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Heat and Martin Kernel estimates for Schrödinger operators with critical Hardy potentials

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Abstract

Let Ω be a bounded domain in $\mathbb{R}^N$ with C^2 boundary and let $K \subset \partial\Omega$ be either a C^2 submanifold of the boundary of codimension $k < N$ or a point. In this article we study various problems related to the Schrödinger operator $L_\mu = -\Delta - \mu d_K^{-2}$ where d_K denotes the distance to K and $\mu \leq k^2/4$. We establish parabolic boundary Harnack inequalities as well as related two-sided heat kernel and Green function estimates. We construct the associated Martin kernel and prove existence and uniqueness for the corresponding boundary value problem with data given by measures. To prove our results we introduce among other things a suitable notion of boundary trace. This trace is different from the one used by Marcus and Nguyen (Math Ann 374(1–2):361–394, 2019) thus allowing us to cover the whole range $\mu \leq k^2/4$.

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1 Introduction

The study of linear Schrödinger operators with singular potentials is central in the theory of parabolic and elliptic partial differential equations. In recent years in particular there has been an intense study of operators with Hardy potentials, see e.g. [2, 4, 6, 9, 10, 15, 20, 22, 23, 31, 40].

Throughout this work we assume that Ω is a bounded C^2 domain; we note however that some of the results presented in this introduction are valid under weaker regularity assumptions.

Consider the problem

$$\begin{cases} u_t = \Delta u + V(x)u, & x \in \Omega, \ t > 0, \\ u = 0, & x \in \partial\Omega, \ t > 0, \\ u(0,x) = u_0(x), & x \in \Omega\,, \end{cases} \tag{1.1}$$

where $V \in L^1_{\text{loc}}(\Omega)$ and set

$$\lambda^* = \inf_{C_c^\infty(\Omega)} \frac{\int_\Omega |\nabla w|^2 dx - \int_\Omega V w^2 dx}{\int_\Omega w^2 dx}.$$

Cabré and Martel [11] have established that if $\lambda^* > -\infty$ then for regular enough initial data there exists a global in time weak solution of (1.1) which in addition satisfies an exponential in time bound. Conversely, the existence of a weak solution which satisfies an exponential bound implies that $\lambda^* > -\infty$. In the prototype case of the Hardy potential $V(x) = c|x|^{-2}$ this has already been studied by Baras and Goldstein [3].

Given the existence of a weak solution one natural question is the existence and asymptotic behaviour of the heat kernel and Green function. If the potential is not too singular then the asymptotic behaviour of the heat kernel for small time is the same as that of the Laplacian, namely

$$
\begin{aligned}
&C^{-1}\Big(\frac{d(x)d(y)}{(d(x)+\sqrt{t})(d(y)+\sqrt{t})}\Big)t^{-\frac{N}{2}}\exp\Big(-C\frac{|x-y|^2}{t}\Big)\\
&\quad\le h(t,x,y)\le C\Big(\frac{d(x)d(y)}{(d(x)+\sqrt{t})(d(y)+\sqrt{t})}\Big)t^{-\frac{N}{2}}\exp\Big(-C^{-1}\frac{|x-y|^2}{t}\Big),
\end{aligned}
$$

where $d(x) = \mathrm{dist}(x, \partial\Omega)$ denotes the distance to the boundary, see e.g. [53].

In the case of a more singular potential such as a Hardy potential, the problem has been studied in [5, 17, 18, 25, 26, 37, 47–49, 51].

A distinction that plays an important role in this context is whether the singularity of the Hardy potential occurs in the interior or on the boundary of the domain. For the potential $\mu|x|^{-2}$, $0 \le \mu \le (\frac{N-2}{2})^2$, where $0 \in \Omega$, for small time we have

$$
\begin{aligned}
&C^{-1}\Big(\frac{d(x)d(y)}{(d(x)+\sqrt{t})(d(y)+\sqrt{t})}\Big)\Big(\frac{|x|\,|y|}{(|x|+\sqrt{t})(|y|+\sqrt{t})}\Big)^{\theta_+}t^{-\frac{N}{2}}\exp\Big(-C\frac{|x-y|^2}{t}\Big)\\
&\le h(t,x,y)\\
&\le C\Big(\frac{d(x)d(y)}{(d(x)+\sqrt{t})(d(y)+\sqrt{t})}\Big)\Big(\frac{|x|\,|y|}{(|x|+\sqrt{t})(|y|+\sqrt{t})}\Big)^{\theta_+}t^{-\frac{N}{2}}\\
&\quad\times\exp\Big(-C^{-1}\frac{|x-y|^2}{t}\Big),
\end{aligned}
$$

where θ_+ is the largest solution to the equation $\theta^2 + (N-2)\theta + \mu = 0$; see [25]. This estimate was generalized in [29] in case where the distance is taken from a closed surface $\Sigma \subset \Omega$ of codimension k, $2 \le k \le N$; see also [27, 28] for more results within this framework.

On the other hand, when the distance is taken from the boundary $\partial\Omega$ the following small time estimate is valid for the heat kernel of the operator $-\Delta - \mu d(x)^{-2}$, $0 \le \mu \le \frac{1}{4}$,

$$
\begin{aligned}
&C^{-1}\Big(\frac{d(x)d(y)}{(d(x)+\sqrt{t})(d(y)+\sqrt{t})}\Big)^{1+\theta_+}t^{-\frac{N}{2}}\exp\Big(-C\frac{|x-y|^2}{t}\Big)\\
&\quad\le h(t,x,y)\ \le C\Big(\frac{d(x)d(y)}{(d(x)+\sqrt{t})(d(y)+\sqrt{t})}\Big)^{1+\theta_+}t^{-\frac{N}{2}}\exp\Big(-C^{-1}\frac{|x-y|^2}{t}\Big),
\end{aligned}
$$

where θ_+ is the largest solution to the equation $\theta^2 + \theta + \mu = 0$, see [25, 26].

Another function that is important in the study of this type of problems is the Martin kernel [1, 35, 46]. Ancona proved the existence of the Martin kernel $K_{\mu,\partial\Omega}(x, y)$ of $L_\mu^{\partial\Omega} = -\Delta - \frac{\mu}{d^2}$, $\mu < \frac{1}{4}$, with pole at y, which is unique up to a normalization (see [1, Theorem 3]). He showed that for any positive solution u of $L_\mu^{\partial\Omega} u = 0$ there exists a unique nonnegative Radon measure ν on $\partial\Omega$ such that

$$u(x) = \int_{\partial\Omega} K_{\mu,\partial\Omega}(x, y) d\nu(y). \tag{1.2}$$

The case $\mu = \frac{1}{4}$ was treated by Gkikas and Véron in [30]. In particular, they showed that the representation formula (1.2) holds true provided the bottom of the spectrum of $L_\mu^{\partial\Omega}$ is positive.

When $K \subset \Omega$ is a closed smooth surface of codimension $k \in \{3, \ldots, N\}$, analogous results where obtained in [29] for the operator $L_\mu^K = -\Delta - \frac{\mu}{d_K^2}$, $\mu \leq \frac{(k-2)^2}{4}$, under the assumption that the bottom of the spectrum of L_μ^K is positive.

Our aim in this article is to study such problems in the case where the Hardy potential involves the distance to a smooth submanifold of the boundary, including the case of a boundary point. In this direction:

- We establish parabolic boundary Harnack inequalities as well as related two-sided heat kernel estimates. For small time, our approach is based on the ideas of Grigoryan and Saloff-Coste [34] (see also [50]), while for large time, we exploit the work of Davies in [16, 17] to obtain sharp- two sided heat kernel estimates; see also [25, 26].
- In the spirit of [12, 35] (see also [29, 30]), we construct the Martin kernel of L_μ in Ω and we prove the uniqueness also in the critical case. Using the heat kernel estimates, we obtain sharp pointwise estimates for the Green function as well as the Martin kernel. We also show that every nonnegative L_μ-harmonic function (i.e. solution of $L_\mu u = 0$ in Ω in the sense of distributions) can be represented as the integral of the Martin kernel with respect to a finite measure on $\partial\Omega$.
- Using the properties of the Green function and Martin kernel we study the boundary value problem with data given by measures. Following Marcus-Véron [44] we prove existence, uniqueness as well as a representation formula for any solution of this problem.

We note that these results are the main tools in the study of semilinear problems for the operator L_μ involving absorption or source terms. In Appendix B we include such results for subcritical absorption. For relevant work see also [7, 8, 13, 21, 27, 28, 30, 32, 33, 41–45] and references therein.

2 Main results

Throughout this article we consider a bounded C^2 domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$, and a C^2 compact submanifold without boundary $K \subset \partial\Omega$ of codimension k, $1 \leq k \leq N$.

For the extreme cases $k = N$ and $k = 1$ we assume that $K = \{0\}$ and $K = \partial\Omega$ respectively. We set $d_K(x) = \mathrm{dist}(x, K)$ and define the operator

$$L_\mu = -\Delta - \frac{\mu}{d_K^2}, \qquad \text{in } \Omega,$$

where μ is a parameter; we shall always assume that $\mu \le \frac{k^2}{4}$ so that L_μ is bounded from below. The study of the parabolic equation $u_t + L_\mu u = 0$ with Dirichlet boundary conditions is strongly related with the minimization problem,

$$C_{\Omega,K} = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega \frac{|u|^2}{d_K^2} dx}.$$

It is well known that $0 < C_{\Omega,K} \le \frac{k^2}{4}$ (see, e.g., [22]).

Let $\mu \le \frac{k^2}{4}$ and let γ_+ (resp. γ_-) denote the largest (resp. the smallest) solution of the equation $\gamma^2 + k\gamma + \mu = 0$. The infimum

$$\lambda_\mu := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx - \mu \int_\Omega \frac{u^2}{d_K^2} dx}{\int_\Omega u^2 dx} \tag{2.1}$$

is finite and, moreover, if $\mu < \frac{k^2}{4}$, then there exists a minimizer $\phi_\mu \in H_0^1(\Omega)$ of (2.1); see [22] for more details. In addition, by [42, Lemma 2.2] the eigenfunction ϕ_μ satisfies

$$\phi_\mu(x) \asymp d(x) d_K^{\gamma_+}(x), \qquad \text{in } \Omega, \tag{2.2}$$

provided $\mu < C_{\Omega,K}$.[1] On the other hand, if $\mu = \frac{k^2}{4}$ then there is no $H_0^1(\Omega)$ minimizer. However, there exists a function $\phi_\mu \in H^1_{loc}(\Omega)$ such that $L_\mu \phi_\mu = \lambda_\mu \phi_\mu$ in Ω in the sense of distributions. In Proposition A.2 in the Appendix we follow ideas of [10, 19, 20, 26] and extend (2.2) to the full range $\mu \le \frac{k^2}{4}$, thus removing the restriction $\mu < C_{\Omega,K}$.

2.1 Heat kernel and boundary Harnack inequality

Let $u \in C^1((0,\infty) : C^2(\Omega))$, setting $u = e^{-\lambda_\mu t}\phi_\mu v$, we can easily see that

$$\frac{u_t + L_\mu u}{\phi_\mu} = v_t - \phi_\mu^{-2} \,\mathrm{div}\left(\phi_\mu^2 \nabla v\right) =: v_t + \mathcal{L}_\mu v. \tag{2.3}$$

[1] Here and below we write $f(x) \asymp g(x)$ in Ω to mean that there exists a constant $c > 1$ such that $c^{-1} f(x) \le g(x) \le c f(x)$ for all $x \in \Omega$.

Hence, instead of studying the properties of the operator L_μ, it is more convenient to study the operator $\frac{\partial}{\partial t} + \mathcal{L}_\mu$. In this direction, we introduce the weighted Sobolev space $H^1(\Omega; \phi_\mu^2)$.

Definition 2.1 Let $D \subset \Omega$ be an open set. We denote by $H^1(D; \phi_\mu^2)$ the weighted Sobolev space

$$H^1(D; \phi_\mu^2) := \{u \in H^1_{loc}(D) : \; |u|\phi_\mu + |\nabla u|\phi_\mu \in L^2(D)\}$$

endowed with the norm

$$\|u\|^2_{H^1(D;\phi_\mu^2)} = \int_D u^2 \phi_\mu^2 dx + \int_D |\nabla u|^2 \phi_\mu^2 dx.$$

We also denote by $H^1_0(D; \phi_\mu^2)$ the closure of $C_c^\infty(D)$ in the norm $\|\cdot\|_{H^1(D;\phi_\mu^2)}$. It is worth mentioning here that $H^1_0(\Omega; \phi_\mu^2) = H^1(\Omega; \phi_\mu^2)$ (see Theorem 4.5).

Next, we normalize ϕ_μ so that $\int_\Omega \phi_\mu^2 dx = 1$. We define the bilinear form $Q : H^1_0(\Omega; \phi_\mu^2) \times H^1_0(\Omega; \phi_\mu^2) \to \mathbb{R}$ by

$$Q(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \phi_\mu^2 dx.$$

The associated operator is the operator $\mathcal{L}_\mu$ defined in (2.3) and generates a contraction semigroup $T(t) : L^2(\Omega; \phi_\mu^2) \to L^2(\Omega; \phi_\mu^2)$, $t \geq 0$, denoted also by $e^{-\mathcal{L}_\mu t}$. This semigroup is positivity preserving and by [17, Lemma 1.3.4] we can easily show that satisfies the conditions of [17, Theorems 1.3.2 and 1.3.3]. Using the logarithmic Sobolev inequality (Theorem 5.1) and some ideas of Davies [16, 17], we shall show that $e^{-\mathcal{L}_\mu t}$ is ultracontractive and therefore has a kernel $k(t, x, y)$. More precisely, we prove the following large time estimates:

Theorem 2.2 *Let $\mu \leq \frac{k^2}{4}$ and $T > 0$. Then there exists $c > 1$ depending only on Ω, K, μ and T such that*

$$c^{-1} \leq k(t, x, y) \leq c$$

for any $t \geq T$ and $x, y \in \Omega$.

For small time the two-sided heat kernel estimate is different. A pivotal ingredient in the proof of this estimate is the boundary Harnack inequality. However, in order to state the boundary Harnack inequality, we first need to give the following definition of weak solution.

Definition 2.3 Let $D \subset \Omega$ be an open set. We say that $v \in C^1((0, T) : H^1(D; \phi_\mu^2))$ is a weak solution of $v_t + \mathcal{L}_\mu v = 0$ in $(0, T) \times D$ if for each $\Phi \in C^1_c((0, T) : C_c^\infty(D))$, we have

$$\int_0^T \int_D (v_t \Phi + \nabla v \cdot \nabla \Phi) \phi_\mu^2 \, dy \, dt = 0.$$

Theorem 2.4 (Boundary Harnack inequality) *Let $\mu \le k^2/4$ and v be a non-negative solution of $v_t + \mathcal{L}_\mu v$ in $(0, r^2) \times \mathcal{B}(x, r) \cap \Omega$. There exist $\beta_1 > 0$ and a positive constant $C = C(\Omega, K, \beta_1, \mu)$ such that for all $r < \beta_1$ there holds*

$$\sup_{(\frac{r^2}{4}, \frac{r^2}{2}) \times \mathcal{B}(x, \frac{r}{2}) \cap \Omega} v \le C \inf_{(\frac{3r^2}{4}, r^2) \times \mathcal{B}(x, \frac{r}{2}) \cap \Omega} v. \tag{2.4}$$

Here $\mathcal{B}(x, r)$ are suitably defined "balls" (see Definition 4.1). Let us briefly explain the proof of the above theorem. We first prove the doubling property for the "balls" $\mathcal{B}(x, r)$ (Lemma 4.2), the Poincaré inequality (Theorem 4.9) and the Moser inequality (Theorem 4.21). The last three results along with the density Theorem 4.5 allow us to apply a Moser iteration argument similar to the one in [34, 50] so that we reach the desired result. Due to the fact that $K \subset \partial\Omega$, the proof of the above theorem is more complicated than the one in [25, 26] and new essential difficulties arise which should be handled in a very delicate way.

Proceeding as in the proof of [50, Theorem 5.4.12], we may deduce that the boundary Harnack inequality (2.4) implies the following sharp two-sided heat kernel estimate for small time.

Theorem 2.5 *Let $\mu \le \frac{k^2}{4}$. There exist $T = T(\Omega, K, \mu) > 0$ and $C = C(\Omega, K, \mu, T) > 1$ such that*

$$\begin{aligned}
&C^{-1}\left((d(x) + \sqrt{t})(d(y) + \sqrt{t})\right)^{-1}\left((d_K(x) + \sqrt{t})(d_K(y) + \sqrt{t})\right)^{-\gamma_+} t^{-\frac{N}{2}} \\
&\quad \times \exp\left(-C\frac{|x-y|^2}{t}\right) \\
&\le k(t, x, y) \\
&\le C\Big((d(x) + \sqrt{t})(d(y) + \sqrt{t})\Big)^{-1}\Big((d_K(x) + \sqrt{t})(d_K(y) + \sqrt{t})\Big)^{-\gamma_+} t^{-\frac{N}{2}} \\
&\quad \times \exp\Big(-C^{-1}\frac{|x-y|^2}{t}\Big),
\end{aligned}$$

for any $0 < t \le T$ and $x, y \in \Omega$.

Let $h(t, x, y)$ denote the Dirichlet heat kernel of L_μ. It is then immediate that $h(t, x, y) = (\phi_\mu(x)\phi_\mu(y))e^{-\lambda_\mu t}k(t, x, y)$. Hence, by Theorems 2.2 and 2.5, we obtain the following theorem.

Theorem 2.6 *Let $\mu \le \frac{k^2}{4}$ and $T > 0$. There exist $C_1 = C_1(\Omega, K, \mu, T, \lambda_\mu) > 1$ and $C_2 = C(\Omega, K, \mu, T) > 1$ such that*

(i)

$$\begin{aligned}
&C_1^{-1}\left(\frac{d(x)}{d(x) + \sqrt{t}}\right)\left(\frac{d(y)}{d(y) + \sqrt{t}}\right)\left(\frac{d_K(x)}{d_K(x) + \sqrt{t}}\right)^{\gamma_+}\left(\frac{d_K(y)}{d_K(y) + \sqrt{t}}\right)^{\gamma_+} t^{-\frac{N}{2}} \\
&\quad \times \exp\left(-C_1\frac{|x-y|^2}{t}\right)
\end{aligned}$$

$$\leq h(t,x,y)$$
$$\leq C_1\left(\frac{d(x)}{d(x)+\sqrt{t}}\right)\left(\frac{d(y)}{d(y)+\sqrt{t}}\right)\left(\frac{d_K(x)}{d_K(x)+\sqrt{t}}\right)^{\gamma_+}\left(\frac{d_K(y)}{d_K(y)+\sqrt{t}}\right)^{\gamma_+} t^{-\frac{N}{2}}$$
$$\times \exp\left(-C_1^{-1}\frac{|x-y|^2}{t}\right),$$

for any $0<t<T$ *and* $x, y \in \Omega$.

(ii)

$$C_2^{-1}\phi_\mu(x)\phi_\mu(y)e^{-\lambda_\mu t} \leq h(t,x,y) \leq C_2\phi_\mu(x)\phi_\mu(y)e^{-\lambda_\mu t},$$

for any $t > T$ *and* $x, y \in \Omega$.

If $\lambda_\mu > 0$, then by the above theorem we can obtain the existence of a minimal Green function $G_\mu(x,y)$ of L_μ as well as precise asymptotic for $G_\mu(x,y)$ (see Sect. 5.2 for more details).

2.2 Martin Kernels and boundary value problems

If $\mu < C_{\Omega,K}$ then the operator $L_\mu = -\Delta - \frac{\mu}{d_K^2}$ is coercive in $H_0^1(\Omega)$. Hence, taking into account the discussion on the first eigenfunction ϕ_μ of (2.1), we may apply Ancona's results in [1] to deduce that any positive solution u of $L_\mu u = 0$ in Ω can be represented like (1.2). If $\mu = C_{\Omega,K} < \frac{k^2}{4}$ then there exists an H_0^1 minimiser of the Hardy quotient and therefore there is no Green function and the operator is not coercive. In the remaining case $\mu = C_{\Omega,K} = \frac{k^2}{4}$, the operator L_μ clearly is not coercive and this case is not covered by Ancona's results in [1]. One of the main goals of this work is to prove that the assumption $\lambda_\mu > 0$ suffices to have a respective representation formula, also in the case $\mu = \frac{k^2}{4}$.

In order to state the main results we first need to give some notations and definitions. For $\beta > 0$ we set

$$K_\beta = \{x \in \mathbb{R}^N \backslash K : \ d_K(x) < \beta\}, \qquad \Omega_\beta = \{x \in \Omega : \ d(x) < \beta\}.$$

We assume that β is small enough so that for any $x \in \Omega_\beta$ there exists a unique $\xi_x \in \partial\Omega$, which satisfies $d(x) = |x - \xi_x|$. Now set

$$\tilde{d}_K(x) = \sqrt{|\mathrm{dist}^{\partial\Omega}(\xi_x, K)|^2 + |x-\xi_x|^2}, \qquad x \in K_\beta, \tag{2.5}$$

where $\mathrm{dist}^{\partial\Omega}(\xi_x, K)$ denotes the distance of ξ_x to K measured on $\partial\Omega$.

Let $\beta_0 > 0$ (this will be determined in Lemma 6.1). We consider a smooth cut-off function $0 \leq \eta_{\beta_0} \leq 1$ with compact support in $K_{\frac{\beta_0}{2}}$ such that $\eta_{\beta_0} = 1$ in $\overline{K}_{\frac{\beta_0}{4}}$. We define

$$W(x)=\begin{cases}(d+\tilde d_K^2)\tilde d_K^{\gamma_-}, & \text{if } \mu<\frac{k^2}{4},\\ (d+\tilde d_K^2)\tilde d_K^{-\frac{k}{2}}(x)|\ln \tilde d_K(x)|, & \text{if } \mu=\frac{k^2}{4},\end{cases}\quad x\in\Omega\cap K_{\beta_0},$$

and

$$\tilde W(x):=(1-\eta_{\beta_0}(x))+\eta_{\beta_0}(x)W(x),\quad x\in\Omega.$$

Let $h\in C(\partial\Omega)$ and $u\in H^1_{loc}(\Omega)\cap C(\Omega)$. We write $\widetilde{\mathrm{tr}}(u)=h$ whenever

$$\lim_{x\in\Omega,\ x\to y\in\partial\Omega}\frac{u(x)}{\tilde W(x)}=h(y)\qquad\text{uniformly for } y\in\partial\Omega. \tag{2.6}$$

In Sect. 6 we prove that for any $h\in C(\partial\Omega)$ the problem

$$\begin{cases}L_\mu v=0, & \text{in }\ \Omega,\\ \widetilde{\mathrm{tr}}(v)=h, & \text{on }\ \partial\Omega,\end{cases}$$

has a unique solution $v=v_h\in H^1_{loc}(\Omega)\cap C(\Omega)$. From this and the accompanying estimate follows that for any $x_0\in\Omega$ the mapping $h\mapsto v_h(x_0)$ is a linear positive functional on $C(\partial\Omega)$. Thus there exists a unique Borel measure on $\partial\Omega$, called L_μ-*harmonic measure* in Ω, denoted by ω^{x_0}, such that

$$v_h(x_0)=\int_{\partial\Omega}h(y)d\omega^{x_0}(y).$$

Thanks to the Harnack inequality the measures ω^x and ω^{x_0}, $x_0,\ x\in\Omega$, are mutually absolutely continuous. Therefore, the Radon–Nikodyn derivative exists and we set

$$K_\mu(x,y):=\frac{dw^x}{dw^{x_0}}(y)\qquad\text{for } \omega^{x_0}\text{- almost all } y\in\partial\Omega.$$

Definition 2.7 Fix $\xi\in\partial\Omega$. A function $\mathcal{K}$ defined in Ω is called a kernel function for L_μ with pole at ξ and basis at $x_0\in\Omega$ if

(i) $\mathcal{K}(\cdot,\xi)$ is L_μ-harmonic in Ω,
(ii) $\frac{\mathcal{K}(\cdot,\xi)}{\tilde W(\cdot)}\in C(\overline{\Omega}\backslash\{\xi\})$ and for any $\eta\in\partial\Omega\backslash\{\xi\}$ we have $\lim_{x\in\Omega,\ x\to\eta}\frac{\mathcal{K}(x,\xi)}{\tilde W(x)}=0$,
(iii) $\mathcal{K}(x,\xi)>0$ for each $x\in\Omega$ and $\mathcal{K}(x_0,\xi)=1$.

Using the ideas in [12], we show the existence and uniqueness of a kernel function with pole at ξ and basis at x_0 (see Proposition 7.3). As a result we obtain the existence of the Martin kernel and moreover

$$K_\mu(x,\xi) = \lim_{y\in\Omega,\ y\to\xi} \frac{G_\mu(x,y)}{G_\mu(x_0,y)}, \quad \forall \xi \in \partial\Omega.$$

In addition, by the estimates on Green function $G_\mu(x,y)$ of L_μ (see Proposition 5.3) we obtain the following result.

Theorem 2.8 *Assume that* $\mu \le \frac{k^2}{4}$ *and* $\lambda_\mu > 0$. *We then have:*
(i) *If* $\mu < \frac{k^2}{4}$ *or* $\mu = \frac{k^2}{4}$ *and* $k < N$ *then*

$$K_\mu(x,\xi) \asymp \frac{d(x)}{|x-\xi|^N} \left(\frac{d_K(x)}{(d_K(x)+|x-\xi|)^2} \right)^{\gamma_+}, \quad \text{in } \Omega\times\partial\Omega. \tag{2.7}$$

(ii) *If* $\mu = \frac{N^2}{4}$ *(so* $k = N$*), then*

$$K_\mu(x,\xi) \asymp \frac{d(x)}{|x-\xi|^N} \left(\frac{|x|}{(|x|+|x-\xi|)^2} \right)^{-\frac{N}{2}} + \frac{d(x)}{|x|^{\frac{N}{2}}} \big| \ln|x-\xi| \big|, \quad \text{in } \Omega\times\partial\Omega. \tag{2.8}$$

When $K = \partial\Omega$, Filippas, Moschini and Tertikas [25] derived sharp two-sided estimate on the associated heat kernel. These estimates where then used in order to obtain sharp estimates on $G_\mu(x,y)$. Chen and Véron [14] studied the operator L_μ with $K = \{0\} \subset \partial\Omega$ and they constructed the corresponding Martin kernel. The case $K \subset \Omega$ was thoroughly studied by Gkikas and Nguyen in [29]. Estimates on the Green kernel of $L_{\mu V} = -\Delta - \mu V$, where V is a singular potential such that $|V(x)| \le c d^{-2}(x)$ in Ω, have been given by Marcus [38, 39]. Marcus and Nguyen [42] used Ancona's result to show that the Martin kernel $K_\mu(x,y)$ is well defined and they applied the results in [39] to the model case L_μ in order to obtain estimates on the Green kernel $G_\mu(x,y)$ and the Martin kernel $K_\mu(x,y)$. However, their results do not cover the critical case $\mu = \frac{k^2}{4}$.

In this work, we follow a different approach which does not use Ancona's result [1] and allows us to study the critical case. In particular our work is inspired by the articles [25, 29, 30]. The main difference here is that $K \subset \partial\Omega$, which has an effect on the value of the optimal Hardy constant $C_{\Omega,K}$ as well as on the behaviour of the eigenfunction ϕ_μ. As a result, this fact yields substantial difficulties and reveals new aspects of the study of L_μ.

We are now ready to state the representation formula.

