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Entire solutions with and without radial symmetry in balanced bistable reaction–diffusion equations

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Abstract
Let $n \geq 2$ be a given integer. In this paper, we assert that an n-dimensional traveling front converges to an $(n-1)$-dimensional entire solution as the speed goes to infinity in a balanced bistable reaction–diffusion equation. As the speed of an n-dimensional axially symmetric or asymmetric traveling front goes to infinity, it converges to an $(n-1)$-dimensional radially symmetric or asymmetric entire solution in a balanced bistable reaction–diffusion equation, respectively. We conjecture that the radially asymmetric entire solutions obtained in this paper are associated with the ancient solutions called the Angenent ovals in the mean curvature flows.

Mathematics Subject Classification 35C07 · 35B08 · 35K57

1 Introduction

In this paper we study a reaction–diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u - W'(u), \qquad \boldsymbol{x} \in \mathbb{R}^n,\ t > 0, \tag{1.1}$$

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^n, \tag{1.2}$$

where $n \geq 2$ is a given integer, and u_0 is a given bounded and uniformly continuous function from $\mathbb{R}^n$ to $\mathbb{R}$. The following is the standing assumptions of $W \in C^3[-1, 1]$ in this paper

$$W'(1) = 0,\ W'(-1) = 0,\ W''(1) > 0,\ W''(-1) > 0, \tag{1.3}$$

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$$W(1)=0,\ W(-1)=0, \tag{1.4}$$
$$W(s)>0 \quad \text{if} \quad -1<s<1. \tag{1.5}$$

Here W' and W'' are the first and second derivatives of W, respectively. Equation (1.1) is called the Allen–Cahn equation, the scalar Ginzburg–Landau equation or the Nagumo equation if

$$W(u)=\frac{(s+1)^2(s-1)^2}{4}, \quad -W'(u)=u-u^3.$$

A nonlinear term $-W'(u)$ with (1.3) is called a bistable one. It is called *balanced* if $W(-1)=W(1)$, and is called *imbalanced* if $W(-1)\neq W(1)$. In this paper we assume that $-W'(u)$ is balanced. We write the solution of (1.1)–(1.2) as $u(x,t;u_0)$. Under the assumption of W stated above, there exists Φ that satisfies

$$\begin{aligned}
&\Phi''(x_1)-W'(\Phi(x_1))=0, \qquad x_1\in\mathbb{R},\\
&-\Phi'(x_1)>0, \qquad x_1\in\mathbb{R},\\
&\Phi(-\infty)=1, \quad \Phi(0)=0, \quad \Phi(\infty)=-1.
\end{aligned}$$

This Φ is called the one-dimensional standing front, and is explicitly given by

$$-x_1=\int_0^{\Phi}\frac{\mathrm{d}s}{\sqrt{2W(s)}}, \qquad -1<\Phi<1.$$

Under assumptions (1.4) and (1.3), there exists Φ if and only if (1.5) holds true. Let

$$\boldsymbol{x}'=(x_1,\ldots,x_{n-1})\in\mathbb{R}^{n-1}, \quad \boldsymbol{x}=(\boldsymbol{x}',x_n)\in\mathbb{R}^n.$$

Now we write $r=|\boldsymbol{x}'|$. Let $c\in(k,\infty)$ be arbitrarily given. We put $z=x_n-ct$ and $w(\boldsymbol{x}',z,t)=u(\boldsymbol{x}',x_n,t)$ for $(\boldsymbol{x}',z)\in\mathbb{R}^n$ and $t>0$. Then we have

$$\frac{\partial w}{\partial t}-\left(\sum_{j=1}^{n-1}\frac{\partial^2}{\partial x_j^2}+\frac{\partial^2}{\partial z^2}\right)w-c\frac{\partial w}{\partial z}+W'(w)=0, \qquad (\boldsymbol{x}',z)\in\mathbb{R}^n,\ t>0,$$
$$w(\boldsymbol{x}',z,0)=u_0(\boldsymbol{x}',z), \qquad (\boldsymbol{x}',z)\in\mathbb{R}^n.$$

We write z simply as x_n. Then we have

$$\frac{\partial w}{\partial t}-\sum_{j=1}^{n}\frac{\partial^2 w}{\partial x_j^2}-c\frac{\partial w}{\partial x_n}+W'(w)=0, \qquad (\boldsymbol{x}',x_n)\in\mathbb{R}^n,\ t>0, \tag{1.6}$$
$$w(\boldsymbol{x}',x_n,0)=u_0(\boldsymbol{x}',x_n), \qquad (\boldsymbol{x}',x_n)\in\mathbb{R}^n. \tag{1.7}$$

We write the solution of (1.6)–(1.7) as $w(\boldsymbol{x},t;u_0)$.

If $v \in C^2(\mathbb{R}^n)$ satisfies

$$\sum_{j=1}^{n} \frac{\partial^2 v}{\partial x_j^2} + c\frac{\partial v}{\partial x_n} - W'(v) = 0, \qquad (\boldsymbol{x}', x_n) \in \mathbb{R}^n \tag{1.8}$$

for $c \in \mathbb{R}$, $v(\boldsymbol{x}', x_n - ct)$ becomes a traveling wave or a traveling front of (1.1). We call (1.8) the profile equation of $v(\boldsymbol{x})$. We call v a traveling profile and call c its speed. Sometimes we denote a traveling front $v(\boldsymbol{x}', x_n - ct)$ simply by (c, v). Now x_n is the traveling direction of (c, v). For a multidimensional traveling front, a traveling direction of (c, v) might not be uniquely determined. We say that a traveling front is axisymmetric if we can choose a traveling direction such that v is axisymmetric with respect to the traveling direction. We say that a traveling front is axially asymmetric if v is axially asymmetric with respect to every traveling direction.

If a function $U(\boldsymbol{x}', t)$ satisfies

$$\frac{\partial U}{\partial t} = \Delta' U - W'(U), \qquad (\boldsymbol{x}', t) \in \mathbb{R}^n, \tag{1.9}$$

$U(\boldsymbol{x}', t)$ is called an entire solution in $\mathbb{R}^{n-1}$, where

$$\Delta' = \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2}. \tag{1.10}$$

A traveling front solution to (1.9) is itself an entire solution to (1.9). Now we say that $U(\boldsymbol{x}', t)$ is radially symmetric or spherically symmetric with respect to $\boldsymbol{a}' \in \mathbb{R}^{n-1}$ if U is a function of $|\boldsymbol{x}' - \boldsymbol{a}'|$ and $t \in \mathbb{R}$. If $U(\boldsymbol{x}', t)$ is not radially symmetric with respect to any $\boldsymbol{a}' \in \mathbb{R}^{n-1}$, we say that $U(\boldsymbol{x}', t)$ is radially asymmetric.

Traveling fronts to (1.1) have been studied by [6, 10, 11, 31–34] in $\mathbb{R}^n$ for $n \geq 2$. Axisymmetric traveling fronts have been studied by [6, 32], and axially asymmetric traveling fronts have been studied by [31]. A one-dimensional entire solution is studied by Chen, Guo and Ninomiya [7] and del Pino and Gkikas [9]. It is an interesting question whether there exists a relation between traveling fronts in $\mathbb{R}^n$ and entire solutions in $\mathbb{R}^{n-1}$. In this paper we show that a traveling profile of (1.8) in $\mathbb{R}^n$ converges to an entire solution of (1.9) in $\mathbb{R}^{n-1}$ as the speed c goes to infinity. Then, using this fact, we show the existence of a radially symmetric entire solution and a radially asymmetric entire solution of (1.9) as the limits of an axisymmetric traveling front and an axially asymmetric traveling front of (1.8), respectively.

Now we define $s_* \in (-1, 1)$ by

$$s_* = \min\left\{s_0 \in (-1, 1) \mid -W'(s) > 0 \quad \text{if} \quad s_0 < s < 1\right\}$$

and fix $\theta_0 \in (s_*, 1)$ with $-W'(\theta_0) > 0$. Now we assert the existence of a radially symmetric entire solution as follows.

Theorem 1 (Radially symmetric entire solution) *Let $R_0 \in (0, \infty)$ be arbitrarily given. One has $U_{\mathrm{sym}}(|\boldsymbol{x}'|, t) = U(\boldsymbol{x}', t)$ for $\boldsymbol{x}' \in \mathbb{R}^{n-1}$ such that one has (1.9) with*

$$\begin{aligned} & -1 < U_{\mathrm{sym}}(r,t) < 1, \qquad (r,t) \in (0,\infty) \times \mathbb{R}, \\ & U_{\mathrm{sym}}(R_0, 0) = \theta_0, \\ & \frac{\partial U_{\mathrm{sym}}}{\partial t}(r,t) > 0, \qquad (r,t) \in (0,\infty) \times \mathbb{R}, \\ & \frac{\partial U_{\mathrm{sym}}}{\partial r}(r,t) > 0, \qquad (r,t) \in (0,\infty) \times \mathbb{R}. \end{aligned}$$

Here $r = |\boldsymbol{x}'|$. One has

$$\lim_{t \to \infty} \inf_{r \in [0,\infty)} U_{\mathrm{sym}}(r,t) = 1.$$

As $t \to -\infty$, $U_{\mathrm{sym}}(r,t)$ converges to -1 on every compact set in $[0, \infty)$. For any fixed $t \in \mathbb{R}$, one has

$$\lim_{r \to \infty} U_{\mathrm{sym}}(r,t) = 1. \tag{1.11}$$

See Fig. 1 for $\{\boldsymbol{x}' \in \mathbb{R}^{n-1} \mid U(\boldsymbol{x}', 0) = \theta_0\}$ for U in Theorem 1. Now we assert the existence of radially asymmetric entire solution.

Theorem 2 (Radially asymmetric entire solution) *Let*

$$0 < R_1 \le R_2 \le \cdots \le R_{n-1} < \infty$$

be arbitrarily given. Then there exists U that satisfies

$$\frac{\partial U}{\partial t}(\boldsymbol{x}', t) = \Delta' U(\boldsymbol{x}', t) - W'(U(\boldsymbol{x}', t)), \qquad (\boldsymbol{x}', t) \in \mathbb{R}^n$$

with

$$\begin{aligned} & -1 < U(\boldsymbol{x}', t) < 1, \qquad (\boldsymbol{x}', t) \in \mathbb{R}^n, \\ & U(0, \ldots, 0, \overset{j}{\smile}{R_j}, 0, \ldots, 0, 0) = \theta_0, \qquad 1 \le j \le n-1, \\ & \frac{\partial U}{\partial t}(\boldsymbol{x}', t) > 0, \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1},\ t \in \mathbb{R}, \\ & \frac{\partial U}{\partial x_j}(\boldsymbol{x}', t) > 0, \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1},\ t \in \mathbb{R},\ 1 \le j \le n-1, \\ & U(x_1, \ldots, \overset{j}{\smile}{-x_j}, \ldots, x_{n-1}, t) = U(x_1, \ldots, \overset{j}{\smile}{x_j}, \ldots, x_{n-1}, t), \\ & \qquad (\boldsymbol{x}', t) \in \mathbb{R}^n,\ 1 \le j \le n-1 \end{aligned}$$

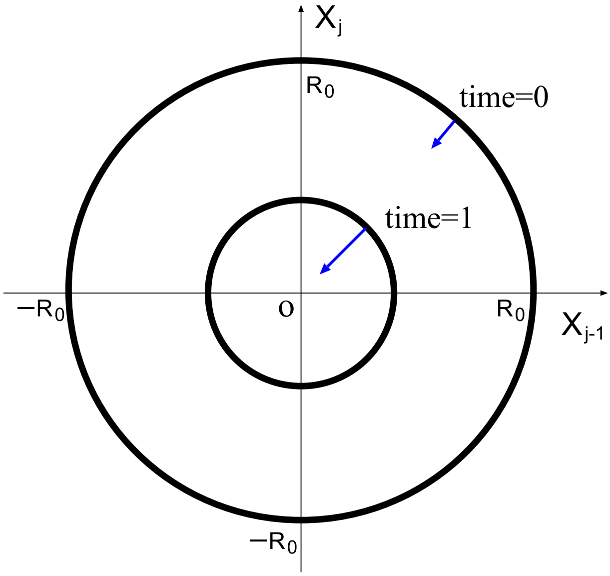

Fig. 1 The evolution of $\{x' \in \mathbb{R}^{n-1} \mid U(x', t) = \theta_0\}$ for $t = 0, 1$, where U is a radially symmetric entire solution in Theorem 1. Here $2 \leq j \leq n$

One has

$$\lim_{t\to\infty} \inf_{x' \in \mathbb{R}^{n-1}} U(x', t) = 1.$$

As $t \to -\infty$, $U(x', t)$ *converges to* -1 *on every compact set in* $\mathbb{R}^{n-1}$. *For any fixed* $t \in \mathbb{R}$, *one has*

$$\lim_{|x'|\to\infty} U(x', t) = 1. \tag{1.12}$$

See Fig. 2 for $\{x' \in \mathbb{R}^{n-1} \mid U(x', 0) = \theta_0\}$ for U in Theorem 2. For a reaction–diffusion equation with an imbalanced bistable reaction term, traveling fronts have been studied by [16–18, 23, 24, 27–30, 34] and so on, and and entire solutions have been studied by [3, 5, 13–15, 21, 22, 37] and so on. See [25] for a relation between traveling fronts in $\mathbb{R}^n$ and entire solutions in $\mathbb{R}^{n-1}$ for $n \geq 2$.

