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Free boundary problems of the incompressible Navier–Stokes equations with non-flat initial surface in the critical Besov space

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Abstract

Global well-posedness of the Navier–Stokes equations with a free boundary condition is considered in the scaling critical homogeneous Besov spaces $\dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+)$ with $n-1 < p < 2n-1$. To show the global well-posedness, we establish end-point maximal L^1-regularity for the initial-boundary value problem of the Stokes equations. Such an estimate is obtained via related estimate for the initial-boundary value problem of the heat equation with the inhomogeneous Neumann data as well as the pressure estimate in the critical Besov space framework. The proof heavily depends on the explicit expression of the fundamental integral kernel of the Lagrange transformed linearized Stokes equations and the almost orthogonal estimates with the space-time Littlewood–Paley dyadic decompositions. Our result here improves the initial space and boundary state than previous results by Danchin–Hieber–Mucha–Tolksdorf (Free boundary problems via Da Prato–Grisvard theory. arXiv:2011.07918v2) and ourselves (Ogawa and Shimizu in J Evol Equ 22(30):67, 2022; Ogawa and Shimizu in J Math Soc Jpn. arXiv:2211.06952v3).

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1 Introduction

1.1 The free boundary problem of the Navier–Stokes system

We consider the initial boundary value problem of the incompressible Navier–Stokes equations with free boundary condition. Let $\Omega(t) \subset \mathbb{R}^n$ be a domain that is occupied by the fluid in the n-dimensional Euclidean space $\mathbb{R}^n$ with $n \geq 2$ and let the initial domain be described by the upper region of a graph of the unknown function $\bar{\eta}(t, y') : \mathbb{R}_+ \times \mathbb{R}^{n-1} \to \mathbb{R}$ as

$$\Omega(t) \equiv \Big\{(t, y', y_n) \in \mathbb{R}_+ \times \mathbb{R}^{n-1} \times \mathbb{R};\ y_n > \bar{\eta}(t, y')\Big\},$$

where $\mathbb{R}^{n-1}$ denotes the $n-1$-dimensional Euclidean space. The velocity of the fluid $\bar{u}(t, y)$ and the pressure $\bar{p}(t, y)$ for $y \in \Omega(t)$ satisfy the incompressible Navier–Stokes equations:

$$\begin{cases} \partial_t \bar{u} + \bar{u} \cdot \nabla \bar{u} - \operatorname{div} T(\bar{u}, \bar{p}) = 0, & t > 0, \quad y \in \Omega(t), \\ \operatorname{div} \bar{u} = 0, & t > 0, \quad y \in \Omega(t), \\ T(\bar{u}, \bar{p})\nu_t = 0, & t > 0, \quad y \in \partial\Omega(t), \\ \frac{\partial_t \bar{\eta}}{\sqrt{1+|\nabla' \bar{\eta}|^2}} = -\bar{u} \cdot \nu_t, & t > 0, \quad y \in \partial\Omega(t), \\ \bar{u}(0, y) = \bar{u}_0(y), & \qquad\quad y \in \Omega(0), \\ \bar{\eta}(0, y') = \eta_0(y'), & \qquad\quad y' \in \mathbb{R}^{n-1}. \end{cases} \tag{1.1}$$

Here, $\partial\Omega(t)$ denotes the boundary of $\Omega(t)$, ν_t is the unit outward normal at a point $y \in \partial\Omega(t)$ given by

$$\nu_t = \frac{(\nabla' \bar{\eta}, -1)}{\sqrt{1 + |\nabla' \bar{\eta}|^2}}, \tag{1.2}$$

$T(\bar{u}, \bar{p})$ is the stress tensor defined by $T(\bar{u}, \bar{p}) = (\nabla \bar{u} + (\nabla \bar{u})^{\mathsf{T}}) - \bar{p} I$, where I is the $n \times n$ identity matrix, $(\nabla_y \bar{u})_{i,j} = \big(\partial \bar{u}_j / \partial y_i\big)_{(1 \leq i, j \leq n)}$, $(\nabla \bar{u})^{\mathsf{T}}$ denotes the transposed matrix of $\nabla \bar{u}$, where $\nabla = \nabla_y = (\partial_{y_1}, \partial_{y_2}, \ldots, \partial_{y_n})^{\mathsf{T}}$ and $\nabla' = \nabla'_y = (\partial_{y_1}, \partial_{y_2}, \ldots, \partial_{y_{n-1}})^{\mathsf{T}}$. $\bar{u}_0$ and $\bar{\eta}_0$ are given initial velocity and initial surface, respectively. Our basic assumption of the dynamics of the boundary of the fluid region $\Omega(t)$ is governed by the kinematic condition (cf. Solonnikov [54]) which is shown from (1.2) by

$$\partial_t \bar{\eta} + \bar{u}' \cdot \nabla' \bar{\eta} = \bar{u}_n. \tag{1.3}$$

In our setting (1.1), we do not take into account of the gravity force nor the surface tension.[1]

Free boundary problems for incompressible fluids were first considered by Solonnikov [54] in the space-time L^2 setting and he proved the time local well-posedness

[1] Practically the natural setting is $\Omega(0) = \mathbb{R}^n_-$ under the gravity circumstance.

of the initial boundary value problem (1.1). It was generalized by Tani–Solonnikov [60], Tani [61, 62], Tani–Tanaka [63], Mucha–Zajączkowski [33], Shibata–Shimizu [51, 52] (see also [41, 48, 49, 55–59]). Beale [5, 6] considered the free surface problem in a semi-infinite domain and Prüss–Simonett [43, 44] proved the local of (1.1) whose initial state $\Omega = \Omega(0)$ is close to the half-space $\mathbb{R}^n_+$ in the class of Sobolev space $W^{1,2}_p((0,T)\times\Omega)$ with $p > n+2$. There are many other contributions on this direction, for instance, [1, 7, 8, 16–18, 25–27, 33, 34, 43–46, 51, 53] and references therein.

It is well-known that the incompressible Navier–Stokes equations are invariant under the scaling transform: For any $\lambda > 0$,

$$\begin{cases} \bar{u}(t,y) \to \bar{u}_\lambda(t,y) \equiv \lambda \bar{u}(\lambda^2 t, \lambda y), \\ \bar{p}(t,y) \to \bar{p}_\lambda(t,y) \equiv \lambda^2 \bar{p}(\lambda^2 t, \lambda y). \end{cases}$$

Subsequently the Cauchy problem of the Navier–Stokes equations can be solved globally in the Bochner class $L^\rho\big(\mathbb{R}_+; \dot{H}^s_p(\mathbb{R}^n;\mathbb{R}^n)\big)$

$$\frac{2}{\rho} + \frac{n}{p} = 1 + s \tag{1.4}$$

by Fujita–Kato [23] (see also the relevant regularity criterion (cf., [40, 42, 47]). Setting $\rho = \infty$, $s = -1 + n/p$ in (1.4), and the critical class at $s = 0$ is given, in particular, Kato [29] by $C_b([0,T); L^n(\mathbb{R}^n))$ and the scaling critical Besov spaces $\dot{B}^{-1+n/p}_{p,\sigma}(\mathbb{R}^n)$, where $1 \le p < \infty$ and $1 \le \sigma \le \infty$ ([3, 11–13, 30]). Meanwhile, ill-posedness of the Cauchy problem was shown in [10, 66, 70], namely the continuous dependence on the initial data in the classes $u_0 \in \dot{B}^{-1}_{\infty,\sigma}(\mathbb{R}^n)$, $1 \le \sigma \le \infty$ breaks down. It is then natural to ask if the free surface problem can also be solvable in such a scaling critical function class.

When $\Omega(0) \equiv \mathbb{R}^n_+$, the problem (1.1) was considered by Danchin–Hieber–Mucha–Tolksdorf [15] in a scaling critical Besov space $\dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+)$ for $n \ge 3$ with $n-1 < p < n$ via maximal L^1-regularity of the linear problem corresponding to (1.8). Their result is based on the Da Prato–Grisvard theory [19] and applied the result for the initial boundary value problem by Danchin–Mucha [16]. Independently the authors consider the free surface problem in [39] for the scaling critical space $\dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+)$ for $n \le p < 2n-1$ with $n \ge 2$ using an explicit form of the Fourier image of the fundamental solutions to the linearized Stokes equations corresponding to (1.8) which has been obtained in Shibata–Shimizu [52]. The argument in the both proofs seems very different from each other and the results are compensated each other when $n \ge 3$.

Under the kinematic boundary condition (1.3), the solution of the Cauchy problem

$$\frac{dy}{dt} = \bar{u}\big(t, y(t)\big), \quad t > 0, \qquad y(0) = \tilde{x} \tag{1.5}$$

induces the problem into a fixed boundary value problem. Namely, the Euler coordinates $y = y_{\bar{u}}(t) \in \Omega(t)$ are transformed into the Lagrangian coordinates $\tilde{x} \in \Omega(0)$

connected by (1.5). If $\bar{u}(t, y)$ is Lipschitz continuous with respect to y, then (1.5) can be solved uniquely by

$$y(t) = \tilde{x} + \int_0^t \bar{u}\big(s, y(s, \tilde{x})\big)ds, \tag{1.6}$$

where $\tilde{x} \in \Omega(0)$ and ν denotes the outer normal at the boundary $\partial\Omega(0)$. Setting

$$\begin{cases} \tilde{u}(t, \tilde{x}) \equiv \bar{u}(t, y(t)), \\ \tilde{p}(t, \tilde{x}) \equiv \bar{p}(t, y(t)), \\ \tilde{\eta}(t, \tilde{x}) \equiv \bar{\eta}(t, y(t)'), \end{cases} \tag{1.7}$$

and applying the Lagrangian coordinate to the original problem (1.1), the system is transformed into a fixed domain problem. We first notice that the kinematic condition (1.3) with (1.6) implies

$$\partial_t\tilde{\eta}(t, \tilde{x}) = \partial_t\bar{\eta}(t, y') + \bar{u}' \cdot \nabla'_y\bar{\eta}(t, y') = \bar{u}_n(t, y) = \tilde{u}_n(t, \tilde{x}), \quad t > 0, \ \ \tilde{x} \in \partial\Omega(0),$$

which ensures us that the transformed domain does not move in time $t > 0$, i.e.,

$$\bar{\eta}(t, y(t)') - y_n(t) = \eta_0(\tilde{x}') - \tilde{x}_n < 0$$

and the fluid region is given by

$$\Omega \equiv \Omega(0) = \Big\{(\tilde{x}', \tilde{x}_n) \in \mathbb{R}^{n-1} \times \mathbb{R};\ \ \tilde{x}_n > \eta_0(\tilde{x}')\Big\}.$$

Hence the dynamics of fluids is governed by the the following intermediate system:

$$\begin{cases} \partial_t\tilde{u} - \Delta\tilde{u} + \nabla\tilde{p} = F_u(\tilde{u}) + F_p(\tilde{u}, \tilde{p}), & t > 0, \ \ \tilde{x} \in \Omega, \\ \operatorname{div}\ \tilde{u} = G_{\mathrm{div}}(\tilde{u}), & t > 0, \ \ \tilde{x} \in \Omega, \\ \Big(\nabla\tilde{u} + (\nabla\tilde{u})^{\mathsf{T}} - \tilde{p}I\Big)\,\nu = H_u(\tilde{u}) + H_p(\tilde{u}, \tilde{p}), & t > 0, \ \ \tilde{x} \in \partial\Omega, \\ \tilde{u}(0, \tilde{x}) = \bar{u}_0(\tilde{x}), & \tilde{x} \in \Omega, \end{cases} \tag{1.8}$$

where ν denotes the outward normal at a point in $\partial\Omega$, $\operatorname{div}\bar{u}_0 = 0$ in the sense of distribution and the nonlinear terms of (1.8) are given by

$$F_u(\tilde{u}) \equiv \operatorname{div}\Big(J(D\tilde{u})^{-1}\big(J(D\tilde{u})^{-1}\big)^{\mathsf{T}}\nabla\tilde{u} - \nabla\tilde{u}\Big), \tag{1.9}$$

$$F_p(\tilde{u}, \tilde{p}) \equiv -\big(J(D\tilde{u})^{-1})^{\mathsf{T}} - I\big)\,\nabla\tilde{p} = -\operatorname{div}\Big(\big(J(D\tilde{u})^{-1} - I\big)\,\tilde{p}\Big), \tag{1.10}$$

$$G_{\mathrm{div}}(\tilde{u}) \equiv -\operatorname{tr}\Big(\big(J(D\tilde{u})^{-1})^{\mathsf{T}} - I\big)\nabla\tilde{u}\Big) = -\operatorname{div}\Big(\big(J(D\tilde{u})^{-1} - I\big)\,\tilde{u}\Big), \tag{1.11}$$

$$\begin{aligned} H_u(\tilde{u}) \equiv &-\Big(\big(J(D\tilde{u})^{-1}\big)^{\mathsf{T}}\ \nabla\tilde{u} + (\nabla\tilde{u})^{\mathsf{T}}\ \big(J(D\tilde{u})^{-1}\big)\Big)\big(J(D\tilde{u})^{-1} - I\big)^{\mathsf{T}}\nu \\ &-\Big(\big(J(D\tilde{u})^{-1} - I\big)^{\mathsf{T}}\ \nabla\tilde{u} + (\nabla\tilde{u})^{\mathsf{T}}\ \big(J(D\tilde{u})^{-1} - I\big)\Big)\nu, \end{aligned} \tag{1.12}$$

$$H_p(\tilde{u}, \tilde{p}) \equiv p\big(J(D\tilde{u})^{-1} - I\big)^{\mathsf{T}}\nu. \tag{1.13}$$

Here $\operatorname{div} K$ denotes $[\nabla^{\mathsf{T}} K]^{\mathsf{T}} \equiv \Big(\sum_{k=1}^{n} \partial_{x_k} K_{kj}(x)\Big)^{\mathsf{T}}$ for an $n \times n$ matrix valued function $K = [K_{kj}(x)]_{1 \le k, j \le n}$, $J(D(\tilde{u}))^{-1}$ denotes the inverse of the Jacobi matrix of the transform. We invoke the divergence-curl structure in (1.10) and (1.11) (see Solonnikov [56], see also (10.2) of Corollary 10.2 in Appendix). By applying the Lagrangian transformation, the free surface problem (1.1) is transformed into the fixed boundary problem and the system is transformed into the quasilinear parabolic equation (1.8) (see e.g., [57]).

In this paper, we discuss the time global existence of a solution of the transformed free surface problem (1.8) with *non-flat initial surface.* We need to discuss the corresponding maximal L^1-regularity for initial-boundary value problems of the Stokes equations with the associated non-stress boundary condition. We extend former results in the homogeneous Besov spaces $\dot{B}^s_{p,1}(\mathbb{R}^n_+)$ with $-1 + 1/p < s < 1/p$ and $1 < p < \infty$ (see for the definition of the homogeneous Besov spaces below) and it naturally extends the well-posedness result to the free boundary problem for the Navier–Stokes equations in the scaling critical setting including both the results in [15] and [39]. Furthermore, we generalize the result into a non-flat initial surface $\partial\Omega = \partial\Omega(0)$, where $\partial\Omega$ is assumed to be described by the graph of a given small function $y_n = \eta_0(y')$. Such an extension enable us to conclude the range of exponent p for the global well-posedness of the free surface problem of the Navier–Stokes equations into $n - 1 < p < 2n - 1$ and hence our result includes former results [15] and [39].

Let us introduce an extension function of the boundary function $\eta_0(\tilde{x}')$ into the whole domain Ω.

Definition. Let $1 \le q < \infty$. For $\eta_0 \in \dot{B}^{1+\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})$, set

$$E(x', x_n) \equiv \big(\operatorname{sech}(x_n|\nabla'|)\eta_0(x'\big) \tag{1.14}$$

so that

$$\begin{aligned}
&\big(\nabla' E(x', x_n), \partial_{x_n} E(x', x_n)\big) \\
&\quad = \big(\operatorname{sech}(x_n|\nabla'|)\nabla'\eta_0(x'), \operatorname{sech}(x_n|\nabla'|)|\nabla'|\eta_0(x')\big), \quad x_n > 0,
\end{aligned} \tag{1.15}$$

where the operator $\operatorname{sech}(x_n|\nabla'|)$ is given by the Fourier multiplier

$$\operatorname{sech}(x_n|\nabla'|) f \equiv \mathcal{F}^{-1}_{\xi'}\big[\operatorname{sech}(x_n|\xi'|)\widehat{f}(\xi')\big],$$

and $\mathcal{F}^{-1}_{\xi'}$ denotes the Fourier inverse transform from $\xi' \in \mathbb{R}^{n-1} \to x' \in \mathbb{R}^{n-1}$. We introduce the domain deformation (flattening) transform $\mathcal{E} : \tilde{x} \in \Omega \mapsto x \in \mathbb{R}^n_+ =$

$\{x = (x', x_n) \in \mathbb{R}^n; x' \in \mathbb{R}^{n-1},\ x_n > 0\}$ given by

$$\begin{cases} \tilde{x}' = x', \\ \tilde{x}_n = x_n + E(x', x_n) \end{cases} \tag{1.16}$$

and the Jacobi matrix $J(DE) = \partial\tilde{x}/\partial x$ of (1.16) with its determinant $1 + \partial_{x_n} E$. Since $\partial_{x_n} E(x', x_n) = \operatorname{sech}(x_n|\nabla'|)|\nabla'|\eta_0(x')$, under the smallness condition on $|\nabla'|\eta_0$, $\partial_{x_n} E > -1$ everywhere (cf. Lemma 8.1 below) and the deformation $\mathcal{E}$ is bijective. If we set $\phi(x_n) = x_n + E(\cdot, x_n)$, then $\partial_{x_n}\phi = 1 + \partial_{x_n} E$ and is strictly positive under the smallness condition for $|\nabla'|\eta_0$, it means that $\phi(x_n)$ is invertible and monotone increasing with respect to x_n. Noting that $\phi(0) = E(x', 0) = \eta_0(x')$, we know that $\mathcal{E}$ maps the domain $\{(\tilde{x}', \tilde{x}_n); \tilde{x}_n \gtrless \eta_0(\tilde{x}')\}$ into

$$\{(x', x_n);\ \phi(x_n) = x_n + E_n(x', x_n) \gtrless \eta_0(x')\} = \{(x', x_n);\ x_n \gtrless 0\},$$

(cf. [53]), and the boundary $\partial\Omega = \{(\tilde{x}', \tilde{x}_n) \in \mathbb{R}^n;\ \tilde{x}_n = \eta_0(\tilde{x}')\}$ is transformed into a new boundary $\partial\mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n;\ x_n = 0\}$. The component of the transposed inverse of the Jacobi matrix is given by using $\partial_j = \partial_{x_j}$ $(j = 1, 2, \ldots, n)$ that

$$(J(DE)^{-1})^{\mathsf{T}} = \begin{pmatrix} 1 & 0 & \cdots & -\frac{\partial_1 E}{1+\partial_n E} \\ 0 & 1 & \cdots & -\frac{\partial_2 E}{1+\partial_n E} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 - \frac{\partial_n E}{1+\partial_n E} \end{pmatrix}. \tag{1.17}$$

The covariant derivatives for a function $K(x) = \tilde{K}(\tilde{x})$ $(1 \le j, k \le n)$ i.e., $\nabla K = (\partial_1 K, \partial_2 K, \ldots \partial_n K)^{\mathsf{T}}$ and a vector field $F(x', x_n) = \tilde{F}(\tilde{x}', \tilde{x}_n) : \mathbb{R}^n_+ \to \mathbb{R}^n$, are expressed from (1.17) by

$$(\nabla_E K)_j \equiv (\nabla K)_j + \big((J(DE)^{-1} - I)^{\mathsf{T}} \nabla K\big)_j = \partial_j K - \frac{\partial_j E}{1 + \partial_n E}\partial_n K, \tag{1.18}$$

$$\operatorname{div}_E F \equiv \operatorname{div} F + \operatorname{tr}\Big((J(DE)^{-1} - I)^{\mathsf{T}} \nabla F\Big) = \operatorname{div} F - \sum_{j=1}^n \frac{\partial_j E}{1 + \partial_n E}\partial_n F_j. \tag{1.19}$$

We also denote $(\partial_E K)_j$ and $D_E K$ the corresponding covariant derivatives and the Jacobi matrix form for any function K, respectively. If E is sufficiently smooth then it follows from (1.19) that

$$(1 + \partial_n E)\operatorname{div}_E F = (1 + \partial_n E)\operatorname{div} F - \nabla E \cdot (\partial_n F)$$

and

$$\begin{aligned}\operatorname{div} F &= \nabla E \cdot (\partial_n F) - (\partial_n E)\operatorname{div} F + (1+\partial_n E)\operatorname{div}_E F \\ &= \partial_n(\nabla E \cdot F) - \operatorname{div}\big((\partial_n E)F\big) + (1+\partial_n E)\operatorname{div}_E F, \end{aligned} \tag{1.20}$$

where the first and the second terms of the right hand side of (1.20) maintain their divergence form.

Introducing new unknown functions;

$$\begin{cases} u(t,x) \equiv \tilde{u}(t,\tilde{x}), \\ p(t,x) \equiv \tilde{p}(t,\tilde{x}), \end{cases}$$

the Jacobi matrix is denoted by

$$J(D_E u)_{1\le j,k\le n} = \left[\delta_{jk} + \int_0^t \Big(\partial_k u_j(s,x) - \frac{\partial_k E(x)}{1+\partial_n E(x)}\partial_n u_j(s,x)\Big) ds\right]_{1\le j,k\le n}. \tag{1.21}$$

Hence applying the boundary flattening operation $\mathcal{E}$ in (1.16) to the problem (1.8), the system is transformed into the following problem on the flat boundary region $\mathbb{R}^n_+$:

$$\begin{cases} \partial_t u - \Delta u + \nabla p = f(u,E) + f(p,E) + F_u(u,E) + F_p(u,p,E), & t>0,\quad x\in\mathbb{R}^n_+, \\ \operatorname{div} u = g(u,E) + (1+\partial_n E)G_{\operatorname{div}}(u,E), & t>0,\quad x\in\mathbb{R}^n_+, \\ \Big(\nabla u + (\nabla u)^{\mathsf{T}} - pI\Big)\nu_n & \\ \quad = h(u,E) + h(p,E) + H_u(u,E) + H_p(u,p,E), & t>0,\quad x'\in\mathbb{R}^{n-1}, \\ u(0,x',x_n) = \bar{u}_0(x', x_n - E(x',x_n)) \equiv u_0(x), & x\in\mathbb{R}^n_+, \end{cases} \tag{1.22}$$

where $\nu_n = (0,\ldots,0,-1)$ denotes the outward normal at a point in $\partial\mathbb{R}^n_+$, the linear variable coefficient terms are given (cf. (1.20)) by y

$$\begin{aligned} f(u,E) \equiv & -\sum_{j=1}^n \partial_j\Big(\frac{\partial_j E}{1+\partial_n E}\partial_n u\Big) - \sum_{j=1}^n \frac{\partial_j E}{1+\partial_n E}\partial_j(\partial_n u) \\ & + \sum_{j=1}^n \frac{\partial_j E}{1+\partial_n E}\partial_n\Big(\frac{\partial_j E}{1+\partial_n E}\partial_n u\Big), \end{aligned} \tag{1.23}$$

$$f(p,E) \equiv -\Big(\big(J(DE)^{-1}\big)^{\mathsf{T}} - I\Big)\nabla p = \frac{\nabla E}{1+\partial_n E}\partial_n p, \tag{1.24}$$

$$g(u,E) \equiv \partial_n(\nabla E\cdot u) - \operatorname{div}\big((\partial_n E)u\big), \tag{1.25}$$

$$\begin{aligned} h(u,E) \equiv & -\big(\nabla_E u + (\nabla_E u)^{\mathsf{T}}\big)\frac{(\nabla' E, -1)^{\mathsf{T}}}{\sqrt{1+|\nabla' E|^2}} + (\nabla u + (\nabla u)^{\mathsf{T}})\nu_n \\ = & -\Big(\nabla u + (\nabla u)^{\mathsf{T}}\Big)\frac{(\nabla' E, \sqrt{1+|\nabla' E|^2} - 1)^{\mathsf{T}}}{\sqrt{1+|\nabla' E|^2}} \end{aligned}$$

$$- \frac{1}{1+\partial_n E}\big(\nabla E \partial_n u_n + \partial_n E \partial_n u\big)$$
$$+ \frac{1}{1+\partial_n E}\Big(\partial_j E \partial_n u_k + \partial_k E \partial_n u_j\Big)_{jk} \frac{(\nabla' E, \sqrt{1+|\nabla' E|^2}-1)^{\mathsf{T}}}{\sqrt{1+|\nabla' E|^2}}, \tag{1.26}$$

$$h(p, E) \equiv \frac{(\nabla' E, \sqrt{1+|\nabla' E|^2}-1)^{\mathsf{T}}}{\sqrt{1+|\nabla' E|^2}} p. \tag{1.27}$$

The nonlinear terms of (1.22) are given by (1.9)–(1.13) and divergence-curl structure by

$$F_u(u, E) \equiv \mathrm{div}_E\Big(J(D_E u)^{-1}\big(J(D_E u)^{-1}\big)^{\mathsf{T}} \nabla_E u - \nabla_E u\Big), \tag{1.28}$$

$$F_p(u, p, E) = -\big(J(DE)^{-1}\big)^{\mathsf{T}} \nabla\Big(\big(J(D_E u)^{-1} - I\big)^{\mathsf{T}} p\Big) = -\mathrm{div}_E\big(J(D_E u)^{-1} - I\big)p\Big), \tag{1.29}$$

$$G_{\mathrm{div}}(u, E) \equiv -\operatorname{tr}\big(J(DE)^{-1}\big)^{\mathsf{T}} \nabla\Big((J(D_E u)^{-1} - I)^{\mathsf{T}} u\Big) = -\mathrm{div}_E\big(J(D_E u)^{-1} - I\big)u\Big), \tag{1.30}$$

$$H_u(u, E) \equiv -\Big(\big(J(D_E u)^{-1}\big)^{\mathsf{T}} \nabla_E u + (\nabla_E u)^{\mathsf{T}} \big(J(D_E u)^{-1}\big)\Big)\big(J(D_E u)^{-1} - I\big)^{\mathsf{T}} \nu_E$$
$$- \Big(\big(J(D_E u)^{-1} - I\big)^{\mathsf{T}} \nabla_E u + (\nabla_E u)^{\mathsf{T}} \big(J(D_E u)^{-1} - I\big)\Big)\nu_E, \tag{1.31}$$

$$H_p(u, p, E) \equiv p\big(J(D_E u)^{-1} - I\big)^{\mathsf{T}} \nu_E. \tag{1.32}$$

Here $\nu_E = (\nabla' E, -1)^{\mathsf{T}}/\sqrt{1+|\nabla' E|^2}$. The initial data u_0 satisfies the natural condition $\operatorname{div} u_0 = g(u, E)\big|_{t=0}$ in the sense of distributions. The notations ∇_E, div_E (and hence D_E) are defined by (1.18) and (1.19), respectively and $J(D_E u)^{-1}$ denotes the inverse of the Jacobi matrix and $J(DE)^{-1}$ is given by (1.17). Hereafter, we denote $\Pi_*^m(A)$ as a polynomial of A of order at most $m = n-1$ or $2n-2$ with $*$ being either u, p, div or bu or bp which indicates the nonlinear terms (1.28)–(1.32). At the above stage, the problem (1.1) is transformed into the fixed and the flat boundary domain with the quasilinear variable coefficient problem.