Theorem 2.9 *Assume that* $\mu \le \frac{k^2}{4}$ *and* $\lambda_\mu > 0$. *Let* u *be a positive* L_μ*-harmonic function in* Ω. *Then* $u \in L^1(\Omega;\phi_\mu)$ *and there exists a unique Radon measure* ν *on* $\partial\Omega$ *such that*

$$u(x) = \int_{\partial\Omega} K_\mu(x,\xi)\,d\nu(\xi) =: \mathbb{K}_\mu[\nu].$$

In order to study the corresponding boundary value problem, we should first introduce the notion of the boundary trace. We will define it in a dynamic way. In this direction, let $\{\Omega_n\}$ be a smooth exhaustion of Ω, that is an increasing sequence of bounded open smooth domains such that $\overline{\Omega_n} \subset \Omega_{n+1}$, $\cup_n \Omega_n = \Omega$ and $\mathcal{H}^{N-1}(\partial\Omega_n) \to \mathcal{H}^{N-1}(\partial\Omega)$. The operator $L_\mu^{\Omega_n}$ defined by

$$L_\mu^{\Omega_n} u = -\Delta u - \frac{\mu}{d_K^2} u$$

is uniformly elliptic and coercive in $H_0^1(\Omega_n)$ and its first eigenvalue $\lambda_\mu^{\Omega_n}$ is larger than λ_μ. For $h \in C(\partial\Omega_n)$ the problem

$$\begin{cases} L_\mu^{\Omega_n} v = 0, & \text{in } \Omega_n \\ v = h, & \text{on } \partial\Omega_n, \end{cases}$$

admits a unique solution which allows to define the $L_\mu^{\Omega_n}$-harmonic measure on $\partial\Omega_n$ by

$$v(x_0) = \int_{\partial\Omega_n} h(y) d\omega_{\Omega_n}^{x_0}(y).$$

Definition 2.10 (L_μ*-boundary trace*) A function $u \in W_{loc}^{1,p}(\Omega)$, $p > 1$, possesses an L_μ*-boundary trace* if there exists a measure $\nu \in \mathfrak{M}(\partial\Omega)$ such that for any smooth exhaustion $\{\Omega_n\}$ of Ω, there holds

$$\lim_{n\to\infty} \int_{\partial\Omega_n} \phi u \, d\omega_{\Omega_n}^{x_0} = \int_{\partial\Omega} \phi \, d\nu, \qquad \forall \phi \in C(\overline{\Omega}).$$

The L_μ-boundary trace of u will be denoted by $\mathrm{tr}_\mu(u)$.

Let $\mathfrak{M}(\partial\Omega)$ denote the space of bounded Borel measures on $\partial\Omega$ and $\mathfrak{M}(\Omega; \phi_\mu)$ the space of Borel measures τ on Ω such that

$$\int_\Omega \phi_\mu d|\tau| < \infty.$$

Arguing as in [45] we obtain in Lemma 8.1 that for any $\nu \in \mathfrak{M}(\partial\Omega)$ we have $\mathrm{tr}_\mu(\mathbb{K}_\mu[\nu]) = \nu$.

Assume now that $\tau \in \mathfrak{M}(\Omega; \phi_\mu)$ and let

$$u = \mathbb{G}_\mu[\tau] := \int_\Omega G_\mu(x, y) d\tau(y).$$

Then $u \in W_{loc}^{1,p}(\Omega)$ for every $1 < p < \frac{N}{N-1}$ and $\mathrm{tr}_\mu(u) = 0$ (see Lemma 8.2).

Next, we give the definition of weak solutions of the following boundary value problem.

Definition 2.11 Let $\tau \in \mathfrak{M}(\Omega; \phi_\mu)$ and $\nu \in \mathfrak{M}(\partial\Omega)$. We say that $u \in L^1(\Omega; \phi_\mu)$ is a weak solution of

$$\begin{cases} L_\mu u = \tau, & \text{in } \Omega, \\ \mathrm{tr}_\mu(u) = \nu, \end{cases} \tag{2.9}$$

if

$$\int_\Omega u\, L_\mu \zeta\, dx = \int_\Omega \zeta\, d\tau + \int_\Omega \mathbb{K}_\mu[\nu] L_\mu \zeta\, dx\;, \qquad \forall \zeta \in \mathbf{X}_\mu(\Omega, K),$$

where

$$\mathbf{X}_\mu(\Omega, K) = \left\{ \zeta \in H^1_{loc}(\Omega) : \phi_\mu^{-1}\zeta \in H^1(\Omega; \phi_\mu^2),\ \phi_\mu^{-1} L_\mu \zeta \in L^\infty(\Omega) \right\}. \tag{2.10}$$

Let us state our main result for problem (2.9).

Theorem 2.12 *Let $\tau \in \mathfrak{M}(\Omega; \phi_\mu)$ and $\nu \in \mathfrak{M}(\partial\Omega)$. There exists a unique weak solution $u \in L^1(\Omega; \phi_\mu)$ of (2.9),*

$$u = \mathbb{G}_\mu[\tau] + \mathbb{K}_\mu[\nu]. \tag{2.11}$$

Furthermore there exists a positive constant $C = C(\Omega, K, \mu)$ such that

$$\|u\|_{L^1(\Omega;\phi_\mu)} \le \frac{1}{\lambda_\mu} \|\tau\|_{\mathfrak{M}(\Omega;\phi_\mu)} + C\|\nu\|_{\mathfrak{M}(\partial\Omega)}. \tag{2.12}$$

If in addition $d\tau = f dx + d\rho$ where $f \in L^1(\Omega; \phi_\mu)$ and $\rho \in \mathfrak{M}(\Omega; \phi_\mu)$, then for any $\zeta \in \mathbf{X}_\mu(\Omega, K)$ with $\zeta \ge 0$, there hold

$$\int_\Omega |u| L_\mu \zeta\, dx \le \int_\Omega \mathrm{sign}(u) f \zeta\, dx + \int_\Omega \zeta\, d|\rho| + \int_\Omega \mathbb{K}_\mu[|\nu|] L_\mu \zeta\, dx, \tag{2.13}$$

$$\int_\Omega u_+ L_\mu \zeta\, dx \le \int_\Omega \mathrm{sign}_+(u) f \zeta\, dx + \int_\Omega \zeta\, d\rho_+ + \int_\Omega \mathbb{K}_\mu[\nu_+] L_\mu \zeta\, dx. \tag{2.14}$$

It is worth mentioning here that Marcus and Nguyen [42] studied problem (2.9) by introducing an alternative normalized boundary trace $\mathrm{tr}^*(u)$ (see [42, Definition 1.2]). However this normalized boundary trace is well defined only if $\mu < \min(C_{\Omega,K}, \frac{2k-1}{4})$. As a consequence they showed that the boundary value problem

$$\begin{cases} L_\mu u = \tau, & \text{in } \Omega, \\ \mathrm{tr}^*(u) = \nu, \end{cases}$$

admits a unique solution provided $\mu < \min(C_{\Omega,K}, \frac{2k-1}{4})$.

3 Hardy–Sobolev type inequalities

In this section we shall prove various Hardy-Sobolev type inequalities that will be essential for our analysis. We start by recalling the following result:

Proposition 3.1 [22, Lemma 2.1] *There exists $\beta_0 = \beta_0(K, \Omega)$ small enough such that, for any $x \in \Omega \cap K_{\beta_0}$, the following estimates hold:*

$$\begin{aligned}
&(a)\ \ \tilde{d}_K^2(x) = d_K^2(x)(1 + g(x))\\
&(b)\ \ \nabla d(x) \cdot \nabla \tilde{d}_K(x) = \frac{d(x)}{\tilde{d}_K(x)}\\
&(c)\ \ |\nabla \tilde{d}_K(x)|^2 = 1 + h(x)\\
&(d)\ \ \tilde{d}_K(x) \Delta \tilde{d}_K(x) = k - 1 + f(x),
\end{aligned}$$

where the functions $g,\ h$ and f satisfy

$$|g(x)| + |h(x)| + |f(x)| \leq C_1(\beta_0, N)\tilde{d}_K(x), \quad \forall x \in \Omega \cap K_{\beta_0}. \tag{3.1}$$

Lemma 3.2 *Assume that $\alpha \neq 0$ and $\gamma + \alpha + k - 1 \neq 0$. There exist $\beta_0 > 0$ and $C = C(\gamma, \alpha, k, \beta_0, N)$ such that for any open $V \subset K_{\beta_0} \cap \Omega$ and for any $u \in C_c^\infty(V)$ there holds*

$$\int_V d^\alpha \tilde{d}_K^{\gamma-1}|u|dx + \int_V d^{\alpha-1}\tilde{d}_K^{\gamma}|u|dx \leq C \int_V d^\alpha \tilde{d}_K^{\gamma}|\nabla u|dx.$$

Proof By Proposition 3.1 we have

$$\begin{aligned}
\gamma \int_V d^\alpha \tilde{d}_K^{\gamma-1}|u|dx + \gamma \int_V d^\alpha \tilde{d}_K^{\gamma-1} h|u|dx &= \int_V d^\alpha \nabla \tilde{d}_K^{\gamma} \cdot \nabla \tilde{d}_K |u|dx\\
= -\alpha \int_V d^{\alpha-1}\tilde{d}_K^{\gamma} \nabla d \cdot \nabla \tilde{d}_K |u|dx &- \int_V d^\alpha \tilde{d}_K^{\gamma} \Delta \tilde{d}_K |u|dx - \int_V d^\alpha \tilde{d}_K^{\gamma} \nabla \tilde{d}_K \cdot \nabla |u|dx\\
= -\alpha \int_V d^\alpha \tilde{d}_K^{\gamma-1}|u|dx &- \int_V d^\alpha \tilde{d}_K^{\gamma-1}(k - 1 + f)|u|dx - \int_V d^\alpha \tilde{d}_K^{\gamma} \nabla \tilde{d}_K \cdot \nabla |u|dx\,,
\end{aligned}$$

that is

$$\begin{aligned}
(\gamma + \alpha + k - 1) \int_V d^\alpha \tilde{d}_K^{\gamma-1}|u|dx = &- \int_V d^\alpha \tilde{d}_K^{\gamma-1}(f + \gamma h)|u|dx\\
&- \int_V d^\alpha \tilde{d}_K^{\gamma} \nabla \tilde{d}_K \cdot \nabla |u|dx\,.
\end{aligned}$$

By the above equality, Proposition 3.1 and (3.1), we can easily prove that

$$\big(|\gamma + \alpha + k - 1| - C(C_1, \gamma)\beta_0\big) \int_V d^\alpha \tilde{d}_K^{\gamma-1}|u|dx \leq (1 + C_1\sqrt{\beta_0}) \int_V d^\alpha \tilde{d}_K^{\gamma}|\nabla u|dx,$$

where $C_1 = C_1(\beta_0, N)$ is the constant in inequality (3.1). Choosing β_0 small enough, we obtain

$$\int_V d^\alpha \tilde{d}_K^{\gamma-1}|u|dx \le C \int_V d^\alpha \tilde{d}_K^\gamma |\nabla u|dx. \tag{3.2}$$

By (3.2) and Proposition 3.1 we have

$$\begin{aligned}\left|\alpha \int_V d^{\alpha-1}\tilde{d}_K^\gamma |u|dx\right| &= \left|\int_V (\nabla d^\alpha \cdot \nabla d)\tilde{d}_K^\gamma |u|dx\right| \\ &\le C\int_V d^\alpha \tilde{d}_K^{\gamma-1}|u|dx + \int_V d^\alpha \tilde{d}_K^\gamma |\nabla u|dx,\end{aligned}$$

provided β_0 is small enough. The result now follows. □

Lemma 3.3 *Assume that $a \neq 0$ and $c + a + k - 1 \neq 0$. Let $1 \le q \le \frac{N}{N-1}$ and $b = a - 1 + N\frac{q-1}{q}$. If β_0 is small enough then there exists $C = C(a, c, k, \beta_0, q, N)$ such that for any open $V \subset \Omega \cap K_{\beta_0}$ and for any $u \in C_c^\infty(V)$ the following inequality is valid*

$$\left(\int_V d^{qb}\tilde{d}_K^{qc}|u|^q dx\right)^{\frac{1}{q}} \le C\int_V d^a \tilde{d}_K^c |\nabla u|dx. \tag{3.3}$$

Proof Let $0 \le \theta_i \le 1$, $i = 1, 2$, be such that $\theta_1 + \theta_2 = 1$ and $\frac{N-1}{N}\theta_1 + \theta_2 = \frac{1}{q}$. By Hölder inequality we have

$$\begin{aligned}\int_V d^{qb}\tilde{d}_K^{qc}|u|^q dx &= \int_V \left(d^{qa\theta_1}\tilde{d}_K^{qc\theta_1}|u|^{\theta_1 q}\right)\left(d^{q(a-1)\theta_2}\tilde{d}_K^{qc\theta_2}|u|^{\theta_2 q}\right)dx \\ &\le \|d^a\tilde{d}_K^c u\|^{\theta_1 q}_{L^{\frac{N}{N-1}}(V)}\|d^{a-1}\tilde{d}_K^c u\|^{\theta_2 q}_{L^1(V)},\end{aligned}$$

and therefore

$$\|d^b\tilde{d}_K^c u\|_{L^q(V)} \le \|d^a\tilde{d}_K^c u\|_{L^{\frac{N}{N-1}}(V)} + \|d^{a-1}\tilde{d}_K^c u\|_{L^1(V)}. \tag{3.4}$$

By the L^1 Sobolev inequality and Lemma 3.2 we have

$$\begin{aligned}\|d^a\tilde{d}_K^c u\|_{L^{\frac{N}{N-1}}(V)} &\le C\left(|c|\int_V d^a\tilde{d}_K^{c-1}|u|dx + |a|\int_V d^{a-1}\tilde{d}_K^c|u|dx + \int_V d^a\tilde{d}_K^c|\nabla u|dx\right) \\ &\le C\int_V d^a\tilde{d}_K^c|\nabla u|dx.\end{aligned}$$

Combining this with Lemma 3.2 and (3.4) concludes the proof. □

Lemma 3.4 *Assume that $a \neq 0$ and $c+a+k-1 \neq 0$. Let $2 < Q \leq \frac{2N}{N-2}$ and $b = a - 1 + N\frac{Q-2}{2Q}$. If β_0 is small enough then there exists $C = C(c, a, k, \beta_0, Q, N)$ such that for any open $V \subset \Omega \cap K_{\beta_0}$ and for any $v \in C_c^\infty(V)$ there holds*

$$\left(\int_V (d^b \tilde{d}_K^c)^{\frac{2Q}{Q+2}} |v|^Q dx\right)^{\frac{2}{Q}} \leq C \int_V d^{2a - \frac{2Qb}{Q+2}} \tilde{d}_K^{\frac{4c}{Q+2}} |\nabla v|^2 dx.$$

Proof Let $s = \frac{Q}{2} + 1$ and write $Q = qs$. Applying (3.3) to the function $u = |v|^s$ we obtain

$$\left(\int_V \left(d^b \tilde{d}_K^c\right)^{\frac{2Q}{Q+2}} |v|^Q dx\right)^{\frac{Q+2}{2Q}} \leq C \int_V d^a \tilde{d}_K^c |v|^{\frac{Q}{2}} |\nabla v| dx. \tag{3.5}$$

Now, by Schwarz inequality, we have

$$\begin{aligned}\int_V d^a \tilde{d}_K^c |v|^{\frac{Q}{2}} |\nabla v| dx &= \int_V d^{b\frac{Q}{Q+2}} \tilde{d}_K^{c\frac{Q}{Q+2}} |v|^{\frac{Q}{2}} d^{a-b\frac{Q}{Q+2}} \tilde{d}_K^{c(1-\frac{Q}{Q+2})} |\nabla v| dx \\ &\leq \left(\int_V \left(d^b \tilde{d}_K^c\right)^{\frac{2Q}{Q+2}} |v|^Q dx\right)^{\frac{1}{2}} \left(\int_V d^{2a-\frac{2Qb}{Q+2}} \tilde{d}_K^{c(2-\frac{2Q}{Q+2})} |\nabla v|^2 dx\right)^{\frac{1}{2}}.\end{aligned}$$

The result follows by (3.5) and the last inequality. □

Corollary 3.5 *Let $\alpha \neq 0$ and assume that $(\alpha+\gamma)\frac{N-1}{N-2} + k - 1 \neq 0$. There exist β_0 small enough and $C > 0$ such that for any open $V \subset \Omega \cap K_{\beta_0}$ and for all $u \in C_c^\infty(V)$ there holds*

$$\left(\int_V \left(d^{\frac{\alpha}{2}} \tilde{d}_K^{\frac{\gamma}{2}} |u|\right)^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}} \leq C \int_V d^\alpha \tilde{d}_K^\gamma |\nabla u|^2 dx.$$

Proof We apply Lemma 3.4 with $Q = \frac{2N}{N-2}$, $a = \alpha\frac{N-1}{N-2}$, $c = \gamma(\frac{N-1}{N-2})$. □

Corollary 3.6 *Let $\alpha > 0$ and $\gamma \geq 0$. There exist $\beta_0 > 0$ and $C > 0$ such that for any open $V \subset \Omega \cap K_{\beta_0}$ and all $u \in C_c^\infty(V)$, the following inequality is valid*

$$\left(\int_V d^\alpha \tilde{d}_K^\gamma |u|^{\frac{2(N+\alpha+\gamma)}{N+\alpha+\gamma-2}} dx\right)^{\frac{N+\alpha+\gamma-2}{N+\alpha+\gamma}} \leq C \int_V d^{\alpha+\frac{2\gamma}{N+\alpha+\gamma}} \tilde{d}_K^{\gamma-\frac{2\gamma}{N+\alpha+\gamma}} |\nabla u|^2 dx.$$

Proof This follows by Lemma 3.4 with $Q = \frac{2(N+\alpha+\gamma)}{N+\alpha+\gamma-2}$, $c = \frac{\gamma}{q}$, $b = \frac{\alpha}{q}$, where $q = \frac{2Q}{Q+2}$. □

Corollary 3.7 *Let $\alpha > 0$, $\gamma < 0$ and assume that $\alpha + \gamma\frac{N+\alpha-1}{N+\alpha} + k - 1 \neq 0$. There exist $\beta_0 > 0$ and $C > 0$ such that for any open $V \subset \Omega \cap K_{\beta_0}$ and all $u \in C_c^\infty(V)$ there holds*

$$\left(\int_V d^\alpha \tilde{d}_K^\gamma |u|^{\frac{2(N+\alpha)}{N+\alpha-2}} dx\right)^{\frac{N+\alpha-2}{N+\alpha}} \leq C \int_V d^\alpha \tilde{d}_K^{\gamma\frac{N+\alpha-2}{N+\alpha}} |\nabla u|^2 dx\,.$$

Proof The proof follows from Lemma 3.4, with $Q = \frac{2(N+\alpha)}{N+\alpha-2}$, $c = \gamma\frac{N+\alpha-1}{N+\alpha}$ and $b = \alpha\frac{N+\alpha-1}{N+\alpha}$. □

4 Heat Kernel estimates for small time

We are now going to introduce some notation and tools that will be useful for our local analysis near K and $\partial\Omega$; see e.g. [36].

Let $x = (x', x'') \in \mathbb{R}^N$, $x' = (x_1, \ldots, x_k) \in \mathbb{R}^k$, $x'' = (x_{k+1}, \ldots, x_N) \in \mathbb{R}^{N-k}$. For $\beta > 0$, we denote by $B_\beta^k(x')$ the ball in $\mathbb{R}^k$ with center x' and radius β. For any $\xi \in K$ we also set

$$V_K(\xi, \beta) = \left\{x = (x', x'') : |x'' - \xi''| < \beta,\ |x_i - \Gamma^\xi_{i,K}(x'')| < \beta,\ \forall i = 1, \ldots, k\right\},$$

for some functions $\Gamma^\xi_{i,K} : \mathbb{R}^{N-k} \to \mathbb{R}$, $i = 1, \ldots, k$.

Since K is a C^2 compact submanifold in $\mathbb{R}^N$ without boundary, there exists $\beta_0 > 0$ such that

- For any $x \in K_{6\beta_0}$, there is a unique $\xi \in K$ satisfying $|x - \xi| = d_K(x)$.
- $d_K \in C^2(K_{4\beta_0})$, $|\nabla d_K| = 1$ in $K_{4\beta_0}$ and there exists $g \in L^\infty(K_{4\beta_0})$ such that

$$\Delta d_K(x) = \frac{k-1}{d_K(x)} + g(x), \qquad \text{in } K_{4\beta_0}.$$

  (See [52, Lemma 2.2] and [21, Lemma 6.2].)
- For any $\xi \in K$, there exist C^2 functions $\Gamma^\xi_{i,K} \in C^2(\mathbb{R}^{N-k}; \mathbb{R})$, $i = 1, \ldots, k$, such that defining

$$V_K(\xi, \beta) := \left\{x = (x', x'') : |x'' - \xi''| < \beta,\ |x_i - \Gamma^\xi_{i,K}(x'')| < \beta,\ i = 1, \ldots, k\right\},$$

  we have (upon relabelling and reorienting the coordinate axes if necessary)

$$V_K(\xi, \beta) \cap K = \left\{x = (x', x'') : |x'' - \xi''| < \beta,\ x_i = \Gamma^\xi_{i,K}(x''),\ i = 1, \ldots, k\right\}.$$

- There exist ξ^j, $j = 1, \ldots, m_0$, ($m_0 \in \mathbb{N}$) and $\beta_1 \in (0, \beta_0)$ such that

$$K_{2\beta_1} \subset \bigcup_{i=1}^{m_0} V_K(\xi^i, \beta_0). \tag{4.1}$$

Now set

$$\delta_K^{\xi}(x) := \left(\sum_{i=1}^{k} |x_i - \Gamma_{i,K}^{\xi}(x'')|^2\right)^{\frac{1}{2}}, \qquad x = (x', x'') \in V_K(\xi, 4\beta_0).$$

Then there exists a constant $C = C(N, K)$ such that

$$d_K(x) \le \delta_K^{\xi}(x) \le C\|K\|_{C^2} d_K(x), \quad \forall x \in V_K(\xi, 2\beta_0), \tag{4.2}$$

where $\xi^j = ((\xi^j)', (\xi^j)'') \in K$, $j = 1, \ldots, m_0$, are the points in (4.1) and

$$\|K\|_{C^2} := \sup\left\{\left\|\Gamma_{i,K}^{\xi^j}\right\|_{C^2(B_{5\beta_0}^{N-k}((\xi^j)''))} : \ i = 1, \ldots, k, \ j = 1, \ldots, m_0\right\} < \infty.$$

For simplicity we shall write δ_K instead of δ_K^{ξ}. Moreover, β_1 can be chosen small enough so that for any $x \in K_{\beta_1}$,

$$B(x, \beta_1) \subset V_K(\xi, \beta_0),$$

where $\xi \in K$ satisfies $|x - \xi| = d_K(x)$.

When $K = \partial\Omega$ we assume that

$$V_{\partial\Omega}(\xi, \beta) \cap \Omega = \left\{x : \sum_{i=2}^{N} |x_i - \xi_i|^2 < \beta^2, \ 0 < x_1 - \Gamma_{1,\partial\Omega}^{\xi}(x_2, \ldots, x_N) < \beta\right\}.$$

Thus, when $x \in K \subset \partial\Omega$ is a C^2 compact submanifold in $\mathbb{R}^N$ without boundary, of co-dimension k, $1 < k \le N$, we have that

$$\Gamma_{1,K}^{\xi}(x'') = \Gamma_{1,\partial\Omega}^{\xi}(\Gamma_{2,K}^{\xi}(x''), \ldots, \Gamma_{k,K}^{\xi}(x''), x''). \tag{4.3}$$

Let $\xi \in K$. For any $x \in V_K(\xi, \beta_0) \cap \Omega$, we define

$$\delta(x) = x_1 - \Gamma_{1,\partial\Omega}^{\xi}(x_2, \ldots, x_N),$$

and

$$\delta_{2,K}(x) = \left(\sum_{i=2}^{k} |x_i - \Gamma_{i,K}^{\xi}(x'')|^2\right)^{\frac{1}{2}}.$$

Then by (4.3), there exists a constant $A > 1$ which depends only on Ω, K and β_0 such that

$$\frac{1}{A}(\delta_{2,K}(x) + \delta(x)) \le \delta_K(x) \le A(\delta_{2,K}(x) + \delta(x)), \tag{4.4}$$

Thus by (4.2) and (4.4) there exists a constant $C = C(\Omega, K, \gamma) > 1$ which depends on $k, N, \Gamma^{\xi}_{i,K}, \Gamma^{\xi}_{1,\partial\Omega}, \gamma$ such that

$$C^{-1}\delta^2(x)(\delta_{2,K}(x) + \delta(x))^{\gamma} \leq d^2(x)d_K^{\gamma}(x) \leq C\delta^2(x)(\delta_{2,K}(x) + \delta(x))^{\gamma}. \quad (4.5)$$

We set

$$\mathcal{V}_K(\xi, \beta_0) = \left\{(x', x'') : |x'' - \xi''| < \beta_0,\ |\delta(x)| < \beta_0,\ |\delta_{2,K}(x)| < \beta_0\right\}.$$

We may then assume that

$$\mathcal{V}_K(\xi, \beta_0) \cap \Omega = \left\{(x', x'') : |x'' - \xi''| < \beta_0, 0 < \delta(x) < \beta_0,\ |\delta_{2,K}(x)| < \beta_0\right\},$$
$$\mathcal{V}_K(\xi, \beta_0) \cap \partial\Omega = \left\{(x', x'') : |x'' - \xi''| < \beta_0, \delta(x) = 0,\ |\delta_{2,K}(x)| < \beta_0\right\},$$

and

$$\mathcal{V}_K(\xi, \beta_0) \cap K = \left\{(x', x'') : |x'' - \xi''| < \beta_0, \delta(x) = 0,\ \delta_{2,K} = 0\right\}.$$

Let $\beta_1 > 0$, $1 < b < 2$, and $0 < r < \beta_1$. For any $x \in V_{\partial\Omega}(\xi, \frac{\beta_0}{16})$ with $d(x) \leq br$, taking β_1 small enough we have

$$\mathcal{D}(x, r) := \left\{y : \sum_{i=2}^{N} |y_i - x_i|^2 < r^2,\ |\delta(y)| < r + d(x)\right\} \subset\subset V_{\partial\Omega}\left(\xi, \frac{\beta_0}{16}\right).$$

In addition there exists $C_\xi = C(\Gamma^\xi, \Omega) > 1$, such that

$$\mathcal{D}(x, r) \subset B(x, C_\xi r). \quad (4.6)$$

Also,

$$\mathcal{D}(x, r) \cap \Omega = \left\{y : \sum_{i=2}^{N} |y_i - x_i|^2 < r^2,\ 0 < \delta(y) < r + d(x)\right\}.$$

Definition 4.1 Let $\beta_1 > 0$ be small enough, $r \in (0, \beta_1)$, $b \in (1, 2)$, $\xi \in K$ and $x \in V(\xi, \frac{\beta_0}{16})$. We define

(i) $\mathcal{B}(x, r) = B(x, r)$, if $d(x) > br$
(ii) $\mathcal{B}(x, r) = \mathcal{D}(x, r)$, if $d(x) \leq br$ and $d_K(x) > bC_\xi r$
(iii) $\mathcal{B}(x, r) = \{y = (y', y'') : |y'' - x''| < r,\ |\delta_{2,K}(y)| < r + d_K(x),\ |\delta(y)| < r + d(x)\}$, if $d(x) \leq br$ and $d_K(x) \leq bC_\xi r$.

Finally we set

$$\overline{\mathcal{M}}_\gamma(x, r) = \int_{\mathcal{B}(x,r)\cap\Omega} d^2(y)d_K^{\gamma}(y)dy.$$

4.1 Doubling property

Lemma 4.2 *Let $\gamma \geq -k$. Let $\xi \in \partial\Omega$ and $x \in V(\xi, \frac{\beta_0}{16})$. Then, there exist $\beta_1 > 0$ and $C = C(\Omega, K, \gamma, \beta_0) > 1$ such that*

$$\frac{1}{C}(r + d(x))^2(r + d_K(x))^\gamma r^N \leq \overline{\mathcal{M}}_\gamma(x, r) \leq C(r + d(x))^2(r + d_K(x))^\gamma r^N, \tag{4.7}$$

for any $0 < r < \beta_1$.

Proof We will consider three cases.

Case 1. $d(x) > br$ Since $d_K(x) \geq d(x)$, we can easily show that for any $y \in B(x, r)$ we have $\frac{b-1}{b}d(x) \leq d(y) \leq \frac{b+1}{b}d(x)$ and $\frac{b-1}{b}d_K(x) \leq d_K(y) \leq \frac{b+1}{b}d_K(x)$. Thus the proof of (4.7) follows easily in this case.

Case 2. $d(x) \leq br$ **and** $d_K(x) > bC_\xi r$. By (4.6), we again have that $\frac{b-1}{b}d_K(x) \leq d_K(y) \leq \frac{b+1}{b}d_K(x)$. Using the last inequality and proceeding as the proof of [25, Lemma 2.2], we obtain the desired result.