This paper and [25] suggest that an n-dimensional traveling fronts converges to an $(n-1)$-dimensional entire solution as the speed goes to infinity in various kind of reaction–diffusion equations.

This paper is organized as follow. In Sect. 3, we summarize the properties of n-dimensional traveling fronts with a speed $c \in (0, \infty)$. In Sect. 4, we study n-dimensional traveling fronts as the speed $c \in (0, \infty)$ goes to infinity and obtain entire solutions as the limits. We prove the existence of radially symmetric entire solutions and radially asymmetric entire solutions to (1.1).

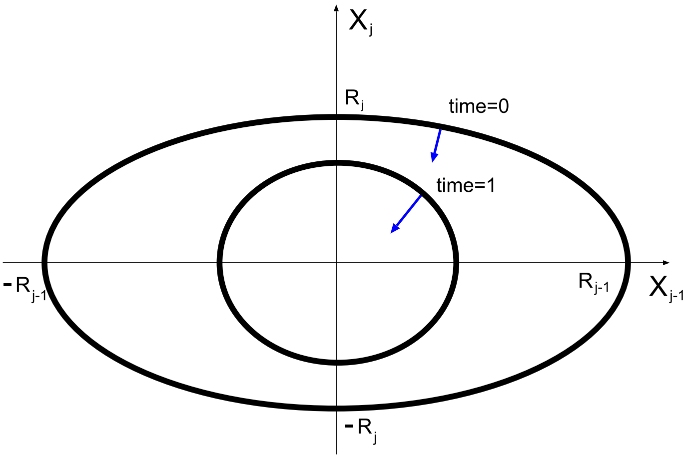

Fig. 2 The evolution of $\{\boldsymbol{x}' \in \mathbb{R}^{n-1} \mid U(\boldsymbol{x}', t) = \theta_0\}$ for $t = 0, 1$. Here U is a radially asymmetric entire solution in Theorem 2, where $2 \leq j \leq n$

2 Discussions

Let $U(\boldsymbol{x}', t)$ be given by Theorem 1 or Theorem 2. When

$$W(u) = \frac{(u+1)^2(u-1)^2}{4\varepsilon^2}, \quad -W'(u) = \frac{u-u^3}{\varepsilon^2} \tag{2.1}$$

for $\varepsilon > 0$, the motion of a level set $\{\boldsymbol{x}' \in \mathbb{R}^{n-1} \mid U(\boldsymbol{x}', t) = 0\}$ is approximated by a mean curvature flow in the limit of $\varepsilon \to 0$. See [4] for instance. Axisymmetric or axially asymmetric traveling fronts in Theorem 4 or in Theorem 8 are closely related to those in mean curvature flows studied by Wang [35]. For a mean curvature flow, a curve or a surface evolves with time. If a curve or a surface is defined for all $t \in (-\infty, t_0)$ with some $t_0 \in \mathbb{R}$, it is called an ancient solution. A typical example is ancient solutions studied by Angenent [1] and Angenent, Daskalopoulos and Sesum [8] for a two-dimensional plane, and they are called the *Angenent ovals* or the *paper clip solutions*. Ancient solutions have been studied by White [36], Haslhofer and Hershkovits [19] and Angenent, Daskalopoulos and Sesum [2] for a space whose dimension is three or more. The author conjectures as follows.

Conjecture 1 *Let W be given by (2.1). Let $U(\boldsymbol{x}', t)$ be an entire solution in Theorem 1 or in Theorem 2. Let $T_0 \in \mathbb{R}$ be uniquely given by $U(\boldsymbol{0}', T_0) = 0$ in Theorem 1 or in Theorem 2. Let $0 < \gamma_1 < \gamma_2 < \infty$ be arbitrarily given. Then, as $\varepsilon \to 0$, a level set $\{\boldsymbol{x}' \in \mathbb{R}^{n-1} \mid U(\boldsymbol{x}', t) = 0\}$ converges to an ancient solution of a mean curvature flow for $t \in [T_0 - \gamma_2, T_0 - \gamma_1]$.*

The study on this convergence will give an important insight on entire solutions in a reaction–diffusion equation and on ancient solutions in a mean curvature flow. Note that a paper clip solution lies between two parallel lines for all time till it extinguishes. As $t \to -\infty$, an axially asymmetric entire solution U in Theorem 2 converges to -1

on every given compact set in $\mathbb{R}^2$ for $n = 3$. Thus $\{\boldsymbol{x}' \in \mathbb{R}^{n-1} \mid U(\boldsymbol{x}', t) = 0\}$ cannot lie between two parallel lines as $t \to -\infty$. This means that axially asymmetric entire solutions in Theorem 2 are novel propagation phenomena.

3 Properties of *n*-dimensional traveling fronts with various speeds

We extend W as a function of class C^3 in an open interval that includes $[-1, 1]$. Let

$$\beta = \frac{1}{2} \min \left\{ W''(1), W''(-1) \right\} > 0.$$

Let $\delta_* \in (0, 1/4)$ be small enough such that $[-1 - 2\delta_*, 1 + 2\delta_*]$ is included in the open interval and one has

$$\min_{|u+1| \le 2\delta_*} W''(u) > \beta, \qquad \min_{|u-1| \le 2\delta_*} W''(u) > \beta.$$

Now we put

$$M = 1 + \max_{|u| \le 1 + 2\delta_*} \left| W''(u) \right| \tag{3.1}$$

and introduce a positive constant σ by

$$\sigma = 1 + \frac{\beta + \max_{|u| \le 1 + 2\delta_*} |W''(u)|}{\beta \min\{-\Phi'(x_1) \mid x_1 \in \mathbb{R},\ -1 + \delta_* \le \Phi(x_1) \le 1 - \delta_*\}}.$$

Throughout this paper we assume

$$-1 < v(\boldsymbol{x}) < 1, \qquad \boldsymbol{x} \in \mathbb{R}^n \tag{3.2}$$

and

$$-1 - \delta_* \le u_0(\boldsymbol{x}) \le 1 + \delta_*, \qquad \boldsymbol{x} \in \mathbb{R}^n.$$

Then $u(\boldsymbol{x}, t) = u(\boldsymbol{x}, t; u_0)$ satisfies (1.1) with

$$-1 - \delta_* \le u(\boldsymbol{x}, t) \le 1 + \delta_*, \qquad \boldsymbol{x} \in \mathbb{R}^n,\ t > 0.$$

Now the Schauder estimates [34, Proposition 2.9, Lemma 2.6] give

$$\max_{1 \le j \le n} \sup_{\boldsymbol{x} \in \mathbb{R}^n,\, t \ge 1} \left| \mathrm{D}_j u(\boldsymbol{x}, t) \right| \le K_*, \tag{3.3}$$

$$\max_{1 \le i \le n, 1 \le j \le n} \sup_{\boldsymbol{x} \in \mathbb{R}^n,\, t \ge 1} \left| \mathrm{D}_{ij} u(\boldsymbol{x}, t) \right| \le K_*, \tag{3.4}$$

$$\sup_{\boldsymbol{x} \in \mathbb{R}^n,\, t \ge 1} |\mathrm{D}_t u(\boldsymbol{x}, t)| \le K_*. \tag{3.5}$$

$$\max_{1\le j_1\le n, 1\le j_2\le n, 1\le j_3\le n} \sup_{\boldsymbol{x}\in\mathbb{R}^n,\, t\ge 1} \left|\mathrm{D}_{j_1}\mathrm{D}_{j_2}\mathrm{D}_{j_3}u(\boldsymbol{x},t)\right| \le K_*. \tag{3.6}$$

Here $K_* \in (0,\infty)$ is a constant depending only on (W, δ_*, n) and is independent of u_0. We use

$$\mathrm{D}_t = \frac{\partial}{\partial t}, \quad \mathrm{D}_j = \frac{\partial}{\partial x_j}, \quad \mathrm{D}_j^2 = \frac{\partial^2}{\partial x_j^2}, \quad \mathrm{D}_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}, \qquad 1 \le i \le n,\ 1 \le j \le n.$$

Lemma 3 *Assume $c \in \mathbb{R}$ and $v \in C^2(\mathbb{R}^n)$ satisfy (1.8) and (3.2). Then one has*

$$\max_{1\le j\le n} \sup_{\boldsymbol{x}\in\mathbb{R}^n} \left|\mathrm{D}_j v(\boldsymbol{x})\right| \le K_*, \qquad \max_{1\le i\le n,\, 1\le j\le n} \sup_{\boldsymbol{x}\in\mathbb{R}^n} \left|\mathrm{D}_{ij} v(\boldsymbol{x})\right| \le K_*, \tag{3.7}$$

$$\max_{1\le j_1\le n,\, 1\le j_2\le n,\, 1\le j_3\le n} \sup_{\boldsymbol{x}\in\mathbb{R}^n} \left|\mathrm{D}_{j_1}\mathrm{D}_{j_2}\mathrm{D}_{j_3} v(\boldsymbol{x})\right| \le K_*. \tag{3.8}$$

Here K_* is a constant in (3.3)–(3.5), and is independent of $(c, v) \in \mathbb{R} \times C^2(\mathbb{R}^n)$.

Proof By putting $u(\boldsymbol{x}', x_n, t) = v(\boldsymbol{x}', x_n - ct)$, u satisfies (1.1) with

$$u(\boldsymbol{x}', x_n, 0) = v(\boldsymbol{x}', x_n), \qquad (\boldsymbol{x}', x_n) \in \mathbb{R}^n.$$

Then (3.3), (3.4) and (3.6) give

$$\max_{1\le j\le n} \sup_{(\boldsymbol{x}',\, x_n)\in\mathbb{R}^n,\, 1\le t} \left|\mathrm{D}_j v(\boldsymbol{x}', x_n - ct)\right| \le K_*,$$

$$\max_{1\le i\le n,\, 1\le j\le n} \sup_{(\boldsymbol{x}',\, x_n)\in\mathbb{R}^n,\, 1\le t} \left|\mathrm{D}_{ij} v(\boldsymbol{x}', x_n - ct)\right| \le K_*,$$

$$\max_{1\le j_1\le n,\, 1\le j_2\le n,\, 1\le j_3\le n} \sup_{(\boldsymbol{x}',\, x_n)\in\mathbb{R}^n,\, 1\le t} \left|\mathrm{D}_{j_1}\mathrm{D}_{j_2}\mathrm{D}_{j_3} v(\boldsymbol{x}', x_n - ct)\right| \le K_*,$$

which give

$$\max_{1\le j\le n} \sup_{(\boldsymbol{x}',\, x_n)\in\mathbb{R}^n} \left|\mathrm{D}_j v(\boldsymbol{x}', x_n - c)\right|$$

$$\le K_*, \quad \max_{1\le i\le n,\, 1\le j\le n} \sup_{(\boldsymbol{x}',\, x_n)\in\mathbb{R}^n} \left|\mathrm{D}_{ij} v(\boldsymbol{x}', x_n - c)\right| \le K_*,$$

$$\max_{1\le j_1\le n,\, 1\le j_2\le n,\, 1\le j_3\le n} \sup_{(\boldsymbol{x}',\, x_n)\in\mathbb{R}^n} \left|\mathrm{D}_{j_1}\mathrm{D}_{j_2}\mathrm{D}_{j_3} v(\boldsymbol{x}', x_n - c)\right| \le K_*.$$

Thus we obtain (3.7) and (3.8). This completes the proof. □

Now we state properties of axisymmetric traveling fronts as follows.

Theorem 4 (Axisymmetric traveling fronts [6, 32]) *Let $c \in (0,\infty)$ be arbitrarily given. There exists $V_c(\boldsymbol{x}', x_n) = V_{\mathrm{sym}}(|\boldsymbol{x}'|, x_n)$ such that (c, V_c) satisfies the profile equation (1.8), $V_{\mathrm{sym}}(0,0) = \theta_0$ and*

$$-1 < V_{\mathrm{sym}}(r, x_n) < 1, \qquad r \ge 0,\ x_n \in \mathbb{R},$$

$$\frac{\partial V_{\text{sym}}}{\partial x_n}(r, x_n) < 0 \quad \text{if} \quad r \geq 0, x_n \in \mathbb{R},$$
$$\frac{\partial V_{\text{sym}}}{\partial r}(r, x_n) > 0 \quad \text{if} \quad r > 0, x_n \in \mathbb{R}. \tag{3.9}$$

For every $\theta \in (-1, 1)$*, one has*

$$\inf_{r \geq 0, x_n \in \mathbb{R}} \left\{ \sqrt{\left(\frac{\partial V_{\text{sym}}}{\partial r}(r, x_n)\right)^2 + \left(\frac{\partial V_{\text{sym}}}{\partial x_n}(r, x_n)\right)^2} \;\middle|\; V_{\text{sym}}(r, x_n) = \theta \right\} > 0. \tag{3.10}$$

Here $r = |\boldsymbol{x}'|$.

