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On uniqueness of mild $L^{3,\infty}$-solutions on the whole time axis to the Navier–Stokes equations in unbounded domains

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Abstract

This paper is concerned with the uniqueness of bounded continuous $L^{3,\infty}$-solutions on the whole time axis $\mathbb{R}$ or the half-line $(-\infty, T)$ to the Navier–Stokes equations in 3-dimensional unbounded domains. When Ω is an unbounded domain, it is known that a small solution in $BC(\mathbb{R}; L^{3,\infty})$ is unique within the class of solutions which have sufficiently small $L^{\infty}(\mathbb{R}; L^{3,\infty})$-norm; i.e., if two solutions u and v exist for the same force f, *both* u and v are small, then the two solutions coincide. There is another type of uniqueness theorem. Farwig et al. (Commun Partial Differ Equ 40:1884–1904, 2015) showed that if two solutions u and v exist for the same force f, u is small and if v has a precompact range $\mathscr{R}(v) := \{v(t); -\infty < t < T\}$ in $L^{3,\infty}$, then the two solutions coincide. However, there exist many solutions which do not have precompact range. In this paper, instead of the precompact range condition, by assuming some decay property of $v(x,t)$ with respect to the spatial variable x near $t = -\infty$, we show a modified version of the above-mentioned uniqueness theorem. As a by-product, in the *half-space* $\mathbb{R}^3_+$, we obtain a non-existence result of backward self-similar $L^{3,\infty}$-solutions sufficiently close to some homogeneous function $Q(x/|x|)/|x|$ in a certain sense.

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1 Introduction

The motion of a viscous incompressible fluid in 3-dimensional domains Ω is governed by the Navier–Stokes equations:

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$$\text{(N-S)} \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = f, & t \in \mathbb{R}, \quad x \in \Omega, \\ \operatorname{div} u = 0, & t \in \mathbb{R}, \quad x \in \Omega, \\ u|_{\partial\Omega} = 0, & t \in \mathbb{R}, \end{cases}$$

where $u = (u^1(x,t), u^2(x,t), u^3(x,t))$ and $p = p(x,t)$ denote the velocity vector and the pressure, respectively, of the fluid at the point $(x,t) \in \Omega \times \mathbb{R}$. Here f is a given external force. In this paper we consider the uniqueness of bounded mild L^3_w-solutions to (N-S) on the whole time axis $(-\infty, \infty)$ or the half-line $(-\infty, T)$ in *unbounded* domains Ω. Typical examples of such solutions are stationary, periodic-in-time and almost periodic-in-time solutions.

In case where $\Omega \subset \mathbb{R}^3$ is bounded, the existence and uniqueness of time-periodic solutions were considered by several authors; see e.g. [12] and references therein. Maremonti [38, 39] was the first to prove the existence of unique time-periodic regular solutions to (N-S) in *unbounded* domains, namely for $\Omega = \mathbb{R}^3$ and $\Omega = \mathbb{R}^3_+$. In the case of more general unbounded domains, the existence of time-periodic solutions was proven by e.g. Kozono–Nakao [30], Maremonti–Padula [40], Salvi [47], Yamazaki [55], Galdi–Sohr [21], Kubo [35], Crispo–Maremonti [6], Kang–Miura–Tsai [28], Geissert–Hieber–Nguyen [22], Okabe–Tsutsui [46], Galdi–Kyed [20], Galdi [19], Eiter–Kyed [8] and Eiter–Kyed–Shibata [9]. Some of them constructed solutions to (N-S) on the whole time axis without any time-periodic condition on the external force. In particular, Kozono–Nakao [30] introduced a new approach to the study of time-periodic solutions by using the following integral equation:

$$u(t) = \int_{-\infty}^{t} e^{-(t-\tau)A} P(-u \cdot \nabla u + f)(\tau)\, d\tau, \tag{1.1}$$

where the definitions of P and A will be mentioned later. In [30], they constructed global L^3-solutions on the whole time axis $(-\infty, \infty)$ to (1.1), which are called mild solutions to (N-S) on $(-\infty, \infty)$. More precisely, Kozono–Nakao [30] showed that if the domain Ω is $\mathbb{R}^n$, $\mathbb{R}^n_+$ $(n \geq 3)$ or an n-dimensional exterior domain $(n \geq 4)$ and if the external force f is small in some function space, then there exists a small L^n-solution to (1.1) in the class

$$\left\{ u \in BC(\mathbb{R}; L^n);\ \sup_{t \in \mathbb{R}} \|u(t)\|_{L^r} + \sup_{t \in \mathbb{R}} \|\nabla u(t)\|_{L^q} < \delta \right\} \quad \left(2 < r < n, \frac{n}{2} < q < n\right), \tag{1.2}$$

where δ is a small number. They also showed that this small solution is unique within the class (1.2). Furthermore, they proved that this small L^n-solution is a strong solution to (N-S) and that if f is a small time-periodic function in some function space, then their small solution u is also time-periodic. In the case where $\Omega \subset \mathbb{R}^n (n \geq 3)$ is a perturbed half space or an aperture domain with $\partial\Omega \in C^\infty$, Kubo [35] proved the same uniqueness and existence theorem as Kozono–Nakao [30]. When $\Omega = \mathbb{R}^3, \mathbb{R}^3_+$, a perturbed half space or an aperture domain with $\partial\Omega \in C^\infty$, Farwig, Nakatsuka and the present author [11] showed that if f is small and has a precompact range in some function space, then the small solution u also has a precompact range in L^3.

In the case where Ω is a 3-dimensional exterior domain with $\partial\Omega \in C^\infty$, Yamazaki [55] succeeded in proving the existence of small mild $L^{3,\infty}$-solutions on the whole time axis, if the external force f is sufficiently small in some sense. More precisely, he showed the existence of small mild $L^{n,\infty}$-solutions in the case Ω is an n-dimensional exterior domain for $n \geq 3$. Here $L^{p,q}$ denotes the Lorentz space and $L^{p,\infty}$ is equivalent to the weak-L^p space (L^p_w). Yamazaki [55] also proved uniqueness of small solutions within the class

$$\left\{ v \in BUC(\mathbb{R}; L^{n,\infty}_\sigma(\Omega));\ \sup_{t\in\mathbb{R}} \|v(t)\|_{L^{n,\infty}} < \delta \right\},$$

where $\delta = \delta(\Omega, n)$ is a small number and $L^{n,\infty}_\sigma$ will be defined in the next section. See Lemma 8 below. Furthermore, if $n = 3$, Kang–Miura–Tsai [28] showed the existence of mild solutions u on the whole time axis with the spatial uniform decay:

$$\sup_t \sup_{|x|>L} |x|^\alpha |v(x,t) - V(x)| < \infty \tag{1.3}$$

for some $\alpha > 1$, $L > 0$ and some function $V(x)$ with $\sup_{|x|>L} |x||V(x)| < \infty$, if f satisfies adequate conditions. Note that $V \in L^{3,\infty}(\Omega)$ and $\sup_t \|v(t) - V\|_{L^{r,\infty}(\Omega)} < \infty$ for some $1 < r < 3$. They also dealt with the inhomogeneous boundary value problem.

Concerning the uniqueness of solutions on the whole time axis, roughly speaking, it was shown in [30, 35, 55] that a small solution in some function spaces (e.g. $BC(\mathbb{R}; L^{3,\infty}(\Omega))$) is unique within the class of solutions which are sufficiently small; i.e., if u and v are solutions for the same force f and if *both of them* are small, then $u = v$. It is notable that, concerning time-periodic solutions, Galdi–Sohr [21] show that a small time-periodic solution is unique within the larger class of all periodic weak solutions v with $\nabla v \in L^2(0, T_{per}; L^2)$, satisfying the energy inequality $\int_0^{T_{per}} \|\nabla v\|^2_{L^2}\, d\tau \leq -\int_0^{T_{per}} (F, \nabla v)\, d\tau$ and mild integrability conditions on the corresponding pressure; here T_{per} is a period of F and $f = \nabla \cdot F$.

Another type of uniqueness theorem for solutions on the whole time axis or the half-line $(-\infty, T)$ was proven by Farwig, Nakatsuka and the present author [10] without time-periodic condition, where it was proven that if u and v are solutions for the same force f, u is small and if v satisfies the precompact range condition (PRC):

$$(PRC)\quad \mathscr{R}(v) := \{v(t) \in L^{3,\infty};\ t \in (-\infty, T)\} \text{ is precompact in } L^{3,\infty},$$

then $u = v$ on $(-\infty, T)$. Note that the smallness condition is assumed only on one of solutions. See also [16, 17, 43, 44, 51].

Since almost periodic-in-time solutions satisfy (PRC), this uniqueness theorem is applicable to almost periodic-in-time solutions. In [10], without (PRC), it was also shown that a similar uniqueness theorem holds under the smallness condition of u, (1.3) and the condition: $u, v \in BC(-\infty, T; L^{3,\infty} \cap L^q)$ for some $q > 3$. On the other hand, there are many mild $L^{3,\infty}$-solutions to (N-S) that satisfy neither (PRC) nor (1.3). For example, traveling solutions $u(x,t) = u(x - a\cdot t, 0)$ ($a \neq 0$, $u \not\equiv 0$) satisfy neither (PRC) nor (1.3), when $\Omega = \mathbb{R}^3$. Other examples are backward self-similar solutions $u(x,t) = \frac{1}{\sqrt{-t}} U(\frac{x}{\sqrt{-t}})$ in $\mathbb{R}^3 \times (-\infty, -1)$, when $U \not\equiv 0$ and $U \in L^3$.

Very recently the present author showed a similar uniqueness theorem without (PRC) or (1.3) in [52, Theorem 3], where the L^r $(r < 3)$ and L^3 integrabilities of solutions are assumed instead of (PRC). These integrabilities, however, are too restrictive for solutions to the 3D exterior problem. Indeed, if Ω is a 3D exterior domain and u is a stationary weak solution to (N-S), then u cannot belong to L^3, excepting in the case where the net force exerted on $\partial\Omega$ is equal to zero. See e.g. [32]. In the present paper, we will improve the uniqueness theorems given in [10, 52] in order to deal with $L^{3,\infty}$-solutions to the 3D exterior problem.

Throughout this paper we impose the following assumption on the domain.

Assumption 1 $\Omega \subset \mathbb{R}^3$ is an exterior domain, the half-space $\mathbb{R}^3_+$, the whole space $\mathbb{R}^3$, a perturbed half-space, or an aperture domain with $\partial\Omega \in C^{2+\nu}$, $0 < \nu < 1$.

Here, the assumption $\partial\Omega \in C^{2+\nu}$ means that for each $x \in \partial\Omega$ there are an open ball B_x centered at x and a function $g_x \in C^{2+\nu}(G)$ for some domain $G \subset \mathbb{R}^2$ such that after a rotation of the Cartesian coordinates, if necessary,

$y_3 > g_x(y_1, y_2)$ for all $y = (y_1, y_2, y_3) \in \Omega \cap B_x$, $y_3 < g_x(y_1, y_2)$ for all $y \in (\mathbb{R}^3 \backslash \bar{\Omega}) \cap B_x$ and $y_3 = g_x(y_1, y_2)$ for all $y \in (\partial\Omega) \cap B_x$.

The definitions of a perturbed half-space and an aperture domain are as follows (See e.g. [13–15]). Let $\mathbb{R}^3_+ := \{x \in \mathbb{R}^3; x_3 > 0\}$ and $\Omega \subset \mathbb{R}^3$ be a domain. If there exists an open ball $B \subset \mathbb{R}^3$ such that $\Omega \cup B = \mathbb{R}^3_+ \cup B$, then Ω is called a perturbed half-space. If there exists an open ball $B \subset \mathbb{R}^3$ such that $\Omega \cup B = \mathbb{R}^3_+ \cup \mathbb{R}^3_- \cup B$ where $d > 0$ and $\mathbb{R}^3_- := \{x \in \mathbb{R}^3; x_3 < -d\}$, then Ω is called an aperture domain. Since the aperture domain Ω should be connected, there are some apertures and one can take two disjoint subdomains $\Omega_\pm$ and a smooth 2-dimensional manifold M such that $\Omega = \Omega_- \cup M \cup \Omega_+$, $\Omega_\pm \backslash B = \mathbb{R}^3_\pm \backslash B$ and $M \cup \partial M = \partial\Omega_- \cap \partial\Omega_+$. We do not need to assume the connectedness of $\mathbb{R}^3 \backslash \bar{\Omega}$.

Let $BC(I; X)$ denote the set of all bounded continuous functions on an interval I with values in a Banach space X. The open ball with center x and radius $R > 0$ will be denoted by $B_R(x)$. Let $\mu(A)$ be the 3-dimensional Lebesgue measure of $A \subset \mathbb{R}^3$. The definition of mild $L^{3,\infty}$-solutions will be written in the next section. Let

$$\tilde{L}^{3,\infty}_\sigma(\Omega) := \overline{L^{3,\infty}_\sigma(\Omega) \cap L^\infty(\Omega)}^{\|\cdot\|_{L^{3,\infty}}},$$

where the definition of $L^{3,\infty}_\sigma(\Omega)$ will be written in the next section. Now our main results on uniqueness of mild $L^{3,\infty}$-solutions reads as follows.

Theorem 1 *Let Ω satisfy Assumption 1. There exists constants $\delta(\Omega), c_*(\Omega) > 0$ with the following property: Let $T \le \infty$, u and v be mild $L^{3,\infty}$-solutions to (N-S) on $(-\infty, T)$ for the same force f,*

$$u, v \in BC((-\infty, T); \tilde{L}^{3,\infty}_{\sigma}(\Omega)) \tag{1.4}$$

and

$$\limsup_{t\to-\infty} \|u(t)\|_{L^{3,\infty}(\Omega)} < \delta. \tag{1.5}$$

Assume that there exists a function $V = V(x) \in L^{3,\infty}(\Omega)$ *such that*

$$\limsup_{t\to-\infty} \|v(t) - V\|_{L^{3,\infty}(\{|v(t)-V|\le\gamma_0|t|^{-1/2}\})} < \delta, \tag{1.6}$$

$$\text{where } \gamma_0 := c_* \cdot \left(\limsup_{t\to-\infty} \|v(t)\|_{L^{3,\infty}(\Omega)} + \|V\|_{L^{3,\infty}(\Omega)} + 1 + \delta\right)^7. \tag{1.7}$$

Then $u \equiv v$ *on* $(-\infty, T)$. *Here* $\{|v(t) - V| \le \gamma_0|t|^{-1/2}\} := \{x \in \Omega\ ;\ |v(x,t) - V(x)| \le \gamma_0|t|^{-1/2}\}$.

Corollary 2 *Let* Ω *satisfy Assumption* 1 *and* $1 < r < 3$. *There exist a constant* $c_{**}(\Omega, r) > 0$ *with the following property: Let* $T \le \infty$, u *and* v *be mild* $L^{3,\infty}$-*solutions to* (N-S) *on* $(-\infty, T)$ *for the same force* f. *Assume that* (1.4) *and* (1.5) *hold. Furthermore assume that there exists a function* $V \in L^{3,\infty}(\Omega)$ *such that*

$$\limsup_{t\to-\infty} \frac{\|v(t) - V\|_{L^{r,\infty}(\Omega)}}{|t|^{\frac{1}{2}(\frac{3}{r}-1)}} < c_{**} \cdot \left(\limsup_{t\to-\infty} \|v(t)\|_{L^{3,\infty}(\Omega)} + \|V\|_{L^{3,\infty}(\Omega)} + 1 + \delta\right)^{-7(\frac{3}{r}-1)}. \tag{1.8}$$

Then $u \equiv v$ *on* $(-\infty, T)$.