Case 3. $d(x) \leq br$ **and** $d_K(x) \leq bC_\xi r$.

Let $\overline{y} = (y_2, \dots, y_k) \in \mathbb{R}^{k-1}$. By (4.5) and the definition of $\mathcal{B}(x, r)$, we have

$$\begin{aligned}
\overline{\mathcal{M}}_\gamma(x, r) &= \int_{\mathcal{B}(x,r)\cap\Omega} d^2(y) d_K^\gamma(y) dy \leq \int_{\mathcal{B}(x,r)\cap\Omega} C\delta^2(y)(\delta_{2,K}(y) + \delta(y))^\gamma dy \\
&\leq C \int_{B^{N-k}(x'',r)} \int_0^{d(x)+r} \int_{|\overline{y}|<d_K(x)+r} (|\overline{y}| + y_1)^\gamma y_1^2 d\overline{y}\, dy_1\, dy'' \\
&= CC(k, N) r^{N-k} \int_0^{d(x)+r} \int_0^{d_K(x)+r} s^{k-2}(s + y_1)^\gamma y_1^2 ds\, dy_1.
\end{aligned} \tag{4.8}$$

Now, if $\gamma > 0$ then

$$\begin{aligned}
&\int_0^{d(x)+r} \int_0^{d_K(x)+r} s^{k-2}(s + y_1)^\gamma y_1^2 ds\, dy_1 \\
&\leq \frac{1}{k-1}(2r + d(x) + d_K(x))^\gamma (d_K(x) + r)^{k-1}(d(x) + r)^3 \\
&\leq \frac{(b+2)^\gamma (bC_\xi + 1)^{k-1}(b+1)}{k-1}(r + d_K(x))^\gamma (d(x) + r)^2 r^k.
\end{aligned}$$

If $-k \leq \gamma \leq 0$, then

$$\begin{aligned}
&\int_0^{d(x)+r} \int_0^{d_K(x)+r} s^{k-2}(s + y_1)^\gamma y_1^2 ds\, dy_1 \\
&\leq \int_0^{d(x)+r} \int_0^{d_K(x)+r} s^{k-2}(s + y_1)^{\gamma+2} ds\, dy_1
\end{aligned}$$

$$\leq \int_0^{d(x)+r} \int_0^{d_K(x)+r} (s+y_1)^{\gamma+k} ds\, dy_1$$
$$\leq (d_K(x)+r)(d(x)+r)(2r+d(x)+d_K(x))^{\gamma+k}$$
$$\leq (2C_\xi+2)(d(x)+r)^2(d_K(x)+r)^\gamma(2r+d(x)+d_K(x))^k$$
$$\leq (2C_\xi+2)(2+b+bC_\xi)^k(d(x)+r)^2(d_K(x)+r)^\gamma r^k.$$

Similarly, for the reverse inequality, we have

$$\int_0^{d(x)+r} \int_0^{d_K(x)+r} s^{k-2}(s+y_1)^\gamma y_1^2 ds\, dy_1$$
$$\geq \int_{\frac{d(x)+r}{2}}^{d(x)+r} \int_{\frac{d_K(x)+r}{2}}^{d_K(x)+r} s^{k-2}(s+y_1)^\gamma y_1^2 ds dy_1$$
$$\geq C(b, C_\xi, k, \gamma)(d(x)+r)^2(d_K(x)+r)^\gamma r^k. \tag{4.9}$$

The desired result follows by (4.8)–(4.9). □

From (2.2) and Lemma 4.2, we have the following corollary.

Corollary 4.3 *Let* $x \in V(\xi, \frac{\beta_0}{16})$ *and*

$$\mathcal{M}(x,r) = \int_{\mathcal{B}(x,r)\cap\Omega} \phi_\mu^2(y) dy.$$

Then, there exist $\beta_1 > 0$ *and* $C = C(\Omega, K, \beta_0) > 1$ *such that*

$$\frac{1}{C}(r+d(x))^2(r+d_K(x))^{2\gamma_+}r^N \leq \mathcal{M}(x,r) \leq C(r+d(x))^2(r+d_K(x))^{2\gamma_+}r^N,$$

for any $0 < r < \beta_1$.

We point out that by (2.2) we have

$$\mathcal{M}(x,r) \asymp \overline{\mathcal{M}}_{2\gamma_+}(x,r)\ , \qquad \text{in } \Omega \times (0, \beta_1).$$

4.2 Density of $C_c^\infty(\Omega)$ functions

Lemma 4.4 *Let* $k \leq N$, $\gamma \geq -k$, $x = (x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_N) = (x_1, \overline{x}, x'')$. *Let*

$$O = (0,1) \times B^{\mathbb{R}^{k-1}}(0,1) \times B^{\mathbb{R}^{N-k}}(0,1)$$

and $u \in H^1(O; x_1^2(x_1 + |\overline{x}|)^\gamma)$. *Assume that there exists* $0 < \varepsilon_0 < 1$ *such that* $u(x) = 0$ *if either* $x_1 > \varepsilon_0$ *or* $|\overline{x}|^2 + |x''|^2 > \varepsilon_0^2$. *Then there exists a sequence* $\{u_n\}_{n=1}^\infty \subset C_c^\infty(O)$ *such that*

$$u_n \to u, \quad \textit{in } H^1(O; x_1^2(x_1 + |\overline{x}|)^\gamma)$$

Proof Let $m \in \mathbb{N}$. Set

$$v_m(x) = \begin{cases} m, & \text{if } u(x) > m, \\ u(x), & \text{if } -m \le u(x) \le m, \\ -m & \text{if } u(x) < -m. \end{cases}$$

Then we can easily prove that $v_m \to u$ in $H^1(O; x_1^2(x_1 + |\overline{x}|)^\gamma)$.

Let $\varepsilon > 0$. There exists $m_0 \in \mathbb{N}$, such that

$$\begin{aligned} \left\| v_{m_0} - u \right\|_{H^1(O; x_1^2(x_1+|\overline{x}|)^\gamma)} &= \left(\int_O x_1^2(x_1 + |\overline{x}|)^\gamma (|v_{m_0} - u|^2 + |\nabla v_{m_0} - \nabla u|^2) dx \right)^{\frac{1}{2}} \\ &< \frac{\varepsilon}{3}. \end{aligned} \tag{4.10}$$

For any $0 < h < 1$, we consider the function

$$\eta_h(x_1) = \begin{cases} 1 & \text{if } x_1 > h, \\ 1 - (\ln h)^{-1} \ln \left(\frac{x_1}{h} \right) & \text{if } h^2 \le x_1 \le h, \\ 0 & \text{if } x_1 < h^2, \end{cases}$$

We will show that $z_h := \eta_h v_{m_0} \to v_{m_0}$ in $H^1(O; x_1^2(x_1 + |\overline{x}|)^\gamma)$, as $h \to 0^+$. We can easily show that $z_h \to v_{m_0}$ in $L^2(O; x_1^2(x_1 + |\overline{x}|)^\gamma)$. Also,

$$\begin{aligned} \int_O x_1^2(x_1 + |\overline{x}|)^\gamma |\nabla(v_{m_0}(1 - \eta_h))|^2 dx \le\ & 2 \int_O x_1^2(x_1 + |\overline{x}|)^\gamma |\nabla v_{m_0}|^2 |(1 - \eta_h)|^2 dx \\ & + 2 \int_O x_1^2(x_1 + |\overline{x}|)^\gamma |v_{m_0}|^2 |\nabla \eta_h|^2 dx \\ \le\ & 2 \int_O x_1^2(x_1 + |\overline{x}|)^\gamma |\nabla v_{m_0}|^2 |(1 - \eta_h)|^2 dx \\ & + C(N, k) m_0^2 (\ln h)^{-2} \\ & \times \int_{h^2}^h \int_0^1 (x_1 + r)^\gamma r^{k-2} dr dx_1 \to 0, \end{aligned}$$

since $\gamma \ge -k$. Thus there exists $h_0 \in (0, 1)$ such that

$$\left\| v_{m_0} - z_{h_0} \right\|_{H^1(O; x_1^2(x_1+|\overline{x}|)^\gamma)} < \frac{\varepsilon}{3}. \tag{4.11}$$

Note that z_{h_0} vanishes outside $\tilde{O}_\sigma = (\sigma, 1) \times B^{\mathbb{R}^{k-1}}(0, 1) \times B^{\mathbb{R}^{N-k}}(0, 1)$, for some $\sigma = \sigma(h_0) \in (0, 1)$. Thus $z_{h_0} \in H_0^1(\tilde{O}_\sigma)$, which implies the existence of a sequence $\{u_n\} \subset C_c^\infty(\tilde{O}_\sigma)$ such that $u_n \to z_{h_0}$ in $H_0^1(\tilde{O}_\sigma)$. Hence, there exists $n_0 \in \mathbb{N}$ such that

$$\left\| z_{h_0} - u_n \right\|_{H^1(O; x_1^2(x_1+|\overline{x}|)^\gamma)} < \frac{\varepsilon}{3}, \quad \forall n \ge n_0. \tag{4.12}$$

The desired result follows by (4.10), (4.11) and (4.12). □

We write a point $x \in \mathbb{R}^N$ as $x = (x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_N) = (x_1, \overline{x}, x'')$. Given $r_1, r_2, r_3 > 0$ we denote

$$O_{r_1,r_2,r_3} = (0, r_1) \times B^{\mathbb{R}^{k-1}}(0, r_2) \times B^{\mathbb{R}^{N-k}}(0, r_3).$$

Theorem 4.5 *Assume that $\gamma \geq -k$. Then $C_c^\infty(\Omega)$ is dense in $H^1(\Omega; d^2 d_K^\gamma)$.*

Proof Let $u \in H^1(\Omega; d^2 d_K^\gamma)$ and $\beta_0 > 0$ be the constant in Lemma 4.2. Let $\xi \in K$ and $0 \leq \phi_\xi \leq 1$ be a smooth function with $\operatorname{supp}(\phi_\xi) \subset \mathcal{V}_K(\xi, \frac{\beta_0}{8})$, and $\phi = 1$ in $\mathcal{V}_K(\xi, \frac{\beta_0}{16})$. Then the function $v = u\phi_\xi$ belongs in $H^1(\Omega; d^2 d_K^\gamma)$.

By (4.5) we have

$$\begin{aligned}
&\int_\Omega d^2(x) d_K^\gamma(x)(|v|^2 + |\nabla v|^2)dx \\
&\asymp C(\Omega, K) \int_{\mathcal{V}_K(\xi, \frac{\beta_0}{8})} \delta^2(x)(\delta_{2,K}(x) + \delta(x))^\gamma (|v|^2 + |\nabla v|^2)dx \\
&\asymp C(\Omega, K) \int_{O_{1, \frac{\beta_0}{8}, \frac{\beta_0}{8}}} y_1^2 (y_1 + |\overline{y}|)^\gamma (|\overline{v}|^2 + |\nabla_y \overline{v}|^2)dy,
\end{aligned}$$

where $\overline{y} = (y_2, \dots, y_k)$ and

$$\begin{aligned}
\overline{v}(y) = v\Big(&y_1 + \Gamma^\xi_{1,\partial\Omega}\Big(y_2 + \Gamma^\xi_{2,K}(y''), \dots, y_k + \Gamma^\xi_{k,K}(y''), y''\Big), y_2 \\
&+ \Gamma^\xi_{2,K}(y''), \dots, y_k + \Gamma^\xi_{k,K}(y''), y''\Big).
\end{aligned}$$

The desired result follows by Lemma 4.4 and a partition of unity argument. □

By Corollaries 3.6 and 3.7, Theorem 4.5 and using a partition of unity argument, we obtain the following two results.

Corollary 4.6 *Let $\gamma \geq 0$. There exists $C = C(\Omega, K, \gamma)$ such that*

$$\left(\int_\Omega d^2 d_K^\gamma |u|^{\frac{2(N+2+\gamma)}{N+\gamma}} dx\right)^{\frac{N+\gamma}{N+2+\gamma}} \leq C\left(\int_\Omega d^2 d_K^\gamma |\nabla u|^2 dx + \int_\Omega d^2 d_K^\gamma u^2 dx\right),$$

for any $u \in H^1(\Omega; d^2 d_K^\gamma)$.

Corollary 4.7 *Let $-k \leq \gamma < 0$. There exists $C = C(\Omega, K, \gamma)$ such that*

$$\left(\int_\Omega d^2 d_K^\gamma |u|^{\frac{2(N+2)}{N}} dx\right)^{\frac{N}{N+2}} \leq C\left(\int_\Omega d^2 d_K^{\gamma\frac{N}{N+2}} |\nabla u|^2 dx + \int_\Omega d^2 d_K^\gamma u^2 dx\right),$$

for any $u \in H^1(\Omega; d^2 d_K^\gamma)$.

4.3 Poincaré inequality

Lemma 4.8 *Let* $1 \leq k \leq N$ *and* $\gamma \geq -k$*. Assume that* $0 < c_0 r_2 < r_3 < r_1 < r_2$*, for some constant* $0 < c_0 < 1$*. Then there exists a positive constant* $C = C(c_0, N, K, \gamma)$ *such that*

$$\inf_{\zeta \in \mathbb{R}} \int_{O_{r_1,r_2,r_3}} |f(x) - \zeta|^2 x_1^2 (x_1 + |\overline{x}|)^\gamma dx \leq C r_2^2 \int_{O_{r_1,r_2,r_3}} |\nabla f(x)|^2 x_1^2 (x_1 + |\overline{x}|)^\gamma dx,$$

for any $f \in C^1(\overline{O}_{r_1,r_2,r_3})$*.*

Proof Let $\zeta \in \mathbb{R}$ and $y_1 = \frac{x_1}{2r_1}$, $\overline{y} = \frac{\overline{x}}{2r_2}$ and $y'' = \frac{x''}{2r_3}$. Set $\overline{f}(y) = f(2r_1 y_1, 2r_2 \overline{y}, 2r_3 y'')$. Then

$$\begin{aligned} &\int_{O_{r_1,r_2,r_3}} |f(x) - \zeta|^2 x_1^2 (x_1 + |\overline{x}|)^\gamma dx \\ &\quad \asymp C(c_0, N, k, \gamma) r_2^{N+\gamma+2} \int_{O_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}} |\overline{f}(y) - \zeta|^2 y_1^2 (y_1 + |\overline{y}|)^\gamma dy. \end{aligned} \tag{4.13}$$

Let

$$\zeta_{\overline{f}} = \left(\int_{O_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}} y_1^2 (y_1 + |\overline{y}|)^\gamma dy \right)^{-1} \int_{O_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}} \overline{f}(y) y_1^2 (y_1 + |\overline{y}|)^\gamma dy.$$

We assert that there exists a positive constant $C > 0$ such that

$$\int_{O_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}} |\overline{f}(y) - \zeta_{\overline{f}}|^2 y_1^2 (y_1 + |\overline{y}|)^\gamma dy \leq C \int_{O_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}} |\nabla \overline{f}(y)|^2 y_1^2 (y_1 + |\overline{y}|)^\gamma dy, \tag{4.14}$$

for any $\overline{f} \in C^1(\overline{O}_{\frac{1}{2},\frac{1}{2},\frac{1}{2}})$.

We will prove this by contradiction. Let $\{\overline{f}_n\} \subset C^1(\overline{O}_{\frac{1}{2},\frac{1}{2},\frac{1}{2}})$ be a sequence such that

$$\int_{O_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}} |\overline{f}_n(y) - \zeta_{\overline{f}_n}|^2 y_1^2 (y_1 + |\overline{y}|)^\gamma dy > n \int_{O_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}} |\nabla \overline{f}_n(y)|^2 y_1^2 (y_1 + |\overline{y}|)^\gamma dy. \tag{4.15}$$

Setting

$$g_n(y) = (\overline{f}_n(y) - \zeta_{\overline{f}_n}) \left(\int_{O_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}} |\overline{f}_n(y) - \zeta_{\overline{f}_n}|^2 y_1^2 (y_1 + |\overline{y}|)^\gamma dy \right)^{-1},$$

(4.15) becomes

$$1=\int_{O_{\frac12,\frac12,\frac12}}|g_n(y)|^2y_1^2(y_1+|\overline{y}|)^\gamma\,dy>n\int_{O_{\frac12,\frac12,\frac12}}|\nabla g_n(y)|^2y_1^2(y_1+|\overline{y}|)^\gamma\,dy$$

and we also have $\zeta_{g_n}=0$.

Let $\varepsilon>0$. There exists an extension $\overline{g}_n$ of g_n such that $\overline{g}_n=g_n$ in $\overline{O}_{\frac12,\frac12,\frac12}$, $\overline{g}_n\in C^1(\overline{O}_{1,1,1})$, $\overline{g}_n=0$ if $y_1>\frac23$ or $|\overline{y}|>\frac23$ or $|y''|>\frac23$ and there exists a positive constant $C_1=C_1(N,k,q)$ such that

$$\int_{O_{1,1,1}}|\overline{g}_n(y)|^qy_1^2(y_1+|\overline{y}|)^\gamma\,dy\le C_1\int_{O_{\frac12,\frac12,\frac12}}|g_n(y)|^qy_1^2(y_1+|\overline{y}|)^\gamma\,dy$$

$$\int_{O_{1,1,1}}|\nabla\overline{g}_n(y)|^qy_1^2(y_1+|\overline{y}|)^\gamma\,dy\le C_1\left(\int_{O_{\frac12,\frac12,\frac12}}|\nabla g_n(y)|^qy_1^2(y_1+|\overline{y}|)^\gamma\,dy\right.$$
$$\left.+\int_{O_{\frac12,\frac12,\frac12}}|g_n(y)|^qy_1^2(y_1+|\overline{y}|)^\gamma\,dy\right),$$

for any $q>1$. Assume first that $-k\le\gamma<0$. Given $\sigma\in(0,1/2)$, by Corollary 3.7 we have that for some $C=C(\gamma,N,k)$,

$$\begin{aligned}
&\int_{O_{\sigma,\frac12,\frac12}}|g_n(y)|^2y_1^2(y_1+|\overline{y}|)^\gamma\,dy\\
&\le C\sigma^{\frac{6}{N+2}}\left(\int_{O_{\frac12,\frac12,\frac12}}|\overline{g}_n(y)|^{\frac{2(N+2)}{N}}y_1^2(y_1+|\overline{y}|)^\gamma\,dy\right)^{\frac{N}{N+2}}\\
&\le C\sigma^{\frac{6}{N+2}}\left(\int_{O_{1,1,1}}|\overline{g}_n(y)|^{\frac{2(N+2)}{N}}y_1^2(y_1+|\overline{y}|)^\gamma\,dy\right)^{\frac{N}{N+2}}\\
&\le C\sigma^{\frac{6}{N+2}}\int_{O_{1,1,1}}|\nabla\overline{g}_n(y)|^2y_1^2(y_1+|\overline{y}|)^\gamma\,dy\\
&\le C\sigma^{\frac{6}{N+2}}\left(\int_{O_{\frac12,\frac12,\frac12}}|\nabla g_n(y)|^2y_1^2(y_1+|\overline{y}|)^\gamma\,dy+\int_{O_{\frac12,\frac12,\frac12}}|g_n(y)|^2y_1^2(y_1+|\overline{y}|)^\gamma\,dy\right)\\
&\le C\sigma^{\frac{6}{N+2}}\left(1+\frac1n\right).
\end{aligned}\tag{4.16}$$

Similarly in case $\gamma\ge0$, by Corollary 3.6 we can show that

$$\int_{O_{\sigma,\frac12,\frac12}}|g_n(y)|^2y_1^2(y_1+|\overline{y}|)^\gamma\,dy\le C(\gamma,N,k)\sigma^{\frac{2(3+\gamma)}{N+2+\gamma}}\left(1+\frac1n\right).\tag{4.17}$$

Since (g_n) is bounded in $H^1((\sigma,\frac{1}{2})\times B^{\mathbb{R}^{k-1}}(0,\frac{1}{2})\times B^{\mathbb{R}^{N-k}}(0,\frac{1}{2}))$ uniformly in $\sigma\in(0,\frac{1}{2})$, by (4.16) and (4.17), we can easily show that there exists a subsequence (g_{n_k}) such that $g_{n_k}\to g$ in $L^2(O_{\frac{1}{2},\frac{1}{2},\frac{1}{2}};y_1^2(y_1+|\overline{y}|)^\gamma)$.

But

$$\lim_{n\to\infty}\int_{O_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}}|\nabla g_n(y)|^2y_1^2(y_1+|\overline{y}|)^\gamma dy=0,$$

which implies that $\nabla g=0$ a.e. in $O_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}$. Hence there exists constant c such that $g=c$ a.e. in $O_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}$. But $\zeta_{g_{n_k}}=0$ and $g_{n_k}\to g$ in $L^2(O_{\frac{1}{2},\frac{1}{2},\frac{1}{2}})$, thus $c=0$, which is clearly a contradiction since

$$\int_{O_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}}|g(y)|^2y_1^2(y_1+|\overline{y}|)^\gamma dy=1.$$

Since

$$\begin{aligned}&\int_{O_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}}|\nabla\overline{f}(y)|^2y_1^2(y_1+|\overline{y}|)^\gamma dy\\&\asymp C(N,k,\gamma)\int_{O_{r_1,r_2,r_3}}r^{-N-\gamma}|\nabla f(x)|^2x_1^2(x_1+|\overline{x}|)^\gamma dx,\end{aligned}\tag{4.18}$$

the result follows by (4.13), (4.14) and (4.18). □

Theorem 4.9 *Assume that* $\gamma\geq -k$. *Let* $\xi\in K$, $x\in V(\xi,\frac{\beta_0}{16})$ *and let* β_1 *be the constant in Lemma 4.2. Then there exists a positive constant* $C=C(C_\xi,\Omega,K,\gamma,b)>0$ *such that*

$$\inf_{\zeta\in\mathbb{R}}\int_{\mathcal{B}(x,r)\cap\Omega}|f(y)-\zeta|^2d^2(y)d_K^\gamma(y)dy\leq Cr^2\int_{\mathcal{B}(x,r)\cap\Omega}|\nabla f(y)|^2d^2(y)d_K^\gamma(y)dy,\tag{4.19}$$

for any $0<r<\beta_1$ *and* $f\in C^1(\overline{\mathcal{B}(x,r)\cap\Omega})$.

Proof **Case 1.** $d(x)\geq br$. Since $d_K(x)\geq d(x)$, we can easily show that for any $y\in B(x,r)$ $\frac{b-1}{b}d(x)\leq d(y)\leq\frac{b+1}{b}d(x)$ and $\frac{b-1}{b}d_K(x)\leq d_K(y)\leq\frac{b+1}{b}d_K(x)$. Thus the proof of (4.19) follows easily in this case.

Case 2. $d(x)\leq br$ **and** $d_K(x)>bC_\xi r$. By (4.6), we again have that $\frac{b-1}{b}d_K(x)\leq d_K(y)\leq\frac{b+1}{b}d_K(x)$. Using the last inequality and proceeding as the proof of [25, Theorem 2.5], we obtain the desired result.

Case 3. $d(x)\leq br$ **and** $d_K(x)\leq bC_\xi r$. By (4.5), it is enough to prove the following inequality

$$\inf_{\zeta\in\mathbb{R}}\int_{\mathcal{B}(x,r)\cap\Omega}|f-\zeta|^2\delta^2(\delta_{2,K}+\delta)^\gamma dy\le Cr^2\int_{\mathcal{B}(x,r)\cap\Omega}|\nabla f|^2\delta^2(\delta_{2,K}+\delta)^\gamma dy.$$

This is a consequence of Lemma 4.8. □

By (2.2) and the above theorem, we can easily prove the following result.

Corollary 4.10 *Let $\mu\le k^2/4$ and let β_1 be the constant in Lemma 4.2. Then there exists a constant $C=C(\Omega,K,\gamma,b)>0$ such that for any $0<r<\beta_1$ any $f\in C^1(\overline{\mathcal{B}(x,r)\cap\Omega})$ and all $x\in\Omega$ there holds*

$$\inf_{\zeta\in\mathbb{R}}\int_{\mathcal{B}(x,r)\cap\Omega}|f(y)-\zeta|^2\phi_\mu^2(y)dy\le Cr^2\int_{\mathcal{B}(x,r)\cap\Omega}|\nabla f(y)|^2\phi_\mu^2(y)dy\,.$$

Proof If $\operatorname{dist}(x,K)<\beta_0/16$ the result follows from Theorem 4.9. In case $\operatorname{dist}(x,K)>\beta_0/16$ the result is well known. □

In view of the proof of Lemma 4.8, Corollaries 4.6 and 4.7 and (2.2), we can prove the following Poincaré inequality in Ω.

Theorem 4.11 *Let $\mu\le k^2/4$. There exists a positive constant $C=C(\Omega,K,\mu)$ such that*

$$\inf_{\zeta\in\mathbb{R}}\int_\Omega|f(y)-\zeta|^2\phi_\mu^2(y)dy\le C\int_\Omega|\nabla f(y)|^2\phi_\mu^2(y)dy,\tag{4.20}$$

for any $f\in C^1(\overline{\Omega})$.

4.4 Moser inequality

Theorem 4.12 *Let $\xi\in K$, $\gamma\ge -k$, $x\in V(\xi,\frac{\beta_0}{16})$ and let β_1 be the constant in Lemma 4.2. Then for any $\nu\ge N+\max\{2,2+\gamma\}$, there exists $C=C(\Omega,K,\nu,\beta_1)$ such that*

$$\begin{aligned}&\int_{\mathcal{B}(x,r)\cap\Omega}|f(y)|^{2(1+\frac{2}{\nu})}d^2(y)d_K^\gamma(y)dy\\&\quad\le Cr^2\overline{\mathcal{M}}_\gamma(x,r)^{-\frac{2}{\nu}}\int_{\mathcal{B}(x,r)\cap\Omega}|\nabla f(y)|^2d^2(y)d_K^\gamma(y)dy\\&\qquad\times\left(\int_{\mathcal{B}(x,r)\cap\Omega}|f(y)|^2d^2(y)d_K^\gamma(y)dy\right)^{\frac{2}{\nu}},\end{aligned}\tag{4.21}$$

for any $0<r<\beta_1$ and all $f\in C_c^\infty(\mathcal{B}(x,r)\cap\Omega)$.

Proof The cases $[d(x)>br]$ and $[d(x)\le br$ and $d_K(x)>bC_\xi r]$ are proved as in [26, Theorem 3.5] and [25, Theorem 2.6] respectively, using also the inequalities already obtained in the proof of Lemma 4.2.

