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Relaxed and logarithmic modules of $\widehat{\mathfrak{sl}_3}$

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Abstract

In Adamović (Commun Math Phys 366:1025–1067, 2019), the affine vertex algebra $L_k(\mathfrak{sl}_2)$ is realized as a subalgebra of the vertex algebra $Vir_c \otimes \Pi(0)$, where Vir_c is a simple Virasoro vertex algebra and $\Pi(0)$ is a half-lattice vertex algebra. Moreover, all $L_k(\mathfrak{sl}_2)$-modules (including, modules in the category KL_k, relaxed highest weight modules and logarithmic modules) are realized as $Vir_c \otimes \Pi(0)$-modules. A natural question is the generalization of this construction in higher rank. In the current paper, we study the case $\mathfrak{g} = \mathfrak{sl}_3$ and present realization of the VOA $L_k(\mathfrak{g})$ for $k \notin \mathbb{Z}_{\geq 0}$ as a vertex subalgebra of $\mathcal{W}_k \otimes \mathcal{S} \otimes \Pi(0)$, where $\mathcal{W}_k$ is a simple Bershadsky–Polyakov vertex algebra and $\mathcal{S}$ is the $\beta\gamma$ vertex algebra. We use this realization to study ordinary modules, relaxed highest weight modules and logarithmic modules. We prove the irreducibility of all our relaxed highest weight modules having finite-dimensional weight spaces (whose top components are Gelfand–Tsetlin modules). The irreducibility of relaxed highest weight modules with infinite-dimensional weight spaces is proved up to a conjecture on the irreducibility of certain $\mathfrak{g}$-modules which are not Gelfand–Tsetlin modules. The next problem that we consider is the realization of logarithmic modules. We first analyse the free-field realization of $\mathcal{W}_k$ from Adamović et al. (Lett Math Phys 111(2), Paper No. 38, arXiv:2007.00396 [math.QA], 2021) and obtain a realization of logarithmic modules for $\mathcal{W}_k$ of nilpotent rank two at most admissible

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levels. Beyond admissible levels, we get realization of logarithmic modules up to a existence of certain $\mathcal{W}_k(\mathfrak{sl}_3, f_{pr})$-modules. Using logarithmic modules for the $\beta\gamma$ VOA, we are able to construct logarithmic $L_k(\mathfrak{g})$-modules of rank three.

1 Introduction

Vertex algebras are rich algebraic structures with interesting applications in mathematics and physics. Given a vertex algebra, usually the major goal is a thorough understanding of its representation theory. While strongly rational vertex operator have semisimple categories of modules, that are in fact modular tensor categories [63], the generic vertex algebra is neither rational nor lisse. Basic vertex operator algebras arise via standard constructions from affine vertex algebras, those associated to affine Lie algebras. One of these standard constructions is quantum Hamiltonian reduction, which associates to the affine vertex algebra $V^k(\mathfrak{g})$ at level k and the nilpotent element f in the simple Lie algebra $\mathfrak{g}$ a $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ [68]. Usually one is then interested in the simple quotient $\mathcal{W}_k(\mathfrak{g}, f)$ of $\mathcal{W}^k(\mathfrak{g}, f)$. Representation categories of the $\mathcal{W}$-algebras are somehow nicer than their parent affine vertex algebras. e.g. simple principal $\mathcal{W}$-algebras at non-degenerate admissible levels are rational and lisse [19, 21].

We are in general interested in the representation theory of the simple affine vertex algebra $L_k(\mathfrak{g})$ at the admissible level k. This topic has received considerable attention in the recent years [1, 3, 9, 20, 22, 27, 35, 43, 44, 58, 69–71]. The only (and only recently) fairly well understood case is the case of $\mathfrak{g} = \mathfrak{sl}_2$ and so our aim is to lift constructions to $\mathfrak{sl}_3$.

Let us start by reviewing the $\mathfrak{sl}_2$ case in order to stress the importance of free field realizations:

One is interested in the category of smooth weight modules at level k of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ that are also modules for $L_k(\mathfrak{sl}_2)$. By weight we mean that the Cartan subalgebra of the horizontal subalgebra $\mathfrak{sl}_2$ acts semisimply. We assume that k is admissible, that is $k + 2$ is a positive rational number and excluding $1/n, n \in \mathbb{Z}$.

The first step is to start with the interesting subcategory of modules whose conformal weight is lower bounded. Simple modules in this category are in one-to-one correspondence with simple modules of Zhu's algebra and they were classified 26 years ago [9]. For general simple Lie algebras see the much more recent [69].

The second step is to twist modules by spectral flow, automorphisms induced from the extended affine Weyl group. In the case of $\mathfrak{sl}_2$ it is an easy computation to verify that every simple smooth module is a spectral flow twist of a lower bounded module and the same statement is true for general simple Lie algebras [20] using a result of [60].

The much harder question is a full classification of indecomposable modules. This will only appear in the forthcoming work [20] and the most important ingredient is the construction of indecomposable modules, in particular logarithmic modules. A logarithmic module has a nilpotent action of the Virasoro zero mode. The single known construction for this is using free field realizatons [3]. It turns out that the full classification of indecomposable modules follows from a classification of simple modules, knowledge of existence of sufficiently many indecomposable modules, in particular logarithmic modules, and a few general criteria on extensions.

Other benefits of the free field realization of [3] are character formula (that were also obtained by different means [70]) and existence of non-trivial intertwining operators as conjectured via Verlinde's formula [43, 44].

Ultimatively one wants to understand also the vertex tensor structure of these categories, however even the existence is open, except for ordinary modules [34].

1.1 Free field realizations and logarithmic modules

Free field realizations of affine vertex algebras and $\mathcal{W}$-algebras provide a good way for understanding their representation theories. It seems that each non-rational simple affine vertex algebra $L_k(\mathfrak{g})$ admits relaxed and logarithmic modules.

But it is still a hard task to prove this using the description of $L_k(\mathfrak{g})$ as a simple quotient of the universal affine vertex algebras $V^k(\mathfrak{g})$.

One simple, but very illustrative example is the admissible affine vertex algebra $L_{-4/3}(\mathfrak{sl}_2)$. The existence of logarithmic modules was predicted by Gaberdiel in [61], while the vertex algebraic construction of these logarithmic modules was obtained by combining the free-field realization from [1], together with a (then) new method of a construction of logarithmic representations from [11]. An immediate corollary of the free field realization of $L_{-4/3}(\mathfrak{sl}_2)$ is that its Heisenberg coset is the $\mathcal{M}(3)$-singlet algebra (cf. [1, Theorem 5.2]). This is useful, since rigid tensor categories of the singlet algebras including all logarithmic modules are understood [37, 40] and the corresponding structure is inherited by $L_{-4/3}(\mathfrak{sl}_2)$ via vertex algebra extension theory [36, 41]. The free field realization of $L_{-4/3}(\mathfrak{sl}_2)$ generalizes to subregular $\mathcal{W}$-algebras of $\mathfrak{sl}_n$ at level $k = -n + \frac{n}{n+1}$ [4, 46] and to many more including $L_{-3/2}(\mathfrak{sl}_3)$ [2, 30]. Another benefit of some free field realizations is that they indicate correspondences (called logarithmic Kazhdan–Lusztig correspondences) to quantum groups, see [26] for the one that involves the subregular $\mathcal{W}$-algebras of $\mathfrak{sl}_n$ at level $k = -n + \frac{n}{n+1}$. The logarithmic Kazhdan–Lusztig correspondence of $L_{-3/2}(\mathfrak{sl}_3)$ is thoroughly studied in the accompanying work [45]. However all these free field realizations only apply to special $\mathcal{W}$-algebras at special levels and for the general case one needs a new idea:

In [3], the affine vertex algebra $L_k(\mathfrak{sl}_2)$, $k \notin \mathbb{Z}_{\geq 0}$, is realized as a subalgebra of the vertex algebra $Vir_{c_k} \otimes \Pi(0)$, where Vir_{c_k} is the simple Virasoro vertex algebra of central charge $c_k = 1 - \frac{6(k+1)^2}{k+2}$, and $\Pi(0)$ is a half-lattice vertex algebra (an extension of a rank two Heisenberg algebra along a rank one lattice). This realization enabled a realization of all modules in KL_k, their intertwining operators, relaxed and logarithmic modules. The construction generalizes straight forwardly to subregular $\mathcal{W}$-algebras of $\mathfrak{sl}_n$ and also $\mathfrak{so}_{2n+1}$ as well as principal $\mathcal{W}$-superalgebras of $\mathfrak{sl}_{n|1}$ and $\mathfrak{spo}_{2n|2}$ via duality [33, 38, 39]. Note, that the subregular $\mathcal{W}$-algebra of $\mathfrak{sl}_2$ is nothing but the affine vertex algebra itself. The subregular $\mathcal{W}$-algebra of $\mathfrak{sl}_3$ is also called the Bershadsky–Polyakov algebra and its representation theory at admissible levels has been studied using the free field realization in [6]. All these examples work, because the $\mathcal{W}$-(super)algebras allow for characterizations as intersections of kernels of screening charges on certain free field algebras. There is a technology to bring the screening charges into suitable form [33] so that one can easily observe that the $\mathcal{W}$-(super)algebra embeds into a principal $\mathcal{W}$-algebra times a half lattice vertex algebra. At non-degenerate admissible levels the simple principal $\mathcal{W}$-algebra is rational and lisse [19, 21] and in particular its representation theory is of a simple form.

The aim of this paper is to lift this construction to encompass also the affine vertex algebra of $\mathfrak{sl}_3$. Our work is motivated from [31]. We explain in section 3 that the free field realization of $V^k(\mathfrak{sl}_3)$ can be brought into such a form that one has an obvious embedding of $V^k(\mathfrak{sl}_3)$ into $\mathcal{W}^k \otimes \mathcal{S} \otimes \Pi(0)$, where $\mathcal{W}^k$ is the universal Bershadsky–Polyakov algebra (the subregular $\mathcal{W}$-algebra of $\mathfrak{sl}_3$) at level k, $\mathcal{S}$ is the $\beta\gamma$ vertex algebra of rank one and $\Pi(0)$ is a lattice type vertex algebra.

Let $\mathcal{W}_k$ be the simple quotient of $\mathcal{W}^k$ and let $k \notin \mathbb{Z}_{\geq 0}$. It has been studied at admissible level [6, 23, 48]. We can use the free field realization to prove the following results:

(1) There is a embedding of the simple affine vertex algebra $L_k(\mathfrak{sl}_3) \hookrightarrow \mathcal{W}_k \otimes \mathcal{S} \otimes \Pi(0)$, and recall that $k \notin \mathbb{Z}_{\geq 0}$ (Theorem 6.2).

(2) Each irreducible module M from the category KL_k is realized as a submodule of $L[x, y] \otimes \mathcal{S} \otimes \Pi(0)^{1/3}$, where $L[x, y]$ is an irreducible $\mathcal{W}_k$-module with $\dim L[x, y]_{top} < \infty$, and $\Pi(0)^{1/3}$ is defined in (7.5).
Conversely each $\mathcal{W}_k$-module $L[x, y]$ with finite dimensional top level is obtained via quantum Hamiltonian reduction from a module in KL_k.
These are the results of Sect. 7.

(3) We construct in Sects. 8 and 10 a family of irreducible relaxed $L_k(\mathfrak{sl}_3)$-module with finite-dimensional weight spaces as tensor product

$$R \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2) \tag{1.1}$$

where R is an irreducible $\mathcal{W}_k$-module, and $\Pi_{-1,-1}(\lambda_1, \lambda_2)$ is a weight $\Pi(0)^{\otimes 2}$-module. Based on the results presented in [22] and [71] in the case of admissible levels, we expect that each irreducible relaxed modules with finite-dimensional weight spaces has the form (1.2). For semi-relaxed modules (cf. [71]) we expect that all of them are realized as

$$R \otimes \mathcal{S} \otimes \Pi_{-1}(\lambda). \tag{1.2}$$

(4) We also construct modules with infinite-dimensional weight spaces $R_M(\lambda) \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2)$, for which we conjecture that they are irreducible (cf. Conjecture 9.2). The proof of irreducibility is reduced to proving that its top component is an irreducible $\mathfrak{sl}_3$ module. Weight $\mathfrak{sl}_3$-modules of a similar type have appeared in the recent work [59], but so far we cannot identify them explicitly.

(5) Finally in Sect. 11 we construct logarithmic modules for $\mathcal{W}_k$. This together with our free field realization is then used in Sect. 12 to get logarithmic modules for $L_k(\mathfrak{sl}_3)$.

1.2 Outlook

It is a combined effort of many researchers in the area to improve our understanding of the representation theory of affine vertex algebras at admissible levels. Past research has focussed on the simplest, but still rather non-trivial, example of $L_k(\mathfrak{sl}_2)$. The understanding is now good enough so that one can lift insights to higher rank, in particular $L_k(\mathfrak{sl}_3)$. This is done in the present work as well as the recent [45, 71].

The free field realization of $L_k(\mathfrak{sl}_2)$ has been employed to obtain some fusion rules [3], but conjecturally there are more [44]. Our next goal is to study fusion rules of $L_k(\mathfrak{sl}_2)$, $L_k(\mathfrak{sl}_3)$ and $\mathcal{W}_k$. For this we will use free field realizations but also the coset realizations of these three algebras [16, 18].

Next one wants to have a full classification of modules. For $L_k(\mathfrak{sl}_2)$ that has been achieved in [20] and it is realistic to hope that $\mathcal{W}_k$ will work in a similar fashion, but we expect $L_k(\mathfrak{sl}_3)$ to be substantially more difficult. The ultimate goal is then a full understanding of the representation category as a vertex tensor category. Here the main issues are existence and rigidity. Two problems that have been solved in the somehow simpler examples of the $\beta\gamma$-vertex algebra $\mathcal{S}$ [15] and $L_k(\mathfrak{gl}_{1|1})$ [42].

Our realization of $L_k(\mathfrak{sl}_3)$ shows that previous results on the intertwining operators and fusion rules for the vertex algebra $\mathcal{S}$ from [14] can be lifted to $L_k(\mathfrak{sl}_3)$.

Our construction also lifts $\mathcal{W}_k$ to $L_k(\mathfrak{sl}_3)$. This means if we have a larger vertex algebra that contains $\mathcal{W}_k$ as a subalgebra, then our construction can be used to provide a vertex algebra that has $L_k(\mathfrak{sl}_3)$. One nice example is nine $\beta\gamma$ vertex algebras that contain $\mathcal{W}^k$ at critical level together with two copies of the affine vertex algebra of $\mathfrak{sl}_3$ at critical level. We expect that our lifting construction will give a realization of $L_{-3}(\mathfrak{e}_6)$. This example and related ones will be investigated further.

1.3 Organization

Both $\mathcal{W}^k$ and $V^k(\mathfrak{sl}_3)$ can be realized as subalgebras of certain free field algebras. These embeddings are characterized by certain screening operators. In Sect. 3 we explain how to massage the free field realization of $V^k(\mathfrak{sl}_3)$ so that it becomes apparent that one has an embedding of $V^k(\mathfrak{sl}_3)$ in $\mathcal{W}^k \otimes \mathcal{S} \otimes \Pi(0)$. Recall that $\mathcal{S}$ is the $\beta\gamma$ vertex algebra and $\Pi(0)$ is an extension of a rank two Heisenberg vertex algebra along a rank one lattice. We call this a half lattice VOA. Also note that $\mathcal{S}$ embeds itself in a copy of $\Pi(0)$. It is important for our purposes that we can express the fields of $V^k(\mathfrak{sl}_3)$ explicitly in terms of those of $\mathcal{W}^k \otimes \mathcal{S} \otimes \Pi(0)$.