By the interpolation inequality:

$$\|g(t)\|_{L^{3,\infty}(\{|g(t)|\le M\})} \le C\|g(t)\|^{1-r/3}_{L^{\infty}(\{|g(t)|\le M\})}\|g(t)\|^{r/3}_{L^{r,\infty}} \le CM^{1-r/3}\|g(t)\|^{r/3}_{L^{r,\infty}},$$

we see that (1.8) implies (1.6) if c_{**} is sufficiently small. Hence, Corollary 2 is a direct consequence of Theorem 1.

Remark 1 (i) When Ω is an aperture domain, all mild $L^{3,\infty}$-solutions are assumed to belong to $L^{3,\infty}_{\sigma}$ and hence satisfy the vanishing flux condition $\phi(u(t)) = 0$ for all $t \in (-\infty, T)$, see the next section.

(ii) Condition (1.8) can be replaced by the following simpler condition:

$$\limsup_{t\to-\infty} \|v(t) - V\|_{L^{r,\infty}(\Omega)} < \infty \tag{1.9}$$

for some $1 < r < 3$ and for some $V \in L^{3,\infty}$, since (1.9) implies that the L.H.S. of (1.8) vanishes. We emphasis that (1.9) does not need any smallness conditions on v and V themselves. Furthermore, when $\Omega = \mathbb{R}^3$, $\mathbb{R}^3_+$, 3-dimensional perturbed half-spaces or aperture domains, letting $V \equiv 0$, we see that Corollary 2 is applicable to mild L^3-solutions on $(-\infty, \infty)$ in the class (1.2) given by Kozono–Nakao [30] and Kubo [35], since $L^3_\sigma(\Omega) \subset \tilde{L}^{3,\infty}_\sigma(\Omega)$.

(iii) For $d(t) := |t|^{1/2}/\log|t|$,

$$\|v(t) - V\|_{L^{3,\infty}(\{|v(t)-V|\le\gamma_0|t|^{-1/2}\}\cap B_{d(t)}(0))} \le C\{\mu(B_{d(t)}(0))\}^{1/3}\gamma_0|t|^{-1/2} \to 0$$

as $t \to -\infty$. Then Condition (1.6) is equivalent to

$$\limsup_{t\to-\infty} \|v(t) - V\|_{L^{3,\infty}(\{|v(t)-V|\le\gamma_0|t|^{-1/2}\}\cap\{|x|\ge d(t)\})} < \delta.$$

Hence, roughly speaking, (1.6) requires only that v is close to some function $V \in L^{3,\infty}$ in the $L^\infty L^{3,\infty}$-topology in the area near $|x| = \infty$ and $t = -\infty$. In other words, v is assumed to behave like $V(x)$ as $|x| \to \infty$ near $t = -\infty$.

(iv) In the case Ω is a 3D exterior domain, Yamazaki [55] proved the existence of bounded continuous mild $L^{3,\infty}$-solutions u on the whole time axis, if f can be written in the form $f = \nabla \cdot F$, $F \in BUC(\mathbb{R}; L^{3/2,\infty})$ and F is sufficiently small. We note that, in addition to this smallness condition on F, if we assume $f \in BC(\mathbb{R}; L^{3,\infty})$, then standard arguments easily prove that Yamazaki's small solution u belongs to $L^\infty(\mathbb{R}; L^9) \cap BC(\mathbb{R}; L^{3,\infty}_\sigma) \subset BC(\mathbb{R}; \tilde{L}^{3,\infty}_\sigma)$; see [16, Remark 2]. Moreover, the existence of small mild solutions with (1.3) was also proven by Kang–Miura–Tsai [28] if f satisfies some conditions. Since (1.3) implies (1.9) and hence (1.8), Theorem 1 is applicable to their solutions. We also note that Corollary 2 is an improvement of our previous uniqueness theorem given in [52, Theorem 3], where the conditions $u, v \in BC((-\infty, T); L^3_\sigma)$, (1.5) and $\limsup_{t\to-\infty} \|v(t)\|_{L^r} < \infty$ $(r < 3)$ were assumed.

In Theorem 1, the function $V = V(x)$ is assumed to be a function of x and independent of time-variable t. When V is assumed to be a function of (x, t), we have:

Theorem 3 *Let Ω satisfy Assumption 1. There exists a constant $\delta(\Omega) > 0$ with the following property: Let $T \le \infty$ and let u and v be mild $L^{3,\infty}$-solutions to (N-S) on $(-\infty, T)$ for the same force f. Assume that (1.4) and (1.5) hold. Furthermore assume that there exists a function $V \in BC((-\infty, T); L^{3,\infty}(\Omega))$ such that*

$$\limsup_{t\to-\infty} \|v(t) - V(t)\|_{L^{3,\infty}(\{|v(t)-V(t)|\le\eta\})} < \delta \quad \text{for some constant } \eta > 0 \tag{1.10}$$

and the range $\mathscr{R}(V) := \{V(t) \in L^{3,\infty}(\Omega);\ t \in (-\infty, T)\}$ can be covered by finitely many open balls of radius $\delta > 0$, i.e., there are finitely many functions $\{V_l\}_{l=1}^N \subset L^{3,\infty}(\Omega)$ satisfying

$$\mathscr{R}(V) \subset \bigcup_{l=1}^N \left\{\theta \in L^{3,\infty}(\Omega);\ \|\theta - V_l\|_{L^{3,\infty}(\Omega)} < \delta\right\}. \tag{1.11}$$

Then $u \equiv v$ *on* $(-\infty, T)$. *Here* $\{|v(t)-V(t)| \le \eta\} := \{x \in \Omega;\ |v(x,t) - V(x,t)| \le \eta\}$.

Remark 2 (i) Condition (1.10) is more restricted compared to Condition (1.6) in some sense, since the constant η does not decay as $t \to -\infty$.

(ii) If $\sup_{t \le t_0} \|v(t) - V(t)\|_{L^{r,\infty}} (=: A) < \infty$ for some $1 < r < 3$ and $t_0 < T$, then

$$\sup_{t \le t_0} \|v(t) - V(t)\|_{L^{3,\infty}(\{|v(t)-V(t)| \le \eta\})} \le C\eta^{1-r/3} A^{r/3} < \delta$$

for all $\eta \in \left(0, \left(\delta/(CA^{r/3})\right)^{3/(3-r)}\right)$. Hence, Condition (1.10) can be replaced by

$$\limsup_{t \to -\infty} \|v(t) - V(t)\|_{L^{r,\infty}(\Omega)} < \infty \quad \text{for some } 1 < r < 3.$$

(iii) If v has a precompact range in $L^{3,\infty}$, then by setting $V = v$ we can see that (1.10) and (1.11) hold. Hence Theorem 3 is an improvement of the uniqueness theorem given in [10] with the precompact range condition.

Remark 3 In Theorems 1 and 3 and Corollary 2, Condition (1.4) can be replaced by the condition:

$$u, v \in \left\{ g = g_1 + g_2 \in BC(I; L^{3,\infty}_\sigma(\Omega)); g_2 \in BC(I; L^{3,\infty} \cap L^\infty), \sup_{t \in I} \|g_1(t)\|_{3,\infty} \le \kappa \right\},$$

where $I = (-\infty, T)$ and κ is a small constant depending only on Ω. See [10, Remark 1(iv)].

In the celebrated papers Nečas–Růžička–Šverák [45] and Tsai [53], the non-existence theorems of backward self-similar solutions in $L^3(\mathbb{R}^3)$ and $L^q(\mathbb{R}^3)$ for $q \in (3, \infty]$ were proven, respectively. More precisely, Nečas–Růžička–Šverák [45] proved that if $\frac{1}{\sqrt{a(T-t)}} v(\frac{x}{\sqrt{a(T-t)}})$, $a > 0$, is a backward self-similar solution to (N-S) for $f = 0$ and if $v \in L^3(\mathbb{R}^3)$, then $v = 0$ in $\mathbb{R}^3$. Tsai [53] proved that if $v \in L^q(\mathbb{R}^3)$ for some $3 < q \le \infty$, then v is constant in $\mathbb{R}^3$ and hence $v = 0$ if $q < \infty$. Moreover, Tsai [53] also proved the non-existence theorem of backward self-similar solutions satisfying local energy estimate. Their results were proven for the case where the domain Ω is the 3D whole space $\mathbb{R}^3$. In the case where Ω is the half-space $\mathbb{R}^3_+$, as a by-product of Corollary 2 we have the following non-existence result.

Corollary 4 *Let* $\Omega = \mathbb{R}^3_+$, $a > 0$, $1 < r < 3$ *and functions* $v, V, R \in L^{3,\infty}(\mathbb{R}^3_+)$ *and* $Q \in C(S^2)$ *satisfy*

$$v = V + R, \tag{1.12}$$

$$v \in \tilde{L}^{3,\infty}_\sigma(\mathbb{R}^3_+), \quad R \in L^{r,\infty}(\mathbb{R}^3_+) \cap L^{3,\infty}(\mathbb{R}^3_+), \tag{1.13}$$

$$V(x) = \frac{Q(x/|x|)}{|x|} \quad \textit{for all } x \in \mathbb{R}^3_+, \tag{1.14}$$

$$\|R\|_{L^{r,\infty}} < c_{**} \cdot (\|v\|_{L^{3,\infty}} + \|V\|_{L^{3,\infty}} + 1 + \delta)^{-7(\frac{3}{r}-1)} a^{-\frac{1}{2}(\frac{3}{r}-1)}. \tag{1.15}$$

Assume $\frac{1}{\sqrt{-at}}v(\frac{x}{\sqrt{-at}})$ *is a mild solution of* (*N-S*) *on* $(-\infty, 0)$ *in* $\mathbb{R}^3_+$ *for* $f = 0$. *Then* $v \equiv 0$. *Here* $c_{**} = c_{**}(\Omega, r)$ *is the constant given in Corollary* 2.

We note that if functions v, V, R, Q satisfy (1.12)–(1.14), then $\frac{1}{\sqrt{-at}}v(\frac{x}{\sqrt{-at}}) \in BC((-\infty, 0); \tilde{L}^{3,\infty}(\mathbb{R}^3_+))$, see Appendix. Since $\frac{1}{\sqrt{-at}}V(\frac{x}{\sqrt{-at}}) = V(x)$, we see that $V(x) - \frac{1}{\sqrt{-at}}v(\frac{x}{\sqrt{-at}})(= \frac{1}{\sqrt{-at}}R(\frac{x}{\sqrt{-at}}))$ satisfies Condition (1.8). Thus, by letting $u = 0$, we see that Corollary 2 directly yields Corollary 4.

Remark 4 (i) For any given $V(x) = Q(x/|x|)/|x| \in L^{3,\infty}_\sigma(\mathbb{R}^3_+)$ with $Q \in C^1(S^2)$ and $1 < r < 3$, we can construct functions $v \in \tilde{L}^{3,\infty}_\sigma(\mathbb{R}^3_+) \cap W^{2,q}(\mathbb{R}^3_+) \cap W^{1,q}_0(\mathbb{R}^3_+)$ $(q > 3)$ and $R \in L^{3,\infty}(\mathbb{R}^3_+) \cap L^{r,\infty}(\mathbb{R}^3_+)$ satisfying (1.12)–(1.15), if we do not assume that $\frac{1}{\sqrt{-at}}v(\frac{x}{\sqrt{-at}})$ is a solution to (N-S) for $f = 0$.

(ii) Corollary 4 also holds for $\Omega = \mathbb{R}^3$. Compared with the results in [45, 53], Condition (1.15) is very restrictive, since (1.15) means that v is sufficiently close to the homogeneous function V.

2 Preliminaries

In this section, we introduce some notations, function spaces and key lemmata. Let $C^\infty_{0,\sigma}(\Omega) = C^\infty_{0,\sigma}$ denote the set of all C^∞-real vector fields $\phi = (\phi^1, \ldots, \phi^n)$ with compact support in Ω such that $\operatorname{div} \phi = 0$. Then $L^r_\sigma(\Omega) = L^r_\sigma$, $1 < r < \infty$, is the closure of $C^\infty_{0,\sigma}$ with respect to the L^r-norm $\|\cdot\|_r$. Concerning Sobolev spaces we use the notations $W^{k,p}(\Omega)$ and $W^{k,p}_0(\Omega)$, $k \in \mathbb{N}$, $1 \le p \le \infty$. Note that very often we will simply write L^r and $W^{k,p}$ instead of $L^r(\Omega)$ and $W^{k,p}(\Omega)$, respectively. Let $L^{p,q}(\Omega)$, $1 < p < \infty, 1 \le q \le \infty$, denote the Lorentz spaces and $\|\cdot\|_{p,q}$ the norm (not quasi-norm) of $L^{p,q}(\Omega)$; for the definition and properties of $L^{p,q}(\Omega)$, see e.g. [1]. The symbol $(\cdot, \cdot)$ denotes the L^2-inner product and the duality pairing between $L^{p,q}$ and $L^{p',q'}$, where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. We note that $L^{p,\infty} = L^p_w$(weak-L^p space) and $L^{p,p} = L^p$ with equivalent norms. Moreover, when $1 < p < \infty$ and $1 \le q < \infty$, then the dual space of $L^{p,q}$ is isometrically isomorphic to $L^{p',q'}$.

In this paper, we denote by C various constants. In particular, $C = C(*, \ldots, *)$ denotes a constant depending on the quantities appearing in the parentheses.