So, let us assume that $d(x) \leq br$ and $d_K(x) \leq bC_\xi r$. We consider first the case where $-k \leq \gamma < 0$. By Hölder inequality, we have

$$\left(\int_{\mathcal{B}(x,r)\cap\Omega} |f(y)|^2 d^2(y) d_K^\gamma(y) dy\right)^{\frac{2(\nu-N-2)}{\nu(N+2)}} \leq \overline{\mathcal{M}}_\gamma(x,r)^{\frac{4(\nu-N-2)}{\nu(N+2)(N+4)}} \left(\int_{\mathcal{B}(x,r)\cap\Omega} |f(y)|^{2(1+\frac{2}{N+2})} d^2(y) d_K^\gamma(y) dy\right)^{\frac{2(\nu-N-2)}{\nu(N+4)}}. \tag{4.22}$$

Moreover

$$\begin{aligned}
&\int_{\mathcal{B}(x,r)\cap\Omega} |f(y)|^{2(1+\frac{2}{\nu})} d^2(y) d_K^\gamma(y) dy \\
&\leq \overline{\mathcal{M}}_\gamma(x,r)^{1-\frac{(\nu+2)(N+2)}{\nu(N+4)}} \left(\int_{\mathcal{B}(x,r)\cap\Omega} |f(y)|^{2(1+\frac{2}{N+2})} d^2(y) d_K^\gamma(y) dy\right)^{\frac{(\nu+2)(N+2)}{\nu(N+4)}} \\
&= \overline{\mathcal{M}}_\gamma(x,r)^{1-\frac{(\nu+2)(N+2)}{\nu(N+4)}} \left(\int_{\mathcal{B}(x,r)\cap\Omega} |f(y)|^{2(1+\frac{2}{N+2})} d^2(y) d_K^\gamma(y) dy\right)^{1-\frac{2(\nu-N-2)}{\nu(N+4)}} \\
&\leq \overline{\mathcal{M}}_\gamma(x,r)^{\frac{2}{N+2}-\frac{2}{\nu}} \int_{\mathcal{B}(x,r)\cap\Omega} |f(y)|^{2(1+\frac{2}{N+2})} d^2(y) d_K^\gamma(y) dy \\
&\quad \times \left(\int_{\mathcal{B}(x,r)\cap\Omega} |f(y)|^2 d^2(y) d_K^\gamma(y) dy\right)^{-\frac{2(\nu-N-2)}{\nu(N+2)}}, \\
&\leq \overline{\mathcal{M}}_\gamma(x,r)^{\frac{2}{N+2}-\frac{2}{\nu}} \left(\int_{\mathcal{B}(x,r)\cap\Omega} |f(y)|^{\frac{2(N+2)}{N}} d^2(y) d_K^\gamma(y) dy\right)^{\frac{N}{N+2}} \\
&\quad \times \left(\int_{\mathcal{B}(x,r)\cap\Omega} |f(y)|^2 d^2(y) d_K^\gamma(y) dy\right)^{\frac{2}{\nu}},
\end{aligned} \tag{4.23}$$

where in the second to last inequality we have used (4.22). By Corollary 3.7 and Proposition 3.1, we have

$$\begin{aligned}
\left(\int_{\mathcal{B}(x,r)\cap\Omega} |f(y)|^{\frac{2(N+2)}{N}} d^2(y) d_K^\gamma(y) dy\right)^{\frac{N}{N+2}} &\leq C \int_{\mathcal{B}(x,r)\cap\Omega} |\nabla f(y)|^2 d^2(y) d_K^{\frac{\gamma N}{N+2}}(y) dy \\
&\leq C r^{-\frac{2\gamma}{N+2}} \int_{\mathcal{B}(x,r)\cap\Omega} |\nabla f(y)|^2 d^2(y) d_K^\gamma(y) dy
\end{aligned} \tag{4.24}$$

Now, by Lemma 4.2

$$\overline{\mathcal{M}}_\gamma(x,r) \asymp C(\Omega, K, \gamma, N, C_\xi, \beta_0) r^{N+\gamma+2}. \tag{4.25}$$

The desired result follows by (4.23), (4.24) and (4.25).

If $\gamma > 0$, the proof of (4.21) is similar, the only difference is that we use Corollary 3.6 instead of Corollary 3.7. □

By (2.2) and the above theorem, we have

Corollary 4.13 *Let $\mu \le k^2/4$ and let β_1 be the constant in Lemma 4.2. Then for any $\nu \ge N + \max\{2, 2+\gamma\}$, there exists $C = C(\Omega, K, \nu, \beta_1)$ such that for any $x \in \Omega$, any $r \in (0, \beta_1)$ and any $f \in H_0^1(\mathcal{B}(x,r) \cap \Omega; \phi_\mu^2)$ there holds*

$$\int_{\mathcal{B}(x,r)\cap\Omega} |f|^{2(1+\frac{2}{\nu})}\phi_\mu^2 dy \le Cr^2 \mathcal{M}(x,r)^{-\frac{2}{\nu}} \left(\int_{\mathcal{B}(x,r)\cap\Omega} |\nabla f|^2 \phi_\mu^2 dy \right) \times \left(\int_{\mathcal{B}(x,r)\cap\Omega} f^2 \phi_\mu^2 dy \right)^{\frac{2}{\nu}}.$$

4.5 Harnack inequality

We consider the problem

$$(\partial_t + \mathcal{L}_\mu)u := u_t - \phi_\mu^{-2} \mathrm{div}(\phi_\mu^2 \nabla u) = 0, \quad \text{in } (0,T) \times \mathcal{B}(x,r) \cap \Omega, \tag{4.26}$$

for any $T > 0$ and $r < \frac{\beta_1}{4}$ where β_1 is the constant in Lemma 4.2. Similarly with Definition 2.3 we have

Definition 4.14 Let $D \subset \Omega$ be an open set. A function $v \in C^1((0,T) : H^1(D; \phi_\mu^2))$ is a weak subsolution of $v_t + \mathcal{L}_\mu v = 0$ in $(0,T) \times D$ if for any non-negative $\Phi \in C_c^1((0,T) : C_c^\infty(D))$ we have

$$\int_0^T \int_D (v_t \Phi + \nabla v \cdot \nabla \Phi)\phi_\mu^2 \, dy \, dt \le 0.$$

We now set

$$Q = (s - r^2, s) \times \mathcal{B}(x,r) \cap \Omega$$
$$Q_\delta = (s - \delta r^2, s) \times \mathcal{B}(x, \delta r) \cap \Omega.$$

Now we are ready to apply the Moser iteration argument in order to prove the Harnack inequality for nonnegative weak solutions. The proof is based on the ideas in the proof of Harnack inequality in noncompact smooth manifold (see [50, Chapter 5]). Let us note here that Theorem 4.5 allows to us to consider test functions in $C_c^\infty(\mathcal{B}(x,r))$ instead of $C_c^\infty(\mathcal{B}(x,r) \cap \Omega)$. Thus we are able to prove boundary Harnack inequalities.

Let us first state the L^p mean value inequality for nonnegative subsolutions of the operator $\partial_t + \mathcal{L}_\mu$.

Theorem 4.15 *Let $\mu \le k^2/4$, $\nu \ge N + \max\{2, 2+2\gamma_+\}$ and $p > 0$. There exists a constant $C(\nu, \lambda, \beta_1, p, \Omega, K)$ such that for any $x \in \Omega$ and for any positive subsolution v of* (4.26) *in Q we have the estimate*

$$\sup_{Q_\delta} |v|^p \le \frac{C}{(\delta' - \delta)^{\nu+2} r^2 \overline{\mathcal{M}}_\gamma(x, r)} \int_{Q_{\delta'}} |v|^p \phi_\mu^2 \, dy \, dt,$$

for each $0 < \delta < \delta' \le 1$.

The proof of the above theorem is similar to the proof of [50, Theorem 5.2.9] and we omit it (see also [25, Theorem 2.12]). Similarly one can establish the proof of the parabolic Harnack inequality up to the boundary of Theorem 2.4.

Let $k(t, x, y)$ be the heat kernel of the problem

$$\begin{cases} v_t = -L_\mu v, & \text{in } (0, T] \times \Omega, \\ v = 0, & \text{on } (0, T] \times \partial\Omega, \\ v(0, x) = v_0(x), & \text{in } \Omega. \end{cases}$$

By the parabolic Harnack inequality (2.4), and following the methods of Grigoryan and Saloff-Coste (see for example [34, Theorem 2.7] and [50, Theorem 5.4.12]) we obtain the following sharp two-sided heat kernel estimate for small time (we recall that $\mathcal{M}(x, r)$ has been defined in Corollary 4.3):

Theorem 4.16 *Let β_1 be the constant of Lemma 4.2. Then there exist positive constants A_1, A_2, C_1 and C_2, such that for all x, $y \in \Omega$ and all $0 < t < \frac{\beta_1^2}{4}$ the heat kernel $k(t, x, y)$ satisfies*

$$\begin{aligned} &\frac{C_1}{\mathcal{M}^{\frac{1}{2}}(x, \sqrt{t}) \mathcal{M}^{\frac{1}{2}}(y, \sqrt{t})} \exp\left(- A_1 \frac{|x-y|^2}{t} \right) \le k(t, x, y) \\ &\quad \le \frac{C_2}{\mathcal{M}^{\frac{1}{2}}(x, \sqrt{t}) \mathcal{M}^{\frac{1}{2}}(y, \sqrt{t})} \\ &\quad \times \exp\left(- A_2 \frac{|x-y|^2}{t} \right). \end{aligned}$$

Proof of Theorem 2.5 This follows easily from Theorem 4.16 and Corollary 4.3. □

5 Heat kernel estimates for large time

5.1 Weighted logarithmic Sobolev inequality

Theorem 5.1 *Let $\mu \le k^2/4$. There exists a positive constant $C = C(\Omega, K, \mu)$ such that for any $\epsilon > 0$ there holds*

$$\int_\Omega u^2 \ln \frac{|u|}{\|u\|_{L^2(\Omega; \phi_\mu^2)}} \phi_\mu^2 dx \le \varepsilon \int_\Omega |\nabla u|^2 \phi_\mu^2 dx + b(\varepsilon) \int_\Omega u^2 \phi_\mu^2 dx, \qquad (5.1)$$

for all $u \in H^1(\Omega;\phi_\mu^2)$*; here* $b(\varepsilon) = C - \frac{N+2+\max(\gamma_+,0)}{4}\min(\ln\varepsilon, 0)$.

Proof We may assume that $\|u\|_{L^2(\Omega;\phi_\mu^2)} = 1$. Assume first that $-\frac{k}{2} \le \gamma_+ < 0$. Then

$$\begin{aligned}
\int_\Omega |u|^2 \ln|u|\phi_\mu^2 dx &= \frac{N}{4}\int_\Omega |u|^2 \ln|u|^{\frac{4}{N}}\phi_\mu^2 dx \\
&\le \frac{N}{4}\ln\left(\int_\Omega |u|^{\frac{2(N+2)}{N}}\phi_\mu^2 dx\right) \\
&= \frac{N+2}{4}\ln\left(\left(\int_\Omega |u|^{\frac{2(N+2)}{N}}\phi_\mu^2 dx\right)^{\frac{N}{N+2}}\right) \\
&\le \frac{N+2}{4}\ln\left(C_0\left(\int_\Omega |\nabla u|^2\phi_\mu^2 dx + \int_\Omega |u|^2\phi_\mu^2 dx\right)\right),
\end{aligned}$$

where in the last inequality, we used Corollary 4.7 and (2.2). Using the fact that $\frac{N+2}{4}\log\theta = \frac{N+2}{4}\ln\frac{4\varepsilon\theta}{C_0(N+2)} + \frac{N+2}{4}\ln\frac{C_0(N+2)}{4\varepsilon}$, $\forall\varepsilon,\theta > 0$, we obtain the desired result with $b(\varepsilon) = 1 + \frac{N+2}{4}(\ln C_0 + \ln\frac{N+2}{4} - \ln\varepsilon)$, if $0 < \varepsilon \le 1$

Similarly, if $\varepsilon \ge 1$ and $-\frac{k}{2} \le \gamma_+ < 0$, we obtain the desired result with $b(\varepsilon) = 1 + \frac{N+2}{4}(\ln C_0 + \ln\frac{N+2}{4})$.

If $\gamma_+ > 0$ we proceed as above and we use Corollary 4.6 instead of Corollary 4.7, in order to obtain (5.1) with $b(\varepsilon) = 1 + \frac{N+2+2\gamma_+}{4}(\ln C_1 + \ln\frac{N+2+2\gamma_+}{4} - \ln\varepsilon)$, where C_1 is the constant in Corollary 4.6. □

Theorem 5.2 *Let* $\mu \le k^2/4$ *and let* $u \in H^1(\Omega;\phi_\mu^2)$ *be such that* $\int_\Omega u\,\phi_\mu^2 dx = 0$. *There exists a positive constant* $C = C(\Omega, K, \mu)$ *such that for any* $\epsilon > 0$ *there holds*

$$\int_\Omega u^2 \ln\frac{|u|}{\|u\|_{L^2(\Omega;\phi_\mu^2)}}\phi_\mu^2 dx \le \varepsilon\int_\Omega |\nabla u|^2\phi_\mu^2 dx + b(\varepsilon)\int_\Omega u^2\phi_\mu^2 dx,$$

where $b(\varepsilon) = C - \frac{N+2+\max(2\gamma_+,0)}{4}\ln\varepsilon$.

Proof By (4.20) and in view of the proof of (5.1) we obtain the desired result. □

Proof of Theorem 2.2 We normalize ϕ_μ so that $\int_\Omega \phi_\mu^2 dx = 1$. We define the bilinear form $Q : H_0^1(\Omega;\phi_\mu^2) \times H_0^1(\Omega;\phi_\mu^2) \to \mathbb{R}$ by

$$Q(u,v) = \int_\Omega \nabla u \cdot \nabla v\,\phi_\mu^2 dx.$$

We recall here that $H^1(\Omega;\phi_\mu^2) = H_0^1(\Omega;\phi_\mu^2)$ by (2.2) and Theorem 4.5.

Let $\mathcal{L}_\mu$ denote the self-adjoint operator on $L^2(\Omega;\phi_\mu^2)$ associated to the form Q, so that formally we may write

$$\mathcal{L}_\mu u = -\phi_\mu^{-2}\,\mathrm{div}\left(\phi_\mu^2\nabla u\right).$$

The operator $\mathcal{L}_\mu$ generates a contraction semigroup $T(t) : L^2(\Omega; \phi_\mu^2) \to L^2(\Omega; \phi_\mu^2)$, $t \geq 0$, denoted also by $e^{-\mathcal{L}_\mu t}$. This semigroup is positivity preserving and by [17, Lemma 1.3.4] we can easily show that satisfies the conditions of [17, Theorems 1.3.2 and 1.3.3]. Thus, by (5.1), we can apply [17, Corollary 2.2.8] to deduce that

$$\|e^{-\mathcal{L}_\mu t} u\|_{L^\infty(\Omega)} \leq C_t \|u\|_{L^2(\Omega;\phi_\mu^2)}, \qquad t > 0, \;\; u \in L^2(\Omega; \phi_\mu^2), \tag{5.2}$$

where

$$C_t = e^{\frac{1}{t}\int_0^t b(\varepsilon)d\varepsilon}.$$

Hence, by [17, Lemma 2.1.2], $e^{-\mathcal{L}_\mu t}$ is ultracontractive and has a kernel $k(t, x, y)$ such that

$$0 \leq k(t, x, y) \leq C_{\frac{t}{2}}^2.$$

By the last inequality, the upper estimate in Theorem 2.2 follows easily. For the lower estimate in Theorem 2.2 we will give two proofs. One using the boundary Harnack inequality (2.4) and the other one proceeding as the proof of [16, Theorem 6].

First proof (as in the proof of [16, Theorem 6]). First we note that since $H^1(\Omega; \phi_\mu^2)$ is compactly embedded in $L^2(\Omega; \phi_\mu^2)$, the operator $\mathcal{L}_\mu$ has compact resolvent. In addition, we have that $\mathcal{L}_\mu 1 = 0$ and hence, by (4.20),

$$\mathrm{sp}(\mathcal{L}_\mu) \subset \{0\} \cup [\lambda, \infty),$$

for some $\lambda > 0$. Thus, using the spectral theorem, we can easily show that for any $f \in L^2(\Omega; \phi_\mu^2)$ such that $\int_\Omega f\phi_\mu^2 dx = 0$ we have

$$\|e^{-\mathcal{L}_\mu t} f\|_{L^2(\Omega;\phi_\mu^2)} \leq e^{-\lambda t} \, \|f\|_{L^2(\Omega;\phi_\mu^2)}, \qquad \forall t \geq 0. \tag{5.3}$$

Now, let $f \in L^1(\Omega; \phi_\mu^2)$ and $\int_\Omega f\phi_\mu^2 dx = 0$. By (5.2) and (5.3), we have

$$\begin{aligned}\|e^{-\mathcal{L}_\mu t} f\|_{L^\infty(\Omega)} &= \|e^{-\mathcal{L}_\mu \frac{t}{3}}\big(e^{-\mathcal{L}_\mu \frac{2t}{3}} f\big)\|_{L^\infty(\Omega)} \leq C_{\frac{t}{3}} \|e^{-\mathcal{L}_\mu \frac{2t}{3}} f\|_{L^2(\Omega;\phi_\mu^2)} \\ &\leq e^{-\frac{\lambda t}{3}} C_{\frac{t}{3}} \|e^{-\mathcal{L}_\mu \frac{t}{3}} f\|_{L^2(\Omega;\phi_\mu^2)}.\end{aligned}$$

Taking adjoints we have

$$\|e^{-\mathcal{L}_\mu \frac{t}{3}} f\|_{L^2(\Omega;\phi_\mu^2)} \leq C_{\frac{t}{3}} \, \|f\|_{L^1(\Omega;\phi_\mu^2)},$$

hence

$$\|e^{-\mathcal{L}_\mu t} f\|_{L^\infty(\Omega)} \leq e^{-\frac{\lambda t}{3}} C_{\frac{t}{3}}^2 \, \|f\|_{L^1(\Omega;\phi_\mu^2)}.$$

Let now $f \in L^1(\Omega; \phi_\mu^2)$. The function $g := f - \int_\Omega f\phi_\mu^2 dx$ satisfies $\int_\Omega g\phi_\mu^2 dx = 0$, thus

$$e^{-\mathcal{L}_\mu t} g = e^{-\mathcal{L}_\mu t} f - \langle f, 1\rangle_{L^2(\Omega;\phi_\mu^2)}.$$

Hence the operator

$$\tilde{T}(t) f = e^{-\mathcal{L}_\mu t} f - \langle f, 1\rangle_{L^2(\Omega;\phi_\mu^2)}$$

satisfies

$$\|\tilde{T}(t) f\|_{L^\infty(\Omega)} = \|e^{-\mathcal{L}_\mu t} g\|_{L^\infty(\Omega)} \le e^{-\frac{\lambda t}{3}} C_{\frac{t}{3}}^2 \|g\|_{L^1(\Omega;\phi_\mu^2)} \le 2e^{-\frac{\lambda t}{3}} C_{\frac{t}{3}}^2 \|f\|_{L^1(\Omega;\phi_\mu^2)}.$$

Therefore the integral kernel $\tilde{k}(t, x, y)$ of $\tilde{T}(t)$ satisfies $\tilde{k}(t, x, y) = k(t, x, y) - 1$ and

$$|\tilde{k}(t, x, y)| \le 2e^{-\frac{\lambda t}{3}} C_{\frac{t}{3}}^2.$$

The desired result follows if we choose t large enough.

Second proof (using the boundary Harnack inequality (2.4)). Let $x_0 \in \Omega$. Then by (2.4) we can show that

$$k(t-1, x, y) \le C(\Omega, K) k(t, x, x_0),$$

for all $t \ge 2$ and $x, y \in \Omega$. Thus,

$$\begin{aligned} 1 = \int_\Omega k(t-1, x, y)\phi_\mu^2(y) dy &\le C(\Omega, K) \int_\Omega k(t, x, x_0)\phi_\mu^2(y) dy \\ &= C(\Omega, K) k(t, x, x_0), \quad \forall t \ge 2. \end{aligned}$$

The desired result follows. □

5.2 Green function estimates

In this subsection we prove the existence of the Green kernel of L_μ along with sharp two-sided estimates.

Proposition 5.3 *Let $\mu \le k^2/4$ and assume that $\lambda_\mu > 0$. For any $y \in \Omega$ there exists a minimal Green function $G_\mu(\cdot, y)$ of the equation*

$$L_\mu u = \delta_y \quad \text{in } \Omega,$$

where δ_y denotes the Dirac measure at y. Furthermore, the following estimates hold

$$G_\mu(x,y) \asymp \begin{cases} |x-y|^{2-N} \min\left\{1, \dfrac{d(x)d(y)}{|x-y|^2}\right\} \left(\dfrac{d_K(x)d_K(y)}{(d_K(x)+|x-y|)\,(d_K(y)+|x-y|)}\right)^{\gamma_+}, \\ \qquad \text{if } \gamma_+ > -\dfrac{N}{2}, \\ |x-y|^{2-N} \min\left\{1, \dfrac{d(x)d(y)}{|x-y|^2}\right\} \left(\dfrac{|x|\,|y|}{(|x|+|x-y|)\,(|y|+|x-y|)}\right)^{-\frac{N}{2}} \\ \quad + \dfrac{d(x)d(y)}{(|x||y|)^{\frac{N}{2}}} \left|\ln\left(\min\left\{\dfrac{1}{|x-y|^2}, \dfrac{1}{d(x)d(y)}\right\}\right)\right|, \quad \text{if } \gamma_+ = -\dfrac{N}{2}. \end{cases} \tag{5.4}$$

Proof First, let $C_1 > 0$ and T be as in Theorem 2.6. We note that

$$\left(\left(\frac{\sqrt{t}}{d(x)}+1\right)\left(\frac{\sqrt{t}}{d(y)}+1\right)\right)^{-1} = \frac{d(x)d(y)}{(\sqrt{t}+d(x))(\sqrt{t}+d(y))} \leq \min\left\{1, \frac{d(x)d(y)}{t}\right\} \tag{5.5}$$

and

$$\left(\left(\frac{\sqrt{t}}{d(x)}+1\right)\left(\frac{\sqrt{t}}{d(y)}+1\right)\right)^{-1} e^{-\frac{C_1|x-y|^2}{t}} = \frac{d(x)d(y)}{(\sqrt{t}+d(x))(\sqrt{t}+d(y))} e^{-\frac{C_1|x-y|^2}{t}} \geq C \min\left\{1, \frac{d(x)d(y)}{t}\right\} e^{-\frac{(1+C_1)|x-y|^2}{t}} \tag{5.6}$$

for all $x, y \in \Omega$ and $0 < t < T$, where $C = C(C_1, T) > 0$.

By Theorem 2.6, (2.2) and estimates (5.5)–(5.6), there exist $C_i = C_i(\Omega, K, \mu) > 0$, $i = 1, 2$ and $T = T(\Omega, K, \mu) > 0$ such that for $t \in (0, T)$ and $x, y \in \Omega$,

$$C_1 \min\left\{1, \frac{d(x)d(y)}{t}\right\}\left(\frac{d_K(x)}{d_K(x)+\sqrt{t}}\right)^{\gamma_+}\left(\frac{d_K(y)}{d_K(y)+\sqrt{t}}\right)^{\gamma_+} t^{-\frac{N}{2}} e^{-\frac{C_2|x-y|^2}{t}} \leq h(t,x,y) \leq C_2 \min\left\{1, \frac{d(x)d(y)}{t}\right\}\left(\frac{d_K(x)}{d_K(x)+\sqrt{t}}\right)^{\gamma_+}\left(\frac{d_K(y)}{d_K(y)+\sqrt{t}}\right)^{\gamma_+} t^{-\frac{N}{2}} e^{-\frac{C_1|x-y|^2}{t}}, \tag{5.7}$$

while

$$C_1 \leq \frac{h(t,x,y)}{d(x)d(y)d_K^{\gamma_+}(x)d_K^{\gamma_+}(y)e^{-\lambda_\mu t}} \leq C_2, \qquad \forall t \geq T, \; x, y \in \Omega. \tag{5.8}$$

By (5.7) and (5.8), we deduce the existence of the minimal Green kernel G_μ of L_μ, given by

$$G_\mu(x,y)=\int_0^\infty h(t,x,y)dt=\int_0^T h(t,x,y)dt+\int_T^\infty h(t,x,y)dt. \tag{5.9}$$

Using (5.8) we easily see that the second integral in (5.9) satisfies the required upper estimate in both cases considered (i.e. $\gamma_+>-\frac{N}{2}$ or $\gamma_+=-\frac{N}{2}$). We now concentrate on the first integral in (5.9).

By the change of variable $s=\frac{|x-y|^2}{t}$, we obtain for $i=1,2$,

$$\begin{aligned}&\int_0^T \min\left\{1,\frac{d(x)d(y)}{t}\right\}\left(\frac{d_K(x)}{d_K(x)+\sqrt{t}}\right)^{\gamma_+}\left(\frac{d_K(y)}{d_K(y)+\sqrt{t}}\right)^{\gamma_+} t^{-\frac{N}{2}}e^{-\frac{C_i|x-y|^2}{t}}dt=|x-y|^{2-N}\\ &\int_{\frac{|x-y|^2}{T}}^\infty \min\left\{1,s\frac{d(x)d(y)}{|x-y|^2}\right\}\left(\left(\frac{|x-y|}{\sqrt{s}d_K(x)}+1\right)\left(\frac{|x-y|}{\sqrt{s}d_K(y)}+1\right)\right)^{-\gamma_+} s^{\frac{N}{2}-2}e^{-C_is}ds\\ &=:|x-y|^{2-N}S_i(x,y)\,.\end{aligned}$$

By (5.7) we therefore have for some $c_1,c_2>0$ that

$$c_1|x-y|^{2-N}S_2(x,y)\le\int_0^T h(t,x,y)dt\le c_2|x-y|^{2-N}S_1(x,y)\,,\qquad x,y\in\Omega. \tag{5.10}$$

In the sequel, we assume that $\frac{|x-y|^2}{T}<\frac{1}{2}$. The proof in the case $\frac{|x-y|^2}{T}>\frac{1}{2}$ is similar, indeed simpler. We write

$$\begin{aligned}S_1=&\int_{\frac{|x-y|^2}{T}}^1 \min\left\{1,s\frac{d(x)d(y)}{|x-y|^2}\right\}\left(\left(\frac{|x-y|}{\sqrt{s}d_K(x)}+1\right)\left(\frac{|x-y|}{\sqrt{s}d_K(y)}+1\right)\right)^{-\gamma_+} s^{\frac{N}{2}-2}e^{-C_1s}ds\\ &+\int_1^\infty \min\left\{1,s\frac{d(x)d(y)}{|x-y|^2}\right\}\left(\left(\frac{|x-y|}{\sqrt{s}d_K(x)}+1\right)\left(\frac{|x-y|}{\sqrt{s}d_K(y)}+1\right)\right)^{-\gamma_+} s^{\frac{N}{2}-2}e^{-C_1s}ds.\end{aligned} \tag{5.11}$$

Concerning the second term in the RHS of (5.11) we have

$$\begin{aligned}&\int_1^\infty \min\left\{1,s\frac{d(x)d(y)}{|x-y|^2}\right\}\left(\left(\frac{|x-y|}{\sqrt{s}d_K(x)}+1\right)\left(\frac{|x-y|}{\sqrt{s}d_K(y)}+1\right)\right)^{-\gamma_+} s^{\frac{N}{2}-2}e^{-C_1s}ds\\ &\le C\min\left\{1,\frac{d(x)d(y)}{|x-y|^2}\right\}\left(\left(\frac{|x-y|}{d_K(x)}+1\right)\left(\frac{|x-y|}{d_K(y)}+1\right)\right)^{-\gamma_+},\end{aligned}$$

and therefore the required estimate is satisfied.

Let $\gamma_+\le 0$. For the first term in the RHS of (5.11) we have

$$\int_{\frac{|x-y|^2}{T}}^1 \min\left\{1,s\frac{d(x)d(y)}{|x-y|^2}\right\}\left(\left(\frac{|x-y|}{\sqrt{s}d_K(x)}+1\right)\left(\frac{|x-y|}{\sqrt{s}d_K(y)}+1\right)\right)^{-\gamma_+} s^{\frac{N}{2}-2}e^{-C_1s}ds$$

$$
\begin{aligned}
&= \int_{\frac{|x-y|^2}{T}}^{1} \min\left\{1, s\frac{d(x)d(y)}{|x-y|^2}\right\} \left(\left(\frac{|x-y|}{d_K(x)}+\sqrt{s}\right)\left(\frac{|x-y|}{d_K(y)}+\sqrt{s}\right)\right)^{-\gamma_+} s^{\frac{N}{2}+\gamma_+-2}e^{-C_1 s}ds \\
&\leq C\Bigg(|x-y|^{-2\gamma_+}\big(d_K(x)d_K(y)\big)^{\gamma_+}\int_{\frac{|x-y|^2}{T}}^{1}\min\left\{1, s\frac{d(x)d(y)}{|x-y|^2}\right\} s^{\frac{N}{2}+\gamma_+-2}e^{-C_1 s}ds \\
&\quad +|x-y|^{-\gamma_+}\big(d_K(x)d_K(y)\big)^{\gamma_+}\int_{\frac{|x-y|^2}{T}}^{1}\min\left\{1, s\frac{d(x)d(y)}{|x-y|^2}\right\} \\
&\quad \times (d_K(x)+d_K(y))^{-\gamma_+} s^{\frac{N}{2}+\frac{\gamma_+}{2}-2}e^{-C_1 s}ds \\
&\quad +\int_{\frac{|x-y|^2}{T}}^{1}\min\left\{1, s\frac{d(x)d(y)}{|x-y|^2}\right\} s^{\frac{N}{2}-2}e^{-C_1 s}ds\Bigg) \\
&=: C(J_1+J_2+J_3) \qquad (5.12)
\end{aligned}
$$

It is easily seen that

$$
J_3 \leq C\min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\}.
$$

Concerning J_1 and J_2 we consider two cases.

