebesgue measure of a measurable set E.

It can be proved that

- μ is a decreasing function in $[0, +\infty)$;
- μ is right-continuous;

- $\mu(0) = |\text{supp} u|$ and $\mu(+\infty) = 0$;
- $\mu(t^-) = \left|\{x \in \Omega : |u(x)| \geq t\}\right|$.

Definition 2.4 Let $u : \Omega \to \mathbb{R}$ be a measurable function, the **decreasing rearrangement** of u is defined as

$$u^* : \mathbb{R}^+ \to \mathbb{R}^+ \qquad u^*(s) = \inf\{t > 0 \,:\, \mu(t) \leq s\}$$

and the **increasing rearrangement** of u as

$$u_* : [0, |\Omega|] \to \mathbb{R}^+ \qquad u_*(s) = u^*(|\Omega| - s)$$

It can be proved that

- u^* (u_*) is a decreasing (increasing) function in $[0, +\infty)$;
- u^* and u_* are lower semi-continuous;
- whenever $u \in L^\infty(\Omega)$ $u^*(0) = \|u\|_{L^\infty(\Omega)}$ and $u^*(t) = 0 \; \forall t \geq |\text{supp} u|$;
- $u_*(|\Omega|) = \|u\|_{L^\infty(\Omega)}$ and $u_*(t) = 0 \; \forall 0 \leq t \leq |\Omega| - |\text{supp} u|$;
- u^* and u_* have the same distribution function as u, so by Cavalieri's principle the L^p norms are equal for every p;
- $u^*(\mu(t)) \leq t$ for every non-negative t, $\mu(u^*(s)) \leq s$ for every non-negative s;
- $u^*(\mu(t)^-) \geq t$ for every non-negative t, $\mu(u^*(s)^-) \geq s$ for every non-negative s;
- the **Hardy-Littlewood inequality:** for any $u, v : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$

$$\int_\Omega |u(x)v(x)|\, dx \leq \int_{\Omega^\sharp} u^*(x)v^*(x)\, dx = \int_\Omega u_*(x)v_*(x)\, dx \tag{2.2}$$

Definition 2.5 Let $u : \Omega \to \mathbb{R}$ be a measurable function. The **Schwarz rearrangement** or the **spherically symmetric decreasing rearrangement** of u is defined as

$$u^\sharp : \mathbb{R}^n \to \mathbb{R}^+ \qquad u^\sharp(x) = u^*(\omega_n |x|^n)$$

where ω_n is the Lebesgue measure of the unit n-dimensional ball.

Moreover the **spherically symmetric increasing rearrangement** of u is defined as

$$u_\sharp : \mathbb{R}^n \to \mathbb{R}^+ \qquad u_\sharp(x) = u_*(\omega_n |x|^n)$$

It can be proved that

- $u^\sharp$ ($u_\sharp$) is non-negative, radial and radially decreasing (increasing);
- $u^\sharp$, $u_\sharp$ and u are equally distributed which means they have the same distribution function;
- the Polya-Szegö inequality holds true [21]: if $u \in W_0^{1,p}(\Omega)$, then $u^\sharp \in W_0^{1,p}(\Omega^\sharp)$ and
$$\|\nabla u^\sharp\|_{L^p(\Omega^\sharp)} \leq \|\nabla u\|_{L^p(\Omega)}.$$

We recall the Theorem of Giarrusso and Nunziante ([17, Theorem 2.2]).

Theorem 2.3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $\Omega^\sharp$ be the centered ball, let $p \geq 1$, let $f : \Omega \to \mathbb{R}$ be a measurable function, let $H : \mathbb{R}^n \to \mathbb{R}$ be measurable non-negative functions and let $K : [0, +\infty) \to [0, +\infty)$ be a strictly increasing real-valued function such that*

$$0 \leq K(|y|) \leq H(y) \qquad \forall y \in \mathbb{R}^n \qquad \text{and } K^{-1}(f) \in L^p(\Omega).$$