Remark 1 As far as the author knows, the uniqueness of V_{sym} in Theorem 4 is yet to be studied. Here we denote a traveling front that satisfies Theorem 4 by V_{sym}. It depends on $c \in (0, \infty)$.

Now we state properties of axially asymmetric traveling fronts in [31] as follows.

Theorem 5 [31] *Let*

$$1 \leq \alpha_2 \leq \cdots \leq \alpha_{n-1} < \infty \tag{3.11}$$

be arbitrarily given and let

$$\boldsymbol{\alpha}' = (1, 00\alpha_2, \ldots, \alpha_{n-1}) \in \mathbb{R}^{n-1}. \tag{3.12}$$

Let $c \in (0, \infty)$ *and* $\zeta \in (0, \infty)$ *be arbitrarily given. There exists* $V(\boldsymbol{x}) = V(\boldsymbol{x}; \boldsymbol{\alpha}', c)$ *that satisfies* (1.8) *with* $V(\mathbf{0}) = \theta_0$ *and*

$$-1 < V(\boldsymbol{x}) < 1, \quad \boldsymbol{x} \in \mathbb{R}^n,$$
$$\mathrm{D}_n V(\boldsymbol{x}', x_n) < 0, \quad (\boldsymbol{x}', x_n) \in \mathbb{R}^n, \tag{3.13}$$

$$V(x_1, \ldots, \underbrace{-x_j}_{j}, \ldots, x_n) = V(x_1, \ldots, \underbrace{x_j}_{j}, \ldots, x_n), \quad (\boldsymbol{x}', x_n) \in \mathbb{R}^n, \; 1 \leq j \leq n-1, \tag{3.14}$$

$$\mathrm{D}_j V(\boldsymbol{x}', x_n) > 0 \quad \text{if} \quad x_j > 0, \; 1 \leq j \leq n-1,$$

$$V(0, \ldots, 0, \underbrace{r_j}_{j}, 0, \ldots, 0, \zeta) = \theta_0, \tag{3.15}$$

where a positive number r_j $(1 \leq j \leq n-1)$ *satisfies*

$$\frac{r_j}{r_1} = \alpha_j, \quad 2 \leq j \leq n-1.$$

For every $\theta \in (-1, 1)$ *one has*

$$\inf_{(\boldsymbol{x}', x_n) \in \mathbb{R}^n} \left\{ |\nabla V(\boldsymbol{x}', x_n)| \mid V(\boldsymbol{x}', x_n) = \theta \right\} > 0. \tag{3.16}$$

Let $c \in (0, \infty)$ be given and let $V_c \in C^2(\mathbb{R}^n)$ satisfy (1.8). Under some condition, we assert that a level set $\{(\boldsymbol{x}', x_n) \mid V_c(\boldsymbol{x}', x_n) = \theta_0\}$ is a graph on the whole space $\mathbb{R}^{n-1}$ in the following proposition.

Proposition 6 *Fix* $\theta_0 \in (s_*, 1)$ *with* $-W'(\theta_0) > 0$ *arbitrarily. For any fixed* $c \in (0, \infty)$*, let* $V_c \in C^2(\mathbb{R}^n)$ *satisfy* $V_c(\mathbf{0}', 0) = \theta_0$*,* (1.8)*,* (3.2)*,* (3.14) *and*

$$\begin{aligned} &\mathrm{D}_n V_c(\boldsymbol{x}', x_n) \le 0, \qquad (\boldsymbol{x}', x_n) \in \mathbb{R}^n, \\ &\mathrm{D}_j V_c(\boldsymbol{x}', x_n) \ge 0 \qquad \text{if} \quad x_j > 0,\ 1 \le j \le n-1. \end{aligned}$$

Then, for arbitrarily given $\mu_0 \in (0, \infty)$*, one has*

$$\lim_{x_n \to \infty} V_c(\mu_0, \ldots, \mu_0, x_n) < \theta_0.$$

Now we will make preparation to prove this proposition. For $\mu_0 \in (0, \infty)$, we define

$$\Omega_{n-1}(\mu_0) = \left\{ (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \,\middle|\, \min_{1 \le j \le n-1} |x_j| \ge \mu_0 \right\}. \tag{3.17}$$

Hereafter we simply write $\Omega_{n-1}(\mu_0)$ as Ω_{n-1}. Let $c \in (0, \infty)$ be arbitrarily given. Taking an initial function

$$w_0(\boldsymbol{x}', x_n) = \begin{cases} \theta_0 & \text{if} \quad \boldsymbol{x}' \in \Omega_{n-1},\ x_n \in \mathbb{R}, \\ -1 & \text{if} \quad \boldsymbol{x}' \in \mathbb{R}^{n-1} \backslash \Omega_{n-1},\ x_n \in \mathbb{R}, \end{cases} \tag{3.18}$$

we consider $w(\boldsymbol{x}', x_n, t; w_0)$ as a solution of (1.6). Since $w(\boldsymbol{x}', x_n, t; w_0)$ is independent of x_n, we simply write $w(\boldsymbol{x}', t; w_0)$. Now $w(\boldsymbol{x}', t; w_0)$ satisfies

$$\begin{aligned} &\frac{\partial w}{\partial t} - \sum_{j=1}^{n-1} \frac{\partial^2 w}{\partial x_j^2} + W'(w) = 0, && \boldsymbol{x}' \in \mathbb{R}^{n-1},\ t > 0, \\ &w(\boldsymbol{x}', 0) = w_0(\boldsymbol{x}'), && \boldsymbol{x}' \in \mathbb{R}^{n-1}. \end{aligned}$$

Note that $w(\boldsymbol{x}', t; w_0)$ is independent of $c \in (0, \infty)$.

Lemma 7 *Let* w_0 *be given by* (3.18)*. Then* $w(\boldsymbol{x}', t; w_0)$ *satisfies*

$$\lim_{t \to \infty} \inf_{\boldsymbol{x}' \in \mathbb{R}^{n-1}} w(\boldsymbol{x}', t; w_0) = 1.$$

Proof First we prove this lemma for $n = 2$. Let $n = 2$. Let $\delta \in (0, \delta_*]$ be given. There exists $T_1 \in (0, \infty)$ such that we have

$$1 - \frac{\delta}{4} < w(\boldsymbol{x}, t; \theta_0), \qquad \boldsymbol{x} \in \mathbb{R},\ t \geq T_1.$$

Note that $w(\boldsymbol{x}, t; \theta_0)$ depends only on t and is independent of $\boldsymbol{x}$. By applying [34, Theorem 5.8], there exists $r_1 \in (0, \infty)$ with

$$\sup_{|x_1| \geq r_1} |w(x_1, T_1; \theta_0) - w(x_1, T_1; w_0)| < \frac{\delta}{4}.$$

Combining these inequalities together, we have

$$1 - \frac{\delta}{2} < \inf_{|x_1| \geq r_1} w(x_1, T_1; w_0). \tag{3.19}$$

Taking $S_1 \in (0, \infty)$ large enough, we have

$$w(x_1, T_1, w_0) \geq \Phi(x_1 - S_1) - \delta, \qquad x_1 \in \mathbb{R},$$

which yields

$$w(x_1, t + T_1; w_0) \geq \Phi(x_1 - S_1 - \sigma\delta(1 - e^{-\beta t})) - \delta e^{-\beta}, \qquad x_1 \in \mathbb{R},\ t \geq 0.$$

Since w_0 is symmetric in x_1, we have

$$\begin{aligned} &\max\{\Phi(x_1 - S_1 - \sigma\delta(1 - e^{-\beta t})) - \delta e^{-\beta}, \Phi(-x_1 + S_1 + \sigma\delta(1 - e^{-\beta t})) - \delta e^{-\beta}\} \\ &\quad \leq w(x_1, t + T_1; w_0) \end{aligned}$$

for $x_1 \in \mathbb{R}$ and $t \geq 0$. Because the left-hand side of the above inequality is a subsolution, it is monotone increasing in $t \geq 0$ and we can define

$$\begin{aligned} v_\infty(x_1) = \lim_{t \to \infty} &\max\{\Phi(x_1 - S_1 - \sigma\delta(1 - e^{-\beta t})) - \delta e^{-\beta}, \\ &\Phi(-x_1 + S_1 + \sigma\delta(1 - e^{-\beta t})) - \delta e^{-\beta}\} \end{aligned}$$

for $x_1 \in \mathbb{R}$. Now v_∞ satisfies

$$\begin{aligned} &v_\infty(x_1) - W'(v_\infty(x_1)) = 0, \qquad x_1 \in \mathbb{R}, \\ &-1 \leq v_\infty(x_1) \leq 1, \qquad x_1 \in \mathbb{R} \end{aligned}$$

due to [26, 34]. Now we will show $v_\infty \equiv 1$. For this purpose, we begin with

$$\max\{\Phi(x_1 - S_1 - \sigma\delta), \Phi(-x_1 + S_1 + \sigma\delta)\} \leq v_\infty(x_1) \leq 1, \qquad x_1 \in \mathbb{R}. \tag{3.20}$$

We define

$$\Lambda = \inf\{\lambda \in \mathbb{R} \mid \Phi(x_1 - \lambda) \leq v_\infty(x_1),\ \ x_1 \in \mathbb{R}\}$$

and will show $\Lambda = -\infty$. We will get a contradiction assuming $\Lambda \in (-\infty, S_1 + \sigma\delta]$. Then we have

$$\Phi(x_1 - \Lambda) \leq v_\infty(x_1), \qquad x_1 \in \mathbb{R}.$$

Using (3.20) and the strong maximum principle, we have

$$\Phi(x_1 - \Lambda) < v_\infty(x_1), \qquad x_1 \in \mathbb{R}.$$

Now we take $R' \in (1 + |\Lambda|, \infty)$ large enough such that we have

$$\sup_{|x_1| \geq R' - 1 - |\Lambda|} |\Phi'(x_1)| < \frac{1}{4\sigma}.$$

Then, we take $h \in (0, 1/(2\sigma))$ small enough such that we have

$$\Phi(x_1 - \Lambda + 2\sigma h) < v_\infty(x_1) \qquad \text{if} \quad |x_1| \leq R'.$$

If $|x_1| \geq R'$, we have

$$\Phi(x_1 - \Lambda + 2\sigma h) - \Phi(x_1 - \Lambda) = 2\sigma h \int_0^1 \Phi'(x_1 - \Lambda + 2\theta\sigma h)\, \mathrm{d}\theta.$$

Using

$$|x_1 - \Lambda + 2\theta\sigma h| \geq |x_1| - |\Lambda - 2\theta\sigma h| \geq R' - 1 - |\Lambda|,$$

we have

$$0 < 2\sigma h \int_0^1 \left(-\Phi'(x_1 - \Lambda + 2\theta\sigma h)\right) \mathrm{d}\theta < h \qquad \text{if} \quad |x_1| \geq R'.$$

Then, using

$$|\Phi(x_1 - \Lambda + 2\sigma h) - \Phi(x_1 - \Lambda)| \leq h \qquad \text{if} \quad |x_1| \geq R',$$

we find

$$\Phi(x_1 - \Lambda) \geq \Phi(x_1 - \Lambda + 2\sigma h) - h \qquad \text{if} \quad |x_1| \geq R'.$$

Thus we get

$$\Phi(x_1 - \Lambda + 2\sigma h) - h \leq \Phi(x_1 - \Lambda) < v_\infty(x_1) \qquad \text{if} \quad |x_1| \geq R'.$$

Combining the two estimates stated above together, we obtain

$$\Phi(x_1 - \Lambda + 2\sigma h) - h < v_\infty(x_1), \qquad x_1 \in \mathbb{R}.$$

Then we have

$$\Phi(x_1 - \Lambda + 2\sigma h - \sigma h e^{-\beta t}) - h e^{-\beta t} < v_\infty(x_1), \qquad x_1 \in \mathbb{R},\ t > 0.$$

Sending $t \to \infty$, we obtain

$$\Phi(x_1 - \Lambda + \sigma h) \le v_\infty(x_1), \qquad x_1 \in \mathbb{R}.$$

This contradicts the definition of Λ. Thus we obtain $\Lambda = -\infty$ and $v_\infty \equiv 1$.