Case I. $-\frac{N}{2} < \gamma_+ \leq 0$. In view of (5.10) and (5.12), it is enough to establish that for $i = 1, 2$ we have

$$
J_i \leq \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\}\left(\frac{d_K(x)d_K(y)}{(d_K(x)+|x-y|)\,(d_K(y)+|x-y|)}\right)^{\gamma_+}, \quad i = 1, 2. \qquad (5.13)
$$

In order to prove (5.13) we shall need to consider additional cases.

Case Ia. $\frac{d(x)d(y)}{|x-y|^2} \leq 1$. In this case it is immediate that

$$
J_1 = C|x-y|^{-2-2\gamma_+}\big(d_K(x)d_K(y)\big)^{\gamma_+}d(x)d(y).
$$

and

$$
J_2 = C|x-y|^{-2-\gamma_+}(d_K(x)d_K(y))^{\gamma_+}\big(d_K(x)+d_K(y)\big)^{-\gamma_+}d(x)d(y).
$$

Hence inequality (5.13) is satisfied.

Case Ib. $\frac{d(x)d(y)}{|x-y|^2} > 1$. In this case we have $\frac{1}{4}d_K(y) \leq d_K(x) \leq 4d_K(y)$. Indeed, suppose that $d_K(x) > 4d_K(y)$. Then, since $d_K(x) \leq |x-y| + d_K(y)$, we easily obtain that $d_K(y) \leq \frac{1}{3}|x-y|$ and $d_K(x) \leq \frac{4}{3}|x-y|$; hence $d(x)d(y) \leq \frac{4}{9}|x-y|^2$, a contradiction.

To proceed we first note that

$$
J_1 \leq |x-y|^{-2\gamma_+}\big(d_K(x)d_K(y)\big)^{\gamma_+}\left(\frac{d(x)d(y)}{|x-y|^2}\int_0^{\frac{|x-y|^2}{d(x)d(y)}} s^{\frac{N}{2}+\gamma_+-1}e^{-C_1 s}ds\right.
$$

$$+\int_{\frac{|x-y|^2}{d(x)d(y)}}^{1} s^{\frac{N}{2}+\gamma_+-2}e^{-C_1 s}ds\Bigg) \tag{5.14}$$

and similarly

$$J_2 \le |x-y|^{-\gamma_+}\left(\frac{d_K(x)d_K(y)}{d_K(x)+d_K(y)}\right)^{\gamma_+}\Bigg(\frac{d(x)d(y)}{|x-y|^2}\int_0^{\frac{|x-y|^2}{d(x)d(y)}} s^{\frac{N}{2}+\frac{\gamma_+}{2}-1}e^{-C_1 s}ds$$

$$+\int_{\frac{|x-y|^2}{d(x)d(y)}}^{1} s^{\frac{N}{2}+\frac{\gamma_+}{2}-2}e^{-C_1 s}ds\Bigg) \tag{5.15}$$

We now consider different subcases.
Case 1. $-\frac{N}{2} < \gamma_+ < -N+2$. From (5.14) and (5.15) we obtain

$$J_1 \le c, \qquad J_2 \le c.$$

It follows that (5.13) is satisfied.
Case 2. $\gamma_+ = -N+2 > -\frac{N}{2}$. In this case (5.14) and (5.15) give

$$J_1 \le c$$

and

$$J_2 \le c|x-y|^{-\gamma_+}\left(\frac{d_K(x)d_K(y)}{d_K(x)+d_K(y)}\right)^{\gamma_+}\left(1+\ln\left(\frac{d(x)d(y)}{|x-y|^2}\right)\right) \le c$$

Again it is easily seen that (5.13) is satisfied.
Case 3. $\max\{-\frac{N}{2}, -N+2\} < \gamma_+ < -\frac{N-2}{2}$. In this case we obtain

$$J_1 \le c, \qquad J_2 \le c|x-y|^{-\gamma_+}\left(\frac{d_K(x)d_K(y)}{d_K(x)+d_K(y)}\right)^{\gamma_+} \le c$$

and (5.13) once again follows.
Case 4. $\gamma_+ = -\frac{N-2}{2} < 0$. In this case we obtain

$$J_1 \le c|x-y|^{-2\gamma_+}\big(d_K(x)d_K(y)\big)^{\gamma_+}\left(1+\ln\left(\frac{d(x)d(y)}{|x-y|^2}\right)\right) \le c,$$

$$J_2 \le c|x-y|^{-\gamma_+}\left(\frac{d_K(x)d_K(y)}{d_K(x)+d_K(y)}\right)^{\gamma_+} \le c$$

and (5.13) once again follows.
Case 5. $-\frac{N-2}{2} < \gamma_+ \le 0$. In this case we obtain

$$J_1 \le c|x-y|^{-2\gamma_+}\big(d_K(x)d_K(y)\big)^{\gamma_+} \le c, \qquad J_2 \le c|x-y|^{-\gamma_+}\left(\frac{d_K(x)d_K(y)}{d_K(x)+d_K(y)}\right)^{\gamma_+} \le c$$

and (5.13) once again follows.

Case II. $\gamma_+ = -\frac{N}{2}$. The proof is very similar to the previous case and for the sake of brevity we shall only consider J_1, where the main difference appears. We note that in this case we have $d_K(x) = |x|$.

We assume that $\frac{|x-y|^2}{T} \leq \frac{1}{2}$. The proof in the case $\frac{|x-y|^2}{T} > \frac{1}{2}$ is similar, indeed simpler.

Case IIa. $\frac{d(x)d(y)}{|x-y|^2} \leq 1$. In this case we easily obtain

$$J_1 \leq c\,|x-y|^{N-2} d(x)d(y)(|x|\,|y|)^{-\frac{N}{2}} \log\left(\frac{T}{|x-y|^2}\right),$$

and this is estimated using the second term in the RHS of (5.4).

Case IIb. $\frac{d(x)d(y)}{|x-y|^2} \geq 1$. We may assume that $\frac{|x-y|^2}{d(x)d(y)} > \frac{|x-y|^2}{T}$, otherwise we need only consider the second of the two integrals below.

We have

$$\begin{aligned} J_1 &= |x-y|^N (|x|\,|y|)^{-\frac{N}{2}} \left(\frac{d(x)d(y)}{|x-y|^2} \int_{\frac{|x-y|^2}{T}}^{\frac{|x-y|^2}{d(x)d(y)}} s^{-1} e^{-C_1 s} ds + \int_{\frac{|x-y|^2}{d(x)d(y)}}^{1} s^{-2} e^{-C_1 s} ds \right) \\ &\leq c|x-y|^{N-2} d(x)d(y)\big(|x|\,|y|\big)^{-\frac{N}{2}} \log\left(\frac{T}{d(x)d(y)}\right), \end{aligned}$$

which satisfies the upper bound in (5.4). Hence the upper bound has been established in all cases.

This concludes the proof of the upper estimate when $\gamma_+ \leq 0$. If $\gamma_+ > 0$ then the proof is essentially similar, indeed simpler, and is omitted.

The proof of the lower bound is much simpler. For example, in case $\gamma_+ \leq 0$ we have from (5.10)

$$G_\mu(x, y) \geq c_1 |x-y|^{2-N} S_2(x, y) \geq c|x-y|^{2-N} J_1(x, y),$$

where J_1 is as above, the only difference being that the exponential factor in the integrand is $e^{-C_2 s}$ instead of $e^{-C_1 s}$. The result then follows easily. □

6 The linear elliptic problem

6.1 Subsolutions and supersolutions

We recall the definition of the function $\tilde{d}_K$ from (2.5). Given parameters $\epsilon > 0$ and $M \in \mathbb{R}$ we define the functions

$$\eta_{\gamma_+,\varepsilon} = e^{-Md}(d+\tilde{d}_K^2)\tilde{d}_K^{\gamma_+} - d\tilde{d}_K^{\gamma_+ +\varepsilon} \qquad \zeta_{\gamma_+,\varepsilon} = e^{Md}(d+\tilde{d}_K^2)\tilde{d}_K^{\gamma_+} + d\tilde{d}_K^{\gamma_+ +\varepsilon}$$
$$\eta_{\gamma_-,\varepsilon} = e^{-Md}(d+\tilde{d}_K^2)\tilde{d}_K^{\gamma_-} + d\tilde{d}_K^{\gamma_- +\varepsilon} \qquad \zeta_{\gamma_-,\varepsilon} = e^{Md}(d+\tilde{d}_K^2)\tilde{d}_K^{\gamma_-} - d\tilde{d}_K^{\gamma_- +\varepsilon}$$
$$\zeta_{+,\varepsilon} = e^{-Md}(-\ln\tilde{d}_K)(d+\tilde{d}_K^2)\tilde{d}_K^{-\frac{k}{2}} - d\tilde{d}_K^{-\frac{k}{2}+\varepsilon} \quad \zeta_{-,\varepsilon} = e^{Md}(-\ln\tilde{d}_K)(d+\tilde{d}_K^2)\tilde{d}_K^{-\frac{k}{2}} + d\tilde{d}_K^{-\frac{k}{2}+\varepsilon}$$

Lemma 6.1 *Let $\mu \le k^2/4$ and $0 < \varepsilon < 1$. There exist positive constants $\beta_0 = \beta_0(\Omega, K, \mu, \varepsilon)$ and $M = M(\Omega, K, \mu, \varepsilon)$ such that the following hold in $K_{\beta_0} \cap \Omega$:*
(i) *The functions $\eta_{\gamma_+,\varepsilon}$ and $\zeta_{\gamma_+,\varepsilon}$ are non-negative in $K_{\beta_0} \cap \Omega$ and satisfy*

$$L_\mu \eta_{\gamma_+,\varepsilon} \ge 0, \qquad L_\mu \zeta_{\gamma_+,\varepsilon} \le 0, \quad \text{in } K_{\beta_0} \cap \Omega.$$

(ii) *If $\mu < k^2/4$ and $\varepsilon < \min\{1, \sqrt{k^2 - 4\mu}\}$) then $\eta_{\gamma_-,\varepsilon}$ and $\zeta_{\gamma_-,\varepsilon}$ are non-negative in $K_{\beta_0} \cap \Omega$ and satisfy*

$$L_\mu \eta_{\gamma_-,\varepsilon} \ge 0, \qquad L_\mu \zeta_{\gamma_-,\varepsilon} \le 0, \quad \text{in } K_{\beta_0} \cap \Omega. \tag{6.1}$$

(iii) *The functions $\zeta_{+,\varepsilon}$ and $\zeta_{-,\varepsilon}$ are non-negative in $K_{\beta_0} \cap \Omega$ and satisfy*

$$L_{\frac{k^2}{4}} \zeta_{+,\varepsilon} \ge 0, \qquad L_{\frac{k^2}{4}} \zeta_{-,\varepsilon} \le 0, \quad \text{in } K_{\beta_0} \cap \Omega.$$

Proof Let $M \in \mathbb{R}$. By Proposition 3.1 we have in $\Omega \cap K_{\beta_0}$,

$$\begin{aligned}
\Delta(d^a \tilde{d}_K^b) &= d^{a-2}\tilde{d}_K^b\big(a(a-1) + ad\Delta d\big) \\
&\quad + d^a\, \tilde{d}_K^{b-2}\big(2ab + b(k-1+f) + b(b-1)(1+h)\big) \\
\nabla e^{Md} \cdot \nabla(d^a \tilde{d}_K^b) &= Me^{Md}(ad^{a-1}\tilde{d}_K^b + bd^{a+1}\tilde{d}_K^{b-2}) \\
\Delta e^{Md} &= e^{Md}(M^2 + M\Delta d)
\end{aligned}$$

Thus

$$\begin{aligned}
L_\mu(e^{Md} d^a \tilde{d}_K^b) &= -e^{Md} d^{a-1}\tilde{d}_K^b \left(M^2 d + Md\Delta d + 2aM + a\Delta d + a(a-1)d^{-1}\right) \\
&- e^{Md} d^a \tilde{d}_K^{b-1}\left(\frac{2Mbd + bf + b(b-1)h + \mu g}{\tilde{d}_K}\right) \\
&- \big(b(k-1) + b(b-1) + 2ab + \mu\big)e^{Md} d^a \tilde{d}_K^{b-2}.
\end{aligned}$$

Now let $M \in \mathbb{R}$ and $0 < \varepsilon < 1$. Using the above formulas we find

$$\begin{aligned}
&L_\mu(e^{Md}(d+\tilde{d}_K^2)\tilde{d}_K^{\gamma_+}) - L_\mu(d\tilde{d}_K^{\gamma_+ +\varepsilon}) \\
&\quad = -e^{Md}\tilde{d}_K^{\gamma_+}\Big((M^2 d + Md\Delta d + 2M + \Delta d) + (M^2 + M\Delta d)\tilde{d}_K^2\Big) \\
&\qquad - e^{Md} d\tilde{d}_K^{\gamma_+ -2}\Big(2M\gamma_+ d + \gamma_+ f + \gamma_+(\gamma_+ - 1)h + \mu g\Big) \\
&\qquad - e^{Md}\tilde{d}_K^{\gamma_+}\Big(2(\gamma_+ + k) + (\gamma_+ + 2)\Big((\gamma_+ + 1)h + f + 2Md\Big)\Big)
\end{aligned}$$

$$+\,\epsilon(2\gamma_+ + k + \epsilon) d\tilde{d}_K^{\gamma_+ + \epsilon - 2}$$
$$+\, d\tilde{d}_K^{\gamma_+ + \epsilon - 2}\Big((\gamma_+ + \epsilon)(\gamma_+ + \epsilon - 1)h + (\gamma_+ + \epsilon) f + \mu g\Big) + (\Delta d)\tilde{d}_K^{\gamma_+ + \epsilon}.$$

The RHS in the last equality consists of six terms. We now choose β_0 small enough and $M < 0$ so that the sum of the first, third and sixth terms is non-negative in $K_{\beta_0} \cap \Omega$. The fourth term is clearly positive, and by taking β_0 smaller if necessary it may also control the second and the fifth terms. Hence $L_\mu \eta_{\gamma_+,\varepsilon} \geq 0$ in $K_{\beta_0} \cap \Omega$.

The proofs of the other cases of the lemma are similar and are omitted. For (iii) we also use the relations

$$\Delta \ln \tilde{d}_K = \frac{\Delta \tilde{d}_K}{\tilde{d}_K} - \frac{|\nabla \tilde{d}_K|^2}{\tilde{d}_K^2}$$
$$\nabla \ln \tilde{d}_K \cdot \nabla(e^{Md} d\tilde{d}_K^b) = \tilde{d}_K^{b-2} e^{Md}\Big(Md^2 + d + b\, d|\nabla \tilde{d}_K|^2\Big)$$

and

$$-\,L_\mu\big((-\ln \tilde{d}_K)e^{Md} d\tilde{d}_K^b\big) = (-\ln \tilde{d}_K)e^{Md}\tilde{d}_K^b\,\Big(M^2 d + Md\Delta d + 2M + \Delta d\Big)$$
$$+\,(-\ln \tilde{d}_K)e^{Md} d\tilde{d}_K^{b-1}\left(\frac{2Mbd + bf + b(b-1)h + \mu g}{\tilde{d}_K}\right)$$
$$+\,(-\ln \tilde{d}_K)\big(b(k+1) + b(b-1) + \mu\big)e^{Md} d\tilde{d}_K^{b-2}$$
$$+\,e^{Md} d\tilde{d}_K^{b-2}\Big(-2Md - f + (1-2b)h - 2b - k\Big).$$

□

Lemma 6.2 *Let $\beta_0 > 0$ be the constant in Lemma 6.1, $\xi \in \partial\Omega$ and $0 < r < \frac{\beta_0}{16}$. We assume that $u \in H^1_{loc}(B_r(\xi) \cap \Omega) \cap C(B_r(\xi) \cap \Omega)$ is L_μ-harmonic in $B_r(\xi) \cap \Omega$ and*

$$\lim_{\mathrm{dist}(x,F)\to 0} \frac{u(x)}{\tilde{W}(x)} = 0, \quad \forall \text{ compact } F \subset B_r(\xi) \cap \partial\Omega. \tag{6.2}$$

Then there exists $C = C(u, \Omega, K, r) > 0$ such that

$$|u| \leq C\phi_\mu, \qquad x \in B_{\frac{r}{4}}(\xi) \cap \Omega. \tag{6.3}$$

Moreover, if $0 \leq \eta_r \leq 1$ is a smooth function with compact support in $B_{\frac{r}{2}}(\xi)$ with $\eta_r = 1$ on $B_{\frac{r}{4}}(\xi)$, then

$$\frac{\eta_r u}{\phi_\mu} \in H^1_0(\Omega; \phi_\mu^2). \tag{6.4}$$

Furthermore, if u is nonnegative there exists $c_1 = c_1(\Omega, K) > 0$ such that

$$\frac{u(x)}{\phi_\mu(x)} \leq c_1 \frac{u(y)}{\phi_\mu(y)}, \quad \forall x, y \in B_{\frac{r}{16}}(\xi) \cap \Omega. \tag{6.5}$$

Proof We will only consider the case $\mu < k^2/4$ and $\xi \in K_{\frac{\beta}{16}} \cap \partial\Omega$; the proof of the other cases is very similar and we omit it.

Since u is L_μ- harmonic in $B_r(\xi) \cap \Omega$, by standard elliptic estimates we have that $u \in C^2(B_r(\xi) \cap \Omega)$. Set $w_l = \max\{u - l\eta_{\gamma_-,\varepsilon}, 0\}$ where $l > 0$ and $\eta_{\gamma_-,\varepsilon}$ is the supersolution in (6.1). Then by Kato's formula we have

$$L_\mu w_l \leq 0, \qquad \text{in } B_r(\xi) \cap \Omega.$$

Setting $v_l = \frac{w_l}{\phi_\mu}$, by straightforward calculations we have

$$-\operatorname{div}(\phi_\mu^2 \nabla v_l) + \lambda_\mu \phi_\mu^2 v_l \leq 0, \qquad \text{in } B_r(\xi) \cap \Omega. \tag{6.6}$$

We note here that $v_l = 0$ if $u \leq l\eta_{\alpha_+,\varepsilon}$, thus by the assumptions we can easily obtain that $v_l \in H^1(B_{\frac{r}{2}}(\xi); \phi_\mu^2)$.

By Theorem 4.15, we can prove the existence of a constant r_{β_0} and $C = C(K) > 0$ such that for any $r' \leq \min\{\frac{r}{2}, r_{\beta_0}\}$ and $p \geq 1$ the following inequality holds

$$\sup_{x \in B_{\frac{r'}{2}}(\xi) \cap \Omega} v_l \leq C \left(\left(\int_{B_{r'}(\xi) \cap \Omega} \phi_\mu^2 dx \right)^{-1} \int_{B_{r'}(\xi) \cap \Omega} |v_l|^p \phi_\mu^2 dx \right)^{\frac{1}{p}}. \tag{6.7}$$

From (6.2) and the definition of w_l, we have

$$w_l \leq u_+ \leq C\tilde{W} = C(d + \tilde{d}_K^2)\tilde{d}_K^{\gamma_-}, \quad \text{in } B_{\frac{r}{2}}(\xi) \cap \Omega.$$

This and (2.2) imply that

$$\int_{B_{r'}(\xi) \cap \Omega} |v_l| \phi_\mu^2 dx \leq \int_{B_{\frac{r}{2}}(\xi) \cap \Omega} |w_l| \phi_\mu dx$$
$$\leq C \int_{B_{\frac{r}{2}}(\xi) \cap \Omega} (d + \tilde{d}_K^2) d \tilde{d}_K^{-k} dx \leq C \int_{B_{\frac{r}{2}}(\xi) \cap \Omega} d_K^{2-k} dx < \infty.$$

Thus by (6.7) and the last inequality we deduce that

$$\sup_{B_{\frac{r'}{2}}(\xi) \cap K} v_l < C_1$$

for some constant $C_1 > 0$ which does not depend on l. Thus

$$w_l \leq C_1\phi_\mu\ , \qquad \text{in } B_{\frac{r'}{2}}(\xi)\cap\Omega.$$

By letting $l \to 0$, we derive

$$u_+ \leq C_1\phi_\mu\ , \qquad \text{in } B_{\frac{r'}{2}}(\xi)\cap\Omega.$$

Thus by a covering argument we can find a constant $C_2 > 0$ such that

$$u_+ \leq C_2\phi_\mu\ , \qquad \text{in } B_{\frac{r}{2}}(\xi)\cap\Omega. \tag{6.8}$$

This implies $v_0 := \frac{u_+}{\phi_\mu} < C_2$ in $B_{\frac{r}{2}}(\xi)\cap\Omega$.

Using $\eta_r^2 v_l$ as a test function in (6.6) we can easily obtain

$$\int_{B_{\frac{r}{2}}(\xi)\cap\Omega} |\nabla(\eta_r v_l)|^2\phi_\mu^2 dx + \lambda_\mu \int_{B_{\frac{r}{2}}(\xi)\cap\Omega} |\eta_r v_l|^2\phi_\mu^2 dx \leq \frac{C}{r^2}\int_{B_{\frac{r}{2}}(\xi)\cap\Omega} |v_l|^2\phi_\mu^2 dx.$$

By (6.8) and by letting $l \to 0$ we obtain that $\eta_r v_0 \in H^1(\Omega;\phi_\mu^2)$, which in turn implies that $\frac{\eta_r u_+}{\phi_\mu} \in H^1(\Omega;\phi_\mu^2)$. Applying the same argument to $-u$ we obtain

$$u_- \leq C_2\phi_\mu \quad \text{in } B_{\frac{r}{2}}(\xi)\cap\Omega,$$

and $\frac{\eta_r u_-}{\phi_\mu} \in H^1(\Omega;\phi_\mu^2)$. By using the fact that $u = u_+ - u_-$, we obtain (6.4) and (6.3).

We next prove the boundary Harnack inequality (6.5). Let u be a nonnegative L_μ-harmonic function and put $v = \frac{u}{\phi_\mu}$. Then $v \in H^1(B_{\frac{r}{4}}(\xi);\phi_\mu^2)$ and v satisfies

$$-\phi_\mu^{-2}\text{div}(\phi_\mu^2\nabla v) + \lambda_\mu v = 0, \qquad \text{in } B_{\frac{r}{4}}(\xi)\cap\Omega.$$

The function $\hat{v}(x,t) := e^{\lambda_\mu t}v(x)$ then satisfies

$$\partial_t\hat{v} - \phi_\mu^{-2}\text{div}(\phi_\mu^2\nabla\hat{v}) = 0, \quad \text{in } B_{\frac{r}{4}}(\xi)\cap\Omega\times\left(0,\frac{r^2}{16}\right).$$

By the Harnack inequality (2.4),

$$\begin{aligned}&\text{ess sup}\left\{\hat{v}(t,x) : (t,x)\in\left(\frac{r^2}{64},\frac{r^2}{32}\right)\times\mathcal{B}\left(\xi,\frac{r}{8}\right)\cap\Omega\right\}\\&\quad\leq C\,\text{ess inf}\left\{\hat{v}(t,x) : (t,x)\in\left(\frac{3r^2}{64},\frac{r^2}{16}\right)\times\mathcal{B}\left(\xi,\frac{r}{8}\right)\cap\Omega\right\}.\end{aligned}$$

This implies (6.5). □

Lemma 6.3 *Let $\mu \le k^2/4$ and assume that $\lambda_\mu > 0$. Let $u \in H^1_{loc}(\Omega) \cap C(\Omega)$ be L_μ-subharmonic in Ω. Assume that*

$$\limsup_{\mathrm{dist}(x,F)\to 0} \frac{u(x)}{\tilde{W}(x)} \le 0, \quad \forall \text{ compact } F \subset \partial\Omega. \tag{6.9}$$

Then $u \le 0$ in Ω.

Proof First we note that $u_+ = \max(u(x), 0)$ is a nonnegative L_μ-subharmonic function in Ω. Let $v = \frac{u_+}{\phi_\mu}$. In view of the proof of (6.4), $v \in H^1_0(\Omega; \phi_\mu^2)$; moreover by a straightforward calculation we have

$$-\operatorname{div}(\phi_\mu^2 \nabla v) + \lambda_\mu \phi_\mu^2 v \le 0 \quad \text{in } \Omega. \tag{6.10}$$

Since $v \in H^1_0(\Omega; \phi_\mu^2)$, we can use it as a test function for (6.10) and obtain

$$\int_\Omega |\nabla v|^2 \phi_\mu^2 dx + \lambda_\mu \int_\Omega |v|^2 \phi_\mu^2 dx \le 0.$$

Hence $v = 0$ and the result follows. □

6.2 Existence and uniqueness

The aim of this subsection is to prove existence and uniqueness of the solution of $L_\mu u = f$, with smooth boundary data. We also prove the boundary Harnack inequalities and maximum principle for the operator L_μ. Let us first define the notion of a weak solution.

Definition 6.4 Let $f \in L^2(\Omega)$. We say that u is a weak solution of

$$L_\mu u = f\,, \quad \text{in } \Omega \tag{6.11}$$

if $\frac{u}{\phi_\mu} \in H^1_0(\Omega; \phi_\mu^2)$ and

$$\int_\Omega \nabla u \cdot \nabla \psi \, dx - \mu \int_\Omega \frac{u\psi}{d_K^2} dx = \int_\Omega f \psi \, dx, \qquad \forall \psi \in C_c^\infty(\Omega).$$

In the next lemma we give the first existence and uniqueness result.

Lemma 6.5 *Let $\mu \le k^2/4$ and assume that $\lambda_\mu > 0$. For any $f \in L^2(\Omega)$ there exists a unique weak solution u of (6.11). Furthermore there holds*

$$\int_\Omega u^2 dx \le C \int_\Omega f^2 dx, \tag{6.12}$$

where $C = C(\lambda_\mu) > 0$.

Proof We first observe that u is a weak solution of (6.11) if and only if $v = \frac{u}{\phi_\mu}$ satisfies

$$\int_\Omega \phi_\mu^2 \nabla v \cdot \nabla \zeta dx + \lambda_\mu \int_\Omega \phi_\mu^2 v \zeta dx = \int_\Omega \phi_\mu f \zeta dx \ , \qquad \forall \zeta \in H_0^1(\Omega; \phi_\mu^2). \tag{6.13}$$

We define on $H_0^1(\Omega; \phi_\mu^2)$ the inner product

$$\langle \psi, \zeta \rangle_{\phi_\mu^2} = \int_\Omega \phi_\mu^2 (\nabla \psi \cdot \nabla \zeta + \lambda_\mu \psi \zeta) dx$$

and consider the bounded linear functional T_f on $H_0^1(\Omega; \phi_\mu^2)$ given by

$$T_f(\zeta) = \int_\Omega \phi_\mu f \zeta dx.$$

Then (6.13) becomes

$$\langle v, \zeta \rangle_{\phi_\mu^2} = T_f(\zeta) \quad \forall \zeta \in H_0^1(\Omega; \phi_\mu^2). \tag{6.14}$$

By Riesz representation theorem there exists a unique function $v \in H_0^1(\Omega; \phi_\mu^2)$ satisfying (6.14). Furthermore, by choosing $\zeta = v$ in (6.13) and then using Young's inequality, we obtain

$$\int_\Omega \phi_\mu^2 |\nabla v|^2 dx + \frac{\lambda_\mu}{2} \int_\Omega \phi_\mu^2 v^2 dx \le C(\lambda_\mu) \int_\Omega f^2 dx. \tag{6.15}$$

By putting $u = \phi_\mu v$, we deduce that u is the unique weak solution of (6.11). Moreover, (6.12) follows from (6.15). □

The next lemma will be useful in order to prove existence and uniqueness of solution for the equation $L_\mu u = f$ with zero boundary data.

Lemma 6.6 [29, Lemma 5.3] *Let $\gamma < N$ and $\alpha \in (0, \min\{k, \gamma\})$. There exists a positive constant $C = C(\alpha, \gamma, \Omega, K)$ such that*

$$\sup_{x \in \Omega} \int_\Omega |x - y|^{-N+\gamma} d_K^{-\alpha}(y) dy < C.$$

In the following lemma we prove the existence of solution for the equation $L_\mu u = f$ with zero boundary data, as well as pointwise estimates.