Let us recall the Helmholtz decomposition: $L^r(\Omega) = L^r_\sigma \oplus G_r$ $(1 < r < \infty)$, where $G_r = \{\nabla p \in L^r;\, p \in L^r_{loc}(\overline{\Omega})\}$, see Miyakawa [42], Simader–Sohr [50], Borchers–Miyakawa [2], and Farwig–Sohr [13, 15]; P_r denotes the projection operator from L^r onto L^r_σ along G_r. The Stokes operator A_r on L^r_σ is defined by $A_r = -P_r\Delta$ with domain $D(A_r) = W^{2,r} \cap W^{1,r}_0 \cap L^r_\sigma$. It is known that $(L^r_\sigma)^*$(the dual space of $L^r_\sigma) = L^{r'}_\sigma$ and A^*_r(the adjoint operator of A_r)$= A_{r'}$, where $1/r + 1/r' = 1$. It is shown by Giga [23], Borchers–Sohr [5], Giga–Sohr [25], Borchers–Miyakawa [2] and Farwig–Sohr [13, 15] that $-A_r$ generates a holomorphic semigroup $\{e^{-tA_r}; t \ge 0\}$ of class C_0 in L^r_σ. Since $P_r u = P_q u$ for all $u \in L^r \cap L^q$ $(1 < r, q < \infty)$ and since $A_r u = A_q u$ for all $u \in D(A_r) \cap D(A_q)$, for simplicity, we shall abbreviate $P_r u$, $P_q u$

as Pu for $u \in L^r \cap L^q$ and $A_r u$, $A_q u$ as Au for $u \in D(A_r) \cap D(A_q)$, respectively. By real interpolation, we define $L^{p,q}_\sigma(\Omega) = L^{p,q}_\sigma$ by

$$L^{p,q}_\sigma := [L^{p_0}_\sigma, L^{p_1}_\sigma]_{\theta,q}$$

where $1 < p_0 < p < p_1 < \infty$, $\theta \in (0,1)$, $q \in [1,\infty]$ satisfy $1/p = (1-\theta)/p_0 + \theta/p_1$. In the case where Ω is an aperture domain, since $L^p_\sigma = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_p}$ is characterized as

$$L^p_\sigma(\Omega) = \{u \in L^p(\Omega);\ \nabla \cdot u = 0,\ u \cdot \nu|_{\partial\Omega} = 0,\ \phi(u) = 0\},$$

see [15], and since $L^{p,q}_\sigma(\Omega) \subset L^{p_1}_\sigma + L^{p_2}_\sigma$, all u belonging to $L^{p,q}_\sigma$ satisfy the vanishing flux condition $\phi(u) = 0$. Here $\phi(u) = \int_M N \cdot u\, dS$ and N is the unit normal vector on M directed to Ω_-.

Recall that, for $1 < p < \infty$ and for a measurable set D, the weak-$L^p(D)$ norm is defined by

$$\|f\|_{L^p_w(D)} := \sup_{\tau > 0} \tau (\mu\{x \in D;\ |f(x)| > \tau\})^{1/p},$$

which is equivalent to $\|f\|_{L^{p,\infty}(D)}$, as mentioned before. It is known that, for $1 < p < \infty$,

$$\|f\|_{L^p_w(D)} \le \sup_{E \subset D,\ 0 < \mu(E) < \infty} \mu(E)^{-1+1/p} \int_E |f(x)| dx \le \frac{p}{p-1} \|f\|_{L^p_w(D)}, \tag{2.1}$$

where the supremum is taken over all measurable subsets E of the domain D with $0 < \mu(E) < \infty$, see e.g. [4, 24].

Now, we define mild $L^{3,\infty}$-solutions to (N-S) according to [55]. A similar definition was introduced in [31] for mild L^3-solutions.

Definition 1 Let $T \le \infty$ and $f \in L^1_{loc}(-\infty, T; D(A_p)^* + D(A_q)^*)$ for some $1 < p, q < \infty$.

A function $v \in C((-\infty, T); L^{3,\infty}_\sigma)$ is called a mild $L^{3,\infty}$-solution to (N-S) on $(-\infty, T)$ if v satisfies

$$\begin{aligned}(v(t), \phi) = \big(e^{-(t-s)A} v(s), \phi\big) + \int_s^t \Big(\big(v(\tau) \cdot \nabla e^{-(t-\tau)A}\phi, v(\tau)\big) \\ + (f(\tau), e^{-(t-\tau)A}\phi)\Big)\, d\tau\end{aligned} \tag{2.2}$$

for all $\phi \in C^\infty_{0,\sigma}$ and all $-\infty < s < t < T$.

Mild $L^{3,\infty}$-solutions to the initial-boundary value problem for (N-S) on the interval $[0, T)$ are defined similarly, so we do not write its definition here. For a moment let us consider the case where $\int_s^t (f(\tau), e^{-(t-\tau)A}\phi) d\tau$ converges as $s \to -\infty$ for all $\phi \in$

$C_{0,\sigma}^{\infty}$. E.g., this holds true by (2.8) below when $f = \nabla \cdot F$ with $F = (F_{ij})_{i,j=1,2,3} \in L^{\infty}(-\infty, T; L^{3/2,\infty})$. Since moreover $\lim_{s \to -\infty} e^{-(t-s)A}\phi = 0$ in $L_{\sigma}^{3/2,1}$, we conclude from Lemma 7 below that in this case (2.2) for $v \in L^{\infty}(-\infty, T; L_{\sigma}^{3,\infty})$ is equivalent to

$$(v(t), \phi) = \int_{-\infty}^{t} \big((v \cdot \nabla e^{-(t-\tau)A}\phi, v)(\tau) + (f(\tau), e^{-(t-\tau)A}\phi)\big)\, d\tau \tag{2.3}$$

for all $\phi \in C_{0,\sigma}^{\infty}$ and all $t < T$. Note that this holds for all $\phi \in L_{\sigma}^{3/2,1}$. Furthermore, we see that (2.3) yields (2.2), if $v \in L^{\infty}(-\infty, T; L_{\sigma}^{3,\infty})$, $f = \nabla \cdot F$ and if $F \in L^{\infty}(-\infty, T; L^{3/2,\infty})$. Hence, Definition 1 is equivalent to the definition given in [55, Definition 1], if we assume $v \in BUC(\mathbb{R}; L_{\sigma}^{3,\infty})$ and $F \in BUC(\mathbb{R}; L^{3/2,\infty})$. We also note that (2.3) is a weak form of (1.1).

In order to prove our main result, we recall properties of the Lorentz spaces, estimates of the Stokes semigroup and several uniqueness theorems for mild solutions.

Lemma 5 (Kozono–Yamazaki [33]) *Let $p_1, p_2 \in (1, \infty)$ with $1/r := 1/p_1 + 1/p_2 < 1$ and let $q \in [1, \infty]$. Then, for all $f \in L^{p_1,\infty}(\Omega)$ and $g \in L^{p_2,q}(\Omega)$, it holds that*

$$\|f \cdot g\|_{r,q} \le C \|f\|_{p_1,\infty} \|g\|_{p_2,q}, \tag{2.4}$$

where $C = C(p_1, p_2, q)$.

For $u \in \dot{W}_0^{1,2}(\Omega) = \overline{C_0^{\infty}(\Omega)}^{\|\nabla \cdot\|_2}$ it holds with an absolute constant $C > 0$ that

$$\|u\|_{6,2} \le C \|\nabla u\|_2. \tag{2.5}$$

Lemma 6 (Shibata [48, 49]) *For all $t > 0$ and $\phi \in L_{\sigma}^{q,s}$, the following inequalities are satisfied*:

$$\|e^{-tA}\phi\|_{p,r} \le C t^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{p})} \|\phi\|_{q,s} \quad \text{when} \begin{cases} 1 < q \le p < \infty, & r = s \in [1, \infty], \\ 1 < q < p < \infty, & r = 1, s = \infty, \end{cases} \tag{2.6}$$

$$\|\nabla e^{-tA}\phi\|_{p,r} \le C t^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{q} - \frac{1}{p})} \|\phi\|_{q,s} \quad \text{when} \begin{cases} 1 < q \le p \le 3, & r = s \in [1, \infty], \\ 1 < q < p \le 3, & r = 1, s = \infty. \end{cases} \tag{2.7}$$

In the case where Ω is an exterior domain, Shibata [48, 49] proved (2.6) and (2.7) for all $r = s$. If $q < p$, his estimates (2.6)–(2.7) with $r = s$ and real interpolation yield (2.6)–(2.7) even for $r = 1$, $s = \infty$. In the restricted case $r = 1$, Yamazaki [55] obtained (2.7) also by a method different from [48, 49]. In the case where Ω is $\mathbb{R}^3$, $\mathbb{R}_+^3$, a perturbed halfspace or an aperture domain, the usual L^q-L^p estimates for the Stokes semigroup and real interpolation directly yield (2.6)–(2.7), since in this case the L^q–L^p estimates hold for all $1 < q \le p < \infty$. For details of L^q–L^p estimates for the Stokes semigroup, see also [2, 3, 25–27, 29, 34, 36, 48, 54].

Lemma 7 (Yamazaki [55]) *The following estimates*

$$\int_s^t \left|(F(\tau), \nabla e^{-(t-\tau)A}\phi)\right| d\tau \le C \left(\sup_{s<\tau<t} \|F\|_{3/2,\infty}\right) \|\phi\|_{3/2,1}, \tag{2.8}$$

$$\int_s^t \left|(u \cdot \nabla e^{-(t-\tau)A}\phi, w)(\tau)\right| d\tau \le C \left(\sup_{s<\tau<t} \|u\|_{3,\infty}\right) \left(\sup_{s<\tau<t} \|w\|_{3,\infty}\right) \|\phi\|_{3/2,1} \tag{2.9}$$

hold for all $F \in L^\infty(s, t; L^{3/2,\infty})$, $u, w \in L^\infty(s, t; L^{3,\infty})$, $\phi \in L^{3/2,1}_\sigma(\Omega)$ *and all* $-\infty \le s < t$, *where the constant* C *depends only on* Ω.

In the case where Ω is an exterior domain, the whole space or halfspace, Yamazaki [55] proved Lemma 7 by real interpolation. His proof is also valid in the case where Ω is a perturbed halfspace or an aperture domain. In the case where $\Omega = \mathbb{R}^3$, Meyer [41] obtained estimates similar to Lemma 7 by a method different from [55].

The following lemma is direct consequence of Lemma 7 using the duality $L^{3,\infty}_\sigma = (L^{3/2,1}_\sigma)^*$.

Lemma 8 (Yamazaki [55]) *There exists a constant* $\epsilon_0 = \epsilon_0(\Omega)$ *with the following property: Let* $T \le \infty$, $u, v, w \in BC((-\infty, T); L^{3,\infty}_\sigma)$ *and let* w *satisfy*

$$(w(t), \phi) = \int_{-\infty}^t \Big(\big(w \cdot \nabla e^{-(t-\tau)A}\phi, u\big)(\tau) + \big(v \cdot \nabla e^{-(t-\tau)A}\phi, w\big)(\tau) \Big) d\tau \tag{2.10}$$

for all $\phi \in L^{3/2,1}_\sigma$ *and all* $-\infty < t < T$. *Assume that*

$$\sup_{-\infty<t<T} \|u\|_{3,\infty} + \sup_{-\infty<t<T} \|v\|_{3,\infty} < \epsilon_0.$$

Then, $w(t) = 0$ *for all* $t \in (-\infty, T)$.

Lemma 9 [10, Lemma 2.6] *There exists a constant* $\epsilon_1(\Omega) > 0$ *such that if* $T \le \infty$, u, v *are mild* $L^{3,\infty}$*-solutions to* (*N-S*) *on* $(-\infty, T)$ *for the same force* f,

$$u, v \in BC((-\infty, T); \tilde{L}^{3,\infty}_\sigma),$$
$$\limsup_{t\to-\infty} \|u(t)\|_{3,\infty} < \epsilon_1 \quad \text{and} \quad \liminf_{t\to-\infty} \|u(t) - v(t)\|_{3,\infty} < \epsilon_1,$$

then

$$u = v \quad \text{on } (-\infty, T).$$

Lemma 9 can be proven by Lemma 8, the uniqueness of mild solutions in $C([0, T); \tilde{L}^{3,\infty})$ to IBVP, see [10, Lemma 2.5], and the continuity of $w(t) := u(t) - v(t)$ with respect to the time-variable t in $\tilde{L}^{3,\infty}$. For the detail, see [10, Lemma 2.6].

Finally, we come to the key lemma of the proof of Theorem 1. If u and v are solutions to the Navier–Stokes equations, then $w := u - v$ satisfies

$$(U) \quad \begin{cases} \partial_t w - \Delta w + w \cdot \nabla u + v \cdot \nabla w + \nabla p' = 0, & t \in (-\infty, T),\ x \in \Omega, \\ \operatorname{div} w = 0, & t \in (-\infty, T),\ x \in \Omega, \\ w|_{\partial\Omega} = 0. \end{cases}$$

Hence, if Ω is a bounded domain and if u, v belong to the Leray–Hopf class, under the hypotheses of Theorem 1, the usual energy method and the Poincaré inequality yield $\|w(t)\|_2^2 \le e^{-c(t-s)}\|w(s)\|_2^2$ for $t > s$. Letting $s \to -\infty$, we get $w(t) = 0$ for all t. Consequently, in the case of *bounded* domains, Theorem 1 is obvious. In the case where Ω is an *unbounded* domain, u and v do not belong to the energy class in general and the Poincaré inequality does not hold in general. Hence, since we cannot use the energy method, we will use the dual equations of the above system. It is notable that several researchers utilized the dual equation argument to prove several uniqueness theorems and a-priori estimates for solutions to (N-S). See e.g. Foias [18], Maremonti [38], Lions–Masmoudi [37]. Very recently, Crispo–Maremonti [7] used the dual equation argument to prove uniqueness theorems for suitable weak solutions to (N-S) in the sense of Caffarelli–Kohn–Nirenberg.

Here we will use a similar argument as in Lions–Masmoudi [37]. We recall the dual equations of the above system (U), namely,

$$\text{(D)} \begin{cases} -\partial_t \Psi - \Delta\Psi - \sum_{i=1}^{3} u^i \nabla \Psi^i - v \cdot \nabla\Psi + \nabla\pi = h, & t \in (-\infty, 0),\ x \in \Omega, \\ \nabla \cdot \Psi = 0, & t \in (-\infty, 0),\ x \in \Omega, \\ \Psi|_{\partial\Omega} = 0, \\ \Psi(0) = 0. \end{cases}$$

Lemma 10 [10] *There exists an absolute constant $\delta_0 > 0$ with the following property: Let $u, v \in BC((-\infty, 0]; \tilde{L}^{3,\infty}_\sigma)$, $h \in L^2_{loc}((-\infty, 0]; L^{6/5} \cap L^2)$ and*

$$\sup_{t \le 0} \|u(t)\|_{3,\infty} \le \delta_0.$$

Then there exists a unique solution $\Psi \in L^2_{\mathrm{loc}}((-\infty, 0]; D(A_2)) \cap W^{1,2}_{\mathrm{loc}}((-\infty, 0]; L^2_\sigma)$ to (D) such that

$$\|\Psi(t)\|_2^2 + \int_t^0 \|\nabla\Psi\|_2^2\, d\tau \le C \int_t^0 \|h\|_{6/5}^2\, d\tau \tag{2.11}$$

for all $t < 0$. Here C is an absolute constant.

Remark 5 In [10, Lemma 2.7], Lemma 10 was proven in the case $h \in BC((-\infty, 0]; L^{6/5} \cap L^2)$. In the same way as in [10], we easily see that this lemma holds even for the case $h \in L^2_{loc}((-\infty, 0]; L^{6/5} \cap L^2)$.

In the rest of this section, we prove some properties of weak L^r-spaces (L^r_w). From now on, for simplicity, we use the following notations:

$$\begin{aligned}\{|f| \leq \sigma\} &:= \{x \in \Omega;\ |f(x)| \leq \sigma\},\\ \{|f| > \rho,\ |g| > \sigma\} &:= \{x \in \Omega;\ |f(x)| > \rho,\ |g(x)| > \sigma\},\\ \{|F(t)| > \sigma\} &:= \{x \in \Omega;\ |F(x,t)| > \sigma\},\end{aligned}$$

etc.