We will prove the lemma by induction. Let N be any integer with $N \ge 3$. We prove this lemma for $n = N$ assuming that it holds true for all n with $n < N$. We have $\boldsymbol{x}' = (x_1, \ldots, x_{N-1})$. Recall

$$w_0(\boldsymbol{x}') = \begin{cases} \theta_0 & \text{if} \quad \boldsymbol{x}' \in \Omega_{N-1}, \\ -1 & \text{if} \quad \boldsymbol{x}' \in \mathbb{R}^{N-1} \backslash \Omega_{N-1}. \end{cases}$$

Now $w(\boldsymbol{x}', t; w_0)$ satisfies

$$\frac{\partial w}{\partial t} - \sum_{j=1}^{N-1} \frac{\partial^2 w}{\partial x_j^2} + W'(w) = 0, \qquad \boldsymbol{x}' \in \mathbb{R}^{N-1},\ t > 0,$$
$$w(\boldsymbol{x}', 0; w_0) = w_0(\boldsymbol{x}'), \qquad \boldsymbol{x}' \in \mathbb{R}^{N-1},$$

where

$$\Omega_{N-1} = \left\{ (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1} \;\middle|\; \min_{1 \le j \le N-1} |x_j| \ge \mu_0 \right\}.$$

Now we have $\boldsymbol{x}'' = (x_1, \ldots, x_{N-2}) \in \mathbb{R}^{N-2}$ and

$$\Omega_{N-2} = \left\{ (x_1, \ldots, x_{N-2}) \in \mathbb{R}^{N-2} \;\middle|\; \min_{1 \le j \le N-2} |x_j| \ge \mu_0 \right\}.$$

If $|x_{N-1}| \ge \mu_0$, we have

$$w(\boldsymbol{x}'', x_{N-1}, 0) = \begin{cases} \theta_0 & \text{if} \quad \boldsymbol{x}'' \in \Omega_{N-2}, \\ -1 & \text{if} \quad \boldsymbol{x}'' \in \mathbb{R}^{N-2} \backslash \Omega_{N-2}. \end{cases}$$

Using

$$\chi_{\Omega_{N-2}}(\boldsymbol{x}'') = \begin{cases} 1 & \text{if} \quad \boldsymbol{x}'' \in \Omega_{N-2}, \\ 0 & \text{if} \quad \boldsymbol{x}'' \in \mathbb{R}^{N-2} \backslash \Omega_{N-2}, \end{cases}$$

we have

$$w(\boldsymbol{x}'', x_{N-1}, 0) = -1 + (1+\theta_0)\chi_{\Omega_{N-2}}(\boldsymbol{x}'').$$

By the assumption of the induction, Lemma 7 holds true for $n = 2, \ldots, N-1$. Let $\delta \in (0, \delta_*]$ be given. There exists $T_2 \in (0, \infty)$ such that we have

$$1 - \frac{\delta}{4} < w(\boldsymbol{x}, t; -1 + (1+\theta_0)\chi_{\Omega_{N-2}}), \qquad \boldsymbol{x} \in \mathbb{R}^{N-1},\ t \geq T_2.$$

By applying [34, Theorem 5.8] again, we have

$$\inf_{|x_{N-1}| \geq r_2} \left| w(\boldsymbol{x}'', x_{N-1}, T_2; w_0) - w(\boldsymbol{x}, t; -1 + (1+\theta_0)\chi_{\Omega_{N-2}}) \right| < \frac{\delta}{2}.$$

by taking $r_2 \in (0, \infty)$ large enough. Thus we obtain

$$1 - \frac{\delta}{2} < \inf_{|x_{N-1}| \geq r_2} w(\boldsymbol{x}'', x_{N-1}, T_2; w_0). \tag{3.21}$$

Then we can start the argument to prove Lemma 7 for $n = 2$ replacing (3.19) by (3.21), and we obtain

$$\lim_{t \to \infty} \inf_{(\boldsymbol{x}'', x_{N-1}) \in \mathbb{R}^{N-1}} w(\boldsymbol{x}'', x_{N-1}, t; w_0) = 1.$$

Thus Lemma 7 holds true for $n = N$. Now it holds true for all $n \geq 2$. This completes the proof. □

Proof of Proposition 6 Assuming the contrary, we have

$$\lim_{x_n \to \infty} V_c(\mu_0, \ldots, \mu_0, x_n) \geq \theta_0$$

for some $\mu_0 \in (0, \infty)$. Since $V_c(\boldsymbol{x}', x_n)$ is monotone non-increasing in x_n, we have

$$w_0(\boldsymbol{x}', x_n) \leq V_c(\boldsymbol{x}', x_n), \qquad (\boldsymbol{x}', x_n) \in \mathbb{R}^n.$$

Here w_0 is given by (3.18). Taking the both sides as initial functions in (1.6), we have

$$w(\boldsymbol{x}', x_n, t; w_0) \leq V_c(\boldsymbol{x}', x_n), \qquad (\boldsymbol{x}', x_n) \in \mathbb{R}^n,\ t > 0.$$

Letting $t \to \infty$ and applying Lemma 7, we obtain $V_c \equiv 1$, which contradicts (3.13), (3.15), and (3.16). Now we complete the proof of Proposition 6. □

Now we modify Theorem 5 in a form that is more useful for our discussion. Let

$$0 < R_1 \leq R_2 \leq \cdots \leq R_{n-1} < \infty \tag{3.22}$$

be arbitrarily given. In [31], we study an imbalanced reaction–diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u - W'(u) + k\sqrt{2W(u)}, \quad \boldsymbol{x} \in \mathbb{R}^n,\ t > 0.$$

Then Φ is the planar front with its speed k. The profile equation for a profile v with its speed $c \in (0, \infty)$ is given by

$$\Delta v + c\frac{\partial v}{\partial x_n} - W'(v) + k\sqrt{2W(v)} = 0, \qquad (\boldsymbol{x}', x_n) \in \mathbb{R}^n. \tag{3.23}$$

For sufficiently small $k > 0$, say, $k \in (0, k_0)$ for $k_0 \in (0, c)$, we define a pyramidal traveling front solution v_k to (3.23) associated with a pyramid

$$p(\boldsymbol{x}') = \frac{\sqrt{c^2 - k^2}}{k} \max\{|x_1|, |x_2| - a_2, \ldots, |x_{n-1}| - a_{n-1}\}$$

for $a_j \in [0, \infty)$ $(2 \le j \le n-1)$. For

$$v_0(\boldsymbol{x}', x_n) = \Phi\left(\frac{k}{c}(x_n - p(\boldsymbol{x}'))\right),$$

one can define

$$v_k(\boldsymbol{x}', x_n) = \lim_{t\to\infty} w(\boldsymbol{x}, t; v_0) \quad \text{on every compact set in } \mathbb{R}^n.$$

For pyramidal traveling fronts, one can see [34] for instance. Now we have $z_k \in \mathbb{R}$ with

$$v_k(\boldsymbol{0}', z_k) = \theta_0.$$

Hereafter we write $v_k(\boldsymbol{x}', x_n + z_k)$ simply as $v_k(\boldsymbol{x}', x_n)$. Now we have

$$v_k(\boldsymbol{0}', 0) = \theta_0$$

and

$$\begin{aligned}
&-1 < v_k(\boldsymbol{x}', x_n) < 1, \qquad (\boldsymbol{x}', x_n) \in \mathbb{R}^n,\\
&\mathrm{D}_n v_k(\boldsymbol{x}', x_n) < 0, \qquad (\boldsymbol{x}', x_n) \in \mathbb{R}^n,\\
&v_k(x_1, \ldots, \underbrace{-x_j}_{j}, \ldots, x_n) = v_k(x_1, \ldots, \underbrace{x_j}_{j}, \ldots, x_n), \quad (\boldsymbol{x}', x_n) \in \mathbb{R}^n,\ 1 \le j \le n-1,\\
&\mathrm{D}_j v_k(\boldsymbol{x}', x_n) > 0 \qquad \text{if} \quad x_j > 0,\ 1 \le j \le n-1.
\end{aligned}$$

For every $\eta \in (0, \infty)$, taking

$$a_j = A_j(\eta), \quad 2 \le j \le n-1$$

given by [31, Lemma 1], we obtain

$$v_k(0, \ldots, 0, \underset{\smile}{\overset{j}{S_j(\eta)}}, 0, \ldots, 0, \eta) = \theta_0, \qquad 1 \leq j \leq n-1$$

with

$$0 < S_1(\eta) \leq S_2(\eta) \leq \cdots \leq S_{n-1}(\eta) < \infty,$$
$$\frac{S_j(\eta)}{S_1(\eta)} = \frac{R_j}{R_1}, \qquad 2 \leq j \leq n-1.$$

Using

$$\lim_{\eta \to 0} S_1(\eta) = 0, \qquad \lim_{\eta \to \infty} S_1(\eta) = \infty,$$

we obtain a positive number ζ_k with

$$S_1 = R_1.$$

Thus we have

$$v_k(0, \ldots, 0, \overset{j}{R_j}, 0, \ldots, 0, \zeta_k) = \theta_0, \qquad 1 \leq j \leq n-1$$

with a positive number ζ_k. Then we define

$$V(\boldsymbol{x}', x_n) = \lim_{k \to 0} v_k(\boldsymbol{x}', x_n) \tag{3.24}$$

for all $(\boldsymbol{x}', x_n)$ in every compact set in $\mathbb{R}^n$. Now we have $V(\boldsymbol{0}', 0) = \theta_0$ and

$$-1 < V(\boldsymbol{x}', x_n) < 1, \qquad (\boldsymbol{x}', x_n) \in \mathbb{R}^n,$$
$$\mathrm{D}_n V(\boldsymbol{x}', x_n) \leq 0, \qquad (\boldsymbol{x}', x_n) \in \mathbb{R}^n,$$

$$V(x_1, \ldots, \overset{j}{-x_j}, \ldots, x_n) = V(x_1, \ldots, \overset{j}{x_j}, \ldots, x_n), \quad (\boldsymbol{x}', x_n) \in \mathbb{R}^n, \ 1 \leq j \leq n-1,$$
$$\mathrm{D}_j V(\boldsymbol{x}', x_n) \geq 0 \quad \text{if} \quad x_j > 0, \ 1 \leq j \leq n-1. \tag{3.25}$$

See [31] for detailed arguments.

Now we state axially asymmetric traveling fronts as follows.

Theorem 8 (Axially asymmetric traveling fronts) *Let $c \in (0, \infty)$ be arbitrarily given. Let $\{R_j\}_{1 \leq j \leq n-1}$ satisfy (3.22). There exists $V(\boldsymbol{x}) = V(\boldsymbol{x}; c)$ that satisfies (1.8) with $V(\boldsymbol{0}) = \theta_0$ and*

$$-1 < V(\boldsymbol{x}) < 1, \qquad \boldsymbol{x} \in \mathbb{R}^n,$$
$$\mathrm{D}_n V(\boldsymbol{x}', x_n) < 0, \qquad (\boldsymbol{x}', x_n) \in \mathbb{R}^n,$$

$$V(x_1, \ldots, \underbrace{-x_j}^{j}, \ldots, x_n) = V(x_1, \ldots, \underbrace{x_j}^{j}, \ldots, x_n), \quad (\boldsymbol{x}', x_n) \in \mathbb{R}^n, \ 1 \le j \le n-1,$$
$$\mathrm{D}_j V(\boldsymbol{x}', x_n) > 0 \quad \textit{if} \ \ x_j > 0, \ 1 \le j \le n-1,$$
$$V(0, \ldots, 0, \underbrace{R_j}^{j}, 0, \ldots, 0, \zeta) = \theta_0$$

with a positive number ζ. *For every* $\theta \in (-1, 1)$ *one has*

$$\inf_{(\boldsymbol{x}', x_n) \in \mathbb{R}^n} \left\{ |\nabla V(\boldsymbol{x}', x_n)| \mid V(\boldsymbol{x}', x_n) = \theta \right\} > 0. \tag{3.26}$$

See Fig. 3 for the level set $\{(\boldsymbol{x}', x_n) \mid V(\boldsymbol{x}', x_n) = 0\}$ of V in Theorem 8.

Remark 2 The uniqueness of V in Theorem 8 is yet to be studied. It is an open problem to show V in Theorem 8 equals V_{sym} in Theorem 4 if $R_1 = R_2 = \cdots = R_{n-1}$.

Proof Using (3.25) and Proposition 6, we obtain

$$\mathrm{D}_n V(\boldsymbol{x}', x_n) < 0, \qquad (\boldsymbol{x}', x_n) \in \mathbb{R}^n.$$

It suffices to show

$$0 < \liminf_{k \to 0} \zeta_k \le \limsup_{k \to 0} \zeta_k < \infty.$$

Assume $\limsup_{k \to 0} \zeta_k = \infty$. Then we have $\zeta = \infty$ and

$$V(R_1, 0, \ldots, 0, x_n) > \theta_0, \qquad x_n \in \mathbb{R}. \tag{3.27}$$

Using

$$V(R_1, 0, \ldots, 0, \eta) < V(R_1, R_1, \ldots, R_1, \eta), \qquad \eta \in \mathbb{R}$$

and applying Proposition 6, we have

$$\lim_{\eta \to \infty} V(R_1, 0, \ldots, 0, \eta) \le \lim_{\eta \to \infty} V(R_1, R_1, \ldots, R_1, \eta) < \theta_0.$$

This contradicts (3.27). Next we assume $\liminf_{k \to 0} \zeta_k = 0$. Then we have

$$V(0, \ldots, 0, \underbrace{R_j}^{j}, 0, \ldots, 0, 0) = \theta_0, \qquad 1 \le j \le n-1.$$

Then we find

$$\mathrm{D}_j V(0, \ldots, 0, \underbrace{R_j/2}^{j}, 0, \ldots, 0, 0) = 0, \qquad 1 \le j \le n-1.$$

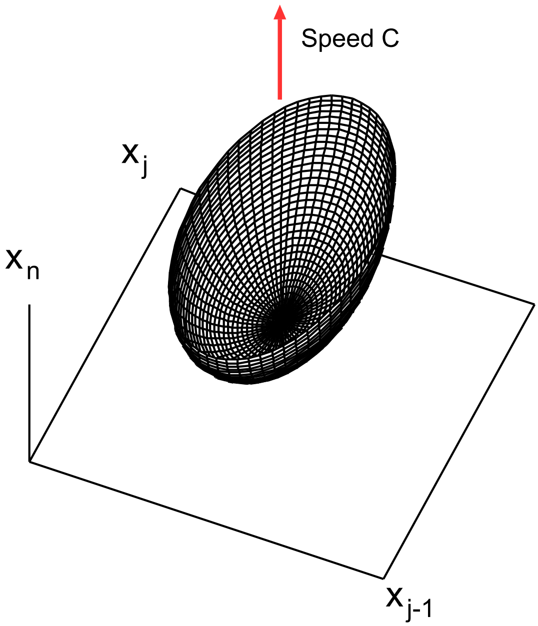

Fig. 3 A level set of axially asymmetric traveling front $V(\boldsymbol{x}; c)$ in Theorem 8. Here $2 \leq j \leq n-1$

Thus the maximum principle gives

$$\mathrm{D}_j V(\boldsymbol{x}', x_n) = 0 \qquad \text{if} \quad x_j > 0$$

for $1 \leq j \leq n-1$. Then V is independent of x_j for $1 \leq j \leq n-1$ and is a function of x_n, that is, $V(x_n - ct)$ is a one-dimensional traveling front solution to (1.1). Since a one-dimensional traveling front solution to (1.1) and its speed is uniquely determined, we obtain $c = 0$. This contradicts $c \in (0, \infty)$.