The explicit free field realization tells us that the free field algebra $S \otimes \Pi(0) \subset \Pi(0)^{\otimes 2}$ contains an affine vertex algebra $V^k(\mathfrak{b})$ for a certain solvable subalgebra $\mathfrak{b} \subset \mathfrak{sl}_3$ (it is not quite a Borel subalgebra). In Sect. 4 we study modules of $\Pi(0)^{\otimes 2}$, list their characters and most importantly study the action of $\widehat{\mathfrak{b}}$ on modules.

The extended affine Weyl group of $\widehat{\mathfrak{sl}_3}$ induces automorphisms on the affine Lie algebra, the spectral flow automorphisms. In Sect. 5 we explain how spectral flow automorphisms of $\mathcal{W}^k$ and $\Pi(0)^{\otimes 2}$ combine into the action of spectral flow on the image of $V^k(\mathfrak{sl}_3)$ in $\mathcal{W}^k \otimes \Pi(0)^{\otimes 2}$.

In Sect. 6 we use the results of the previous sections together with explicit computations to show that $L_k(\mathfrak{sl}_3)$ embeds into $\mathcal{W}_k \otimes S \otimes \Pi(0)$ if and only if k is not a non-negative integer. This tells us that an $\mathcal{W}_k \otimes S \otimes \Pi(0)$-module is automatically also a module for $L_k(\mathfrak{sl}_3)$.

Next we study singular vectors for the affine subalgebras of $\mathcal{W}^k$ and $\Pi(0)^{\otimes 2}$ and in particular use them to show that every ordinary module, i.e. every module in $KL_k(\mathfrak{sl}_3)$ appears as a submodule of $\mathcal{W}^k$ and $\Pi(0)^{\otimes 2}$, see Sect. 7.

We then turn in Sect. 8 to relaxed modules. We show that simple modules of the free field algebra are almost simple modules for $L_k(\mathfrak{sl}_3)$. Almost simple means that every submodule intersects the top level subspace non-trivially. In particular we then show that such an $L_k(\mathfrak{sl}_3)$ is simple if and only if the top level is simple as a $\mathfrak{sl}_3$-module.

If we start with a simple $\mathcal{W}_k$-module L whose top level is infinite dimensional, then L times the appropriate $\Pi(0)^{\otimes 2}$-module has as top-level an $\mathfrak{sl}_3$-module whose weight spaces are of infinite dimension, i.e. they are not of Gelfand–Tsetlin type. We determine the explicit action of $\mathfrak{sl}_3$ on these modules in Sect. 9. This leads to a Conjecture on their simplicity. We don't see an isomorphism to the modules constructed by Futorny, Liu, Lu and Zhao [59] and so we leave it as an open question whether our modules are new.

On the other hand, if we start with a simple $\mathcal{W}_k$-module L whose top level is finite dimensional, then L times the appropriate $\Pi(0)^{\otimes 2}$-module has as top-level an $\mathfrak{sl}_3$-module whose weight spaces are of finite dimension, i.e. they are of Gelfand–Tsetlin type. In this case, we obtain relaxed modules with finite dimensional weight spaces and we can determine a condition on the weights that implies simplicity. This is the content of Sect. 10.

Finally, we use the screening charges to deform the action of $L_k(\mathfrak{sl}_3)$ on modules of the free field algebra. This yields logarithmic modules: indecomposable modules with nilpotent action of the Virasoro zero-mode. This is done in Sect. 12 and as a preparation we need to do a similar construction of logarithmic modules for $\mathcal{W}_k$, which is interesting in its own right, see Sect. 11.

2 Preliminaries

We assume that the reader is familiar with the notion of vertex (super)algebra (cf. [29, 54, 66]) and of simple Lie algebras (see [64]) and their affinizations (see [65]).

2.1 Vertex algebras and intertwining operators

Let V be a conformal vertex algebra with the conformal vector ω and let $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$ so that $L(0)$ acts on V semisimply and $L(-1)$ coincides with the translation operator ∂ of V. A V-module (cf. [72]) is a pair (M, Y_M), where M is a vector space and Y_M a linear map from V to the space of $\mathrm{End}(M)$-valued fields

$$a \mapsto Y_M(a, z) = \sum_{n\in\mathbb{Z}} a^M_{(n)} z^{-n-1}$$

such that:

(1) $Y_M(|0\rangle, z) = \mathrm{Id}_M$,
(2) for $a, b \in V$,

$$z_0^{-1}\delta\Big(\frac{z_1 - z_2}{z_0}\Big)Y_M(a, z_1)Y_M(b, z_2) - z_0^{-1}\delta\Big(\frac{z_2 - z_1}{-z_0}\Big)Y_M(b, z_2)Y_M(a, z_1)$$
$$= z_2^{-1}\delta\Big(\frac{z_1 - z_0}{z_2}\Big)Y_M(Y(a, z_0)b, z_2).$$

Definition 2.1 A module (M, Y_M) for a conformal vertex algebra V with conformal vector ω is called a logarithmic module of rank $m \in \mathbb{Z}_{>0}$ if

$$(L(0) - L_{ss}(0))^m = 0, \quad (L(0) - L_{ss}(0))^{m-1} \neq 0,$$

where $L_{ss}(0)$ is the semi-simple part of $L(0)$.

Given three V-modules M_1, M_2, M_3, an *intertwining operator of type* $\begin{pmatrix} M_3 \\ M_1\ M_2 \end{pmatrix}$ (cf. [53, 55]) is a linear map $I : a \mapsto I(a, z) = \sum_{n\in\mathbb{Z}} a^I_{(n)}z^{-n-1}$ from M_1 to the space of $\mathrm{Hom}(M_2, M_3)$-valued fields such that:

(1) for $a \in V$, $b \in M_1$, $c \in M_2$, the following Jacobi identity holds:

$$z_0^{-1}\delta\Big(\frac{z_1 - z_2}{z_0}\Big)Y_{M_3}(a, z_1)I(b, z_2)c - z_0^{-1}\delta\Big(\frac{z_2 - z_1}{-z_0}\Big)I(b, z_2)Y_{M_2}(a, z_1)c$$
$$= z_2^{-1}\delta\Big(\frac{z_1 - z_0}{z_2}\Big)I(Y_{M_1}(a, z_0)b, z_2)c,$$

(2) for every $a \in M_1$,

$$I(L(-1)a, z) = \frac{d}{dz}I(a, z).$$

Let $I\begin{pmatrix} M_3 \\ M_1\ M_2 \end{pmatrix}$ denotes the vector space of all intertwining operators of type $\begin{pmatrix} M_3 \\ M_1\ M_2 \end{pmatrix}$.

2.2 Affine vertex algebras and affine *W*-algebras

Let $\mathfrak{g}$ be a simple Lie algebra with a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Denote by $(\cdot\,|\,\cdot)$ the non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$ such that $(\theta\,|\,\theta) = 2$ for the highest root θ of $\mathfrak{g}$. We will often identify $\mathfrak{h}^*$ with $\mathfrak{h}$ via $(\cdot\,|\,\cdot)$. For $\mu \in \mathfrak{h}^*$, denote by $V(\mu)$ the irreducible highest weight $\mathfrak{g}$-module with the highest weight μ.

Let $\widehat{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}d$ be the affinization of $\mathfrak{g}$ with the usual commutation relations (cf [65]). The triangular decomposition is $\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+$ where

$$\widehat{\mathfrak{n}}_\pm = \mathfrak{n}_\pm \oplus \mathfrak{g} \otimes t^{\pm 1}\mathbb{C}[t^\pm], \quad \widehat{\mathfrak{h}} = \mathfrak{h} + \mathbb{C}K + \mathbb{C}d.$$

For $\lambda \in \widehat{\mathfrak{h}}^*$, let $L(\lambda)$ be the irreducible highest weight $\widehat{\mathfrak{g}}$-module with the highest weight λ.

Denote by $V^k(\mathfrak{g})$ the universal affine vertex algebra associated to $\widehat{\mathfrak{g}}$ of level $k \in \mathbb{C}$. We shall assume that $k \neq -h^\vee$. Then (see e.g. [66]) $V^k(\mathfrak{g})$ is a conformal vertex algebra with Segal–Sugawara conformal vector $\omega_\mathfrak{g}$. Let $Y(\omega_\mathfrak{g}, z) = \sum L_\mathfrak{g}(n) z^{-n-2}$ be the corresponding Virasoro field. Denote by $L_k(\mathfrak{g})$ the (unique) simple quotient of $V^k(\mathfrak{g})$. Clearly, $L_k(\mathfrak{g}) \cong L_\mathfrak{g}(k\Lambda_0)$ as $\widehat{\mathfrak{g}}$-modules.

If M is a restricted module of level k for $\widehat{\mathfrak{g}}$ then it is a weak module for $V^k(\mathfrak{g})$; conversely, letting d act on weak modules by $-L_\mathfrak{g}(0)$ yields restricted modules for $\widehat{\mathfrak{g}}$.

We say that M is a weight module if $\hat{\mathfrak{h}}$ acts semi-simply on M. A vector $v \in M$ is called singular of weight λ if

$$\widehat{\mathfrak{n}}_+ . v = 0, \quad hv = \lambda(h)v, \quad \forall h \in \hat{\mathfrak{h}}.$$

We denote by $KL^k(\mathfrak{g})$ the category of weak modules for $V^k(\mathfrak{g})$, which are locally finite as $\mathfrak{g}$-modules and and admit a decomposition into generalized eigenspaces for $L_\mathfrak{g}(0)$ whose eigenvalues are bounded below. Let $KL_k(\mathfrak{g})$ be the category of all modules in $KL^k(\mathfrak{g})$ which are $L_k(\mathfrak{g})$-modules (cf. [7, 8, 47]).

Denote by $\mathcal{W}^k(\mathfrak{g}, \theta)$ the (affine) $\mathcal{W}$-algebra obtained from $V^k(\mathfrak{g})$ by the Hamiltonian reduction associated to a minimal nilpotent element $e_{-\theta}$ in $\mathfrak{g}$, which is the root vector of $-\theta$ (cf. [67]). The central charge is

$$c(\mathfrak{g}, k) = \frac{k \dim\mathfrak{g}}{k + h^\vee} - 6k + h^\vee - 4. \tag{2.1}$$

More generally, for a $V^k(\mathfrak{g})$-module M, denote by $H_\theta(M)$ the associated $W^k(\mathfrak{g}, \theta)$-module. Denote by $\mathcal{W}_k(\mathfrak{g}, \theta)$ the simple quotient of $\mathcal{W}^k(\mathfrak{g}, \theta)$.

2.3 The $\beta\gamma$ vertex algebra $\mathcal{S}(m)$

The $\beta\gamma$ vertex algebra $\mathcal{S}(m)$ is freely generated by the $2m$ bosonic fields

$$\beta_i(z) = \sum_{n \in \mathbb{Z}} \beta_i(n) z^{-n-1}, \gamma_i(z) = \sum_{n \in \mathbb{Z}} \gamma_i(n) z^{-n}, \quad i = 1, \ldots, m$$

that satisfy the OPE relations

$$\beta_i(z)\gamma_j(w) \sim \frac{\delta_{i,j}}{z - w}, \quad \beta_i(z)\beta_j(w) \sim 0 \sim \gamma_i(z)\gamma_j(w).$$

$\mathcal{S}(m)$ is conformal of central charge $c = -2m$ with respect to the Virasoro vector

$$\omega = \sum_{i=1}^{m} \beta_i(-1)\gamma_i(-1)|0\rangle.$$

2.4 The Bershadsky–Polyakov vertex algebra

The Bershadsky–Polyakov vertex algebra $\mathcal{W}^k = \mathcal{W}^k(\mathfrak{sl}_3, \theta)$ [28, 75] is the $\mathcal{W}$-algebra obtained from the affine vertex algebra of $\mathfrak{sl}_3$ at level k via the quantum Hamiltonian reduction associated to a minimal nilpotent element f in $\mathfrak{sl}_3$. $\mathcal{W}^k$ is freely and strongly generated by four fields $J(z)$, $G^{\pm}(z)$, $L^{\mathrm{BP}}(z)$. For non-critical k, that is $k \neq -3$, the operator products are

$$\begin{aligned}
J(z)J(w) &\sim \frac{2k+3}{3(z-w)^2}, \quad G^{\pm}(z)G^{\pm}(w) \sim 0,\\
J(z)G^{\pm}(w) &\sim \frac{\pm G^{\pm}(w)}{z-w},\\
L^{\mathrm{BP}}(z)L^{\mathrm{BP}}(w) &\sim -\frac{(2k+3)(3k+1)}{2(k+3)(z-w)^4} + \frac{2L^{\mathrm{BP}}(w)}{(z-w)^2} + \frac{\partial L^{\mathrm{BP}}(w)}{z-w},\\
L^{\mathrm{BP}}(z)G^{\pm}(w) &\sim \frac{3G^{\pm}(w)}{2(z-w)^2} + \frac{\partial G^{\pm}(w)}{z-w},\\
L^{\mathrm{BP}}(z)J(w) &\sim \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w},\\
G^{+}(z)G^{-}(w) &\sim \frac{(k+1)(2k+3)}{(z-w)^3} + \frac{3(k+1)J(w)}{(z-w)^2}\\
&\quad + \frac{1}{z-w}\left(3 : J(w)^2 : + \frac{3(k+1)}{2}\partial J(w) - (k+3)L^{\mathrm{BP}}(w)\right).
\end{aligned}$$

In the case $k = -3$, we need to replace the Virasoro field $L^{\mathrm{BP}}(z)$ with the commutative field $T^{\mathrm{BP}}(z)$ in the set of the generators and OPE relations above by $(k+3)L^{\mathrm{BP}}(z) \mapsto T^{\mathrm{BP}}(z)$. The central charge of $\mathcal{W}^k$ is

$$c_k = c(\mathfrak{sl}_3, k) = -\frac{(2k+3)(3k+1)}{(k+3)}.$$

Note that $\widetilde{L}(z) = L^{\mathrm{BP}}(z) + \frac{1}{2}\partial J(z)$ also defines a Virasoro field on $\mathcal{W}^k$, but now with the central charge

$$\tilde{c}_k = -\frac{4(k+1)(2k+3)}{k+3},$$

which gives a $\mathbb{Z}_{\geq 0}$-grading on $\mathcal{W}^k$ such that conformal weights of $J(z), G^+(z), G^-(z)$ are 1, 1, 2 respectively. Set

$$J(z) = \sum_{n\in\mathbb{Z}} J(n)z^{-n-1},\ \widetilde{L}(z) = \sum_{n\in\mathbb{Z}} \widetilde{L}(n)z^{-n-2},$$
$$G^+(z) = \sum_{n\in\mathbb{Z}} G^+(n)z^{-n-1},\ G^-(z) = \sum_{n\in\mathbb{Z}} G^-(n)z^{-n-2}.$$

Assume that W is a weight $\mathcal{W}^k$-module. A vector $w \in W$ is called singular, or highest weight vector of weight $(x, y) \in \mathbb{C}^2$ if

$$J(n)w = x\delta_{n,0}w,\ \widetilde{L}(n)w = y\delta_{n,0}w,\ G^+(n+1)w = G^-(n)w = 0,\ n \geq 0. \tag{2.2}$$

Let $L[x, y]$ denote the irreducible highest weight $\mathcal{W}^k$-module with highest weight $(x, y) \in \mathbb{C}^2$, generated by the highest weight vector $w = v_{x,y}$ satisfying (2.2).