Lemma 11 *Let $\Omega \subset \mathbb{R}^3$ be a measurable set and $1 < r < \infty$. If $f \in L^r_w(\Omega)$ and a constant $A > 0$ satisfy*

$$\|f\|_{L^r_w(\{|f| \leq 2^{-j}\})} > A \quad \text{for all } j \in \mathbb{N}, \tag{2.12}$$

then there exists a sequence $\{j_n\}_{n=1}^{\infty}$ of natural numbers such that

$$j_{n+1} > j_n (\geq n), \quad \|f\|_{L^r_w(\{2^{-j_n-1} < |f| \leq 2^{-j_n}\})} > L(r)A \quad \text{for all } n \in \mathbb{N},$$

where $L = L(r) = \frac{1}{4}\left(\frac{2^r-1}{2}\right)^{1/r}$.

Proof We use a proof by contradiction. Assume that there exists $n_0 \in \mathbb{N}$ such that

$$\|f\|_{L^r_w(\{2^{-j-1} < |f| \leq 2^{-j}\})} \leq L \cdot A \quad \text{for all natural numbers } j \geq n_0. \tag{2.13}$$

Let

$$m(s, f) := \mu\{|f| > s\}.$$

Since

$$\begin{aligned}&2^{-j-1}\Big(m(2^{-j-1}, f) - m(2^{-j}, f)\Big)^{1/r}\\ &= 2^{-j-1}\left(\mu\{2^{-j-1} < |f| \leq 2^{-j}\}\right)^{1/r}\\ &= 2^{-j-1}\left(\mu\{x \in \Omega;\ |f(x)| > 2^{-j-1},\ 2^{-j-1} < |f(x)| \leq 2^{-j}\}\right)^{1/r}\\ &\leq \sup_{s>0} s \left(\mu\{x \in \Omega;\ |f(x)| > s,\ 2^{-j-1} < |f(x)| \leq 2^{-j}\}\right)^{1/r}\\ &= \|f\|_{L^r_w(\{2^{-j+1} < |f| \leq 2^{-j}\})} \leq L \cdot A \quad \text{for all } j \geq n_0,\end{aligned}$$

we have

$$m(2^{-j-1}, f) - m(2^{-j}, f) \leq (2^{j+1} L \cdot A)^r \quad \text{for all } j \geq n_0.$$

Hence, for $N > n_0$,

$$\begin{aligned} m(2^{-N}, f) &= m(2^{-n_0}, f) + \sum_{j=n_0}^{N-1} (m(2^{-j-1}, f) - m(2^{-j}, f)) \\ &\le m(2^{-n_0}, f) + (2L \cdot A)^r \sum_{j=n_0}^{N-1} 2^{jr} \\ &= m(2^{-n_0}, f) + (2L \cdot A)^r \cdot \frac{2^{rN} - 2^{n_0 r}}{2^r - 1} \\ &\le 2^{n_0 r} \|f\|^r_{L^r_w(\Omega)} + (2L \cdot A)^r \cdot \frac{2^{rN}}{2^r - 1}, \end{aligned}$$

which implies

$$2^{-Nr} m(2^{-N}, f) \le 2^{(n_0 - N)r} \|f\|^r_{L^r_w(\Omega)} + \frac{(2L \cdot A)^r}{2^r - 1}. \tag{2.14}$$

Letting $N_0 (\ge n_0)$ be sufficiently large so that

$$2^{(n_0 - N_0)r} \|f\|^r_{L^r_w(\Omega)} \le \frac{(2L \cdot A)^r}{2^r - 1},$$

by (2.14) we have

$$2^{-Nr} m(2^{-N}, f) \le \frac{2(2L \cdot A)^r}{2^r - 1} \quad \text{for all} \;\; N \ge N_0. \tag{2.15}$$

This implies

$$2^{-N} (\mu\{|f| > 2^{-N}\})^{1/r} \le \left(\frac{2}{2^r - 1} \right)^{1/r} 2L \cdot A = \frac{A}{2} \quad \text{for all } N \ge N_0. \tag{2.16}$$

On the other hand, from (2.12) with $j = N_0$ we obtain

$$\begin{aligned} A &< \sup_{s>0} s \, (\mu\{|f| > s, \; |f| \le 2^{-N_0}\})^{1/r} \\ &= \sup_{0<s<2^{-N_0}} s \, (\mu\{|f| > s, \; |f| \le 2^{-N_0}\})^{1/r} \\ &= \sup_{k \ge N_0} \sup_{2^{-k-1} \le s < 2^{-k}} s \, (\mu\{|f| > s, \; |f| \le 2^{-N_0}\})^{1/r} \\ &\le \sup_{k \ge N_0} \sup_{2^{-k-1} \le s < 2^{-k}} 2^{-k} \, (\mu\{|f| > 2^{-k-1}, \; |f| \le 2^{-N_0}\})^{1/r} \\ &\le \sup_{k \ge N_0} 2^{-k} \, (\mu\{|f| > 2^{-k-1}\})^{1/r} = 2 \sup_{N \ge N_0 + 1} 2^{-N} \, (\mu\{|f| > 2^{-N}\})^{1/r}. \end{aligned}$$

This contradicts (2.16). Therefore, (2.13) cannot be true and we conclude that for each natural number n, there is a natural number $j_n(\geq n)$ such that

$$\|f\|_{L^r_w(\{2^{-j_n-1}<|f|\leq 2^{-j_n}\})} > L \cdot A,$$

which proves Lemma 11. □

Lemma 12 *Let $\Omega(\subset \mathbb{R}^3)$ be a measurable set and $U \in L^3_w(\Omega)$. Assume that there exist a real number $\omega > 0$, $p_0 \in \mathbb{N}$ and a sequence $\{g_n\}$ such that*

$$\|g_n\|_{L^3_w(\{|U|>2^{-n}\})} \leq \omega \quad \textit{for all } n \geq p_0, \tag{2.17}$$

$$\limsup_{n\to\infty} 2^{-n-2}(\mu\{2^{-n-2} < |g_{n+1} - U| < 2^{-n+1}\})^{1/3} < \omega. \tag{2.18}$$

Then, there exists $k_0 \in \mathbb{N}$ such that

$$\|U\|_{L^3_w(\{|U|\leq 2^{-k_0}\})} \leq C_2\omega. \tag{2.19}$$

Here $C_2 := 2^{1/3}4/L(3)$.

Proof We use a proof by contradiction. Assume that

$$\|U\|_{L^3_w(\{|U|\leq 2^{-j}\})} > C_2\omega \quad \text{for all } j \in \mathbb{N}.$$

Then, from Lemma 11 we observe that there exists a sequence $\{j_n\}$ of natural numbers such that

$$j_n \nearrow \infty \quad \text{as } n \to \infty, \tag{2.20}$$

$$\|U\|_{L^3_w(\{2^{-j_n-1}<|U|\leq 2^{-j_n}\})} > L(3)C_2\omega \quad \text{for all } n \in \mathbb{N}. \tag{2.21}$$

From (2.21) we obtain

$$\begin{aligned} L(3)C_2\omega &< \sup_{\tau>0} \tau\, \mu\{|U| > \tau,\ 2^{-j_n-1} < |U| \leq 2^{-j_n}\}^{1/3} \\ &= \sup_{0<\tau\leq 2^{-j_n}} \tau\, \mu\{|U| > \tau,\ 2^{-j_n-1} < |U| \leq 2^{-j_n}\}^{1/3} \\ &\leq 2^{-j_n}\mu\{2^{-j_n-1} < |U| \leq 2^{-j_n}\}^{1/3}, \end{aligned} \tag{2.22}$$

which yields

$$2^{3j_n}\big(L(3)C_2\omega\big)^3 < \mu\{2^{-j_n-1} < |U| \leq 2^{-j_n}\} \quad \text{for all } n \in \mathbb{N}. \tag{2.23}$$

On the other hand, by using (2.17) with n replaced by $k+1$, we have for all $\tau > 0$ and all $k \geq p_0 - 1$

$$\tau\ (\mu\{|g_{k+1}| > \tau,\ 2^{-k-1} < |U|\})^{1/3} \leq \omega.$$

Letting $\tau = 2^{-k-2}$, we obtain

$$\mu\{|g_{k+1}| > 2^{-k-2},\ 2^{-k-1} < |U| \le 2^{-k}\} \le 4^3 2^{3k}\omega^3 \quad \text{for all } k \ge p_0 - 1.$$

Since $j_n \ge p_0$ for all sufficiently large n, we have

$$\mu\{|g_{j_n+1}| > 2^{-j_n-2},\ 2^{-j_n-1} < |U| \le 2^{-j_n}\} \le 4^3 2^{3j_n}\omega^3 \quad \text{for all large } n. \tag{2.24}$$

Then from (2.23) and (2.24) we obtain

$$\begin{aligned}&\mu\Big(\{2^{-j_n-1} < |U| \le 2^{-j_n}\}\backslash\{|g_{j_n+1}| > 2^{-j_n-2},\ 2^{-j_n-1} < |U| \le 2^{-j_n}\}\Big)\\ &\quad \ge 2^{3j_n}\omega^3((L(3))^3 C_2^3 - 4^3) = 2^{3j_n}\omega^3 4^3 \quad \text{for all large } n.\end{aligned} \tag{2.25}$$

Since

$$\begin{aligned}&\{2^{-j_n-1} < |U| \le 2^{-j_n}\}\backslash\{|g_{j_n+1}| > 2^{-j_n-2},\ 2^{-j_n-1} < |U| \le 2^{-j_n}\}\\ &\quad = \{|g_{j_n+1}| \le 2^{-j_n-2},\ 2^{-j_n-1} < |U| \le 2^{-j_n}\}\\ &\quad \subset \{2^{-j_n-2} < |g_{j_n+1} - U| < 2^{-j_n+1}\},\end{aligned}$$

by (2.25) we see

$$2^{-j_n-2}\mu\{2^{-j_n-2} < |g_{j_n+1} - U| < 2^{-j_n+1}\}^{1/3} \ge \omega \quad \text{for all large } n. \tag{2.26}$$

This and (2.20) yield

$$\limsup_{n\to\infty} 2^{-n-2}(\mu\{2^{-n-2} < |g_{n+1} - U| < 2^{-n+1}\})^{1/3} \ge \omega,$$

which contradicts (2.18). This proves Lemma 12. □

Recall that, for any measurable set D, $L^3_w(D) = L^{3,\infty}(D)$ and

$$\|f\|_{L^{3,\infty}(D)} \le c\|f\|_{L^3_w(D)} \le c'\|f\|_{L^{3,\infty}(D)} \quad \text{for } f \in L^{3,\infty}(D),$$

where c and c' are absolute constants. Then, since $2^{-n-2}(\mu\{2^{-n-2} < |g_{n+1} - U| < \lambda\})^{1/3} \le \|g_{n+1} - U\|_{L^3_w(\{|g_{n+1}-U|<\lambda\})}$ for all $\lambda > 2^{-n-2}$, Lemma 12 directly yields the following lemma with L^3_w replaced by $L^{3,\infty}$:

Lemma 13 *Let $\Omega (\subset \mathbb{R}^3)$ be a measurable set and $U \in L^{3,\infty}(\Omega)$. Assume that there exist a real number $\omega > 0$, $p_0 \in \mathbb{N}$ and a sequence $\{g_n\}$ such that*

$$\|g_n\|_{L^{3,\infty}(\{|U|>2^{-n}\})} \le \omega \quad \text{for all } n \ge p_0, \tag{2.27}$$

$$\limsup_{n\to\infty} \|g_{n+1} - U\|_{L^{3,\infty}(\{|g_{n+1}-U|<2^{-n+1}\})} < \omega. \tag{2.28}$$

Then, there exists $k_0 \in \mathbb{N}$ such that

$$\|U\|_{L^{3,\infty}(\{|U|\leq 2^{-k_0}\})} \leq C_3\omega, \tag{2.29}$$

where C_3 is an absolute constant.

3 Proof of Theorem 1

In this section, we prove Theorem 1 by using a similar argument given in [10] and Lemma 13.

Proof of Theorem 1 Let $\delta := \frac{\min\{\epsilon_1, \delta_0\}}{6+3C_3}$, where ϵ_1, δ_0 and C_3 are the constants given in Lemmata 9, 10 and 13. By (1.5) and (1.7), there exists $s_0 \in (-\infty, T)$ such that

$$\sup_{t\leq s_0} \|u(t)\|_{3,\infty} \leq \delta, \quad \gamma_0 \geq \frac{c_*}{2}\left(\sup_{t\leq s_0} \|v(t)\|_{3,\infty} + \|V\|_{3,\infty} + 1 + \delta\right)^7.$$

Since $\lim_{t\to-\infty} \frac{|t|^{1/2}}{|t-s_0|^{1/2}} = 1$, by (1.6) we have

$$\limsup_{t\to-\infty} \|v(t) - V\|_{L^{3,\infty}(\{|v(t)-V|\leq \frac{\gamma_0}{2|t-s_0|^{1/2}}\})} \leq \delta.$$

Without loss of generality, we may assume $T > 0$ and $s_0 = 0$, i.e.,

$$\sup_{t\leq 0} \|u(t)\|_{3,\infty} \leq \delta, \tag{3.1}$$

$$\limsup_{t\to-\infty} \|v(t) - V\|_{L^{3,\infty}(\{|v(t)-V|\leq \frac{\gamma_0}{2|t|^{1/2}}\})} \leq \delta, \tag{3.2}$$

$$\gamma_0 \geq \frac{c_*}{2}\left(\sup_{t\leq 0} \|v(t)\|_{3,\infty} + \|V\|_{3,\infty} + 1 + \delta\right)^7. \tag{3.3}$$

Let $\epsilon \in (0, 1]$ be an arbitrary fixed number, which will be chosen suitably small later on,

$$K := \sup_{t\leq 0} \|v(t)\|_{L^{3,\infty}(\Omega)} + 1 + \delta \quad \text{and let}$$
$$w := u - v.$$

Since $w = u - v \in BC((-\infty, 0]; L^{3,\infty})$, there exists a sequence $\{t_k\}$ such that

$$0 = t_0 > t_1 > t_2 > \cdots, \quad |t_k - t_{k+1}| < 1, \quad t_k \to -\infty \text{ as } k \to \infty,$$
$$\|w(t) - w(t_k)\|_{3,\infty} < \epsilon \quad \text{for all } t \in [t_{k+1}, t_k] \text{ and all } k = 0, 1, 2, \ldots. \tag{3.4}$$

Indeed, since $w \in BUC([-n-1,-n]; L^{3,\infty})$, for each $n = 0, 1, 2, \dots$ there exist a number $m = m(n)$ and a finite sequence $\{t_i^n\}_{i=1}^m \subset [-n-1, -n]$ such that

$$-n = t_0^n > t_1^n > t_2^n > \cdots > t_m^n = -n-1, \quad |t_i^n - t_{i+1}^n| < 1,$$
$$\|w(t) - w(t_i^n)\|_{3,\infty} < \epsilon \quad \text{for all } t \in [t_{i+1}^n, t_i^n] \text{ and all } i = 0, 1, 2, \dots, m-1. \tag{3.5}$$