Then, taking a subsequence if necessary, we can define $\zeta \in (0, \infty)$ with

$$\zeta = \lim_{k \to 0} \zeta_k.$$

Then V given by (3.24) satisfies Theorem 8. See [31] for detailed arguments. □

Let $\theta \in (-1, 1)$ be arbitrarily given. We define $R = R_\theta$ by

$$R = 1 + \frac{(n-1)K_*(1+\theta)}{W(\theta)} \tag{3.28}$$

and have

$$(n-1)K_*(1+\theta) < W(\theta)R.$$

For any given $(\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$, we define $\mathcal{D} = \mathcal{D}_\theta$ by

$$\mathcal{D} = (\xi_1 - R, \xi_1 + R) \times (\xi_2 - R, \xi_2 + R) \times \ldots \\ \times (\xi_{n-1} - R, \xi_{n-1} + R) \subset \mathbb{R}^{n-1}. \tag{3.29}$$

We have

$$\mathcal{D} \subset B(\boldsymbol{\xi}'; \sqrt{n-1}R),$$

where

$$B(\boldsymbol{\xi}'; \sqrt{n-1}R) = \{\boldsymbol{x}' \in \mathbb{R}^{n-1} \mid |\boldsymbol{x}' - \boldsymbol{\xi}'| < \sqrt{n-1}R\}.$$

Let $c \in (0, \infty)$ be arbitrarily given and let $V \in C^2(\mathbb{R}^n)$ satisfy (1.8), that is,

$$\sum_{j=1}^{n} \frac{\partial^2 V}{\partial x_j^2} + c\frac{\partial V}{\partial z} - W'(V) = 0, \qquad (\boldsymbol{x}', x_n) \in \mathbb{R}^n, \tag{3.30}$$

with (3.2), (3.13), (3.15), (3.14) and (3.16). Now V in Theorem 8 satisfies these assumptions. For any $\theta \in (-1, 1)$, we define $q_\theta(\boldsymbol{x}')$ by

$$V(\boldsymbol{x}', q_\theta(\boldsymbol{x}')) = \theta, \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1}. \tag{3.31}$$

Then we have $q_\theta \in C^1(\mathbb{R}^{n-1})$. If $z_0 \in \mathbb{R}$ satisfies $q_\theta(\boldsymbol{0}') < z_0$, we can uniquely determine $x_n^\theta \in (0, \infty)$ with

$$q_\theta(0, \ldots, 0, x_{n-1}^\theta) = z_0$$

and have

$$x_{n-1}^\theta = \max\left\{|\boldsymbol{x}'| \;\middle|\; \boldsymbol{x}' \in \mathbb{R}^{n-1},\ q_\theta(\boldsymbol{x}') = z_0\right\}.$$

The following proposition plays an important role when we take the limits of traveling fronts as $c \to \infty$.

Proposition 9 *Let $c \in (0, \infty)$ be arbitrarily given and let $\theta \in (-1, 1)$ be arbitrarily given. Assume $V \in C^2(\mathbb{R}^n)$ satisfies (3.30), (3.2), (3.13), (3.14), (3.15) and (3.16). Then one has*

$$\int_{\mathcal{D}} \left|\nabla V(\boldsymbol{x}', q_\theta(\boldsymbol{x}'))\right|^2 \mathrm{d}\boldsymbol{x}' \geq 4(2R)^{n-2}\left[W(\theta)R - (n-1)K_*(1+\theta)\right] > 0. \tag{3.32}$$

where R is defined by (3.28). The right-hand side is independent of $c \in (0, \infty)$.

We write the right-hand side of (3.32) as $A(\theta, R)^2\,|\mathcal{D}|$ with

$$A(\theta, R) = \sqrt{\frac{2}{R}\,[W(\theta)R - (n-1)K_*(1+\theta)]}. \tag{3.33}$$

Then (3.32) is written as

$$\int_{\mathcal{D}} \left|\nabla V(\boldsymbol{x}', q_\theta(\boldsymbol{x}'))\right|^2 \,\mathrm{d}\boldsymbol{x}' \geq A(\theta, R)^2|\mathcal{D}| > 0.$$

Let s_1 be arbitrarily given with

$$\begin{aligned} -1 &< s_1 < \theta < 1, \\ 0 &< W(s_1) < W(\theta) \end{aligned}$$

The volume of $\mathcal{D}$ is given by $(2R)^{n-1}$, and the surface area of the boundary of $\mathcal{D}$ is given by $2(n-1)(2R)^{n-2}$. Using (3.28), we have

$$K_*(1+\theta)\,|\partial\mathcal{D}| < W(\theta)\,|\mathcal{D}|$$

for every $(\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$.

We define

$$\Omega = \{(\boldsymbol{x}', x_n) \,|\, \boldsymbol{x}' \in \mathcal{D}, s_1 < V(\boldsymbol{x}', x_n) < \theta\}.$$

Let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$ be the outward normal vector on $\partial\Omega$. We have

$$\partial\Omega = \Gamma_\theta \cup \Gamma_1 \cup \Gamma_{\mathrm{f}},$$

where

$$\begin{aligned} \Gamma_\theta &= \{(\boldsymbol{x}', x_n) \,|\, \boldsymbol{x}' \in \mathcal{D}, V(\boldsymbol{x}', x_n) = \theta\}, \\ \Gamma_1 &= \{(\boldsymbol{x}', x_n) \,|\, \boldsymbol{x}' \in \mathcal{D}, V(\boldsymbol{x}', x_n) = s_1\}, \\ \Gamma_{\mathrm{f}} &= \{(\boldsymbol{x}', x_n) \,|\, \boldsymbol{x}' \in \partial\mathcal{D}, s_1 \leq V(\boldsymbol{x}', x_n) \leq \theta\}. \end{aligned}$$

Now we have

$$\mathrm{div}\left(\frac{\partial V}{\partial x_n}\nabla V\right) = \frac{\partial V}{\partial x_n}\Delta V + \frac{1}{2}\frac{\partial}{\partial x_n}\left(|\nabla V|^2\right).$$

Multiplying (3.30) by $-\mathrm{D}_n V$, we have

$$-\,\mathrm{div}\left(\frac{\partial V}{\partial x_n}\nabla V\right) + \frac{1}{2}\frac{\partial}{\partial x_n}\left(|\nabla V|^2\right) - c\left(\frac{\partial V}{\partial x_n}\right)^2 + W'(V)\frac{\partial V}{\partial x_n} = 0.$$

Integrating the both hand sides over Ω and using the Gauss divergence theorem, we get

$$\int_{\partial\Omega}\left(-\frac{\partial V}{\partial x_n}(\nabla V,\nu)+\frac{1}{2}|\nabla V|^2\nu_n\right)\mathrm{d}S-c\int_{\Omega}\left(\frac{\partial V}{\partial x_n}\right)^2\mathrm{d}\boldsymbol{x}+\int_{\Omega}\frac{\partial}{\partial x_n}\left(W(V)\right)\mathrm{d}\boldsymbol{x}=0.$$

Here $\mathrm{d}S$ is the surface element of $\partial\Omega$. Using

$$\boldsymbol{\nu}=\frac{\nabla V}{|\nabla V|}\qquad\text{on }\Gamma_\theta,$$

we get

$$-\frac{\partial V}{\partial x_n}(\nabla V,\boldsymbol{\nu})+\frac{1}{2}|\nabla V|^2\nu_n=-\frac{1}{2}|\nabla V|\frac{\partial V}{\partial x_n}\qquad\text{on }\Gamma_\theta.$$

Similarly, using

$$\boldsymbol{\nu}=-\frac{\nabla V}{|\nabla V|}\qquad\text{on }\Gamma_1,$$

we get

$$-\frac{\partial V}{\partial x_n}(\nabla V,\boldsymbol{\nu})+\frac{1}{2}|\nabla V|^2\nu_n=\frac{1}{2}|\nabla V|\frac{\partial V}{\partial x_n}\qquad\text{on }\Gamma_1.$$

Using $\nu_n=0$ on Γ_{f}, we have

$$-\frac{\partial V}{\partial x_n}(\nabla V,\boldsymbol{\nu})+\frac{1}{2}|\nabla V|^2\nu_n=-\frac{\partial V}{\partial x_n}(\nabla V,\boldsymbol{\nu})\qquad\text{on }\Gamma_{\mathrm{f}}.$$

We have

$$\int_{\Omega}\mathrm{D}_n\left(W(V)\right)\mathrm{d}\boldsymbol{x}=\int_{\mathcal{D}}\left(W(s_1)-W(\theta)\right)\mathrm{d}\boldsymbol{x}=\left(W(s_1)-W(\theta)\right)|\mathcal{D}|\,.$$

Now we calculate

$$\left|\int_{\Gamma_{\mathrm{f}}}(\nabla V,\boldsymbol{\nu})\frac{\partial V}{\partial x_n}\,\mathrm{d}S\right|\le\left(\max_{\mathbb{R}^n}|\nabla V|\right)\int_{\Gamma_{\mathrm{f}}}\left(-\frac{\partial V}{\partial x_n}\right)\mathrm{d}S.$$

Using

$$\int_{\Gamma_{\mathrm{f}}}\left(-\frac{\partial V}{\partial x_n}\right)\mathrm{d}S=\int_{\partial\mathcal{D}}(\theta-s_1)\,\mathrm{d}S\le(\theta-s_1)|\partial\mathcal{D}|.$$

Then we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_{\Gamma_\theta} |\nabla V| \left(-\frac{\partial V}{\partial x_n}\right) \mathrm{d}S \\
&\quad \geq \frac{1}{2}\int_{\Gamma_1} |\nabla V| \left(-\frac{\partial V}{\partial x_n}\right) \mathrm{d}S + c\int_{\Omega} \left(\frac{\partial V}{\partial x_n}\right)^2 \mathrm{d}\boldsymbol{x} \\
&\qquad + (W(\theta) - W(s_1))\,|\mathcal{D}| - K_*(\theta - s_1)|\partial\mathcal{D}| \\
&\quad \geq (W(\theta) - W(s_1))\,|\mathcal{D}| - K_*(\theta - s_1)|\partial\mathcal{D}|.
\end{aligned}
$$

Sending $s_1 \to -1$, we obtain

$$
\frac{1}{2}\int_{\Gamma_\theta} |\nabla V| \left(-\frac{\partial V}{\partial x_n}\right) \mathrm{d}S \geq W(\theta)|\mathcal{D}| - K_*(\theta + 1)|\partial\mathcal{D}| > 0. \tag{3.34}
$$

Now we use the following lemma.