Let $v \in W_0^{1,p}(\Omega)$ be a function that satisfies

$$\begin{cases} H(\nabla v) = f(x) & \text{a.e. in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases},$$

denoting by $z \in W_0^{1,p}(\Omega^\sharp)$ the unique spherically decreasing symmetric solution to

$$\begin{cases} K(|\nabla z|) = f_\sharp(x) & \text{a.e. in } \Omega^\sharp \\ z = 0 & \text{on } \partial\Omega^\sharp \end{cases},$$

then

$$\|v\|_{L^1(\Omega)} \leq \|z\|_{L^1(\Omega^\sharp)}.$$

Moreover, in [20] the following uniqueness result is proved:

Theorem 2.4 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $v \in W_0^{1,1}(\Omega)$ be a non-negative function. Denote by $f(x) = |\nabla v|(x)$ and by $w \in W_0^{1,1}(\Omega^\sharp)$ the decreasing spherically symmetric solution to*

$$|\nabla w| = f_\sharp.$$

If $\|v\|_{L^1} = \|w\|_{L^1}$ then there exists $x_0 \in \mathbb{R}^n$ such that $\Omega = x_0 + \Omega^\sharp$, $f = f_\sharp(\cdot + x_0)$ and $v = w(\cdot + x_0)$.

From now on $\Omega \subset \mathbb{R}^n$ is a bounded open set with finite perimeter. Let us consider

$$\mathrm{BV}_0(\Omega) := \left\{ u \in \mathrm{BV}(\mathbb{R}^n) : \ u \equiv 0 \text{ in } \mathbb{R}^n \setminus \Omega \right\},$$

and u a non-negative function belonging to $\mathrm{BV}_0(\Omega)$. Let us define

$$f(x, s) = \big(u - u^*(s)\big)_+(x) \qquad x \in \mathbb{R}^n,\ s \in [0, +\infty). \tag{2.3}$$

The function $f(\cdot, s)$ belongs to $\mathrm{BV}_0(\Omega)$ for every $s \in [0, +\infty)$ since it is a truncation of u (See [6, Theorem 3.96]). Moreover, for every $s \in [0, +\infty)$ we denote by

$$G(s) = |Df(\cdot, s)|(\mathbb{R}^n) = |D^a f(\cdot, s)|(\mathbb{R}^n) + |D^s f(\cdot, s)|(\mathbb{R}^n) = G_1(s) + G_2(s), \tag{2.4}$$

where $D^a f$ and $D^s f$ are, respectively, the absolutely continuous part and singular part of the measure Df.

The following corollary holds.

Corollary 2.5 *Let u be a non-negative function belonging to $BV_0(\Omega)$ and let $G(s)$ be the function defined as in (2.4). Then for a.e. $s \in [0,+\infty)$:*

$$G(s) = \int_{u^*(s)}^{+\infty} Per(\{u > \xi\})\, d\xi. \tag{2.5}$$

Proof For a.e. $s \in [0,+\infty)$, applying 2.2 with $E = \mathbb{R}^n$ to the function $f(\cdot, s)$ defined in (2.3), we have

$$G(s) = \left|D\big((u - u^*(s))_+\big)\right|(\mathbb{R}^n) = \int_{-\infty}^{+\infty} \mathrm{Per}\left(\left\{\big(u - u^*(s)\big)_+ > \xi\right\}\right) d\xi. \tag{2.6}$$

Moreover, we have

$$\int_{-\infty}^{+\infty} \mathrm{Per}\left(\left\{\big(u - u^*(s)\big)_+ > \xi\right\}\right) d\xi = \int_{0}^{+\infty} \mathrm{Per}\left(\left\{u - u^*(s) > \xi\right\}\right) d\xi,$$

and a change of variables gives (2.5).