Then, arranging all members of $\{t_i^n\}_{n,i}$ in order from the largest, we have the sequence $\{t_k\}$ satisfying (3.4). Then, letting

$$\tilde{w}(t) := w(t_k) \text{ for } t \in (t_{k+1}, t_k],\ k \in \mathbb{N} \cup \{0\},$$
$$i.e.\ \tilde{w}(t) := \sum_{k=0}^{\infty} w(t_k) 1_{(t_{k+1}, t_k]}(t) \quad \text{for all } t \le 0,$$

where 1_S denotes the characteristic function of a set S, we have

$$\sup_{t \le 0} \|w(t) - \tilde{w}(t)\|_{L^{3,\infty}(\Omega)} \le \epsilon \quad \text{and} \quad \sup_{t \le 0} \|\tilde{w}(t)\|_{L^{3,\infty}(\Omega))} \le K. \tag{3.6}$$

Let $\{D_k\}_{k=0}^{\infty} (\subset \Omega)$ be an arbitrary sequence of measurable sets with $0 < \mu(D_k) < \infty$ for $k = 0, 1, 2, \dots,$ which will be suitably defined later on. Using this sequence, we define $\tilde{D}(t) \subset \Omega$ for each $t \le 0$ as follows:

$$\tilde{D}(t) := D_k \quad \text{for } t \in (t_{k+1}, t_k],\ k = 0, 1, 2, \dots. \tag{3.7}$$

(Step 1) We will first show that

$$\int_{-j}^{0} \|w(\tau)\|^2_{L^{3,\infty}(\tilde{D}(\tau))} d\tau \le C_0 \left\{ \frac{K^3}{j^{3/4}} \left(\int_{-j}^{0} \mu(\tilde{D}(\tau))^{1/3}\, d\tau \right)^{1/2} + K\epsilon \right\} \tag{3.8}$$

for all $j = 1, 2, \dots,$ where $C_0 = C_0(\Omega)$ is a constant depending only on Ω. Note that the functions $\mu(\tilde{D}(t))$ and $\|w(\tau)\|^2_{L^{3,\infty}(\tilde{D}(\tau))}$ are continuous on $(t_{k+1}, t_k]$ for each $k \in \mathbb{N} \cup \{0\}$ and hence these functions are piecewise continuous on $[-j, 0]$, so that both sides of (3.8) are well-defined. Also note that, as we will show below, (3.8) holds for an arbitrary choice of measurable sets $\{D_k\}$ with $0 < \mu(D_k) < \infty$ for $k = 0, 1, 2 \dots.$

By (2.1) and the Hölder inequality, it holds that, for all measurable sets $F \subset \Omega$,

$$\begin{aligned}\|w(t)\|^2_{L^{3,\infty}(F)} &\le C^* \|w(t)\|^2_{L^3_w(F)} \\ &\le C^* \left(\sup_{E \subset F,\ 0<\mu(E)<\infty} \mu(E)^{-2/3} \int_E |w(x,t)| dx \right)^2 \\ &\le C^* \sup_{E \subset F,\ 0<\mu(E)<\infty} \mu(E)^{-1/3} \int_E |w(x,t)|^2 dx, \end{aligned} \tag{3.9}$$

where C^* is an absolute constant. Hence, for each $k \in \mathbb{N}\cup\{0\}$, there exists a measurable set $E_k \subset \tilde{D}(t_k)(= D_k)$ such that

$$\|w(t_k)\|^2_{L^{3,\infty}(\tilde{D}(t_k))} \leq C^*\mu(E_k)^{-1/3}\int_{E_k}|w(x,t_k)|^2dx + \epsilon, \quad 0 < \mu(E_k) < \infty. \tag{3.10}$$

Using the sequence $\{E_k\}$, we define a set $\tilde{E}(t) \subset \Omega$ for each $t \leq 0$ as follows:

$$\tilde{E}(t) := E_k \quad \text{for } t \in (t_{k+1}, t_k],\ k = 0, 1, 2, \ldots.$$

Then $\tilde{E}(t) \subset \tilde{D}(t)$ for each $t \leq 0$. Let, for $\tau \leq 0$,

$$h(x,\tau) = \tilde{w}(x,\tau)\mu(\tilde{E}(\tau))^{-\frac{1}{3}}1_{\tilde{E}(\tau)}(x)$$
$$i.e.\quad h(x,\tau) = \sum_{k=0}^{\infty} w(x,t_k)\mu(E_k)^{-\frac{1}{3}}1_{E_k}(x)\cdot 1_{(t_{k+1},t_k]}(\tau). \tag{3.11}$$

Note that for $t_{k+1} < t \leq t_k$, $h(t) = h(t_k) = w(t_k)\mu(E_k)^{-\frac{1}{3}}1_{E_k}$. Also note that

$$\mu(E_k)^{-1/3}\int_{E_k}|w(x,t_k)|^2dx = (w(t_k), h(t_k)). \tag{3.12}$$

By Lemma 5 and (3.6), we have

$$\begin{aligned}\|h(t)\|_{L^{6/5,1}(\Omega)} &= \mu(\tilde{E}(t))^{-1/3}\|\tilde{w}(t)1_{\tilde{E}(t)}\|_{6/5,1}\\ &\leq C\mu(\tilde{E}(t))^{-1/3}\|\tilde{w}(t)\|_{3,\infty}\|1_{\tilde{E}(t)}\|_{2,1} \leq C\mu(\tilde{E}(t))^{1/6}K,\end{aligned} \tag{3.13}$$

where we used the interpolation inequality:

$$\|1_{\tilde{E}(t)}\|_{L^{q,r}} \leq C\|1_{\tilde{E}(t)}\|^{1/q}_{L^1}\|1_{\tilde{E}(t)}\|^{1-1/q}_{L^\infty} \leq C\mu(\tilde{E}(t))^{1/q}$$

for $1 < q < \infty$ and $1 \leq r \leq \infty$. Similarly, by Lemma 5 we have

$$\begin{aligned}\|h(t)\|_{L^2(\Omega)} &= \mu(\tilde{E}(t))^{-1/3}\|\tilde{w}(t)1_{\tilde{E}(t)}\|_2\\ &\leq C\mu(\tilde{E}(t))^{-1/3}\|\tilde{w}(t)\|_{3,\infty}\|1_{\tilde{E}(t)}\|_{6,2}\\ &\leq C\mu(\tilde{E}(t))^{-1/6}\|\tilde{w}(t)\|_{3,\infty} \leq C\mu(\tilde{E}(t))^{-1/6}K.\end{aligned} \tag{3.14}$$

Since $0 < \mu(\tilde{E}(t)) = \mu(E_k) < \infty$ for all $t_{k+1} < t \leq t_k$ and all $k = 0, 1, \ldots,$ by (3.13) and (3.14) we see $h \in L^\infty_{loc}((-\infty, 0]; L^2 \cap L^{6/5,1})$.

Furthermore, by the interpolation inequality, (3.13) and (3.14), we have

$$\|h(t)\|_{L^{3/2,1}(\Omega)} \leq C\|h(t)\|^{1/2}_{L^{6/5,1}(\Omega)}\|h(t)\|^{1/2}_{L^2(\Omega)} \leq CK. \tag{3.15}$$

Since for $t_{k+1} < t \le t_k$ it holds that

$$\begin{aligned}\|w(t)\|^2_{L^{3,\infty}(\tilde{D}(t))} &= \|w(t)\|^2_{L^{3,\infty}(\tilde{D}(t_k))} \\ &\le \left(\|w(t_k)\|_{L^{3,\infty}(\tilde{D}(t_k))} + \|w(t) - w(t_k)\|_{L^{3,\infty}}\right)^2 \\ &\le (\|w(t_k)\|_{L^{3,\infty}(\tilde{D}(t_k))} + \epsilon)^2 \\ &\le 2\|w(t_k)\|^2_{L^{3,\infty}(\tilde{D}(t_k))} + 2\epsilon^2,\end{aligned}$$

from (3.4), (3.10), (3.12), (3.15) and $L^{3,\infty} = (L^{3/2,1})^*$, we obtain for $t_{k+1} < t \le t_k$

$$\begin{aligned}&\|w(t)\|^2_{L^{3,\infty}(\tilde{D}(t))} \le 2C^*\mu(E_k)^{-1/3}\int_{E_k}|w(x,t_k)|^2dx + 2\epsilon + 2\epsilon^2 \\ &\le 2C^* \cdot (w(t_k), h(t_k)) + 4\epsilon \\ &= 2C^* \cdot (w(t), h(t_k)) + 2C^* \cdot (w(t_k) - w(t), h(t_k)) + 4\epsilon \\ &\le 2C^* \cdot (w(t), h(t_k)) + C\|w(t_k) - w(t)\|_{3,\infty}\|h(t_k)\|_{3/2,1} + 4\epsilon \\ &\le 2C^* \cdot (w(t), h(t_k)) + CK\epsilon + 4\epsilon \\ &= 2C^* \cdot (w(t), h(t)) + CK\epsilon + 4\epsilon.\end{aligned}$$

Since the above estimate holds for all $t \in (t_{k+1}, t_k]$ and all $k = 0, 1, 2, \ldots$, we have for all $j \in \mathbb{N}$

$$⨍_{-j}^{0} \|w(\tau)\|^2_{L^{3,\infty}(\tilde{D}(\tau))}d\tau \le 2C^* ⨍_{-j}^{0}(w(\tau), h(\tau))d\tau + CK\epsilon + 4\epsilon. \tag{3.16}$$

Hence in order to show (3.8), it suffices to show

$$⨍_{-j}^{0}(w(\tau), h(\tau))d\tau \le \frac{CK^3}{j^{3/4}}\left\{\int_{-j}^{0}\mu(\tilde{D}(\tau))^{1/3}\,d\tau\right\}^{1/2}. \tag{3.17}$$

Let $j \in \mathbb{N}$ be fixed. For $-3j < t < 0$, let

$$\begin{aligned}w_0(t) &:= e^{-(t+3j)A}w(-3j) \\ w_1(t) &:= w(t) - w_0(t).\end{aligned} \tag{3.18}$$

Then, it holds that

$$(w_1(t), \phi) = \int_{-3j}^{t}\left((w \cdot \nabla e^{-(t-s)A}\phi, u) + (v \cdot \nabla e^{-(t-s)A}\phi, w)\right)ds$$

for all $\phi \in C^\infty_{0,\sigma}$. Since $C^\infty_{0,\sigma}$ is dense in $L^{3/2,1}_\sigma$, from Lemma 7, we see that the above equality holds for all $\phi \in L^{3/2,1}_\sigma$. By the duality $L^{3/2,\infty} = (L^{3,1})^*$, Lemma 5 and Lemma 6, we have for $\varphi \in L^{3/2,1} \cap L^2$

$$
\begin{aligned}
|(w_1(t),\varphi)| &= |(w_1(t), P\varphi)| \\
&\le \int_{-3j}^{t} \left\| \nabla e^{-(t-s)A} P\varphi \right\|_{3,1} \| w\otimes u + v\otimes w\|_{3/2,\infty}\, ds \\
&\le C\int_{-3j}^{t} (t-s)^{-\frac{3}{4}} \|\varphi\|_{2,\infty} \| w\otimes u + v\otimes w\|_{3/2,\infty}\, ds \\
&\le C(t+3j)^{\frac{1}{4}} \sup_{-\infty<s<0} \|w(s)\|_{3,\infty}(\|u(s)\|_{3,\infty} + \|v(s)\|_{3,\infty})\|\varphi\|_2,
\end{aligned}
\tag{3.19}
$$

which implies $w_1(t) \in L^2$ and

$$
\|w_1(t)\|_2 \le C(\Omega)K^2(t+3j)^{\frac{1}{4}} \quad \text{for } -3j < t < 0. \tag{3.20}
$$

Furthermore we observe that w_1 satisfies

$$
\begin{aligned}
&\int_{-j}^{0} \big((w_1, -\partial_t\psi - \Delta\psi) - (w\cdot\nabla\psi, u) - (v\cdot\nabla\psi, w)\big)\, ds \\
&\quad = (w_1(-j), \psi(-j)) - (w_1(0), \psi(0))
\end{aligned}
\tag{3.21}
$$

for all $\psi \in W^{1,2}(-j,0; L^2_\sigma) \cap L^2(-j,0; D(A_2))$. For the detail of the proof of (3.21), see [10, Proof of (3.8)].

In order to show (3.17), since $w = w_0 + w_1$, we decompose $⨍_{-j}^{0} (w(\tau), h)\, d\tau$ into two terms as follows:

$$
⨍_{-j}^{0} (w(\tau), h(\tau))\, d\tau = ⨍_{-j}^{0} (w_0(\tau), h(\tau))\, d\tau + ⨍_{-j}^{0} (w_1(\tau), h(\tau))\, d\tau =: I_0 + I_1.
$$

We estimate I_0 and I_1 separately. By $L^{6,\infty} = (L^{6/5,1})^*$, Lemma 6 and (3.13), we obtain

$$
\begin{aligned}
|I_0| &\le ⨍_{-j}^{0} \|w_0(\tau)\|_{6,\infty}\|h(\tau)\|_{6/5,1}\, d\tau \\
&\le Cj^{-1}\int_{-j}^{0} \left\| e^{-(\tau+3j)A} w(-3j)\right\|_{6,\infty} \mu(\tilde{E}(\tau))^{1/6} K\, d\tau \\
&\le CKj^{-1}\int_{-j}^{0} (\tau+3j)^{-\frac{1}{4}} \|w(-3j)\|_{3,\infty} \mu(\tilde{D}(\tau))^{1/6}\, d\tau \\
&\le \frac{CK^2}{j^{5/4}} \int_{-j}^{0} \mu(\tilde{D}(\tau))^{1/6}\, d\tau \le \frac{CK^2}{j^{3/4}} \left\{ \int_{-j}^{0} \mu(\tilde{D}(\tau))^{1/3}\, d\tau \right\}^{1/2}.
\end{aligned}
\tag{3.22}
$$

Let Ψ be the solution to (D) with right-hand side $h(x,\tau) = \tilde{w}(x,\tau)\mu(\tilde{E}(\tau))^{-\frac{1}{3}} 1_{\tilde{E}(\tau)}(x)$ and initial value $\Psi(0) = 0$, cf. Lemma 10. Note that $h \in L^\infty_{loc}((-\infty, 0]; L^{6/5} \cap L^2)$.