Lemma 6.7 *Let $\mu \le k^2/4$ and assume that $\lambda_\mu > 0$, $\gamma_- - 1 < b < 0$ and $f \in L^\infty(\Omega)$. Then there exists a unique $u \in H^1_{loc}(\Omega) \cap C(\Omega)$ which satisfies $L_\mu u = f d_K^b$ in the sense of distributions as well as (6.9). Moreover, for any $\gamma \in (-\infty, \gamma_+] \cap (-\infty, b + 1) \cap (-\infty, 0]$ there exists a positive constant $C = C(\Omega, K, b, \mu, \gamma)$ such that*

$$|u(x)| \le C \|f\|_{L^\infty(\Omega)} d(x) d_K^\gamma(x), \qquad x \in \Omega. \tag{6.16}$$

Proof We assume first that $f \geq 0$. Set $f_n = \min\{f d_K^b, n\}$. By Lemma 6.5, there exists a unique solution u_n of $L_\mu v = f_n$ in Ω. Furthermore, a standard argument yields the representation formula

$$u_n(x) = \int_\Omega G_\mu(x, y) f_n(y) dy.$$

We assume first that $0 < \mu < \frac{k^2}{4}$. By (5.4) we have

$$\begin{aligned}
0 &\leq \int_\Omega G_\mu(x, y) f_n(y) dy \\
&\leq C_1 \int_\Omega \min\left\{\frac{1}{|x-y|^{N-2}}, \frac{d(x)d(y)}{|x-y|^N}\right\} \\
&\quad \times \left(\frac{d_K(x) d_K(y)}{(d_K(x) + |x-y|)(d_K(y) + |x-y|)}\right)^{\gamma_+} f_n(y) dy \\
&\leq C d_K^{\gamma_+}(x) \int_\Omega |x-y|^{-N+2-2\gamma_+} \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\} d_K^{\gamma_+}(y) f_n(y) dy \\
&\quad + C \int_\Omega |x-y|^{-N+2-\gamma_+} \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\} d_K^{\gamma_+}(y) f_n(y) dy \\
&\quad + C d_K^{\gamma_+}(x) \int_\Omega |x-y|^{-N+2-\gamma_+} \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\} f_n(y) dy \\
&\quad + C \int_\Omega |x-y|^{-N+2} \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\} f_n(y) dy \\
&= C(I_1 + I_2 + I_3 + I_4).
\end{aligned}$$

First we note that if $d_K(y) \leq \frac{1}{4} d_K(x)$ then $|x-y| \geq \frac{3}{4} d_K(x)$. Thus for $\gamma \leq \gamma_+$, we have

$$\begin{aligned}
I_1 &= d_K^{\gamma_+}(x) \int_{\Omega \cap \{d_K(y) \leq \frac{1}{4} d_K(x)\}} |x-y|^{-N+2-2\gamma_+} \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\} d_K^{\gamma_+}(y) f_n(y) dy \\
&\quad + d_K^{\gamma_+}(x) \int_{\Omega \cap \{d_K(y) > \frac{1}{4} d_K(x)\}} |x-y|^{-N+2-2\gamma_+} \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\} d_K^{\gamma_+}(y) f_n(y) dy \\
&\leq C \|f\|_{L^\infty(\Omega)} d_K^\gamma(x) \int_{\Omega \cap \{d_K(y) \leq \frac{1}{4} d_K(x)\}} |x-y|^{-N+2-\gamma-\gamma_+} \\
&\quad \times \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\} d_K^{b+\gamma_+}(y) dy \\
&\quad + C \|f\|_{L^\infty(\Omega)} d_K^\gamma(x) \int_{\Omega \cap \{d_K(y) > \frac{1}{4} d_K(x)\}} |x-y|^{-N+2-2\gamma_+} \\
&\quad \times \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\} d_K^{b-\gamma+2\gamma_+}(y) dy
\end{aligned}$$

$$\leq C\|f\|_{L^\infty(\Omega)} d_K^{\gamma}(x) d(x) \int_{\Omega\cap\{d_K(y)\leq \frac{1}{4}d_K(x)\}} |x-y|^{-N-\gamma-\gamma_+} d_K^{b+\gamma_++1}(y)dy$$
$$+ C\|f\|_{L^\infty(\Omega)} d_K^{\gamma}(x) d(x) \int_{\Omega\cap\{d_K(y)> \frac{1}{4}d_K(x)\}} |x-y|^{-N-2\gamma_+} d_K^{b-\gamma+2\gamma_++1}(y)dy$$
$$\leq C\|f\|_{L^\infty(\Omega)} d_K^{\gamma}(x) d(x)$$

where in the last inequalities we have used Lemma 6.6.

Similarly we can prove that

$$I_1 + I_2 + I_3 + I_4 \leq C\|f\|_{L^\infty(\Omega)} d_K^{\gamma}(x) d(x).$$

Combining the above estimates, we deduce that for any $\gamma \in (-\infty, \gamma_+] \cap (-\infty, b+1)$, there exists a positive constant $C = C(\Omega, K, \mu, b, \gamma)$ such that

$$|u_n(x)| \leq C\|f\|_{L^\infty(\Omega)} d(x) d_K^{\gamma}(x), \qquad x \in \Omega. \tag{6.17}$$

If we choose $\gamma \in (\gamma_-, \gamma_+] \cap (\gamma_-, b+1)$, then we can show that

$$\lim_{\mathrm{dist}(x,F)\to 0} \frac{d(x) d_K^{\gamma}(x)}{\tilde{W}(x)} = 0, \quad \forall \text{ compact } F \subset \partial\Omega. \tag{6.18}$$

Thus by the above inequality, (6.17) and applying Lemma 6.3, we can easily show that $u_n \nearrow u$ locally uniformly in Ω and in $H^1_{loc}(\Omega)$. Furthermore, by standard elliptic theory $u \in C^1(\Omega)$ and, by (6.17),

$$|u(x)| \leq C\|f\|_{L^\infty(\Omega)} d(x) d_K^{\gamma}(x), \qquad x \in \Omega. \tag{6.19}$$

The uniqueness follows by (6.18), (6.19) and Lemma 6.3.

For the general case, we set $u = u_+ - u_-$ where $u_\pm$ are the unique solutions of $L_\mu v = f_\pm d_K^{-b}$ in Ω respectively, which satisfy (6.16). Thus u satisfies (6.16) and the result follows in the case $0 < \mu < \frac{k^2}{4}$.

The proof in the cases $\mu = \frac{k^2}{4}$ and $\mu \leq 0$ is similar and is omitted. □

The following lemma is the main result of this subsection.

Lemma 6.8 *Let $\mu \leq k^2/4$ and assume that $\lambda_\mu > 0$. For any $h \in C(\partial\Omega)$ there exists a unique L_μ-harmonic function $u \in H^1_{loc}(\Omega) \cap C(\Omega)$ satisfying*

$$\lim_{x\in\Omega,\ x\to y\in\partial\Omega} \frac{u(x)}{\tilde{W}(x)} = h(y) \qquad \text{uniformly in } y \in \partial\Omega.$$

Furthermore there exists a constant $c = c(\Omega, K) > 0$

$$\left\| \frac{u}{\tilde{W}} \right\|_{L^\infty(\Omega)} \leq c\|h\|_{C(\partial\Omega)}.$$

Proof Uniqueness is a consequence of Lemma 6.3.

Existence. We will only consider the case $0 < \mu < \frac{k^2}{4}$, the proof in the other cases is very similar. First we assume that $h \in C^2(\overline{\Omega})$. Then a function $u \in C^2(\Omega)$ is L_μ-harmonic if and only if $v := \tilde{W}h - u$ is a solution of

$$L_\mu v = L_\mu(\tilde{W}h) = h(L_\mu \tilde{W}) - 2\nabla\tilde{W}\cdot\nabla h - \tilde{W}\Delta h, \qquad \text{in } \Omega\,. \tag{6.20}$$

Arguing as in the proof of Lemma 6.1 we see that there exists $C = C(\Omega, K, \mu, \beta_0)$ such that

$$|L_\mu \tilde{W}| \le C d_K^{\gamma_-}, \qquad \text{in } \Omega.$$

Hence (6.20) can be written as

$$L_\mu v = f d_K^{\gamma_-}, \qquad \text{in } \Omega,$$

with $\|f\|_{L^\infty(\Omega)} \le C(\gamma_-, \Omega, K)\,\|h\|_{C^2(\overline{\Omega})}$.

By Lemma 6.7 there exists a unique solution v of (6.20) that satisfies

$$|v(x)| \le C\|h\|_{C^2(\overline{\Omega})} d(x) d_K^{\gamma}(x), \qquad x \in \Omega,$$

for any $\gamma \in (\gamma_-, \gamma_+] \cap (\gamma_-, \gamma_- + 1)$. Thus

$$\left|\frac{u(x)}{\tilde{W}(x)} - h(x)\right| \le C\|h\|_{C^2(\overline{\Omega})}\frac{d(x)d_K^{\gamma}(x)}{\tilde{W}(x)}, \qquad x \in \Omega, \tag{6.21}$$

and the desired result follows in this case, since

$$\lim_{\mathrm{dist}(x,F)\to 0}\frac{d(x)d_K^{\gamma}(x)}{\tilde{W}(x)} = 0, \qquad \forall \text{ compact } F \subset \partial\Omega.$$

for any $\gamma \in (\gamma_-, \gamma_+] \cap (\gamma_-, \gamma_- + 1)$.

Suppose now that $h \in C(\partial\Omega)$. We can then find a sequence $\{h_n\}_{n=1}^{\infty}$ of smooth functions in $\partial\Omega$ such that $h_n \to h$ in $L^\infty(\partial\Omega)$. Then there exist $H_n \in C^2(\overline{\Omega})$ with value $H_n|_{\partial\Omega} = h_n$ and $\|H_n\|_{L^\infty(\overline{\Omega})} \le C\|h_n\|_{L^\infty(\partial\Omega)}$ where C does not depend on n or h_n. By the previous case there exists a unique weak solution u_n of $L_\mu u = 0$ satisfying

$$\left|\frac{u_n(x)}{\tilde{W}(x)} - H_n(x)\right| \le C\|H_n\|_{C^2(\overline{\Omega})}\frac{d(x)d_K^{\gamma}(x)}{\tilde{W}(x)}, \qquad \forall x \in \Omega,$$

for some C which does not depend on n and h_n.

By (6.21) and Lemma 6.3, we can easily show that

$$\left|\frac{u_n(x) - u_m(x)}{\tilde{W}(x)}\right| \le C\|h_n - h_m\|_{L^\infty(\partial\Omega)}, \qquad x \in \Omega;$$

thus $u_n \to u$ locally uniformly in Ω.

Now, let $y \in \partial\Omega$. Then

$$\left|\frac{u(x)}{\tilde{W}(x)} - h(y)\right| \le \left|\frac{u(x) - u_n(x)}{\tilde{W}(x)}\right| + \left|\frac{u_n(x)}{\tilde{W}(x)} - h_n(y)\right| + |h_n(y) - h(y)|$$

and the result follows by letting successively $x \to y$ and $n \to \infty$. □

7 Martin kernel

7.1 L_μ-harmonic measure

Let $x_0 \in \Omega$, $h \in C(\partial\Omega)$ and denote $L_{\mu,x_0}(h) := v_h(x_0)$ where v_h is the solution of the Dirichlet problem (see Lemma 6.8)

$$\begin{cases} L_\mu v = 0, & \text{in } \Omega, \\ \widetilde{\mathrm{tr}}(v) = h, & \text{in } \partial\Omega, \end{cases}$$

where $\widetilde{\mathrm{tr}}(v) = h$ is understood in the sense of Lemma 6.8 (cf. also (2.6)). By Lemma 6.3, the mapping $h \mapsto L_{\mu,x_0}(h)$ is a positive linear functional on $C(\partial\Omega)$. Thus there exists a unique Borel measure on $\partial\Omega$, called *L_μ-harmonic measure* in Ω, denoted by ω^{x_0}, such that

$$v_h(x_0) = \int_{\partial\Omega} h(y) d\omega^{x_0}(y).$$

Thanks to the Harnack inequality the measures ω^x and ω^{x_0}, $x_0,\ x \in \Omega$, are mutually absolutely continuous. For every fixed x we denote the Radon–Nikodyn derivative by

$$K_\mu(x, y) := \frac{dw^x}{dw^{x_0}}(y), \qquad \text{for } \omega^{x_0}\text{- almost all } y \in \partial\Omega. \tag{7.1}$$

Let $\xi \in \partial\Omega$. We set $\Delta_r(\xi) = \partial\Omega \cap B_r(\xi)$ and denote by $x_r = x_r(\xi)$ the point in Ω determined by $d(x_r) = |x_r - \xi| = r$. We recall here that $\beta_0 = \beta_0(\Omega, K, \mu) > 0$ is small enough and has been defined in Lemma 6.1.

Lemma 7.1 *Let $\mu \le k^2/4$ and assume that $\lambda_\mu > 0$. Let $0 < r \le \beta_0$. We assume that u is a positive L_μ-harmonic function in Ω such that*

(i) $\frac{u}{\tilde{W}} \in C(\overline{\Omega \setminus B_r(\xi)})$,

(ii) $\lim_{x \in \Omega, x \to x_0} \frac{u(x)}{\tilde{W}(x)} = 0, \quad \forall x_0 \in \partial\Omega \setminus \overline{B_r(\xi)}$, *uniformly with respect to x_0.*

Then

$$\begin{aligned} c^{-1}\frac{u(x_r(\xi))}{G_\mu(x_r(\xi),x_{\frac{r}{16}}(\xi))}G_\mu(x,x_{\frac{r}{16}}(\xi)) &\le u(x) \\ \le c\frac{u(x_r(\xi))}{G_\mu(x_r(\xi),x_{\frac{r}{16}}(\xi))}G_\mu(x,x_{\frac{r}{16}}(\xi)), &\qquad \forall x\in\Omega\backslash\overline{B_{2r}(\xi)}, \end{aligned} \tag{7.2}$$

with $c>1$ *depending only on* Ω, K *and* μ.

Proof It follows from Lemma 6.2 that there exists $c>1$ such that

$$\begin{aligned} c^{-1}\frac{u(x_{2r}(\xi))}{G_\mu(x_{2r}(\xi),x_{\frac{r}{16}}(\xi))}G_\mu(x,x_{\frac{r}{16}}(\xi)) &\le u(x) \\ \le c\frac{u(x_{2r}(\xi))}{G_\mu(x_{2r}(\xi),x_{\frac{r}{16}}(\xi))}G_\mu(x,x_{\frac{r}{16}}(\xi)), &\qquad \forall x\in\Omega\cap\partial B_{2r}(\xi), \end{aligned}$$

Applying Harnack inequality between $x_{2r}(\xi)$ and $x_r(\xi)$ we obtain

$$\begin{aligned} c^{-1}\frac{u(x_r(\xi))}{G_\mu(x_r(\xi),x_{\frac{r}{16}}(\xi))}G_\mu(x,x_{\frac{r}{16}}(\xi)) &\le u(x) \\ \le c\frac{u(x_r(\xi))}{G_\mu(x_r(\xi),x_{\frac{r}{16}}(\xi))}G_\mu(x,x_{\frac{r}{16}}(\xi)), &\qquad \forall x\in\Omega\cap\partial B_{2r}(\xi). \end{aligned}$$

For $\varepsilon>0$ let

$$u_\varepsilon(x)=u(x)-c\frac{u(x_r(\xi))}{G_\mu(x_r(\xi),x_{\frac{r}{16}}(\xi))}G_\mu(x,x_{\frac{r}{16}}(\xi))-\varepsilon v_1(x),$$

where c is as above. Then u_ε is L_μ-harmonic and the function $u_\varepsilon^+=\max(u_\varepsilon,0)$ has compact support in $\Omega\backslash\overline{B_{2r}(\xi)}$. Set $v_\varepsilon=\frac{u_\varepsilon}{\phi_\mu}$ and $v_\varepsilon^+=\frac{u_\varepsilon^+}{\phi_\mu}$. Using u_ε^+ as a test function we obtain

$$\int_{\Omega\backslash\overline{B_{2r}(\xi)}}\nabla v_\varepsilon\cdot\nabla v_\varepsilon^+\phi_\mu^2dx+\lambda_\mu\int_{\Omega\backslash\overline{B_{2r}(\xi)}}v_\varepsilon v_\varepsilon^+\phi_\mu^2dx=0.$$

Letting $\varepsilon\to 0$ in the above equation we get

$$\lambda_\mu\int_\Omega|v^+|^2\phi_\mu^2dx\le 0,$$

hence $u(x)-c\frac{u(x_r(\xi))}{G_\mu(x_r(\xi),x_{\frac{r}{16}}(\xi))}G_\mu(x,x_{\frac{r}{16}}(\xi))\le 0$ for all $x\in\Omega\backslash\overline{B_{2r}(\xi)}$. The proof of the lower estimate in (7.2) is similar and we omit it. □

7.2 The Poisson kernel of L_μ

In this section we establish some properties of the Poisson kernel associated to L_μ.

Definition 7.2 A function $\mathcal{K}$ defined in Ω is called a kernel function for L_μ with pole at $\xi \in \partial\Omega$ and basis at $x_0 \in \Omega$ if

(i) $\mathcal{K}(\cdot,\xi)$ is L_μ-harmonic in Ω,
(ii) $\frac{\mathcal{K}(\cdot,\xi)}{\tilde{W}(\cdot)} \in C(\overline{\Omega}\setminus\{\xi\})$ and for any $\eta \in \partial\Omega\setminus\{\xi\}$ we have $\lim_{x\in\Omega,\, x\to\eta} \frac{\mathcal{K}(x,\xi)}{\tilde{W}(x)} = 0$,
(iii) $\mathcal{K}(x,\xi) > 0$ for each $x \in \Omega$ and $\mathcal{K}(x_0,\xi) = 1$.

Proposition 7.3 *Assume that $\lambda_\mu > 0$. There exists a unique kernel function for L_μ with pole at ξ and basis at x_0.*

Proof The proof is similar to that of [12, Theorem 3.1] and we include it for the sake of completeness.

Existence. We shall prove that the function $K_\mu(x,\xi)$ defined by (7.1) has the required properties.

Fix $\xi \in \partial\Omega$. Set

$$u_n(x) = \frac{\omega^x(\Delta_{2^{-n}}(\xi))}{\omega^{x_0}(\Delta_{2^{-n}}(\xi))}, \qquad \forall n \in \mathbb{N}.$$

Clearly $u_n(x) \to K_\mu(x,\xi)$, $x \in \Omega$. Since $u_n \geq 0$, $L_\mu u_n = 0$ in Ω and $u_n(x_0) = 1$ the sequence $\{u_n\}$ is locally bounded in Ω by Harnack inequality. Hence we can find a subsequence, again denoted by $\{u_n\}$, which converges to $K_\mu(\cdot,\xi)$ locally uniformly in Ω.

Let $\eta \in \partial\Omega\setminus\{\xi\}$ and let $n_1 \in \mathbb{N}$ be such that $\eta \in \partial\Omega\setminus\overline{B_{2^{-n+1}}(\xi)}$, $\forall n \geq n_1$. By Lemma 7.1 we have

$$u_n(x) \leq c\frac{u_n(x_{2^{-n_1}}(\xi))}{G_\mu(x_{2^{-n_1}}, x_{2^{-n_1-4}}(\xi))} G_\mu(x, x_{2^{-n_1-4}}(\xi)), \qquad \forall x \in \Omega\setminus\overline{B_{2^{-n_1+1}}(\xi)},$$

which implies

$$K_\mu(x,\xi) \leq c\frac{u_n(x_{2^{-n_1}}(\xi))}{G_\mu(x_{2^{-n_1}}, x_{2^{-n_1-4}}(\xi))} G_\mu(x, x_{2^{-n_1-4}}(\xi)), \qquad \forall x \in \Omega\setminus\overline{B_{2^{-n_1+1}}(\xi)}.$$

It follows that

$$\lim_{x\in\Omega,\, x\to\eta} \frac{K_\mu(x,\xi)}{\tilde{W}(x)} = 0,$$

hence $K_\mu(x,\xi)$ is a kernel function for L_μ with pole at ξ and basis at x_0.
Uniqueness. Assume f and g are two kernel functions for L_μ in Ω with pole at ξ and basis at x_0. Let $0 < r < \beta_0$. By Lemma 7.1 and the properties of f and g there holds

$$\frac{1}{c'}\frac{f(x_r(\xi))}{g(x_r(\xi))} \leq \frac{f(x)}{g(x)} \leq c'\frac{f(x_r(\xi))}{g(x_r(\xi))}, \qquad \forall x \in \Omega\setminus\overline{B_{2r}(\xi)}.$$

In particular we can obtain if we take $x = x_0$

$$\frac{f(x_r(\xi))}{g(x_r(\xi))} \leq c',$$

and hence

$$\frac{f(x)}{g(x)} \leq c'^2 =: c\,, \qquad \forall x \in \Omega.$$

We derive that for any two kernel functions f and g for L_μ with pole at ξ and basis at x_0 there holds

$$f(x) \leq cg(x) \leq c^2 f(x)\,, \qquad x \in \Omega.$$

Obviously $c \geq 1$. If $c = 1$ the result is proved. If $c > 1$ then we set $A = \frac{1}{c-1}$ and $f + A(f - g)$ is also a kernel function for L_μ with pole at ξ and basis at x_0. Repeating the argument for the functions $f + A(f - g)$ and g we obtain that

$$f + A(f - g) + A\big(f - g + A(f - g)\big),$$

is also a kernel function with pole at ξ and basis at x_0. Proceeding in this manner we conclude that for each positive integer k there exist nonnegative numbers $a_{1k}, \ldots, a_{kk}$ such that

$$f + \left(kA + \sum_{i=1}^{k} a_{ik}\right)(f - g)$$

is a kernel function with pole at ξ and basis at x_0. Hence

$$f + \left(kA + \sum_{i=1}^{k} a_{ik}\right)(f - g) \leq cf.$$

This last inequality can hold for all k only if $f \equiv g$. □

Proposition 7.4 *Assume that $\lambda_\mu > 0$. For any $x \in \Omega$, the function $\xi \mapsto K_\mu(x, \xi)$ is continuous on $\partial\Omega$.*

Proof The proof is an adaptation of that of [12, Corollary 3.2]. Suppose that $\{\xi_n\}$ is a sequence converging to ξ. Then the sequence $\{K_\mu(\cdot, \xi_n)\}$ of positive solutions of $L_\mu u = 0$ in Ω has a subsequence which converges locally uniformly in Ω to a positive L_μ-harmonic function. Moreover, for any $r > 0$, $\frac{K_\mu(x,\xi_n)}{\tilde{W}(x)}$ converges to zero uniformly in n as $x \to \eta \in \partial\Omega \setminus B_r(\xi)$. Hence the limit function of the subsequence is the kernel function $K_\mu(x, \xi)$. By the uniqueness of the kernel function we conclude that the convergence

$$K_\mu(x, \xi_n) \to K_\mu(x, \xi)$$

holds for the entire sequence $\{\xi_n\}$. □

We can now identify the Martin boundary and topology with their classical analogues. We begin by recalling the definitions of the Martin boundary and related concepts.

Let $x_0 \in \Omega$ be fixed. For $x,\ y \in \Omega$ we set

$$\mathcal{K}_\mu(x, y) := \frac{G_\mu(x, y)}{G_\mu(x_0, y)}.$$

Consider the family of sequences $\{y_k\}_{k\geq 1}$ of points of Ω without cluster points in Ω for which $\mathcal{K}_\mu(x, y_k)$ converges in Ω to a L_μ-harmonic function, denoted by $\mathcal{K}_\mu(x, \{y_k\})$. Two such sequences y_k and y_k' are called equivalent if $\mathcal{K}_\mu(x, \{y_k\}) = \mathcal{K}_\mu(x, \{y_k'\})$ and each equivalence class is called an element of the Martin boundary Γ. If Y is such an equivalence class (i.e., $Y \in \Gamma$) then $\mathcal{K}_\mu(x, Y)$ will denote the corresponding harmonic limit function. Thus each $Y \in \Omega \cup \Gamma$ is associated with a unique function $\mathcal{K}_\mu(x, Y)$. The Martin topology on $\Omega \cup \Gamma$ is given by the metric

$$\rho(Y, Y') = \int_A \frac{|\mathcal{K}_\mu(x, Y) - \mathcal{K}_\mu(x, Y')|}{1 + |\mathcal{K}_\mu(x, Y) - \mathcal{K}_\mu(x, Y')|} dx, \qquad Y, Y' \in \Omega \cup \Gamma,$$

where A is a small enough neighbourhood of x_0. The function $\mathcal{K}_\mu(x, Y)$ is a ρ-continuous function of $Y \in \Omega \cup \Gamma$ for any fixed $x \in \Omega$. Moreover $\Omega \cup \Gamma$ is compact and complete with respect to ρ, $\Omega \cup \Gamma$ is the ρ-closure of Ω and the ρ-topology is equivalent to the Euclidean topology in Ω.

Proposition 7.5 *Assume that $\lambda_\mu > 0$. There is a one-to-one correspondence between the Martin boundary of Ω and the Euclidean boundary $\partial\Omega$. If $Y \in \Gamma$ corresponds to $\xi \in \partial\Omega$ then $\mathcal{K}_\mu(x, Y) = K_\mu(x, \xi)$. The Martin topology on $\Omega \cup \Gamma$ is equivalent to the Euclidean topology on $\Omega \cup \partial\Omega$.*

Proof The proof is similar as the one of Theorem 4.2 in [35] and we include it for the sake of completeness. By uniqueness of the kernel function we have that

$$\mathcal{K}_\mu(x, \{y_k\}) = K_\mu(x, \xi),$$

where $\{y_k\}$ is a sequence in Ω such that $y_k \to \xi \in \partial\Omega$. It follows that each point of Γ may be associated with a point of $\partial\Omega$. Lemma 7.1 clearly shows that $K_\mu(\cdot, \xi) \neq K_\mu(\cdot, \xi')$ if $\xi \neq \xi'$. Hence, the functions $\mathcal{K}_\mu(x, y_k)$ cannot converge if the sequence $\{y_k\}$ has more than one cluster point on $\partial\Omega$ and different points of $\partial\Omega$ must be associated with different points of Γ. This gives a one-to-one correspondence between $\partial\Omega$ and Γ with $\mathcal{K}_\mu(x, Y) = K_\mu(x, \xi)$ when $Y \in \Gamma$ corresponds to $\xi \in \partial\Omega$. If $\xi_k \to \xi$ in the Euclidean topology then $\mathcal{K}_\mu(x, Y_k) \to \mathcal{K}_\mu(x, Y)$ and, therefore, $Y_k \to Y$ in the ρ-topology by Lebesgue's dominated convergence theorem. On the other hand suppose that $Y_k \to Y$ in the ρ-topology. If ξ_k does not converge to ξ in the Euclidean topology there is a subsequence ξ_{k_j} such that $\xi_{k_j} \to \xi' \neq \xi$ in the Euclidean topology. Then $Y_{k_j} \to Y'$ and $Y_{k_j} \to Y$ in the ρ- topology with $Y \neq Y'$, which is impossible. Therefore, the Martin ρ-topology on $\Omega \cup \Gamma$ is equivalent to the Euclidean topology on $\Omega \cup \partial\Omega$. □

Proof of Theorem 2.8 The result follows immediately by Propositions 5.3 and 7.5. □

The next lemma will be used to prove the representation formula of Theorem 2.9.

Lemma 7.6 *Assume that $\lambda_\mu > 0$. Let $F \subset \partial\Omega$ and D be an open smooth neighbourhood of F. We assume $\Omega \cap D \subset \Omega_\beta$ for some $\beta > 0$. Let u be a positive L_μ-harmonic function in Ω. There exists a L_μ-superharmonic function V such that*

$$V(x) = \begin{cases} v(x), \ in \ \ \Omega \backslash D, \\ u(x), \ in \ \ \Omega \cap \overline{D}, \end{cases}$$

where v satisfies

$$\begin{cases} L_\mu v = 0, & in \ \ \Omega\backslash\overline{D}, \\ \lim_{x \in \Omega\backslash\overline{D},\ x \to y} v(x) = u(y), & \forall y \in \partial D \cap \Omega, \\ \lim_{x \in \Omega\backslash\overline{D},\ x \to y} \frac{v(x)}{\tilde{W}(x)} = 0, & \forall y \in \partial\Omega\backslash\overline{D}. \end{cases}$$

Proof The function u is C^2 in Ω since it is L_μ-harmonic. We assume that $\{r_n\}_{n=0}^\infty$ is a decreasing sequence $r_n \searrow 0$ and $r_1 < \frac{\beta_0}{16}$. We set $D_{r_n} = \{\xi \in \partial D \cap \Omega : d(\xi) > 2r_n\}$.