2 Main results

Before stating our results, we define the Besov space and the Lizorkin–Triebel space in the half-space and on the half-line, respectively (see for details Peetre [42], Triebel [65]).

Definition (The Besov spaces). Let $s \in \mathbb{R}$, $1 \le p, \sigma \le \infty$. Let $\{\phi_j\}_{j\in\mathbb{Z}}$ be the Littlewood–Paley dyadic decomposition of unity for $x \in \mathbb{R}^n$, i.e., $\widehat{\phi}$ is the Fourier transform of a smooth radial function ϕ satisfying $\widehat{\phi}(\xi) \ge 0$, $\operatorname{supp}\widehat{\phi} \subset \{\xi \in \mathbb{R}^n \mid 2^{-1} < |\xi| < 2\}$, and $\widehat{\phi_j}(\xi) = \widehat{\phi}(2^{-j}\xi)$, $\sum_{j\in\mathbb{Z}} \widehat{\phi_j}(\xi) = 1$ for any $\xi \in \mathbb{R}^n \setminus \{0\}$ for $j \in \mathbb{Z}$, and $\widehat{\phi_0}(\xi) + \sum_{j\ge 1} \widehat{\phi_j}(\xi) = 1$ for any $\xi \in \mathbb{R}^n$, where $\widehat{\phi_{\widehat{0}}}(\xi) \equiv \widehat{\zeta}(|\xi|)$ with a low frequency cut-off $\widehat{\zeta}(r) = 1$ for $0 \le r < 1$ and $\widehat{\zeta}(r) = 0$ for $2 < r$ (see [9]). For

$s \in \mathbb{R}$ and $1 \le p, \sigma \le \infty$, let $\dot{B}^s_{p,\sigma}(\mathbb{R}^n)$ be the homogeneous Besov space with the norm

$$\|\tilde{f}\|_{\dot{B}^s_{p,\sigma}} \equiv \begin{cases} \Big(\sum_{j\in\mathbb{Z}} 2^{s\sigma j}\|\phi_j * \tilde{f}\|_p^\sigma\Big)^{1/\sigma}, & 1 \le \sigma < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{sj}\|\phi_j * \tilde{f}\|_p, & \sigma = \infty \end{cases}$$

and $B^s_{p,\sigma}(\mathbb{R}^n)$ be the inhomogeneous Besov space with the norm

$$\|\tilde{f}\|_{B^s_{p,\sigma}} \equiv \begin{cases} \Big(\|\phi_{\hat{0}} * \tilde{f}\|_p + \sum_{j\in\mathbb{Z}} 2^{s\sigma j}\|\phi_j * \tilde{f}\|_p^\sigma\Big)^{1/\sigma}, & 1 \le \sigma < \infty, \\ \|\phi_{\hat{0}} * \tilde{f}\|_p + \sup_{j\in\mathbb{Z}} 2^{sj}\|\phi_j * \tilde{f}\|_p, & \sigma = \infty. \end{cases}$$

We introduce the homogeneous Besov space on the half-Euclidean space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n; x = (x', x_n), x_n > 0, x' \in \mathbb{R}^{n-1}\}$: $\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)$ as the set of all measurable functions f in $\mathbb{R}^n_+$ satisfying

$$\|f\|_{\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)} \equiv \inf \left\{ \|\tilde{f}\|_{\dot{B}^s_{p,\sigma}(\mathbb{R}^n)} < \infty; \quad \begin{array}{c} \tilde{f} = \begin{Bmatrix} f(x', x_n) & (x_n > 0) \\ \text{a proper extension} & (x_n \le 0) \end{Bmatrix}, \\ \tilde{f} = c_n^{-1} \sum_{j\in\mathbb{Z}} \phi_j * \tilde{f} \text{ in } \mathcal{S}'(\mathbb{R}^n) \end{array} \right\}, \tag{2.1}$$

where $c_n^{-1} = (2\pi)^{n/2}$. The inhomogeneous version $B^s_{p,\sigma}(\mathbb{R}^n_+)$ is analogously defined.

Definition (The Bochner–Lizorkin–Triebel spaces). Let $s \in \mathbb{R}$ and $X(\mathbb{R}^n_+)$ be a Banach space on $\mathbb{R}^n_+$ with the norm $\|\cdot\|_X$. Let $\{\psi_k\}_{k\in\mathbb{Z}}$ be the Littlewood–Paley dyadic decomposition of unity for $t \in \mathbb{R}$. For a Banach space X, let $\dot{F}^s_{1,1}(\mathbb{R}; X)$ be the Bochner–Lizorkin–Triebel space ([31, 64]) with the norm

$$\|\tilde{f}\|_{\dot{F}^s_{1,1}(\mathbb{R};X)} \equiv \Big\| \sum_{k\in\mathbb{Z}} 2^{s\sigma k}\|\psi_k * \tilde{f}(t,\cdot)\|_X \Big\|_{L^1(\mathbb{R}_t)}.$$

Analogously as above, we define the Bochner–Lizorkin–Triebel spaces $\dot{F}^s_{1,1}(I; X)$ for an interval $I = (0, T)$ $(T \le \infty)$ as the set of all measurable functions f on X satisfying

$$\|f\|_{\dot{F}^s_{1,1}(I;X)} \equiv \inf \left\{ \|\tilde{f}\|_{\dot{F}^s_{1,1}(\mathbb{R};X)} < \infty; \quad \tilde{f} = \begin{Bmatrix} f(t,x) & (t \in I) \\ \text{a proper extension} & (t \in \mathbb{R}\setminus I) \end{Bmatrix} \right\}.$$

We should like to notice that $\dot{F}^s_{1,1}(\mathbb{R}_+; X)$ is equivalent to $\dot{B}^s_{1,1}(\mathbb{R}_+; X)$ from its definition.

Let $\mathcal{D}'(\Omega)$ denote the distributions over Ω and let $C_b(I; X)$ be a set of all bounded continuous functions from an interval I to a Banach space X. We also use $C_v(\mathbb{R}^n_+)$ (or $C_v(\mathbb{R}^{n-1})$) as a set of all continuous functions vanishing at $|x| \to \infty$.

Theorem 2.1 (Global well-posedness of the transformed problem) *Let $n \geq 2$, $n-1 < p < 2n-1$ and $1 \leq q \leq p(n-1)/(n-p)$ $(n-1 < p < n)$ and $1 \leq q < p(n-1)/(p-n)$ $(n \leq p < 2n-1)$. If the initial data $u_0 \in \dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+)$ with the condition $\operatorname{div} u_0 = g(u, E)\big|_{t=0}$ in $\mathcal{D}'(\mathbb{R}^n_+)$ and the initial boundary $\eta_0 \in \dot{B}^{1+(n-1)/q}_{q,1}(\mathbb{R}^{n-1})$ satisfy for some small $\varepsilon_0 > 0$ that*

$$\|u_0\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)} + \|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})} \leq \varepsilon_0, \tag{2.2}$$

then the initial boundary value problem (1.22) admits a unique global solution

$$u \in C_b(\overline{\mathbb{R}_+}; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)) \cap \dot{W}^{1,1}(\mathbb{R}_+; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)), \quad \Delta u, \nabla p \in L^1(\mathbb{R}_+; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)),$$
$$p|_{x_n=0} \in \dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1})) \cap L^1(\mathbb{R}_+; \dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))$$

with the estimate

$$\begin{aligned}\left\|\partial_t u\right\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \left\|D^2 u\right\|_{L^1(\mathbb{R}_+;\dot{B}^{1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \left\|\nabla p\right\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\ + \left\|p|_{x_n=0}\right\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))} + \left\|p|_{x_n=0}\right\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))} \leq \varepsilon_1,\end{aligned} \tag{2.3}$$

where $D^2 u = \partial_i \partial_j u$ $(i, j = 1, \ldots n)$ and $\varepsilon_1 = \varepsilon_1(n, p, \varepsilon_0)$ is a constant.

Remark Since our regularity class is the scaling invariant, the Fujita–Kato principle (cf. [23]) implies that the local well-posedness for the problem (1.22) also holds for the condition (2.2) being assumed only for the surface function η_0. We also note that our result above includes the case $n = 2$ that does not seem to be included in the earlier result [15] on the free surface problem in an unbounded domain.

For the regularity of the initial surface $\eta_0 \in \dot{B}^{1+(n-1)/q}_{q,1}(\mathbb{R}^{n-1})$, the exponent q can be taken independently of the regularity exponent p for the velocity field and the pressure and restricted by the limitation of the boundary bilinear estimate (see Proposition 10.3 (3) in Appendix). Under the restriction of η_0, its mean value over $\mathbb{R}^{n-1}$ of η_0 is vanishing if it is integrable and $\nabla'\eta_0 \in \dot{B}^{(n-1)/q}_{q,1}(\mathbb{R}^{n-1}) \subset C_v(\mathbb{R}^{n-1})$ (cf. [15]).

Accordingly the original problem is considered to be solvable in the corresponding critical space if we introduce the space of a pull back of functions by observing the ordinary differential equation (1.5) is uniquely solvable. Let $\mathcal{E}$ be defined from $\Omega \to \mathbb{R}^n_+$ by (1.16).

Corollary 2.2 (Global well-posedness of the non-flat fixed boundary problem) *Let $n \geq 2$ and $n-1 < p < 2n-1$. For the same ε_0 in Theorem 2.1 and $\bar{u}_0 \circ \mathcal{E}^{-1} = u_0 \in \dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+)$ with* $\operatorname{div} \bar{u}_0 = 0$ *in $\mathcal{D}'(\Omega)$ satisfying (2.2), let (u, p) be the global solution of (1.22) obtained in Theorem 2.1. Then the pull-back $(\tilde{u}, \tilde{p})$ of (u, p) via the transformation (1.16) with the estimate (2.3) satisfies (1.8).*

Corollary 2.3 (Global well-posedness of the free boundary problem (1.1)) *Let $n \geq 2$ and $n-1 < p < 2n-1$. For the same ε_0 in Theorem 2.1 and $\bar{u}_0 \circ \mathcal{E}^{-1} \in \dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+)$ with* $\operatorname{div} \bar{u}_0 = 0$ *in $\mathcal{D}'(\Omega)$ satisfying (2.2), let $(\tilde{u}, \tilde{p}, \tilde{\eta})$ be the global solution of (1.8) obtained in Corollary 2.2. Then the pull-back $(\bar{u}, \bar{p}, \bar{\eta})$ of $(\tilde{u}, \tilde{p}, \tilde{\eta})$ given in (1.7) via the transformation (1.6) uniquely solves the original problem (1.1).*

2.1 Maximal L^1-regularity for the linearized Stokes equations

In order to show the global well-posedness of the Navier–Stokes equations by the Lagrange coordinate form (1.22), maximal L^1-regularity for the heat equation with the Neumann boundary condition plays a crucial role. We consider a corresponding regularity estimate to the initial-boundary value problem of the Stokes equations with free stress boundary condition:

$$\begin{cases} \partial_t u - \Delta u + \nabla p = f, & t > 0, \quad x \in \mathbb{R}^n_+, \\ \operatorname{div} u = g, & t > 0, \quad x \in \mathbb{R}^n_+, \\ \big(\nabla u + (\nabla u)^{\mathsf{T}} - pI\big)\, \nu_n = h, & t > 0, \quad x' \in \mathbb{R}^{n-1}, \\ u(0, x) = u_0(x), & \qquad\quad x \in \mathbb{R}^n_+, \end{cases} \tag{2.4}$$

where u_0, f, g and h are given initial, external and boundary data, respectively and ν_n denotes the outer normal on $\partial\mathbb{R}^n_+$. The following theorem improves the former result on maximal L^1-regularity with a free boundary value problem in Ogawa–Shimizu [39].

Theorem 2.4 (Maximal L^1-regularity for the Stokes system) *Let $1 < p < \infty$ and $-1 + 1/p < s < 1/p$. The problem (2.4) admits a unique solution (u, p) with*

$$\begin{aligned} &u \in C_b(\overline{\mathbb{R}_+}; \dot{B}^s_{p,1}(\mathbb{R}^n_+)) \cap \dot{W}^{1,1}(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+)), \quad \Delta u, \nabla p \in L^1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+)), \\ &p\big|_{x_n=0} \in \dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^{n-1})) \cap L^1(\mathbb{R}_+; \dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1})) \end{aligned}$$

if and only if the data in (2.4) satisfy

$$\begin{aligned} &u_0 \in \dot{B}^s_{p,1}(\mathbb{R}^n_+), \quad f \in L^1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+)), \\ &\operatorname{div} u_0 = g\big|_{t=0} \ \text{in } \mathcal{D}'(\mathbb{R}^n_+), \\ &\nabla g \in L^1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+)), \quad \nabla(-\Delta)^{-1} g \in \dot{W}^{1,1}(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+)), \\ &h \in \dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^{n-1})) \cap L^1(\mathbb{R}_+; \dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1})), \end{aligned}$$

where $(-\Delta)^{-1}$ *denotes the inverse operator of the Laplacian with 0-Dirichlet boundary condition on* $\partial\mathbb{R}^n_+$*. Besides the solution* (u, p) *satisfies the following estimate for some constant* $C_M > 0$ *depending only on* p*,* s *and* n

$$\begin{aligned} &\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|D^2 u\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|\nabla p\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \\ &\qquad + \|p|_{x_n=0}\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))} + \|p|_{x_n=0}\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))} \\ &\quad \le C_M\Big(\|u_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)} + \|f\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \\ &\qquad + \|\nabla g\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|\partial_t \nabla(-\Delta)^{-1} g\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \\ &\qquad + \|h\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))} + \|h\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big). \end{aligned} \tag{2.5}$$

The above theorem is that the range of the differentiability exponent s is enlarged than our previous results [38] and [39]. Namely Theorem 2.4 includes the case $0 < s < 1/p$. Such an extension is established by reconsidering the detailed estimate for the linear heat equations. Indeed, there is no limitation for the upper bound of s from our explicit analysis in the subsequent theorems (Theorems 2.5, 4.1). The limitation is posed in order to make clear the condition of the homogeneous Besov setting (see Proposition 3.2 below). After establishing maximal L^1-regularity in the range $-1 + 1/p < s < 1/p$, the proof of the global well-posedness Theorem 2.1 follows by a reasonable structure of the Besov space that $\dot{B}^{n/p}_{p,1}(\mathbb{R}^n_+)$ is a Banach algebra and all the nonlinear terms can be estimated by such a structure, which can be seen in [15] and [39]. Note that the upper range of $p < 2n - 1$ is caused by the worst nonlinear estimate which arose from the boundary nonlinearity.

To establish maximal regularity on the half-space problem (2.4), we decompose the problem (2.4) into several partial components of the data and reduce the problem into the inhomogeneous problem with only boundary data being provided as we presented in the previous works [38, 39]. First we remove the divergence data g as in the proof of Theorem 2.1 in [39]. Introducing properly extended data $\widetilde{f} = (\bar{f}^o_1, \bar{f}^o_2, \ldots, \bar{f}^e_n)$ with $\bar{f}$ being the divergence term correction and $\bar{f}^o_\ell, \bar{f}^e_n$ $(\ell = 1, 2, \ldots n-1)$ denote the odd and even extension to $\mathbb{R}^n$, respectively, and $\widetilde{u_0}$ into $\mathbb{R}^n$ in the similar manner, we consider the Cauchy problem of the Stokes flow:

$$\begin{cases} \partial_t \widetilde{u} - \Delta\widetilde{u} + \nabla\widetilde{p} = \widetilde{f}, & t > 0, \quad x \in \mathbb{R}^n, \\ \operatorname{div} \widetilde{u} = 0, & t > 0, \quad x \in \mathbb{R}^n, \\ \widetilde{u}(0, x) = \widetilde{u_0}(x), & \qquad\quad\; x \in \mathbb{R}^n. \end{cases} \tag{2.6}$$

Thanks to extension of $\widetilde{f}$ and $\widetilde{u_0}$ we notice that $\widetilde{p}(x', 0) = 0$ by the setting of the problem (2.6) (cf. [50, (4.21)]). Then by subtracting the solution of (2.6) from the the original problem (2.4), one can reduce the problem to the following initial boundary value problem for (v, q) with inhomogeneous boundary data:

$$\begin{cases} \partial_t v - \Delta v + \nabla q = 0, & t > 0, \quad x \in \mathbb{R}^n_+, \\ \operatorname{div} v = 0, & t > 0, \quad x \in \mathbb{R}^n_+, \\ \big((\nabla v) + (\nabla v)^{\mathsf{T}} - qI\big)\, \nu_n = H, & t > 0, \quad x' \in \mathbb{R}^{n-1}, \\ v(0, x) = 0, & x \in \mathbb{R}^n_+, \end{cases} \tag{2.7}$$

where we set

$$H \equiv \widetilde{h} - \big(\nabla\widetilde{u} + (\nabla\widetilde{u})^{\mathsf{T}}\big)\nu_n|_{x_n=0}. \tag{2.8}$$

In order to prove Theorem 2.4, it is essential to show maximal L^1-regularity for (2.7). The following estimate is obtained partially in [15] and [39].

Theorem 2.5 *Let* $1 < p < \infty$ *and* $-1 + 1/p < s < 1/p$. *The problem* (2.7) *admits a unique solution*

$$v \in C_b(\overline{\mathbb{R}_+}; \dot{B}^s_{p,1}(\mathbb{R}^n_+)) \cap \dot{W}^{1,1}(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+)), \quad \Delta v,\ \nabla q \in L^1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+)),$$
$$q|_{x_n=0} \in \dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^{n-1})) \cap L^1(\mathbb{R}_+; \dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))$$

if and only if the data in (2.7) *satisfy*

$$H \in \dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^{n-1})) \cap L^1(\mathbb{R}_+; \dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1})). \tag{2.9}$$

Besides the solution (v, q) *satisfies the following estimate for some constant* $C_M > 0$ *depending only on* p, s *and* n*:*

$$\begin{aligned} &\|\partial_t v\|_{L^1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|D^2 v\|_{L^1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|\nabla q\|_{L^1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+))} \\ &\quad + \|q|_{x_n=0}\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^{n-1}))} + \|q|_{x_n=0}\|_{L^1(\mathbb{R}_+; \dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))} \\ &\le C\Big(\|H\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^{n-1}))} + \|H\|_{L^1(\mathbb{R}_+; \dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big). \end{aligned} \tag{2.10}$$

The function class connected to the x-variable in Theorem 2.5 is restricted in $\dot{B}^s_{p,1}(\mathbb{R}^n_+) \subsetneq \dot{W}^{s,p}(\mathbb{R}^n_+)$ and such a restriction is necessary for maximal L^1-regularity; maximal L^1-regularity fails for the Lebesgue spaces L^p even over the whole space $\mathbb{R}^n$ in general (see [35]. See also a possible estimate Giga–Saal [24]). On the other hand, the condition (2.9) is sufficient to conclude the estimate (2.10) in Theorem 2.5 even for the end-point spatial exponent $p = 1$. This end-point is excluded because the trace estimate fails when $p = 1$.

The rest of this paper is organized as follows. After preparing basic relations in the Besov space in the half-space $\mathbb{R}^n_+$ in the next section, we present a basic formulation for the proof in particular the reduction to the boundary value problems of the heat equations with the Neumann boundary condition and the Stokes equations with the non-stress boundary condition in Sect. 4. We construct an explicit solution formula of the fundamental solutions in Sect. 5. In Sect. 6, we recall the linear boundary estimate of inhomogeneous Neumann type and in Sect. 7 maximal L^1 regularity for the Stokes system is shown. Section 8 is devoted to the linear and nonlinear perturbation estimates and Sect. 9 shows the proof of the global well-posedness for the transformed Navier–Stokes equations. The final section Appendix includes various bilinear estimates.

Throughout this paper, we use the following notations. For $x \in \mathbb{R}^n$, $\langle x\rangle \equiv (1+|x|^2)^{1/2}$. The boundary $\partial\mathbb{R}^n_+$ is denoted by $\mathbb{R}^{n-1}$ for the variables $x' = (x_1, x_2, \dots, x_{n-1})$. The transpose of a matrix A is denoted by A^{T}. The Fourier and the inverse Fourier transforms of $f \in \mathcal{S}(\mathbb{R}^n)$ are defined with $c_n = (2\pi)^{-n/2}$ by

$$\widehat{f}(\xi) = \mathcal{F}[f](\xi) \equiv c_n \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx, \quad \mathcal{F}^{-1}[f](x) \equiv c_n \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)d\xi.$$

For any functions $f = f(t, x', x_n)$ and $g = g(t, x', x_n)$, $f \underset{(t)}{*} g$, $f \underset{(t,x')}{*} g$ and $f \underset{(x_n)}{*} g$ stand for the convolution between f and g with respect to the variable indicated under $*$, respectively. If both f and g are vector field functions, $f \underset{(t,x')}{\cdot *} g$ denotes the convolution in x' as well as the inner product of f and g, i.e.,

$$f \underset{(t,x')}{\cdot *} g = \sum_{\ell=1}^{n-1} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} f_\ell(t-s, x'-y') g_\ell(s, y') dy' ds. \tag{2.11}$$

In the summation $\sum_{k\in\mathbb{Z}}$, the parameter k runs for all integers $k \in \mathbb{Z}$ and for $\sum_{k\le j}$, k runs for all integers less than or equal to $j \in \mathbb{Z}$. We denote the norm of $L^p(\mathbb{R}^{n-1})$ with $x' \in \mathbb{R}^{n-1}$ variable by $\|\cdot\|_{L^p_{x'}}$. Let $L^\rho(I; X)$ denotes the ρ-th powered Lebesgue–Bochner space upon a Banach space X. The norm for the Bochner–Lizorkin–Triebel spaces on $\dot{F}^s_{p,\rho}\big(I; X(\mathbb{R}^{n-1})\big)$ we use

$$\|f\|_{\dot{F}^s_{p,\rho}(I;X)} = \|f\|_{\dot{F}^s_{p,\rho}(I;X(\mathbb{R}^{n-1}))}$$

unless it may cause any confusion. For the Besov spaces, we abbreviate $\mathbb{R}^n$ for $\dot{B}^s_{p,\sigma} = \dot{B}^s_{p,\sigma}(\mathbb{R}^n)$ and its norm $\|\cdot\|_{\dot{B}^s_{p,\sigma}}$. For $a \in \mathbb{R}^n$, we denote $B_R(a)$ as the open ball centered at a with its radius $R > 0$. We also denote the complement of $B_R(0)$ by B_R^c. $\Gamma(\cdot)$ denotes the Gamma function. Various constants are simply denoted by C unless otherwise stated.

3 The homogeneous Besov space in the half-space

3.1 The homogeneous Besov spaces on the half-space

We recall the summary for the Besov spaces over a domain Ω near the half-Euclidean space $\mathbb{R}^n_+$. Let $\ell_0 = \{\{a_k\}_k; k \in \mathbb{Z}, \;\; a_k \in \mathbb{R}, \;\; \lim_{|k|\to\infty} |a_k| = 0, \;\; \|\{a_k\}_k\|_{\ell_0} = \max_k |a_k|\} \subsetneq \ell_\infty$. It is well known that $(\ell_0)^* \simeq \ell_1$.

Definition Let $\sigma = 0$ or $1 \le \sigma < \infty$ with $s \in \mathbb{R}$. Let

$$\dot{\mathcal{B}}^s_{\infty,\sigma}(\mathbb{R}^n) \equiv \overline{C_0^\infty(\mathbb{R}^n_+)}^{\dot{B}^s_{\infty,\sigma}(\mathbb{R}^n)},$$
$$\dot{\mathcal{B}}^s_{\infty,0}(\mathbb{R}^n) \equiv \overline{C_0^\infty(\mathbb{R}^n_+)}^{\dot{B}^s_{\infty,0}(\mathbb{R}^n)}, \quad \text{where } \|f\|_{\dot{B}^s_{\infty,0}} \equiv \left\|2^{sk}\|\phi_k * f\|_\infty\right\|_{\ell_0}.$$

Definition Let $1 \le p < \infty$ and $1 \le \sigma < \infty$ with $s \in \mathbb{R}$.

$$\overset{\circ}{B}{}^s_{p,\sigma}(\mathbb{R}^n_+) \equiv \overline{C_0^\infty(\mathbb{R}^n_+)}^{\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)}$$

by the Besov norm $\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)$ (see Bahouri-Chemin-Danchin [4] and Bergh–Löfström [9]). It is shown that the above defined space coincides the space $\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)$ defined by the restriction in (2.1). Namely, the following proposition is shown by Triebel [65] and Danchin–Mucha [16] (see also [28, 38]).

Proposition 3.1 [16, 65] *Let* $1 \le p < \infty$. (1) *For* $0 \le s$, $1 \le \sigma < \infty$,

$$\dot{B}^{-s}_{p',\sigma'}(\mathbb{R}^n_+) \simeq \big(\overset{\circ}{B}{}^s_{p,\sigma}(\mathbb{R}^n_+)\big)^*,$$
$$\dot{B}^{-s}_{1,1}(\mathbb{R}^n_+) \simeq \big(\dot{\mathcal{B}}^s_{\infty,0}(\mathbb{R}^n_+)\big)^*.$$

(2) *For* $-\infty < s \le 1/p$ *and for* $1 < \sigma < \infty$,

$$\overset{\circ}{B}{}^s_{p,\sigma}(\mathbb{R}^n_+) \simeq \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+).$$

(3) *For* $-\infty < s < 1/p$ *and* $\sigma = 1$,

$$\overset{\circ}{B}{}^s_{p,1}(\mathbb{R}^n_+) \simeq \dot{B}^s_{p,1}(\mathbb{R}^n_+).$$

We consider the restriction operator R_0 by multiplying a cut-off function $\chi_{\mathbb{R}^n_+}(x) = 1$ over $\mathbb{R}^n_+$ and otherwise 0, i.e., for $f \in \dot{B}^s_{p,\sigma}(\mathbb{R}^n)$ with setting $R_0 f = \chi_{\mathbb{R}^n_+}(x) f(x)$ in $\dot{B}^s_{p,\sigma}(\mathbb{R}^n)$ if $s > 0$ and it is understood in a distributional sense. Let E_0 be the zero extension operator from $\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)$ to $\dot{B}^s_{p,\sigma}(\mathbb{R}^n)$. Using Proposition 3.1, the following statement is a variant introduced by Triebel [65, p. 228].

Proposition 3.2 *Let $1 \le p < \infty$, $1 \le \sigma < \infty$ and $-1 + 1/p < s < 1/p$. It holds that*

$$\begin{aligned} R_0 &: \dot{B}^s_{p,\sigma}(\mathbb{R}^n) \to \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+), \\ E_0 &: \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+) \to \dot{B}^s_{p,\sigma}(\mathbb{R}^n), \end{aligned} \tag{3.1}$$

are linear bounded operators. Besides it holds that

$$R_0 E_0 = Id : \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+) \to \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+),$$

where Id denotes the identity operator. Namely E_0 and R_0 are a retraction and a co-retraction, respectively.