The following properties hold:

1. G is an increasing function on $(0,+\infty)$ by (2.5), constant in $(|\Omega|,+\infty)$, it belongs to $\mathrm{BV}_{\mathrm{loc}}([0,+\infty))$. Then, there exists a positive measure σ such that

$$G(s) = \int_{(0,s]} d\sigma(\tau) \qquad \forall s \in [0,+\infty); \tag{2.7}$$

2. $G_1(s) = \displaystyle\int_{\{u>u^*(s)\}} |\nabla^{\mathrm{a}} u|\, dx$ is increasing and AC on $[0,+\infty)$, then there exists a function F_1 belonging to $L^1([0,+\infty))$:

$$G_1(s) = \int_0^s F_1(\tau)\, d\tau \qquad \forall s \in [0,+\infty);$$

3. G_2 is an increasing function belonging to $\mathrm{BV}_{\mathrm{loc}}([0,+\infty))$, so there exists a positive measure σ_2 such that

$$G_2(s) = \int_{(0,s]} d\sigma_2(\tau) \qquad \forall s \in [0,+\infty).$$

Then, $\forall s \geq 0$

$$G(s) = \sigma((0,s]) = \int_{(0,s]} d\sigma(\tau) = \int_0^s F_1(\tau)\, d\tau + \int_{(0,s]} d\sigma_2(\tau) \tag{2.8}$$

We will need the following technical lemma which can be proved by arguing as [3, Lemma 2.1].

Lemma 2.6 *Let Ω be a bounded open set in $\mathbb{R}^n$. If $g \in L^1([0, |\Omega|))$, then there exists a sequence of functions $\{g_k\}$ such that $g_k^* = g^*$ and*

$$\lim_k \int_0^{|\Omega|} g_k(s)\varphi(s)\,dx = \int_0^{|\Omega|} g(s)\varphi(s)\,ds, \qquad \forall \varphi \in BV\big([0, |\Omega|)\big). \tag{2.9}$$

3 Proof of Theorem 1.2

Let us define the following function

$$v(s) := \int_s^{+\infty} \frac{1}{n\omega_n^{\frac{1}{n}}\tau^{1-\frac{1}{n}}}\,d\sigma(\tau) \qquad \forall s \in [0, +\infty), \tag{3.1}$$

where σ is defined in (2.7). We observe that, since $\operatorname{supp}(\sigma) \subseteq [0, |\Omega|]$, v is identically 0 on $(|\Omega|, +\infty)$, hence $v \in \mathrm{BV}_0([0, |\Omega|])$.

As intermediate step towards Theorem 1.2, we prove the following proposition.

Proposition 3.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with finite perimeter and assume that u is a non-negative function belonging to $BV_0(\Omega)$. If $v(s)$ is the function defined as in (3.1), then*

$$u^*(s) \le v(s) \qquad \textit{for a.e. } s \in [0, +\infty). \tag{3.2}$$

Proof The isoperimetric inequality implies

$$n\omega_n^{\frac{1}{n}}\mu(t)^{1-\frac{1}{n}} \le \mathrm{Per}(\{u > t\}) \qquad \forall t \in [0. + \infty),$$

by (2.5) and (2.8) we have

$$G(s) = \int_{u^*(s)}^{+\infty} \mathrm{Per}(\{u > \xi\})\,d\xi = \int_{(0,s]} d\sigma(\tau) \qquad \text{for a.e. } s \in [0, +\infty).$$

Hence, for all $0 \le s_1 < s_2 < +\infty$ we have

$$\begin{aligned}
\sigma\big((s_1, s_2)\big) = \int_{s_1}^{s_2} d\sigma(\tau) &= \lim_{s \to s_2^-} G(s) - G(s_1) \\
&= \lim_{s \to s_2^-} \int_{u^\star(s)}^{u^\star(s_1)} \mathrm{Per}(\{u > \xi\})\,d\xi \\
&\ge \lim_{s \to s_2^-} \int_{u^\star(s)}^{u^\star(s_1)} n\omega_n^{\frac{1}{n}}\mu(\xi)^{1-\frac{1}{n}}\,d\xi = D\big[H(u^*)\big]\big((s_1, s_2)\big),
\end{aligned}$$

where

$$H(\tau) = \int_\tau^{+\infty} n\omega_n^{\frac{1}{n}}\mu(\xi)^{1-\frac{1}{n}}\,d\xi.$$

Since this holds for every open interval (s_1, s_2), we have

$$\sigma(A) \geq D\big[H(u^*)\big](A) \qquad \forall A \subseteq [0, +\infty) \text{ Borel set.} \tag{3.3}$$