Then,

$$I_1 = \int_{-j}^{0} (w_1(\tau), h(\tau))\, d\tau$$
$$= \int_{-j}^{0} \left(w_1(\tau), -\partial_t \Psi - \Delta \Psi - \sum_{i=1}^{3} u^i \nabla \Psi^i - v \cdot \nabla \Psi + \nabla \pi \right) d\tau.$$

Since $\Psi(0) = 0$ and since $w_1 \in L^2(-j, 0; L^2_\sigma)$ implies that $\int_{-j}^{0} (w_1, \nabla \pi) d\tau = 0$, by (3.21) we observe that

$$I_1 = \frac{1}{j}(w_1(-j), \Psi(-j))$$
$$+ \int_{-j}^{0} \left((w \cdot \nabla \Psi, u) + (v \cdot \nabla \Psi, w) - \left(w_1, \sum_{i=1}^{3} u^i \nabla \Psi^i + v \cdot \nabla \Psi \right) \right) d\tau$$
$$= \frac{1}{j}(w_1(-j), \Psi(-j)) + \int_{-j}^{0} (w_0 \cdot \nabla \Psi, u)\, d\tau + \int_{-j}^{0} (v \cdot \nabla \Psi, w_0)\, d\tau$$
$$=: J_0 + J_1 + J_2.$$

By (2.11), (3.13) and (3.20), we have

$$|J_0| = \frac{1}{j} \left| (w_1(-j), \Psi(-j)) \right| \le \frac{1}{j} \|w_1(-j)\|_2 \|\Psi(-j)\|_2$$
$$\le \frac{1}{j} \cdot C K^2 j^{1/4} \cdot \left\{ \int_{-j}^{0} \|h\|_{6/5}^2\, d\tau \right\}^{1/2}$$
$$\le \frac{1}{j} \cdot C K^2 j^{1/4} \cdot \left\{ \int_{-j}^{0} \mu(\tilde{E}(\tau))^{1/3} K^2\, d\tau \right\}^{1/2}$$
$$\le \frac{1}{j^{3/4}} \cdot C K^3 \left\{ \int_{-j}^{0} \mu(\tilde{D}(\tau))^{1/3}\, d\tau \right\}^{1/2}.$$

Furthermore, by Lemmata 5 and 6, (2.11), (3.13) and the duality $L^{6,2} = (L^{6/5,2})^*$, we have

$$|J_1| = \left| \int_{-j}^{0} (w_0(\tau) \cdot \nabla \Psi(\tau), u(\tau))\, d\tau \right| = \left| \int_{-j}^{0} \left(e^{-(\tau+3j)A} w(-3j) \cdot \nabla \Psi, u \right) d\tau \right|$$
$$\le \int_{-j}^{0} \left\| e^{-(\tau+3j)A} w(-3j) \right\|_{6,2} \left\| |\nabla \Psi(\tau)| |u(\tau)| \right\|_{6/5,2}\, d\tau$$
$$\le C \int_{-j}^{0} (\tau + 3j)^{-\frac{1}{4}} \|w(-3j)\|_{3,\infty} \|\nabla \Psi(\tau)\|_2 \|u(\tau)\|_{3,\infty}\, d\tau$$

$$
\begin{aligned}
&\le CK^2 j^{-1}\left\{\int_{-j}^{0}(\tau+3j)^{-1/2}d\tau\right\}^{1/2}\left\{\int_{-j}^{0}\|\nabla\Psi\|_2^2\,d\tau\right\}^{1/2}\\
&\le CK^2 j^{-3/4}\left\{\int_{-j}^{0}\|h\|_{6/5}^2\,d\tau\right\}^{1/2}\le \frac{1}{j^{3/4}}\cdot CK^3\left\{\int_{-j}^{0}\mu(\tilde D(\tau))^{1/3}\,d\tau\right\}^{1/2}.
\end{aligned}
$$

Similarly, we observe that

$$
|J_2|\le \frac{1}{j^{3/4}}\cdot CK^3\left\{\int_{-j}^{0}\mu(\tilde D(\tau))^{1/3}\,d\tau\right\}^{1/2}.
$$

Hence, we obtain $|I_1|=|J_0+J_1+J_2|\le \frac{CK^3}{j^{3/4}}\left\{\int_{-j}^{0}\mu(\tilde D(\tau))^{1/3}\,d\tau\right\}^{1/2}$ so that by (3.22)

$$
\left|\int_{-j}^{0}(w,h)\,d\tau\right| = |I_0+I_1|\le \frac{CK^3}{j^{3/4}}\left\{\int_{-j}^{0}\mu(\tilde D(\tau))^{1/3}\,d\tau\right\}^{1/2},
$$

which is the desired estimate (3.17). Thus from (3.16) and (3.17) we get (3.8).

(Step 2) Here we will show $\liminf_{t\to-\infty}\|w(t)\|_{L^{3,\infty}(\Omega)}<\epsilon_1$. Let us define $\{D_k\}$. Let

$$
a_k=\frac{(K+\|V\|_{3,\infty})^5}{\epsilon^2(|t_k|+1)^{1/2}}, \tag{3.23}
$$

$y_0\in\Omega$ be a fixed arbitrary point and

$$
\begin{aligned}
D_k^0 &:= \{x\in\Omega;\ |V(x)|\ge a_k\},\\
D_k^1 &:= \{x\in\Omega;\ |v(x,t_k)-V(x)|\ge a_k\},\\
D_k^2 &:= \{x\in\Omega;\ |x-y_0|<1/a_k\},\\
D_k &:= D_k^0\cup D_k^1\cup D_k^2 \quad \text{for } k=0,1,2,\dots.
\end{aligned} \tag{3.24}
$$

Note that $\mu(D_k)\ge\mu(D_k^2)>0$. Then, since $\mu(\{x\in\Omega\,;\,|f(x)|>s\})\le Cs^{-3}\|f\|_{3,\infty}^3$, we have

$$
\begin{aligned}
\mu(D_k)&\le Ca_k^{-3}(\|V\|_{3,\infty}^3+\|v(t_k)-V\|_{3,\infty}^3+1)\\
&\le C\left(\frac{\epsilon^2(|t_k|+1)^{1/2}}{(K+\|V\|_{3,\infty})^5}\right)^3(K+\|V\|_{3,\infty})^3\le C\frac{\epsilon^6(|t_k|+1)^{3/2}}{K^{12}}.
\end{aligned}
$$

Recall $\tilde D(t)=D_k$ for $t_{k+1}<t\le t_k$. Since $|t_k|\le|t|$ for $t_{k+1}<t\le t_k$, it holds that

$$
\mu(\tilde D(t))\le\frac{C\epsilon^6(|t|+1)^{3/2}}{K^{12}} \tag{3.25}
$$

for all $t \le 0$. Then, (3.8) implies

$$\int_{-j}^{0} \|w(\tau)\|_{L^{3,\infty}(\tilde{D}(\tau))}^2 d\tau \le \tilde{C}_0 K\epsilon \quad \text{for all } j \in \mathbb{N}, \tag{3.26}$$

where $\tilde{C}_0$ is a constant depending only on Ω.

Now we choose ϵ such that

$$\epsilon := \frac{\delta^2}{2\tilde{C}_0 K + \delta + \delta^2} \left(\le \min\left\{ \frac{\delta^2}{2\tilde{C}_0 K}, \delta, 1 \right\} \right). \tag{3.27}$$

Then, by (3.26), for all $j \in \mathbb{N}$ it holds that

$$\begin{aligned} \frac{1}{j}\int_{-2j}^{-j} \|w(\tau)\|_{L^{3,\infty}(\tilde{D}(\tau))}^2 d\tau &\le 2 \cdot \frac{1}{2j}\int_{-2j}^{0} \|w(\tau)\|_{L^{3,\infty}(\tilde{D}(\tau))}^2 d\tau \\ &\le 2\tilde{C}_0 K\epsilon \le \delta^2. \end{aligned} \tag{3.28}$$

Thus, it is straightforward to see that there exists a sequence $\{s_j\}$ such that

$$-2j \le s_j \le -j, \quad \|w(s_j)\|_{L^{3,\infty}(\tilde{D}(s_j))} \le \delta \tag{3.29}$$

for all $j \in \mathbb{N}$.

Next, we will estimate $\|w(s_j)\|_{L^{3,\infty}(\Omega\setminus\tilde{D}(s_j))}$. Since $t_k > t_{k+1} \to -\infty$, we can choose a sequence $\{k(j)\}_{j=1}^{\infty} \subset \mathbb{N}$ such that

$$t_{k(j)+1} < s_j \le t_{k(j)} (\le 0). \tag{3.30}$$

By (3.7), we see $\tilde{D}(s_j) = D_{k(j)}$. Thus, by $\sup_{s\le 0}\|u(s)\|_{3,\infty} \le \delta$,

$$\begin{aligned} \|w(s_j)\|_{L^{3,\infty}(\Omega\setminus\tilde{D}(s_j))} &\le \|w(s_j) - w(t_{k(j)})\|_{L^{3,\infty}(\Omega\setminus D_{k(j)})} \\ &\quad + \|w(t_{k(j)})\|_{L^{3}(\Omega\setminus D_{k(j)})} \\ &\le \|w(s_j) - w(t_{k(j)})\|_{L^{3,\infty}(\Omega\setminus D_{k(j)})} \\ &\quad + \|u(t_{k(j)})\|_{L^{3,\infty}(\Omega\setminus D_{k(j)})} \\ &\quad + \|v(t_{k(j)}) - V\|_{L^{3,\infty}(\Omega\setminus D_{k(j)})} + \|V\|_{L^{3,\infty}(\Omega\setminus D_{k(j)})} \\ &\le \epsilon + \delta + \|v(t_{k(j)}) - V\|_{L^{3,\infty}(\Omega\setminus D_{k(j)})} + \|V\|_{L^{3,\infty}(\Omega\setminus D_{k(j)})} \\ &\le 2\delta + \|v(t_{k(j)}) - V\|_{L^{3,\infty}(\{|v(t_{k(j)})-V|<a_{k(j)}\})} + \|V\|_{L^{3,\infty}(\Omega\setminus D_{k(j)})}, \end{aligned} \tag{3.31}$$

since (3.4) and (3.30) imply $\|w(s_j) - w(t_{k(j)})\|_{L^{3,\infty}(\Omega\setminus D_{k(j)})} \le \epsilon \le \delta$.

Now let

$$c_* := \frac{2^5(2\tilde{C}_0 + 1 + \delta)^2}{\delta^4}. \tag{3.32}$$

Note that, since $K > \delta$, we see $(2\tilde{C}_0 + 1 + \delta)K > 2\tilde{C}_0 K + \delta + \delta^2$ and hence by (3.27)

$$c_* \cdot (K + \|V\|_{3,\infty})^2 \geq c_* K^2 > \frac{2^5(2\tilde{C}_0 K + \delta + \delta^2)^2}{\delta^4} = \frac{2^5}{\epsilon^2}. \tag{3.33}$$

Then (3.3),(3.23) and (3.33) imply

$$\frac{1}{2}\gamma_0 |t_{k(j)}|^{-1/2} > a_{k(j)} \quad \text{for all } j \in \mathbb{N}.$$

Thus, by (3.2) and (3.31) yield

$$\limsup_{j\to\infty} \|w(s_j)\|_{L^{3,\infty}(\Omega\setminus\tilde{D}(s_j))} \leq 3\delta + \limsup_{j\to\infty} \|V\|_{L^{3,\infty}(\Omega\setminus D_{k(j)})}. \tag{3.34}$$

We turn to estimate the last term of (3.34) i.e. $\limsup_{j\to\infty} \|V\|_{L^{3,\infty}(\Omega\setminus D_{k(j)})}$ by using Lemma 13, (3.2) and (3.29). Choose $m \in \mathbb{N}$ such that

$$2^{m-1} < \frac{(K + \|V\|_{3,\infty})^5}{\epsilon^2} \leq 2^m, \quad i.e., \ 2^m \sim \frac{(K + \|V\|_{3,\infty})^5}{\epsilon^2}. \tag{3.35}$$

Since $|t_{k(j)}| + 1 \geq |t_{k(j)}| + |t_{k(j)+1} - t_{k(j)}| \geq |t_{k(j)+1}| \geq |s_j|$,

$$\begin{aligned} \tilde{D}(s_j) = D_{k(j)} \supset D^0_{k(j)} &= \left\{ x \in \Omega;\ |V(x)| \geq \frac{(K + \|V\|_{3,\infty})^5}{\epsilon^2(|t_{k(j)}| + 1)^{1/2}} \right\} \\ \supset \left\{ x \in \Omega\,;\ |V(x)| \geq \frac{(K + \|V\|_{3,\infty})^5}{\epsilon^2 |s_j|^{1/2}} \right\} &\supset \left\{ x \in \Omega\,;\ |V(x)| \geq \frac{2^m}{|s_j|^{1/2}} \right\}. \end{aligned} \tag{3.36}$$

Since $j \leq |s_j| \leq 2j$ for all $j \in \mathbb{N}$, we have $2^i \leq |s_{2^i}| \leq 2^{i+1}$ for all $i \in \mathbb{N}$. Let $\{\tilde{s}_n\}$ be the following subsequence of $\{s_j\}$:

$$\tilde{s}_n := s_{2^{2n+2m}} \quad \text{for } n \in \mathbb{N},$$

then

$$2^{n+m} \leq |\tilde{s}_n|^{1/2} \leq \sqrt{2} \cdot 2^{n+m}. \tag{3.37}$$

Thus,

$$\tilde{D}(\tilde{s}_n) \supset \left\{ x \in \Omega;\ |V(x)| > \frac{1}{2^n} \right\}. \tag{3.38}$$

Set

$$g_n := v(\tilde{s}_n), \quad U := V \quad \text{and} \quad \omega := 3\delta.$$

Since (3.29) and (3.38) implies

$$\begin{aligned}\|v(\tilde{s}_n)\|_{L^{3,\infty}(\{|V|>2^{-n}\})} &\le \|w(\tilde{s}_n)\|_{L^{3,\infty}(\{|V|>2^{-n}\})} + \|u(\tilde{s}_n)\|_{L^{3,\infty}(\{|V|>2^{-n}\})}\\ &\le \|w(\tilde{s}_n)\|_{L^{3,\infty}(\tilde{D}(\tilde{s}_n))} + \delta \le 2\delta \le \omega\end{aligned}$$

for all $n \in \mathbb{N}$, we see that $g_n (= v(\tilde{s}_n))$ and ω satisfy (2.27).

Since (3.3), (3.33),(3.35) and (3.37) imply

$$\frac{1}{2}\gamma_0 |\tilde{s}_{n+1}|^{-1/2} \ge 2^{-n+1},$$

by (3.2) we see that $g_n (= v(\tilde{s}_n))$ and $U (= V)$ satisfy (2.28) with $\omega = 3\delta$.

Thus, since $a_{k(j)} \searrow 0$ as $j \to \infty$, from Lemma 13 we observe

$$\begin{aligned}\limsup_{j\to\infty} \|V\|_{L^{3,\infty}(\Omega\setminus D_{k(j)})} &\le \limsup_{j\to\infty} \|V\|_{L^{3,\infty}(\{x\in\Omega;\ |V(x)|<a_{k(j)}\})}\\ &\le \limsup_{i\to\infty} \|V\|_{L^{3,\infty}(\{x\in\Omega;\ |V(x)|\le 2^{-i}\})}\\ &\le C_3\omega = 3C_3\delta. \end{aligned} \tag{3.39}$$

Here C_3 is the constant given in Lemma 13.