The proof of Proposition 3.2 is along the same line of the proof in [65] (cf. [38]). Note that the spaces are homogeneous Besov spaces and then the arrangement appears in Proposition 3 in Danchin–Mucha [16] is required. Furthermore, Triebel [65, Theorem 2.9.1] states that

Proposition 3.3 (cf. [15, 65]) *Let $1 \le p < \infty$ and $s \in \mathbb{R}$, $f \in \dot{B}^{s+1}_{p,1}(\mathbb{R}^n_+)$ then $\nabla f \in \dot{B}^s_{p,1}(\mathbb{R}^n_+)$ and hence $\nabla f \in \overset{\circ}{B}{}^s_{p,1}(\mathbb{R}^n_+)$ if $s < 1/p$. Conversely if $\nabla f \in \dot{B}^s_{p,1}(\mathbb{R}^n_+)$ then $f \in \dot{B}^{s+1}_{p,1}(\mathbb{R}^n_+)$ if $-1 + 1/p < s \le -1 + n/p$.*

In what follows, we restrict ourselves to the regularity range of the Besov spaces $\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)$ in $-1 + 1/p < s < 1/p$ for $1 < p < \infty$ unless otherwise stated. According to Proposition 3.2, we may regard that any distribution in $\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)$ under such restriction on s and p can be extended into a distribution over whole space $\mathbb{R}^n$ and conversely.

3.2 The L-P decomposition with a separation of variables

In order to split the variables $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}_+$, we introduce an x'-parallel decomposition and an x_n-parallel decomposition by Littlewood–Paley type. We introduce $\{\overline{\Phi_m}\}_{m\in\mathbb{Z}}$ as a Littlewood–Paley dyadic frequency decomposition of unity in separated variables $(\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

Definition (The Littlewood–Paley decomposition of separated variables). For $m \in \mathbb{Z}$, let

$$\widehat{\zeta_m}(\xi_n) = \begin{cases} 1, & 0 \le |\xi_n| \le 2^m, \\ \text{smooth}, & 2^m \le |\xi_n| \le 2^{m+1}, \\ 0, & 2^{m+1} \le |\xi_n|, \end{cases} \qquad \widehat{\zeta_m}(\xi_n) = \widehat{\zeta_{m-1}}(\xi_n) + \widehat{\phi_m}(\xi_n) \tag{3.2}$$

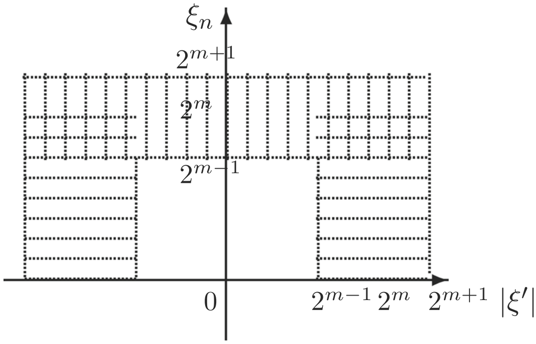


Fig. 1 The support of Littlewood–Paley decomposition $\{\overline{\Phi_m}\}_{m\in\mathbb{Z}}$

(one can choose $\widehat{\zeta_m}(r) = \sum_{\ell\le m-1}\widehat{\phi_\ell}(r) + \widehat{\phi_{-\infty}}(r)$ with a correction distribution $\widehat{\phi_{-\infty}}(r)$ at $r=0$) and set

$$\widehat{\overline{\Phi_m}}(\xi) \equiv \widehat{\phi_m}(\xi') \otimes \widehat{\zeta_{m-1}}(\xi_n) + \widehat{\zeta_m}(|\xi'|) \otimes \widehat{\phi_m}(\xi_n). \tag{3.3}$$

Then it is obvious from Fig. 1 (restricted on the upper half region in $\mathbb{R}^n$) that

$$\sum_{m\in\mathbb{Z}} \widehat{\overline{\Phi_m}}(\xi) \equiv 1, \quad \xi = (\xi', \xi_n) \in \mathbb{R}^n \setminus \{0\}. \tag{3.4}$$

Definition (Various kinds of the Littlewood–Paley dyadic decompositions).

Let $(\tau, \xi', \xi_n) \in \mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}$ be Fourier adjoint variables corresponding to $(t, x', x_n) \in \mathbb{R}_+ \times \mathbb{R}^{n-1} \times \mathbb{R}_+$. Let $\{\Phi_m(x)\}_{m\in\mathbb{Z}}$ be the standard (supported in annulus) Littlewood–Paley dyadic decomposition by
$x = (x', x_n) \in \mathbb{R}^n_+$.

- $\{\overline{\Phi_m}(x)\}_{m\in\mathbb{Z}}$: the Littlewood–Paley dyadic decomposition over $x = (x', x_n) \in \mathbb{R}^n_+$ given by (3.3).
- $\{\psi_k(\tilde{t})\}_{k\in\mathbb{Z}}$: the Littlewood–Paley dyadic decompositions in $\tilde{t} \in \mathbb{R}$.
- $\{\phi_j(x')\}_{j\in\mathbb{Z}}$ and $\{\phi_j(\tilde{x}_n)\}_{j\in\mathbb{Z}}$: the standard (annulus type) Littlewood–Paley dyadic decompositions in $x' \in \mathbb{R}^{n-1}$ and $\tilde{x}_n \in \mathbb{R}$, respectively.
- $\{\zeta_m(x')\}_{m\in\mathbb{Z}}$ and $\{\zeta_m(\tilde{x}_n)\}_{m\in\mathbb{Z}}$: the lower frequency smooth cut-off given by (3.2), respectively.
- For the Littlewood–Paley decompositions $\{\phi_j(x')\}_{j\in\mathbb{Z}}$ and $\{\psi_k(t)\}_{k\in\mathbb{Z}}$, we set

$$\begin{cases} \widetilde{\phi_j} = \phi_{j-1} + \phi_j + \phi_{j+1}, \\ \widetilde{\psi_k} = \psi_{k-1} + \psi_k + \psi_{k+1} \end{cases} \tag{3.5}$$

  that stands for the j-neighborhood of $\phi_j(x')$ and the k-neighborhood of $\psi_k(t)$, respectively.

Since all the above defined decompositions are even functions, we identify $\tilde{t} \in \mathbb{R}$ and $\tilde{x}_n \in \mathbb{R}$ with $|\tilde{t}| = t > 0$ and $|\tilde{x}_n| = x_n > 0$, respectively. Then it is easy to see that the norm of the Besov spaces on $\mathbb{R}^n_+$ defined by $\{\Phi_m\}_m$ is equivalent to the one from the Littlewood–Paley decomposition of direct sum type $\{\overline{\Phi_m}\}_m$.

4 The initial boundary value problem for the Stokes equations

In this section, we study the solution formula for the initial boundary value problem of the heat equation with the Neumann boundary value problem. The formula is a basis to consider the solution formula to the initial boundary value problem of the Stokes equations.

4.1 The initial Neumann boundary value problem for the heat equation

For $I = (0, T)$ with $0 < T \le \infty$, let u be a solution of the initial-boundary value problem of the second-order parabolic equation with variable coefficients and the inhomogeneous Neumann boundary condition in the half-space $\mathbb{R}^n_+ = \{x = (x', x_n); x' \in \mathbb{R}^{n-1}, x_n > 0\}$:

$$\begin{cases} \partial_t u - \Delta u = f, & t \in I, \quad x \in \mathbb{R}^n_+, \\ \partial_n u\big|_{x_n=0} = g, & t \in I, \quad x' \in \mathbb{R}^{n-1}, \\ u(t,x)\big|_{t=0} = u_0(x), & \qquad\quad x \in \mathbb{R}^n_+, \end{cases} \tag{4.1}$$

where ∂_t and $\partial_i = \partial_{x_i}$ are partial derivatives with respect to t and x_i, $u = u(t,x)$ denotes the unknown function, $u_0 = u_0(x)$, $f = f(t,x)$ and $g = g(t,x')$ are given initial, external force and boundary data, respectively.

In this context, Weidemaier [69] and Denk–Hieber–Prüss [20, 21] obtained maximal regularity in general settings. Let $I = (0, T)$ for $T < \infty$, $1 < \rho, p < \infty$ and $1/2 - 1/(2p) \neq 1/\rho$. The initial-boundary value problem (4.1) has a unique solution u in $W^{1,\rho}(\mathbb{R}_+; L^p(\mathbb{R}^n_+)) \cap L^\rho(\mathbb{R}_+; W^{2,p}(\mathbb{R}^n_+))$ with the compatibility condition

$$(\partial_n u_0)(x', x_n)|_{x_n=0} = g(t,x')|_{t=0}, \text{ under } \frac{1}{2} - \frac{1}{2p} > \frac{1}{\rho}. \tag{4.2}$$

and the solution fulfills the estimate:

$$\begin{aligned} &\|\partial_t u\|_{L^\rho(I;L^p(\mathbb{R}^n_+))} + \|D^2 u\|_{L^\rho(I;L^p(\mathbb{R}^n_+))} \\ &\quad \le C_T \Big(\|u_0\|_{B^{2(1-1/2p)}_{p,\rho}(\mathbb{R}^n_+)} + \|f\|_{L^\rho(I;L^p(\mathbb{R}^n_+))} + \|g\|_{F^{1/2-1/2p}_{\rho,p}(I;L^p(\mathbb{R}^{n-1}))} + \|g\|_{L^\rho(I;B^{1-1/p}_{p,p}(\mathbb{R}^{n-1}))} \Big), \end{aligned}$$

where $B^{2-1/p}_{p,p}(\mathbb{R}^{n-1})$ and $F^{1-1/2p}_{\rho,p}(I; X)$ denote the interpolation spaces of the Besov and Lizorkin–Triebel type, respectively. The end-point case $\rho = 1$ is considered in Ogawa–Shimizu [38] both with Dirichlet and Neumann boundary value problems in $-1 + 1/p < s \le 0$.

Theorem 4.1 (The Neumann boundary condition) *Let $1 < p < \infty$, $-1 + 1/p < s < 1/p$. Then the problem (4.1) admits a unique solution*

$$u \in C_b\big([0,T); \dot{B}^s_{p,1}(\mathbb{R}^n_+)\big) \cap \dot{W}^{1,1}(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+)), \quad \Delta u \in L^1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+)),$$

if and only if the external, initial and boundary data in (4.1) satisfy

$$\begin{aligned} &f \in L^1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+)), \quad u_0 \in \dot{B}^s_{p,1}(\mathbb{R}^n_+),\\ &g \in \dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^{n-1})) \cap L^1(\mathbb{R}_+; \dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1})), \end{aligned}$$

respectively. Moreover end-point maximal L^1-regularity holds:

$$\begin{aligned} &\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|D^2 u\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))}\\ &\quad \le C_M\big(\|u_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)} + \|f\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|g\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))} + \|g\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\big), \end{aligned}$$

where C_M is depending only on p, s and n.

Remarks. (i) The linear evolution generated by the Laplacian generates C_0-semigroup in $\dot{B}^s_{p,1}(\mathbb{R}^n_+)$ for $1 \le p < \infty$ and the estimate of maximal L^1-regularity ensures that the absolute continuity of the solution in t-variable.
(ii) Since $1/2 - 1/(2p) < 1$ for all $1 < p < \infty$, the pointwise compatibility condition (4.2) is not required.
(iii) If $p = \infty$, the corresponding result holds for the homogeneous Besov space

$$\dot{\mathcal{B}}^s_{\infty,1}(\mathbb{R}^n_+) \equiv \overline{C^\infty_0(\mathbb{R}^n_+)}^{\dot{B}^s_{\infty,1}(\mathbb{R}^n_+)}$$

instead of the Besov space $\dot{B}^s_{\infty,1}(\mathbb{R}^n_+)$. Note that $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^n_+) \subset C_{v,0}(\mathbb{R}^n_+) \equiv \{f \in C(\mathbb{R}^n_+);\ \mathrm{supp}\, f \subset \mathbb{R}^n_+,\ |f(x)| \to 0, \text{ as } |x| \to \infty, x \in \mathbb{R}^n_+\}$ for the endpoint case $(s, p) = (0, \infty)$.

We only show the estimate for the full time interval $\mathbb{R}_+$ but a similar estimate for the finite time interval $I = (0, T)$ with $T < \infty$ is also available. In such a case, the restriction on the initial data u_0 can be relaxed into the inhomogeneous Besov space $B^s_{p,1}(\mathbb{R}^n_+) \supset \dot{B}^s_{p,1}(\mathbb{R}^n_+)$ and the constant appeared in the estimate can be estimated as $C_M \simeq O(\log T)$ as $T \to \infty$.

For the proof of Theorem 4.1, the principal arugment is reduced into the following problem:

$$\begin{cases} \partial_t u - \Delta u = 0, & t \in I,\ x \in \mathbb{R}^n_+,\\ \partial_n u(t, x', x_n)\big|_{x_n=0} = h(t, x'), & t \in I,\ x' \in \mathbb{R}^{n-1},\\ u(t,x)\big|_{t=0} = 0, & x \in \mathbb{R}^n_+. \end{cases} \tag{4.3}$$

Then the following result yields our main result for the Neumann problem Theorem 4.1.

Theorem 4.2 (Maximal L^1-regularity by the Neumann boundary data) *Let $1 < p < \infty$ and $-1 + 1/p < s < 1/p$. There exists a unique solution*

$$u \in \dot{W}^{1,1}(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+)), \quad \Delta u \in L^1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$$

to (4.3) *if and only if*

$$h \in \dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}\big(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^{n-1})\big) \cap L^1\big(\mathbb{R}_+; \dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1})\big).$$

Besides it holds the estimate:

$$\begin{aligned}\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} &+ \|D^2 u\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))}\\ &\le C\big(\|h\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))} + \|h\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\big),\end{aligned}$$

where C is depending only on p, s and n.

When $p = \infty$, the analogous result holds under arranging the function class as in the remark after Theorem 4.1.

To show Theorem 4.2, we extend the boundary data $h(t, x')$ into $t < 0$ by the zero extension and apply the Laplace transform with respect to t, the partial Fourier transform with respect to x' and we obtain the solution formula of (4.3) as

$$\partial_t u(t, x', x_n) = \int_{\mathbb{R}_+}\int_{\mathbb{R}^n} \Psi_N(t-s, x'-y', x_n) h(s, y') dy' ds \tag{4.4}$$

by using the boundary potential term:

$$\Psi_N(t, x', x_n) = -\,\text{p.v.} c_{n+1} \int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} e^{it\tau + ix'\cdot\xi'} \frac{i\tau}{\sqrt{i\tau + |\xi'|^2}} e^{-\sqrt{\lambda+|\xi'|^2}x_n}\, d\xi' d\tau, \tag{4.5}$$

where $c_{n+1} = (2\pi)^{-(n+1)/2}$ and Γ is a pass parallel to the imaginary axis.

4.2 The solution formula for the Stokes equations

We construct the solution formula of (2.7) following the method by Shibata–Shimizu [50] and [52].

Let $H = H(t, x') \equiv (H'(t, x'), H_n(t, x'))$ be the boundary data extended into $t \le 0$ by the zero extension. Besides we assume that they are smooth and decay sufficiently fast at $|x'| \to \infty$. The solution formula for the ℓ-th component of the velocity and the pressure is obtained by Shibata–Shimizu [52, (5.19)] as follows: Letting

$$B(\tau, \xi') = \sqrt{i\tau + |\xi'|^2}, \tag{4.6}$$

$$D(\tau,\xi') = B(\tau,\xi')^3 + |\xi'|B(\tau,\xi')^2 + 3|\xi'|^2 B(\tau,\xi') - |\xi'|^3, \tag{4.7}$$

For any smooth rapidly decreasing boundary data $(\widehat{H'}, \widehat{H_n})$ in both (τ, ξ') variables, we consider that

$$\begin{aligned} &v_n(t,x',x_n) \\ &= c_{n+1}\text{p.v.}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} e^{it\tau+ix'\cdot\xi'}\left\{\frac{|\xi'|}{i\tau}\widehat{q}(\tau,\xi,x_n) + \frac{|\xi'|}{(B(\tau,\xi')-|\xi'|)D(\tau,\xi')}\right. \\ &\quad\times \left(2|\xi'|^2 - (B^2(\tau,\xi')+|\xi'|^2)\right)\begin{pmatrix} 0 & e^{-B(\tau,\xi')x_n} \\ e^{-B(\tau,\xi')x_n} & 0\end{pmatrix}\left.\begin{pmatrix}\frac{i\xi'}{|\xi'|}\cdot\widehat{H}' \\ \widehat{H}_n\end{pmatrix}\right\}d\xi' d\tau, \end{aligned} \tag{4.8}$$

$$\begin{aligned} &q(t,x',x_n) \\ &= c_{n+1}\text{p.v.}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} e^{it\tau+ix'\cdot\xi'}\left\{\frac{B(\tau,\xi')+|\xi'|}{D(\tau,\xi')}\right. \\ &\quad\times \left(2|\xi'|^2 - (B^2(\tau,\xi')+|\xi'|^2)\right)\begin{pmatrix} B(\tau,\xi')|\xi'|^{-1}e^{-|\xi'|x_n} & 0 \\ 0 & e^{-|\xi'|x_n}\end{pmatrix}\left.\begin{pmatrix}\frac{i\xi'}{|\xi'|}\cdot\widehat{H}' \\ \widehat{H}_n\end{pmatrix}\right\}d\xi' d\tau, \end{aligned} \tag{4.9}$$

where we take a limit of the integral pass avoiding the singularity at $(\tau, \xi') = (0, 0)$. All the other components of the velocity fields $v_\ell(t,x)$ $(\ell = 1, 2, \dots, n-1)$ are given by the above two components (v_n, q) and the boundary data $H = (H', H_n)$ from the Eq. (2.7) (see [52] for the detail of their derivation).

Our main task is to prove maximal L^1-regularity of the velocity v_n and the pressure term q of (2.7) which is directly obtained from the inhomogeneous boundary data. We also set symbols of the singular integral operator by the following Fourier multipliers: $m_*(\tau, \xi') : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^n$ as

$$\begin{aligned} m_\Psi(\tau,\xi') &= \left(m'_\Psi(\tau,\xi'), m_{\Psi,n}(\tau,\xi')\right) \\ &\equiv \frac{B(\tau,\xi')}{i\tau}\,\frac{(B(\tau,\xi')+|\xi'|)}{D(\tau,\xi')}\Big(-2(B^2(\tau,\xi')+|\xi'|^2)i\xi',\ 2|\xi'|^3\Big), \end{aligned} \tag{4.10}$$

$$\begin{aligned} m_\pi(\tau,\xi') &= \left(m'_\pi(\tau,\xi'), m_{\pi n}(\tau,\xi')\right) \\ &\equiv \frac{B(\tau,\xi')+|\xi'|}{D(\tau,\xi')}\Big(2i\xi' B(\tau,\xi'),\ -(B^2(\tau,\xi')+|\xi'|^2)\Big). \end{aligned} \tag{4.11}$$

Using the potential expression (4.8), we obtain a desired pressure estimate by the boundary data H. For any smooth data $(\widehat{H'}, \widehat{H_n})$ in both (τ, ξ') variables, we see the explicit expression of ∇q and the n-th component $\partial_t v_n$ can be expressed as

$$\begin{aligned} &\partial_t v_n(t,x',x_n) \\ &= c_{n+1}\text{p.v.}\iint_{\mathbb{R}^n} e^{it\tau+ix'\cdot\xi'}\Big\{ - \widehat{\partial_n q}(\tau,\xi,x_n) \end{aligned}$$

$$+\frac{i\tau}{B(\tau,\xi')}e^{-B(\tau,\xi')x_n}\big(m_\Psi(\tau,\xi')\cdot\widehat{H}(\tau,\xi')\big)\Big\}d\tau d\xi', \tag{4.12}$$

$$\begin{aligned}&\nabla q(t,x',x_n)\\&=c_{n+1}p.v.\iint_{\mathbb{R}^n}e^{it\tau+ix'\cdot\xi'}(i\xi',-|\xi'|)^{\mathsf{T}}\big(m_\pi(\tau,\xi')\cdot\widehat{H}(\tau,\xi')\big)e^{-|\xi'|x_n}d\tau d\xi',\end{aligned} \tag{4.13}$$

where $B=B(\tau,\xi')$ and $D(\tau,\xi')$ are defined by (4.6) and (4.7), respectively. Hence from (4.10), (4.12), (4.13), the term operated by the Laplacian is given by

$$\begin{aligned}&\Delta v_n(t,x',x_n)=c_{n+1}\text{p.v.}\\&\iint_{\mathbb{R}^n}e^{it\tau+ix'\cdot\xi'}\frac{i\tau}{B(\tau,\xi')}e^{-B(\tau,\xi')x_n}\big(m_\Psi(\tau,\xi')\cdot\widehat{H}(\tau,\xi')\big)d\tau d\xi'.\end{aligned} \tag{4.14}$$

For the construction of the explicit expression of the solution of (2.7) in [50, (4.24), (4.25)] and [52, (5.19)], the other components of the velocity fields $v'=(v_1(t,x),v_2(t,x),\dots,v_{n-1}(t,x))$ satisfy the initial boundary value problem of the heat equations as the pressure and the n-th component velocity as the external force and boundary condition as follows: For $\ell=1,2\dots,n-1$,

$$\begin{cases}\partial_t v_\ell-\Delta v_\ell=-\partial_\ell q, & t>0,\ x\in\mathbb{R}^n_+,\\ \partial_n v_\ell=-H_\ell-\partial_\ell v_n, & t>0,\ x\in\partial\mathbb{R}^n_+,\\ v_\ell(0,x)=0, & x\in\mathbb{R}^n_+.\end{cases} \tag{4.15}$$

Here we remark that v' in (4.15) and v_n in (4.8) satisfy the divergence free condition $\operatorname{div} v=0$. Since $\Delta q(t,x',x_n)=0$ by (4.9), we see from the problem (4.15) that

$$\begin{aligned}&v_\ell(t,x',x_n)\\&=c_{n+1}\text{p.v.}\iint_{\mathbb{R}^n}e^{it\tau+ix'\cdot\xi'}\Big\{-\frac{i\xi_\ell}{i\tau}\widehat{q}(\tau,\xi',x_n)+\frac{\widehat{H_\ell}}{B(\tau,\xi')}e^{-B(\tau,\xi')x_n}\\&\quad+\frac{i\xi_\ell}{(B(\tau,\xi')-|\xi'|)D(\tau,\xi')}\big(2B(\tau,\xi')|\xi'|\ \ B(\tau,\xi')^{-1}|\xi'|\big(B(\tau,\xi')^2+|\xi'|^2\big)-4|\xi'|^2\big)\\&\quad\times\begin{pmatrix}0 & e^{-B(\tau,\xi')x_n}\\ e^{-B(\tau,\xi')x_n} & 0\end{pmatrix}\begin{pmatrix}\frac{i\xi'}{|\xi'|}\cdot\widehat{H}'\\ \widehat{H}_n\end{pmatrix}\Big\}d\tau d\xi',\quad \ell=1,2,\dots,n-1,\end{aligned} \tag{4.16}$$

where we use the formulas (4.8)–(4.9) with a view of (4.5). Hence maximal L^1-regularity for the velocity v_ℓ can be reduced to the maximal regularity estimate for the initial Neumann boundary value problem of the heat equation in the half-space (4.1). We then turn into our attention to the initial boundary value problem of the heat equation with the Neumann boundary condition (cf. [38] and [37]).

5 The potential of boundary term and almost orthognality

5.1 The boundary potential

In this subsection, we derive the exact solution formula of (4.3) which is a bases of the solution formula to the velocity $v_n(t)$. Let $h = h(t, x')$ be the boundary data extended into $t < 0$ by the zero extension. The solution to the problem (4.3) is expressed by

$$u(t, x) = G_N(t, x) \underset{(x')}{*} h(t, x'), \tag{5.1}$$

where G_N denotes the Green's function of the initial-boundary value problem (4.3) identified by

$$G_N(t, x', x_n) = -\text{p.v.}c_{n+1} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{it\tau + ix'\cdot\xi'} \frac{1}{B(\tau, \xi')} e^{-B(\tau,\xi')x_n} d\xi' d\tau, \tag{5.2}$$

where $B(\tau, \xi') = \sqrt{i\tau + |\xi|^2}$ (cf. [38]). From (4.5) $\Psi_N(t, x', x_n) = \partial_t G_N(t, x', x_n)$, we regard x_n as if it is *a spectral parameter* like (λ, ξ'), we then decompose this boundary potential (4.5) by a combination of two families of the Littlewood–Paley dyadic decomposition of unity. Here we notice that from (5.1)–(5.2), the potential Ψ_N represents the solution operated by the Laplace operator as in (4.4).

5.2 Almost orthogonality of the Neumann boundary potential

In this section we recall the almost orthogonality estimates that are shown in Ogawa–Shimizu [38] and [39] and are mentioned in Sect. 4.1. The estimate is in between the boundary potential term Ψ_N for the Neumann boundary problem of the heat equation and the time and space Littlewood–Paley decompositions $\{\psi_k\}_{k\in\mathbb{Z}}$ and $\{\phi_j\}_{j\in\mathbb{Z}}$. For the symbol of the gradient of the pressure, we introduce the useful notation for a part of the symbol defined by (4.6); $B(\tau, \xi') = \sqrt{i\tau + |\xi'|^2}$.

Lemma 5.1 (Almost orthogonality I [38]) *For $k, j, \ell \in \mathbb{Z}$ let $\{\psi_k(t)\}_{k\in\mathbb{Z}}$ and $\{\phi_j(x')\}_{j\in\mathbb{Z}}$ be the time and the space Littlewood–Paley dyadic decomposition and let $\Psi_N(t, x', x_n)$ be the boundary potential defined in* (4.5)*. Set*

$$\Psi_{N,k,j}(t, x', x_n) \equiv \big(\Psi_N \underset{(t)}{*} \psi_k \underset{(x')}{*} \phi_j\big)(t, x', x_n), \tag{5.3}$$

where $\Psi_N(t, x', x_n)$ is given by (4.5)*. Then there exists a constant $C_n > 0$ depending only on the dimension n satisfying*

$$\|\Psi_{N,k,j}(t, \cdot, x_n)\|_{L^1_{x'}} \le \begin{cases} C_n 2^{\frac{k}{2}} \big(1 + (2^{\frac{k}{2}} x_n)^{n+2}\big) e^{-2^{\frac{k}{2}-1} x_n} \dfrac{2^k}{\langle 2^k t\rangle^2}, & k \ge 2j, \\ C_n 2^{\frac{k}{2}} \big(1 + (2^j x_n)^{n+2}\big) e^{-2^{j-1} x_n} \dfrac{2^k}{\langle 2^k t\rangle^2}, & k < 2j. \end{cases} \tag{5.4}$$

For estimating the term involving the grand Littlewood–Paley decomposition, the proof involves an x_n-convolution between the potential Ψ_N and $\phi_m(x_n)$. Concerning the related estimate, we show the following second orthogonal estimate:

Lemma 5.2 (Almost orthogonality II [38]) *Let $k, j, m \in \mathbb{Z}$ and assume $j \le m+1$. Let $\Psi_N(t,x',x_n)$ be the potential of the solution for the Neumann data defined by (4.5) and let $\{\psi_k(t)\}_{k\in\mathbb{Z}}$ and $\{\phi_j(x')\}_{j\in\mathbb{Z}}$ be a spatial and time Littlewood–Paley decomposition. Let $\Psi_{N,k,j}(t,x',x_n)$ be defined by (5.3). Then for any $N \in \mathbb{N}$, there exists a constant $C_N > 0$ such that for $\{\phi_m(x_n)\}_{m\in\mathbb{Z}}$,*

$$\|(\phi_m \underset{(x_n)}{*} \Psi_{N,k,j})(t,\cdot,x_n)\|_{L^1_{x'}} \le \begin{cases} C_N 2^{\frac{k}{2}} \dfrac{2^{-|\frac{k}{2}-m|}}{\langle 2^{\min(\frac{k}{2},m)} x_n\rangle^N} \dfrac{2^k}{\langle 2^k t\rangle^2}, & k \ge 2j, \\ C_N 2^{\frac{k}{2}} \dfrac{2^{-|j-m|}}{\langle 2^j x_n\rangle^N} \dfrac{2^k}{\langle 2^k t\rangle^2}, & k < 2j. \end{cases} \tag{5.5}$$

The proof of Lemmas 5.1 and 5.2 are very similar to the case for the Dirichlet boundary condition obtained in [38]. Indeed, the estimate for the Neumann boundary condition is stated in [38]. The only difference between those two boundary condition is the factor (4.6) in the formula (4.5) and the difference simply reflects the difference of regularity. See [38] for the details.