Observing that H is a Lipschitz function, $D\big[H(u^*)\big]$ is given by (see [5])

$$D\big[H(u^*)\big] = \begin{cases} -n\omega_n^{\frac{1}{n}} s^{1-\frac{1}{n}} Du^* & \text{on } [0, +\infty) \setminus J_{u^*} \\ -n\omega_n^{\frac{1}{n}} s^{1-\frac{1}{n}} \big((u^*)^+ - (u^*)^-\big), & \text{on } J_{u^*} \end{cases}$$

since $\mu(u^*(s)) = s$ a.e. with respect Du^* (by the properties of the rearrangements) and since for $s \in J_{u^*}$

$$\begin{aligned} H\big(((u^*)^+(s)\big) - H\big(((u^*)^-(s)\big) &= \int_{u^*(s)}^{u^*(s^-)} n\omega_n^{\frac{1}{n}} \mu(\xi)^{1-\frac{1}{n}} \, d\xi \\ &= -n\omega_n^{\frac{1}{n}} s^{1-\frac{1}{n}} \big((u^*)^+(s) - (u^*)^-(s)\big). \end{aligned}$$

Then we can write

$$\frac{dD\big[H(u^*)\big]}{dDu^*} = -n\omega_n^{\frac{1}{n}} s^{1-\frac{1}{n}}. \tag{3.4}$$

Therefore, by means of (3.3), (3.4), we have

$$u^*(s) = -\int_s^{+\infty} d(Du^*)(\tau) = \int_s^{+\infty} \frac{dD\big[H(u^*)\big](\tau)}{n\omega_n^{\frac{1}{n}} \tau^{1-\frac{1}{n}}} \leq \int_s^{+\infty} \frac{d\sigma(\tau)}{n\omega_n^{\frac{1}{n}} \tau^{1-\frac{1}{n}}} = v(s).$$

Now we are in position to prove the main theorem.

Proof of Theorem 1.2 First of all, let us emphasize that the decreasing rearrangement of $u^\star$, defined in (1.3), is

$$(u^\star)^*(s) = \int_s^{+\infty} \frac{|\nabla^{\mathrm{a}} u|_*(t)}{n\omega_n^{\frac{1}{n}} t^{1-\frac{1}{n}}} \, dt + \frac{1}{\mathrm{Per}(\Omega^\sharp)} \big|D^s u\big|(\mathbb{R}^n) \, \chi_{[0,|\Omega|]}(s) \qquad \forall s \in [0, +\infty).$$

Now, let us integrate (3.2) between 0 and $+\infty$ and let us use Fubini's Theorem to obtain

$$\begin{aligned} \int_0^{+\infty} u^*(s) \, ds &\leq \int_0^{+\infty} v(s) \, ds \\ &= \frac{1}{n\omega_n^{\frac{1}{n}}} \int_0^{+\infty} \left(\int_s^{+\infty} \frac{d\sigma(t)}{t^{1-\frac{1}{n}}} \right) ds \\ &= \frac{1}{n\omega_n^{\frac{1}{n}}} \int_0^{+\infty} \left(\int_0^t \frac{ds}{t^{1-\frac{1}{n}}} \right) d\sigma(t) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n\omega_n^{\frac{1}{n}}} \int_0^{+\infty} t^{\frac{1}{n}}\, d\sigma(t) \\
&= \frac{1}{n\omega_n^{\frac{1}{n}}} \left[\int_0^{+\infty} t^{\frac{1}{n}} F_1(t)\, dt + \int_0^{+\infty} t^{\frac{1}{n}}\, d\sigma_2(t) \right].
\end{aligned}
$$