Therefore, from (3.29), (3.34) and (3.39) we obtain

$$\begin{aligned}\limsup_{j\to\infty} \|w(s_j)\|_{L^{3,\infty}(\Omega)} &\le \limsup_{j\to\infty} \|w(s_j)\|_{L^{3,\infty}(\tilde{D}(s_j))}\\ &\quad + \limsup_{j\to\infty} \|w(s_j)\|_{L^{3,\infty}(\Omega\setminus\tilde{D}(s_j))} \le 4\delta + 3C_3\delta,\end{aligned} \tag{3.40}$$

which implies

$$\liminf_{t\to-\infty} \|w(t)\|_{L^{3,\infty}(\Omega)} \le (4 + 3C_3)\delta. \tag{3.41}$$

Since $\delta = \frac{\min\{\epsilon_1,\delta_0\}}{6+3C_3}$, we conclude that

$$\liminf_{t\to-\infty} \|w(t)\|_{L^{3,\infty}(\Omega)} < \epsilon_1,$$

which with the help of Lemma 9 yields

$$u = v \quad \text{on } (-\infty, T].$$

This proves Theorem 1. □

4 Proof of Theorem 3

Proof In this section we will prove Theorem 3, by using the same methods as in the proof of Theorem 1. Let δ be the same number given in the proof of Theorem 1. Without loss of generality, we may assume $T > 0$ and

$$\sup_{t\le 0}\|u(t)\|_{3,\infty}\le\delta, \tag{4.1}$$
$$\sup_{t\le 0}\|v(t)-V(t)\|_{L^{3,\infty}(\{|V(t)-v(t)|\le\eta\})}\le\delta. \tag{4.2}$$

Let $\epsilon\in(0,1]$ be an arbitrary fixed number, K be the same number given in the proof of Theorem 1, i.e. $K:=\sup_{t\le 0}\|v(t)\|_{3,\infty}+1+\delta$, and $\{t_k\}$ be a sequence such that

$$\begin{aligned}&0=t_0>t_1>t_2>\cdots,\quad |t_k-t_{k+1}|<1,\quad t_k\to-\infty\ \text{ as } k\to\infty,\\&\sup_{t_{k+1}\le t\le t_k}\|w(t)-w(t_k)\|_{3,\infty}+\sup_{t_{k+1}\le t\le t_k}\|V(t)-V(t_k)\|_{3,\infty}<\epsilon\\&\qquad\text{for all } k=0,1,2,\ldots.\end{aligned}\tag{4.3}$$

Let $y_0\in\Omega$ be a fixed arbitrary point and

$$b_k:=\frac{(K+\sup_{s\le 0}\|V(s)\|_{3,\infty}+\sum_{l=1}^N\|V_l\|_{3,\infty})^5}{\epsilon^2(|t_k|+1)^{1/2}}.$$

Then we define a sequence $\{D_k\}$ of measurable subsets of Ω as follows.

$$\begin{aligned}D_k^{0,l}&:=\{x\in\Omega;\ |V_l(x)|\ge b_k\},\quad l=1,2,\ldots,N,\\D_k^1&:=\{x\in\Omega;\ |v(x,t_k)-V(x,t_k)|\ge b_k\},\\D_k^2&:=\{x\in\Omega;\ |x-y_0|<1/b_k\},\\D_k&:=\left(\bigcup_{l=1}^N D_k^{0,l}\right)\cup D_k^1\cup D_k^2,\quad k=0,1,2,\ldots\end{aligned}\tag{4.4}$$

and also define $\tilde D(t)$ in the same way as in the proof of Theorem 1:

$$\tilde D(t)=D_k\quad\text{if } t\in(t_{k+1},t_k],\ k=0,1,2\ldots.$$

Note that, as we have proven in (Step 1) of the proof of Theorem 1, (3.8) holds. In the same way as in the proofs of (3.25)–(3.26), we see that

$$\mu(\tilde D(t))\le\frac{C(N)\epsilon^6(|t|+1)^{3/2}}{K^{12}},\quad \int_{-j}^0\|w(\tau)\|^2_{L^3(\tilde D(\tau))}d\tau\le\tilde C_0(N)K\epsilon,$$

where the constants $C(N)$ and $\tilde{C}_0(N)$ are depending only on N and Ω. Now we choose ϵ such that

$$\epsilon := \frac{\delta^2}{2\tilde{C}_0(N)K + \delta + \delta^2} \left(\leq \min \left\{ \frac{\delta^2}{2\tilde{C}_0(N)K}, \delta, 1 \right\} \right). \tag{4.5}$$

Then, in the same way as in (3.29)–(3.30), there exist sequences $\{k(j)\} \subset \mathbb{N}$ and $\{s_j\}$ such that

$$-2j \leq s_j \leq -j, \quad t_{k(j)+1} < s_j \leq t_{k(j)}, \quad \|w(s_j)\|_{L^{3,\infty}(\tilde{D}(s_j))} \leq \delta \tag{4.6}$$

for all $j \in \mathbb{N}$. Since

$$\begin{aligned} &\tilde{D}(s_j) = D_{k(j)} \supset \{|V_l| \geq b_{k(j)}\} \quad \text{for all } l = 1, 2, \ldots, N \text{ and all } j \in \mathbb{N}, \\ &b_{k(j)} \searrow 0 \quad \text{as } j \to \infty, \text{ and} \\ &\mathscr{R}(V) = \{V(t) \in L^{3,\infty};\ t \in (-\infty, 0]\} \subset \bigcup_{l=1}^{N} \{\theta \in L^{3,\infty};\ \|\theta - V_l\|_{3,\infty} < \delta\}, \end{aligned} \tag{4.7}$$

we can choose subsequences $\{s_{j(n)}\}_{n=1}^{\infty}$ and $\{t_{k(j(n))}\}_{n=1}^{\infty}$ of $\{s_j\}$ and $\{t_{k(j)}\}$ respectively and find a suitable $l_0 \in \{1, 2, \ldots, N\}$ such that

$$\begin{aligned} &\|V(t_{k(j(n))}) - V_{l_0}\|_{3,\infty} < \delta, \\ &\quad t_{k(j(n))+1} < s_{j(n)} \leq t_{k(j(n))}, \\ &\quad \tilde{D}(s_{j(n)}) = D_{k(j(n))} \supset \{|V_{l_0}| > 2^{-n}\}, \quad n = 1, 2, \ldots. \end{aligned} \tag{4.8}$$

For simplicity, we denote $\{k(j(n))\}$ by $\{k(n)\}$. Note $t_{k(n)+1} < s_{j(n)} \leq t_{k(n)}$, $t_{k(n)} \to -\infty$ and $b_{k(n)} \to 0$ as $n \to \infty$. Since

$$\begin{aligned} &\|w(s_{j(n)})\|_{L^{3,\infty}(\Omega \setminus \tilde{D}(s_{j(n)}))} \\ &\leq \|w(s_{j(n)}) - w(t_{k(n)})\|_{L^{3,\infty}(\Omega \setminus D_{k(n)})} + \|w(t_{k(n)})\|_{L^3(\Omega \setminus D_{k(n)})} \\ &\leq \|w(s_{j(n)}) - w(t_{k(n)})\|_{L^{3,\infty}(\Omega \setminus D_{k(n)})} \\ &\quad + \|u(t_{k(n)})\|_{L^{3,\infty}(\Omega \setminus D_{k(n)})} + \|v(t_{k(n)}) - V(t_{k(n)})\|_{L^{3,\infty}(\Omega \setminus D_{k(n)})} \\ &\quad + \|V(t_{k(n)}) - V_{l_0}\|_{L^{3,\infty}(\Omega \setminus D_{k(n)})} + \|V_{l_0}\|_{L^{3,\infty}(\Omega \setminus D_{k(n)})} \\ &\leq \delta + \delta + \|v(t_{k(n)}) - V(t_{k(n)})\|_{L^{3,\infty}(\Omega \setminus D_{k(n)})} + \delta + \|V_{l_0}\|_{L^{3,\infty}(\Omega \setminus D_{k(n)})} \end{aligned} \tag{4.9}$$

and since

$$\begin{aligned} \|v(t_{k(n)}) - V(t_{k(n)})\|_{L^{3,\infty}(\Omega \setminus D_{k(n)})} &\leq \|v(t_{k(n)}) - V(t_{k(n)})\|_{L^{3,\infty}(\Omega \setminus D^1_{k(n)})} \\ &\leq \|v(t_{k(n)}) - V(t_{k(n)})\|_{L^{3,\infty}(\{|v(t_{k(n)}) - V(t_{k(n)})| \leq \eta\})} \end{aligned} \tag{4.10}$$

for sufficiently large n, by (4.2) we have

$$\limsup_{n\to\infty} \|w(s_{j(n)})\|_{L^{3,\infty}(\Omega\setminus\tilde{D}(s_{j(n)}))} \le 4\delta + \limsup_{n\to\infty} \|V_{l_0}\|_{L^{3,\infty}(\Omega\setminus D_{k(n)})}. \tag{4.11}$$

Set

$$g_n := v(s_{j(n)}) - V(s_{j(n)}) + V_{l_0}, \quad U := V_{l_0}, \quad \text{and} \quad \omega := 4\delta.$$

Then, by (4.1), (4.3), (4.5), (4.6) and (4.8) we have

$$\begin{aligned} \|g_n\|_{L^{3,\infty}(\{|V_{l_0}|>2^{-n}\})} &= \| - w(s_{j(n)}) + u(s_{j(n)}) \\ &\quad -(V(s_{j(n)}) - V(t_{k(n)})) + (V_{l_0} - V(t_{k(n)}))\|_{L^{3,\infty}(\{|V_{l_0}|>2^{-n}\})} \\ &\le \|w(s_{j(n)})\|_{L^{3,\infty}(\tilde{D}(s_{j(n)}))} + \|u(s_{j(n)})\|_{L^{3,\infty}(\tilde{D}(s_{j(n)}))} \\ &\quad +\|V(s_{j(n)}) - V(t_{k(n)})\|_{L^{3,\infty}(\tilde{D}(s_{j(n)}))} \\ &\quad +\|V_{l_0} - V(t_{k(n)})\|_{L^{3,\infty}(\tilde{D}(s_{j(n)}))} \\ &\le 4\delta = \omega. \end{aligned} \tag{4.12}$$

Since $g_n - U = v(s_{j(n)}) - V(s_{j(n)})$, by (4.2) and (4.12) we see that $\{g_n\}$ and $U(= V_{l_0})$ satisfy (2.27) and (2.28). Thus, by Lemma 13 we have

$$\begin{aligned} \limsup_{n\to\infty} \|V_{l_0}\|_{L^{3,\infty}(\Omega\setminus D_{k(n)})} &\le \limsup_{n\to\infty} \|V_{l_0}\|_{L^{3,\infty}(\{|V_{l_0}|<b_{k(n)}\})} \\ &\le \limsup_{i\to\infty} \|V_{l_0}\|_{L^{3,\infty}(\{|V_{l_0}|\le 2^{-i}\})} \le C_3\omega = 4C_3\delta. \end{aligned} \tag{4.13}$$

Hence, from (4.6), (4.11) and (4.13) we obtain

$$\liminf_{t\to-\infty} \|w(t)\|_{L^{3,\infty}(\Omega)} \le \limsup_{n\to\infty} \|w(s_{j(n)})\|_{L^{3,\infty}(\Omega)} \le 5\delta + 4C_3\delta. \tag{4.14}$$

Therefore, since $\delta = \frac{\min\{\epsilon_1,\delta_0\}}{6+4C_3}$, we conclude that

$$\liminf_{t\to-\infty} \|w(t)\|_{L^{3,\infty}(\Omega)} < \epsilon_1,$$

which with the help of Lemma 9 proves Theorem 3. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Appendix

Here we will show that (1.12)–(1.14) guarantee $\frac{1}{\sqrt{-t}}v(\frac{x}{\sqrt{-t}}) \in BC((-\infty, 0); \tilde{L}^{3,\infty}_{\sigma}(\mathbb{R}^3_+))$. Let

$$S_t f := \frac{1}{\sqrt{-t}} f\left(\frac{\cdot}{\sqrt{-t}}\right)$$

for $t < 0$ and let

$$\tilde{L}^{3,\infty}(\mathbb{R}^3_+) := \overline{L^{3,\infty}(\mathbb{R}^3_+) \cap L^{\infty}(\mathbb{R}^3_+)}^{\|\cdot\|_{3,\infty}}. \tag{A.1}$$

Since $\|S_t v\|_{3,\infty} = \|v\|_{3,\infty}$ and $S_t v \in \tilde{L}^{3,\infty}(\mathbb{R}^3_+)$ for $v \in \tilde{L}^{3,\infty}(\mathbb{R}^3_+)$ and for $t < 0$, it suffices to show the following lemma:

Lemma 14 *Let functions v, R, V, Q satisfy*

$$\begin{aligned} &v \in \tilde{L}^{3,\infty}(\mathbb{R}^3_+),\\ &R \in L^{3,\infty}(\mathbb{R}^3_+) \cap L^{r,\infty}(\mathbb{R}^3_+) \quad \textit{for some } r \in (1,3),\\ &V(x) = \frac{Q(x/|x|)}{|x|} \quad \textit{for all } x \in \mathbb{R}^3_+,\ Q \in C(S^2),\\ &v = V + R. \end{aligned} \tag{A.2}$$

Then

$$S_t v \in C((-\infty, 0); L^{3,\infty}(\mathbb{R}^3_+)). \tag{A.3}$$

Proof (Case 1) We first consider the case $v \in L^{3,\infty}(\mathbb{R}^3_+) \cap L^{\infty}(\mathbb{R}^3_+)$. Since $S_t V = V$, we have

$$S_{t_1} v - S_{t_2} v = S_{t_1} R - S_{t_2} R \quad \text{for all } t_1, t_2 < 0. \tag{A.4}$$

Let $\epsilon > 0$ be an arbitrary fixed number and let $r < q < 3$. Since $R \in L^{3,\infty}(\mathbb{R}^3_+) \cap L^{r,\infty}(\mathbb{R}^3_+)$, we see that $R \in L^q(\mathbb{R}^3_+)$ and hence there exists a function $\psi_\epsilon \in C_0^\infty(\mathbb{R}^3_+)$ such that

$$\|R - \psi_\epsilon\|_q < \epsilon.$$

It is straightforward to see that $S_t\psi_\epsilon \in C((-\infty,0); L^q(\mathbb{R}^3_+))$. Then, since

$$\begin{aligned}\|S_{t_1}R - S_{t_2}R\|_q &\le \|S_{t_1}(R-\psi_\epsilon)\|_q + \|S_{t_2}(R-\psi_\epsilon)\|_q + \|S_{t_1}\psi_\epsilon - S_{t_2}\psi_\epsilon\|_q \\ &\le \left(|t_1|^{\frac{1}{2}(\frac{3}{q}-1)} + |t_2|^{\frac{1}{2}(\frac{3}{q}-1)}\right)\epsilon + \|S_{t_1}\psi_\epsilon - S_{t_2}\psi_\epsilon\|_q,\end{aligned}$$

we have $\limsup_{t_1\to t_2}\|S_{t_1}R - S_{t_2}R\|_q \le 2|t_2|^{\frac{1}{2}(\frac{3}{q}-1)}\epsilon$ and, letting $\epsilon \to 0$,

$$\limsup_{t_1\to t_2}\|S_{t_1}R - S_{t_2}R\|_q = 0 \tag{A.5}$$

for all $t_2 < 0$. Since $\|S_{t_1}v - S_{t_2}v\|_\infty \le (\frac{1}{|t_1|^{1/2}} + \frac{1}{|t_2|^{1/2}})\|v\|_\infty$, by the interpolation, (A.4) and (A.5), we have

$$\begin{aligned}&\limsup_{t_1\to t_2}\|S_{t_1}v - S_{t_2}v\|_{3,\infty} \\ &\le C\limsup_{t_1\to t_2}\left(\|S_{t_1}R - S_{t_2}R\|_q^{q/3}\|S_{t_1}v - S_{t_2}v\|_\infty^{1-q/3}\right) = 0\end{aligned} \tag{A.6}$$

for all $t_2 < 0$. Therefore, (A.3) holds.