Lemma 10 *One has*

$$
\int_{\mathcal{D}} |\nabla V(\boldsymbol{x}', q_\theta(\boldsymbol{x}'))|^2 \,\mathrm{d}\boldsymbol{x}' = \int_{\Gamma_\theta} |\nabla V(\boldsymbol{x})| \left(-\frac{\partial V}{\partial x_n}\right) \mathrm{d}S.
$$

Proof Differentiating

$$
V(\boldsymbol{x}', q_\theta(\boldsymbol{x}')) = \theta
$$

by x_j, we have

$$
\mathrm{D}_j V(\boldsymbol{x}', q_\theta(\boldsymbol{x}')) = -\mathrm{D}_n V(\boldsymbol{x}', q_\theta(\boldsymbol{x}'))\mathrm{D}_j q_\theta(\boldsymbol{x}'), \qquad 1 \leq j \leq n-1.
$$

Then we have

$$
\begin{aligned}
\nabla V(\boldsymbol{x}', q_\theta(\boldsymbol{x}')) &= \begin{pmatrix} -\mathrm{D}_n V(\boldsymbol{x}', q_\theta(\boldsymbol{x}'))\mathrm{D}_1 q_\theta(\boldsymbol{x}') \\ \vdots \\ -\mathrm{D}_n V(\boldsymbol{x}', q_\theta(\boldsymbol{x}'))\mathrm{D}_{n-1} q_\theta(\boldsymbol{x}') \\ \mathrm{D}_n V(\boldsymbol{x}', q_\theta(\boldsymbol{x}')) \end{pmatrix} \\
&= \mathrm{D}_n V(\boldsymbol{x}', q_\theta(\boldsymbol{x}')) \begin{pmatrix} -\mathrm{D}_1 q_\theta(\boldsymbol{x}') \\ \vdots \\ -\mathrm{D}_{n-1} q_\theta(\boldsymbol{x}') \\ 1 \end{pmatrix}.
\end{aligned}
$$

Then we have

$$
\left|\nabla V(\boldsymbol{x}', q_\theta(\boldsymbol{x}'))\right| = -\mathrm{D}_n V(\boldsymbol{x}', q_\theta(\boldsymbol{x}'))\sqrt{1 + |\nabla q_\theta(\boldsymbol{x}')|^2},
$$

where

$$\nabla q_\theta(\boldsymbol{x}') = \left(\frac{\partial q_\theta}{\partial x_1}(\boldsymbol{x}'), \dots, \frac{\partial q_\theta}{\partial x_{n-1}}(\boldsymbol{x}') \right).$$

Since $\boldsymbol{\nu}$ is the outward normal vector at $\partial\Omega$, we have

$$\boldsymbol{\nu} = \frac{\nabla V(\boldsymbol{x}', q_\theta(\boldsymbol{x}'))}{|\nabla V(\boldsymbol{x}', q_\theta(\boldsymbol{x}'))|} = -\frac{1}{\sqrt{1+|\nabla q_\theta(\boldsymbol{x}')|^2}} \begin{pmatrix} -\nabla q_\theta(\boldsymbol{x}') \\ 1 \end{pmatrix} \qquad \text{on } \Gamma_\theta.$$

Thus we obtain

$$\begin{aligned}
&\int_{\Gamma_\theta} |\nabla V(\boldsymbol{x}', q_\theta(\boldsymbol{x}'))| \left(-\frac{\partial V}{\partial x_n}(\boldsymbol{x}', q_\theta(\boldsymbol{x}')) \right) \mathrm{d}S \\
&= \int_{\mathcal{D}} |\nabla V(\boldsymbol{x}', q_\theta(\boldsymbol{x}'))| \left(-\frac{\partial V}{\partial x_n}(\boldsymbol{x}', q_\theta(\boldsymbol{x}')) \right) \sqrt{1+|\nabla q_\theta(\boldsymbol{x}')|^2}\, \mathrm{d}\boldsymbol{x}' \\
&= \int_{\mathcal{D}} |\nabla V(\boldsymbol{x}', q_\theta(\boldsymbol{x}'))|^2\, \mathrm{d}\boldsymbol{x}'.
\end{aligned}$$

Now we complete the proof. □

Now we give a proof for Proposition 9.

Proof of Proposition 9 Combining (3.34) and Lemma 10, we have

$$\begin{aligned}
\frac{1}{2}\int_{\mathcal{D}} |\nabla V(\boldsymbol{x}', q_\theta(\boldsymbol{x}'))|^2\, \mathrm{d}\boldsymbol{x}' &\ge W(\theta)|\mathcal{D}| - K_*(1+\theta)|\partial\mathcal{D}| \\
&\ge (2R)^{n-1} W(\theta) - 2(n-1)(2R)^{n-2} K_*(1+\theta) \\
&\ge 2(2R)^{n-2}\left[W(\theta)R - (n-1)K_*(1+\theta)\right] > 0.
\end{aligned}$$

This completes the proof. □

Now we show the following assertion.

Lemma 11 *Under the same assumption of Proposition 9, one has*

$$\lim_{\nu\to\infty} \inf\{V(\boldsymbol{x}', x_n) \mid (\boldsymbol{x}', x_n) \in \mathbb{R}^n,\, V(\boldsymbol{x}', x_n) \ge \theta_0,\, \mathrm{dist}((\boldsymbol{x}', x_n), \Gamma_0) \ge \nu\} = 1.$$

Here

$$\Gamma_0 = \{(\boldsymbol{x}', x_n) \mid V(\boldsymbol{x}', x_n) = \theta_0\}.$$

Proof Let $\delta \in (0, \delta_*]$ be given. As was mentioned in the proof of Lemma 7, there exists $T_1 \in (0, \infty)$ such that we have

$$1 - \frac{\delta}{4} < w(\boldsymbol{x}, T_1; \theta_0), \qquad \boldsymbol{x} \in \mathbb{R}.$$

Let $(\boldsymbol{x}_0', z_0)$ belongs to

$$\{(\boldsymbol{x}', x_n) \in \mathbb{R}^n \mid V(\boldsymbol{x}', x_n) \geq \theta_0, \mathrm{dist}((\boldsymbol{x}', x_n), \Gamma_0) \geq \nu\}.$$

Then we have

$$\theta_0 \leq V(\boldsymbol{x}', x_n), \qquad \text{if} \quad (\boldsymbol{x}', x_n) \in \overline{B((\boldsymbol{x}_0', z_0); \nu)}$$

By applying [34, Theorem 5.7],

$$w(\boldsymbol{x}, T_1; \theta_0) \leq w(\boldsymbol{x}, T_1; V) \qquad \text{if} \quad (\boldsymbol{x}', x_n) \in \overline{B((\boldsymbol{x}_0', z_0); \nu/2)}$$

if $\nu \in (0, \infty)$ is large enough. Then we have

$$1 - \frac{\delta}{4} < w((\boldsymbol{x}_0', z_n), T_1; V) = V(\boldsymbol{x}_0', z_n)$$

if $\nu \in (0, \infty)$ is large enough. Since we can take $\delta \in (0, \delta_*]$ arbitrarily small, the lemma follows from this inequality. □

The following proposition asserts that Proposition 6 holds true for every $\theta \in (-1, 1)$. That is, for a traveling front V_c, every level set $\{(\boldsymbol{x}', x_n) \mid V_c(\boldsymbol{x}', x_n) = \theta\}$ is a graph on the whole space $\mathbb{R}^{n-1}$.

Proposition 12 *Let $c \in (0, \infty)$ be arbitrarily fixed. Let $V_c(\boldsymbol{x})$ satisfy*

$$V_c(\mathbf{0}', 0) = \theta,$$

(1.8), (3.2), (3.13), (3.14) *and* (3.15). *Then, for any given $\mu_0 \in (0, \infty)$ and for any given $\theta \in (-1, 1)$, one has*

$$\lim_{x_n \to \infty} V_c(\mu_0, \ldots, \mu_0, x_n) < \theta.$$

Proof Now we define

$$\Theta = \left\{ \theta \in (-1, 1) \;\middle|\; \lim_{x_n \to \infty} \sup_{c \,\in (0,\infty)} V_c(\mu_0, \ldots, \mu_0, x_n) < \theta \quad \text{for every } \mu_0 \in (0, \infty) \right\}.$$

Proposition 6 implies $(s_*, 1) \subset \Theta$. We define $\theta_\infty = \inf \Theta$, and will show $\theta_\infty = -1$. To do this, we will get a contradiction assuming $\theta_\infty \in (-1, s_*]$. Then we choose $\{\theta_j\}_{j \geq 1} \subset \Theta$ with

$$\theta_\infty < \theta_j < \theta_i \qquad \text{if} \quad i < j,$$
$$\lim_{j \to \infty} = \theta_\infty.$$

Now we define

$$R_\infty = \max_{\theta \in [\theta_\infty, s_*]} R_\theta \in (0, \infty).$$

Now V_c satisfies Proposition 9. We write the right-hand side of (3.32) as $A(\theta, R_\theta)^2 |\mathcal{D}_\theta|$, that is,

$$A(\theta, R_\theta) = \sqrt{\frac{2}{R_\theta} \left[W(\theta) R_\theta - (n-1) K_* (1+\theta) \right]} > 0. \tag{3.35}$$

We define

$$A_\infty = \min_{\theta \in [\theta_\infty, s_*]} A(\theta, R_\infty) \in (0, \infty).$$

We set

$$\boldsymbol{\xi}' = \left(\mu_0 + 1, \ldots, \mu_0 + 1\right) \in \mathbb{R}^{n-1}. \tag{3.36}$$

Using Proposition 9, we have $\boldsymbol{\eta}'_j \in \mathcal{D}_{\theta_j}$ with

$$\left| \nabla V_c(\boldsymbol{\eta}'_j) \right| \geq A_\infty > 0.$$

Here $\mathcal{D}_{\theta_j}$ is given by (3.29) with $\boldsymbol{\xi}'$ in (3.36) and $R = R_{\theta_j}$. Using Lemma 3, we can have $\epsilon_0 \in (0, 1/2)$ with

$$\left| \nabla V_c(\boldsymbol{x}') \right| \geq \frac{A_\infty}{2} > 0 \quad \text{ if } \quad \left| \boldsymbol{x}' - \boldsymbol{\eta}'_j \right| \leq \varepsilon_0,$$

where ε_0 is independent of $j \geq 1$. Then we define $\boldsymbol{x}(t)$ by

$$\frac{d\boldsymbol{x}}{dt}(t) = -\frac{\nabla V_c(\boldsymbol{x}(t))}{|\nabla V_c(\boldsymbol{x}(t))|}, \qquad 0 < t < \varepsilon_0,$$
$$\boldsymbol{x}(0) = \boldsymbol{\eta}'_j.$$

Then, using

$$\frac{d}{dt} \left(V_c(\boldsymbol{x}(t)) \right) = \left(\nabla V_c(\boldsymbol{x}(t)), \frac{d\boldsymbol{x}}{dt}(t) \right) = - \left| \nabla V_c(\boldsymbol{x}(t)) \right| \leq -\frac{A_\infty}{2}, \qquad t \in (0, \varepsilon_0),$$

we have

$$V_c(\boldsymbol{x}(\varepsilon_0)) < \theta_j - \frac{\varepsilon_0 A_\infty}{2}.$$

Now we have

$$\boldsymbol{x}(\varepsilon_0) \in \Omega_{n-1},$$

where Ω_{n-1} is given by (3.17). Using an assumption (3.15), we have

$$V_c(\mu_0, \ldots, \mu_0, x_n) < \theta_j - \frac{\varepsilon_0 A_\infty}{2}$$

with $x_n \in \mathbb{R}$ for $j \geq 1$. This contradicts the definition of θ_∞. Thus we obtain $\theta_\infty = -1$. This completes the proof. □

4 Proof of Theorems 1 and 2, and the limits of traveling fronts as the speeds go to infinity

Let $\boldsymbol{\alpha}'$ in (3.12) be arbitrarily fixed with (3.11). Let $R_1 \in (0, \infty)$ be arbitrarily fixed. Let $V_c(\boldsymbol{x}) = V(\boldsymbol{x}; \boldsymbol{\alpha}', c)$ be given by Theorem 8 for every $c \in (0, \infty)$. Now V_c satisfies

$$\Delta V_c + c\mathrm{D}_n V_c - W'(V_c) = 0, \qquad \boldsymbol{x} \in \mathbb{R}^n$$

with

$$V_c(\mathbf{0}', 0) = \theta_0.$$

Now we take $\zeta_c \in (0, \infty)$ that depends on $c \in (0, \infty)$ such that we have

$$V_c(R_1, 0, \ldots, 0, \zeta_c) = \theta_0, \qquad 1 \leq j \leq n-1.$$

When $R_0 \in (0, \infty)$ is arbitrarily given and we consider V_{sym} given by Theorem 4, we define $\zeta_c \in (0, \infty)$ by

$$V_{\mathrm{sym}}(R_0, \zeta_c) = \theta_0.$$

We take the limit of V_c as $c \to \infty$. We have

$$\sup_{\boldsymbol{x} \in \mathbb{R}^n} \left| W'(V_c(\boldsymbol{x})) \right| \leq \|W'\|_{C[-1,1]} < \infty,$$

where

$$\|W'\|_{C[-1,1]} = \max_{|s| \leq 1} |W'(s)|.$$

Using Lemma 3, we have

$$\sup_{\boldsymbol{x} \in \mathbb{R}^n} |\Delta V_c(\boldsymbol{x})| \leq n K_* < \infty.$$

Using

$$0 < -\mathrm{D}_n V_c(\boldsymbol{x}) \le \frac{\Delta V_c(\boldsymbol{x}) - W'(V_c(\boldsymbol{x}))}{c} \le \frac{nK_* + \|W'\|_{C[-1,1]}}{c}, \qquad \boldsymbol{x} \in \mathbb{R}^n,$$

we obtain

$$\lim_{c\to\infty} \sup_{\boldsymbol{x}\in\mathbb{R}^n} |\mathrm{D}_n V_c(\boldsymbol{x})| = 0. \tag{4.1}$$

Using Lemma 3, we have

$$\sup_{\boldsymbol{x}\in\mathbb{R}^n} \left|\mathrm{D}_n^3 V_c(\boldsymbol{x})\right| \le K_*, \tag{4.2}$$

where $K_* \in (0, \infty)$ is independent of (c, V_c). Combining (4.1) and (4.2), we obtain

$$\lim_{c\to\infty} \sup_{\boldsymbol{x}\in\mathbb{R}^n} \left|\mathrm{D}_n^2 V_c(\boldsymbol{x})\right| = 0. \tag{4.3}$$

Indeed, we define $m_c \in [0, \infty)$ by

$$3K_* m_c = \sup_{(\boldsymbol{x}', x_n)\in\mathbb{R}^n} \left|\mathrm{D}_n^2 V_c(\boldsymbol{x}', x_n)\right|,$$

and have

$$\left|\mathrm{D}_n^2 V_c(\boldsymbol{y}', y_n)\right| \ge 2K_* m_c$$

for some $(\boldsymbol{y}', y_n) \in \mathbb{R}^n$. Using (4.2), we have

$$\left|\mathrm{D}_n^2 V_c(\boldsymbol{y}', z_n)\right| \ge K_* m_c \qquad \text{if} \quad |z_n - y_n| \le m_c,$$

and have

$$\left|\mathrm{D}_n V_c(\boldsymbol{y}', y_n + m_c) - \mathrm{D}_n V_c(\boldsymbol{y}', y_n - m_c)\right| \ge 2K_* (m_c)^2 .$$

Combining this inequality and (4.1), we get $\lim_{c\to\infty} m_c = 0$, that is, (4.3).