5.3 Almost orthogonality of the pressure potential

We derive almost orthogonality concerning the pressure term which is shown in Ogawa–Shimizu [39].

Definition (The pressure potentials). For $j, k \in \mathbb{Z}$, let $\{\psi_k(t)\}_{k\in\mathbb{Z}}$, $\{\phi_j(x')\}_{j\in\mathbb{Z}}$ be the Littlewood–Paley decompositions for $t \in \mathbb{R}$ and $x' \in \mathbb{R}^{n-1}$ valuables, respectively. We set for $x_n = x_n > 0$,

$$\begin{cases} \pi(t,x',x_n) \equiv c_{n+1} \displaystyle\iint_{\mathbb{R}\times\mathbb{R}^{n-1}} e^{it\tau+ix'\cdot\xi'} (i\xi', -|\xi'|)^{\mathsf{T}} m_\pi(\tau,\xi') e^{-|\xi'|x_n} d\tau d\xi', \\ \pi_{k,j}(t,x',x_n) \equiv \psi_k \underset{(t)}{*} \phi_j \underset{(x')}{*} \pi(t,x',x_n) \\ \qquad\qquad\qquad = \big(\pi'_{k,j}(t,x',x_n), \pi_{n,k,j}(t,x,x_n)\big), \end{cases} \tag{5.6}$$

where $m_\pi : \mathbb{R}\times\mathbb{R}^{n-1} \to \mathbb{R}^n$ is defined in (4.11). We extend the potential $\pi(t,x',x_n)$ into all $x_n \in \mathbb{R}$ by the even extension (i.e. exchange x_n into $|x_n|$).

Recalling the notation $\widetilde{\phi_j}$, $\widetilde{\psi_k}$ defined in (3.5) and noting that

$$\sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \widehat{\psi_k}(\tau)\widehat{\phi_j}(\xi') \equiv 1, \quad (\tau,\xi') \ne (0,0),$$

we have for $x_n > 0$ that

$$
\begin{aligned}
&\nabla q(t, x', x_n) \\
&= c_{n+1} \iint_{\mathbb{R}^n} e^{it\tau + ix'\cdot\xi'} (i\xi', -|\xi'|)^{\mathsf{T}} \Big(m'(\tau, \xi') \cdot \widehat{H'} + m_n(\tau, \xi') \widehat{H_n} \Big) \\
&= e^{-|\xi'| x_n} \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \widehat{\psi_k}(\tau) \widehat{\phi_j}(\xi') d\tau d\xi' \\
&\equiv \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \Big({\pi'}_{k,j} \underset{(t,x')}{\cdot *} \Big(\widetilde{\psi_k} \underset{(t)}{*} \widetilde{\phi_j} \underset{(x')}{*} H' \Big) + \pi_{n k,j} \underset{(t,x')}{*} \Big(\widetilde{\psi_k} \underset{(t)}{*} \widetilde{\phi_j} \underset{(x')}{*} H_n \Big) \Big),
\end{aligned}
\tag{5.7}
$$

where $(\pi'_{k,j}, \pi_{n,k,j})$ denote the potential for the derivative of the pressure with $\mathbb{R}^{n-1}$ direction and x_n direction given in (5.6), respectively and we use the notion of the inner product-convolution (2.11) and the data is extended by the zero extension for $t \le 0$. We show the almost orthogonality and its variation in the following.

Lemma 5.3 (Pressure almost orthogonality I [39]) *For $k, j \in \mathbb{Z}$, let $\pi_{k,j}(t, x', x_n)$ be the pressure potentials defined by (5.6) and let $\{\psi_k(t)\}_{k\in\mathbb{Z}}$ and $\{\phi_j(x')\}_{j\in\mathbb{Z}}$ be the Littlewood–Paley decompositions for time and space, respectively.*

(1) For the time-dominated region $k \ge 2j$, there exists $C_n > 0$ such that for any $x_n \in \mathbb{R}_+$ and $t \in \mathbb{R}$,

$$
\big\| \pi_{k,j}(t, \cdot, x_n) \big\|_{L^1_{x'}} \le C_n 2^j \big(1 + (2^j x_n)^{n+2} \big) e^{-2^{(j-1)} x_n} \frac{2^k}{\langle 2^k t \rangle^2}, \tag{5.8}
$$

where $\|\cdot\|_{L^1_{x'}}$ denotes the $L^1(\mathbb{R}^{n-1})$ norm in x'-variable.

(2) For the space-dominated region $k < 2j$, there exists $C_n > 0$ such that for any $x_n \in \mathbb{R}_+$ and $t \in \mathbb{R}$,

$$
\Big\| \sum_{k<2j} \pi_{k,j}(t, \cdot, x_n) \Big\|_{L^1_{x'}} \le C_n 2^j \big(1 + (2^j x_n)^{n+2} \big) e^{-2^{(j-1)} x_n} \frac{2^{2j}}{\langle 2^{2j} t \rangle^2}. \tag{5.9}
$$

The estimates are extended to $x_n \in \mathbb{R}$ by the even extensions.

We consider the almost orthogonality estimate of second type which will be used for the triumphal arch type Littlewood–Paley dyadic decomposition.

Lemma 5.4 (Pressure almost orthogonality II [39]) *Let $k, j, m \in \mathbb{Z}$ and $\pi_{k,j}(t, x', x_n)$ be the pressure potential given by (5.6) and let $\{\psi_k(t)\}_{k\in\mathbb{Z}}$ and $\{\phi_j(x')\}_{j\in\mathbb{Z}}$ be the Littlewood–Paley decompositions for time and space, respectively. Assume that $j \le m$, then for any $N \in \mathbb{N}$ and for $\{\phi_m(x_n)\}_{m\in\mathbb{Z}}$, there exists a constant $C_{n,N} > 0$ depending on n and N such that the following estimates hold:*

(1) For the time-dominated region $k \ge 2j$,

$$\big\|\phi_m \underset{(x_n)}{*} \pi_{k,j}(t,\cdot,x_n)\big\|_{L^1_{x'}} \le C_{n,N}\frac{2^j 2^{-(m-j)}}{\langle 2^j x_n\rangle^N}\frac{2^k}{\langle 2^k t\rangle^2}. \tag{5.10}$$

(2) For the space-dominated region $k < 2j$, it holds that

$$\Big\|\sum_{k<2j}\phi_m \underset{(x_n)}{*} \pi_{k,j}(t,\cdot,x_n)\Big\|_{L^1_{x'}} \le C_{n,N}\frac{2^j 2^{-(m-j)}}{\langle 2^j x_n\rangle^N}\frac{2^{2j}}{\langle 2^{2j} t\rangle^2}. \tag{5.11}$$

See [39] for the proof of Lemmas 5.3 and 5.4.

6 Estimates for the inhomogeneous Neumann boundary condition

6.1 The space-time splitting argument

For $k, j \in \mathbb{Z}$ let $\{\psi_k\}_{k\in\mathbb{Z}}$ and $\{\phi_j(x')\}_{j\in\mathbb{Z}}$ be the Littlewood–Paley dyadic decomposition of time and space variables, respectively and we introduce the decomposed boundary potential defined by (5.3). Since the support of the Fourier image of Φ_m only survives where $m \simeq j$, we see that

$$\begin{aligned}
\overline{\Phi_m} \underset{(x',x_n)}{*} \big(\Psi_N(t,x',x_n)\big) &= \overline{\Phi_m} \underset{(x',x_n)}{*} \sum_{k,j\in\mathbb{Z}}\Psi_{N,k,j}(t,x',x_n)\big)\\
&= \sum_{k\in\mathbb{Z}}\sum_{|j-m|\le 1}\zeta_{m-1}(|x_n|) \underset{(x_n)}{*} \Big(\phi_m(x') \underset{(x')}{*} \big(\Psi_{N,k,j}(t,x',x_n)\big)\Big)\\
&\quad+\sum_{k\in\mathbb{Z}}\sum_{|j-m|\le 1}\phi_m(x_n) \underset{(x_n)}{*} \Big(\zeta_m(|x'|) \underset{(x')}{*} \big(\Psi_{N,k,j}(t,x',x_n)\big)\Big),
\end{aligned} \tag{6.1}$$

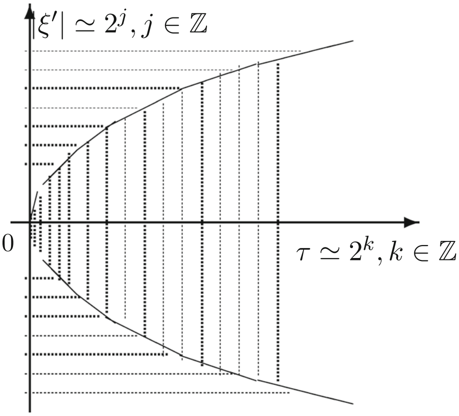
where Ψ_N and $\Psi_{N,k,j}$ are defined by (4.5) and (5.3), respectively. The $L^p(\mathbb{R}^n_+)$ norm of the first term of the right hand side of (6.1) is estimated by the Hausdorff–Young inequality of x_n-variable and the term $\zeta_{m-1}(|x_n|)$ can be treated as the following:

$$\begin{aligned}
&\Big\|\zeta_{m-1} \underset{(x_n)}{*} \Big(\phi_m(x') \underset{(x')}{*} \Psi_{N,k,j}(t,x',x_n)\Big)\Big\|_{L^p(\mathbb{R}_{+,x_n};L^p(\mathbb{R}^{n-1}_{x'}))}\\
&\quad\le \|\zeta_{m-1}\|_{L^1(\mathbb{R}_{+,x_n})}\big\|\phi_m(x') \underset{(x')}{*} \Psi_{N,k,j}(t,x',x_n)\big\|_{L^p(\mathbb{R}_{+,x_n};L^p(\mathbb{R}^{n-1}_{x'}))}\\
&\quad= C\big\|\phi_m(x') \underset{(x')}{*} \Psi_{N,k,j}(t,x',x_n)\big\|_{L^p(\mathbb{R}_{+,x_n};L^p(\mathbb{R}^{n-1}_{x'}))}.
\end{aligned}$$

The term $\zeta_m(|x'|)$ in the second term of the right hand side of (6.1) is canceling by the $\phi_j(x')$-convolution in $\Psi_{N,k,j}$ (cf. (5.3)).

Regarding the relation (4.4) and applying the Hausdorff–Young inequality to the right hand side of (6.1), it follows that

Fig. 2 The space-time splitting



$$\begin{aligned}
&\int_0^\infty \|\Delta u(t)\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)} dt \\
&\le C \Big\| \sum_{m\in\mathbb{Z}} 2^{sm} \Big(\int_{\mathbb{R}_+} \Big\| \phi_m(x') \underset{(x')}{*} \sum_{k\in\mathbb{Z}} \sum_{|j-m|\le 1} \Psi_{N,k,j}(t,x',x_n) \underset{(t,x')}{*} h(t,x') \Big\|^p_{L^p(\mathbb{R}^{n-1}_{x'})} dx_n \Big)^{1/p} \Big\|_{L^1_t(\mathbb{R}_+)} \\
&\quad + C \Big\| \sum_{m\in\mathbb{Z}} 2^{sm} \Big(\int_{\mathbb{R}_+} \Big\| \phi_m(x_n) \underset{(x_n)}{*} \sum_{k\in\mathbb{Z}} \sum_{|j-m|\le 1} \Psi_{N,k,j}(t,x',x_n) \underset{(t,x')}{*} h(t,y') \Big\|^p_{L^p(\mathbb{R}^{n-1}_{x'})} dx_n \Big)^{1/p} \Big\|_{L^1_t(\mathbb{R}_+)} \\
&\equiv \|P_1^N\|_{L^1_t(\mathbb{R}^+)} + \|P_2^N\|_{L^1_t(\mathbb{R}^+)},
\end{aligned} \tag{6.2}$$

where the first term of the right hand side of (6.2) includes $\phi_m(x')$, once the outer decomposition $\sum_{m\in\mathbb{Z}}$ is fixed then the inner decomposition $\{\phi_j(x')*\}_{j\in\mathbb{Z}}$ is restricted into only $|j-m| \le 1$ and the summation for j disappears. This is the one of the main differences from the result shown in [38] and [39].

We separate the estimate of (6.2) into two regions; one is time-dominated area and the other is space-dominated area. The relation between each variables is illustrated in Fig. 2.

In order to prove Theorem 4.2, it is enough to prove the following lemma.

Lemma 6.1 *Let $1 \le p < \infty$ and $s \in \mathbb{R}$. The terms P_1^N and P_2^N defined in (6.2) are estimated as follows:*

$$\|P_1^N\|_{L^1_t(\mathbb{R}_+)} \le C\Big(\big\|h\big\|_{\dot{F}^{1-1/2p}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))} + \big\|h\big\|_{L^1(\mathbb{R}_+;\dot{B}^{s+2-1/p}_{p,1}(\mathbb{R}^{n-1}))} \Big), \tag{6.3}$$

$$\|P_2^N\|_{L^1_t(\mathbb{R}_+)} \le C\Big(\big\|h\big\|_{\dot{F}^{1-1/2p}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))} + \big\|h\big\|_{L^1(\mathbb{R}_+;\dot{B}^{s+2-1/p}_{p,1}(\mathbb{R}^{n-1}))} \Big). \tag{6.4}$$

Remark The above estimates are crucial to extend the regularity range of maximal regularity into higher range $s > 0$.

The most of the estimates are very similar to the case of the proof of the Dirichlet boundary case appeared in Ogawa–Shimizu [38, Lemma 4.3]. However, the detailed proof for the Neumann boundary case was not given there. Since the above estimates are crucial for showing our new result, we give a full proof of the estimates.

Proof of Lemma 6.1 We split the boundary data h into the time-dominated region and the space-dominated region. Let $\widetilde{\psi_k}$ and $\widetilde{\phi_j}$ be defined in (3.5). Since

$$\begin{aligned} h(t,x') &= 3^{-2}\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \widetilde{\psi_k}(t) \underset{(t)}{*} \widetilde{\phi_j}(x') \underset{(x')}{*} h(t,x') \\ &= 3^{-2}\sum_{k\in\mathbb{Z}}\sum_{k\ge 2j} \widetilde{\psi_k}(t) \underset{(t)}{*} \widetilde{\phi_j}(x') \underset{(x')}{*} h(t,x') \\ &\quad + 3^{-2}\sum_{k\in\mathbb{Z}}\sum_{2j>k} \widetilde{\psi_k}(t) \underset{(t)}{*} \widetilde{\phi_j}(x') \underset{(x')}{*} h(t,x'). \end{aligned} \tag{6.5}$$

and letting $h_m(t,x') \equiv \widetilde{\phi_m}(x') \underset{(x')}{*} h(t,x')$ $(m\in\mathbb{Z})$, we proceed

$$\begin{aligned} P_1^N(t) \le{}& C\sum_{m\in\mathbb{Z}} 2^{sm}\Big(\int_{\mathbb{R}_+} \Big\|\phi_m(x') \underset{(x')}{*} \sum_{k\in\mathbb{Z}}\ \sum_{|j-m|\le 1, 2j\le k} \Psi_{N,k,j}(t,x',x_n) \underset{(t,x')}{*} \\ &\times \widetilde{\psi_k}(t) \underset{(t)}{*} \widetilde{\phi_j}(x') \underset{(x')}{*} h(t,x')\Big\|^p_{L^p(\mathbb{R}^{n-1}_{x'})} dx_n\Big)^{1/p} \\ &+ C\sum_{m\in\mathbb{Z}} 2^{sm}\Big(\int_{\mathbb{R}_+} \Big\|\phi_m(x') \underset{(x')}{*} \sum_{j\in\mathbb{Z}}\ \sum_{|j-m|\le 1, k<2j} \Psi_{N,k,j}(t,x',x_n) \underset{(t,x')}{*} \\ &\times \widetilde{\psi_k}(t) \underset{(t)}{*} \widetilde{\phi_j}(x') \underset{(x')}{*} h(t,x')\Big\|^p_{L^p(\mathbb{R}^{n-1}_{x'})} dx_n\Big)^{1/p} \\ \le{}& C\sum_{m\in\mathbb{Z}} 2^{sm}\Big(\int_{\mathbb{R}_+} \Big\|\sum_{k\ge 2m} \Psi_{N,k,m}(t,x',x_n) \underset{(t,x')}{*} \widetilde{\psi_k}(t) \underset{(t)}{*} h_m(t,x')\Big\|^p_{L^p_{x'}} dx_n\Big)^{1/p} \\ &+ C\sum_{m\in\mathbb{Z}} 2^{sm}\Big(\int_{\mathbb{R}_+} \Big\|\sum_{k<2m} \Psi_{N,k,m}(t,x',x_n) \underset{(t,x')}{*} h_m(t,x')\Big\|^p_{L^p_{x'}} dx_n\Big)^{1/p} \equiv L_1 + L_2, \end{aligned} \tag{6.6}$$

where $\Psi_{N,k,m}(t,x',x_n) \equiv \psi_k(t) \underset{(t)}{*} \phi_m(x') \underset{(x')}{*} \Psi_N(t,x',x_n)$. We see that L_1 is the time-dominated region and applying the Minkowski and the Hausdorff–Young inequality with using (5.3), we have

$$L_1 \le C\sum_{m\in\mathbb{Z}} 2^{sm}\Big(\int_{\mathbb{R}_+}\Big\{\sum_{k\ge 2m}\int_{\mathbb{R}_+} \Big\|\Psi_{N,k,m}(t-s,\cdot,x_n)\Big\|_{L^1_{x'}} \Big\|\widetilde{\psi_k}(s) \underset{(s)}{*} h_m(s,\cdot)\Big\|_{L^p_{x'}} ds\Big\}^p dx_n\Big)^{1/p}. \tag{6.7}$$

Then by the almost orthogonal estimate between the boundary potential Ψ_N and the Littlewood–Paley decomposition ψ_k in time, namely we invoke Lemma 5.1. Noting the restriction $k\ge 2m$ on the time-dominated region and $\psi_k(s) \underset{(s)}{*} \widetilde{\psi_k}(s) = \psi_k(s)$, we apply the first estimate in (5.4) to (6.7) and obtain that

$$
\begin{aligned}
&\|L_1\|_{L^1_t(\mathbb{R}_+)}\\
&\le C\Big\|\sum_{m\in\mathbb{Z}}2^{sm}\Big(\int_{\mathbb{R}_+}\Big\{\sum_{k\ge 2m}(2^{\frac{k}{2}}e^{-2^{\frac{k}{2}-1}x_n})\int_{\mathbb{R}}\frac{2^k}{\langle 2^k(t-s)\rangle^2}\|\psi_k\underset{(s)}{*}h_m(s,\cdot)\|_{L^p_{x'}}ds\Big\}^p dx_n\Big)^{1/p}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&= C\Big\|\sum_{m\in\mathbb{Z}}2^{sm}\Big\{\sum_{k\ge 2m}2^{\frac{k}{2}}\int_{\mathbb{R}}\frac{2^k}{\langle 2^k(t-s)\rangle^2}\|\psi_k\underset{(s)}{*}h_m(s,\cdot)\|_{L^p_{x'}}ds\Big(\int_{\mathbb{R}_+}\exp(-p2^{\frac{k}{2}-1}x_n)dx_n\Big)^{1/p}\Big\}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\le C\sum_{k\in\mathbb{Z}}2^{(\frac12-\frac{1}{2p})k}\sum_{m\in\mathbb{Z}}2^{sm}\Big\|\int_{\mathbb{R}}\frac{2^k}{\langle 2^k(t-s)\rangle^2}\|\psi_k\underset{(s)}{*}h_m(s,\cdot)\|_{L^p_{x'}}ds\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\le C\sum_{k\in\mathbb{Z}}2^{(\frac12-\frac{1}{2p})k}\sum_{m\in\mathbb{Z}}2^{sm}\Big\|\|\psi_k\underset{(s)}{*}h_m(s,\cdot)\|_{L^p_{x'}}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\le C\|h\|_{\dot F^{1/2-1/2p}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1}_{x'}))}.
\end{aligned}
\tag{6.8}
$$

Meanwhile the second term L_2 is the space-dominated region and letting $h_m(t,x')\equiv\widetilde{\phi_m}(x')*h(t,x')$, we apply again the Minkowski inequality, the Hausdorff–Young inequality and (5.3),

$$
\begin{aligned}
&\|L_2\|_{L^1_t(\mathbb{R}_+)}\\
&\le C\Big\|\sum_{m\in\mathbb{Z}}2^{sm}\Big(\int_{\mathbb{R}_+}\Big\{\sum_{k<2m}(2^{\frac{k}{2}}e^{-2^{m-1}x_n})\int_{\mathbb{R}}\frac{2^k}{\langle 2^k(t-s)\rangle^2}\|h_m(s,\cdot)\|_{L^p_{x'}}ds\Big\}^p dx_n\Big)^{1/p}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&= C\Big\|\sum_{m\in\mathbb{Z}}2^{sm}\sum_{k<2m}2^{\frac{k}{2}}\\
&\qquad\int_{\mathbb{R}}\frac{2^k}{\langle 2^k(t-s)\rangle^2}\|h_m(s,\cdot)\|_{L^p_{x'}}ds\Big(\int_{\mathbb{R}_+}\exp(-p2^{m-1}x_n)dx_n\Big)^{1/p}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\le C\sum_{m\in\mathbb{Z}}2^{(s+1-\frac{1}{p})m}\sum_{k<2m}2^{\frac{k}{2}-m}\Big\|\int_{\mathbb{R}}\frac{2^k}{\langle 2^k(t-s)\rangle^2}\|h_m(s,\cdot)\|_{L^p_{x'}}ds\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\qquad\le C\|h\|_{L^1(\mathbb{R}_+;\dot B^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}_{x'}))}.
\end{aligned}
\tag{6.9}
$$

From (6.6), (6.8) and (6.9), the estimate (6.3) is shown. [2]

We then prove (6.4). Similar way to (6.6) from (6.5), we split P_2^N into the time-like region and the space-like region;

$$
\begin{aligned}
P_2^N(t)\le\ & C\sum_{m\in\mathbb{Z}}2^{sm}\Bigg(\int_{\mathbb{R}_+}\Big\|\phi_m(x_n)\underset{(x_n)}{*}\sum_{j\in\mathbb{Z}}\sum_{|j-m|\le 1,k<2j}\Psi_{N,k,j}(t,x',x_n)\underset{(t,x')}{*}\\
&\times\widetilde{\psi_k}(t)\underset{(t)}{*}\widetilde{\phi_j}(x')\underset{(x')}{*}h(t,x')\Big\|^p_{L^p(\mathbb{R}^{n-1}_{x'})}dx_n\Bigg)^{1/p}\\
\le\ & C\sum_{m\in\mathbb{Z}}2^{sm}\Big\|\sum_{k\ge 2m-2}\sum_{|j-m|\le 1}\\
&\Big\|\phi_m(x_n)\underset{(x_n)}{*}\Psi_{N,k,j}(t,x',x_n)\underset{(t,x')}{*}\widetilde{\psi_k}(t)\underset{(t)}{*}h_j(t,x')\Big\|_{L^p_{x'}}\Big\|_{L^p(\mathbb{R}^+_{x_n})}
\end{aligned}
$$

[2] Up to this level there is no restriction on p nor s.

$$
\begin{aligned}
&+C\sum_{m\in\mathbb{Z}}2^{sm}\Big\|\sum_{k<2m+2}\sum_{|j-m|\le 1}\\
&\Big\|\phi_m(x_n)\underset{(x_n)}{*}\Psi_{N,k,j}(t,x',x_n)\underset{(t,x')}{*}\widetilde{\psi_k}(t)\underset{(t)}{*}h_j(t,x')\Big\|_{L^p_{x'}}\Big\|_{L^p(\mathbb{R}^+_{x_n})}\\
&\equiv M_1+M_2.
\end{aligned}
\tag{6.10}
$$

The first term M_1 of the right hand side is the time-dominated part, letting $h_j(t,x')\equiv\widetilde{\phi}_j(x')*h(t,x')$, we apply the Minkowski inequality and the Hausdorff–Young inequality with (5.3) as well as the almost orthogonal estimate (5.5) between ϕ_m and $\Psi_{N,k,j}$ in Lemma 5.2 with $m\simeq j$. Then setting $2^m x_n=\tilde{x}_n$, the first term of the right hand side of (6.10) can be estimated as follows:

$$
\begin{aligned}
&\|M_1\|_{L^1_t(\mathbb{R}_+)}\\
&\le C\Big\|\sum_{m\in\mathbb{Z}}2^{sm}\Big(\int_{\mathbb{R}_+}\Big\{\sum_{k\in\mathbb{Z}}\Big(\frac{2^{-|\frac{k}{2}-m|}}{\langle 2^{\min(\frac{k}{2},m)}|x_n|\rangle^N}\Big)\\
&\quad\times 2^{\frac{k}{2}}\int_{\mathbb{R}}\frac{2^k}{\langle 2^k(t-s)\rangle^2}\Big\|\psi_k\underset{(s)}{*}h_m(s,\cdot)\Big\|_{L^p_{x'}}ds\Big\}^p dx_n\Big)^{1/p}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\le C\Big\|\sum_{m\in\mathbb{Z}}2^{sm}\Big(\int_{\mathbb{R}_+}\Big\{\Big(\sum_{k\in\mathbb{Z}}2^{-|\frac{k}{2}-m|}2^{\frac{k}{2}}\int_{\mathbb{R}}\frac{2^k}{\langle 2^k(t-s)\rangle^2}\Big\|\psi_k\underset{(s)}{*}h_m(s,\cdot)\Big\|_{L^p_{x'}}ds\Big)\frac{1}{\langle|\tilde{x}_n|\rangle^N}\Big\}^p\\
&\quad\times 2^{-\min(\frac{k}{2},m)}d\tilde{x}_n\Big)^{1/p}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\le C\Big\|\sum_{k\in\mathbb{Z}}2^{\frac{k}{2}}\sum_{m\in\mathbb{Z}}2^{-|\frac{k}{2}-m|}2^{-\frac{1}{p}\min(\frac{k}{2},m)}2^{sm}\int_{\mathbb{R}}\frac{2^k}{\langle 2^k(t-s)\rangle^2}\Big\|\psi_k\underset{(s)}{*}h_m(s,\cdot)\Big\|_{L^p_{x'}}ds\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\le C\Big\|\sum_{k\in\mathbb{Z}}2^{\frac{k}{2}}2^{-\frac{1}{2p}k}\Big(\sum_{m\ge\frac{k}{2}}2^{-(m-\frac{k}{2})}+\sum_{m<\frac{k}{2}}2^{-(\frac{k}{2}-m)}2^{\frac{1}{p}(\frac{k}{2}-m)}\Big)\\
&\quad\times 2^{sm}\int_{\mathbb{R}}\frac{2^k}{\langle 2^k(t-s)\rangle^2}\Big\|\psi_k\underset{(s)}{*}h_m(s,\cdot)\Big\|_{L^p_{x'}}ds\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\le C\sum_{k\in\mathbb{Z}}2^{(\frac{1}{2}-\frac{1}{2p})k}\sum_{m\in\mathbb{Z}}2^{-(1-\frac{1}{p})|m-\frac{k}{2}|}2^{sm}\Big\|\int_{\mathbb{R}}\frac{2^k}{\langle 2^k(t-s)\rangle^2}\Big\|\psi_k\underset{(s)}{*}h_m(s,\cdot)\Big\|_{L^p_{x'}}ds\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\le C\Big\|\sum_{k\in\mathbb{Z}}2^{(\frac{1}{2}-\frac{1}{2p})k}\Big\|\psi_k\underset{(s)}{*}h(s,\cdot)\Big\|_{\dot{B}^s_{p,1}(\mathbb{R}^{n-1}_{x'})}\Big\|_{L^1_t(\mathbb{R}_+)}=C\|h\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}_{x'}))},
\end{aligned}
\tag{6.11}
$$

where the estimate is valid even for $p=1$.