(Case 2) Next we consider the case $v \in \tilde{L}^{3,\infty}(\mathbb{R}^3_+)$. In this case, it holds that

$$\lim_{n\to\infty}\|v1_{\{|v|>n\}}\|_{3,\infty} = 0. \tag{A.7}$$

Indeed, by (A.1), there exists a sequence $\{\phi_k\}$ such that

$$\phi_k \in L^{3,\infty}(\mathbb{R}^3_+) \cap L^\infty(\mathbb{R}^3_+), \quad \|v-\phi_k\|_{3,\infty} < 1/k \qquad \text{for all } k \in \mathbb{N}.$$

Then

$$\begin{aligned}\|v1_{\{|v|>n\}}\|_{3,\infty} &\le \|(v-\phi_k)1_{\{|v|>n\}}\|_{3,\infty} + \|\phi_k 1_{\{|v|>n\}}\|_{3,\infty} \\ &\le 1/k + C\|\phi_k\|_\infty\|1_{\{|v|>n\}}\|_{L^3} \le 1/k + C\|\phi_k\|_\infty(\mu\{|v|>n\})^{1/3} \\ &\le 1/k + C\|\phi_k\|_\infty\frac{\|v\|_{3,\infty}}{n} \quad \text{for all } k \in \mathbb{N},\end{aligned} \tag{A.8}$$

which implies $\limsup_{n\to\infty}\|v1_{\{|v|>n\}}\|_{3,\infty} \le 1/k$ for all $k \in \mathbb{N}$ and hence (A.7). Now let

$$v_n(x) := v(x)1_{\{|v|\le n\}}(x) \quad \text{and} \quad R_n(x) := R(x) - v(x)1_{\{|v|>n\}}(x).$$

Clearly,

$$v_n \in L^{3,\infty}(\mathbb{R}^3_+) \cap L^\infty(\mathbb{R}^3_+) \quad \text{and} \quad R_n \in L^{3,\infty}(\mathbb{R}^3_+). \tag{A.9}$$

Since

$$\begin{aligned}\|v1_{\{|v|>n\}}\|_{r,\infty} &\le C\sup_{s>0} s\,(\mu\{|v|>s,|v|>n\})^{1/3}\,(\mu\{|v|>s,|v|>n\})^{1/r-1/3}\\ &\le C\sup_{s>0} s\,(\mu\{|v|>s\})^{1/3}\left(\frac{\|v\|_{3,\infty}}{n}\right)^{3/r-1}\\ &\le C\|v\|_{3,\infty}^{3/r}\frac{1}{n^{3/r-1}}.\end{aligned}$$

Then,

$$R_n \in L^{r,\infty}(\mathbb{R}^3_+). \tag{A.10}$$

Since $v_n - R_n = v - R = V$, we have

$$v_n = V + R_n. \tag{A.11}$$

Hence, as demonstrated in (Case 1), (A.9), (A.10) and (A.11) imply

$$S_t v_n \in C((-\infty,0); L^{3,\infty}(\mathbb{R}^3_+)).$$

Thus,

$$\begin{aligned}\limsup_{t_1\to t_2}\|S_{t_1}v - S_{t_2}v\|_{3,\infty} &\le \limsup_{t_1\to t_2}\big(\|S_{t_1}(v-v_n)\|_{3,\infty}\\ &\qquad +\|S_{t_2}(v-v_n)\|_{3,\infty} + \|S_{t_1}v_n - S_{t_2}v_n)\|_{3,\infty}\big)\\ &= 2\|v-v_n\|_{3,\infty} = 2\|v1_{\{|v|>n\}}\|_{3,\infty}\end{aligned} \tag{A.12}$$

for all $t_2 < 0$. Therefore, letting $n \to \infty$, from (A.7) we conclude that $\lim_{t_1\to t_2}\|S_{t_1}v - S_{t_2}v\|_{3,\infty} = 0$ and (A.3) holds. □

References

1. Bergh, J., Löfström, J.: Interpolation Spaces, An Introduction. Springer, Berlin (1976)
2. Borchers, W., Miyakawa, T.: L^2 decay for the Navier–Stokes flow in halfspaces. Math. Ann. **282**, 139–155 (1988)
3. Borchers, W., Miyakawa, T.: Algebraic L^2 decay for Navier–Stokes flows in exterior domains. Acta Math. **165**, 189–227 (1990)
4. Borchers, W., Miyakawa, T.: On stability of exterior stationary Navier–Stokes flows. Acta Math. **174**, 311–382 (1995)
5. Borchers, W., Sohr, H.: On the semigroup of the Stokes operator for exterior domains in L^q-spaces. Math. Z. **196**, 415–425 (1987)
6. Crispo, F., Maremonti, P.: Navier–Stokes equations in aperture domains: global existence with bounded flux and time-periodic solutions. Math. Methods Appl. Sci. **31**, 249–277 (2008)
7. Crispo, F., Maremonti, P.: On the uniqueness of a suitable weak solution to the Navier–Stokes Cauchy problem. Partial Differ. Equ. Appl. **2**, 35 (2021). https://doi.org/10.1007/s42985-021-00073-z
8. Eiter, T., Kyed, M.: Viscous flow around a rigid body performing a time-periodic motion. J. Math. Fluid Mech. **23**, 28 (2021)

9. Eiter, T., Kyed, M., Shibata, Y.: Periodic L_p estimates by $\mathscr{R}$-boundedness: Applications to the Navier–Stokes equations. Preprint
10. Farwig, R., Nakatsuka, T., Taniuchi, Y.: Uniqueness of solutions on the whole time axis to the Navier–Stokes equations in unbounded domains. Commun. Partial Differ. Equ. **40**, 1884–1904 (2015)
11. Farwig, R., Nakatsuka, T., Taniuchi, Y.: Existence of solutions on the whole time axis to the Navier–Stokes equations with precompact range in L^3. Arch. Math. **104**, 539–550 (2015)
12. Farwig, R., Okabe, T.: Periodic solutions of the Navier–Stokes equations with inhomogeneous boundary conditions. Ann. Univ. Ferrara Sez. VII Sci. Mat. **56**, 249–281 (2010)
13. Farwig, R., Sohr, H.: Generalized resolvent estimates for the Stokes system in bounded and unbounded domains. J. Math. Soc. Jpn. **46**, 607–643 (1994)
14. Farwig, R., Sohr, H.: On the Stokes and Navier–Stokes system for domains with noncompact boundary in L^q-spaces. Math. Nachr. **170**, 53–77 (1994)
15. Farwig, R., Sohr, H.: Helmholtz decomposition and Stokes resolvent system for aperture domains in L^q-spaces. Analysis **16**, 1–26 (1996)
16. Farwig, R., Taniuchi, Y.: Uniqueness of almost periodic-in-time solutions to Navier–Stokes equations in unbounded domains. J. Evol. Equ. **11**, 485–508 (2011)
17. Farwig, R., Taniuchi, Y.: Uniqueness of backward asymptotically almost periodic-in-time solutions to Navier–Stokes equations in unbounded domains. Discret. Contin. Dyn. Syst. Ser. S **6**, 1215–1224 (2013)
18. Foias, C.: Une remarque sur l'unicité des solutions des équations de Navier–Stokes en dimension n. Bull. Soc. Math. Fr. **89**, 1–8 (1961)
19. Galdi, G.P.: Viscous flow past a body translating by time-periodic motion with zero average. Arch. Ration. Mech. Anal. **237**, 1237–1269 (2020)
20. Galdi, G.P., Kyed, M.: Time-periodic solutions to the Navier–Stokes equations. In: Giga, Y., Novotny, A. (eds.) Handbook of Mathematical Analysis in Mechanics of Viscous Fluids. Springer, Berlin (2017)
21. Galdi, G.P., Sohr, H.: Existence and uniqueness of time-periodic physically reasonable Navier–Stokes flow past a body. Arch. Ration. Mech. Anal. **172**, 363–406 (2004)
22. Geissert, M., Hieber, M., Nguyen, T.H.: A general approach to time periodic incompressible viscous fluid flow problems. Arch. Ration. Mech. Anal. **220**, 1095–1118 (2016)
23. Giga, Y.: Analyticity of the semigroup generated by the Stokes operator in L_r spaces. Math. Z. **178**, 297–329 (1981)
24. Giga, Y., Miyakawa, T.: Navier–Stokes flow in R^3 with measures as initial vorticity and Morrey spaces. Commun. Partial Differ. Equ. **14**, 577–618 (1989)
25. Giga, Y., Sohr, H.: On the Stokes operator in exterior domains. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **36**, 103–130 (1989)
26. Hishida, T.: The nonstationary Stokes and Navier–Stokes flows through an aperture. In: Galdi, G.P., Heywood, J.G., Rannacher, R. (eds.) Contributions to Current Challenges in Mathematical Fluid Mechanics. Advances in Mathematical Fluid Mechanics, pp. 79–123. Birkhäuser, Basel (2004)
27. Iwashita, H.: $L_q - L_r$ estimates for solutions of the nonstationary Stokes equations in an exterior domain and the Navier–Stokes initial value problems in L_q spaces. Math. Ann. **285**, 265–288 (1989)
28. Kang, K., Miura, H., Tsai, T.-P.: Asymptotics of small exterior Navier–Stokes flows with non-decaying boundary data. Commun. Partial Differ. Equ. **37**, 1717–1753 (2012)
29. Kobayashi, T., Shibata, Y.: On the Oseen equation in the three dimensional exterior domains. Math. Ann. **310**, 1–45 (1998)
30. Kozono, H., Nakao, M.: Periodic solutions of the Navier–Stokes equations in unbounded domains. Tohoku Math. J. **48**, 33–50 (1996)
31. Kozono, H., Ogawa, T.: On stability of Navier–Stokes flows in exterior domains. Arch. Ration. Mech. Anal. **128**, 1–31 (1994)
32. Kozono, H., Sohr, H., Yamazaki, M.: Representation formula, net force and energy relation to the stationary Navier–Stokes equations in 3-dimensional exterior domains. Kyushu J. Math. **51**, 239–260 (1997)
33. Kozono, H., Yamazaki, M.: Uniqueness criterion of weak solutions to the stationary Navier–Stokes equations in exterior domains. Nonlinear Anal. Ser. A: Theory Methods **38**, 959–970 (1999)
34. Kubo, T.: The Stokes and Navier–Stokes equations in an aperture domain. J. Math. Soc. Jpn. **59**, 837–859 (2007)
35. Kubo, T.: Periodic solutions of the Navier–Stokes equations in a perturbed half-space and an aperture domain. Math. Methods Appl. Sci. **28**, 1341–1357 (2005)

36. Kubo, T., Shibata, Y.: On some properties of solutions to the Stokes equation in the half-space and perturbed half-space. In: Dispersive Nonlinear Problems in Mathematical Physics. Quad. Mat., vol. 15, pp. 149–220. Department of Mathematics, Seconda Univ. Napoli, Caserta (2004)
37. Lions, P.-L., Masmoudi, N.: Uniqueness of mild solutions of the Navier–Stokes system in L^N. Commun. Partial Differ. Equ. **26**, 2211–2226 (2001)
38. Maremonti, P.: Existence and stability of time-periodic solutions to the Navier–Stokes equations in the whole space. Nonlinearity **4**, 503–529 (1991)
39. Maremonti, P.: Some theorems of existence for solutions of the Navier–Stokes equations with slip boundary conditions in half-space. Ric. Mat. **40**, 81–135 (1991)
40. Maremonti, P., Padula, M.: Existence, uniqueness, and attainability of periodic solutions of the Navier–Stokes equations in exterior domains. J. Math. Sci. (N. Y.) **93**, 719–746 (1999)
41. Meyer, Y.: Wavelets, paraproducts, and Navier–Stokes equations. In: Bott, R., Jaffe, A., Jerison, D., Lusztig, G., Singer, I., Yau, S.T. (eds.) Current Developments in Mathematics, 1996 (Cambridge, MA), pp. 105–212. Int. Press, Boston (1997)
42. Miyakawa, T.: On nonstationary solutions of the Navier–Stokes equations in an exterior domain. Hiroshima Math. J. **12**, 115–140 (1982)
43. Nakatsuka, T.: On uniqueness of stationary solutions to the Navier–Stokes equations in exterior domains. Nonlinear Anal. **75**, 3457–3464 (2012)
44. Nakatsuka, T.: Uniqueness of steady Navier–Stokes flows in exterior domains. Funkcial. Ekvac. **56**, 323–337 (2013)
45. Nečas, J., Růžička, M., Šverák, V.: On Leray's self-similar solutions of the Navier–Stokes equations. Acta Math. **176**, 283–294 (1996)
46. Okabe, T., Tsutsui, Y.: Time-periodic strong solutions to the incompressible Navier–Stokes equations with external forces of non-divergence form. J. Differ. Equ. **263**, 8229–8263 (2017)
47. Salvi, R.: On the existence of periodic weak solutions on the Navier–Stokes equations in exterior regions with periodically moving boundaries. In: Sequeira, A. (ed.) Navier–Stokes Equations and Related Nonlinear Problems (Funchal, 1994), pp. 63–73. Plenum, New York (1995)
48. Shibata, Y.: On a stability theorem of the Navier–Stokes equations in a three dimensional exterior domain. Tosio Kato's method and principle for evolution equations in mathematical physics (Sapporo, 2001). Surikaisekikenkyusho Kokyuroku **1234**, 146–172 (2001)
49. Shibata, Y.: On some stability theorems about viscous fluid flow. Quad. Sem. Mat. Brescia **01**, 1–72 (2003)
50. Simader, C.G., Sohr, H.: A new approach to the Helmholtz decomposition and the Neumann problem in L^q-spaces for bounded and exterior domains. In: Galdi, G.P. (ed.) Mathematical Problems Relating to the Navier–Stokes Equation. Series on Advances in Mathematics for Applied Sciences, vol. 11, pp. 1–35. World Scientific, River Edge (1992)
51. Taniuchi, Y.: On the uniqueness of time-periodic solutions to the Navier–Stokes equations in unbounded domains. Math. Z. **261**, 597–615 (2009)
52. Taniuchi, Y.: A remark on the uniqueness of Kozono–Nakao's mild L^3-solutions on the whole time axis to the Navier–Stokes equations in unbounded domains. Partial Differ. Equ. Appl. **2**, 68 (2021). https://doi.org/10.1007/s42985-021-00121-8
53. Tsai, T.-P.: On Leray's self-similar solutions of the Navier–Stokes equations satisfying local energy estimate. Arch. Ration. Mech. Anal **143**, 29–51 (1998)
54. Ukai, S.: A solution formula for the Stokes equation in $\mathbb{R}^n_+$. Commun. Pure Appl. Math. **40**, 611–621 (1987)
55. Yamazaki, M.: The Navier–Stokes equations in the weak-L^n space with time-dependent external force. Math. Ann. **317**, 635–675 (2000)