2.5 The vertex algebra $\mathcal{Z}^k = \mathcal{W}^k(\mathfrak{sl}_3, f_{pr})$

Let $\mathcal{Z}^k$ denotes the principal affine W-algebra $\mathcal{W}^k(\mathfrak{sl}(3), f_{pr})$ of central charge $c = c_k^Z = 2 - 24\frac{(k+2)^2}{k+3}$ (see e.g. [17]). $\mathcal{Z}^k$ is generated by the Virasoro field $T(z) = \sum_{m\in\mathbb{Z}} T_m z^{-m-2}$ and another field of conformal weight 3:

$$W(z) = \sum_{m\in\mathbb{Z}} W_m z^{-m-3}$$

OPEs:

$$\begin{aligned}
T(z)T(w) &\sim \frac{c_k^Z}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w},\\
T(z)W(w) &\sim \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w},\\
W(z)W(w) &\sim \frac{(k+3)^3}{3}\left[\frac{2\Lambda(w)}{(z-w)^2} + \frac{\partial\Lambda(w)}{z-w}\right]\\
&\quad + A\left[\frac{c_k^Z}{3(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} + \frac{\frac{3}{10}\partial^2 T(w)}{(z-w)^2} + \frac{\frac{1}{15}\partial^3 T(w)}{z-w}\right],
\end{aligned} \tag{2.3}$$

where

$$A = -\frac{(k+3)^2(3k+4)(5k+12)}{6} \quad \text{and} \quad \Lambda = T^2 : -\frac{3}{10}\partial^2 T. \tag{2.4}$$

Let $L^W(c,h,h_W)$ denotes the irreducible highest weight $\mathcal{Z}^k$-module of the highest weight (h,h_W) with respect to (T_0,W_0). Let $\mathcal{Z}_k$ be the simple quotient of $\mathcal{Z}^k$.

3 Realization of the universal affine VOA $V^k(\mathfrak{sl}_3)$

Recall that $\mathcal{S}(n)$ is the n-fold tensor product of the $\beta\gamma$ vertex algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{sl}_3$ and π^k the Heisenberg vertex algebra associated to $\mathfrak{h}^*$ at level k with respect to $(\cdot|\cdot)$ and π^k_λ the Fock module of π^k with the highest weight $\lambda\in\mathfrak{h}^*$, that is, π^k_λ is a π^k-module generated by a non-zero vector e^λ satisfying that $\alpha(n)e^\lambda=\delta_{n,0}k(\alpha|\lambda)e^\lambda$ for all $\alpha\in\mathfrak{h}^*$ and $n\geq 0$. Here $\alpha(z)=\sum_{n\in\mathbb{Z}}\alpha(n)z^{-n-1}$ for $\alpha\in\mathfrak{h}^*$ are generating fields on π^k. Then e^λ is called the highest weight vector of π^k_λ (unique up to scalar multiples). Fix simple roots $\alpha_1,\alpha_2\in\mathfrak{h}^*$ of $\mathfrak{sl}_3$. Recall Wakimoto modules of $V^k(\mathfrak{sl}_3)$ in [49, 52, 78]. Let N_+ be the Lie group corresponding to the upper nilpotent subalgebra $\mathfrak{n}_+$ of $\mathfrak{sl}_3$. Then the natural representation $\mathfrak{sl}_3\hookrightarrow\mathfrak{gl}_3=\mathrm{End}(\mathbb{C}^3)$ implies a matrix representation of N_+. We take an affine coordinate system on N_+ by

$$N_+=\left\{\begin{pmatrix}1&-x_1&-x_3\\0&1&-x_2\\0&0&1\end{pmatrix}\,\middle|\,x_i\in\mathbb{C}\right\}.$$

Using the formulae (1.4) and Theorem 5.1 in [52], $V^k(\mathfrak{sl}_3)$ has a Wakimoto free field realization

$$V^k(\mathfrak{sl}_3)\hookrightarrow\mathcal{S}(3)\otimes\pi^{k+3}$$

defined by

$$\begin{aligned}
e_1&\mapsto\beta_1-:\gamma_2\beta_3:,\\
e_2&\mapsto\beta_2,\\
e_3&\mapsto\beta_3,\\
h_1&\mapsto -2:\gamma_1\beta_1:+:\gamma_2\beta_2:-:\gamma_3\beta_3:+\alpha_1,\\
h_2&\mapsto \gamma_1\beta_1:-2:\gamma_2\beta_2:-:\gamma_3\beta_3:+\alpha_2,\\
f_1&\mapsto -:\gamma_1^2\beta_1:-:\gamma_3\beta_2:+(k+1)\partial\gamma_1+\alpha_1\gamma_1,\\
f_2&\mapsto (\gamma_1\gamma_2+\gamma_3)\beta_1:-:\gamma_2^2\beta_2:-:\gamma_2\gamma_3\beta_3:+k\partial\gamma_2+\alpha_2\gamma_2,\\
f_3&\mapsto -:\gamma_1(\gamma_1\gamma_2+\gamma_3)\beta_1-:\gamma_2\gamma_3\beta_2:-:\gamma_3^2\beta_3:+k\partial\gamma_3\\
&\qquad +(k+1)(\partial\gamma_1)\gamma_2+:\alpha_1(\gamma_1\gamma_2+\gamma_3):+\alpha_2\gamma_3,
\end{aligned}$$

where (β_i,γ_i) is the $\beta\gamma$-pair of the i-th $\beta\gamma$-system in $\mathcal{S}(3)$, $h_1=e_{1,1}-e_{2,2}$, $h_2=e_{2,2}-e_{3,3}$, $e_1=e_{1,2}$, $e_2=e_{2,3}$, $e_3=e_{1,3}$, $f_1=e_{2,1}$, $f_2=e_{3,2}$, $f_3=e_{3,1}$ under the natural representation $\mathfrak{sl}_3\hookrightarrow\mathfrak{gl}_3=\mathrm{End}(\mathbb{C}^3)$, and $\{e_{i,j}\}_{i,j=1}^3$ denotes the standard basis of $\mathfrak{gl}_3$. Thanks to Proposition 8.2 in [52], the image of the map above is characterized by the screening operators if k is generic:

$$V^k(\mathfrak{sl}_3)\cong\mathrm{Ker}\int\beta_1(z)\mathrm{e}^{-\frac{\alpha_1}{k+3}}(z)dz\,\cap\,\mathrm{Ker}\int(\gamma_1\beta_3-\beta_2)(z)\mathrm{e}^{-\frac{\alpha_2}{k+3}}(z)dz.$$

where $e^{\lambda}(z)$ is an intertwining operator (see e.g. [72] for the definition of intertwining operators) between Heisenberg Fock modules defined by

$$e^{\lambda}(z) = s_{\lambda} z^{\lambda(0)} \exp\left(-\sum_{n<0} \lambda(n) \frac{z^{-n}}{n}\right) \exp\left(-\sum_{n>0} \lambda(n) \frac{z^{-n}}{n}\right)$$

and s_{λ} is the shift operator mapping the highest weight vector to the highest weight vector by shifting λ, and commuting with all $\mu(n)$, $n \neq 0$ for $\mu \in \mathfrak{h}^*$. Let Z be the lattice spanned by x_i, y_i for $i = 1, \dots, n$ with $(x_i|x_j) = \delta_{i,j} = -(y_i|y_j)$ and $(x_i|y_j) = 0$, and $\mathcal{V}(n)$ be the vertex subalgebra of V_Z generated by $x_i, y_i, e^{x_i+y_i}, e^{-x_i-y_i}$ for $i = 1, \dots, n$. The bosonization of $\mathcal{S}(n)$ [56] is the embedding

$$\mathcal{S}(n) \hookrightarrow \mathcal{V}(n), \quad \beta_i \mapsto e^{x_i+y_i}, \quad \gamma_i \mapsto -:x_i e^{-x_i-y_i}:,$$

whose image is characterized by the screening operators

$$\mathcal{S}(n) \cong \bigcap_{i=1}^{n} \operatorname{Ker}\left(\int e^{x_i}(z)dz \colon \mathcal{V}(n) \to V_Z\right).$$

Using the bosonization of β_3, γ_3

$$\mathcal{S}(1) \hookrightarrow \mathcal{V}(1), \quad \beta_3 \mapsto e^{x+y}, \quad \gamma_3 \mapsto -:xe^{-x-y}:,$$

we have

$$\rho_0 \colon V^k(\mathfrak{sl}_3) \hookrightarrow \mathcal{S}(2) \otimes \mathcal{V}(1) \otimes \pi^{k+3}$$

such that, for generic k,

$$\begin{aligned}\operatorname{Im} \rho_0 \ &= \operatorname{Ker} \int \beta_1(z) e^{-\frac{\alpha_1}{k+3}}(z)dz \ \cap \ \operatorname{Ker} \int (\gamma_1 e^{x+y} \\ &\quad -\beta_2)(z) e^{-\frac{\alpha_2}{k+3}}(z)dz \ \cap \ \operatorname{Ker} \int e^{x}(z)dz.\end{aligned}$$

Set

$$\begin{aligned}&\widetilde{\beta} = \beta_1, \quad \widetilde{\gamma} = \gamma_1 - \beta_2 e^{-x-y}, \quad \beta = \beta_2, \quad \gamma = \gamma_2 - \beta_1 e^{-x-y}, \\ &\widetilde{\alpha}_1 = \alpha_1, \quad \widetilde{\alpha}_2 = \alpha_2 - (k+3)(x+y), \quad c = x+y, \\ &d = -\frac{2k+3}{3}x - \frac{2k+9}{3}y - 2\beta_1\beta_2 e^{-x-y} + \frac{2}{3}(\alpha_1 + 2\alpha_2).\end{aligned}$$

Let $\widetilde{\mathcal{S}}$ be a vertex algebra generated by $\widetilde{\beta}, \widetilde{\gamma}$, $\mathcal{S}$ a vertex algebra generated by β, γ, $\widetilde{\pi}^{k+3}$ a vertex algebra generated by $\widetilde{\alpha}_1, \widetilde{\alpha}_2$ and $\Pi(0)$ a vertex algebra generated by

$c, d, \mathrm{e}^{\pm c}$. Then

$$\mathcal{S}(2) \otimes \mathcal{V}(1) \otimes \pi^{k+3} = \widetilde{\mathcal{S}} \otimes \mathcal{S} \otimes \widetilde{\pi}^{k+3} \otimes \Pi(0),$$

and for generic k,

$$\operatorname{Im}\rho_0 = \operatorname{Ker}\int \widetilde{\beta}(z)\mathrm{e}^{-\frac{\widetilde{\alpha}_1}{k+3}}(z)dz \cap \operatorname{Ker}\int \widetilde{\gamma}(z)\mathrm{e}^{-\frac{\widetilde{\alpha}_2}{k+3}}(z)dz \cap \operatorname{Ker}\int \mathrm{e}^{x}(z)dz. \tag{3.1}$$

From [50], $\mathcal{W}^k$ may be defined as a vertex subalgebra of $\widetilde{\mathcal{S}} \otimes \widetilde{\pi}^{k+3}$ by

$$\begin{aligned}
&\rho_1 \colon \mathcal{W}^k \hookrightarrow \widetilde{\mathcal{S}} \otimes \widetilde{\pi}^{k+3},\\
&J \mapsto -\frac{1}{3}\widetilde{\alpha}_1 + \frac{1}{3}\widetilde{\alpha}_2 + :\widetilde{\beta}\widetilde{\gamma}:,\\
&G^+ \mapsto \widetilde{\alpha}_1\widetilde{\gamma} - :\widetilde{\beta}\widetilde{\gamma}^2: + (k+1)\partial\widetilde{\gamma}, \quad G^- \mapsto \widetilde{\alpha}_2\widetilde{\beta} + :\widetilde{\beta}^2\widetilde{\gamma}: + (k+1)\partial\widetilde{\beta},\\
&L^{\mathrm{BP}} \mapsto \frac{1}{k+3}\left(\frac{1}{3}:(\widetilde{\alpha}_1^2 + \widetilde{\alpha}_1\widetilde{\alpha}_2 + \widetilde{\alpha}_2^2): + \frac{k+1}{2}(\partial\widetilde{\alpha}_1 + \partial\widetilde{\alpha}_2)\right)\\
&\quad + \frac{1}{2}:\widetilde{\beta}\partial\widetilde{\gamma}: - \frac{1}{2}:(\partial\widetilde{\beta})\widetilde{\gamma}:,
\end{aligned}$$

such that, for generic k,

$$\operatorname{Im}\rho_1 = \operatorname{Ker}\int \widetilde{\beta}(z)\mathrm{e}^{-\frac{\widetilde{\alpha}_1}{k+3}}(z)dz \cap \operatorname{Ker}\int \widetilde{\gamma}(z)\mathrm{e}^{-\frac{\widetilde{\alpha}_2}{k+3}}(z)dz.$$

Since ρ_1 consists of the Miura map of $\mathcal{W}^k$ and the Wakimoto free field realization of $V^{k+1}(\mathfrak{sl}_2)$, ρ_1 is injective for all k, where we will replace $(k+3)L^{\mathrm{BP}}$ by T^{BP} in case that $k = -3$. Thus, for generic k, by forgetting the screening operator $\int \mathrm{e}^{x}(z)dz$ in (3.1), we obtain an embedding

$$\Phi_0 \colon V^k(\mathfrak{sl}_3) \to \mathcal{W}^k \otimes \mathcal{S} \otimes \Pi(0) \tag{3.2}$$

such that

$$\rho_0 = (\rho_1 \otimes \mathrm{id}) \circ \Phi_0. \tag{3.3}$$

Then the map Φ_0 is well-defined for all k. Indeed, Φ_0 is defined by

$$\begin{aligned}
&e_1 \mapsto -\gamma\mathrm{e}^{c}, \quad e_2 \mapsto \beta, \quad e_3 \mapsto \mathrm{e}^{c},\\
&h_1 \mapsto -2J + :\gamma\beta: - \frac{2k+9}{6}c + \frac{1}{2}d, \quad h_2 \mapsto J - 2:\gamma\beta: + \frac{4k+9}{6}c + \frac{1}{2}d,\\
&f_1 \mapsto G^+ - :\left(2J + \frac{8k+9}{6}c - \frac{1}{2}d\right)\beta\mathrm{e}^{-c}: + (k+1)(\partial\beta)\mathrm{e}^{-c},
\end{aligned}$$

$$
\begin{aligned}
f_2 &\mapsto G^- \mathrm{e}^{-c} + \left(J + \frac{4k+9}{6}c + \frac{1}{2}d\right)\gamma + k\partial\gamma - :\gamma^2\beta:, \\
f_3 &\mapsto G^+\gamma - G^-\beta\mathrm{e}^{-2c} + :\Big\{(k+3)L^{\mathrm{BP}} - J\left(J + \frac{2k-9}{6}c - \frac{1}{2}d\right) \\
&\quad + \frac{k+1}{2}\partial\left(J + \frac{2}{3}kc - d\right) - 2\gamma\beta\left(J + \frac{8k+9}{12}c - \frac{1}{4}d\right) \\
&\quad + (k+1)\gamma(\partial\beta) - \frac{4k^2-18k-27}{36}c^2 + \frac{k}{3}cd - \frac{1}{4}d^2\Big\}\mathrm{e}^{-c}: .
\end{aligned}
$$

Since ρ_0 and ρ_1 are injective, so is Φ_0 for all k by (3.3). The Sugawara construction gives a Virasoro field L_{sug} on $V^k(\mathfrak{sl}_3)$ with the central charge $8k/(k+3)$ if $k \neq -3$. Then the image of L_{sug} is

$$
L_{sug} \mapsto L^{\mathrm{BP}} + \frac{1}{2}\partial J + :(\partial\gamma)\beta: + \frac{1}{2}:cd: + \frac{k}{3}\partial c - \frac{1}{2}\partial d. \tag{3.4}
$$