By (2.9) applied to F_1 and the Hardy-Littlewood inequality (2.2), we have

$$
\begin{aligned}
\int_0^{+\infty} t^{\frac{1}{n}} F_1(t)\, dt &= \int_0^{|\Omega|} t^{\frac{1}{n}} F_1(t)\, dt = \lim_k \int_0^{|\Omega|} t^{\frac{1}{n}} (F_1)_k(t)\, dt \\
&\le \int_0^{|\Omega|} t^{\frac{1}{n}} |\nabla^{\mathrm{a}} u|_*(t)\, dt = \int_0^{+\infty} t^{\frac{1}{n}} |\nabla^{\mathrm{a}} u|_*(t)\, dt,
\end{aligned}
$$

then

$$
\begin{aligned}
\int_0^{+\infty} u^*(s)\, ds &\le \frac{1}{n\omega_n^{\frac{1}{n}}} \left[\int_0^{+\infty} t^{\frac{1}{n}} |\nabla^{\mathrm{a}} u|_*(t)\, dt + \int_0^{+\infty} t^{\frac{1}{n}}\, d\sigma_2(t) \right] \\
&\le \frac{1}{n\omega_n^{\frac{1}{n}}} \left[\int_0^{+\infty} t^{\frac{1}{n}} |\nabla^{\mathrm{a}} u|_*(t)\, dt + |\Omega|^{\frac{1}{n}} \int_0^{+\infty} d\sigma_2(t) \right],
\end{aligned}
\tag{3.5}
$$

since $F_2(A) = 0$ for all $A \subset (|\Omega|, +\infty)$.

Using again Fubini's Theorem, we can compute

$$
\int_0^{+\infty} |\nabla^{\mathrm{a}} u|_*(t) t^{\frac{1}{n}}\, dt = \int_0^{+\infty} \frac{|\nabla^{\mathrm{a}} u|_*(t)}{t^{1-\frac{1}{n}}} \int_0^t ds = \int_0^{+\infty} \left(\int_s^{+\infty} \frac{|\nabla^{\mathrm{a}} u|_*(t)}{t^{1-\frac{1}{n}}}\, dt \right) ds,
$$

and

$$
\begin{aligned}
\frac{|\Omega|^{\frac{1}{n}}}{n\omega_n^{\frac{1}{n}}} \int_0^{+\infty} dF_2(t) &= |\Omega| \frac{1}{\mathrm{Per}(\Omega^\sharp)} \left| D^s u \right|(\mathbb{R}^n) \\
&= \int_0^{+\infty} \frac{1}{\mathrm{Per}(\Omega^\sharp)} \left| D^s u \right|(\mathbb{R}^n) \chi_{[0,|\Omega|]}(s)\, ds.
\end{aligned}
$$

Hence, (3.5) can be written as

$$
\begin{aligned}
\|u\|_{L^1(\Omega)} &\le \int_0^{+\infty} \left[\int_s^{+\infty} \frac{|\nabla^{\mathrm{a}} u|_*(t)}{n\omega_n^{\frac{1}{n}} t^{1-\frac{1}{n}}}\, dt + \frac{1}{\mathrm{Per}(\Omega^\sharp)} \left| D^s u \right|(\mathbb{R}^n) \chi_{[0,|\Omega|]}(s) \right] ds \\
&= \left\| u^\star \right\|_{L^1(\Omega^\sharp)}.
\end{aligned}
$$

Remark 3.1 We stress the following facts:

$$
\left| D^{\mathrm{a}} u \right|(\mathbb{R}^n) = \int_{\mathbb{R}^n} \left| \nabla^{\mathrm{a}} u \right| dx = \int_{\Omega^\sharp} \left| \nabla^{\mathrm{a}} u^\star \right| dx \quad \text{and} \quad \left| D^s u \right|(\mathbb{R}^n) = \left| D^s u^\star \right|(\mathbb{R}^n),
$$

and then

$$|Du|(\mathbb{R}^n) = \left|Du^\star\right|(\mathbb{R}^n).$$

4 Two versions of the torsional rigidity

For a given $\Lambda > 0$ we consider

$$\mathcal{F}_\Lambda(\psi) := \frac{1}{2}\int_\Omega |\nabla\psi|^2\,dx - \int_\Omega \psi\,dx + \Lambda|\{|\nabla\psi| \neq 0\}| \qquad \psi \in H_0^1(\Omega), \tag{4.1}$$

and the associated minimum problem:

$$T_{\mathcal{F}}(\Omega, \Lambda) := -\inf_{\psi\in H_0^1(\Omega)} \mathcal{F}_\Lambda(\psi). \tag{4.2}$$

First of all, let us observe that the minimum can be found among non-negative functions. Indeed, passing from ψ to $|\psi|$ it holds $\mathcal{F}(\psi) \geq \mathcal{F}(|\psi|)$.