On the other hand for the estimate M_2, we proceed a similar way to treat M_1. Exchanging the order of the summation of j and k and setting $h_j(t,x')\equiv\phi_j(x')*h(t,x')$, it follows by changing $m-j\to m$ and (6.10) that

$$
\begin{aligned}
&\|M_2\|_{L^1} \\
&\le C\Big\| \sum_{m\in\mathbb{Z}} 2^{sm}\Big(\int_{\mathbb{R}_+}\Big\{\sum_{k<2m+2} 2^{\frac{k}{2}}\Big(\frac{C_N}{\langle 2^m|x_n|\rangle^N}\int_{\mathbb{R}}\frac{2^k}{\langle 2^k(t-s)\rangle^2}\|h_m(s,\cdot)\|_{L^p_{x'}}ds\Big\}^p dx_n\Big)^{1/p}\Big\|_{L^1_t(\mathbb{R}_+)} \\
&\le C\Big\| \sum_{m\in\mathbb{Z}} 2^{sm}\int_{\mathbb{R}_+}\sum_{k<2m+2} 2^{\frac{k}{2}}\int_{\mathbb{R}}\frac{2^k}{\langle 2^k(t-s)\rangle^2}\|h_m(s,\cdot)\|_{L^p_{x'}}ds\Big(\int_{\mathbb{R}_+}\frac{1}{\langle 2^m|x_n|\rangle^{pN}}dx_n\Big)^{1/p}\Big\|_{L^1_t(\mathbb{R}_+)} \\
&\quad\text{(changing the variable } 2^m x_n=\tilde{x}_n \text{ nd choosing} pN>1) \\
&\le C\Big\| \sum_{m\in\mathbb{Z}} 2^{sm}2^{-\frac{m}{p}}2^m\sum_{k<2m+2} 2^{\frac{k}{2}-m}\int_{\mathbb{R}}\frac{2^k}{\langle 2^k(t-s)\rangle^2}\Big\|h_m(s,\cdot)\Big\|_{L^p_{x'}}ds\Big(\int_{\mathbb{R}_+}\frac{1}{\langle|\tilde{x}_n|\rangle^{pN}}d\tilde{x}_n\Big)^{1/p}\Big\|_{L^1_t(\mathbb{R}_+)} \\
&\le C\Big\| \sum_{j\in\mathbb{Z}} 2^{(s+1-\frac{1}{p})m}\Big(\sum_{k<2m} 2^{\frac{k}{2}-m}\Big)\|h_m(s,\cdot)\|_{L^p_{x'}}\Big\|_{L^1_t(\mathbb{R}_+)} = C\|h\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}_{x'}))}.
\end{aligned}
\tag{6.12}
$$

Here we notice that there is no restriction on p nor s. In the last estimate, we exchange the order of the integration in time and the summation of m and k and use the Hausdorff–Young inequality to remove the convolution with the time potential term and then recovers the time integration out side. From (6.10), (6.11) and (6.12) the estimate (6.4) is shown. This completes the proof of Lemma 6.1. □

6.2 The boundary trace estimates

We show the optimality for the boundary trace estimate which is required for establishing maximal regularity. This shows that the condition on the boundary data in those theorems are not only sufficient but also a necessary condition (see for more detailed estimates for the boundary trace [28, 32])

Proposition 6.2 (Sharp boundary derivative trace) *For* $1<p<\infty$ *and* $-1+1/p<s$, *there exists a constant* $C>0$ *such that for all function* $u=u(t,x',x_n)\in \dot{W}^{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))$, $\Delta u\in L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))$ *with* $\partial_{x_n}u(0,x',x_n)=0$, *it holds for all* $\ell=1,2,\dots,n$ *that*

$$
\sup_{x_n\in\mathbb{R}_+}\big\|\partial_{x_\ell}u(\cdot,\cdot,x_n)\big\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))} \le C\Big(\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))}+\|\Delta u\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))}\Big). \tag{6.13}
$$

$$
\sup_{x_n\in\mathbb{R}_+}\big\|\partial_{x_\ell}u(\cdot,\cdot,x_n)\big\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))} \le C\|\Delta u\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))}. \tag{6.14}
$$

Remark If a frequency projection upon the time-dominated region

$$
P_{2j\le k}\partial_{x_\ell}f(t,x',x_n)\equiv\sum_{k\in\mathbb{Z}}\sum_{2j\le k}\psi_k \underset{(t)}{*} \phi_j \underset{(x')}{*} \partial_{x_\ell}f(t,x',x_n) \tag{6.15}
$$

is operated to the left hand side of (6.13), then the spatial end-point exponent $p=1$ is included in the above statement, while the estimate (6.14) is valid for $p=1$ (cf. [39]). Hence if we combine the both regularities of the trace side, $p=1$ is available.

See for the proof of Proposition 6.2 in [38, Theorem 7.1].

7 Maximal regularity for the Stokes equations

To show Theorem 2.5, we show maximal L^1-regularity for the pressure term.

7.1 The estimate for the pressure

We recall the notations for the potential (5.6) for the pressure ∇q that is shown in Ogawa–Shimizu [39].

Proposition 7.1 [39] *Let $1 \le p < \infty$ and $s \in \mathbb{R}$. For given data*

$$H \in \dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^{n-1})) \cap L^1(\mathbb{R}_+; \dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1})),$$

there exists $C > 0$ independent of H such that the pressure part q of the problem (2.7) satisfies the estimate

$$\|\nabla q\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \le C\Big(\|H\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))} + \|H\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big). \tag{7.1}$$

To show the pressure estimate (7.1), we use the potential expression $\pi_{k,j}(t, x', x_n)$ in (5.6) and the Littlewood–Paley decomposition of unity (3.3);

$$\begin{aligned}
&\overline{\Phi_m} \underset{(x',x_n)}{*} \big(\pi(t, x', x_n)\big) \\
&\quad \equiv \zeta_{m-1}(x_n) \underset{(x_n)}{*} \phi_m(x') \underset{(x')}{*} \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \pi_{k,j}(t, x', x_n) \\
&\qquad + \phi_m(x_n) \underset{(x_n)}{*} \zeta_m(x') \underset{(x')}{*} \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \pi_{k,j}(t, x', x_n).
\end{aligned} \tag{7.2}$$

Concerning the first term of the right-hand side of (7.2), we estimate that the convolution with $\zeta_{m-1}(x_n)$ can be treated by the Hausdorff–Young inequality in x_n-variable. Note that the potential $\pi(t, x', x_n)$ has the even extension in $x_n \in \mathbb{R}$ and hence the $L^p(\mathbb{R}^n_+)$ norm of the term is estimated as follows:

$$\begin{aligned}
&\Big\|\zeta_{m-1} \underset{(x_n)}{*} \Big(\phi_m(x') \underset{(x')}{*} \pi_{k,j}(t, x', x_n)\Big)\Big\|_{L^p(\mathbb{R}_{+,x_n};L^p(\mathbb{R}^{n-1}_{x'}))} \\
&\quad \le \|\zeta_{m-1}\|_{L^1(\mathbb{R}_{+,x_n})} \Big\|\phi_m(x') \underset{(x')}{*} \pi_{k,j}(t, x', x_n)\Big\|_{L^p(\mathbb{R}_{+,x_n};L^p(\mathbb{R}^{n-1}_{x'}))} \\
&\quad \le C\Big\|\phi_m(x') \underset{(x')}{*} \pi_{k,j}(t, x', x_n)\Big\|_{L^p(\mathbb{R}_{+,x_n};L^p(\mathbb{R}^{n-1}_{x'}))}
\end{aligned} \tag{7.3}$$

and we apply Lemma 5.3. Concerning the second term of the right-hand side of (7.2), the number of overlapping supports of the kernel $\zeta_m(x') \underset{(x')}{*} \phi_j(x')$ is limited in finite numbers, i.e., $|m-j|\le 1$ and we apply the almost orthogonality of the second type stated in Lemma 5.4.

Proof of Proposition 7.1 Let the boundary data $H(t,x')=(H'(t,x'),H_n(t.x'))$ is extended into $t<0$ by the zero extension. From (6.1),

$$\begin{aligned}\overline{\Phi_m}\underset{(x',x_n)}{*}\nabla q(t,x',x_n)&\equiv\overline{\Phi_m}\underset{(x',x_n)}{*}\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\Big(\pi_{k,j}\underset{(t,x')}{\cdot *}H\Big)\\&=\overline{\Phi_m}\underset{(x',x_n)}{*}\sum_{k\in\mathbb{Z}}\sum_{|j-m|\le 1}\Big(\pi_{k,j}\underset{(t,x')}{\cdot *}H\Big),\end{aligned}$$

and observing the estimate (7.3) we divide the term into two terms.

$$\begin{aligned}&\|\nabla q\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\&\le C\Big\|\sum_{m\in\mathbb{Z}}2^{sm}\Big(\int_{\mathbb{R}}\Big\|\phi_m(x')\underset{(x')}{*}\sum_{k\in\mathbb{Z}}\sum_{|j-m|\le 1}\pi_{k,j}(t,x',x_n)\underset{(t,x')}{\cdot *}H(t,x')\Big\|^p_{L^p(\mathbb{R}^{n-1}_{x'})}d\tilde x_n\Big)^{1/p}\Big\|_{L^1_t(\mathbb{R}_+)}\\&+C\Big\|\sum_{m\in\mathbb{Z}}2^{sm}\Big(\Big(\int_{\mathbb{R}}\Big\|\Big(\phi_m(x_n)\underset{(\tilde\eta)}{*}\zeta_m(|x'|)\underset{(x')}{*}\\&\times\sum_{k\in\mathbb{Z}}\sum_{|j-m|\le 1}\pi_{k,j}(t,x',x_n)\Big)\underset{(t,x')}{\cdot *}H(t,x')\Big\|^p_{L^p(\mathbb{R}^{n-1}_{x'})}dx_n\Big)^{1/p}\Big\|_{L^1_t(\mathbb{R}_+)}\\&\equiv\|P_1\|_{L^1_t(\mathbb{R}_+)}+\|P_2\|_{L^1_t(\mathbb{R}_+)},\end{aligned}\tag{7.4}$$

where we use the inner product-convolution $\cdot *$ defined by (2.11). Noting that the data H is divided into the time-dominated region $k\ge 2j$ and the space-dominated region $k<2j$, respectively, as

$$H(t,x')=\sum_{k\in\mathbb{Z}}\sum_{2j\le k}H_{k,j}(t,x')+\sum_{k\in\mathbb{Z}}\sum_{2j>k}H_{k,j}(t,x'),$$

where we set

$$\begin{aligned}H_{k,j}(t,x')&=\widetilde{\psi_k}(t)\underset{(t)}{*}\widetilde{\phi_j}(x')\underset{(x')}{*}H(t,x'),\\H_j(t,x')&=\widetilde{\phi_j}(x')\underset{(x')}{*}H(t,x'),\end{aligned}$$

and we use $\widetilde{\phi_j} = \phi_{j-1} + \phi_j + \phi_{j+1}$ and $\widetilde{\psi_k}$ with a similar arrangement. Then noticing $\widetilde{\phi_j} \underset{(x')}{*} \phi_j = \phi_j$, $\widetilde{\psi_k} \underset{(t)}{*} \psi_k = \psi_k$, and Proposition 3.2, we divide $P_1(t)$ into L_1 and L_2 to have the following:

$$\begin{aligned} P_1(t) \le C \sum_{m\in\mathbb{Z}} 2^{sm} \Big\| \Big\| \phi_m(x') \underset{(x')}{*} \sum_{k\ge 2m} \sum_{|j-m|\le 1} \pi_{k,j}(t,x',x_n) \underset{(t,x')}{\cdot *} H_{k,j}(t,x') \Big\|_{L^p(\mathbb{R}^{n-1}_{x'})} \Big\|_{L^p(\mathbb{R}_{+,x_n})} \\ + C \sum_{m\in\mathbb{Z}} 2^{sm} \Big\| \Big\| \phi_m(x') \underset{(x')}{*} \sum_{k< 2m} \sum_{|j-m|\le 1} \pi_{k,j}(t,x',x_n) \underset{(t,x')}{\cdot *} H_{k,j}(t,x') \Big\|_{L^p(\mathbb{R}^{n-1}_{x'})} \Big\|_{L^p(\mathbb{R}_{+,x_n})} \\ \equiv L_1 + L_2, \end{aligned} \tag{7.5}$$

where $\{\pi_{k,m}\}_{k,m\in\mathbb{Z}}$ is defined in (5.6). For the time-dominated part L_1, since $k \ge 2m$, we apply the almost orthogonality estimate (5.8) in Lemma 5.3, and the estimate can be obtained in a very similar way to (6.8) and (6.9). By the change of variable $2^m x_n = \bar{x}_n$, it holds that

$$\begin{aligned} \|L_1\|_{L^1_t(\mathbb{R}_+)} &\le C \Big\| \sum_{m\in\mathbb{Z}} 2^{sm} \Big(\int_{\mathbb{R}_+} \Big\{ \sum_{k\ge 2m} \int_{\mathbb{R}} \big\| \pi_{k,m}(t-s,x',x_n) \big\|_{L^1_{x'}} \big\| H_{k,m}(s,x') \big\|_{L^p_{x'}} ds \Big\}^p dx_n \Big)^{1/p} \Big\|_{L^1_t(\mathbb{R}_+)} \\ &= C \Big\| \sum_{m\in\mathbb{Z}} 2^{sm} \Big(\Big\{ \sum_{k\ge 2m} 2^m \int_{\mathbb{R}} \frac{2^k}{\langle 2^k(t-s)\rangle^2} \big\| \widetilde{\psi_k} \underset{(t)}{*} H_m(s,\cdot) \big\|_{L^p_{x'}} ds \Big\}^p \\ &\quad \times 2^{-\frac{1}{p}m} \Big(\int_{\mathbb{R}_+} \Big((1+\bar{x}_n^{n+2}) e^{-\bar{x}_n/2} \Big)^p d\bar{x}_n \Big) \Big)^{1/p} \Big\|_{L^1_t(\mathbb{R}_+)} \\ &\le C \Big\| \sum_{m\in\mathbb{Z}} 2^{sm} 2^{(1-\frac{1}{p})m} \sum_{k\ge 2m} \int_{\mathbb{R}} \frac{2^k}{\langle 2^k(t-s)\rangle^2} \big\| \widetilde{\psi_k} \underset{(t)}{*} H_m(s,\cdot) \big\|_{L^p_{x'}} ds \Big\|_{L^1_t(\mathbb{R}_+)} \\ &\le C \sum_{k\in\mathbb{Z}} 2^{\frac{k}{2}(1-\frac{1}{p})} \Big\| \int_{\mathbb{R}} \frac{2^k}{\langle 2^k(t-s)\rangle^2} \sum_{m\le k/2} 2^{sm} \big\| \widetilde{\psi_k} \underset{(t)}{*} H_m(s,\cdot) \big\|_{L^p_{x'}} ds \Big\|_{L^1_t(\mathbb{R}_+)} \\ &\le C \sum_{k\in\mathbb{Z}} 2^{\frac{k}{2}(1-\frac{1}{p})} \Big\| \sum_{m\in\mathbb{Z}} 2^{sm} \big\| \widetilde{\psi_k} \underset{(t)}{*} H_m(s,\cdot) \big\|_{L^p_{x'}} \Big\|_{L^1_t(\mathbb{R}_+)} \\ &\le C \|H\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+; B^s_{p,1}(\mathbb{R}^{n-1}))}. \end{aligned} \tag{7.6}$$

On the other hand, when $k < 2m$, for the space-dominated part L_2, applying the almost orthogonality estimate (5.9) in Lemma 5.3 with using the Minkowski inequality, the Hausdorff–Young inequality, we obtain

$$\begin{aligned} \|L_2\|_{L^1_t(\mathbb{R}_+)} &\le C \Big\| \sum_{m\in\mathbb{Z}} 2^{sm} \Big(\int_{\mathbb{R}_+} \Big\{ \big(2^m (1+(2^m x_n)^{n+2}) e^{-(2^{m-1}x_n)} \big) \\ &\quad \times \int_{\mathbb{R}} \frac{2^{2m}}{\langle 2^{2m}(t-s)\rangle^2} \|H_m(s,\cdot)\|_{L^p_{x'}} ds \Big\}^p dx_n \Big)^{1/p} \Big\|_{L^1_t(\mathbb{R}_+)} \\ &\le C \Big\| \sum_{m\in\mathbb{Z}} 2^{sm} 2^m 2^{-\frac{m}{p}} \Big(\int_{\mathbb{R}_+} \Big((1+\bar{x}_n^{n+2}) e^{-\bar{x}_n/2} \Big)^p d\bar{x}_n \end{aligned}$$

$$
\begin{aligned}
&\times \left\{ \int_{\mathbb{R}} \frac{2^{2m}}{\langle 2^{2m}(t-s)\rangle^2} \|H_m(s,\cdot)\|_{L^p_{x'}} ds \right\}^p \Big)^{1/p} \Big\|_{L^1_t(\mathbb{R}_+)} \\
&\le C \sum_{m\in\mathbb{Z}} 2^{(s+1-\frac{1}{p})m} \Big\| \frac{2^{2m}}{\langle 2^{2m}t\rangle^2} \Big\|_{L^1_t(\mathbb{R})} \Big\| \|H_m(s,\cdot)\|_{L^p_{x'}} \Big\|_{L^1_t(\mathbb{R}_+)} \\
&\le C \|H\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))}.
\end{aligned} \tag{7.7}
$$

In the same way for $P_1(t)$, we decompose $P_2(t)$ into the time-dominated region and the space-dominated region.

$$
\begin{aligned}
P_2(t) \le\,& C \sum_{m\in\mathbb{Z}} 2^{sm} \Big(\int_{\mathbb{R}_+} \Big\| \zeta_m(|x'|) \underset{(x')}{*} \sum_{k\ge 2m} \phi_m(x_n) \underset{(x_n)}{*} \pi_{k,m} \underset{(t,x')}{\cdot *} H_{k,m}(t,x') \Big\|^p_{L^p(\mathbb{R}^{n-1}_{x'})} dx_n \Big)^{1/p} \\
&+ C \sum_{m\in\mathbb{Z}} 2^{sm} \Big(\int_{\mathbb{R}_+} \Big\| \zeta_m(|x'|) \underset{(x')}{*} \sum_{k<2m} \phi_m(x_n) \underset{(x_n)}{*} \pi_{k,m} \underset{(t,x')}{\cdot *} H_m(t,x') \Big) \Big\|^p_{L^p(\mathbb{R}^{n-1}_{x'})} dx_n \Big)^{1/p} \\
\equiv\,& M_1 + M_2.
\end{aligned} \tag{7.8}
$$

For the time-dominated part M_1, using the Minkowski inequality, the Hausdorff–Young inequality, and also using (5.10) in Lemma 5.4 (1) (the second almost orthogonality), we have

$$
\begin{aligned}
\|M_1\|_{L^1_t(\mathbb{R}_+)} \le\,& C \Big\| \sum_{m\in\mathbb{Z}} 2^{sm} \\
&\Big(\int_{\mathbb{R}_+} \Big\{ \sum_{k\ge 2m} \frac{2^m}{\langle 2^m x_n\rangle^N} \int_{\mathbb{R}} \frac{2^k}{\langle 2^k(t-s)\rangle^2} \Big\| \psi_k \underset{(s)}{*} H_m(s,\cdot) \Big\|_{L^p_{x'}} ds \Big\}^p dx_n \Big)^{1/p} \Big\|_{L^1_t(\mathbb{R}_+)} \\
=\,& C \Big\| \sum_{m\in\mathbb{Z}} 2^{sm} \\
&\Big(\Big\{ \sum_{k\ge 2m} 2^m \int_{\mathbb{R}} \frac{2^k}{\langle 2^k(t-s)\rangle^2} \Big\| \psi_k \underset{(s)}{*} H_m(s,\cdot) \Big\|_{L^p_{x'}} ds \Big\}^p \int_{\mathbb{R}_+} \frac{C_N}{\langle \bar{x}_n\rangle^N} 2^{-m} d\bar{x}_n \Big)^{1/p} \Big\|_{L^1_t(\mathbb{R}_+)} \\
\le\,& C \sum_{k\in\mathbb{Z}} \sum_{k\ge 2m} 2^{(1-\frac{1}{p})m} 2^{sm} \Big\| \int_{\mathbb{R}} \frac{2^k}{\langle 2^k(t-s)\rangle^2} \Big\| \psi_k \underset{(s)}{*} H_m(s,\cdot) \Big\|_{L^p_{x'}} ds \Big\|_{L^1_t(\mathbb{R}_+)} \\
\le\,& C \Big\| \sum_{k\in\mathbb{Z}} 2^{(1-\frac{1}{p})\frac{k}{2}} \sum_{k\ge 2m} 2^{sm} \Big\| \psi_k \underset{(s)}{*} H_m(s,\cdot) \Big\|_{L^p_{x'}} \Big\|_{L^1_t(\mathbb{R}_+)} \\
\le\,& C \|H\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))}.
\end{aligned} \tag{7.9}
$$

The space-dominated part M_2 is estimated in the similar way as M_1. Applying the Minkowski inequality, the Hausdorff–Young inequality, and using the almost orthogonality (5.11) in Lemma 5.4 (2) for $k < 2m$, we have

$$
\|M_2\|_{L^1_t(\mathbb{R}_+)} \le \Big\| \sum_{m\in\mathbb{Z}} 2^{sm}
$$

$$\begin{aligned}
&\Big(\int_{\mathbb{R}_+}\Big\{\int_{\mathbb{R}}\Big\|\phi_m(x_n)\underset{(x_n)}{*}\sum_{k<2m}\pi_{k,m}(t-s,x',x_n)\Big\|_{L^1_{x'}}\|H_m(s,x')\|_{L^p_{x'}}ds\Big\}^p dx_n\Big)^{1/p}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\le C\Big\|\sum_{m\in\mathbb{Z}}2^{sm}\Big(\int_{\mathbb{R}_+}\Big\{\frac{2^m}{\langle 2^m x_n\rangle^N}\int_{\mathbb{R}}\frac{2^{2m}}{\langle 2^{2m}(t-s)\rangle^2}\Big\|H_m(s)\Big\|_{L^p_{x'}}ds\Big\}^p dx_n\Big)^{1/p}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\le C\Big\|\sum_{m\in\mathbb{Z}}2^{sm}2^m\Big(\int_{\mathbb{R}}\frac{2^{2m}}{\langle 2^{2m}(t-s)\rangle^2}\Big\|H_m(s)\Big\|_{L^p_{x'}}ds\Big)\Big(\int_{\mathbb{R}_+}\frac{1}{\langle 2^m x_n\rangle^{pN}}dx_n\Big)^{1/p}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\le C\sum_{m\in\mathbb{Z}}2^{sm}2^{(1-\frac{1}{p})m}\Big\|\int_{\mathbb{R}}\frac{2^{2m}}{\langle 2^{2m}(t-s)\rangle^2}\Big\|H_m(s)\Big\|_{L^p_{x'}}ds\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\le C\Big\|\sum_{m\in\mathbb{Z}}2^{(s+1-\frac{1}{p})m}\Big\|\phi_m\underset{(x')}{*}H(t)\Big\|_{L^p_{x'}}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&= C\|H\|_{L^1(\mathbb{R}_+;\dot B^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))}.
\end{aligned}\tag{7.10}$$

Combining all the estimates (7.4)–(7.10), we obtain

$$\|\nabla q\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}\le C_M\Big(\|H\|_{\dot F^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1}))}+\|H\|_{L^1(\mathbb{R}_+;\dot B^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big).$$

The restriction on the regularity exponent s stems from the structure of the homogeneous Besov space stated in Propositions 3.1–3.3. □

The following estimate is the sharp trace estimate and it is required for showing maximal regularity for the velocity part of the Stokes equation.

Proposition 7.2 *Let* $1\le p<\infty$ *and* $-(n-1)/p'<s\le(n-1)/p$. *Given boundary data*

$$H\in\dot F^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1}))\cap L^1(\mathbb{R}_+;\dot B^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1})),$$

let q *be the pressure term defined by* (4.9). *Then there exists a constant* $C>0$ *such that the following estimates hold:*

$$\big\|q|_{x_n=0}\big\|_{\dot F^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1}))}\le C\big\|H\big\|_{\dot F^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1}))},\tag{7.11}$$

$$\big\|q|_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot B^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\le C\big\|H\big\|_{L^1(\mathbb{R}_+;\dot B^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))}.\tag{7.12}$$

Proof of Proposition 7.2 Let $\{\psi_k\}_{k\in\mathbb{Z}}$ and $\{\phi_j\}_{j\in\mathbb{Z}}$ be the Littlewood–Paley dyadic decomposition of the unity in $t\in\mathbb{R}$ and $x'\in\mathbb{R}^{n-1}$ variables, respectively. For simplicity, we assume that $q\in\mathcal{S}_0(\mathbb{R}^{n-1})$ and show the estimates (7.11) and (7.12). The results follows by the density $\mathcal{S}_0(\mathbb{R}^{n-1})\subset\dot B^s_{p,1}(\mathbb{R}^{n-1})$, where $\mathcal{S}_0(\mathbb{R}^{n-1})$ denotes the rapidly decreasing functions with vanishing at the origin of their Fourier images.