Consider the bosonization of $\mathcal{S} \otimes \Pi(0)$:

$$
\begin{aligned}
&\mathcal{S} \otimes \Pi(0) \hookrightarrow \Pi(0)^{\otimes 2}, \\
&\beta \mapsto \mathrm{e}^{c_1}, \quad \gamma \mapsto -\frac{1}{2}:(c_1+d_1)\mathrm{e}^{-c_1}:, \quad c \mapsto c_2, \quad d \mapsto d_2,
\end{aligned}
$$

where $\Pi(0)^{\otimes 2}$ is a vertex algebra generated by $c_1, c_2, d_1, d_2, \mathrm{e}^{\pm c_1}, \mathrm{e}^{\pm c_2}$ with OPE relations

$$
c_i(z)d_j(w) \sim \frac{2\delta_{i,j}}{(z-w)^2}, \quad c_i(z)c_j(w) \sim 0 \sim d_i(z)d_j(w).
$$

Then we have an embedding

$$
\Phi_1 \colon V^k(\mathfrak{sl}_3) \hookrightarrow \mathcal{W}^k \otimes \Pi(0)^{\otimes 2} \tag{3.5}
$$

defined by

$$
\begin{aligned}
e_1 &\mapsto \frac{1}{2} : (c_1+d_1)\mathrm{e}^{-c_1+c_2} :, \quad e_2 \mapsto \mathrm{e}^{c_1}, \quad e_3 \mapsto \mathrm{e}^{c_2}, \\
h_1 &\mapsto -2J + \frac{1}{2}c_1 - \frac{1}{2}d_1 - \frac{2k+9}{6}c_2 + \frac{1}{2}d_2, \\
h_2 &\mapsto J - c_1 + d_1 + \frac{4k+9}{6}c_2 + \frac{1}{2}d_2, \\
f_1 &\mapsto G^+ - :\left(2J - (k+1)c_1 + \frac{8k+9}{6}c_2 - \frac{1}{2}d_2\right)\mathrm{e}^{c_1-c_2} :, \\
f_2 &\mapsto G^-\mathrm{e}^{-c_2} - \frac{k+1}{2} : (\partial c_1 + \partial d_1)\mathrm{e}^{-c_1} :
\end{aligned}
$$

$$
\begin{aligned}
&\quad -\frac{1}{2} : \left(J - \frac{2k+3}{2}c_1 + \frac{1}{2}d_1 + \frac{4k+9}{6}c_2 + \frac{1}{2}d_2\right)(c_1+d_1)\mathrm{e}^{-c_1} :, \\
&f_3 \mapsto -\frac{1}{2} : G^+(c_1+d_1)\mathrm{e}^{-c_1} : - G^-\mathrm{e}^{c_1-2c_2} \\
&\quad + : \left((k+3)L^{\mathrm{BP}} + \frac{k+1}{2}\partial\left(J + c_1 + \frac{2}{3}kc_2 - d_2\right)\right)\mathrm{e}^{-c_2} : \\
&\quad + : \left\{-J\left(J + c_1 - d_1 + \frac{2k-9}{6}c_2 - \frac{1}{2}d_2\right) - \frac{1}{12}(c_1-d_1)\left((8k+9)c_2 - 3d_2\right)\right. \\
&\quad \left. - \frac{k+2}{2}c_1d_1 - \frac{4k^2-18k-27}{36}(c_2)^2 + \frac{k}{3}c_2d_2 - \frac{1}{4}(d_2)^2\right\}\mathrm{e}^{-c_2} : .
\end{aligned}
$$

The image of L_{sug} is

$$
L_{sug} \mapsto L^{\mathrm{BP}} + \frac{1}{2}\partial J + \frac{k}{3}\partial c_2 - \frac{1}{2}\partial d_1 - \frac{1}{2}\partial d_2 + \frac{1}{2} : (c_1d_1 + c_2d_2) : .
$$

4 The vertex algebra $\Pi(0)^{\otimes 2}$ and its weight modules

We choose the Virasoro vector in $\Pi(0)^{\otimes 2}$ to be

$$
L^{\Pi} = \frac{k}{3}\partial c_2 - \frac{1}{2}\partial d_1 - \frac{1}{2}\partial d_2 + \frac{1}{2} : (c_1d_1 + c_2d_2) : .
$$

It has central charge $c_\Pi = 4 - 12(-\frac{2k}{3}) = 4 + 8k$. Define the $\Pi(0)^{\otimes 2}$-module

$$
\Pi_{r_1,r_2}(\lambda_1,\lambda_2) := \Pi(0)^{\otimes 2} e^{r_1\frac{d_1}{2} + r_2\frac{d_2}{2} + \lambda_1 c_1 + \lambda_2 c_2}, \tag{4.1}
$$

where $r_1, r_2 \in \mathbb{Z}$, $\lambda_1, \lambda_2 \in \mathbb{C}$. Note that $\Pi_{r_1,r_2}(\lambda_1,\lambda_2) \cong \Pi_{r_1,r_2}(\lambda_1 + n_1, \lambda_2 + n_2)$ for $n_1, n_2 \in \mathbb{Z}$. We have

$$
L^{\Pi}(0)\, e^{r_1\frac{d_1}{2} + r_2\frac{d_2}{2} + \lambda_1 c_1 + \lambda_2 c_2} = \left(-\frac{k}{3}r_2 + (1+r_1)\lambda_1 + (1+r_2)\lambda_2\right) e^{r_1\frac{d_1}{2} + r_2\frac{d_2}{2} + \lambda_1 c_1 + \lambda_2 c_2}.
$$

This implies that for $r_1 = r_2 = -1$ and all $\lambda_1, \lambda_2 \in \mathbb{C}$ we have

$$
L^{\Pi}(0)\, e^{-\frac{d_1}{2} - \frac{d_2}{2} + \lambda_1 c_1 + \lambda_2 c_2} = \frac{k}{3} e^{-\frac{d_1}{2} - \frac{d_2}{2} + \lambda_1 c_1 + \lambda_2 c_2}. \tag{4.2}
$$

Let

$$
\mu_1 = \frac{1}{2}c_1 - \frac{1}{2}d_1 - \frac{2k+9}{6}c_2 + \frac{1}{2}d_2, \quad \mu_2 = -c_1 + d_1 + \frac{4k+9}{6}c_2 + \frac{1}{2}d_2. \tag{4.3}
$$

Define the character of a $\Pi(0)^{\otimes 2}$-module M to be

$$\mathrm{ch}[M] := \mathrm{tr}_M \left(q^{L^{\Pi}(0) - \frac{c_{\Pi}}{24}} z_1^{\mu_1(0)} z_2^{\mu_2(0)} \right)$$

for formal variables q, z_1, z_2. In particular, the graded trace of $\Pi_{-1,-1}(\lambda_1, \lambda_2)$ is

$$\begin{aligned} \mathrm{ch}[\Pi_{-1,-1}(\lambda_1, \lambda_2)] &= q^{-\frac{k}{3}+\frac{k}{3}} z_1^{-\frac{1}{2}+\frac{2k+9}{6}-\lambda_1+\lambda_2} z_2^{1-\frac{4k+9}{6}+2\lambda_1+\lambda_2} \\ \frac{\sum_{n_1,n_2\in\mathbb{Z}} z_1^{n_2-n_1} z_2^{2n_1+n_2}}{\eta(q)^4} &= z_1^{1+\frac{k}{3}-\lambda_1+\lambda_2} z_2^{-\frac{1}{2}-\frac{2k}{3}+2\lambda_1+\lambda_2} \frac{\delta(z_1^{-1} z_2^2, z_1 z_2)}{\eta(q)^4}, \end{aligned} \tag{4.4}$$

where

$$\delta(z_1, z_2) = \sum_{n_1,n_2\in\mathbb{Z}} z_1^{2n_1} z_2^{2n_2}$$

the formal delta distribution in two variables, in the sense that it satisfies

$$f(z_1, z_2)\delta\left(\frac{z_1}{w_1}, \frac{z_2}{w_2}\right) \frac{1}{w_1 w_2} = f(w_1, w_2)\delta\left(\frac{z_1}{w_1}, \frac{z_2}{w_2}\right) \frac{1}{w_1 w_2} \tag{4.5}$$

for any formal power series $f(z_1, z_2)$.

4.1 Partitions

Recall that a partition is a finite sequence of positive integers $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ of length $\ell = \ell(\mu) \in \mathbb{Z}_{>0}$ satisfying

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_\ell. \tag{4.6}$$

The weight of the partition μ is defined to be $|\mu| = \mu_1 + \mu_2 + \cdots + \mu_\ell$. Let $\mathcal{P}$ denote the set of all partitions.

Given a partition $\mu \in \mathcal{P}$ of length ℓ and an element A of a vertex algebra, we introduce (whenever it makes sense) the convenient notation

$$\begin{aligned} A_\mu &= A_{\mu_\ell} \cdots A_{\mu_2} A_{\mu_1}, \qquad & A_{-\mu} &= A_{-\mu_1} A_{-\mu_2} \cdots A_{-\mu_\ell}, \\ A_{\mu-1} &= A_{\mu_\ell - 1} \cdots A_{\mu_2 - 1} A_{\mu_1 - 1}. \end{aligned} \tag{4.7}$$

4.2 $\Pi_{0,0}(\lambda_1, \lambda_2)$ as a $\widehat{\mathfrak{b}}$-module

Let $\widehat{\mathfrak{b}}$ be the Lie algebra generated by $e_1(n), e_2(n), e_3(n), \bar{h}(n), n \in \mathbb{Z}$ where $\bar{h}(n) = h_1(n) + 2h_2(n)$.

Note that $\widehat{\mathfrak{b}}$ is a Lie subalgebra of $\widehat{\mathfrak{sl}}_3$.

Using realization we see that $U(\widehat{\mathfrak{b}}).\mathbf{1}$ is a vertex subalgebra of $\mathcal{S}\otimes\Pi(0)\subset\Pi(0)^{\otimes 2}$ generated by

$$\begin{aligned}
e_1 &= -\gamma \mathrm{e}^{c_2} = \frac{1}{2}(c_1+d_1)e^{-c_1+c_2},\\
e_2 &= \beta = e^{c_1},\\
e_3 &= e^{c_2},\\
\bar{h} &= h_1+2h_2 = -3:\gamma\beta:+\frac{2k+3}{2}c_2+\frac{3}{2}d_2\\
&= \frac{2k+3}{2}c_2+\frac{3}{2}(-c_1+d_1+d_2).
\end{aligned}$$

Let $\bar{c}_1=-c_1+c_2$. Set $\bar{\ell}_i=\ell_i+\lambda_i$, for $i=1,2$ and $\ell_i\in\mathbb{Z}$. Then the set:

$$\begin{aligned}
&\mathcal{B}_{\Pi_{0,0}(\lambda_1,\lambda_2)}\\
&\quad=\{(d_1)_{-\mu_1}(d_2)_{-\mu_2}(\bar{c}_1)_{-\nu_1}(c_2)_{-\nu_2}e^{\bar{\ell}_1c_1+\bar{\ell}_2c_2}\mid \mu_1,\mu_2,\nu_1,\nu_2\in\mathcal{P},\ell_1,\ell_2\in\mathbb{Z}\}
\end{aligned}$$

is a basis of $\Pi(0)^{\otimes 2}$.

By direct calculation and the relations

$$\begin{aligned}
[e_1(n),\bar{c}_1(m)] &= -me^{\bar{c}}_{n+m-1},\\
[e_2(n),d_1(m)] &= -e^{c_1}_{n+m},\\
[e_3(n),d_2(m)] &= -e^{c_2}_{n+m},
\end{aligned}\tag{4.8}$$

we get the following important technical lemma:

Lemma 4.1 *We have*

$$\begin{aligned}
(e_2)_{\mu_1-1}(d_1)_{-\mu_1}(d_2)_{-\mu_2}(\bar{c}_1)_{-\nu_1}(c_2)_{-\nu_2}e^{\bar{\ell}_1c_1+\bar{\ell}_2c_2} &= A_1(d_2)_{-\mu_2}(\bar{c}_1)_{-\nu_1}(c_2)_{-\nu_2}e^{(\bar{\ell}_1+\ell(\mu_1))c_1+\bar{\ell}_2c_2}\\
(e_2)_{\bar{\mu}_1-1}(d_1)_{-\mu_1}(d_2)_{-\mu_2}(\bar{c}_1)_{-\nu_1}(c_2)_{-\nu_2}e^{\bar{\ell}_1c_1+\bar{\ell}_2c_2} &= 0 \quad \textit{if } \mu_1<\bar{\mu}_1\\
(e_3)_{\mu_2-1}(d_2)_{-\mu_2}(\bar{c}_1)_{-\nu_1}(c_2)_{-\nu_2}e^{\bar{\ell}_1c_1+\bar{\ell}_2c_2} &= A_3(\bar{c}_1)_{-\nu_1}(c_2)_{-\nu_2}e^{\bar{\ell}_1c_1+(\bar{\ell}_2+\ell(\mu_2))c_2}\\
(e_3)_{\bar{\mu}_2-1}(d_2)_{-\mu_2}(\bar{c}_1)_{-\nu_1}(c_2)_{-\nu_2}e^{\bar{\ell}_1c_1+\bar{\ell}_2c_2} &= 0 \quad \textit{if } \mu_2<\bar{\mu}_2\\
(\bar{h})_{\nu_2}(\bar{c}_1)_{-\nu_1}(c_2)_{-\nu_2}e^{\bar{\ell}_1c_1+\bar{\ell}_2c_2} &= A_3(\bar{c}_1)_{-\nu_1}e^{\bar{\ell}_1c_1+\bar{\ell}_2c_2}\\
(\bar{h})_{\bar{\nu}_2}(\bar{c}_1)_{-\nu_1}(c_2)_{-\nu_2}e^{\bar{\ell}_1c_1+\bar{\ell}_2c_2} &= 0 \quad \textit{if } \nu_2<\bar{\nu}_2\\
(e_1)_{\nu_1}(\bar{c}_1)_{-\nu_1}e^{\bar{\ell}_1c_1+\bar{\ell}_2c_2} &= A_4e^{(\bar{\ell}_1-\ell(\nu_1))c_1+(\bar{\ell}_2+\ell(\nu_1))c_2}\\
(e_1)_{\bar{\nu}_1}(\bar{c}_1)_{-\nu_1}e^{\bar{\ell}_1c_1+\bar{\ell}_2c_2} &= 0 \quad \textit{if } \nu_1<\bar{\nu}_1,
\end{aligned}$$

for some nonzero constants A_i, $i=1,2,3,4$.

Using the previous lemma and arguments analogous to those in [6, Proposition 5.6] we get:

Proposition 4.2 *For each $w\in\Pi_{0,0}(\lambda_1,\lambda_2)$ there is $y\in U(\widehat{\mathfrak{b}})$ such that*

$$y.w\in\Pi_{0,0}(\lambda_1,\lambda_2)_{top}=\mathbb{C}[(\mathbb{Z}+\lambda_1)c_1+(\mathbb{Z}+\lambda_2)c_2].$$

4.3 $\Pi_{r_1,r_2}(\lambda_1,\lambda_2)$ as $U(\widehat{\mathfrak{b}})$-module

Consider again the irreducible $\Pi(0)^{\otimes 2}$-module:

$$\Pi_{r_1,r_2}(\lambda_1,\lambda_2) := \Pi(0)^{\otimes 2} e^{r_1 d_1/2 + r_2 d_2/2 + \lambda_1 c_1 + \lambda_2 c_2},$$

where $r_1, r_2 \in \mathbb{Z}$, $\lambda_1, \lambda_2 \in \mathbb{C}$.