Now we introduce

$$t = -\frac{x_n - \zeta_c}{c},$$

that is, $x_n = \zeta_c - ct$. Then we define

$$u_c(\boldsymbol{x}', t) = V_c(\boldsymbol{x}', \zeta_c - ct), \qquad (\boldsymbol{x}', t) \in \mathbb{R}^n.$$

Now u_c satisfies

$$
\begin{aligned}
&-1 < u_c(\boldsymbol{x}', t) < 1, \qquad (\boldsymbol{x}', t) \in \mathbb{R}^n,\\
&u_c(0, \ldots, 0, \underbrace{R_j}^{j}, 0, \ldots, 0, 0) = \theta_0, \qquad 1 \le j \le n-1,\\
&\mathrm{D}_t u_c(\boldsymbol{x}', t) > 0, \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1},\ t \in \mathbb{R},\\
&\mathrm{D}_j u_c(\boldsymbol{x}', t) > 0, \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1},\ t \in \mathbb{R},\ 1 \le j \le n-1,\\
&u_c(x_1, \ldots, \underbrace{-x_j}^{j}, \ldots, x_{n-1}, t) = u_c(x_1, \ldots, \underbrace{x_j}^{j}, \ldots, x_{n-1}, t),\ \boldsymbol{x}' \in \mathbb{R}^{n-1},\\
&\quad t \in \mathbb{R},\ 1 \le j \le n-1.
\end{aligned}
$$

Now we have

$$
\frac{\partial V_c}{\partial x_n}(\boldsymbol{x}', x_n) = -\frac{1}{c}\frac{\partial u_c}{\partial t}(\boldsymbol{x}', t), \qquad (\boldsymbol{x}', t) \in \mathbb{R}^n.
$$

Then we find

$$
\sum_{j=1}^{n-1} \frac{\partial^2 u_c}{\partial x_j^2}(\boldsymbol{x}', t) + \frac{\partial^2 V_c}{\partial x_n^2}(\boldsymbol{x}', \zeta_c - ct) - \frac{\partial u_c}{\partial t}(\boldsymbol{x}', t) - W'(u_c(\boldsymbol{x}', t)) = 0,
\quad (\boldsymbol{x}', t) \in \mathbb{R}^n. \tag{4.4}
$$

Now we introduce

$$
U(\boldsymbol{x}', t) = \lim_{c \to \infty} u_c(\boldsymbol{x}', t) \tag{4.5}
$$

on every compact set in $\mathbb{R}^n$. The heat kernel in $\mathbb{R}^{n-1}$ is given by

$$
G(\boldsymbol{x}', t) = \frac{1}{(4\pi t)^{\frac{n-1}{2}}} \exp\left(-\frac{|\boldsymbol{x}'|^2}{4t}\right), \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1},\ t > 0.
$$

Let $t_{\mathrm{init}} \in \mathbb{R}$ be arbitrarily given. Using (4.4), we get

$$
\begin{aligned}
&u_c(\boldsymbol{x}', t)\\
&\quad = \int_{\mathbb{R}^{n-1}} G(\boldsymbol{x}' - \boldsymbol{y}', t - t_{\mathrm{init}}) u_c(\boldsymbol{y}', t_{\mathrm{init}})\, \mathrm{d}\boldsymbol{y}'\\
&\qquad + \int_{t_{\mathrm{init}}}^{t} \left(\int_{\mathbb{R}^{n-1}} G(\boldsymbol{x}' - \boldsymbol{y}', t - s) \left(-W'(u_c(\boldsymbol{y}', s)) + \mathrm{D}_n^2 V_c(\boldsymbol{y}', \zeta_c - cs)\right) \mathrm{d}\boldsymbol{y}' \right) \mathrm{d}s
\end{aligned}
$$

for $t > t_{\rm init}$. Taking the limit of $c \to \infty$ for the both sides, we find

$$\begin{aligned} U(\boldsymbol{x}', t) &= \int_{\mathbb{R}^{n-1}} G(\boldsymbol{x}' - \boldsymbol{y}', t - t_{\rm init}) U(\boldsymbol{y}', t_{\rm init})\, \mathrm{d}\boldsymbol{y}' \\ &\quad + \int_{t_{\rm init}}^{t} \left(\int_{\mathbb{R}^{n-1}} G(\boldsymbol{x}' - \boldsymbol{y}', t - s) \left(-W'(u_c(\boldsymbol{y}', s)) \right) \mathrm{d}\boldsymbol{y}' \right) \mathrm{d}s \end{aligned}$$

for $t > t_{\rm init}$, which gives

$$\frac{\partial U}{\partial t}(\boldsymbol{x}', t) = \sum_{j=1}^{n-1} \frac{\partial^2 U}{\partial x_j^2}(\boldsymbol{x}', t) - W'(U(\boldsymbol{x}', t)), \qquad (\boldsymbol{x}', t) \in \mathbb{R}^n$$

for $t > t_{\rm init}$. Since $t_{\rm init} \in \mathbb{R}$ is arbitrary, we obtain

$$\frac{\partial U}{\partial t}(\boldsymbol{x}', t) = \sum_{j=1}^{n-1} \frac{\partial^2 U}{\partial x_j^2}(\boldsymbol{x}', t) - W'(U(\boldsymbol{x}', t)), \qquad (\boldsymbol{x}', t) \in \mathbb{R}^n$$

with

$$U(0, \ldots, 0, \underbrace{R_j}_{j}, 0, \ldots, 0, 0) = \theta_0, \qquad 1 \le j \le n-1.$$

Thus the limit of an n-dimensional traveling front V_c gives an $(n-1)$-dimensional entire solution U as $c \to \infty$. The gradient in $\mathbb{R}^{n-1}$ is given by

$$\nabla' = (\mathrm{D}_1, \ldots, \mathrm{D}_{n-1}).$$

The properties of U is as follows.

Proposition 13 *Let*

$$0 < R_1 \le R_2 \le \ldots R_{n-1} < \infty$$

be arbitrarily given. Then U given by (4.5) satisfies

$$\frac{\partial U}{\partial t}(\boldsymbol{x}', t) = \sum_{j=1}^{n-1} \frac{\partial^2 U}{\partial x_j^2}(\boldsymbol{x}', t) - W'(U(\boldsymbol{x}', t)), \qquad (\boldsymbol{x}', t) \in \mathbb{R}^n$$

with

$$\begin{aligned} &-1 < U(\boldsymbol{x}', t) < 1, \qquad (\boldsymbol{x}', t) \in \mathbb{R}^n, \\ &U(0, \ldots, 0, \underbrace{R_j}_{j}, 0, \ldots, 0, 0) = \theta_0, \qquad 1 \le j \le n-1, \end{aligned} \tag{4.6}$$

$$\mathrm{D}_t U(\boldsymbol{x}', t) \geq 0, \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1},\ t \in \mathbb{R}, \tag{4.7}$$

$$\mathrm{D}_j U(\boldsymbol{x}', t) \geq 0 \quad \text{if } x_j > 0,\ t \in \mathbb{R},\ 1 \leq j \leq n-1, \tag{4.8}$$

$$\mathrm{D}_j U(\boldsymbol{x}', t) = 0 \quad \text{if } x_j = 0,\ t \in \mathbb{R},\ 1 \leq j \leq n-1,$$

$$U(x_1, \ldots, \overset{j}{\smile}{-x_j}, \ldots, x_{n-1}, t) = U(x_1, \ldots, \overset{j}{\smile}{x_j}, \ldots, x_{n-1}, t),\ (\boldsymbol{x}', t) \in \mathbb{R}^n,\ 1 \leq j \leq n-1. \tag{4.9}$$

One has

$$\sup_{(\boldsymbol{x}', t) \in \mathbb{R}^n} \left| \mathrm{D}_t U(\boldsymbol{x}', t) \right| \leq L_*, \tag{4.10}$$

$$\sup_{(\boldsymbol{x}', t) \in \mathbb{R}^n} \left| \mathrm{D}_j U(\boldsymbol{x}', t) \right| \leq L_*, \qquad \sup_{(\boldsymbol{x}', t) \in \mathbb{R}^n} \left| \mathrm{D}_j^2 U(\boldsymbol{x}', t) \right| \leq L_*, \qquad 1 \leq j \leq n-1. \tag{4.11}$$

Here $L_* \in (0, \infty)$ is a constant depending only on (W, n).

Proof The proof follows from the argument stated above. For the proof of the Schauder estimate (4.10) and (4.11), see [34, Proposition 2.9] for instance. □

Now we introduce the following useful lemma.

Lemma 14 (Parabolic Harnack inequality) *Let t_1 and t_2 satisfy $-\infty < t_1 < t_2 < \infty$. Let $a \in (0, \infty)$ be arbitrarily given. Let $\mathcal{D}$ be given by (3.29). Assume*

$$\mathcal{D} \subset \{\boldsymbol{x}' \in \mathbb{R}^{n-1} \mid x_j \geq a\}$$

for some $1 \leq j \leq n-1$. Then one has

$$\sup_{\boldsymbol{x}' \in \mathcal{D}} \mathrm{D}_j U(\boldsymbol{x}', t_1) \leq C \inf_{\boldsymbol{x}' \in \mathcal{D}} \mathrm{D}_j U(\boldsymbol{x}', t_2),$$

where a constant C depends only on $(R, n, a, M, t_2 - t_1)$ and is independent of $(\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$ and $t_1 \in \mathbb{R}$.

Proof For every $1 \leq j \leq n-1$, we have

$$\begin{aligned} \left(\mathrm{D}_t - \Delta' + W''(U)\right) \mathrm{D}_j U &= 0, & \boldsymbol{x}' &\in \mathbb{R}^{n-1},\ t > 0, \\ \mathrm{D}_j U &\geq 0, & \text{if } x_j &> 0,\ t \in \mathbb{R}, \\ \mathrm{D}_j U &= 0, & \text{if } x_j &= 0,\ t \in \mathbb{R}, \end{aligned}$$

where Δ' is defined by (1.10). Then, using (4.8), we obtain

$$\begin{aligned} \left(\mathrm{D}_t - \Delta' + M\right) \mathrm{D}_j U &\geq 0, & \boldsymbol{x}' &\in \mathbb{R}^{n-1},\ t > 0, \\ \mathrm{D}_j U &\geq 0, & \text{if } x_j &> 0,\ t \in \mathbb{R}, \\ \mathrm{D}_j U &= 0, & \text{if } x_j &= 0,\ t \in \mathbb{R}, \end{aligned}$$

for every $1 \leq j \leq n-1$. Here M is given by (3.1). Then this lemma follows from a general theory on the parabolic Harnack inequality [12, Chapter 7, Theorem 10]. □

Now we show that U converges to 1 uniformly in $\boldsymbol{x}' \in \mathbb{R}^{n-1}$ as $t \to \infty$.