Then the resulting estimates follows from the following bounds.

$$\Big\| \|\psi_k \underset{(t)}{*} \phi_j \underset{(x')}{*} q\big|_{x_n=0}\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R}_+)} \le C\Big\| \|\psi_k \underset{(t)}{*} \phi_j \underset{(x')}{*} H\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R}_+)}, \tag{7.13}$$

$$\Big\| \|\phi_j \underset{(x')}{*} q\big|_{x_n=0}\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R}_+)} \le C\Big\| \|\phi_j \underset{(x')}{*} H\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R}_+)}. \tag{7.14}$$

Indeed, admitting the above estimate (7.13), the Minkowski inequality yields

$$\begin{aligned}
&\|q\big|_{x_n=0}\|_{\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1}))}\\
&\le C\sum_{k\in\mathbb{Z}} 2^{(\frac12-\frac1{2p})k}\sum_{j\in\mathbb{Z}} 2^{sj}\Big\| \|\psi_k \underset{(t)}{*} \phi_j \underset{(x')}{*} H\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\le C\|H\|_{\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1}))},
\end{aligned}$$

which implies (7.11). The estimate (7.12) also follows from (7.14) in the similar way as

$$\begin{aligned}
\|q\ |_{x_n=0}\|_{L^1(\mathbb{R}_+;\dot B^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))} &\le C\,\Big\| \sum_{j\in\mathbb{Z}} 2^{(s+1-\frac1p)j}\|\phi_j \underset{(x')}{*} H\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\le C\,\|H\|_{L^1(\mathbb{R}_+;\dot B^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}.
\end{aligned}$$

To see (7.13), from (4.9), it follows

$$\begin{aligned}
&\psi_k \underset{(t)}{*} \phi_j \underset{(x')}{*} q(t,x',x_n)\big|_{x_n=0}\\
&= c_{n+1}\iint_{\mathbb{R}^n} e^{it\tau+ix'\cdot\xi'}\left\{\frac{B+|\xi'|}{D(\tau,\xi')}\Big(2B(i\xi'\cdot\widehat{H}') - (|\xi'|^2+B^2)\widehat{H_n}\Big)\right\}\\
&\quad \widehat{\psi_k}(\tau)\widehat{\phi_j}(\xi')d\tau d\xi',
\end{aligned} \tag{7.15}$$

where symbols $B(\tau,\xi')$ and $D(\tau,\xi')$ are given in (4.6) and (4.7) and the support of the symbol on the right hand side is in an annulus domain and hence there is no singular point in both τ, $|\xi'|$-variables and it gives a smooth symbol.

For the symbol of the gradient of the pressure, we recall the symbol $B(\tau,\xi') = \sqrt{i\tau+|\xi'|^2}$ defined by (4.6). □

Lemma 7.3 *Let* $\sigma\in\mathbb{R}$, $\zeta'\in\mathbb{R}^{n-1}$ *and* $k, j, \ell\in\mathbb{Z}_+$.
(1) *For the time-dominated region* $k-2j\ge 0$,

$$2^{\frac k2-\frac12} \le |B(2^k\sigma, 2^j\zeta')| \le (20)^{1/4}2^{\frac k2}. \tag{7.16}$$

(2) *For the space-dominated region* $k-2j<0$, *there exist constants* $1<C$ *independent of* j *and* k *such that*

$$2^{j-1}\le |B(2^k\sigma,2^j\zeta')|\le C2^{j+4}. \tag{7.17}$$

In particular, there exists a constant $c>0$ *such that*

$$c\le Re\ B(\tau,\xi'). \tag{7.18}$$

(3) *Let* $D(\tau,\xi')$ *be given by* (4.7) *and let* $k,j\in\mathbb{Z}$ *and* $2^{-1}<\sigma,|\zeta'|<2$. *Then it holds that*

$$\left|B(2^k\sigma,2^j\zeta')+2^j|\zeta'|\right|\ge\begin{cases}2^{\frac{k}{2}-\frac{1}{2}}, & k-2j\ge 0,\\ 2^{j-1}, & k-2j<0.\end{cases} \tag{7.19}$$

Proof of Lemma 7.3 (1) In the case when $k-2j\ge\ell\ge 2$, by using $2^{-1}<|\sigma|,\ |\zeta'|<2$, it holds that

$$\begin{aligned}B(2^k\sigma,2^j\zeta')&=2^{\frac{k}{2}}b_T(\sigma,\zeta',a)\big|_{a=2^{\frac{k}{2}-j}}=2^{\frac{k}{2}}\sqrt{i\sigma+(2^{j-\frac{k}{2}})^2|\zeta'|^2}\\&=2^{\frac{k}{2}}\cdot\sqrt[4]{\sigma^2+(2^{j-\frac{k}{2}}|\zeta'|)^4}\exp\left(\frac{i}{2}\tan^{-1}\frac{2^k\sigma}{2^{2j}|\zeta'|^2}\right),\end{aligned}$$

and (7.16) follows immediately.
(2) In the case when $2j-k>\ell\ge 1$, it holds that

$$\begin{aligned}2^{-1}2^j&\le 2^j\cdot\sqrt[4]{|\zeta'|^4}\le|B(2^k\sigma,2^j\zeta')|\\&=2^j\cdot\sqrt[4]{2^{2(k-2j)}\sigma^2+|\eta'|^4}\le(2^{2-2\ell}+2^4)^{1/4}\cdot 2^j\le 5^{1/4}\cdot 2^j.\end{aligned}$$

The constants c and C can be taken as $c=1/2$ and $C=\sqrt{5}$.
(3) In particular, the argument of $B(\tau,\xi')$ is less than $\frac{\pi}{4}$, (7.19) follows immediately. □

In (4.10) and (4.11), we see that the common factor of the both symbols contains

$$\frac{B+|\xi'|}{D}=\frac{B+|\xi'|}{\big(B-|\xi'|\big)^3+4|\xi'|B^2}=\frac{(B+|\xi'|)^4}{(i\tau)^3+4|\xi'|B^2(B+|\xi'|)^3} \tag{7.20}$$

and the only zero-point of the denominator is $\tau=\xi'=0$ and properly away from 0 under the support of Littlewood–Paley cut-off functions (see [50, Lemma 4.4]). Hence in $k-2j\ge 0$, we see that

$$\begin{aligned}&\left|\frac{B(2^k\sigma,2^{2j}\zeta')+2^j|\zeta'|}{D(2^k\sigma,2^j\zeta')}\Big(-2\cdot 2^j|\zeta'|B(2^k\sigma,2^{2j}|\zeta'|^2)\frac{i\zeta'}{|\zeta'|}\Big)\right|\\&=\left|\frac{B(\sigma,2^{2j-k}\zeta')+2^{j-\frac{k}{2}}|\zeta'|}{D(\sigma,2^{j-\frac{k}{2}}\zeta')}\Big(-2\cdot 2^j|\zeta'|B(2^ki\sigma,2^{2j}|\zeta'|^2)\frac{i\zeta'}{|\zeta'|}\Big)\right|\simeq O(1).\end{aligned}\tag{7.21}$$

Analogously for the space-like region $k-2j<0$, we see from (7.17) that

$$\left|\frac{B(2^ki\sigma,2^{2j}|\zeta'|^2)+2^j|\zeta'|}{D(2^k\sigma,2^j\zeta')}\Big((i2^k\sigma+2\cdot 2^{2j}|\zeta'|^2)\Big)\right|\simeq O(1).\tag{7.22}$$

Those bounds enable us to treat the operator given by (7.29) is $L^p(\mathbb{R}^{n-1})$ bounded in x' and L^1 bound in t-variable. Thus the estimate (7.13) holds for all $1\le p\le\infty$. This completes the proof of Proposition 7.2. □

7.2 Estimate for the velocity

Once we obtain the estimates for the pressure ∇q to (2.7), the required estimates for the velocity v_n of the solution to (2.7) can be obtained by establishing the bounded estimate for the singular integral part of the fundamental solution in (4.12) and then applying maximal regularity in Theorem 2.1 for the initial boundary value of the heat equations (4.1). Then the estimates for the rest of the velocity components v_ℓ follow from the estimate for (4.14) and the pressure with (4.15) (cf. [37–39]). To this end, we prepare the following estimate.

Proposition 7.4 *Let $1\le p<\infty$ and $s\in\mathbb{R}$. Let $m_\Psi(\tau,\xi')$ be the symbol defined in (4.10) and let M_Ψ be the Fourier multiplier operator defined by*

$$M_\Psi H\equiv p.v.c_{n+1}\iint_{\mathbb{R}^n}e^{it\tau+ix'\cdot\xi'}(m_\Psi\cdot\widehat{H})d\tau d\xi'$$

for any $H\in\dot F^{1/2-1/(2p)}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1}))\cap L^1(\mathbb{R}_+;\dot B^{s+1-1/p}_{p,1}(\mathbb{R}^{n-1}))$. Then it satisfies the following estimates:

$$\|M_\Psi H\|_{\dot F^{1/2-1/(2p)}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1}))}\le C\|H\|_{\dot F^{1/2-1/(2p)}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1}))},\tag{7.23}$$

$$\|M_\Psi H\|_{L^1(\mathbb{R}_+;\dot B^{s+1-1/p}_{p,1}(\mathbb{R}^{n-1}))}\le C\|H\|_{L^1(\mathbb{R}_+;\dot B^{s+1-1/p}_{p,1}(\mathbb{R}^{n-1}))}.\tag{7.24}$$

Proof of Proposition 7.4 The proof is shown in an analogous way seen in the proof of Proposition 7.2. Noting

$$\psi_k\underset{(t)}{*}\phi_j\underset{(x')}{*}M_\Psi H=\mathrm{p.v.}c_{n+1}\iint_{\mathbb{R}^n}e^{it\tau+ix'\cdot\xi'}(\widehat{\psi_k}(\tau)\widehat{\phi_j}(\xi')m_\Psi\cdot\widehat{H})d\tau d\xi'$$

$$= \text{p.v.} c_{n+1} \iint_{\mathbb{R}^n} e^{it\tau + ix'\cdot\xi'} (\widehat{\psi_k}(\tau)\widehat{\phi_j}(\xi') m_\Psi \cdot (\widehat{\widetilde{\psi}_k}(\tau)\widehat{\widetilde{\phi}_j}(\xi')\widehat{H}) d\tau d\xi',$$

where $\widetilde{\psi}$ and $\widetilde{\phi}$ are defined by (3.5), it suffices to show that

$$\Big\| \| M_\Psi \big(\psi_k \otimes \phi_j\big) \|_{L^1(\mathbb{R}^{n-1})} \Big\|_{L^1_t(\mathbb{R}_+)} \leq C, \tag{7.25}$$

$$\Big\| \| M_\Psi \phi_j \|_{L^1(\mathbb{R}^{n-1})} \Big\|_{L^1_t(\mathbb{R}_+)} \leq C, \tag{7.26}$$

then immediately by the Hausdorff–Young inequality, we obtain

$$\Big\| \| M_\Psi \cdot \big(\psi_k \underset{(t)}{*} \phi_j \underset{(x')}{*} H\big) \|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^1_t(\mathbb{R}_+)} \leq C \Big\| \| \psi_k \underset{(t)}{*} \phi_j \underset{(x')}{*} H \|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^1_t(\mathbb{R}_+)}, \tag{7.27}$$

$$\Big\| \| M_\Psi \cdot \big(\phi_j \underset{(x')}{*} H\big) \|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^1_t(\mathbb{R}_+)} \leq C \Big\| \| \phi_j \underset{(x')}{*} H \|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^1_t(\mathbb{R}_+)} \tag{7.28}$$

and the estimates (7.23) and (7.24) follow. To see (7.27), it follows from (4.10) that

$$m_\Psi(\tau, \xi') = \frac{B}{i\tau} \frac{(B + |\xi'|)}{D} \Big(-2(B^2 + |\xi'|^2) i\xi', 2|\xi'|^3 \Big) \tag{7.29}$$

where symbols $B = B(\tau, \xi')$ and $D = D(\tau, \xi')$ are given by (4.6) and (4.7) and the support of the symbol on the right hand side is in an annulus domain and hence there is no singular point in both τ, $|\xi'|$-variables and it gives a smooth symbol. For $\sigma \in \mathbb{R}$ and $\zeta' \in \mathbb{R}^n$ with $1/2 < |\sigma|, |\zeta'| < 2$. For $a > 0, \sigma \in \mathbb{R}$ and $\zeta' \in \mathbb{R}^{n-1}$, the estimates (7.16) and (7.17) give the bounds when $k \geq 2j + 4$ that

$$\Big| \partial_\tau^\alpha m_\Psi(\tau, \xi') \Big| \simeq O(2^{-k|\alpha|}), \quad \Big| \partial_{\xi'}^\beta m_\Psi(\tau, \xi') \Big| \simeq O(2^{-\frac{k}{2}|\beta|}) \tag{7.30}$$

for $|\alpha| \leq 2$. Analogously for the space-like region, we see from (7.17) that

$$\Big| \partial_\tau^\alpha m_\Psi(\tau, \xi') \Big| \simeq O(2^{-2j|\alpha|}), \quad \Big| \partial_{\xi'}^\beta m_\Psi(\tau, \xi') \Big| \simeq O(2^{-j|\beta|}) \tag{7.31}$$

for $|\beta| \le n$. Those bounds enable us to obtain the estimates (7.25) and (7.26) by integration by parts and (7.30)–(7.31), we see that $|x'| \ge 1$ and $|t| \ge 1$,

$$
\begin{aligned}
\Big|M_\Psi\big(\psi_k \otimes \phi_j\big)\Big| &\le \frac{C}{t^2|x'|^n}\left|\iint_{\mathbb{R}^n} e^{it\tau+ix'\cdot\xi'} \sum_{\alpha=2,|\beta|=n} \big((\partial_\tau)^\alpha(\partial_{\xi'})^\beta\big(m_\Psi(\tau,\xi')(\widehat{\psi_k}(\tau)\widehat{\phi_j}(\xi')\big)d\tau d\xi'\right| \\
&\le \frac{C}{t^2|x'|^n}\bigg|\iint_{\mathbb{R}^n} e^{2^k it\sigma+2^j ix'\cdot\zeta'} \\
&\quad\times \sum_{\alpha=2,|\beta|=n} 2^{-2k-nj}\big((\partial_\sigma)^\alpha(\partial_{\zeta'})^\beta\big(m_\Psi(\sigma,\zeta')(\widehat{\psi_0}(\sigma)\widehat{\phi_0}(\zeta')\big)2^{k+(n-1)j}d\sigma d\zeta'\bigg| \\
&\le \frac{C2^{k+(n-1)j}}{|2^k t|^2|2^j x'|^n}\iint_{\mathbb{R}^n}\left|\sum_{\alpha=2,|\beta|=n}\big((\partial_\sigma)^\alpha(\partial_{\zeta'})^\beta m_\Psi(\sigma,\zeta')(\widehat{\psi_0}(\sigma)\widehat{\phi_0}(\zeta')\big)\right| d\sigma d\zeta' \\
&\le \frac{C2^{k+(n-1)j}}{|2^k t|^2|2^j x'|^n}.
\end{aligned}
\tag{7.32}
$$

Thus the estimates (7.27) hold for all $1 \le p \le \infty$. A similar argument implies the estimate and (7.28) also follows. This shows the proof of Proposition 7.4. □

Proof of Theorem 2.5 Let the boundary data satisfy the regularity assumption (2.9). First we consider the n-th component of the unknown velocity that satisfies the initial boundary value problem (4.14). The direct application of Proposition 7.1–7.4 and Theorem 4.1 yields that the solution $v_n(t,x)$ to the problem (4.12) (and hence (4.14)) fulfills the following estimate:

$$
\begin{aligned}
&\|\partial_t v_n\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))} + \|D^2 v_n\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))} \\
&\quad\le C\Big(\|\partial_n q\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))} + \|H\|_{\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1}))} + \|H\|_{L^1(\mathbb{R}_+;\dot B^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}\Big) \\
&\quad\le C\big(\|H\|_{\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1}))} + \|H\|_{L^1(\mathbb{R}_+;\dot B^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}\big).
\end{aligned}
\tag{7.33}
$$

The other components of the velocity fields $v' = (v_1(t,x), v_2(t,x), \dots, v_{n-1}(t,x))$ satisfy the initial boundary value problem (4.15) by the pressure and the n-th component velocity as the external force and boundary condition. Similarly to the above estimate, we have from Proposition 7.1, 6.2, Theorem 4.1 and the estimate (7.33) that the solution $v_\ell(t,x)$ to the problem (4.15) has the estimate

$$
\begin{aligned}
&\|\partial_t v_\ell\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|D^2 v_\ell\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \\
&\quad \le C\big(\|\partial_\ell q\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|H_\ell\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))} + \|H_\ell\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))} \\
&\qquad + \big\|\partial_\ell v_n\big|_{x_n=0}\big\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))} + \big\|\partial_\ell v_n\big|_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\big) \\
&\quad \le C\big(\|\nabla q\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|H_\ell\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))} + \|H_\ell\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))} \\
&\qquad + \|\partial_t v_n\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|\nabla^2 v_n\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))}\big) \\
&\quad \le C\big(\|H\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))} + \|H\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\big).
\end{aligned}
\tag{7.34}
$$

In fact, one can apply the analogous arugment to obtain the above estimate for the velocity v_ℓ as the way of v_n with using the expression (4.16). Combining the estimates (7.33) and (7.34) for all $\ell = 1, 2, \dots, n-1$ as well as the pressure estimate (7.1) in Proposition 7.1, we conclude that the desired estimate (2.10) holds.

Conversely, if the solution (v, q) to the problem (2.7) exists, then it holds by letting f by v in the trace estimate (6.13) of Proposition 6.2 that

$$
\begin{aligned}
&\|H\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))} + \|H\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))} \\
&\quad \le 2\|\nabla v\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1})))} + 2\|\nabla v\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))} \\
&\qquad + \big\|q\ |_{x_n=0}\big\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))} + \big\|q\ |_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))} \\
&\quad \le C\Big(\|\partial_t v\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|\nabla^2 v\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|\nabla q\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \\
&\qquad + \big\|q\ |_{x_n=0}\big\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))} + \big\|q\ |_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big).
\end{aligned}
$$

This shows regularity for the boundary data is necessary. This proves Theorem 2.5. □

Proof of Theorem 2.4 Applying the maximal L^1-regularity result to the initial-boundary value problem of the Stokes equations with the boundary condition, we obtain end-point maximal L^1-maximal regularity from (7.33), (7.34). Hence by combining the maximal regularity estimates for the problems (2.6) in [36] (see also [35]), (2.7), (2.8) and the estimate (2.10) in Theorem 2.5, we obtain (2.5).

Conversely, by using (7.11)–(7.12) in Proposition 7.2, (6.13)–(6.14) in Proposition 6.2, we conclude that regularity for data is necessary for the existence of the solution (u, p) to the Stokes system (2.4).

Concerning the uniqueness, we invoke the standard argument (see [52, Theorem 4.3 and 5.7]) for the half-space. Under the assumption $-1+1/p < s < 1/p$, let (v, q) be a solution of the Stokes system (2.4) with vanishing data with regularity given in Theorem 2.4. Let $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^n_+)$ be supported in $(-1, T) \times \mathbb{R}^n_+$ with its extention $\widetilde{\phi}$ to $\mathbb{R} \times \mathbb{R}^n$ satisfying $\widehat{\widetilde{\phi}}(t, 0) = 0$ and let (v_*, q_*) be a solution of the adjoint Stokes system except the pressure sign with external force ϕ with the regularity class

$$
\begin{aligned}
v_* &\in \dot{W}^{1,1}(I; \dot{B}^{-s}_{p',1}(\mathbb{R}^n_+)) \cap L^1(I; \dot{B}^{-s+2}_{p',1}(\mathbb{R}^n_+)) \\
&\subset C_b(I; \dot{B}^{-s}_{p',1}(\mathbb{R}^n_+)) \quad \text{for } -1+\frac{1}{p} < s < \frac{1}{p}.
\end{aligned}
$$

The regularity (7.35) is ensured by our existence proof. Here we note that $\dot{B}^{-s}_{p',1}(\mathbb{R}^n_+) \subset \dot{B}^{-s}_{p',\infty}(\mathbb{R}^n_+) \simeq (\dot{B}^{s}_{p,1}(\mathbb{R}^n_+))^*$, where $-1+1/p' < -s < 1/p'$ with the subset of the dual space

$$
v_* \in C_b(I; \dot{B}^{-s}_{p',1}(\mathbb{R}^n_+)) \subset L^\infty(I; \dot{B}^{-s}_{p',\infty}(\mathbb{R}^n_+)). \tag{7.35}
$$

Let $\chi(x)$ be a smooth non-negative cut-off function with supp $\chi(x) \subset \{x = (x', x_n) \in \mathbb{R}^n_+, x' \in \mathbb{R}^{n-1}, 1 < x_n < 2\}$ and set $\chi_R(x) \equiv R^{-1}\chi(R^{-1}x)$ for $R > 0$. Let $I = (-1, T)$ and $-1+1/p < s < 1/p$ (i.e., $-1+1/p' < -s < 1/p'$). Using the mean value theorem we see that

$$
\begin{aligned}
&\Big| \int_I \int_{\mathbb{R}^n_+} q(t,x)\chi_R(x) v_*(t,x)\, dx\, dt \Big| \\
&\quad \le C\Big(\big\| q|_{x_n=0} \big\|_{L^1(I; \dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))} + \|\nabla q\|_{L^1(I;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \Big) \\
&\qquad \times \sup_{t \in I} \sum_{j \in \mathbb{Z}} 2^{-sj} \Big\| \|\phi_j \underset{(x')}{*} v_*(t)\|_{L^{p'}(\mathbb{R}^{n-1})} \Big\|_{L^{p'}(R,2R)}
\end{aligned} \tag{7.36}
$$

and similarly

$$
\begin{aligned}
&\Big| \int_I \int_{\mathbb{R}^n_+} q_*(t,x)\chi_R(x) v(t,x)\, dx\, dt \Big| \\
&\quad \le C\Big(\|q_*|_{x_n=0}\|_{L^1(I; \dot{B}^{-s+1-\frac{1}{p'}}_{p',1}(\mathbb{R}^{n-1}))} + \|\nabla q_*\|_{L^1(I;\dot{B}^{-s}_{p',1}(\mathbb{R}^n_+))} \Big) \\
&\qquad \times \sup_{t \in I} \sum_{j \in \mathbb{Z}} 2^{sj} \Big\| \|\phi_j \underset{(x')}{*} v(t)\|_{L^{p}(\mathbb{R}^{n-1})} \Big\|_{L^{p}(R,2R)}.
\end{aligned} \tag{7.37}
$$

We then claim that

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}} 2^{-sj} \Big\| \|\phi_j \underset{(x')}{*} v_*\|_{L^{p'}(\mathbb{R}^{n-1})} \Big\|_{L^{p'}(R,2R)} \\
&\le \sum_{j\in\mathbb{Z}} 2^{-sj} \Big\| \| \sum_{|\ell-j|\le 1} \bar{\Phi}_\ell \underset{(x)}{*} \phi_j \underset{(x')}{*} v_*\|_{L^{p'}(\mathbb{R}^{n-1})} \Big\|_{L^{p'}(R,2R)} \\
&\le C \sum_{j\in\mathbb{Z}} 2^{-sj} \Big\| \|\bar{\Phi}_j \underset{(x)}{*} v_*\|_{L^{p'}(\mathbb{R}^{n-1})} \Big\|_{L^{p'}(R,2R)} \\
&\le C\|v_*\|_{\dot{B}^{-s}_{p',1}(\mathbb{R}^n_+)}
\end{aligned}
\tag{7.38}
$$

and the left hand side of (7.38) vanishes as $R \to \infty$, since $-s < \frac{1}{p'}$, v_* can be approximated by $C^\infty_{0,0}(\mathbb{R}^n_+) = \big\{ f \in C^\infty_0(\mathbb{R}^n_+);\ \tilde{f}(x) = f(x)(x_n > 0)$, properly extended into $x_n \le 0$, $\widehat{\tilde{f}}(0) = 0 \big\}$ functions $\{v_{*k}\}_k$ in the norm $\dot{B}^{-s}_{p',1}(\mathbb{R}^n_+)$, pointwisely over I. Maximal L^1-regularity for the solution $v_* \in \dot{W}^{1,1}(I; \dot{B}^{-s}_{p',1}(\mathbb{R}^n_+))$ gives translation invariant in t-variable and it provides that v_* is uniformly continuous, the approximation and the convergence can be uniform on I. Hence the right hand side of (7.36)–(7.37) converges to 0 as $R \to \infty$ which justify the integration by parts (see [39] for the case $-1 + 1/p < s \le 0$). Analogously, one can find that

$$
\Big|(q|_{x_n=0}, v_*|_{x_n=0})_{\mathbb{R}\times\mathbb{R}^{n-1}}\Big| \le C\|q|_{x_n=0}\|_{L^1(I;\dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))} \|v_*\|_{C_b(I;\dot{B}^{-s}_{p',1}(\mathbb{R}^n_+))},
\tag{7.39}
$$

$$
\Big|(q_*|_{x_n=0}, v|_{x_n=0})_{\mathbb{R}\times\mathbb{R}^{n-1}}\Big| \le C\|q_*|_{x_n=0}\|_{L^1(I;\dot{B}^{-s+1-\frac{1}{p'}}_{p',1}(\mathbb{R}^{n-1}))} \|v\|_{C_b(I;\dot{B}^{s}_{p,1}(\mathbb{R}^n_+))}
\tag{7.40}
$$

for all range of $-1 + 1/p < s < 1/p$ and the dual coupling of the boundary trace is also justified. The above relations ensure the following argument remains valid: Using (7.35) –(7.40),

$$
\begin{aligned}
\langle v, \phi\rangle_{\mathbb{R}\times\mathbb{R}^n_+} &= \langle v, -\partial_t v_* - \Delta v_* + \nabla q_*\rangle_{\mathbb{R}\times\mathbb{R}^n_+} \\
&= \langle \partial_t v, v_*\rangle_{\mathbb{R}\times\mathbb{R}^n_+} + \langle \nabla v + (\nabla v)^\top - q, \nabla v_*\rangle_{\mathbb{R}\times\mathbb{R}^n_+} \\
&= \langle \partial_t v, v_*\rangle_{\mathbb{R}\times\mathbb{R}^n_+} + \langle \Delta v + \nabla q, v_*\rangle_{\mathbb{R}\times\mathbb{R}^n_+} \\
&\qquad + \big(T(v,q)\cdot e_n|_{x_n=0}, v_*|_{x_n=0}\big)_{\mathbb{R}\times\mathbb{R}^{n-1}} \\
&= \langle \partial_t v - \Delta v + \nabla q, v_*\rangle_{\mathbb{R}\times\mathbb{R}^n_+} = 0,
\end{aligned}
$$

from which and the Hahn–Banach extension theorem, we conclude $v = 0$ and hence $q = 0$ by $\nabla q = 0$ in $\mathbb{R}^n_+$ and $q(\cdot, 0) = 0$ by (2.7).

This completes the proof of Theorem 2.4. □

8 The linear and nonlinear perturbation estimates

8.1 Estimate for the extension function of initial surface

First we give an auxiliary estimate for the extension function given by the initial surface η_0.

First we show the estimate for the extension function E defined in (1.15).

Lemma 8.1 *Let* $1 \le q < \infty$ *and* $\eta_0 \in \dot{B}^{1+(n-1)/q}_{q,1}(\mathbb{R}^{n-1})$. *Then there exists a constant* $C > 0$ *such that*

$$\|\nabla E\|_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)} \le C\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}. \tag{8.1}$$

The above estimate is one of maximal regularity estimates for the half Laplacian heat semi-group in view of (1.15).