Let $0 \le \eta_n \le 1$ be a smooth function such that $\eta_n = 1$ in $\overline{D}_{r_n}$ with compact support in $D_{\frac{r_n}{2}}$. In view of the proof of Lemmas 6.5 and 6.8, for $m > n$, we can find a unique solution $v_{n,m}$ of

$$\begin{cases} L_\mu v = 0, & \text{in } (\Omega\backslash\overline{\Omega}_{\frac{r_m}{2}})\backslash\overline{D}, \\ \lim_{x\to y} v(x) = \eta_n(y)u(y), & \forall y \in \partial D \cap (\Omega\backslash\overline{\Omega}_{\frac{r_m}{2}}), \\ \lim_{x \to y} v(x) = 0, & \forall y \in (\partial\Omega_{\frac{r_m}{2}})\backslash\overline{D}. \end{cases}$$

By comparison principle we have $0 \le v_{n,m} \le u$ and $v_{n,m} \le v_{n,m+1}$. In addition, there exists a constant $c_n = c_n(\|u\|_{L^\infty(D_{\frac{r_n}{2}})}, \inf_{x \in D_{\frac{r_n}{2}}} \phi_\mu)$ such that

$$0 \le v_{n,m}(x) \le \min\{u(x), c_n\phi_\mu(x)\}, \qquad x \in (\Omega\backslash\overline{\Omega}_{\frac{r_m}{2}})\backslash\overline{D}.$$

Thus $v_{n,m}$ converges to some function v_n as $m \to \infty$ locally uniformly in $\Omega\backslash\overline{D}$ and

$$0 \le v_n(x) \le \min\{u(x), c_n\phi_\mu(x)\}, \qquad x \in \Omega\backslash\overline{D}, \quad n \in \mathbb{N}. \tag{7.3}$$

Let $\xi \in \partial\Omega\backslash\overline{D}$. By (7.3) and (6.5) there exists $r_0 < \frac{\text{dist}(\xi,\partial D)}{4}$ such that

$$\frac{v_n(x)}{\phi_\mu(x)} \le c\frac{v_n(y)}{\phi_\mu(y)} \le c\frac{u(y)}{\phi_\mu(y)}, \quad \forall x, y \in B_{\frac{r_0}{4}}(\xi) \cap \Omega.$$

Thus v_n converges to some function v locally uniformly in Ω. The desired result now follows easily. □

We consider a *smooth exhaustion* of Ω, that is an increasing sequence of bounded open smooth domains $\{\Omega_n\}$ such that $\overline{\Omega_n} \subset \Omega_{n+1}$, $\cup_n \Omega_n = \Omega$ and $\mathcal{H}^{N-1}(\partial\Omega_n) \to \mathcal{H}^{N-1}(\partial\Omega)$. The operator $L_\mu^{\Omega_n}$ defined by

$$L_\mu^{\Omega_n} u = -\Delta u - \frac{\mu}{d_K^2} u \tag{7.4}$$

is uniformly elliptic and coercive in $H_0^1(\Omega_n)$ and its first eigenvalue $\lambda_\mu^{\Omega_n}$ is larger than λ_μ. For $h \in C(\partial\Omega_n)$ the problem

$$\begin{cases} L_\mu^{\Omega_n} v = 0, \text{ in } \Omega_n, \\ v = h, \qquad \text{on } \partial\Omega_n, \end{cases}$$

admits a unique solution which allows to define the $L_\mu^{\Omega_n}$-harmonic measure on $\partial\Omega_n$ by

$$v(x_0) = \int_{\partial\Omega_n} h(y) d\omega_{\Omega_n}^{x_0}(y).$$

Thus the Poisson kernel of $L_\mu^{\Omega_n}$ is

$$K_{L_\mu^{\Omega_n}}(x, y) = \frac{d\omega_{\Omega_n}^{x}}{d\omega_{\Omega_n}^{x_0}}(y), \qquad x \in \Omega_n, \quad y \in \partial\Omega_n. \tag{7.5}$$

Proposition 7.7 *Assume that* $\lambda_\mu > 0$ *and* $x_0 \in \Omega_1$. *Then for every* $Z \in C(\overline{\Omega})$,

$$\lim_{n\to\infty} \int_{\partial\Omega_n} Z(x)\tilde{W}(x) d\omega_{\Omega_n}^{x_0}(x) = \int_{\partial\Omega} Z(x) d\omega^{x_0}(x). \tag{7.6}$$

Proof Let $n_0 \in \mathbb{N}$ be such that

$$\mathrm{dist}(\partial\Omega_n, \partial\Omega) < \frac{\beta_0}{16}, \qquad \forall n \geq n_0.$$

For $n \geq n_0$ let w_n be the solution of

$$\begin{cases} L_\mu^{\Omega_n} w_n = 0, \text{ in } \Omega_n, \\ w_n = \tilde{W}, \qquad \text{on } \partial\Omega_n. \end{cases}$$

In view of the proof of Lemma 6.8, there exists a positive constant $c = c(\Omega, K, \mu)$ such that

$$\left\| \frac{w_n}{\tilde{W}} \right\|_{L^\infty(\Omega_n)} \leq c, \qquad \forall n \geq n_0.$$

Furthermore

$$w_n(x_0) = \int_{\partial\Omega_n} \tilde{W}(x) d\omega_{\Omega_n}^{x_0}(x) < c. \tag{7.7}$$

We extend $\omega_{\Omega_n}^{x_0}$ to a Borel measure on $\overline{\Omega}$ by setting $\omega_{\Omega_n}^{x_0}(\overline{\Omega}\setminus\Omega_n) = 0$, and keep the notation $\omega_{\Omega_n}^{x_0}$ for the extension. Because of (7.7) the sequence $\{\tilde{W}\omega_{\Omega_n}^{x_0}\}$ is bounded in the space $\mathfrak{M}_b(\overline{\Omega})$ of bounded Borel measures in $\overline{\Omega}$. Thus there exists a subsequence, still denoted by $\{\tilde{W}\omega_{\Omega_n}^{x_0}\}$, which converges narrowly to some positive measure, say $\widetilde{\omega}$, which is clearly supported on $\partial\Omega$ and satisfies $\|\widetilde{\omega}\|_{\mathfrak{M}_b} \leq c$ by (7.7). Thus for every $Z \in C(\overline{\Omega})$ there holds

$$\lim_{n\to\infty} \int_{\partial\Omega_n} Z\,\tilde{W} d\omega_{\Omega_n}^{x_0} = \int_{\partial\Omega} Z d\widetilde{\omega}.$$

Setting $\zeta = Z\lfloor_{\partial\Omega}$ and

$$z(x) := \int_{\partial\Omega} K_\mu(x, y)\zeta(y) d\omega^{x_0}(y)$$

we then have

$$\lim_{d(x)\to 0} \frac{z(x)}{\tilde{W}(x)} = \zeta \qquad \text{and} \qquad z(x_0) = \int_{\partial\Omega} \zeta d\omega^{x_0}.$$

By Lemma 6.8, $\frac{z}{\tilde{W}} \in C(\overline{\Omega})$. Since $\frac{z}{\tilde{W}}\lfloor_{\partial\Omega_n}$ converges uniformly to ζ as $n \to \infty$, there holds

$$z(x_0) = \int_{\partial\Omega_n} z\lfloor_{\partial\Omega_n} d\omega_{\Omega_n}^{x_0} = \int_{\partial\Omega_n} \tilde{W} \frac{z\lfloor_{\partial\Omega_n}}{\tilde{W}} d\omega_{\Omega_n}^{x_0} \to \int_{\partial\Omega} \zeta d\tilde{\omega}, \qquad \text{as } n \to \infty.$$

It follows that

$$\int_{\partial\Omega} \zeta d\widetilde{\omega} = \int_{\partial\Omega} \zeta d\omega^{x_0}, \qquad \forall \zeta \in C(\partial\Omega).$$

Consequently $d\widetilde{\omega} = d\omega^{x_0}$. Because the limit does not depend on the subsequence it follows that the whole sequence $\tilde{W}(x) d\omega_{\Omega_n}^{x_0}$ converges weakly to ω^{x_0}. This implies (7.6). □

Proof of Theorem 2.9 The proof which is presented below follows the ideas of the one of [35, Th. 4.3]. Let B be a relatively closed subset of Ω. We define

$$R_u^B(x) := \inf\left\{\psi(x) : \ \psi \text{ is a nonnegative supersolution in } \Omega \text{ with } \psi \geq u \text{ on } B\right\}.$$

For a closed subset F of $\partial\Omega$, we define

$$\nu^x(F) := \inf\left\{R_u^{\Omega\cap\overline{G}}(x):\ F\subset G,\ G \text{ open in } \mathbb{R}^N\right\}.$$

The set function ν^x defines a regular Borel measure on $\partial\Omega$ for each fixed $x\in\Omega$. Since $\nu^x(F)$ is a positive L_μ-harmonic function in Ω the measures ν^x, $x\in\Omega$, are mutually absolutely continuous by Harnack inequality. Hence,

$$\nu^x(F) = \int_F d\nu^x(y) = \int_F \frac{d\nu^x}{d\nu^{x_0}}d\nu^{x_0}(y).$$

We assert that $\frac{d\nu^x}{d\nu^{x_0}} = K_\mu(x,y)$ for ν^{x_0}-a.e. $y\in\partial\Omega$. By Besicovitch's theorem,

$$\frac{d\nu^x}{d\nu^{x_0}}(y) = \lim_{r\to 0}\frac{\nu^x(\Delta_r(y))}{\nu^{x_0}(\Delta_r(y))},$$

for ν^{x_0}-a.e. $y\in\partial\Omega$. In view of the proof of Proposition 7.3, we can prove that the function $\nu^x(\Delta_r(y))$ is L_μ-harmonic and

$$\lim_{x\in\Omega,\ x\to\xi}\frac{\nu^x(\Delta_r(y))}{\tilde{W}(x)} = 0, \qquad \forall\xi\in\partial\Omega\setminus\overline{\Delta}_r(y).$$

Proceeding as in the proof of Proposition 7.3, we may prove that $\frac{d\nu^x}{d\nu^{x_0}}$ is a kernel function, and by the uniqueness of kernel functions the assertion follows. Hence

$$\nu^x(A) = \int_A K_\mu(x,y)d\nu^{x_0}(y),$$

for all Borel $A\subset\partial\Omega$ and in particular

$$u(x) = \nu^x(\partial\Omega) = \int_{\partial\Omega} K_\mu(x,y)d\nu^{x_0}(y).$$

Suppose now that

$$u(x) = \int_{\partial\Omega} K_\mu(x,y)d\nu(y),$$

for some nonnegative Borel measure ν on $\partial\Omega$. We will show that $\nu(F) = \nu^{x_0}(F)$ for any closed set $F\subset\partial\Omega$.

Choose a sequence of open sets $\{G_\ell\}$ in $\mathbb{R}^N$ such that $\cap_{\ell=1}^{\infty}G_\ell = F$ and

$$\nu^x(F) = \lim_{l\to\infty} R_u^{\Omega\cap\overline{G}_\ell}(x).$$

Since

$$R_u^B(x) \leq R_u^A(x), \qquad \text{if } B \subset A,$$

we can choose $\{G_\ell\}$ so that $\overline{G}_{\ell+1} \subset G_\ell$, $\forall \ell \geq 1$ and G_ℓ to be a C^2 domain for all $\ell \geq 1$. In view of the proof of Lemma 7.6, we may assume that $R_u^{\Omega\cap\overline{G}_\ell}(x) = V_\ell$ where V_ℓ is the L_μ-superharmonic function in Lemma 7.6 for $D = G_\ell$. Furthermore we have that $R_u^{\Omega\cap\overline{G}_\ell}(x) = u(x)$ in $\Omega \cap \overline{G}_\ell$ and $R_u^{\Omega\cap\overline{G}_\ell}(x) \leq u(x)$ for all $x \in \Omega$.

We consider an increasing sequence of smooth domains $\{\Omega_\ell\}$ such that $\overline{\Omega_\ell} \subset \Omega_{\ell+1}$, $\cup_{\ell=1}^\infty \Omega_\ell = \Omega$, $G_\ell \cap \Omega \subset \overline{\Omega} \backslash \Omega_\ell$, $\mathcal{H}^{N-1}(\partial\Omega_\ell) \to \mathcal{H}^{N-1}(\partial\Omega)$. Let $w_{\Omega_n}^{x_0}$ be the L_μ-harmonic measure in $\partial\Omega_n$ (see (7.4)–(7.5)). Let $n > \ell$ and let v_n be the unique solution of

$$\begin{cases} L_\mu v = 0, & \text{in } \Omega_n, \\ v = R_u^{\Omega\cap\overline{G}_\ell}, & \text{on } \partial\Omega_n. \end{cases}$$

Since $R_u^{\Omega\cap\overline{G}_\ell}(x)$ is a supersolution in Ω we have $R_u^{\Omega\cap\overline{G}_\ell}(x) \geq v_n(x)$, $x \in \Omega_n$. Hence

$$R_u^{\Omega\cap\overline{G}_\ell}(x_0) \geq v_n(x_0) = \int_{\partial\Omega_n} R_u^{\Omega\cap\overline{G}_\ell}(y)dw_{\Omega_n}^{x_0}(y) \geq \int_{\partial\Omega_n\cap G_\ell} R_u^{\Omega\cap\overline{G}_\ell}(y)dw_{\Omega_n}^{x_0}(y).$$

Now, by Lemma 7.6,

$$\begin{aligned} \int_{\partial\Omega_n\cap G_\ell} R_u^{\Omega\cap\overline{G}_\ell}(y)dw_{\Omega_n}^{x_0}(y) &= \int_{\partial\Omega_n\cap G_\ell} u(y)dw_{\Omega_n}^{x_0}(y) \\ &= \int_{\partial\Omega_n\cap G_\ell}\int_{\partial\Omega} K_\mu(y,\xi)d\nu(\xi)dw_{\Omega_n}^{x_0}(y) \\ &= \int_{\partial\Omega}\int_{\partial\Omega_n\cap G_\ell} K_\mu(y,\xi)dw_{\Omega_n}^{x_0}(y)d\nu(\xi) \\ &\geq \int_F\int_{\partial\Omega_n\cap G_\ell} K_\mu(y,\xi)dw_{\Omega_n}^{x_0}(y)d\nu(\xi). \end{aligned}$$

Let $\xi \in F$. We have

$$1 = K_\mu(x_0,\xi) = \int_{\partial\Omega_n\cap G_\ell} K_\mu(y,\xi)dw_{\Omega_n}^{x_0}(y) + \int_{\partial\Omega_n\backslash G_\ell} K_\mu(y,\xi)dw_{\Omega_n}^{x_0}(y).$$

But

$$K_\mu(y,\xi) \leq cd(y)d_K^{\gamma_+}(y), \qquad \forall y \in \partial\Omega_n\backslash G_\ell,$$

thus by Proposition 7.7 we have that

$$\lim_{n\to\infty}\int_{\partial\Omega_n\backslash G_\ell} K_\mu(y,\xi)dw_{\Omega_n}^{x_0}(y) = 0.$$

Combining all the above inequalities and using Lebesgue's dominated convergence theorem we obtain

$$R_u^{\Omega\cap\overline{G}_\ell}(x_0) \geq \lim_{n\to\infty} \int_F \int_{\partial\Omega_n \cap G_\ell} K_\mu(y,\xi)\,dw_{\Omega_n}^{x_0}(y)\,d\nu(\xi) = \nu(F),$$

which implies

$$\nu^{x_0}(F) \geq \nu(F).$$

For the opposite inequality, let $m < \ell$. Then

$$\begin{aligned} R_u^{\Omega\cap\overline{G}_\ell}(x_0) &= \int_{\partial\Omega_\ell} R_u^{\Omega\cap\overline{G}_\ell}(y)\,dw_{\Omega_\ell}^{x_0}(y) \\ &= \int_{\partial\Omega_\ell\cap G_m} R_u^{\Omega\cap\overline{G}_\ell}(y)\,dw_{\Omega_\ell}^{x_0}(y) + \int_{\partial\Omega_\ell\setminus G_m} R_u^{\Omega\cap\overline{G}_\ell}(y)\,dw_{\Omega_\ell}^{x_0}(y). \end{aligned}$$

In view of the proof of Lemma 7.6, we have that

$$R_u^{\Omega\cap\overline{G}_\ell}(x) \leq C d(x) d_K^{\gamma_+}(x), \qquad \forall x \in \Omega\setminus G_m.$$

Thus by Proposition 7.7 we have

$$\lim_{l\to\infty} \int_{\partial\Omega_\ell\setminus G_m} R_u^{\Omega\cap\overline{G}_\ell}(y)\,dw_{\Omega_\ell}^{x_0}(y) = 0,$$

and

$$\begin{aligned} \int_{\partial\Omega_\ell\cap G_m} R_u^{\Omega\cap\overline{G}_\ell}(y)\,dw_{\Omega_\ell}^{x_0}(y) &\leq \int_{\partial\Omega_\ell\cap G_m} u(y)\,dw_{\Omega_\ell}^{x_0}(y) \\ &= \int_{\partial\Omega_\ell\cap G_m} \int_{\partial\Omega} K_\mu(y,\xi)\,d\nu(\xi)\,dw_{\Omega_\ell}^{x_0}(y) \\ &= \int_{\partial\Omega} \int_{\partial\Omega_\ell\cap G_m} K_\mu(y,\xi)\,dw_{\Omega_\ell}^{x_0}(y)\,d\nu(\xi). \end{aligned}$$

If $\xi \in \partial\Omega\setminus G_{m-1}$ we have again by Proposition 7.7 that

$$\lim_{\ell\to\infty} \int_{\partial\Omega_\ell\cap G_m} K_\mu(y,\xi)\,dw_{\Omega_\ell}^{x_0}(y) = 0.$$

If $\xi \in \partial\Omega \cap G_m$, then

$$\int_{\partial\Omega_\ell\cap G_m} K_\mu(y,\xi)\,dw_{\Omega_\ell}^{x_0}(y) \leq K_\mu(x_0,\xi) = 1.$$

Combining all the above inequalities, we obtain

$$\nu^{x_0}(F) = \lim_{\ell\to\infty} R_u^{\Omega\cap\overline{G}_\ell}(x_0) \le \int_{\partial\Omega\cap\overline{G}_{m-1}} K_\mu(x_0,\xi)d\nu(\xi) = \nu(\partial\Omega\cap\overline{G}_{m-1}),$$

which implies

$$\nu^{x_0}(F) \le \nu(F).$$

Thus we get the desired result. □

8 Boundary value problem for linear equations

8.1 Boundary trace

We first examine the boundary trace of $\mathbb{K}_\mu[\nu]$.

Lemma 8.1 *Let $\mu \le k^2/4$ and assume that $\lambda_\mu > 0$. Then for any $\nu \in \mathfrak{M}(\partial\Omega)$ we have $\mathrm{tr}_\mu(\mathbb{K}_\mu[\nu]) = \nu$.*

Proof The proof is the similar to the proof of Lemma 2.2 in [45] and we omit it. □

Lemma 8.2 *Let $\mu \le k^2/4$ and assume that $\lambda_\mu > 0$. For $\tau \in \mathfrak{M}(\Omega;\phi_\mu)$ we set $u = \mathbb{G}_\mu[\tau]$. Then $u \in W^{1,p}_{loc}(\Omega)$ for every $1 < p < \frac{N}{N-1}$ and $\mathrm{tr}_\mu(u) = 0$ for any $p \in [1, \frac{N}{N-1})$.*

Proof By [44, Theorem 1.2.2], $u \in W^{1,p}_{loc}(\Omega)$ for every $1 < p < \frac{N}{N-1}$. Let $\{\Omega_n\}$ be a smooth exhaustion of Ω (cf. (7.4)) and v_n be the unique solution of

$$\begin{cases} L_\mu^{\Omega_n} v = 0, & \text{in } \Omega_n,\\ v = u, & \text{on } \partial\Omega_n. \end{cases}$$

We note here that $v_n(x_0) = \int_{\partial\Omega_n} u(y)d\omega^{x_0}_{\Omega_n}(y)$. We first assume that $\tau \ge 0$. Let $G_\mu^{\Omega_n}$ be the Green kernel of L_μ in Ω_n. Then $G_\mu^{\Omega_n}(x,y) \nearrow G_\mu(x,y)$ for any $x, y \in \Omega$, $x \ne y$. Putting $\tau_n = \tau|_{\Omega_n}$ and $u_n = \mathbb{G}_\mu^{\Omega_n}[\tau_n]$ we then have $u_n \nearrow u$ a.e. in Ω. By uniqueness we have that $u = u_n + v_n$ a.e. in Ω_n. In particular, $u(x_0) = u_n(x_0) + v_n(x_0)$ and therefore $\lim_{n\to\infty} v_n(x_0) = 0$. Consequently, $\mathrm{tr}_\mu(u) = 0$.

In the general case, the result follows by linearity. □

Theorem 8.3 *Let $\mu \le k^2/4$ and assume that $\lambda_\mu > 0$. We then have*
(i) *Let u be a positive L_μ-superharmonic function in the sense of distributions in Ω. Then $u \in L^1(\Omega;\phi_\mu)$ and there exist $\tau \in \mathfrak{M}^+(\Omega;\phi_\mu)$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$ such that*

$$u = \mathbb{G}_\mu[\tau] + \mathbb{K}_\mu[\nu]. \tag{8.1}$$

In particular, $u \ge \mathbb{K}_\mu[\nu]$ in Ω and $\mathrm{tr}_\mu(u) = \nu$.

(ii) *Let u be a positive L_μ-subharmonic function in the sense of distributions in Ω. Assume that there exists a positive L_μ-superharmonic function w such that $u \le w$ in Ω. Then $u \in L^1(\Omega; \phi_\mu)$ and there exist $\tau \in \mathfrak{M}^+(\Omega; \phi_\mu)$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$ such that*

$$u + \mathbb{G}_\mu[\tau] = \mathbb{K}_\mu[\nu]. \tag{8.2}$$

In particular, $u \le \mathbb{K}_\mu[\nu]$ in Ω and $\mathrm{tr}_\mu(u) = \nu$.

Proof (i) Since $L_\mu u \ge 0$ in the sense of distributions in Ω, there exists a nonnegative Radon measure τ in Ω such that $L_\mu u = \tau$ in the sense of distributions. By [44, Lemma 1.5.3], $u \in W^{1,p}_{loc}(\Omega)$ for any $p \in [1, \frac{N}{N-1})$.

Let $\{\Omega_n\}$ be a smooth exhaustion of Ω (cf. (7.4)). Denote by $G^{\Omega_n}_\mu$ and $P^{\Omega_n}_\mu$ the Green kernel and the Poisson kernel of L_μ in Ω_n respectively (recalling that $P^{\Omega_n}_\mu = -\partial_{\mathbf{n}} G^{\Omega_n}_\mu$). Then $u = \mathbb{G}^{\Omega_n}_\mu[\tau] + \mathbb{P}^{\Omega_n}_\mu[u]$, where $\mathbb{G}^{\Omega_n}_\mu$ and $\mathbb{P}^{\Omega_n}_\mu$ are the Green operator and the Poisson operator for Ω_n respectively.

Since τ and $\mathbb{P}^{\Omega_n}_\mu[u]$ are nonnegative and $G^{\Omega_n}_\mu(x,y) \nearrow G_\mu(x,y)$ for any $x, y \in \Omega$, $x \ne y$, we obtain $0 \le \mathbb{G}_\mu[\tau] \le u$ a.e. in Ω. In particular, $0 \le \mathbb{G}_\mu[\tau](x_0) \le u(x_0)$ where $x_0 \in \Omega$ is a fixed reference point. This, together with the estimate $G_\mu(x_0, \cdot) \ge c\phi_\mu$ a.e. in Ω, implies $\tau \in \mathfrak{M}(\Omega; \phi_\mu)$.

Moreover, we see that $u - \mathbb{G}_\mu[\tau]$ is a nonnegative L_μ- harmonic function in Ω. Thus by Theorem 2.9 there exists a unique $\nu \in \mathfrak{M}^+(\partial\Omega)$ such that (8.1) holds.

(ii) Since $L_\mu u \le 0$ in the sense of distributions in Ω, there exists a nonnegative Radon measure τ in Ω such that $L_\mu u = -\tau$ in the sense of distributions. By [44, Lemma 1.5.3], $u \in W^{1,p}_{loc}(\Omega)$ for any $p \in [1, \frac{N}{N-1})$. Let Ω_n and $\mathbb{P}^{\Omega_n}_\mu$ be as in (i). Then $u + \mathbb{G}^{\Omega_n}_\mu[\tau] = \mathbb{P}^{\Omega_n}_\mu[u]$. This, together with the fact that $u \ge 0$ and $\mathbb{P}_\mu[u] \le w$, implies $\mathbb{G}^{\Omega_n}_\mu[\tau] \le w$. By using a similar argument as in (i), we deduce that $\tau \in \mathfrak{M}(\Omega; \phi_\mu)$ and there exists $\nu \in \mathfrak{M}^+(\partial\Omega)$ such that (8.2) holds. □

8.2 Boundary value problem for linear equations

We recall (cf. (2.10)) that for $\mu \le k^2/4$ we have defined

$$\mathbf{X}_\mu(\Omega, K) := \left\{ \zeta \in H^1_{loc}(\Omega) : \phi_\mu^{-1}\zeta \in H^1(\Omega; \phi_\mu^2),\ \phi_\mu^{-1} L_\mu \zeta \in L^\infty(\Omega) \right\}.$$

Lemma 8.4 *Let $\mu \le k^2/4$ and assume that $\lambda_\mu > 0$. Then any $\zeta \in \mathbf{X}_\mu(\Omega, K)$ satisfies $|\zeta| \le c\phi_\mu$ in Ω.*

Proof Let $\zeta \in \mathbf{X}_\mu(\Omega, K)$ and $g = L_\mu \zeta$. Then there exist $C = C(\|g\phi_\mu^{-1}\|_{L^\infty(\Omega)}, \lambda_\mu)$ such that $|g| \le C\lambda_\mu \phi_\mu$ in Ω. Set $\tilde{\zeta} = C^{-1}\phi_\mu^{-1}\zeta$. Then,

$$\int_\Omega \phi_\mu^2 \nabla\tilde{\zeta} \cdot \nabla\psi \, dx + \lambda_\mu \int_\Omega \phi_\mu^2 \tilde{\zeta}\psi \, dx = \frac{1}{C}\int_\Omega \phi_\mu g \psi \, dx \le \lambda_\mu \int_\Omega \phi_\mu^2 \psi \, dx \,, \quad \forall 0 \le \psi \in H^1_0(\Omega; \phi_\mu^2).$$

By taking $\psi = (\tilde{\zeta} - 1)_+$ as test function in the above inequality, we obtain that $\tilde{\zeta} \le 1$, which implies $\zeta \le C\phi_\mu$ in Ω. Applying the same argument to $-\zeta$ completes the proof. □

Lemma 8.5 *Let $\mu \le k^2/4$ and assume that $\lambda_\mu > 0$. Given $\tau \in \mathfrak{M}(\Omega; \phi_\mu)$ there exists a unique weak solution u of (2.9) with $\nu = 0$. Furthermore $u = \mathbb{G}_\mu[\tau]$ and there holds*

$$\|u\|_{L^1(\Omega;\phi_\mu)} \le \frac{1}{\lambda_\mu} \|\tau\|_{\mathfrak{M}(\Omega;\phi_\mu)}. \tag{8.3}$$

Proof *A priori estimate.* Assume $u \in L^1(\Omega; \phi_\mu)$ is a weak solution of (2.9) with $\nu = 0$. Let $\zeta \in \mathbf{X}_\mu(\Omega, K)$ be such that $L_\mu \zeta = \mathrm{sign}(u)\phi_\mu$. By Kato's inequality,

$$L_\mu|\zeta| \le \mathrm{sign}(\zeta) L_\mu \zeta \le \phi_\mu = L_\mu\left(\frac{1}{\lambda_\mu}\phi_\mu\right).$$

Hence by Lemmas 6.3 and 8.4 we deduce that $|\zeta| \le \frac{1}{\lambda_\mu}\phi_\mu$ in Ω. This, combined with (2.9) (for $\nu = 0$) implies (8.3).