Lemma 15 *Under the same assumption of Proposition 13, one has*

$$\lim_{t\to\infty} \inf_{\boldsymbol{x}'\in\mathbb{R}^{n-1}} U(\boldsymbol{x}', t) = 1.$$

Proof Using Proposition 13, we have

$$U(R_{n-1}, \dots, R_{n-1}) \geq \theta_0.$$

Let $\Omega_{n-1}(R_{n-1})$ be defined by (3.17). Then we have

$$U(\boldsymbol{x}') \geq \theta_0 \qquad \text{if} \quad \boldsymbol{x}' \in \Omega_{n-1}(R_{n-1}).$$

Now Lemma 7 gives

$$\lim_{t\to\infty} \inf_{\boldsymbol{x}'\in\mathbb{R}^{n-1}} w(\boldsymbol{x}', t; U) = 1.$$

This completes the proof. □

Combining Proposition 13 and Lemma 15, we obtain

$$\mathrm{D}_t U(\boldsymbol{x}', t) > 0, \qquad (\boldsymbol{x}', t) \in \mathbb{R}^n.$$

Lemma 16 *Under the same assumption of Proposition 13, let $\mu_0 \in (0, \infty)$ be arbitrarily given. Then one has*

$$\lim_{t\to-\infty} U(\mu_0, \dots, \mu_0, t) < \theta_0$$

Proof Assume the contrary. Then we have

$$U(\mu_0, \dots, \mu_0, t) \geq \theta_0, \qquad t \in \mathbb{R}.$$

Let Ω_{n-1} be given by (3.17). We have

$$U(\boldsymbol{x}', t) \geq \theta_0, \qquad \boldsymbol{x}' \in \Omega_{n-1},\ t \in \mathbb{R}.$$

Let $t_0 \in \mathbb{R}$ be arbitrarily given. We have

$$U(\boldsymbol{x}', t_0) \geq \theta_0, \qquad \boldsymbol{x}' \in \Omega_{n-1}.$$

Lemma 7 implies

$$U(\boldsymbol{x}', t + t_0) \geq w(\boldsymbol{x}', t; w_0) \qquad \boldsymbol{x}' \in \Omega_{n-1},\ t > 0.$$

Thus, for any given $\delta \in (0, 1 - \theta_0)$, we have $T \in (0, \infty)$ that depends only on δ such that we have

$$\inf_{\boldsymbol{x}' \in \mathbb{R}^{n-1}} U(\boldsymbol{x}', T + t_0) \geq 1 - \delta.$$

Since $t_0 \in \mathbb{R}$ can be chosen arbitrarily, we have

$$U(\boldsymbol{x}', t) \geq 1 - \delta, \qquad \boldsymbol{x}' \in \Omega_{n-1},\ t \in \mathbb{R}.$$

This contradicts (4.7). Now we complete the proof. □

Let $\theta \in (-1, 1)$ be arbitrarily given. For $\boldsymbol{x}' \in \mathbb{R}^{n-1}$, we define $h_\theta(\boldsymbol{x}')$ by

$$U(\boldsymbol{x}', h_\theta(\boldsymbol{x}')) = \theta$$

if it exists. Lemma 15 and Lemma 16 imply that $h_\theta(\boldsymbol{x}')$ exists for every $\boldsymbol{x}' \in \mathbb{R}^{n-1}$ and every $\theta \in [\theta_0, 1)$.

Lemma 17 *Under the same assumption of Proposition 13, let R be given by (3.28), and let $\mathcal{D}$ be given by (3.29) for every $\boldsymbol{\xi}' \in \mathbb{R}^{n-1}$. Then one has*

$$\int_{\mathcal{D}} \left|\nabla' U(\boldsymbol{x}', h_{\theta_0}(\boldsymbol{x}'))\right|^2 \mathrm{d}\boldsymbol{x}' \geq A(\theta_0, R)|\mathcal{D}|^2 > 0. \tag{4.12}$$

Let $\theta \in (-1, \theta_0)$ be arbitrarily given. If $h_\theta(\boldsymbol{x}')$ is defined for all $\boldsymbol{x}' \in \mathcal{D}$, one has

$$\int_{\mathcal{D}} \left|\nabla' U(\boldsymbol{x}', h_{\theta}(\boldsymbol{x}'))\right|^2 \mathrm{d}\boldsymbol{x}' \geq A(\theta, R)|\mathcal{D}|^2 > 0. \tag{4.13}$$

Proof Proposition 9 implies

$$\int_{\mathcal{D}} \left|\nabla V_c(\boldsymbol{x}', q_{\theta_0}(\boldsymbol{x}'))\right|^2 \mathrm{d}\boldsymbol{x}' \geq A(\theta_0, R)|\mathcal{D}|^2 > 0$$

for any $c \in (0, \infty)$. Combining this inequality and (4.3), we obtain (4.12) as $c \to \infty$. Consequently, Proposition 9 implies

$$\int_{\mathcal{D}} \left|\nabla V_c(\boldsymbol{x}', q_{\theta}(\boldsymbol{x}'))\right|^2 \mathrm{d}\boldsymbol{x}' \geq A(\theta, R)|\mathcal{D}|^2 > 0$$

for any $c \in (0, \infty)$. Combining this inequality and (4.3), we obtain (4.13) as $c \to \infty$. □

Now we assert

$$\mathrm{D}_j U(\boldsymbol{x}', t) > 0 \qquad \text{if} \quad x_j > 0,\ t \in \mathbb{R} \tag{4.14}$$

for any given $1 \leq j \leq n-1$. Indeed, in view of (4.8), we have $\mathrm{D}_j U \equiv 0$ in $\mathbb{R}^n$ if $\mathrm{D}_j U$ takes zero at some point with $x_j > 0$ by the maximum principle. Then, using (4.6), we have $U(\mathbf{0}', 0) = \theta_0$. Combining this equality and (4.6) for all $1 \leq j \leq n-1$, we have $\nabla' U \equiv 0$ in $\mathbb{R}^n$. This contradicts (4.12) in Lemma 17. Thus (4.14) holds true.

Now we assert the following lemma.

Lemma 18 *For every $\theta \in (-1, \theta_0)$, one has*

$$\lim_{t \to -\infty} U(\boldsymbol{x}', t) < \theta$$

for every $\boldsymbol{x}' \in \mathbb{R}^{n-1}$. That is, $h_\theta(\boldsymbol{x}')$ is defined for every $\boldsymbol{x}' \in \mathbb{R}^{n-1}$.

Proof For arbitrarily given $\mu_4 \in (0, \infty)$, we consider $(\mu_4, \ldots, \mu_4) \in \mathbb{R}^{n-1}$. We take $\boldsymbol{\xi}' \in \mathbb{R}^{n-1}$ such that $\mathcal{D}$ given by (3.29) satisfies

$$\mathcal{D} \subset \Omega_{n-1}(\mu_4).$$

Here $\Omega_{n-1}(\mu_4)$ is given by (3.17). Lemma 17 and (4.11) imply that there exists $\varepsilon_0 \in (0, \theta_0)$ such that we have $\boldsymbol{x}'_0 \in \mathcal{D}$ with

$$\lim_{t \to -\infty} U(\boldsymbol{x}'_0, t) < \theta_0 - \varepsilon_0.$$

Then, using Proposition 13, we have

$$\lim_{t \to -\infty} U(\mu_4, \ldots, \mu_4, t) < \theta_0 - \varepsilon_0.$$

Since $\mu_4 \in (0, \infty)$ can be taken arbitrarily large, we complete the proof. □

Lemma 18 implies that, as $t \to -\infty$, $U(\boldsymbol{x}', t)$ converges to -1 on every compact set in $\mathbb{R}^{n-1}$. Now we state the following assertion.

Lemma 19 *Let U be given by (4.5) and let $\theta \in (-1, 1)$ be arbitrarily given. For any $(\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$, let $\mathcal{D}$ be given by (3.29). Then one has*

$$\int_{\mathcal{D}} \left|\nabla' U(\boldsymbol{x}', h_\theta(\boldsymbol{x}'))\right|^2 \, \mathrm{d}\boldsymbol{x}' \geq A(\theta, R)^2 |\mathcal{D}| > 0, \tag{4.15}$$

$$\max_{\boldsymbol{x}' \in \mathcal{D}} |\nabla' U(\boldsymbol{x}', h_\theta(\boldsymbol{x}'))| \geq A(\theta, R) > 0. \tag{4.16}$$

Let $a \in (0, \infty)$ be arbitrarily given. One has

$$\inf \left\{ |\nabla' U(\boldsymbol{x}', t)| \;\Big|\; (\boldsymbol{x}', t) \in \mathbb{R}^n,\ \min_{1 \leq j \leq n-1} |x_j| \geq a,\ U(\boldsymbol{x}', t) = \theta \right\} > 0. \tag{4.17}$$

Proof Equation (4.15) follows from Lemmas 17 and 18. Equation (4.16) follows from (4.15). It suffices to prove (4.17) in

$$Q = \{(x_1, \ldots, x_{n-1}) \mid \min_{1 \le j \le n-1} x_j \ge a\}.$$

Let $R = R_\theta$ be given by (3.28). We define

$$R_{\max} = \max_{\rho \in [(\theta-1)/2, (1+\theta)/2]} R_\rho \in (0, \infty).$$

Let $\mathcal{D}_{\max}$ be

$$\mathcal{D}_{\max} = (\xi_1 - R_{\max}, \xi_1 + R_{\max}) \times \cdots \times (\xi_{n-1} - R_{\max}, \xi_{n-1} + R_{\max}) \subset Q.$$

Let $\tau \in \mathbb{R}$ satisfy

$$U(\xi_1, \ldots, \xi_{n-1}, \tau) = \theta.$$

Let L_* be as in Proposition 13. We have

$$-1 < \frac{\theta - 1}{2} \le U\left(\xi_1, \ldots, \xi_{n-1}, \tau - \frac{\min\{1+\theta, 1-\theta\}}{2L_*}\right) \le \frac{1+\theta}{2} < 1.$$

For some

$$\theta' \in [(\theta - 1)/2, (1+\theta)/2],$$

we have

$$U\left(\xi_1, \ldots, \xi_{n-1}, \tau - \frac{\min\{1+\theta, 1-\theta\}}{2L_*}\right) = \theta',$$

that is,

$$h_{\theta'}(\boldsymbol{\xi}') = \tau - \frac{\min\{1+\theta, 1-\theta\}}{2L_*}.$$

Here

$$\boldsymbol{\xi}' = (\xi_1, \ldots, \xi_{n-1}).$$

We find

$$\begin{aligned}
&\max_{\boldsymbol{x}' \in \mathcal{D}_{\max}} |\nabla' U(\boldsymbol{x}', h_{\theta'}(\boldsymbol{x}'))| \\
&\quad \ge A(\theta', R_{\theta'}) \\
&\quad \ge \min\left\{A(\rho, R_\rho) \mid \rho \in [(\theta-1)/2, (1+\theta)/2]\right\} > 0.
\end{aligned}$$

Thus, for some $1 \leq j_0 \leq n-1$, we have

$$\begin{aligned}&\sqrt{n-1} \max_{\boldsymbol{x}' \in \mathcal{D}_{\max}} \mathrm{D}_{j_0} U(\boldsymbol{x}', h_{\theta'}(\boldsymbol{x}')) \\ &\geq \min \left\{ A(\rho, R_\rho) \mid \rho \in [(\theta-1)/2, (1+\theta)/2] \right\} > 0.\end{aligned}$$

Now we have $(\xi_1, \ldots, \xi_{n-1}) \in \mathcal{D}_{\max}$. Using the parabolic Harnack inequality in Lemma 14, we obtain

$$\begin{aligned}0 &< \frac{1}{\sqrt{n-1}} \min \left\{ A(\rho, R_\rho) \mid \rho \in [(\theta-1)/2, (1+\theta)/2] \right\} \\ &\leq \max_{\boldsymbol{x}' \in \mathcal{D}_{\max}} \mathrm{D}_{j_0} U(\boldsymbol{x}', h_{\theta'}(\boldsymbol{\xi}')) \\ &= \max_{\boldsymbol{x}' \in \mathcal{D}_{\max}} \mathrm{D}_{j_0} U\left(\boldsymbol{x}', \tau - \frac{\min\{1+\theta, 1-\theta\}}{2L_*}\right) \\ &\leq C_0 \min_{\boldsymbol{x}' \in \mathcal{D}_{\max}} \mathrm{D}_{j_0} U(\boldsymbol{x}', \tau) \\ &\leq C_0 \, \mathrm{D}_{j_0} U(\xi_1, \ldots, \xi_{n-1}, \tau) \\ &\leq C_0 \left| \nabla' U(\xi_1, \ldots, \xi_{n-1}, \tau) \right|.\end{aligned}$$

Here $C_0 \in (0, \infty)$ is a constant depending only on $(M, n, R_{\max}, \theta, L_*)$ and is independent of $(\xi_1, \ldots, \xi_{n-1}, \tau) \in \mathbb{R}^n$. Now we proved (4.17) and this completes the proof. □

Proofs of Theorem 1 and Theorem 2 Now we prove (1.11) and (1.12). Let U be given by (4.5). Let $a \in (0, \infty)$ be arbitrarily given. Using Lemma 19, we have

$$\lim_{\mu \to \infty} \inf \left\{ U(\boldsymbol{x}', t) \,\middle|\, |\boldsymbol{x}'| \geq \mu, \min_{1 \leq j \leq n-1} |x_j| \geq a \right\} \geq \theta_0$$

for every $t \in \mathbb{R}$. By combining Lemma 7 and [34, Theorem 5.8], there exists $T_\varepsilon \in (0, \infty)$ for any given $\varepsilon \in (0, 1)$ such that we have

$$\lim_{\mu \to \infty} \inf \left\{ U(\boldsymbol{x}', t + T_\varepsilon) \,\middle|\, |\boldsymbol{x}'| \geq \mu \right\} \geq 1 - \varepsilon$$

for every $t \in \mathbb{R}$. Thus we obtain

$$\lim_{|\boldsymbol{x}'| \to \infty} U(\boldsymbol{x}', t) \geq 1 - \varepsilon$$

for every $t \in \mathbb{R}$. Since $\varepsilon \in (0, 1)$ can be taken arbitrarily, we obtain (1.11) and (1.12). The other assertions in these two theorems follow from Proposition 13, Lemmas 15, 18 and 19. □

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Declarations

Conflict of interest The author states that there is no conflict of interest.

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