Lemma 4.3 *We have:*

$$\Pi_{-1,-1}(\lambda_1,\lambda_2) = U(\widehat{\mathfrak{b}}).\Pi_{-1,-1}(\lambda_1,\lambda_2)_{top}.$$

Proof The proof follows from the fact that the set

$$\mathcal{B}_{\Pi(\lambda_1,\lambda_2)} = \{\bar{h}_{-\mu}(e_1)_{-\nu_1}(e_2)_{-\nu_2}(e_3)_{-\nu_3} e^{(\lambda_1+\ell_1)c_1+(\lambda_2+\ell_2)c_2} \mid \mu, \nu_1, \nu_2, \nu_3 \in \mathcal{P}, \ell_1, \ell_2 \in \mathbb{Z}\}$$

is a basis of $\Pi_{-1,-1}(\lambda_1,\lambda_2)$. Let us prove this claim.

We have the following natural basis of $\Pi_{-1,-1}(\lambda_1,\lambda_2)$

$$\begin{aligned} & d_2(-k_1)\cdots d_2(-k_i) d_1(-l_1)\cdots d_1(-l_j) c_1(-n_1)\cdots c_1(-n_r) \\ & c_2(-m_1)\cdots c_2(-m_s) e^{-\frac{1}{2}d_1 - \frac{1}{2}d_2 + (\lambda_1+\ell_1)c_1 + (\lambda_2+\ell_2)c_2} \end{aligned} \tag{4.9}$$

where $\ell_1, \ell_2 \in \mathbb{Z}$ and

$$k_1 \geq \cdots \geq k_i \geq 1, l_1 \geq \cdots \geq l_j \geq 1, n_1 \geq \cdots \geq n_r \geq 1, m_1 \geq \cdots \geq m_s \geq 1.$$

Since each element

$$\begin{aligned} & \bar{h}(-k_1)\cdots \bar{h}(-k_i) e_1(-l_1)\cdots e_1(-l_j) e_2(-n_1)\cdots e_2(-n_r) \\ & e_3(-m_1)\cdots e_3(-m_s) e^{-\frac{1}{2}d_1 - \frac{1}{2}d_2 + (\lambda_1+\ell_1+j-r)c_1 + (\lambda_2+\ell_2-j-s)c_2} \end{aligned} \tag{4.10}$$

contains a unique summand which is non-trivial scalar multiple of basis vector (4.9), we conclude that all vectors (4.10) with

$$k_1 \geq \cdots \geq k_i \geq 1, l_1 \geq \cdots \geq l_j \geq 1, n_1 \geq \cdots \geq n_r \geq 1, m_1 \geq \cdots \geq m_s \geq 1$$

form a new basis of $\Pi_{-1,-1}(\lambda_1,\lambda_2)$.

The proof follows. □

The proof of the following result is analogous to that of Proposition 4.2. Alternatively it follows directly from Proposition 4.2 together with spectral flow, which we will discuss in the next section, see in particular (5.2).

Proposition 4.4 *For each $w \in \Pi_{-1,-1}(\lambda_1,\lambda_2)$ there is $y \in U(\widehat{\mathfrak{b}})$ such that $y.w \in \Pi_{-1,-1}(\lambda_1,\lambda_2)_{top}$.*

4.4 $\mathcal{S} \otimes \Pi(0)$-modules as $\widehat{\mathfrak{b}}$-modules

Consider the following $\mathcal{S} \otimes \Pi(0)$-module

$$\mathcal{S}_{r_2}(\lambda_2) := \mathcal{S} \otimes \Pi(0).e^{r_2 d_2/2+\lambda_2 c_2}.$$

Proposition 4.5 *Assume that* $w \in \mathcal{S}_{r_2}(\lambda_2)$ *and* $r_2 \in \{-1, 0\}$. *Then there exist* $y \in U(\widehat{\mathfrak{b}})$ *and* $\ell \in \mathbb{Z}$ *such that*

$$y.w = e^{r_2 d_2/2+(\ell+\lambda_2)c_2}.$$

Proof By using Propositions 4.2 and 4.4 we construct $y_1 \in U(\widehat{\mathfrak{b}})$ such that $y_1 w = e^{r_2 d_2/2+\ell_1 c_1+\bar{\ell}_2 c_2}$ for certain $\ell_1, \ell_2 \in \mathbb{Z}$. Since $y_1 w \in \mathcal{S}_{r_2}(\lambda_2)$, we conclude that $e^{\ell_1 c_1} \in \mathcal{S}$, which implies that $\ell_1 \geq 0$, and therefore $e^{\ell_1 c} =: \beta^{\ell_1}$:. By applying the action of $e_1 = -\gamma e^{c_2}$, we get

$$\begin{aligned} e_1(-1)^{\ell_1} y_1 w &= \nu_1 e^{r_2 d_2/2+(\bar{\ell}_2+\ell_1)c_2} \quad \text{if } r_2 = 0, \\ e_1(0)^{\ell_1} y_1 w &= \nu_2 e^{r_2 d_2/2+(\bar{\ell}_2+\ell_1)c_2} \quad \text{if } r_2 = -1, \end{aligned}$$

where $\nu_1, \nu_2 \neq 0$. The proof follows.

□

5 Spectral flow

Let V be a vertex algebra and let

$$h(z) = \sum_{n\in\mathbb{Z}} h(n) z^{-n-1}$$

be the generator of a Heisenberg vertex subalgebra of V. Define the Li Δ-operator [73][1]

$$\Delta(h, z) := z^{-h(0)} \prod_{n=1}^{\infty} \exp\left(\frac{(-1)^n h(n)}{n}\right)$$

and denote by

$$\gamma^h(A(z)) = Y(\Delta(h, z)A, z)$$

the field $A(z)$ twisted by $\Delta(h, z)$. In particular for a V-module M one defines the spectrally flow twisted module $\gamma^{-h}(M)$ to be M with the action of V given by $\gamma^h(A(z))$ for $A \in V$. The following two special cases are important:

[1] Note that our convention differs by a sign from the one used in [73].

(1) If $h(z)A(w) = \alpha(z-w)^{-2}$, then $\gamma^h(A(z)) = A(z) - \alpha z^{-1}$.
(2) If $h(z)A(w) = \alpha A(w)(z-w)^{-1}$, then $\gamma^h(A(z)) = z^{-\alpha}A(z)$.

The first one is obtained in the proof of Proposition 3.4 of [73] and the second one just means that if $h(0)$ acts by multiplication with α then $\Delta(h, z)$ acts by $z^{-\alpha}$. In particular if V is a lattice vertex algebra V_L and $h(0) = \mu$ in L' then $\gamma^{-h}(V_{\lambda+L}) = V_{\lambda-\mu+L}$ for any $\lambda \in L'$. This follows since the first case tells us that the action of the zero-modes of the Heisenberg subalgebra is shifted by $-\beta$ and it has been first proven in Proposition 3.4 of [73]. Shifting the zero-mode of the Heisenberg vertex algebra is an automorphism of the Heisenberg vertex algebra. In general, for a V-module M and an automorphism γ, we denote by $\gamma(M)$ the V-module whose underlying vector space is isomorphic to M, the isomorphism is denoted by γ as well and the action of V is given by

$$A\gamma(w) = \gamma\left(\gamma^{-1}(A)w\right), \qquad w \in M.$$

We consider now $\mathcal{W}^k \otimes \Pi(0)^{\otimes 2}$. Recall that we set μ_1 and μ_2 by the equation (4.3). Define $\lambda^{a,b}$ as the spectral flow corresponding to $a\mu_1 + b\mu_2$ for integers a, b. From above discussion we obtain

$$\begin{aligned}
\lambda^{a,b}(c_1(z)) &= c_1(z) - (2b-a)z^{-1},\\
\lambda^{a,b}(d_1(z)) &= d_1(z) + (2b-a)z^{-1},\\
\lambda^{a,b}(c_2(z)) &= c_2(z) - (a+b)z^{-1},\\
\lambda^{a,b}(d_2(z)) &= d_2(z) + \left(\frac{2k+9}{6}a - \frac{4k+9}{6}b\right)z^{-1}.
\end{aligned} \tag{5.1}$$

From this we see that the action of $c_1(0)$ is shifted by $2b-a$, the one of $d_1(0)$ by $-(2b-a)$ and similar for $c_2(0)$, $d_2(0)$, in particular we can identify

$$\lambda^{a,b}\left(\Pi_{r_1,r_2}(\lambda_1,\lambda_2)\right) \cong \Pi_{r_1+2b-a, r_2+a+b}(\lambda_1 + \frac{a}{2}, \lambda_2 + \frac{k}{6}(2b-a) + \frac{1}{4}(a-b)). \tag{5.2}$$

as Virasoro field we take

$$L^{\Pi} = \frac{k}{3}\partial c_2 - \frac{1}{2}\partial d_1 - \frac{1}{2}\partial d_2 + \frac{1}{2} :(c_1 d_1 + c_2 d_2):$$

and its image under spectral flow is obtained from the spectral flow action on the Heisenberg fields.

$$\lambda^{a,b}(L^{\Pi}) = L^{\Pi} - z^{-1}\mu_{a,b} + z^{-2}\left(-\frac{(2b-a)^2}{2} - (a+b)\left(\frac{2k+9}{6}a - \frac{4k+9}{6}b\right)\right.$$
$$\left.+\frac{k}{3}(a+b)+\frac{2b-a}{2}+\left(\frac{2k+9}{6}a - \frac{4k+9}{6}b\right)\right) = L^{\Pi} - z^{-1}\mu_{a,b}+z^{-2}$$
$$\times\left(k(a^2+b^2-ab) - \frac{(2k+3)(b-2a)(b-2a+1)}{6}\right). \tag{5.3}$$

Denote by σ^{ℓ} the spectral flow corresponding to ℓJ. From Sect. 2 of [6] we obtain that

$$\begin{aligned}\sigma^{\ell}(G^{\pm}(z)) &= z^{\mp\ell}G^{\pm}(z),\\ \sigma^{\ell}(J(z)) &= J - \frac{2k+3}{3}\ell z^{-1},\\ \sigma^{\ell}\left(L^{\mathrm{BP}}(z) + \frac{1}{2}\partial J(z)\right) &= L^{\mathrm{BP}}(z) + \frac{1}{2}\partial J(z) - \ell z^{-1}J(z) + \frac{2k+3}{3}\frac{\ell(\ell+1)}{2}z^{-2}.\end{aligned} \tag{5.4}$$

Note that their Virasoro field is $\widetilde{L}(z) = L^{\mathrm{BP}}(z) + \frac{1}{2}\partial J(z)$. Let $\gamma^{a,b}$ be the spectral flow corresponding to $ah_1 + bh_2$. Then via the embedding in $\mathcal{W}^k \otimes \Pi(0)^{\otimes 2}$ we have

$$\gamma^{a,b} = \sigma^{b-2a} \otimes \lambda^{a,b}$$

since $h_1 = -2J + \mu_1$ and $h_2 = J + \mu_2$. It acts on generators and the Virasoro field $L(z) = L^{\mathrm{BP}}(z) + \frac{1}{2}\partial J(z) + L^{\Pi}(z)$ as follows

$$\begin{aligned}\gamma^{a,b}(e_1(z)) &= z^{-2a+b}e_1(z),\\ \gamma^{a,b}(e_2(z)) &= z^{a-2b}e_2(z),\\ \gamma^{a,b}(e_3(z)) &= z^{-a-b}e_3(z),\\ \gamma^{a,b}(h_1(z)) &= h_1(z) - k(2a-b)z^{-1},\\ \gamma^{a,b}(h_2(z)) &= h_2(z) - k(-a+2b)z^{-1},\\ \gamma^{a,b}(f_1(z)) &= z^{2a-b}f_1(z),\\ \gamma^{a,b}(f_2(z)) &= z^{-a+2b}f_2(z),\\ \gamma^{a,b}(f_3(z)) &= z^{a+b}f_3(z),\\ \gamma^{a,b}(L(z)) &= L(z) - z^{-1}(ah_1(z) + bh_2(z)) + z^{-2}k(a^2+ab+b^2).\end{aligned} \tag{5.5}$$

Let M be a module for $\mathcal{W}^k \otimes \Pi(0)^{\otimes 2}$ and define its character as

$$\mathrm{ch}[M](q, z_1, z_2) := \mathrm{tr}_M\left(q^{L(0)-\frac{c}{24}}z_1^{h_1(0)}z_2^{h_2(0)}\right).$$

Then

$$\mathrm{ch}[\gamma^{a,b}(M)](q,z_1,z_2)=q^{k(a^2+ab+b^2)}z_1^{k(2a-b)}z_2^{k(2b-a)}\mathrm{ch}[M](q,z_1q^a,z_2q^b).$$

6 Realization of the simple affine VOA $L_k(\mathfrak{sl}_3)$

Using Proposition 4.5 and the action of $U(\widehat{\mathfrak{b}})$ we get immediately:

Lemma 6.1 *Assume that U is any $V^k(\mathfrak{g})$-submodule of $\mathcal{W}_k\otimes\mathcal{S}\otimes\Pi(0)$. Then U contains vector $Z\otimes e^{\ell_2 c_2}$ where $Z\in\mathcal{W}_k$ and $\ell_2\in\mathbb{Z}$.*

Recall that $\mathcal{W}_k$ is the simple Bershadsky–Polyakov algebra at level k as in Sect. 2.4. We choose the Virasoro field $\widetilde{L}=L^{\mathrm{BP}}+\frac{1}{2}\partial J$, so that. G^+ (resp. G^-) is a primary vector of weight 1 (resp. 2). Also recall that $\mathcal{W}_k$ is generated by fields

$$J(z)=\sum_{n\in\mathbb{Z}}J(n)z^{-n-1},\ \widetilde{L}(z)=\sum_{n\in\mathbb{Z}}\widetilde{L}(n)z^{-n-2},$$
$$G^+(z)=\sum_{n\in\mathbb{Z}}G^+(n)z^{-n-1},\ G^-(z)=\sum_{n\in\mathbb{Z}}G^-(n)z^{-n-2}.$$

Let

$$\Phi: V^k(\mathfrak{sl}_3)\to\mathcal{W}_k\otimes\mathcal{S}(1)\otimes\Pi(0)\subset\mathcal{W}_k\otimes\Pi(0)^{\otimes 2}$$

as in Sect. 3 and $\widetilde{L}_k(\mathfrak{sl}_3)=\mathrm{Im}(\Phi)$.

Theorem 6.2 *We have: $\widetilde{L}_k(\mathfrak{sl}_3)$ is simple if and only if $k\notin\mathbb{Z}_{\geq 0}$. In particular, for $k\notin\mathbb{Z}_{\geq 0}$, there exist an embedding $L_k(\mathfrak{sl}_3)\hookrightarrow\mathcal{W}_k\otimes\mathcal{S}(1)\otimes\Pi(0)$.*

Proof Assume that $\widetilde{L}_k(\mathfrak{sl}_3)$ is not simple. Then it contains a non-trivial ideal $\mathcal{I}$, which, by Lemma 6.1, contains vector

$$Z\otimes e^{\ell_2 c_2}\in\mathcal{I},\quad (Z\in\mathcal{W}_k,\ell_2\in\mathbb{Z}).$$

We claim that

$$\mathbf{1}\otimes e^{\ell_2 c_2}\in\mathcal{I}. \tag{6.1}$$

In the proof of (6.1) we use the simplicity of $\mathcal{W}_k$ and the action of elements $f_j(k)$, $h_1(k)$, $k\geq 1$. Let us prove this claim. First we notice that since $\mathcal{W}_k$ is simple, then either $Z\in\mathbb{C}\mathbf{1}$ or Z can not be a singular vector for $\mathcal{W}_k$.

Assume that $Z\notin\mathbb{C}\mathbf{1}$. Then one of the following holds for certain $n\in\mathbb{Z}_{\geq 1}$:

(a) $J(n)Z\neq 0$,
(b) $G^+(n)Z\neq 0$,
(c) $G^-(n)Z\neq 0$,

(d) $L(n)^{\mathrm{BP}} Z \neq 0$.
(Note that the case (d) doesn't need to be considered at the critical level $k = -3$ since then $L(n)^{\mathrm{BP}}$ acts as a scalar on $\mathcal{W}_k \otimes \mathcal{S} \otimes \Pi(0)$.)