Uniqueness. The uniqueness follows directly from (8.3).

Existence. Assume $\tau = f dx$ with $f \in L^\infty(\Omega)$ with compact support in Ω. The existence of a solution u follows by Lemma 6.5.

Since $f \in L^\infty(\Omega)$ has compact support in Ω, there exists a positive constant $c = c(\mathrm{supp}(f), \|f\|_\infty, \Omega, K, \mu)$ such that $|f| \le c\phi_\mu$. It follows that $u \in \mathbf{X}_\mu(\Omega)$ and therefore $|u(x)| \le C\phi_\mu(x)$, $x \in \Omega$, by Lemma 8.4.

Next we will show that $u = \mathbb{G}_\mu[f]$. Set $w = \mathbb{G}_\mu[f]$. We can easily show that w satisfies $L_\mu w = f$ in the sense of distributions in Ω and by (5.4) there exists a positive constant C such that $|w(x)| \le C\phi_\mu(x)$ for all $x \in \Omega$. Therefore,

$$\lim_{\mathrm{dist}(x,F)\to 0} \frac{|u(x) - w(x)|}{\tilde{W}(x)} \le C \lim_{\mathrm{dist}(x,F)\to 0} \frac{\phi_\mu(x)}{\tilde{W}(x)} = 0$$

for any compact set $F \subset \partial\Omega$. Furthermore, we note that $|u - w|$ is L_μ-subharmonic in Ω. Hence from Lemma 6.3, we deduce that $|u - w| = 0$, i.e. $u = w$ in Ω.

Now assume that $\tau = f dx$ with $f \in L^1(\Omega; \phi_\mu)$. Let $\{\Omega_n\}$ be a smooth exhaustion of Ω (see (7.4)). Set $f_n = \chi_{\Omega_n} g_n(f) \in L^\infty(\Omega)$, where

$$g(t) = \begin{cases} n, & \text{if } t \ge n, \\ t, & \text{if } -n < t < n, \\ -n, & \text{if } t \le -n. \end{cases}$$

Then $f_n \to f$ in $L^1(\Omega; \phi_\mu)$. Put $u_n := \mathbb{G}_\mu[f_n]$. Then

$$\int_\Omega u_n L_\mu \zeta \, dx = \int_\Omega f_n \zeta \, dx, \qquad \forall \xi \in \mathbf{X}_\mu(\Omega, K).$$

By (8.3) we can easily prove that $u_n = \mathbb{G}_\mu[f_n] \to \mathbb{G}_\mu[f] := u$ in $L^1(\Omega; \phi_\mu)$. Then by letting $n \to \infty$ and using Lemma 8.4, we deduce the desired result when $f \in L^1(\Omega; \phi_\mu)$.

Assume finally that $\tau \in \mathfrak{M}(\Omega; \phi_\mu)$. Let $\{f_n\}$ be a sequence in $L^1(\Omega; \phi_\mu)$ such that $f_n \rightharpoonup \tau$ in $C_{\phi_\mu}(\Omega)$, where $C_{\phi_\mu}(\Omega) = \{\zeta \in C(\Omega) : \phi_\mu^{-1}\zeta \in L^\infty(\Omega)\}$. Then proceeding as above we can prove that $u_n = \mathbb{G}_\mu[f_n] \to \mathbb{G}_\mu[\tau] := u$ in $L^1(\Omega; \phi_\mu)$ and u satisfies (2.9) with $\nu = 0$. □

Proof of Theorem 2.12 First we note that by Theorem 2.8, we can easily show that

$$\|\mathbb{K}_\mu[|\nu|]\|_{L^1(\Omega;\phi_\mu)} \leq c\|\nu\|_{\mathfrak{M}(\partial\Omega)}. \tag{8.4}$$

Existence. The existence and (2.11) follow from Lemma 8.5 and (8.4).

A priori estimate (2.12). This follows from (8.4), (8.3) and (2.11).

Uniqueness. Uniqueness follows from (2.12).
Proof of estimates (2.13)–(2.14). Assume $d\tau = f\,dx + d\rho$ and let $\{\Omega_n\}$ be a smooth exhaustion of Ω. Let v_τ^n be the solution of

$$\begin{cases} L_\mu^{\Omega_n} v = 0, & \text{in } \Omega_n \\ v = \mathbb{G}_\mu[\tau], & \text{on } \partial\Omega_n, \end{cases}$$

and $w_\nu = \mathbb{K}_\mu[\nu]$. Then, by uniqueness, $u = \mathbb{G}_\mu^{\Omega_n}[\tau|_{\Omega_n}] + v_\tau + w_\nu$ and $|u| \leq \mathbb{G}_\mu[|\tau|] + w_{|\nu|}$ $\mathcal{H}^{N-1}$-a.e. on $\partial\Omega_n$.

Let $\eta \in C_c^2(\overline{\Omega_n})$ be non-negative and such that $\eta = 0$ on $\partial\Omega_n$. By [44, Proposition 1.5.9],

$$\int_{\Omega_n} |u| L_\mu \eta\, dx \leq \int_{\Omega_n} \operatorname{sign}(u) f \eta\, dx + \int_{\Omega_n} \eta d|\rho| - \int_{\partial\Omega_n} |u| \frac{\partial \eta}{\partial \mathbf{n}^n} dS$$

where $\mathbf{n}^n$ is the unit outer normal vector on $\partial\Omega_n$. Since $|u| \leq \mathbb{G}_\mu[|\tau|] + w_{|\nu|}$ a.e. on $\partial\Omega_n$ and $\frac{\partial\eta}{\partial\mathbf{n}^n} \leq 0$ on $\partial\Omega_n$, using integration by parts we obtain

$$-\int_{\partial\Omega_n} |u| \frac{\partial\eta}{\partial\mathbf{n}^n} dS \leq -\int_{\partial\Omega_n} (\mathbb{G}_\mu[|\tau|] + w_{|\nu|}) \frac{\partial\eta}{\partial\mathbf{n}^n} dS = \int_{\Omega_n} (v_{|\tau|}^n + w_{|\nu|}) L_\mu \eta\, dx.$$

Hence

$$\int_{\Omega_n} |u| L_\mu \eta dx \leq \int_{\Omega_n} \operatorname{sign}(u) f \eta\, dx + \int_{\Omega_n} \eta d|\rho| + \int_{\Omega_n} (v_{|\tau|}^n + w_{|\nu|}) L_\mu \eta\, dx. \tag{8.5}$$

Let $\zeta \in \mathbf{X}_\mu(\Omega, K)$, $\zeta > 0$ in Ω. Let z_n and ζ_n be respectively solutions of

$$\begin{cases} L_\mu z_n = L_\mu \zeta, & \text{in } \Omega_n, \\ z_n = 0, & \text{on } \partial\Omega_n, \end{cases} \qquad \begin{cases} L_\mu \zeta_n = \operatorname{sign}(z_n) L_\mu \zeta, & \text{in } \Omega_n, \\ \zeta_n = 0, & \text{on } \partial\Omega_n. \end{cases}$$

By Kato's inequality, $L_\mu|z_n| \le \mathrm{sign}(z_n)L_\mu z_n$ in the sense of distributions in Ω_n. Hence by a comparison argument, we have that $|z_n| \le \zeta_n$ in Ω_n. Furthermore it can be checked that $z_n \to \zeta$ and $\zeta_n \to \zeta$ in $L^1(\Omega; \phi_\mu)$ and locally uniformly in Ω.

Now note that (8.5) is valid for any nonnegative solution $\eta \in C_c^2(\Omega_n)$. Thus we can use ζ_n as a test function in (8.5) to obtain

$$\begin{aligned}\int_{\Omega_n} |u|\mathrm{sign}(z_n)L_\mu\zeta\,dx \le \int_{\Omega_n} \mathrm{sign}(u) f\zeta_n dx + \int_{\Omega_n} \zeta_n d|\rho| \\ + \int_{\Omega_n} (v^n_{|\tau|} + w_{|\nu|})\mathrm{sign}(z_n)L_\mu\zeta\,dx.\end{aligned} \tag{8.6}$$

Also, since $\mathbb{G}_\mu[|\tau|] = \mathbb{G}_\mu^{\Omega_n}[|\tau||_{\Omega_n}] + v^n_{|\tau|}$ a.e. in Ω_n, we deduce that $v^n_{|\tau|} \to 0$ in $L^1(\Omega; \phi_\mu)$ as $n \to \infty$. Thus sending $n \to \infty$ in (8.6) we obtain (2.13) since $\zeta > 0$ in Ω. Estimate (2.14) follows by adding (2.13) and (2.9). Thus the proof is complete when ζ is positive.

If ζ is nonnegative we set $\zeta_\varepsilon = \zeta + \varepsilon\phi_\mu$. Then estimates (2.13) and (2.14) are valid for ζ_ε for any $\varepsilon > 0$. The desired result follows by letting $\varepsilon \to 0$. □

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Declarations

conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Appendix A: Pointwise estimates on eigenfunctions

In this appendix, we prove sharp two-sided pointwise estimates for eigenfunctions of (2.1). Let $\beta > 0$ be small enough and $\Gamma = \partial\Omega$ or K. Let $\eta_{\beta,\Gamma} \in C_c^\infty(\Gamma_\beta)$ be such that

$0 \le \eta_{\beta,\Gamma} \le 1$ in $\mathbb{R}^N$ and $\eta = 1$ in $\overline{\Gamma}_{\frac{\beta}{2}}$. We set

$$\zeta_\beta = (1 - \eta_{4\beta,\partial\Omega}) + \eta_{4\beta,\partial\Omega} d\left((1 - \eta_{\beta,K}) + \eta_{\beta,K} \tilde{d}_K^{\gamma_+}\right) \quad \text{in } \Omega.$$

Setting $u = \zeta_\beta v$ in (2.1) we obtain that

$$\lambda_\mu = \inf_{v \in C_c^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega \zeta_\beta^2 |\nabla v|^2 dx - \int_\Omega v^2 (\zeta_\beta \Delta \zeta_\beta + \mu \frac{\zeta_\beta^2}{d_K^2}) dx}{\int_\Omega \zeta_\beta^2 u^2 dx}. \tag{A.1}$$

By [22, Lemma 3.1] there exists β_0 and a positive constant $C = C(\Omega, K, \beta_0)$ such that

$$\int_{K_{\beta_0}\cap\Omega} |\nabla u|^2 dx - \frac{k^2}{4} \int_{K_{\beta_0}\cap\Omega} \frac{u^2}{d_K^2} dx \ge C \int_{K_{\beta_0}\cap\Omega} \frac{|u|^2}{d_K^2 |\ln d_K|^2} dx, \quad \forall u \in C_c^\infty(K_{\beta_0} \cap \Omega). \tag{A.2}$$

In view of the proof of Lemma 6.1, for $\varepsilon > 0$ there exist positive constants $M = M(\Omega, K, \varepsilon)$ and $\beta_1 = \beta_1(\Omega, K, \varepsilon)$ such that the function

$$\tilde{\phi} := e^{Md} d \tilde{d}_K^{\gamma_+} + d \tilde{d}_K^{\gamma_+ + \varepsilon} \asymp d \tilde{d}_K^{\gamma_+}$$

satisfies $L_\mu \tilde{\phi} \le 0$ in $K_{\beta_1} \cap \Omega$.

Now let $u \in C_c^\infty(K_{\beta_1} \cap \Omega)$. Setting $u = \tilde{\phi} v$, by (A.2) we have

$$\int_{K_{\beta_1}\cap\Omega} d^2 \tilde{d}_K^{2\gamma_+} |\nabla v|^2 dx \ge C \int_{K_{\beta_1}\cap\Omega} \frac{d^2 v^2}{\tilde{d}_K^{2-2\gamma_+} |\ln \tilde{d}_K|^2} dx, \quad \forall v \in C_c^\infty(K_{\beta_1} \cap \Omega). \tag{A.3}$$

Now, by [24, Theorem 3.2], there exists $\beta_2 = \beta_2(\Omega) > 0$ such that

$$\int_{\Omega_{\beta_2}} |\nabla u|^2 dx \ge \frac{1}{4} \int_{\Omega_{\beta_2}} \frac{u^2}{d^2} dx, \qquad \forall u \in C_c^\infty(\Omega_{\beta_2}).$$

Setting $u = dv$, we have that there exists a positive constant $\beta_3 = \beta_3(\Omega) < \beta_2$ such that

$$\int_{\Omega_{\beta_3}} d^2 |\nabla v|^2 dx \ge \frac{1}{8} \int_{\Omega_{\beta_3}} v^2 dx, \qquad \forall v \in C_c^\infty(\Omega_{\beta_3}). \tag{A.4}$$

We denote by $H_0^1(\Omega; d^2 \tilde{d}_K^{2\gamma_+})$ the closure of $C_c^\infty(\Omega)$ in the norm

$$\|u\|^2_{H^1(\Omega; d^2 \tilde{d}_K^{2\gamma_+})} = \int_\Omega u^2 d^2 \tilde{d}_K^{2\gamma_+} dx + \int_\Omega |\nabla u|^2 d^2 \tilde{d}_K^{2\gamma_+} dx.$$

Proposition A.1 *Let* $\mu \le \frac{k^2}{4}$ *and* $\beta \le \frac{1}{16}\min(\beta_3, \beta_1)$. *Then there exists a minimizer* $v_\mu \in H_0^1(\Omega; d^2\tilde{d}_K^{2\gamma_+})$ *of* (A.1).

Proof Let $\{w_k\}_k \subset C_c^\infty(\Omega)$ be a minimizing sequence of (A.1) normalized by

$$\int_\Omega \zeta_\beta^2 w_k^2 dx = 1, \qquad k \in \mathbb{N}.$$

First we note that $\zeta_\beta^2 \asymp d^2\tilde{d}_K^{2\gamma_+}$ in Ω and

$$\left|\zeta_\beta \Delta\zeta_\beta + \mu\frac{\zeta_\beta^2}{d_K^2}\right| \le C d\tilde{d}_K^{2\gamma_+}, \qquad \text{in } K_{\frac{\beta}{2}}, \tag{A.5}$$

where C depends only on Ω, K and β_0. For any $v \in C_c^\infty(K_{\beta_5} \cap \Omega)$ we have

$$\int_{K_{\beta_5}\cap\Omega} d\tilde{d}_K^{2\gamma_+-\frac{1}{2}} v^2 dx = \frac{1}{2}\int_{K_{\beta_5}\cap\Omega} d_K^{2\gamma_+-\frac{1}{2}}(\nabla d^2 \cdot \nabla d) v^2 dx,$$

so by integration by parts, Hölder inequality, Proposition 3.1 (b) and (A.3), we find that for any $\varepsilon > 0$ there exits $\beta_5 = \beta_5(\Omega, K, \varepsilon)$ such that

$$\int_{K_{\beta_5}\cap\Omega} d\tilde{d}_K^{2\gamma_+-\frac{1}{2}} v^2 dx \le \varepsilon \int_{K_{\beta_5}\cap\Omega} |\nabla v|^2 d^2\tilde{d}_K^{2\gamma_+} dx, \tag{A.6}$$

Now, there holds

$$\left|\zeta_\beta \Delta\zeta_\beta + \mu\frac{\zeta_\beta^2}{d_K^2}\right| \le C d, \qquad \text{in } \Omega\backslash K_{\frac{\beta}{2}},$$

where C depends only on Ω, K and β_0.

Let $r > 0$. By (A.4) and proceeding as in the proof of (A.6), we have that for any $\varepsilon > 0$ there exists $\beta_6 = \beta_6(\Omega, K, \varepsilon, r)$ such that

$$\int_{\Omega_{\beta_6}\backslash K_r} d|v|^2 dx \le \varepsilon \int_{\Omega_{\beta_6}\backslash K_r} |\nabla v|^2 d^2\tilde{d}_K^{2\gamma_+} dx, \qquad \forall v \in C_c^\infty(\Omega_{\beta_6}\backslash K_r).$$

Combining all above, we may deduce that for any $\varepsilon > 0$ there exists $M(\varepsilon, \beta)$ such that

$$\left|\int_\Omega w_k^2\left(\zeta_\beta \Delta\zeta_\beta + \mu\frac{\zeta_\beta^2}{d_K^2}\right) dx\right| \le \varepsilon \int_\Omega \zeta_\beta^2 |\nabla w_k|^2 dx + M.$$

Hence, the sequence $\{w_k\}$ is uniformly bounded in $H_0^1(\Omega; d^2\tilde{d}_K^{2\gamma_+})$. Thus there exists $v_\mu \in H_0^1(\Omega; d^2\tilde{d}_K^{2\gamma_+})$ and a subsequence w_k, denoted by the same index k, such that $w_k \rightharpoonup v_\mu$ in $H_0^1(\Omega; d^2\tilde{d}_K^{2\gamma_+})$; it follows that $w_k \to v_\mu$ in $L^2_{loc}(\Omega)$ and a.e. in Ω.

By compactness we have that $w_k \to v_\mu$ in $L^2(\Omega; \zeta_\beta^2)$. Moreover, from (A.6) and (A.4) we have

$$\int_\Omega w_k^2 \left(\zeta_\beta \Delta \zeta_\beta + \mu \frac{\zeta_\beta^2}{d_K^2} \right) dx \to \int_\Omega v_\mu^2 \left(\zeta_\beta \Delta \zeta_\beta + \mu \frac{\zeta_\beta^2}{d_K^2} \right) dx.$$

The desired result now follows by the lower semicontinuity of the gradient term. □

Proposition A.2 *Let $\mu \le \frac{k^2}{4}$ and $\beta \le \frac{1}{16}\min(\beta_3, \beta_1)$. The function $\phi_\mu = v_\mu \zeta_\beta$ satisfies*

$$L_\mu \phi_\mu = \lambda_\mu \phi_\mu, \qquad in\ \Omega.$$

and has the asymptotics

$$\phi_\mu \asymp d\tilde{d}_K^{\gamma_+}, \qquad in\ \ \Omega.$$

Proof First we note that $\zeta_\beta \asymp d\tilde{d}_K^{\gamma_+}$. Furthermore $(1-\eta_{\beta,K})\phi_\mu \in H_0^1(\Omega)$ for small $\beta > 0$. Hence by standard elliptic theory, we have that for any $r > 0$ there exists $C = C(r, \Omega, K, \mu)$ such that

$$\phi_\mu \asymp Cd \quad \text{in } \Omega \setminus K_r,$$

which implies

$$v_\mu \asymp C \quad \text{in } \ \Omega \setminus K_r.$$

We will show that $v_\mu \ge c$ in Ω. Let $\Lambda > -\lambda_\mu$. For any $\varepsilon \in (0,1)$, there exists $\beta_0 < \frac{\beta}{4}$ such that the function

$$\tilde{\phi} = e^{Md} d\tilde{d}_K^{\gamma_+} + d\tilde{d}_K^{\gamma_+ + \varepsilon} \asymp d\tilde{d}_K^{\gamma_+} \quad \text{in } K_{\beta_0} \cap \Omega$$

satisfies

$$L_\mu \tilde{\phi} + \Lambda \tilde{\phi} \le 0, \qquad \text{in } K_{\beta_0} \cap \Omega. \tag{A.7}$$

Set $\phi = C\zeta_\beta^{-1}\tilde{\phi} = C(e^{Md} + \tilde{d}_K^{\epsilon})$, where $C > 0$ is a constant such that $\phi \le \frac{1}{2} v_\mu$ in $\partial K_{\beta_0} \cap \Omega$. By (A.7) and because v_μ satisfies the Euler equation for (A.1), we have

$$-\mathrm{div}(\zeta_\beta^2 \nabla(\phi - v_\mu)) - (\phi - v_\mu)\left(\zeta_\beta \Delta \zeta_\beta + \mu \frac{\zeta_\beta^2}{d_K^2} \right) + \Lambda \zeta_\beta^2 (\phi - v_\mu) \le 0, \quad \text{in } \ K_{\beta_0} \cap \Omega.$$

By Theorem 4.5, we may take $g = (\phi - v_\mu)_+$ as test function in the above inequality. Therefore,

$$\int_{K_{\beta_0}\cap\Omega} \zeta_\beta^2 |\nabla g|^2 dx - \int_{K_{\beta_0}\cap\Omega} g^2 \left(\zeta_\beta \Delta\zeta_\beta + \mu \frac{\zeta_\beta^2}{d_K^2}\right) dx + \Lambda \int_{K_{\beta_0}\cap\Omega} g^2 \zeta_\beta^2 dx \le 0, \tag{A.8}$$

But, by (A.1) we have

$$\int_{K_{\beta_0}\cap\Omega} \zeta_\beta^2 |\nabla g|^2 dx - \int_{K_{\beta_0}\cap\Omega} g^2 \left(\zeta_\beta \Delta\zeta_\beta + \mu \frac{\zeta_\beta^2}{d_K^2}\right) dx \ge \lambda_\mu \int_{K_{\beta_0}\cap\Omega} g^2 \zeta_\beta^2 dx.$$

This, together with (A.8), implies $g = 0$ since $\Lambda > -\lambda_\mu$. Hence $v_\mu \ge c$ in Ω.

Next we will similarly prove that $v_\mu \le c$ in Ω. As in the proof of Lemma 6.1, for $\varepsilon \in (0, 1)$ there exists $\beta_0 < \frac{\beta}{4}$ such that the function

$$\tilde{\zeta} = e^{-Md} d\tilde{d}_K^{\gamma_+} - d\tilde{d}_K^{\gamma_+ + \varepsilon} \asymp d\tilde{d}_K^{\gamma_+} \quad \text{in } K_{\beta_0} \cap \Omega$$

satisfies $L_\mu \tilde{\zeta} - \lambda_\mu \tilde{\zeta} \ge 0$ in $K_{\beta_0} \cap \Omega$. Set $\zeta = C\zeta_\beta^{-1}\tilde{\zeta}$, where $C > 0$ is a constant such that

$$\zeta \ge 2v_\mu\,, \qquad \text{in } \partial K_{\beta_0} \cap \Omega.$$

This time we have

$$-\mathrm{div}\left(\zeta_\beta^2 \nabla(v_\mu - \zeta)\right) - (v_\mu - \zeta)\left(\zeta_\beta \Delta\zeta_\beta + \mu \frac{\zeta_\beta^2}{d_K^2}\right) \le \lambda_\mu \zeta_\beta^2 (v_\mu - \zeta), \qquad \text{in } K_{\beta_0} \cap \Omega.$$

Hence, we may take $g = (v_\mu - \zeta)_+$ as test function in the above inequality. Therefore,

$$\int_{K_{\beta_0}\cap\Omega} \zeta_\beta^2 |\nabla g|^2 dx + \int_{K_{\beta_0}\cap\Omega} g^2 \left(\zeta_\beta \Delta\zeta_\beta + \mu \frac{\zeta_\beta^2}{d_K^2}\right) dx \le \lambda_\mu \int_{K_{\beta_0}\cap\Omega} g^2 \zeta_\beta^2 dx.$$

By (A.3), (A.5), (A.6) and the above inequality we obtain

$$C \int_{K_{\beta_0}\cap\Omega} \frac{d^2 g^2}{\tilde{d}_K^{2-2\gamma_+} |\ln \tilde{d}_K|^2} dx \le \lambda_\mu \int_{K_{\beta_0}\cap\Omega} g^2 \zeta_\beta^2 dx,$$

which implies that $g = 0$, provided β_0 is small enough. Hence, $v_\mu \le c$ in Ω and the result follows. □

Appendix B: Applications to nonlinear problems

We present here some consequences of our results on the operator L_μ to the study of the semilinear problem

$$\begin{cases} L_\mu u + g(u) = 0, & \text{in } \Omega, \\ \mathrm{tr}_\mu(u) = \nu, \end{cases} \tag{B.1}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing continuous function such that $g(0) = 0$. The above problem was treated by Marcus and Nguyen who consider a normalized boundary trace $\mathrm{tr}^*(u)$ (see [42, Definition 1.2]) instead of $\mathrm{tr}_\mu(u)$. The proofs of the following theorems can be found in the first version of the present article which is available in arXiv.

Theorem B.1 *Let $\mu \le k^2/4$. We set $p = \min\left(\frac{N+1}{N-1}, \frac{N+\gamma_+ +1}{N+\gamma_+ -1}\right)$ and in addition assume that $\lambda_\mu > 0$. Then there exists a positive constant $C = C(\Omega, K, \mu)$ such that*

$$\left\| \mathbb{K}_\mu[\nu] \right\|_{L_w^p(\Omega;\phi_\mu)} \le C \left\| \nu \right\|_{\mathfrak{M}(\partial\Omega)}$$

for any measure $\nu \in \mathfrak{M}(\partial\Omega)$.

Theorem B.2 *Let $\mu \le k^2/4$ and assume that $\lambda_\mu > 0$. We set $p_{\partial\Omega} = \frac{N+1}{N-1}$ and $p_K = \frac{N+\gamma_+ +1}{N+\gamma_+ -1}$. We then have*
(i) *Let $\nu \in \mathfrak{M}(\partial\Omega)$ with compact support F, where $F \subset \partial\Omega \setminus K$. Then there exists a positive constant $C = C(\Omega, K, \mu, \mathrm{dist}(F, K))$ such that*

$$\left\| \mathbb{K}_\mu[\nu] \right\|_{L_w^{p_{\partial\Omega}}(\Omega;\phi_\mu)} \le C \left\| \nu \right\|_{\mathfrak{M}(\partial\Omega)}.$$

(ii) *Assume in addition that $\mu < \frac{N^2}{4}$. There exists a positive constant $C = C(\Omega, K, \mu)$ such that for any $\nu \in \mathfrak{M}(\partial\Omega)$ with compact support in K there holds*

$$\left\| \mathbb{K}_\mu[\nu] \right\|_{L_w^{p_K}(\Omega;\phi_\mu)} \le C \left\| \nu \right\|_{\mathfrak{M}(\partial\Omega)}.$$

(iii) *Let $\mu = \frac{N^2}{4}$. For any $0 < \gamma < 2$ there exists a positive constant $C = C(\Omega, \mu, \gamma)$ such that for any $\nu \in \mathfrak{M}(\partial\Omega)$ which is concentrated at $0 \in \partial\Omega$ there holds*

$$\left\| \mathbb{K}_\mu[\nu] \right\|_{L_w^{\frac{N+2}{N-\gamma}}(\Omega;\phi_\mu)} \le C \left\| \nu \right\|_{\mathfrak{M}(\partial\Omega)}.$$

The above weak estimates lead to the following existence results.

Theorem B.3 *Let $\mu \le k^2/4$, $\lambda_\mu > 0$, $\nu \in \mathfrak{M}(\partial\Omega)$ and $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing continuous function such that $g(0) = 0$. Assume that $g(\pm\mathbb{K}_\mu[\nu_\pm]) \in L^1(\Omega; \phi_\mu)$. Then there exists a unique weak solution u of (B.1). Furthermore, there holds*

$$u + \mathbb{G}_\mu[g(u)] = \mathbb{K}_\mu[\nu], \quad a.e. \text{ in } \Omega.$$

Theorem B.4 *Let $\mu \leq k^2/4$ and let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing continuous function such that $g(0) = 0$. Assume that for some $p > 1$ there holds*

$$\int_1^\infty t^{-1-p}(g(t) - g(-t))dt < +\infty. \tag{B.2}$$

Let $\nu \in \mathfrak{M}(\partial\Omega)$. Then

(a) *If (B.2) holds true with $p = \min\left(\frac{N+1}{N-1}, \frac{N+\gamma_+ +1}{N+\gamma_+ -1}\right)$ then there exists a unique weak solution u of (B.1).*
(b) *Assume that either $k < N$ or $k = N$ and $\mu < N^2/4$. If ν has support in K and (B.2) holds true with $p = \frac{N+\gamma_+ +1}{N+\gamma_+ -1}$ then there exists a unique weak solution u of (B.1).*
(c) *If ν has compact support in $\partial\Omega \setminus K$ and (B.2) holds true with $p = \frac{N+1}{N-1}$ then there exists a unique weak solution u of (B.1).*

Moreover in all three cases the weak solution u satisfies

$$u + \mathbb{G}_\mu[g(u)] = \mathbb{K}_\mu[\nu], \quad a.e. \text{ in } \Omega.$$

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