Assuming that problem (4.2) admits a minimum $u \in H_0^1(\Omega)$, then it is also a maximum for the torsional rigidity defined by Diaz, Polya and Weinstein in [12, 22] of a multiply-connected cross-section with fixed measure of the holes, that is

$$T(\Omega) = \max_{\substack{\psi\in C_0(D)\cap C^1(\Omega)\\ \psi \text{ constant}\\ \text{in every } A_i}} \frac{\left(\displaystyle\int_D \psi\,dx\right)^2}{\displaystyle\int_D |\nabla\psi|^2\,dx},$$

where A_i are the connected component of $\{|\nabla u| = 0\}$ and $D = \Omega \cup \bigcup_i A_i$.

Functionals with penalizing terms are very common in the mathematical modelling of physical problems. The bibliography is very wide and some cornerstones are [1, 11].

However, in the literature, penalizing terms of the form $|\{|\nabla\psi| \neq 0\}|$ are quite unusual. The main difficulty in the study of (4.2) is to prove the existence of a minimizer because of the lack of the lower semicontinuity of the functional.

For this reason, we prove the existence of a minimizer in the case when Ω is a ball.

Proposition 4.1 *Let $\Lambda, R > 0$ and let B_R be the centered ball with radius R. Then the functional $\mathcal{F}_\Lambda$ defined in (4.1) admits a minimizer v belonging to $H_0^1(\Omega)$. Such a minimizer is unique up to a sign, it is radially symmetric and $|\nabla v|$ is radially increasing.*

Proof We divide the proof in 3 steps.

1. Boundness from below.
   First of all, let us prove that the functional $\mathcal{F}_\Lambda$ is bounded from below for every choice of Λ and for every $R > 0$. For all $\psi \in H_0^1(B_R)$, sing Young and Poincaré

inequalities, we get

$$\begin{aligned}\mathcal{F}_\Lambda(\psi) &= \frac{1}{2}\int_{B_R} |\nabla\psi|^2\,dx - \int_{B_R} \psi\,dx + \Lambda|\{\nabla\psi \neq 0\}| \\ &\geq \frac{1}{2}\int_{B_R} |\nabla\psi|^2\,dx - \varepsilon\int_{B_R} \frac{\psi^2}{2} - \frac{|B_R|}{2\varepsilon} \\ &\geq \frac{1}{2}\int_{B_R} |\nabla\psi|^2\,dx - \frac{\varepsilon C(n, B_R)}{2}\int_{B_R} |\nabla\psi|^2\,dx - \frac{|B_R|}{2\varepsilon} \\ &= \frac{(1-\varepsilon C(n, B_R))}{2}\int_{B_R} |\nabla\psi|^2\,dx - \frac{|B_R|}{2\varepsilon}.\end{aligned}$$

Chosing ε sufficiently small such that

$$0 < \varepsilon \leq \frac{1}{C(n, B_R)}$$

then

$$\mathcal{F}_\Lambda(\psi) \geq -\frac{|B_R|}{2C(n, B_r)} \geq -C(n, B_R) > -\infty$$

so

$$T(B_R, \Lambda) = -\inf_{\psi\in H_0^1(B_R)} \mathcal{F}_\Lambda(\psi) < \infty.$$

2. Compactness and semicontinuity.
Now we consider a minimizing sequence $\{\psi_k\}$ for $T_\mathcal{F}(B_R, \Lambda)$ and we prove that it is bounded in $H_0^1(B_R)$. We can assume that $\mathcal{F}_\Lambda(\psi_k) \leq -T_\mathcal{F}(B_R, \Lambda) + 1$ and by Proposition 2.3 we can assume that ψ_k are radial function with $|\nabla\psi_k|$ radially symmetric increasing.
Using Young and Poincaré inequalities, we obtain