Let us analyze these cases

(a) If $J(n)Z \neq 0$. Then since $e^{\ell_2 c_2}$ is a highest-weight vector for c_1, c_2, d_1 and d_2 and since

$$h_1 = -2J + \frac{1}{2}c_1 - \frac{1}{2}d_1 - \frac{2k+9}{6}c_2 + \frac{1}{2}d_2$$

and

$$h_2 = J - c_1 + d_1 + \frac{4k+9}{6}c_2 + \frac{1}{2}d_2$$

we have

$$\begin{aligned} h_i(n).(Z \otimes e^{\ell_2 c_2}) &= \nu_i J(n)Z \otimes e^{\ell_2 c_2} \\ &= Z_1 \otimes e^{\ell_1 c_1 + \ell_2 c_2} \end{aligned}$$

for certain Z_1, $\mathrm{wt}(Z_1) < \mathrm{wt}(Z)$ and $\nu_1 = -2, \nu_2 = 1$.

(b) If $G^+(n)Z \neq 0$ and $J(i)Z = 0$ for all $i \geq 1$. Then since

$$f_1 = G^+ - :\left(2J - (k+1)c_1 + \frac{8k+9}{6}c_2 - \frac{1}{2}d_2\right)\mathrm{e}^{c_1 - c_2}:$$

we have

$$f_1(n).(Z \otimes e^{\ell_2 c_2}) = G^+(n)Z \otimes e^{\ell_2 c_2} = Z_2 \otimes e^{\ell_2 c_2}$$

for certain Z_2, $\mathrm{wt}(Z_2) < \mathrm{wt}(Z)$.

(c) Let $G^-(n)Z \neq 0$ and $J(i)Z = G^-(n+i)Z = 0$ for all $i \geq 1$. Then since

$$\begin{aligned} f_2 = G^-\mathrm{e}^{-c_2} &- \frac{k+1}{2} :(\partial c_1 + \partial d_1)\mathrm{e}^{-c_1}: \\ &- \frac{1}{2} :\left(J - \frac{2k+3}{2}c_1 + \frac{1}{2}d_1 + \frac{4k+9}{6}c_2 + \frac{1}{2}d_2\right)(c_1 + d_1)\mathrm{e}^{-c_1}: \end{aligned}$$

we have

$$\begin{aligned} f_2(n).(Z \otimes e^{\ell_2 c_2}) &= G^-(n)Z \otimes e^{-c_2}_{-1} e^{\ell_2 c_2} \\ &= Z_3 \otimes e^{(\ell_2 - 1)c_2} \end{aligned}$$

for certain Z_3, $\mathrm{wt}(Z_3) < \mathrm{wt}(Z)$.

(d) Let $G^{\pm}(i)Z = J(i)Z = L(n+i)Z = 0$ for all $i > 0$ and $L(n)Z \neq 0$. Then since

$$
\begin{aligned}
f_3 = &-\frac{1}{2} : G^{+}(c_1+d_1)\mathrm{e}^{-c_1} : -G^{-}\mathrm{e}^{c_1-2c_2} \\
&+ : \left((k+3)L^{\mathrm{BP}} + \frac{k+1}{2}\partial\left(J + c_1 + \frac{2}{3}kc_2 - d_2\right)\right)\mathrm{e}^{-c_2} : \\
&+ : \Bigg\{-J\left(J + c_1 - d_1 + \frac{2k-9}{6}c_2 - \frac{1}{2}d_2\right) \\
&- \frac{1}{12}(c_1-d_1)\left((8k+9)c_2 - 3d_2\right) - \frac{k+2}{2}c_1 d_1 \\
&- \frac{4k^2-18k-27}{36}(c_2)^2 + \frac{k}{3}c_2 d_2 - \frac{1}{4}(d_2)^2\Bigg\}\mathrm{e}^{-c_2} :,
\end{aligned}
$$

we have

$$
\begin{aligned}
f_3(n+1).(Z \otimes e^{\ell_2 c_2}) &= (k+3)L^{\mathrm{BP}}(n)Z \otimes e^{-c_2}_{-1} e^{\ell_2 c_2} \\
&= Z_4 \otimes e^{(\ell_2-1)c_2}
\end{aligned}
$$

for certain Z_4, $\mathrm{wt}(Z_4) < \mathrm{wt}(Z)$.

By applying (a)–(d) and simplicity of $\mathcal{W}_k$ we conclude that $\mathbf{1} \otimes e^{\ell_2 c_2} \in \mathcal{I}$. Using the operator $e_3(-1)$ we can conclude that $\mathbf{1} \otimes e^{\ell_2 c_2} \in \mathcal{I}$ for certain $\ell_2 \in \mathbb{Z}_{\geq 0}$. This implies that $e_3(-1)^{\ell_2}\mathbf{1}$ belongs to the maximal submodule of $V^k(\mathfrak{sl}_3)$. But this can only happen if $k \in \mathbb{Z}_{\geq 0}$.

□

Remark 6.3 Note that the proof of the previous theorem also works at the critical level $k = -3$. For $\mathcal{W}_k$ we can take any simple quotient of $\mathcal{W}^k$. Only the case (d) doesn't need to be considered.

Corollary 6.4 *If R is a $\mathcal{W}_k$-module and $k \notin \mathbb{Z}_{\geq 0}$, then $R \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2)$ is an $L_k(\mathfrak{sl}_3)$-module.*

Remark 6.5 Note the following:

- We can choose R so that R_{top} is finite-dimensional. Then all weight spaces for $\mathfrak{sl}_3$-module $R_{top} \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2)_{top}$ are finite-dimensional too. In particular, this happens for the collapsing level $k = -3/2$ and the levels $k = -3/2 + n$ and n in $\mathbb{Z}_{\geq 0}$ where $\mathcal{W}_k$ is strongly rational [23].
- For $k = -9/4$ (B_p algebra for $p = 4$ case [46]), when R_{top} is finite-dimensional, one shows that then R_{top} is 1-dimensional [5].
- But for $k = -9/4$ (and more generally for k non-degenerate principal admissible) we have irreducible modules R such that R_{top} is infinite-dimensional. Then all weight spaces of $\mathfrak{sl}_3$-module $R_{top} \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2)_{top}$ will be infinite-dimensional. So we can not apply character argument. Our proof covers this case too.

Affine vertex algebras at the critical level are of course interesting in its own right, e.g. since they have a large center. In addition, vertex algebras that are extensions of three copies of the affine vertex algebra at the critical level play the crucial role in the Moore–Tachikawa conjecture on vertex algebras of class S [74], proven by Arakawa [25]. These extensions are such that the center is identified. In the case $\mathfrak{g} = \mathfrak{sl}_2$ this algebra is just four $\beta\gamma$ vertex algebras and in fact in general n^2 $\beta\gamma$ vertex algebras are an extension of two affine vertex algebras of type $\mathfrak{sl}_n$ at the critical level times a subregular $\mathcal{W}$-algebra of $\mathfrak{sl}_n$ at the critical level [32]. For $n = 3$ we expect that our procedure lifts nine $\beta\gamma$ vertex algebras to $L_{-3}(\mathfrak{e}_6)$.

7 Singular vectors and realizations of $V^k(\mathfrak{sl}_3)$-modules in KL_k

In this section we shall give a realisation of all irreducible modules in KL_k for $\mathfrak{sl}_3$ inside the $\mathcal{W}^k \otimes \mathcal{S} \otimes \Pi(0)$-modules of type $L[x, y] \otimes S_0(\lambda_2)$, where $L[x, y]$ is an irreducible, highest weight $\mathcal{W}^k$-module such that $L[x, y]_{top}$ is finite-dimensional. For that purpose we shall first study singular vectors in $L[x, y] \otimes S_0(\lambda_2)$.

The vector $w \in \Pi_{0,0}(\lambda_1, \lambda_2)$ is called singular for $\widehat{\mathfrak{b}}$ if

$$\bar{h}(n+1)w = e_i(n)w = 0 \quad \text{for } i = 1, 2, 3, n \geq 0.$$

Again Proposition 4.5 gives us immediately

Lemma 7.1 *If u is a singular vector in $\Pi_{0,0}(\lambda_1, \lambda_2)$, then $\lambda_1 \in \mathbb{Z}$ and $u = Ae^{(m+\lambda_2)c_2}$ for $A \neq 0$ and $m \in \mathbb{Z}$.*

Assume that Z is any weight $\mathcal{W}^k$-module. Recall that a vector $v \in Z$ is a singular vector of weight (x, y) if $J(0)v = xv$, $\widetilde{L}(0)v = yv$ and

$$G^+(n+1)v = G^-(n)v = \widetilde{L}(n+1)v = J(n+1)v = 0 \quad (n \in \mathbb{Z}_{\geq 0}).$$

Proposition 7.2 *Assume that Z is any weight $\mathcal{W}^k$-module. Then $w \in Z \otimes \Pi_{0,0}(0, \lambda_2)$ is a singular vector for $\widehat{\mathfrak{sl}_3}$ if and only if w has the form*

$$w = v \otimes e^{\bar{m}c_2} \tag{7.1}$$

where v is a singular vector for $\mathcal{W}^k$ of highest weight (x, y), $\bar{m} = m + \lambda_2$ and

$$(k+3)(y + \tfrac{x}{2}) - \tfrac{k+1}{2}(x - 2\bar{m}) - x(x - \bar{m}) - \bar{m}^2 = 0. \tag{7.2}$$

Proof By using Lemma 7.1 and the same arguments as in the proof of Theorem 6.2 we get that any singular vector for $\widehat{\mathfrak{sl}_3}$ must be of the form (7.1). Let us now find a sufficient condition that $v \otimes e^{\bar{m}c_2}$ is a singular vector. Assume that v is a highest weight vector of the highest weight (x, y).

Since we already know that w is a singular for $\widehat{\mathfrak{b}}$, it remains to check conditions $f_i(1)w = 0$ for $i = 1, 2, 3$.

One easily see that $f_1(1)w = f_2(1)w = 0$ without any further condition.

By direct calculation we get

$$f_3(1)w = \left((k+3)(y+\tfrac{x}{2}) - \tfrac{k+1}{2}(x-2\bar{m}) - x(x-\bar{m}) - \bar{m}^2\right) v \otimes e^{(\bar{m}-1)c_2}.$$

This implies that w is a singular vector if and only if

$$(k+3)(y+\tfrac{x}{2}) - \tfrac{k+1}{2}(x-2\bar{m}) - x(x-\bar{m}) - \bar{m}^2 = 0.$$

□

Let $a, b \in \mathbb{C}$. Consider the highest weight $\mathcal{W}^k$-module $L[x, y]$ with highest weight vector $v[x, y]$ and highest weight

$$x = \frac{b-a}{3}, \tag{7.3}$$

$$y = \frac{(b-a)^2 - 3(a+b)(2(k+1)-a-b)}{12(k+3)} - \frac{b-a}{6}. \tag{7.4}$$

Then the equation (7.2) has two solutions:

$$m_1 := \frac{a+2b}{3}, \quad m_2 := \frac{3-2a-b}{3} + k.$$

Then

$$v[x, y] \otimes e^{m_1 c_2}, \quad v[x, y] \otimes e^{m_2 c_2}$$

are singular vectors for $V^k(\mathfrak{sl}_3)$.

Define the following simple current extension of $\Pi(0)$:

$$\Pi(0)^{1/3} = \Pi(0) \oplus \Pi(0).e^{\frac{c_2}{3}} \oplus \Pi(0).e^{\frac{2c_2}{3}}. \tag{7.5}$$

Theorem 7.3 *Assume that $a, b \in \mathbb{Z}_{\geq 0}$, $k \notin \mathbb{Z}_{\geq 0}$. Then we have:*

(1) $L_k^{\mathfrak{sl}_3}(a\omega_1 + b\omega_2) = V^k(\mathfrak{sl}_3).(v[x, y] \otimes e^{m_1 c_2}) \subset L[x, y] \otimes \mathcal{S}_0(m_1)$.

(2) $L_k^{\mathfrak{sl}_3}(a\omega_1 + b\omega_2) = \left(L[x, y] \otimes \mathcal{S} \otimes \Pi(0)^{1/3}\right)^{int_{\mathfrak{sl}_3}}$.

Proof Using realization we get that the $\mathfrak{sl}_3$-highest weights of

- $v[x, y] \otimes e^{m_1 c_2}$ is $a\omega_1 + b\omega_2$,
- $v[x, y] \otimes e^{m_2 c_2}$ is $(k+1-b)\omega_1 + (k+1-a)\omega_2$.

Let $\widetilde{L}_k^{\mathfrak{sl}_3}(a\omega_1 + b\omega_2) = V^k(\mathfrak{sl}_3).(v[x, y] \otimes e^{m_1 c_2})$.

Since there exist at most two linearly independent singular vectors in $L[x, y] \otimes \mathcal{S}_0(m_1)$, and the $\mathfrak{sl}_3$-weight of the singular vector $v[x, y] \otimes e^{m_2 c_2}$ is not dominant integral, we conclude:

$$\begin{aligned} &\widetilde{L}_k^{\mathfrak{sl}_3}(a\omega_1 + b\omega_2) \quad \text{is irreducible} \\ &\quad \Longleftrightarrow \; v[x, y] \otimes e^{m_2 c_2} \notin \widetilde{L}_k^{\mathfrak{sl}_3}(a\omega_1 + b\omega_2) \\ &\quad \Longleftrightarrow \; \widetilde{L}_k^{\mathfrak{sl}_3}(a\omega_1 + b\omega_2)_{top} = L_k^{\mathfrak{sl}_3}(a\omega_1 + b\omega_2)_{top}. \end{aligned}$$

Next we recall that

$$L[x, y]_{top} = i \iff G^+(0)^i v[x, y] = 0 \iff h_i(x, y) = 0.$$

In our case we get $h_i(x, y) = (1 + a - i)(b + i - k - 2)$, which implies that $h_{a+1}(x, y) = 0$. So $\dim L[x, y]_{top} = a + 1 < \infty$.

Using the automorphism ρ of $\mathcal{W}^k$ such that $G^+ \mapsto G^-$ we get that in our case $\rho(L[x, y]) = L[-x, y + x]$. Since $h_j(-x, y + x) = (1 + b - i)(a + i - k - 2)$, we conclude that $\dim L[-x, y + x]_{top} = b + 1 < \infty$, implying that in $L[x, y]$

$$G^-(-1)^{b+1} v[x, y] = 0.$$

We get

$$\begin{aligned} f_1(0)^{a+1}(v[x, y] \otimes e^{m_1 c_2}) &= G^+(0)^{a+1} v[x, y] \otimes e^{m_1 c_2} = 0. \\ f_2(0)^{b+1}(v[x, y] \otimes e^{m_1 c_2}) &= G^-(-1)^{b+1} v[x, y] \otimes e^{m_1 c_2 - (b+1) c_2} = 0. \end{aligned}$$

So $\widetilde{L}_k^{\mathfrak{sl}_3}(a\omega_1 + b\omega_2)_{top} \cong L_k^{\mathfrak{sl}_3}(a\omega_1 + b\omega_2)_{top}$, and therefore $\widetilde{L}_k^{\mathfrak{sl}_3}(a\omega_1 + b\omega_2)$ is irreducible. This proves assertion (1). The proof of assertion (2) follows from (1) and the fact that $v \otimes e^{\frac{a+2b}{3}}$ is, up to a scalar factor, unique singular vector in $L[x, y] \otimes \mathcal{S} \otimes \Pi(0)^{1/3}$ whose weight is dominant integral with respect to $\mathfrak{sl}_3$. $\square$

We also note the following important consequence of the previous theorem.