Proof of Lemma 8.1 Let us extend $\eta_0(x')$ into the whole space $\mathbb{R}^n$ by regarding $x_n \le 0$ as

$$\nabla\tilde{E}(x', x_n) = \big(\mathrm{sech}(x_n|\nabla'|)\nabla'\eta_0(x'),\ \mathrm{sech}(x_n|\nabla'|)|\nabla'|\eta_0(x')\big)$$

where $\epsilon_0 > 0$ is chosen properly small, which is one of a proper extension in $x_n \in \mathbb{R}$. Then the above estimate can be proven by the restriction of the estimate for $\tilde{E}$. To see the estimate (8.1), we employ maximal trace regularity. Since $\nabla'\eta_0 \in \dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})$, $\nabla'\eta_0 = \sum_{m\in\mathbb{Z}} \phi_m * \nabla'\eta_0$ holds in $\mathcal{S}'$, where ϕ_m denotes the Littlewood–Paley dyadic decomposition in $\mathbb{R}^{n-1}$ and it follows by the relation between the supports of the Fourier images of $\overline{\Phi_j}$ and ϕ_m that

$$\begin{aligned}
\|\nabla\tilde{E}\|_{\dot{B}^{\frac{n}{q}}_{q,1}} &= \sum_{j\in\mathbb{Z}} 2^{\frac{n}{q}j}\Big\|\overline{\Phi_j} \underset{(x',x_n)}{*} (\mathrm{sech}(x_n|\nabla'|)\sum_{m\in\mathbb{Z}}\phi_m \underset{(x')}{*} (\nabla', |\nabla'|)\eta_0\Big\|_q \\
&= \sum_{j\in\mathbb{Z}} 2^{\frac{n}{q}j}\Big\|\overline{\Phi_j} \underset{(x',x_n)}{*} (\mathrm{sech}(x_n|\nabla'|)\sum_{|m-j|\le 1}\phi_m \underset{(x')}{*} (\nabla', |\nabla'|)\eta_0)\Big\|_q \\
&= \sum_{j\in\mathbb{Z}}\sum_{|j-m|\le 1} 2^{\frac{n}{q}j}\Big\|\zeta_{j-1} \underset{(x_n)}{*} \mathrm{sech}(x_n|\nabla'|)(\phi_m \underset{(x')}{*} \phi_j \underset{(x')}{*} (\nabla', |\nabla'|)\eta_0)\Big\|_q \\
&\quad + \sum_{j\in\mathbb{Z}}\sum_{|j-m|\le 1} 2^{\frac{n}{q}j}\Big\|\phi_j \underset{(x_n)}{*} (\mathrm{sech}(x_n|\nabla'|)(\phi_m \underset{(x')}{*} \zeta_j \underset{(x')}{*} (\nabla', |\nabla'|)\eta_0))\Big\|_q \\
&\equiv I + II.
\end{aligned} \tag{8.2}$$

Then for the ℓ-th component of the first term of the right hand side of (8.2) can be seen for all $\ell = 1, 2\dots, n-1$ that

$$I_\ell \le 2\sum_{j\in\mathbb{Z}} 2^{\frac{n}{q}j}\|\zeta_{j-1}\|_{L^1_{x_n}(\mathbb{R}_+)}\Big(\int_{\mathbb{R}} \|\mathrm{sech}(x_n|\nabla'|)(\phi_j \underset{(x')}{*} \partial_\ell\eta_0)\|^q_{L^q(\mathbb{R}^{n-1})} dx_n\Big)^{1/q}$$

$$
\begin{aligned}
&\le C\sum_{j\in\mathbb{Z}} 2^{\frac{n}{q}j}\Big(\int_{\mathbb{R}} \|(\mathrm{sech}(x_n|\nabla'|)\widetilde{\phi}_j) \underset{(x')}{*} \phi_j \underset{(x')}{*} \partial_\ell \eta_0\|^q_{L^q(\mathbb{R}^{n-1})} dx_n\Big)^{1/q}\\
&\le C\sum_{j\in\mathbb{Z}} 2^{\frac{n}{q}j}\Big(\int_{\mathbb{R}} \|\mathrm{sech}(x_n|\nabla'|)\widetilde{\phi}_j\|^q_{L^1(\mathbb{R}^{n-1})} \|\phi_j \underset{(x')}{*} \partial_\ell \eta_0\|^q_{L^q(\mathbb{R}^{n-1})} dx_n\Big)^{1/q}\\
&\le C\sum_{j\in\mathbb{Z}} 2^{\frac{n}{q}j}\Big(\int_{\mathbb{R}} e^{-2^j q|x_n|} dx_n\Big)^{1/q} \|\phi_j \underset{(x')}{*} \partial_\ell \eta_0\|_{L^q(\mathbb{R}^{n-1})}\\
&\le C\sum_{j\in\mathbb{Z}} 2^{\frac{n}{q}j} 2^{-\frac{1}{q}j} \|\phi_j \underset{(x')}{*} \partial_\ell \eta_0\|_{L^q(\mathbb{R}^{n-1})} = C\|\partial_\ell \eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})},
\end{aligned}
$$

where we set $\widetilde{\phi}_j = \phi_{j-1} + \phi_j + \phi_{j+1}$. The estimate for the second term II is along the similar way.

Finally, we confirm that

$$
\nabla \tilde{E}(x', x_n) = c_n^{-1} \sum_{j\in\mathbb{Z}} \phi_j \underset{(x',x_n)}{*} \nabla \tilde{E}(x', x_n) \quad \text{in } \mathcal{S}',
$$

which is justified by the argument found in [35, Proposition 2.1]. Indeed, noticing $\mathcal{F}[\mathrm{sech} ax](\xi) = a^{-1}\mathrm{sech} a^{-1}\xi$ for $a > 0$ and $\mathrm{sech} a^{-1}\xi$ is bounded and converging to 0 around $a \simeq 0$, by making a coupling with $\varphi \in \mathcal{S}$ that

$$
\begin{aligned}
&{}_{\mathcal{S}'}\Big\langle c_n^{-1} \sum_{j\in\mathbb{Z}} \phi_j \underset{(x',x_n)}{*} \nabla \tilde{E}(x', x_n), \quad \varphi \Big\rangle_{\mathcal{S}}\\
&= -{}_{\mathcal{S}'}\Big\langle \sum_{j\in\mathbb{Z}} \widehat{\phi_j}(\xi', \xi_n)(|\xi'|)^{-1}\mathrm{sech}(\xi_n|\xi'|^{-1})\widehat{\eta_0}(\xi'), \quad c_n^{-1}\mathcal{F}^{-1}_{\xi'}\mathcal{F}^{-1}_{\xi_n}\Big[\nabla\varphi\Big]\Big\rangle_{\mathcal{S}}\\
&= -{}_{\mathcal{S}'}\Big\langle (|\xi'|)^{-1}\mathrm{sech}(\xi_n|\xi'|^{-1})\widehat{\eta_0}(\xi'), \quad c_n^{-1} \sum_{j\in\mathbb{Z}} \widehat{\phi_j}(\xi', \xi_n)\mathcal{F}^{-1}_{\xi'}\mathcal{F}^{-1}_{\xi_n}\Big[\nabla\varphi\Big]\Big\rangle_{\mathcal{S}}\\
&= {}_{\mathcal{S}'}\Big\langle c_n^{-1}\mathcal{F}^{-1}_{\xi'}\Big[\mathrm{sech}(x_n|\xi'|)\widehat{\eta_0}(\xi')\Big], \nabla\varphi\Big\rangle_{\mathcal{S}} = {}_{\mathcal{S}'}\Big\langle \nabla \tilde{E}(x', x_n), \varphi\Big\rangle_{\mathcal{S}}.
\end{aligned}
$$

□

8.2 Estimates for the linear perturbation

We now consider the estimates for the linear variable coefficient terms defined in (1.24)–(1.27). All the estimate is based on the bilinear estimate in the homogeneous Besov space Proposition 10.3. See Appendix below.

To show the estimates for the linear variable coefficient terms, we prepare the following basic lemma.

Lemma 8.2 *For $1 \le q < \infty$, let $E(x, x_n)$ is given by (1.15) and assume that for some small $\varepsilon_0 > 0$*

$$\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})} \le \varepsilon_0.$$

Then there exists a constant $C > 0$ such that

$$\begin{aligned}&\left\|\frac{\nabla E}{1+\partial_n E}\right\|_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)}, \left\|\frac{\nabla' E}{\sqrt{1+|\nabla' E|^2}}\right\|_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)}, \left\|\frac{\sqrt{1+|\nabla' E|^2}-1}{\sqrt{1+|\nabla' E|^2}}\right\|_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)}\\ &\le C\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})},\end{aligned} \tag{8.3}$$

where $\nabla' = (\partial_1, \partial_2, \ldots, \partial_{n-1})^{\mathsf{T}}$.

Proof of Lemma 8.2 To see (8.3), we use the Taylor expansion of

$$\frac{x}{1+x} = \sum_{k=1}^{\infty}(-1)^{k-1}x^k$$

and noticing that $\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)$ is the Banach algebra, it follows from Lemma 8.1 that

$$\left\|\frac{\nabla E}{1+\partial_n E}\right\|_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)} \le \sum_{k=1}^{\infty}\|\nabla E\|^k_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)} \le C\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}.$$

The second and third estimates follow in a similar way. □

Proposition 8.3 (Estimates for linear variable coefficient terms) *Let $n \ge 2$ and $1 \le p < 2n$. For $u \in C(\overline{\mathbb{R}_+}; \dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+))$, $\partial_t u$, $D^2 u$, $\nabla p \in L^1(\mathbb{R}_+; \dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+))$ and E defined in* (1.15)*, let $f(u, p, E) \equiv f(u, E) + f(p, E)$, $g(u, E)$ and $h(u, p, E) \equiv h(u, E) + h(p, E)$ be the terms defined in* (1.23), (1.24), (1.25), (1.26) *and* (1.27) *respectively. Under the assumption $\|\nabla'\eta_0\|_{\dot{B}^{(n-1)/q}_{q,1}(\mathbb{R}^{n-1})}$ is small enough, the following estimates hold: For $1 \le q < pn/|p-n|$,*

$$\begin{aligned}&\|f(u,p,E)\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\ &\qquad\le C\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\Big(\|D^2u\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \|\nabla p\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big),\end{aligned} \tag{8.4}$$

$$\begin{aligned}&\|\nabla g(u,E)\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\ &\qquad\le C\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\|D^2u\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))},\end{aligned} \tag{8.5}$$

$$\begin{aligned}&\left\|\partial_t \nabla(-\Delta)^{-1} g(u,E)\right\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\&\quad\le C\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\Big(\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}+\|D^2u\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big).\end{aligned} \tag{8.6}$$

For $1 \le q \le p(n-1)/(n-p)$ $(1 \le p < n)$ *and* $1 \le q < p(n-1)/(p-n)$ $(n \le p < \infty)$,

$$\begin{aligned}&\left\|h(u,p,E)\right\|_{\dot{F}^{\frac12-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}\\&\quad\le C\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\Big(\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}+\|D^2u\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\&\qquad+\left\|p|_{x_n=0}\right\|_{\dot{F}^{\frac12-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}+\left\|p|_{x_n=0}\right\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big),\end{aligned} \tag{8.7}$$

$$\begin{aligned}&\left\|h(u,p,E)\right\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n}{p}-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\\&\quad\le C\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\Big(\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}+\|D^2u\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\&\qquad+\left\|p|_{x_n=0}\right\|_{\dot{F}^{\frac12-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}+\left\|p|_{x_n=0}\right\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big).\end{aligned} \tag{8.8}$$

Proof of Proposition 8.3 Recalling the definition of $f(u,E)$ and $f(p,E)$, and the covariant derivative (1.18), we show (8.4) by

$$\begin{aligned}&\|f(u,p,E)\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\&\quad\le\left\|\operatorname{div}\Big(\frac{\nabla E}{1+\partial_n E}\partial_n u\Big)\right\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}+\left\|\frac{\nabla E}{1+\partial_n E}\cdot\partial_n\big(\nabla_E u\big)\right\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\&\qquad+\left\|\frac{\nabla E}{1+\partial_n E}\partial_n p\right\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\&\quad\le C\left\|\frac{\nabla E}{1+\partial_n E}\right\|_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)}\Big(\left\|\partial_n u\right\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\&\qquad+\left\|\nabla_E u\right\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}+\left\|\partial_n p\right\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big)\\&\quad\le C\left\|\frac{\nabla E}{1+\partial_n E}\right\|_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)}\Big(\left\|D^2 u\right\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\&\qquad+\left\|\frac{\nabla E}{1+\partial_n E}\right\|_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)}\left\|D^2 u\right\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}+\left\|\partial_n p\right\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big)\\&\quad\le C\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\Big(\left\|D^2 u\right\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}+\left\|\nabla p\right\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big),\end{aligned}$$

where we apply Lemma 8.1 and notice that $\dot{B}^{n/p}_{p,1}(\mathbb{R}^n_+)$ is the Banach algebra and no restriction on the exponents p nor q. The estimates (8.5) and (8.6) follow in a similar way.

To see the boundary terms, we recall the term into the velocity part and the pressure part such as (1.26) and (1.27).

For those terms, we prepare the boundary bilinear estimate of space-time type. We introduce auxiliary norms of the Chemin–Lerner type (cf. [14]).

Definition. For $1 \le p, \rho \le \infty$ and $r, s \in \mathbb{R}$, the Besov space and the Bochner space of Chemin–Lerner type $\widetilde{\dot{B}^r_{\rho,1}\big(\mathbb{R}_+;\, \dot{B}^s_{p,1}}(\mathbb{R}^{n-1})\big)$ and $\widetilde{L^\rho\big(\mathbb{R}_+;\, \dot{B}^s_{p,1}}(\mathbb{R}^{n-1})\big)$ are defined by the following norms:

$$
\begin{aligned}
\|f\|_{\widetilde{\dot{B}^r_{\rho,1}\left(\mathbb{R}_+;\dot{B}^s_{p,1}\right.}(\mathbb{R}^{n-1})\big)} &\equiv \sum_{k\in\mathbb{Z}} 2^{rk} \sum_{j\in\mathbb{Z}} 2^{sj} \big\|\psi_k \underset{(t)}{*} \phi_j \underset{(x')}{*} f(t,x')\big\|_{L^\rho_t(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}_{x'}))},\\
\|f\|_{\widetilde{L^\rho\left(\mathbb{R}_+;\dot{B}^s_{p,1}\right.}(\mathbb{R}^{n-1})\big)} &\equiv \sum_{j\in\mathbb{Z}} 2^{sj} \big\|\phi_j \underset{(x')}{*} f(t,x')\big\|_{L^\rho_t(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}_{x'}))}.
\end{aligned}
\tag{8.9}
$$

□

Lemma 8.4 (Multiple estimates for boundary terms) *Let $n \ge 2$, $1 \le p < 2n-1$, $1 \le q \le p(n-1)/(n-p)$ $(1 \le p < n)$ and $1 \le q < p(n-1)/(p-n)$ $(n \le p < 2n-1)$ and assume that functions F and G over $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ satisfy $F \in \dot{F}^{1/2-1/2p}_{1,1}(\mathbb{R}_+;\, \dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^{n-1})) \cap L^1(\mathbb{R}_+;\, \dot{B}^{(n-1)/p}_{p,1}(\mathbb{R}^{n-1}))$ and $G \in \widetilde{\dot{B}^{1/2-1/2p}_{\infty,1}}(\mathbb{R}_+;\, \dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^{n-1})) \cap \widetilde{L^\infty(\mathbb{R}_+;}\, \dot{B}^{(n-1)/q}_{q,1}(\mathbb{R}^{n-1}))$. Then the following estimate holds:*

$$
\begin{aligned}
&\|FG\|_{\dot{F}^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1}))}\\
&\le C\Big(\|F\|_{\dot{F}^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1}))} + \|F\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big)\\
&\quad\times\Big(\|G\|_{\widetilde{\dot{B}^{\frac12-\frac1{2p}}_{\infty,1}}(\mathbb{R}_+;\dot{B}^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1}))} + \|G\|_{\widetilde{L^\infty(\mathbb{R}_+;}\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1}))}\Big).
\end{aligned}
\tag{8.10}
$$

The proof of Lemma 8.4 directly follows from Proposition 10.5 shown in Appendix below (cf. [39]).

Since ∇E is independent of t (using the fact that the average of ψ_k vanishes) we notice that

$$
\begin{aligned}
&\Big\|\frac{\nabla' E}{\sqrt{1+|\nabla' E|^2}}\Big|_{x_n=0}\Big\|_{\widetilde{\dot{B}^{\frac12-\frac1{2p}}_{\infty,1}}(\mathbb{R}_+;\dot{B}^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1}))}\\
&\quad= \Big\|\frac{\sqrt{1+|\nabla' E|^2}-1}{\sqrt{1+|\nabla' E|^2}}\Big|_{x_n=0}\Big\|_{\widetilde{\dot{B}^{\frac12-\frac1{2p}}_{\infty,1}}(\mathbb{R}_+;\dot{B}^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1}))} \equiv 0
\end{aligned}
$$

and hence applying the bilinear estimate (8.10), we obtain the following estimates: Since $\partial_n E\big|_{x_n=0} = 0$ at the boundary $\partial\mathbb{R}^n_+$,

$$\begin{aligned}
&\big\|h(u,E)\big\|_{\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1}))}\\
&\le C\bigg(\Big\|\frac{\nabla' E}{\sqrt{1+|\nabla' E|^2}}\Big|_{x_n=0}\Big\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot B^{\frac{n-1}{q}}_{q,1}}(\mathbb{R}^{n-1})}+\Big\|\frac{\sqrt{1+|\nabla' E|^2}-1}{\sqrt{1+|\nabla' E|^2}}\Big|_{x_n=0}\Big\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot B^{\frac{n-1}{q}}_{q,1}}(\mathbb{R}^{n-1})}\\
&\quad+\Big\|\frac{\nabla E}{(1+\partial_n E)\sqrt{1+|\nabla' E|^2}}\Big|_{x_n=0}\Big\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot B^{\frac{n-1}{q}}_{q,1}}(\mathbb{R}^{n-1})}\bigg)\\
&\quad\times\bigg(\big\|\nabla_E u|_{x_n=0}\big\|_{\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1}))}+\big\|\nabla_E u|_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\bigg)\\
&\le C\big\|\nabla'\eta_0\big\|_{\dot B^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\bigg(\big\|\partial_t u\big\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^n_+))}+\big\|D^2 u\big\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^n_+))}\bigg).
\end{aligned}\tag{8.11}$$

In very much similar way, we find that

$$\begin{aligned}
&\big\|h(p,E)\big\|_{\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1}))}\\
&\le C\big\|\nabla E\big|_{x_n=0}\big\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot B^{\frac{n-1}{q}}_{q,1}}(\mathbb{R}^{n-1})}\bigg(\big\|p\big|_{x_n=0}\big\|_{\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1}))}\\
&\quad+\big\|p\big|_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\bigg).
\end{aligned}\tag{8.12}$$

The estimates (8.11) and (8.12) yield the resulting estimate (8.7).

The other estimate (8.8) can be shown in much straightforward way: Because $\dot B^{(n-1)/q}_{q,1}(\mathbb{R}^{n-1})$ is the Banach algebra, it follows directly that

$$\begin{aligned}
&\big\|h(u,E)\big\|_{L^1(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\\
&\le\ C\big\|\nabla'\eta_0\big|_{x_n=0}\big\|_{\dot B^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\big\|\nabla_E u|_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\\
&\le\ C\big\|\nabla'\eta_0\big\|_{\dot B^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\bigg(\big\|\partial_t u\big\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^n_+))}+\big\|D^2 u\big\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^n_+))}\bigg).
\end{aligned}$$

The case for $h(p,E)$ also follows in a similar way. This shows the proof of Proposition 8.3. □

8.3 The nonlinear estimates

The perturbation terms for the Navier–Stokes equations in the Lagrangian coordinate, it holds that the following multilinear estimates.

Proposition 8.5 (Nonlinear estimates for $F_p(u, p, E)$ and $G_{\mathrm{div}}(u, E)$) *Let $n \geq 2$, $1 \leq p < 2n$ and $1 \leq q < np/|p-n|$. For $u \in C(\overline{\mathbb{R}_+}; \dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+))$, $\partial_t u$, $D^2 u$, $\nabla p \in L^1(\mathbb{R}_+; \dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+))$ and E defined in (1.15), let $F_p(u, p, E)$ and $G_{\mathrm{div}}(u, E)$ be the nonlinear terms defined in (1.10) and (1.11), respectively. Then the following estimates hold provided $\|\nabla' \eta_0\|_{\dot{B}^{(n-1)/q}_{q,1}(\mathbb{R}^{n-1})}$ is small enough,*

$$\begin{aligned}
&\|F_p(u, p, E)\|_{L^1(\mathbb{R}_+; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\
&\qquad \leq C\big(1 + \|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\big) \\
&\qquad\qquad \times \sum_{k=1}^{n-1} \|D^2 u\|^k_{L^1(\mathbb{R}_+; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \|\nabla p\|_{L^1(\mathbb{R}_+; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}, \qquad (8.13)\\
&\big\|\nabla\big((1+\partial_n E) G_{\mathrm{div}}(u, E)\big)\big\|_{L^1(\mathbb{R}_+; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\
&\qquad \leq C\big(1 + \|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\big) \sum_{k=1}^{n-1} \|D^2 u\|^{k+1}_{L^1(\mathbb{R}_+; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \qquad (8.14)
\end{aligned}$$

and

$$\begin{aligned}
&\big\|\partial_t \nabla(-\Delta)^{-1}\big((1+\partial_n E) G_{\mathrm{div}}(u, E)\big)\big\|_{L^1(\mathbb{R}_+; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\
&\qquad \leq C\big(1 + \|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\big) \\
&\qquad\qquad \times \sum_{k=1}^{n-1} \|D^2 u\|^k_{L^1(\mathbb{R}_+; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \|\partial_t u\|_{L^1(\mathbb{R}_+; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}. \qquad (8.15)
\end{aligned}$$

Proof of Proposition 8.5 To show the estimate (8.13), we see the form (1.29) that for any $1 \leq p < \infty$,

$$\begin{aligned}
&\|F_p(u, p, E)\|_{L^1(\mathbb{R}_+; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\
&\leq C\Big\|\Big(J(DE)^{-1}\Big)^{\mathsf{T}}\Big\|_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)} \Big\|\nabla\Big(\Pi^{n-1}_p\Big(\int_0^t D_E u ds\Big) p\Big)\Big\|_{L^1(\mathbb{R}_+; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\
&\leq C\big(1 + \|\nabla E\|_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)}\big) \sum_{k=1}^{n-1} \Big\|\int_0^t D_E u\, ds\Big\|^k_{L^\infty(\mathbb{R}_+; \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \|p\|_{L^1(\mathbb{R}_+; \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\
&\leq C\big(1 + \|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\big) \sum_{k=1}^{n-1} \|D_E u\|^k_{L^1(\mathbb{R}_+; \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \|\nabla p\|_{L^1(\mathbb{R}_+; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}.
\end{aligned} \qquad (8.16)$$

Here we estimate $D_E u$ term by its definition and it follows that

$$\begin{aligned}\|D_E u\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} &\le \|Du\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \left\|\frac{\nabla E}{1+\partial_n E}\partial_n u\right\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\ &\le \|Du\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \left\|\frac{\nabla E}{1+\partial_n E}\right\|_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)}\|D^2 u\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\ &\le C\big(1+\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\big)\|D^2 u\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}.\end{aligned} \tag{8.17}$$

Thus we conclude from (8.16) and (8.17) that

$$\begin{aligned}&\|F_p(u,p,E)\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\ &\le C\big(1+\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\big)\sum_{k=1}^{n-1}\|D^2 u\|^k_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\|\nabla p\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))},\end{aligned}$$

provided $\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}$ is small enough.

Secondly, we proceed in a similar manner for (8.14) by observing (1.30), we have for all $1\le p<\infty$ that

$$\begin{aligned}&\|\nabla\big((1+\partial_n E)G_{div}(u,E)\big)\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\ &\le C\big(1+\|\partial_n E\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}\big)\left\|\operatorname{tr}\left(\Pi^{n-1}_{\mathrm{div}}\left(\nabla E,\int_0^t D_E u ds\right)Du\right)\right\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\ &\le C\big(1+\|\partial_n E\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}\big)\sum_{k=1}^{n-1}\sigma_k(\|\nabla E\|_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)})\|D_E u\|^k_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\|Du\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))},\end{aligned} \tag{8.18}$$

where $\sigma_k(\|\nabla E\|_{\dot{B}^{n/q}_{q,1}(\mathbb{R}^n_+)})$ denotes a term involving $\|\nabla E\|_{\dot{B}^{n/q}_{q,1}(\mathbb{R}^n_+)}$ of order at most 1. Using Lemmas 8.1, 8.2 and (8.17), we conclude from (8.18),

$$\begin{aligned}&\|\nabla\big((1+\partial_n E)G_{div}(u,E)\big)\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\ &\le C\left(1+\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\right)\\ &\sum_{k=1}^{n-1}\|D^2 u\|^{k+1}_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}.\end{aligned}$$

The proof of (8.15) can be done by a quite analogous way. □

Proposition 8.6 (Nonlinear estimate for $F_u(u,E)$) *Let $n\ge 2$, $1\le p<\infty$. For $D^2u\in L^1(\mathbb{R}_+;\dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+))$, let $F_u(u,E)$ be defined by* (1.9)*. Then the following* *estimate holds:*

$$\|F_u(u,E)\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \le C\sum_{k=1}^{2n-2}\|D^2u\|^{k+1}_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}. \tag{8.19}$$

The proof of Proposition 8.6 is very similar to the proof of Proposition 8.5 (cf. [39, Proposition 5.6]). Since

$$\begin{aligned}
&F_u(u,E) = \\
&\operatorname{div}\Big(J(D_Eu)^{-1}\big(J(D_Eu)^{-1}-I\big)^{\mathsf{T}}\big(J(DE)^{-1}\big)^{\mathsf{T}}\nabla u\Big) + \operatorname{div}\Big(\big(J(D_Eu)^{-1}-I\big)\big(J(DE)^{-1}\big)^{\mathsf{T}}\nabla u\Big)\\
&-\frac{\nabla E}{1+\partial_n E}\cdot\partial_n\Big(J(D_Eu)^{-1}\big(J(D_Eu)^{-1}-I\big)^{\mathsf{T}}\big(J(DE)^{-1}\big)^{\mathsf{T}}\nabla u\Big)\\
&-\frac{\nabla E}{1+\partial_n E}\cdot\partial_n\Big(\big(J(D_Eu)^{-1}-I\big)\big(J(DE)^{-1}\big)^{\mathsf{T}}\nabla u\Big),
\end{aligned}$$

those terms are divergence form and the estimates are reduced into the multilinear estimate over the Banach algebra $\dot{B}^{n/p}_{p,1}(\mathbb{R}^n_+)$.

We finally treat the boundary nonlinearities as follows.