$$\begin{aligned}\mathcal{F}_\Lambda(\psi_k) &= \frac{1}{2}\int_{B_R} |\nabla\psi_k|^2\,dx - \int_{B_R} \psi_k\,dx + \Lambda|\{\nabla\psi_k \neq 0\}| \\ &\geq \frac{1}{2}\int_{B_R} |\nabla\psi_k|^2\,dx - \int_{B_R} \psi_k\,dx \\ &\geq \frac{1}{2}\int_{B_R} |\nabla\psi_k|^2\,dx - \varepsilon\int_{B_R} \frac{\psi_k^2}{2} - \frac{|B_R|}{2\varepsilon} \\ &\geq \frac{1}{2}\int_{B_R} |\nabla\psi_k|^2\,dx - \frac{\varepsilon C(n, B_r)}{2}\int_{B_R} |\nabla\psi_k|^2\,dx - \frac{|B_R|}{2\varepsilon} \\ &= \frac{1-\varepsilon C(n, B_R)}{2}\int_{B_R} |\nabla\psi_k|^2\,dx - \frac{|B_R|}{2\varepsilon}.\end{aligned}$$

Choosing $\varepsilon < \dfrac{1}{C(n, B_R)}$ we have

$$-T_{\mathcal{F}}(B_R, \Lambda) + 1 \geq \mathcal{F}_\Lambda(\psi_k) \geq \frac{1}{4}\int_{B_R} |\nabla \psi_k|^2 \, dx - C(B_R)$$

then by Poincaré inequality, the sequence $\{\psi_k\}$ is bounded in $H_0^1(B_R)$.

This implies that there exists a subsequence (still denoted by ψ_k) and a function $v \in H_0^1(B_R)$ such that $\psi_k \to v$ strongly in $L^2(\Omega)$, a.e. in Ω and $\nabla\psi_k \rightharpoonup \nabla v$ weakly in L^2. Let us show that v is a minimum for $\mathcal{F}_\Lambda$.
The lower semicontinuity of the norms gives

$$\liminf_k \left[\frac{1}{2}\int_{B_R} |\nabla \psi_k|^2 \, dx - \int_{B_R} \psi_k \, dx\right] \geq \frac{1}{2}\int_{B_R} |\nabla v|^2 \, dx - \int_{B_R} v \, dx. \tag{4.3}$$

Let us deal with the last term of $\mathcal{F}_\Lambda$ and let us prove that

$$\liminf_k |\{|\nabla u_k| \neq 0\}| \geq |\{|\nabla u| \neq 0\}|.$$

Denoting by r_k the radius of the ball where $|\nabla \psi_k| = 0$, we can assume that r_k converges to some $r \geq 0$. Therefore

$$\liminf_k \; |\{|\nabla \psi_k| \neq 0\}| = \lim_k [\omega_n(R^n - r_k^n)] = \omega_n(R^n - r^n).$$

So we have just to prove that $|\nabla v| = 0$ in B_r. Since $\{\psi_k\}$ are radial functions, obviously v is radial too.
If $r = 0$ there is nothing to prove.
If $r > 0$, assume by contradiction that there exists $A \subset B_r$ with $|A| > 0$ and that $|\nabla v| \neq 0$ in A. Clearly there exists $\varepsilon > 0$ such that $|A \cap B_{r-\varepsilon}| > 0$.

Since $r_k \to r$ if we choose a function $g \in C_C^\infty(B_R, \mathbb{R}^n)$ with support included in $A \cap B_{r-\varepsilon}$ we have

$$\int_{B_R} \langle \nabla v, g\rangle \, dx = \lim_k \int_{B_R} \langle \nabla \psi_k, g\rangle \, dx = 0.$$

Since this must be true for every $g \in C_C^\infty(A \cap B_{r-\varepsilon}, \mathbb{R}^n)$, we get a contradiction.

Then in any case

$$\liminf_k |\{|\nabla \psi_k| \neq 0\}| \geq |\{|\nabla v| \neq 0\}|. \tag{4.4}$$

By (4.3) and(4.4), we get

$$-T_{\mathcal{F}}(B_R, \Lambda) = \liminf_k \mathcal{F}_\Lambda(\psi_k) \geq \mathcal{F}_\Lambda(v) \geq -T_{\mathcal{F}}(B_R, \Lambda)$$

so v is a minimum of $\mathcal{F}_\Lambda$ in B_R.