Corollary 7.4 *For $a, b \in \mathbb{Z}_{\geq 0}$ and $k \notin \mathbb{Z}_{\geq 0}$ we have:*

- $H_{f_{min}}\left(L_k^{\mathfrak{sl}_3}(a\omega_1 + b\omega_2)\right) = L[x, y]$, *where the weight (x, y) is defined by (7.3)–(7.4).*
- $L_k^{\mathfrak{sl}_3}(a\omega_1 + b\omega_2)$ *is a $L_k(\mathfrak{sl}_3)$-module if and only if $L[x, y]$ is a $\mathcal{W}_k$-module.*

8 Relaxed $L_k(\mathfrak{sl}_3)$-modules

Theorem 8.1 *Assume that R is an irreducible $\mathbb{Z}_{\geq 0}$-graded $\mathcal{W}_k$-module with top component R_{top}. Then $R \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2)$ is an almost irreducible $L_k(\mathfrak{sl}_3)$-module in the sense that any submodule W of $L \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2)$ intersects $R_{top} \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2)_{top}$ non-trivially.*

Proof Assume that W is any submodule of $R \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2)$, take any weight vector $w \in W$. Using the action of the Borel subalgebra $\widehat{\mathfrak{b}}$ and Lemma 4.3 we get a non-zero weight vector

$$z = u \otimes e^{-\frac{1}{2}d_1 - \frac{1}{2}d_1 + t_1 c_1 + t_2 c_2} \in W,$$

where u is element of R, $t_1 - \lambda_1, t_2 - \lambda_2 \in \mathbb{Z}$.

Now we proceed as in the proof of Theorem 6.2.

- Assume that $u \notin R_{top}$. Then one of the following holds for certain $n \in \mathbb{Z}_{\geq 1}$:

$$G^+(n)u \neq 0, \; G^-(n)u \neq 0, \; L(n)u \neq 0, \; J(n)u \neq 0.$$

- Let us analyze these cases

  (a) Let $J(n)u \neq 0$. Then $(h_i, -\frac{1}{2}d_1 - \frac{1}{2}d_1 + t_1 c_1 + t_2 c_2) \neq 0$ for certain $i \in \{1, 2\}$. This implies that

$$\begin{aligned} h_i(n).(u \otimes e^{-\frac{1}{2}d_1 - \frac{1}{2}d_1 + t_1 c_1 + t_2 c_2}) &= \nu J(n)u \otimes e^{-\frac{1}{2}d_1 - \frac{1}{2}d_1 + t_1 c_1 + t_2 c_2} \\ &= u_1 \otimes e^{-\frac{1}{2}d_1 - \frac{1}{2}d_1 + t_1 c_1 + t_2 c_2} \quad (\nu \neq 0) \end{aligned}$$

  for certain u_1, $\mathrm{wt}(u_1) < \mathrm{wt}(u)$.

  (b) Let $G^+(n)u \neq 0$ and $J(i)u = 0$ for all $i \geq 0$. Then

$$\begin{aligned} & f_1(k).(u \otimes e^{-\frac{1}{2}d_1 - \frac{1}{2}d_1 + t_1 c_1 + t_2 c_2}) \\ &= (G^+(n) + J(n)e^{c_1 - c_2}_{-1})u \otimes e^{-\frac{1}{2}d_1 - \frac{1}{2}d_1 + t_1 c_1 + t_2 c_2} \\ &= u_2 \otimes e^{-\frac{1}{2}d_1 - \frac{1}{2}d_1 + t_1 c_1 + t_2 c_2} \end{aligned}$$

  for certain u_2, $\mathrm{wt}(u_2) < \mathrm{wt}(u)$.

  (c) Let $G^-(n)u \neq 0$ and $J(i)u = G^-(n+i)u = 0$ for all $i > 0$. Then

$$\begin{aligned} f_2(k+1).(u \otimes e^{-\frac{1}{2}d_1 - \frac{1}{2}d_1 + t_1 c_1 + t_2 c_2}) &= G^-(k)u \otimes e^{-c_2}_{-2} e^{-\frac{1}{2}d_1 - \frac{1}{2}d_1 + t_1 c_1 + t_2 c_2} \\ &= u_3 \otimes e^{-\frac{1}{2}d_1 - \frac{1}{2}d_1 + t_1 c_1 + (t_2 - 1) c_2} \end{aligned}$$

  for certain u_3, $\mathrm{wt}(u_3) < \mathrm{wt}(u)$.

  (d) Let $G^{\pm}(i)u = J(i)u = L(n+i)u = 0$ for all $i > 0$ and $L(n)u \neq 0$. Then

$$\begin{aligned} & f_3(n).(u \otimes e^{-\frac{1}{2}d_1 - \frac{1}{2}d_1 + t_1 c_1 + t_2 c_2}) \\ &= (k+3)L(n)u \otimes e^{-c_2}_{-2} e^{-\frac{1}{2}d_1 - \frac{1}{2}d_1 + t_1 c_1 + (t_2 - 1) c_2} \\ &= u_4 \otimes e^{-\frac{1}{2}d_1 - \frac{1}{2}d_1 + t_1 c_1 + (t_2 - 1) c_2} \end{aligned}$$

  for certain u_4, $\mathrm{wt}(u_4) < \mathrm{wt}(u)$.

- By applying (a)–(d) and simplicity of the $\mathcal{W}_k$-module R we conclude that $\bar{u} \otimes e^{-\frac{1}{2}d_1-\frac{1}{2}d_1+t_1c_1+(t_2-1)c_2} \in W$ for certain $\bar{u} \in R_{top}$.

The proof follows. □

Remark 8.2 The proof of the Theorem 8.1 also hold for $k = -3$. The main difference with the non-critical case is that as in the proof of Theorem 6.2 the case (d) doesn't need to be considered.

8.1 Irreducibility of relaxed modules

Theorem 8.3 *Assume that R is an irreducible $\mathbb{Z}_{\geq 0}$-graded $\mathcal{W}_k$-module with the top component R_{top}. Assume that $R_{top} \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2)_{top}$ is irreducible $\mathfrak{sl}_3$-module. Then $R \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2)$ is an irreducible $L_k(\mathfrak{sl}_3)$-module.*

Proof Assume first that $k \neq -3$. We need to prove that every vector $w \in R \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2)$ is cyclic. Let $W = L_k(\mathfrak{sl}_3).w$. By Theorem 8.1, we have that there is a vector $0 \neq w_1 \in W \cap R_{top} \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2)_{top}$. By assumption, $U(\mathfrak{sl}_3).w_1 = L_{top} \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2)_{top}$.

Let

$$\mathcal{U} = L_k(\mathfrak{sl}_3).\left(R_{top} \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2)_{top}\right).$$

Therefore for the proof of irreducibility, it suffices to prove the following.

$$(*) \quad R \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2) = \mathcal{U}$$

Define the following vectors in $L_k(\mathfrak{sl}_3)$:

$$\begin{aligned} R_J &= h_1 \\ R_{G^+} &= f_1 \\ R_{G^-} &= e_3(-1)f_2 \\ R_L &= L_{sug} = L + \frac{1}{2}\partial J + \frac{k}{3}\partial c_2 - \frac{1}{2}\partial d_1 - \frac{1}{2}\partial d_2 + \frac{1}{2} :(c_1d_1 + c_2d_2):. \end{aligned}$$

We have the following spanning set for R:

$$L(-k_1)\cdots L(-k_i)G^-(-l_1)\cdots G^{\cdot}(-l_j)G^+(-n_1)\cdots G^+(-n_r)J(-m_1)\cdots J(-m_s)v$$

where $v \in L_{top}$ and

$$k_1 \geq \cdots \geq k_i \geq 2, l_1 \geq \cdots \geq l_j \geq 2, n_1 \geq \cdots \geq n_r \geq 1, m_1 \geq \cdots \geq m_s \geq 1.$$

Using induction, Theorem 8.1 and Lemma 4.3 we get:

$$L_{sug}(-k_1)\cdots L_{sug}(-k_i)R_{G^-}(-l_1)\cdots R_{G^{\cdot}}(-l_j)$$

$$
\begin{aligned}
& R_{G^+}(-n_1)\cdots R_{G^+}(-n_r)R_J(-m_1)\cdots R_J(-m_s)(v\otimes e^{-\frac{1}{2}d_1-\frac{1}{2}d_1+t_1c_1+t_2c_2})\\
&\quad = \nu L(-k_1)\cdots L(-k_i)G^-(-l_1)\cdots G^{\cdot}(-l_j)G^+(-n_1)\cdots G^+(-n_r)\\
&\qquad \cdot J(-m_1)\cdots J(-m_s)v\otimes e^{-\frac{1}{2}d_1-\frac{1}{2}d_1+t_1c_1+t_2c_2}+w_1
\end{aligned}
$$

where $\nu \neq 0$, $w_1 \in \mathcal{U}$.

This implies that

$$
w\otimes e^{-\frac{1}{2}d_1-\frac{1}{2}d_1+t_1c_1+t_2c_2}\in\mathcal{U}
$$

for every $w \in L$, $t_1, t_2 \in \mathbb{C}$, $t_i - \lambda_i \in \mathbb{Z}$, $i = 1, 2$.

Now we apply again Lemma 4.3 and get $(*)$. This proves the theorem in the case $k \neq -3$.

For the proof of the critical level case $k = -3$ is similar. The difference is that then the spanning set for L is "smaller"

$$
G^-(-l_1)\cdots G^{\cdot}(-l_j)G^+(-n_1)\cdots G^+(-n_r)J(-m_1)\cdots J(-m_s)v
$$

where $v \in L_{top}$ and

$$
l_1\geq\cdots\geq l_j\geq 2, n_1\geq\cdots\geq n_r\geq 1, m_1\geq\cdots\geq m_s\geq 1.
$$

□

Corollary 8.4 *Assume that $U \subset R_{top}\otimes\Pi_{-1,-1}(\lambda_1,\lambda_2)_{top}$ is irreducible $\mathfrak{sl}_3$-module. Then $\mathcal{L}(U) := V^k(\mathfrak{sl}_3).U$ is an irreducible $V^k(\mathfrak{sl}_3)$-module.*

9 The structure of the $\mathfrak{sl}_3$-module $L_{top}\otimes\Pi_{-1,-1}(\lambda_1,\lambda_2)_{top}$

9.1 The relaxed $\mathcal{W}^k$-modules $R_M(\lambda)$ [6].

Let $\mathcal{Z}^k$ denotes the simple Zamolodchikov algebra with generators T, W and central charge $c_k^Z = -\frac{2(3k+5)(4k+9)}{k+3}$. The relaxed $\mathcal{W}^k$-modules $R_M(\lambda)$ is constructed in [6] as $R_M(\lambda) = M\otimes\Pi_{-1}(\lambda)$ where M is an irreducible highest weight Z^k-module with highest weight vector u of T_0 eigenvalue Δ and W_0 eigenvalue w. Define the polynomial

$$
p^{w,\Delta}(x) = w-(k+2)(k+3)\Delta+[(k+3)\Delta-2(k+2)^2]x+3(k+2)x^2-x^3
$$

(see the formula (5.17) in [6]). Then $R_M(\lambda)_{top}\cong \mathrm{span}_{\mathbb{C}}\{u\otimes e^{-j+\bar{n}c}\mid n\in\mathbb{Z}\}$ and the action of Zhu's algebra $A(\mathcal{W}_k)$ is uniquely determined by

$$
\begin{aligned}
G^+(0)(u\otimes e^{-j+\bar{n}c}) &= u\otimes e^{-j+\overline{n+1}c},\\
G^-(0)(u\otimes e^{-j+\bar{n}c}) &= p^{w,\Delta}(\bar{n})u\otimes e^{-j+\overline{n-1}c},
\end{aligned}
$$

$$J(0)(u \otimes e^{-j+(\lambda+n)c}) = \left(\overline{n} - \frac{2k+3}{3}\right)(u \otimes e^{-j+\overline{n}c})$$
$$\widetilde{L}(0)(u \otimes e^{-j+\overline{c}}) = \left(\Delta + \frac{2k+3}{3}\right)(u \otimes e^{-j+\overline{n}c}).$$

where $\overline{n} = n + \lambda$. It is proved in [6, Theorem 5.12] that $R_M(\lambda)$ is irreducible $\mathcal{W}^k$-module if and only if $p^{w,\Delta}(\overline{n}) \neq 0$ for any $n \in \mathbb{Z}$. When $2k+3 \notin \mathbb{Z}_{\geq 0}$ and M is Z_k-module, then $R_M(\lambda)$ is a module for the simple vertex algebra $\mathcal{W}_k$.

9.2 The $\mathfrak{sl}_3$-action on $R_M(\lambda)_{top} \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2)_{top}$

A basis of $R_M(\lambda)_{top} \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2)_{top}$ consists of the following elements

$$v[n, m, p] := u \otimes e^{-j+\overline{n}c} \otimes e^{-\frac{d_1}{2}-\frac{d_2}{2}+\overline{m}c_1+\overline{p}c_2}, \quad (m, n, p \in \mathbb{Z})$$

where $\overline{m} = m + \lambda_1$, $\overline{n} = n + \lambda$, $\overline{p} = p + \lambda_2$. Identify $x = x(0)$ for $x \in \mathfrak{sl}_3$. We have

$$\begin{aligned}
e_1 v[n, m, p] &= (\overline{m} - \tfrac{1}{2}) v[n, m-1, p+1], \\
e_2 v[n, m, p] &= v[n, m+1, p], \\
e_3 v[n, m, p] &= v[n, m, p+1], \\
h_1 v[n, m, p] &= \left(-2\overline{n} - (\overline{m} - \tfrac{1}{2}\right) + \overline{p} + \tfrac{5(2k+3)}{6}) v[n, m, p], \\
h_2 v[n, m, p] &= \left(\overline{n} + 2\overline{m} + \overline{p} - \tfrac{8k+9}{6}\right) v[n, m, p], \\
f_1 v[n, m, p] &= v[n+1, m, p] - \left(2\overline{n} - \overline{p} - \tfrac{5(2k+3)}{6}\right) v[n, m+1, p-1], \\
f_2 v[n, m, p] &= p^{w,\Delta}(\overline{n}) v[n-1, m, p-1] + (\overline{m} - \tfrac{1}{2}) \left(\tfrac{2(2k+3)}{3} - \overline{n} - \overline{m} - \overline{p}\right) v[n, m-1, p] \\
f_3 v[n, m, p] &= A v[n, m, p-1] + B v[n+1, m-1, p] + C v[n-1, m+1, p-2] \\
A &= p^{w,\Delta}(\overline{n}+1) - p^{w,\Delta}(\overline{n}) - \left(2\overline{n} - \overline{p} - \tfrac{5(2k+3)}{6}\right)\left(\tfrac{2(2k+3)}{3} - \overline{n} - \overline{m} - \overline{p}\right) \\
B &= -\left(\overline{m} - \tfrac{1}{2}\right) \\
C &= -p^{w,\Delta}(\overline{n}).
\end{aligned}$$

Let $\mathcal{U} = U(\mathfrak{sl}_3).v[n, m, p]$. By direct calculation we get

$$\begin{aligned}
e_1 f_1 v[n, m, p] &= A_1 v[n, m, p] + \left(\overline{m} - \tfrac{1}{2}\right) v[n+1, m-1, p+1] \\
A_1 &= -\left(\overline{m} + \tfrac{1}{2}\right)\left(2\overline{n} - \overline{p} - \tfrac{5}{6}(2k+3)\right), \\
e_2 f_2 v[n, m, p] &= A_2 v[n, m, p] + p^{w,\Delta}(\overline{n}) v[n-1, m+1, p-1] \\
A_2 &= \left(\overline{m} - \tfrac{1}{2}\right)\left(\tfrac{2(2k+3)}{3} - \overline{n} - \overline{m} - \overline{p}\right), \\
e_3 f_3 v[n, m, p] &= A v[n, m, p] - \left(\overline{m} - \tfrac{1}{2}\right) v[n+1, m-1, p+1] \\
&\quad - p^{w,\Delta}(\overline{n}) v[n-1, m+1, p-1],
\end{aligned}$$

which implies that $v[n-1, m+1, p-1]$, $v[n+1, m-1, p+1] \in \mathcal{U}$. Using induction we get:

$$Z_i := v[n-i, m+i, p-i] \in \mathcal{U} \quad \forall i \in \mathbb{Z}. \tag{9.1}$$

Note that all vectors in (9.1) have the weight $(r_1', r_2') = (r_1 + \frac{1}{2} + \frac{5(2k+3)}{6}, r_2 - \frac{8k+9}{6})$, where $r_1 = -2n - m + p$, $r_2 = n + 2m + p$.