Proposition 8.7 (Multiple estimates for boundary nonlinearity) *Let $n\ge 2$, $1\le p<2n-1$, $1\le q\le p(n-1)/(n-p)$ $(1\le p<n)$ and $1\le q<p(n-1)/(p-n)$ $(n\le p<\infty)$, and assume that functions u and p satisfy $u\in C(\overline{\mathbb{R}_+};\dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+))$, $\partial_t u$, D^2u $\nabla p\in L^1(\mathbb{R}_+;\dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+))$ and $\nabla'\eta_0\in\dot{B}^{(n-1)/q}_{q,1}(\mathbb{R}^{n-1})$, $p|_{x_n=0}\in\dot{F}^{1/2-1/2p}_{1,1}(\mathbb{R}_+;\dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^{n-1}))\cap L^1(\mathbb{R}_+;\dot{B}^{(n-1)/p}_{p,1}(\mathbb{R}^{n-1}))$. Let $H_u(u,E)$ and $H_p(u,p,E)$ be the boundary terms defined by (1.31) and (1.32), respectively. Then the following estimates hold provided $\|\nabla'\eta_0\|_{\dot{B}^{(n-1)/q}_{q,1}(\mathbb{R}^{n-1})}$ is small enough:*

$$\begin{aligned}
&\|H_u(u,E)\|_{\dot{F}^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}\\
&\quad\le C\big(1+\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\big)\sum_{k=2}^{2n-1}\Big(\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}+\|D^2u\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big)^k,
\end{aligned}\tag{8.20}$$

$$\begin{aligned}
&\|H_u(u,E)\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\\
&\quad\le C\big(1+\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\big)\sum_{k=2}^{2n-1}\|D^2u\|^k_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))},
\end{aligned}\tag{8.21}$$

$$\begin{aligned}
&\|H_p(u,p,E)\|_{\dot{F}^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}\\
&\quad\le C\big(1+\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\big)\Big(\big\|p|_{x_n=0}\big\|_{\dot{F}^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}+\big\|p|_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big)\\
&\qquad\times\sum_{k=1}^{n-1}\Big(\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}+\|D^2u\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big)^k,
\end{aligned}\tag{8.22}$$

$$\|H_p(u,p,E)\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}$$

$$\leq C\big(1+\|\nabla'\eta_0\|_{\dot B^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\big)\big\|p|_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}$$
$$\times\sum_{k=1}^{n-1}\Big(\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}+\|D^2u\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big)^k. \tag{8.23}$$

In order to prove of Proposition 8.7, we prepare some lemmas. First we introduce auxiliary norms of the Chemin–Lerner type (cf. [14]) for the proof Proposition 8.7.

Lemma 8.8 *For any* $1\leq p<\infty$,

$$\Big\|\int_0^t D_E u(s)\,ds\,\Big|_{x_n=0}\Big\|_{\widetilde{\dot B^{\frac12-\frac{1}{2p}}_{\infty,1}(\mathbb{R}_+};\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}$$
$$\leq\ C\big\|D_E u|_{x_n=0}\big\|_{\dot F^{\frac12-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}, \tag{8.24}$$

$$\Big\|\int_0^t D_E u(s)\,ds\,\Big|_{x_n=0}\Big\|_{\widetilde{L^\infty(\mathbb{R}_+};\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\leq\ C\big\|D_E u|_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}. \tag{8.25}$$

Proof of Lemma 8.8 The estimates are shown in [39, Lemma 5.9]. We give an outlined proof here. The first estimate (8.24) follows by using $\widetilde{\psi}_k(t)=\psi_{k-1}(t)+\psi_k(t)+\psi_{k+1}(t)$ and noticing $\|\partial_t^{-1}\psi_k\|_{L^\infty(\mathbb{R}_+)}\leq\|\psi_k\|_{L^1(\mathbb{R}_+)}$, where $\partial_t^{-1}\psi_k$ is defined as

$$\partial_t^{-1}\psi_k(t-s)\equiv\int_0^s\psi_k(t-r)dr,$$

that

$$\Big\|\int_0^t D_E u(s)\,ds\,\Big|_{x_n=0}\Big\|_{\widetilde{\dot B^{\frac12-\frac{1}{2p}}_{\infty,1}(\mathbb{R}_+};\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}$$
$$\leq\sum_{j\in\mathbb{Z}}2^{(-1+\frac{n}{p})j}\sum_{k\in\mathbb{Z}}2^{(\frac12-\frac{1}{2p})k}\Big\|\Big\|(\partial_t^{-1}\psi_k)\underset{(t)}{*}\widetilde{\psi_k}\underset{(t)}{*}\phi_j\underset{(x')}{*}\partial_t\Big(\int_0^t D_E u(s)\,|_{x_n=0}\,ds\Big)\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\infty_t(\mathbb{R}_+)}$$
$$\leq\sum_{j\in\mathbb{Z}}2^{(-1+\frac{n}{p})j}\sum_{k\in\mathbb{Z}}2^{(\frac12-\frac{1}{2p})k}\|\psi_k\|_{L^1_t(\mathbb{R}_+)}\Big\|\Big\|\widetilde{\psi_k}\underset{(t)}{*}\phi_j\underset{(x')}{*}D_E u\,|_{x_n=0}\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R}_+)}$$
$$\leq C\big\|D_E u\,|_{x_n=0}\big\|_{\dot F^{\frac12-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}.$$

The second inequality (8.25) follows from the following estimate:

$$\Big\|\int_0^t D_E u(s)\,ds\,\Big|_{x_n=0}\Big\|_{\widetilde{L^\infty(\mathbb{R}_+};\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}$$
$$\leq\sum_{j\in\mathbb{Z}}2^{\frac{n-1}{p}j}\Big\|\Big\|\int_0^t\phi_j\underset{(x')}{*}D_E u(s)\,|_{x_n=0}ds\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\infty_t(\mathbb{R}_+)}$$
$$\leq\sum_{j\in\mathbb{Z}}2^{\frac{n-1}{p}j}\Big\|\|\phi_j\underset{(x')}{*}D_E u\,|_{x_n=0}\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R}_+)}$$

$$= \Big\| \sum_{j\in\mathbb{Z}} 2^{\frac{n-1}{p}j} \|\phi_j \underset{(x')}{*} D_E u\,|_{x_n=0}\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^1_t(\mathbb{R}_+)} = \big\| D_E u\,|_{x_n=0} \big\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}.$$

□

Proof of Proposition 8.7 From (1.12) and (1.13) and from the regularity assumptions; we notice that the sharp trace estimate implies

$$Du|_{x_n=0},\ p|_{x_n=0} \in \dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\ \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1})) \cap L^1(\mathbb{R}_+;\ \dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1})). \tag{8.26}$$

We first prove the estimate (8.22) holds. Setting

$$F(t,x') \equiv p(t,x',x_n)|_{x_n=0}, \quad G(t,x') \equiv \Pi^{n-1}_{bp}\Big(\int_0^t D_E u(s,x',x_n)ds\Big)\Big|_{x_n=0}$$

in Lemma 8.4 with regarding (8.26), we find that

$$\begin{aligned}
&\|H_p(u,p,E)\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))} \\
&\le C\Big(\|p|_{x_n=0}\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))} + \|p|_{x_n=0}\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big) \\
&\quad\times\Big(\Big\|\big((J(D_E u)^{-1})^{\mathsf{T}} - I\big)\Big|_{x_n=0}\Big\|_{\widetilde{\dot{B}^{\frac{1}{2}-\frac{1}{2p}}_{\infty,1}}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))} \\
&\quad+\Big\|\big((J(D_E u)^{-1})^{\mathsf{T}} - I\big)\Big|_{x_n=0}\Big\|_{\widetilde{L^\infty(\mathbb{R}_+}; \dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big).
\end{aligned} \tag{8.27}$$

The polynomial terms can be estimated as the following way: First the space $\widetilde{L^\infty(\mathbb{R}_+};\dot{B}^{(n-1)/p}_{p,1}(\mathbb{R}^{n-1}))$ is the Banach algebra (see (10.9) in Lemma 10.6 below) and by the estimate (8.25), we have for $Du|_{x_n=0} \in L^1(\mathbb{R}_+;\ \dot{B}^{(n-1)/p}_{p,1}(\mathbb{R}^{n-1}))$ that

$$\begin{aligned}
&\Big\|(J(D_E u)^{-1} - I)^{\mathsf{T}}\,\Big|_{x_n=0}\Big\|_{\widetilde{L^\infty(\mathbb{R}_+};\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))} \\
&\le \sum_{k=1}^{n-1} \Big\|\int_0^t D_E u(s)ds\,\Big|_{x_n=0}\Big\|^k_{\widetilde{L^\infty(\mathbb{R}_+};\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))} \\
&\le C\sum_{k=1}^{n-1} \|D_E u\,|_{x_n=0}\|^k_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1})} \\
&\le C\big(1+\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\big)\sum_{k=1}^{n-1}\|D^2u\|^k_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}.
\end{aligned} \tag{8.28}$$

Secondly by Proposition 10.5 in Appendix, Lemma 8.8 and the boundary bilinear estimate (10.3) in Proposition 10.3, we see that

$$\begin{aligned}
&\left\|((J(D_E u)^{-1})^{\mathsf{T}} - I)\Big|_{x_n=0}\right\|_{\widetilde{\dot B^{\frac12-\frac{1}{2p}}_{\infty,1}}(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))} \\
&\le C\sum_{k=1}^{n-1}\left(\left\|\int_0^t D_E u(s)ds\,\Big|_{x_n=0}\right\|_{\widetilde{\dot B^{\frac12-\frac{1}{2p}}_{\infty,1}}(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))} + \left\|\int_0^t D_E u(s)ds\,\Big|_{x_n=0}\right\|_{\widetilde{L^\infty}(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}\right)^k \\
&\le C\sum_{k=1}^{n-1}\left(\left\|D_E u\,|_{x_n=0}\right\|_{\dot F^{\frac12-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))} + \left\|D_E u\,|_{x_n=0}\right\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}\right)^k \\
&\le C\Big(1+\|\nabla'\eta_0\|_{\dot B^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\Big)\sum_{k=1}^{n-1}\left(\left\|\partial_t u\right\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))} + \left\|D^2 u\right\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}\right)^k
\end{aligned} \tag{8.29}$$

by the sharp trace estimate Proposition 6.2. Combining the estimates (8.27)–(8.29), we obtain (8.22). The estimate (8.23) can also be done in a very similar way. This completes the proof of Proposition 8.7. □

9 The global well-posedness

In this section, we show an outlined proof of Theorem 2.1 (cf. [39]).

Proof of Theorem 2.1 We define the complete metric space

$$X = \left\{\begin{aligned} &u \in C(\overline{\mathbb{R}_+};\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)),\quad \partial_t u, D^2 u, \nabla p \in L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)),\\ &\quad p|_{x_n=0} \in \dot F^{1/2-1/2p}_{1,1}(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1})) \cap L^1(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1})),\quad \|(u,p)\|_X \le M \end{aligned}\right\},$$

where

$$\begin{aligned}
&\|(u,p)\|_X \equiv \\
&\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \|D^2 u\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \|\nabla p\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\
&+ \left\|p|_{x_n=0}\right\|_{\dot F^{\frac12-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))} + \left\|p|_{x_n=0}\right\|_{L^1(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}.
\end{aligned}$$

The constant $M > 0$ is chosen to be small enough depending on the norm of the initial data. Given $(\tilde u, \tilde p) \in X$, we consider the liner inhomogeneous initial boundary value problem:

$$\begin{cases}
\partial_t u - \Delta u + \nabla p = f(\tilde u, E) + f(\tilde p, E) + F_u(\tilde u, E) + F_p(\tilde u, \tilde p, E), & t>0,\ x \in \mathbb{R}^n_+,\\
\qquad \operatorname{div} u = g(\tilde u, E) + (1+\partial_n E)G_{\mathrm{div}}(\tilde u, E), & t>0,\ x\in\mathbb{R}^n_+,\\
\left(\nabla u + (\nabla u)^{\mathsf{T}} - pI\right)\nu_n & \\
\quad = h(\tilde u, E) + h(\tilde p, E) + H_u(\tilde u, E) + H_p(\tilde u, \tilde p, E), & t>0,\ x\in\partial\mathbb{R}^n_+,\\
\quad u(0, x', x_n) = u_0(x), & x\in\mathbb{R}^n_+,
\end{cases} \tag{9.1}$$

where $u_0(x) = \bar{u}_0(x', x_n - E(x', x_n))$, the linear variable coefficient terms are given by (1.24)–(1.27) and the nonlinear terms are (1.28)–(1.32).

We define the map $\Phi : X \to X$ by $(\tilde{u}, \tilde{p}) \to (u, p) \equiv \Phi\big[\tilde{u}, \tilde{p}\big]$ and prove that Φ is contraction on X.

First we show that a priori estimate of $\Phi[u, p]$ in $L^1(\mathbb{R}_+; \dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n))$. Let (u, p) solve (9.1). Applying Theorem 2.4 to the Eq. (9.1), we have by (2.5), Propositions 8.5–8.7 to the nonlinear terms that

$$\big\|\Phi[\tilde{u}, \tilde{p}]\big\|_X \leq C_1\bigg(\|u_0\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)} + \|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\big\|\Phi[\tilde{u}, \tilde{p}]\big\|_X + \sum_{k=1}^{2n-1} M^{k+1}\bigg). \tag{9.2}$$

Therefore if we choose the initial data small enough

$$C_1\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})} < \frac{1}{2}, \qquad 2C_1\sum_{k=1}^{2n-1} M^k < \frac{1}{2}, \qquad 2C_1\|u_0\|_{\dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+)} < \frac{1}{2}M,$$

then we obtain from (9.2) that

$$\|\Phi[\tilde{u}, \tilde{p}]\|_X \leq M.$$

Moreover, for all $(u_1, p_1), (u_2, p_2) \in X$, we know that the difference

$$w = u_1 - u_2, \qquad q = p_1 - p_2$$

satisfy the same estimate (9.2) without $\|u_0\|_{\dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+)}$, i.e.,

$$\big\|\Phi[w, q]\big\|_X \leq C_2\bigg(\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})} + \sum_{k=1}^{2n-1} M^k\bigg)\big\|(w, q)\big\|_X.$$

Therefore if we choose

$$C_2\bigg(\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})} + \sum_{k=1}^{2n-1} M^k\bigg) \leq \frac{1}{2},$$

then it holds that

$$\|\Phi[w, q]\|_X \leq \frac{1}{2}\|(w, q)\|_X,$$

which shows the map

$$\Phi : X \to X$$

is contraction. By the fixed point theorem of Banach–Caccioppoli, there exists a unique fixed point (u, p) of the map Φ in X.

Then the unique fixed point (u, p) satisfies (9.1) with the all right members changed into (u, p) and it is a time global strong solution of (1.22). This completes the proof of Theorem 2.1. $\square$

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Declarations

Conflict of interest The authors declare that there is no conflict of interest.

10 Appendix

10.1 Null-Lagrangian structure

According to Evans [22, section 8.1], we recall the null Lagrangian structure for the Jacobi of a Lipschitz continuous function u.

For $n \in \mathbb{N}$, let A be a $n \times n$ matrix whose components are denoted by $\{a_{kj}\}$ and consider its $\ell \times \ell$ sub-matrix $A^{[\ell]}$ given by

$$A^{[\ell]} = \begin{pmatrix} a_{\sigma_1\tau_1} & \cdots & a_{\sigma_1\tau_\ell} \\ \vdots & \ddots & \vdots \\ a_{\sigma_\ell\tau_1} & \cdots & a_{\sigma_\ell\tau_\ell} \end{pmatrix}, \tag{10.1}$$

where $\sigma_k, \tau_j \in \{1, 2, \ldots, n\}$ with $1 \le \sigma_1 < \sigma_2 < \cdots < \sigma_\ell \le n$ and $1 \le \tau_1 < \tau_2 < \cdots < \tau_\ell \le n$.

Lemma 10.1 [22] *Let $1 \le \ell \le n$ and let $u : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz continuous function and $J(Du)^{[\ell]}$ denotes the $\ell \times \ell$ sub-matrix of the Jacobi matrix $J(Du)$ defined by (10.1), $\mathrm{cof}\,(J(Du)^{[\ell]})_{kj}$ denotes the (k, j)- cofactor and $\mathrm{cof}\,(J(Du)^{[\ell]})$*

be the cofactor matrix. Then for any $x \in \mathbb{R}^n$ *with* $\det(J(Du)(x) \neq 0$ *it holds that*

$$\operatorname{div}_j \big(\operatorname{cof}\,(J(Du)^{[\ell]})\big)_{kj} = 0$$

Naturally the divergence-curl structure leads to the following corollary.

Corollary 10.2 *Let* $1 \le p < \infty$ *and* $1 \le q < \infty$, *For* $D\tilde{u}$, $\tilde{p} \in L^1(\mathbb{R}_+; \dot{B}^{\frac{n}{p}}_{p,1}(\Omega))$, *let* $F_u(\tilde{u})$, $F_p(\tilde{u}, \tilde{p})$, $G_{\text{div}}(\tilde{u})$, $H_u(\tilde{u})$ *and* $J(D\tilde{u})$ *be given by* (1.10)–(1.13). *Then the following identities hold:*

$$F_p(\tilde{u}, \tilde{p}) = -\operatorname{div}\Big(\big(J(D\tilde{u})^{-1} - I\big)\,\tilde{p}\Big), \tag{10.2}$$

$$G_{\text{div}}(\tilde{u}) = -\operatorname{div}\Big(\big(J(D\tilde{u})^{-1} - I\big)\,\tilde{u}\Big). \tag{10.3}$$

Proof of Corollary 10.2 Since $\operatorname{div} \bar{u}(t, y) = 0$, the Liouville theorem implies $\det J(D\tilde{u}) = 1$ and hence $\big(J(D\tilde{u})^{-1}\big)^{\mathsf{T}} = \big((\det J(D\tilde{u}))^{-1}\operatorname{cof} J(D\tilde{u})^{\mathsf{T}}\big)^{\mathsf{T}} = \operatorname{cof} J(D\tilde{u})$. From Lemma 10.1,

$$F_p(\tilde{u}, \tilde{p}) = -\Big(\big(J(D\tilde{u})^{-1}\big)^{\mathsf{T}} - I\Big)\nabla\tilde{p}\Big) = -\operatorname{div}\Big(\big(J(D\tilde{u})^{-1} - I\big)\,\tilde{p}\Big).$$

The other terms follow in a similar observation. □

10.2 Bilinear estimates

The following bilinear estimates over the whole space $\mathbb{R}^n$ are obtained by Abidi–Paicu [2] (cf. [39]).

Proposition 10.3 [2] *Let* $1 \le p, p_1, p_2, \sigma, \lambda_1, \lambda_2 \le \infty$, $1/p \le 1/p_1 + 1/p_2$, $p_1 \le \lambda_2$, $p_2 \le \lambda_1$ *and*

$$\frac{1}{p} \le \frac{1}{p_1} + \frac{1}{\lambda_1} \le 1, \quad \frac{1}{p} \le \frac{1}{p_2} + \frac{1}{\lambda_2} \le 1.$$

and $s_1 + s_2 + n\inf(0, 1 - 1/p_1 - 1/p_2) > 0$ *with* $s_1 + n/\lambda_2 \le n/p_1$, $s_2 + n/\lambda_1 \le n/p_2$ (*and hence* $r = s_1 + s_2 - n\big(1/p_1 + 1/p_2 - 1/p\big) > -n + n/p$).

(1) There exists $C > 0$ *such that for any* $f \in \dot{B}^{s_1}_{p_1,1}$ *and* $g \in \dot{B}^{s_2}_{p_2,1}$ *the following estimate holds*

$$\|fg\|_{\dot{B}^r_{p,1}} \le C\|f\|_{\dot{B}^{s_1}_{p_1,1}}\|g\|_{\dot{B}^{s_2}_{p_2,1}}.$$

(2) If $1 \le p < \infty$ *and* $1 \le q \le pn/(n-p)$ $(1 \le p < n)$ *and* $1 \le q < pn/(p-n)$ $(n \le p < \infty)$, *then for any* $f \in \dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)$ *and* $g \in \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)$,

$$\|f\,g\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)} \le C\|f\|_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)}\|g\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}. \tag{10.4}$$

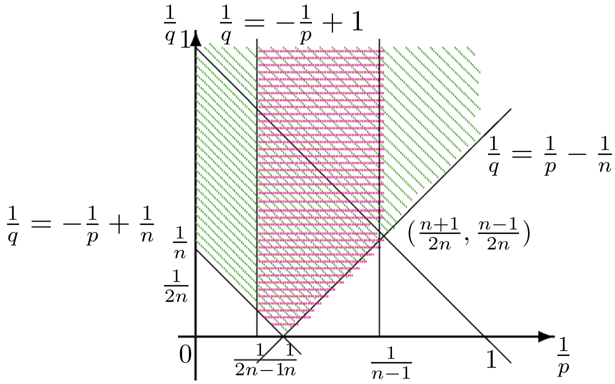

Fig. 3 The possible range of exponents (p, q) in the bilinear estimate (10.4) (the green area)

In particular, for any $f \in \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)$ *and* $g \in \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)$ *with* $1 \le p < 2n$*, the following estimate holds*

$$\|f\, g\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)} \le C\|f\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}\|g\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}. \tag{10.5}$$

(3) *For the boundary case,* $n \ge 2$, $1 \le p < \infty$ *and* $1 \le q \le p(n-1)/(n-p)$ $(1 \le p < n)$ *and* $1 \le q < p(n-1)/(p-n)$ $(n \le p < \infty)$*. Then it holds that*

$$\|f\, g\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1})} \le C\|f\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\|g\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1})}. \tag{10.6}$$

See the possible range of exponents (p, q) to the estimate (10.4) in Fig. 4 below. Hence regarding the region of possible choice of $1/q$, we see that

$$\begin{cases} \dfrac{1}{q} \ge \dfrac{1}{p} - \dfrac{1}{n}, & 1 \le p < n, \\ \dfrac{1}{q} > -\dfrac{1}{p} + \dfrac{1}{n}, & n \le p < \infty. \end{cases}$$

On the other hand, the restriction of the exponent (p, q) for the boundary estimate is given by

$$\begin{cases} \frac{1}{q} \ge \frac{n}{n-1}\frac{1}{p} - \frac{1}{n-1}, & 1 \le p < n, \\ \frac{1}{q} > -\frac{n}{n-1}\frac{1}{p} + \frac{1}{n-1}, & n \le p < \infty \end{cases}.$$

The red lined area in both Figs. 3 and 4 are the possible range of the exponents with boundary and inner nonlinear estimates. If $p = q$ then the possible range for q is limited $p = q < 2n - 1$ as is seen in the Fig. 4.

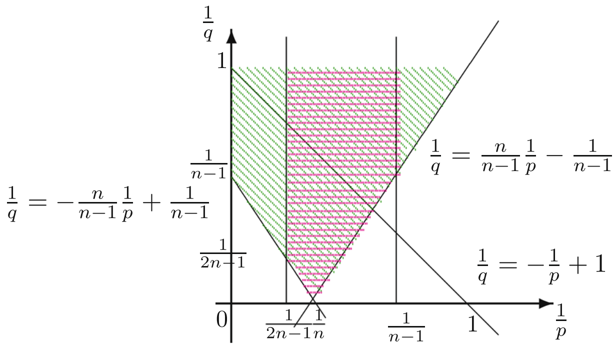

Fig. 4 The possible range of exponents (p, q) at the boundary (the green area)

Since Danchin–Mucha [17] treats the equations depending on the density, the restriction on the exponent p in the solution space $\dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n)$ stems from the restriction on $1 \le p < 2n$ for the above bilinear estimate (10.5). One may improve the restriction by using the divergence-curl free structure of nonlinear terms.

The bilinear estimates as above hold for the case when the two functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ has the divergence structure condition:

$$f \cdot D_x g = D_x(f \cdot g),$$

where D_x denotes any combination of partial derivatives by $x = (x_1, x_2, \dots, x_n)$ of the first order. A typical case is given by the form when f and g satisfies *divergence free-rotation free structure* as $\operatorname{div} f = 0$ and $\operatorname{rot} g = 0$.

Proposition 10.4 (Bilinear estimate under divergence structure) *Let* $1 \le p < \infty$ *and* $f \in \dot{B}^{-1+n/p}_{p,1}$ *and* $g \in \dot{B}^{n/p}_{p,1}$.

(1) If there exists $F = F(x)$ *such that* $f \cdot g = D_x(F \cdot g)$ *with* $f = D_x F(x)$ *in the sense of distribution, where* D_x *is any combination of the first differentiation in* x*. Then*

$$\|f \cdot g\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}} \le C\|f\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}} \|g\|_{\dot{B}^{\frac{n}{p}}_{p,1}}. \tag{10.7}$$

(2) In particular, with additional conditions $\operatorname{div} f = 0$, $\operatorname{rot} g = 0$ *in the distribution sense, it holds*

$$\|f \cdot g\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}} \le C\|f\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}} \|g\|_{\dot{B}^{\frac{n}{p}}_{p,1}}. \tag{10.8}$$

See for the proof, [36, 39].

Proposition 10.5 (The space-time bilinear estimate) *Let* $1 \le \rho \le \infty$ *and* $1 \le p < 2n - 1$ *and* $1 \le q \le p(n-1)/(n-p)$ $(1 \le p < n)$ *and* $1 \le q < p(n -$

$1)/(p-n)$ $(n \le p < 2n-1)$. *Then for* $F \in \dot{B}^{1/2-1/(2p)}_{\rho,1}\widetilde{(\mathbb{R}_+; \dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^{n-1}))} \cap \widetilde{L^{\rho}(\mathbb{R}_+; \dot{B}^{(n-1)/p}_{p,1}(\mathbb{R}^{n-1}))}$ *and*
$G \in \dot{B}^{1/2-1/(2p)}_{\infty,1}\widetilde{(\mathbb{R}_+; \dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^{n-1}))} \cap \widetilde{L^{\infty}(\mathbb{R}_+; \dot{B}^{(n-1)/q}_{q,1}(\mathbb{R}^{n-1}))}$, *it holds that*

$$\begin{aligned}\|FG\|_{\dot{B}^{\frac12-\frac1{2p}}_{\rho,1}\widetilde{(\mathbb{R}_+;\dot{B}^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1}))}} &\le C\Big(\|F\|_{\dot{B}^{\frac12-\frac1{2p}}_{\rho,1}\widetilde{(\mathbb{R}_+;\dot{B}^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1}))}} + \|F\|_{\widetilde{L^{\rho}(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}}\Big)\\ &\quad\times\Big(\|G\|_{\dot{B}^{\frac12-\frac1{2p}}_{\infty,1}\widetilde{(\mathbb{R}_+;\dot{B}^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1}))}} + \|G\|_{\widetilde{L^{\infty}(\mathbb{R}_+;\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1}))}}\Big),\end{aligned}$$

where we recall the definition of the function class defined in (8.9).

The proof of Proposition 10.5 can be shown in a very similar way to the proof of the related boundary bilinear estimate shown in [39, Proposition 7.6] with an aid of the estimate (10.3) above.

Lemma 10.6 $\widetilde{L^{\infty}(\mathbb{R}_+; \dot{B}^{(n-1)/p}_{p,1}(\mathbb{R}^{n-1}))}$ *is the Banach algebra, namely for any* $f, g \in \widetilde{L^{\infty}(\mathbb{R}_+; \dot{B}^{(n-1)/p}_{p,1}(\mathbb{R}^{n-1}))}$ *it holds*

$$\|fg\|_{\widetilde{L^{\infty}(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}} \le C\|f\|_{\widetilde{L^{\infty}(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}}\|g\|_{\widetilde{L^{\infty}(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}}. \quad (10.9)$$

See for the proof of above Proposition 10.5 and Lemma 10.6,[39].

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