3. Uniqueness.
   Let us suppose that v is a minimum of $\mathcal{F}_\Lambda(\psi)$. By Theorem 2.3, it exists $\overline{v} \in H_0^1(B_R)$ such that

$$\mathcal{F}_\Lambda(v) \geq \mathcal{F}_\Lambda(\overline{v})$$

   and since v is minimum, it holds

$$\mathcal{F}_\Lambda(v) = \mathcal{F}_\Lambda(\overline{v}).$$

   Since $|\nabla v|$ is equally distributed with $|\nabla \overline{v}|$, the previous equality implies

$$\|v\|_{L^1} = \|\overline{v}\|_{L^1}$$

   so Theorem 2.4 gives that $|v| = \overline{v}$.

Remark 4.1 We highlight that Theorem 2.3 ensures us that the minimum when Ω is a ball has gradient equal to zero only in a ball B_r centered at the origin with $0 \leq r \leq R$.

Now, as already mention in the introduction, we prove a Saint-Venant type inequality for $T_{\mathcal{F}}(\Omega, \Lambda)$.

Corollary 4.2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with finite perimeter and let $\Omega^\sharp$ be the centered ball. If $\Lambda > 0$, then*

$$T_{\mathcal{F}}(\Omega, \Lambda) \leq T_{\mathcal{F}}(\Omega^\sharp, \Lambda).$$

Proof For every function $\psi \in H_0^1(\Omega)$, by Theorem 2.3 or 1.2, there exists $\overline{\psi} \in H_0^1(\Omega^\sharp)$ that satisfies

$$\mathcal{F}_\Lambda(\psi) \geq \mathcal{F}_\Lambda(\overline{\psi}) \geq -T_{\mathcal{F}}(\Omega^\sharp, \Lambda)$$

and then

$$T_{\mathcal{F}}(\Omega, \Lambda) \leq T_{\mathcal{F}}(\Omega^\sharp, \Lambda).$$

Now we deal with the functional

$$\mathcal{G}(\psi) := \frac{\displaystyle\int_\Omega |\nabla \psi|^2\, dx + \frac{1}{m}\left(\int_{\partial\Omega} |\psi|\, d\mathcal{H}^{n-1}\right)^2}{\displaystyle\left(\int_\Omega |\psi|\, dx\right)^2} \qquad \psi \in H^1(\Omega).$$

with $m > 0$.

The interest in this type of functional is related to the problem of optimal insulation in a given domain. Indeed, the minimum of $\mathcal{G}$ gives the long-time distribution of temperature of the domain Ω and the displacement around Ω of a thin layer of insulator with total mass equal to m. We refer to [8] for more details.

If Ω is a Lipschitz domain, $\mathcal{G}(\psi)$ achieves its minimum among all $H^1(\Omega)$ functions. So we define

$$\frac{1}{T_{\mathcal{G}}(\Omega, m)} := \min_{\psi \in H^1(\Omega)} \mathcal{G}(\psi).$$

It is easy to check that the Euler-Lagrange equation of this functional is

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \dfrac{1}{m} \displaystyle\int_{\partial\Omega} |u| \, d\mathcal{H}^{n-1} = 0 & \text{on } \partial\Omega. \end{cases}$$

So Theorem 1.2 gives us the following Saint-Venant type inequality for $T_{\mathcal{G}}(\Omega)$.

Corollary 4.3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with finite perimeter and let $\Omega^\sharp$ be the centered ball. If $m > 0$, then*

$$T_{\mathcal{G}}(\Omega, m) \le T_{\mathcal{G}}(\Omega^\sharp, m).$$

Proof For every function $\psi \in H^1(\Omega)$, by 1.2, there exists $\overline{\psi} \in H^1(\Omega^\sharp)$ that satisfies

$$\mathcal{G}(\psi) \ge \mathcal{G}(\overline{\psi}) \ge \frac{1}{T_{\mathcal{G}}(\Omega^\sharp, m)}$$

and then

$$T_{\mathcal{G}}(\Omega, m) \le T_{\mathcal{G}}(\Omega^\sharp, m).$$

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