Let $V[r_1, r_2]$ be set of vectors of $R_M(\lambda)_{top} \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2)_{top}$ having weight (r_1', r_2'). We have proved that $V[r_1, r_2] \subset \mathcal{U}$.

Remark 9.1 One sees that:

- $\mathcal{U}$ has all infinite-dimensional weight spaces;
- $\mathcal{U}$ is not a Gelfand–Tsetlin module.
- So our modules are of the same type as modules in [59] (i.e., infinite-dimensional weight spaces, not Gelfand–Tsetlin module). But we don't see how to construct an isomorphism (or embedding), so it is possible that we discovered new type of modules with infinite-dimensional weight spaces. This deserves further investigation.

Conjecture 9.2 *Assume that*

$$p^{w,\Delta}(\overline{n}) \neq 0, \overline{m} - \tfrac{1}{2} \neq 0, 2\overline{n} - \overline{p} - \tfrac{5(2k+3)}{6} \neq 0, \tfrac{2(2k+3)}{3} - \overline{n} - \overline{m} - \overline{p} \neq 0$$

for all $n, m, p \in \mathbb{Z}$.

Then $R_M(\lambda)_{top} \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2)_{top}$ *is an irreducible* $\mathfrak{sl}_3$*-module.*

10 The case dim $R_{top} < \infty$

Let us consider the case of highest weight $\mathcal{W}^k$-modules R such that R_{top} is finite-dimensional.

The Zhu algebra $A(\mathcal{W}^k)$ is generated by $[G^+]$, $[G^-]$, $[J]$, $[\widetilde{L}]$ and it is isomorphic to a quotient of Smith algebra (cf. [5, 23]). We have

$$G^-(0)G^+(0)^i v_{x,y} = h_i(x, y)G^+(0)^{i-1} v_{x,y},$$

where (cf. [5, 23])

$$h_i(x, y) = -i^2 + ki - 3xi + 3i - 3x^2 - k + 2kx + 6x + ky + 3y - 2.$$

Assume that $L[x, y]_{top} = N < \infty$, then

$$L[x, y]_{top} = \mathrm{span}_{\mathbb{C}}\{G^+(0)^i v_{x,y} \mid 0 \leq i \leq N-1\},$$

and $h_N(x, y) = 0$.

This implies that $L[x,y]_{top} \otimes \Pi_{-1,-1}(\lambda_1,\lambda_2)_{top}$ has the following basis:

$$w[i,m,p] := G^+(0)^i v_{x,y} \otimes e^{-\frac{d_1}{2}-\frac{d_2}{2}+\overline{m}c_1+\overline{p}c_2},$$

where $i = 0,\dots,N-1, m, p \in \mathbb{Z}$. The $\mathfrak{sl}_3$ action on $L[x,y]_{top} \otimes \Pi_{-1,-1}(\lambda_1,\lambda_2)_{top}$ is given by

$$\begin{aligned}
e_1 w[i,m,p] &= \left(\overline{m}-\tfrac{1}{2}\right) w[i,m-1,p+1],\\
e_2 w[i,m,p] &= w[i,m+1,p],\\
e_3 w[i,m,p] &= w[i,m,p+1],\\
h_1 w[i,m,p] &= (-2(x+i) - \left(\overline{m}-\tfrac{1}{2}\right) + \overline{p} + \tfrac{5(2k+3)}{6}) w[i,m,p],\\
h_2 w[n,m,p] &= \left(x+i+2\overline{m}+\overline{p}-\tfrac{8k+9}{6}\right) w[i,m,p],\\
f_1 w[i,m,p] &= v[i+1,m,p] - \left(2(x+i)-\overline{p}-\tfrac{5(2k+3)}{6}\right) w[i,m+1,p-1],\\
&\quad (0 \le i \le N-2),\\
f_1 w[N-1,m,p] &= -\left(2(x+N-1)-\overline{p}-\tfrac{5(2k+3)}{6}\right) w[N-1,m+1,p-1]\\
f_2 w[i,m,p] &= h_i(x,y) v[i-1,m,p-1] +\\
&\quad \left(\overline{m}-\tfrac{1}{2}\right)\left(\tfrac{2(2k+3)}{3}-(x+i)-\overline{m}-\overline{p}\right) w[i,m-1,p],\\
&\quad (1 \le i \le N-1)\\
f_2 w[0,m,p] &= \left(\overline{m}-\tfrac{1}{2}\right)\left(\tfrac{2(2k+3)}{3}-x-\overline{m}-\overline{p}\right) w[0,m-1,p]\\
f_3 w[n,m,p] &= [f_2,f_1] w[n,m,p]
\end{aligned}$$

In the special, but very important case $N = 1$, we get a realization of a family of irreducible, relaxed modules for arbitrary level k:

Theorem 10.1 *Assume that*

$$h_1(x,y) = (3+2k)x - 3x^2 + (3+k)y = 0,$$

and for all $m, p \in \mathbb{Z}$:

$$\overline{m}-\tfrac{1}{2} \neq 0, \ \tfrac{2(2k+3)}{3} - x - \overline{m} - \overline{p} \neq 0, \ 2x - \overline{p} - \tfrac{5(2k+3)}{6} \neq 0.$$

Then we have:

- $L[x,y]_{top} \otimes \Pi_{-1,-1}(\lambda_1,\lambda_2)_{top}$ *is an irreducible* $\mathfrak{sl}_3$*-module.*
- $L[x,y] \otimes \Pi_{-1,-1}(\lambda_1,\lambda_2)$ *is an irreducible* $V^k(\mathfrak{sl}_3)$*-module.*
- *If* $L[x,y]$ *is an* $\mathcal{W}_k$*-module and* $k \notin \mathbb{Z}_{\ge 0}$*, then* $L[x,y] \otimes \Pi_{-1,-1}(\lambda_1,\lambda_2)$ *is an irreducible* $L_k(\mathfrak{sl}_3)$*-module.*

Proof Note that in the case $N = 1$, the actions of f_1 and f_2 are simplified:

$$f_1 w[0,m,p] = -\left(2x - \overline{p} - \tfrac{5(2k+3)}{6}\right) w[0,m+1,p-1]$$

$$f_2 w[0,m,p] = \left(\overline{m}-\tfrac{1}{2}\right)\left(\tfrac{2(2k+3)}{3}-x-\overline{m}-\overline{p}\right) w[0,m-1,p]$$

The proof of irreducibility of $L[x,y]_{top}\otimes\Pi_{-1,-1}(\lambda_1,\lambda_2)_{top}$ is now easy. The second and third assertions follow from Theorem 8.3. $\square$

Next we should look at the case $N\geq 2$. Consider the vector space

$$W[r_1,r_2] = \mathrm{span}_{\mathbb{C}}\{w[i,m-i,p+i], i=0,\ldots,N-1\}$$

of vectors of weight

$$(r_1',r_2') = \left(r_1-2x-\lambda_2+\lambda_1+\tfrac{1}{2}+\tfrac{5(2k+3)}{6}, r_2+2\lambda_1-\lambda_2-\tfrac{8k+9}{6}\right),$$

where $r_1=-m+p$, $r_2=2m+p$.

Denote by $\mathbf{C}(\mathfrak{h})$ the centralizator of the Cartan subalgebra $\mathfrak{h}$ in $U(\mathfrak{sl}_3)$.

Lemma 10.2 *Let $N>1$. Assume that*

$$h_N(x,y)=0,\ h_j(x,y)\neq 0\quad (1\leq j\leq N-1),$$

and that for all $m,p\in\mathbb{Z}$:

$$\overline{m}-\tfrac{1}{2}\neq 0, \tfrac{2(2k+3)}{3}-x-\overline{m}-\overline{p}\neq 0, 2x-\overline{p}-\tfrac{5(2k+3)}{6}\neq 0.$$

Then $W[r_1,r_2]$ is an irreducible $\mathbf{C}(\mathfrak{h})$-module.

Proof Define $Z_i := w[i,m-i,p+i]$, $i=0,\ldots,N-1$, and elements $u_i := e_i f_i\in\mathbf{C}(\mathfrak{h})$, $i=1,2$. We have:

$$u_1 Z_{N-1} = \underbrace{-(\overline{m}-N+1)\left(2x+N-1-\overline{p}-\tfrac{5(2k+3)}{6}\right)}_{=a_{N-1}} Z_{N-1}$$

$$u_1 Z_i = \underbrace{-\left(\overline{m}+\tfrac{1}{2}\right)\left(2x+i-\overline{p}-\tfrac{5(2k+3)}{6}\right)}_{=a_i} Z_i - \left(\overline{m}-\tfrac{1}{2}\right) Z_{i+1}$$
$$(0\leq i\leq N-2).$$

Similarly we get

$$u_2 Z_0 = \underbrace{\left(\overline{m}-\tfrac{1}{2}\right)\left(\tfrac{2(2k+3)}{3}-x-\overline{m}-\overline{p}\right)}_{=b_0} Z_0$$

$$u_2 Z_i = \underbrace{\left(\overline{m}-i-\tfrac{1}{2}\right)\left(\tfrac{2(2k+3)}{3}-(x+i)-\overline{m}-\overline{p}\right)}_{=b_i} Z_i + h_i(x,y) Z_{i-1}$$
$$(1\leq i\leq N-1).$$

(1) First we see that each Z_i is cyclic.
Let $U = \mathbf{C}(\mathfrak{h}).Z_i$. If $i < N-1$, we take any $N-1 \geq j > i$. Then

$$(u_1 - b_{j-1}) \cdots (u_1 - b_i) Z_i = \nu Z_j \quad (\nu \neq 0).$$

This implies that $Z_i, \ldots, Z_{N-1} \in U$. If $i > 0$, we take any $0 \leq j < i$ and get

$$(u_1 - b_0) \cdots (u_1 - b_{i-1}) Z_i = \nu Z_j \quad (\nu \neq 0).$$

This way we get $U = W[r_1, r_2]$.

(2) Now we prove that $W[r_1, r_2]$ is irreducible.
Let $W \subset W[r_1, r_2]$ by any non-zero submodule.
Let

$$w_1 = c_1 Z_{i_1} + \cdots + c_\ell Z_{i_\ell} \in W$$

where all $c_i \neq 0$ and $0 \leq i_1 < \cdots < i_\ell \leq N$. By applying $u_1 - a_{i_1}$, we get

$$w_2 = (u_1 - a_{i_1}) w_1 = c'_1 Z_{j_1} + \cdots + c'_{\ell'} Z_{j_{\ell'}} \in W$$

where $c'_1 \neq 0$, $j_1 > i_1$.
By continuing this approach, after finitely many steps, we get that $Z_i \in W$ for certain $0 \leq i \leq N-1$, implying by (1) that $W = W[r_1, r_2]$.
This proves the irreducibility.

□

Theorem 10.3 *Assume that the conditions of Lemma 10.2 are satisfied. Then $L[x, y]_{top} \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2)_{top}$ is an irreducible $\mathfrak{sl}_3$-module and $L[x, y] \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2)$ is an irreducible $V^k(\mathfrak{sl}_3)$-module.*

If $L[x, y]$ is an $\mathcal{W}_k$-module and $k \notin \mathbb{Z}_{\geq 0}$, then $L[x, y] \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2)$ is an irreducible $L_k(\mathfrak{sl}_3)$-module.

Proof One sees $L[x, y]_{top} \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2)_{top}$ is a direct sum of N-dimensional weight spaces $W[r_1, r_2]$ as above, and all operators $e_i(0)$, $f_i(0)$, $i = 1, 2$ act injectively on $W[r_1, r_2]$. Then applying Lemma 10.2 we get that $L[x, y]_{top} \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2)_{top}$ is irreducible $\mathfrak{sl}_3$-module. The irreducibility of $L[x, y] \otimes \Pi_{-1,-1}(\lambda_1, \lambda_2)$ follows by Theorem 8.3. □

11 Screening operators and logarithmic modules for $\mathcal{W}_k$

Recall that $T(z)$, $W(z)$ are generators of $\mathcal{Z}^k = \mathcal{W}^k(\mathfrak{sl}_3, f_{\text{prin}})$ with conformal weight 2, 3 such that W is primary to T, and T, W satisfy the same OPE relations as in [6], see also Sect. 2.5. Also recall that

$$T(z) = \sum_{n \in \mathbb{Z}} T_n z^{-n-2}, \quad W(z) = \sum_{n \in \mathbb{Z}} W_n z^{-n-3}.$$

Proposition 11.1 *There exist a highest weight $\mathcal{Z}^k$-module $\widetilde{L}$ with the highest weight vector v_0 such that*

$$T_0 v_0 = \frac{4k+9}{3} v_0, \quad W_0 v_0 = -\frac{(k+3)(4k+9)(5k+12)}{27} v_0,$$

and that

$$V = \int (v_0 \otimes \mathrm{e}^{-\frac{2k+3}{6}c+\frac{1}{2}d})(z)dz : \mathcal{Z}^k \otimes \Pi(0) \to \widetilde{L} \otimes \Pi(0).\mathrm{e}^{-\frac{2k+3}{6}c+\frac{1}{2}d}$$

is a screening operator of $\mathcal{W}^k \hookrightarrow \mathcal{Z}^k \otimes \Pi(0)$.

Proof We have

$$\begin{aligned}\mathcal{W}^k &= \mathrm{Ker} \int \mathrm{e}^{-\alpha_1/(k+3)}(z)dz \cap \mathrm{Ker} \int \beta(z)\mathrm{e}^{-\alpha_2/(k+3)}(z)dz \\ &= \mathrm{Ker} \int \mathrm{e}^{-\widetilde{\alpha}_1/(k+3)}(z)dz \cap \mathrm{Ker} \int \mathrm{e}^{-\widetilde{\alpha}_2/(k+3)}(z)dz \cap \mathrm{Ker} \int \mathrm{e}^{x}(z)dz,\end{aligned}$$

where we use $\beta \mapsto \mathrm{e}^{x+y}$, $\gamma \mapsto -:x\mathrm{e}^{-x-y}:$ and

$$\widetilde{\alpha}_1 = \alpha_1, \quad \widetilde{\alpha}_2 = \alpha_2 - (k+3)(x+y).$$

Let $\mathcal{Z}^k = \mathcal{W}^k(\mathfrak{sl}_3, f_{\mathrm{prin}})$. Since

$$\mathcal{Z}^k = \mathrm{Ker} \int \mathrm{e}^{-\widetilde{\alpha}_1/(k+3)}(z)dz \cap \mathrm{Ker} \int \mathrm{e}^{-\widetilde{\alpha}_2/(k+3)}(z)dz,$$

we get

$$\mathcal{W}^k \hookrightarrow \mathcal{Z}^k \otimes \Pi(0) \tag{11.1}$$

by forgetting the screening operator $\int \mathrm{e}^{x}(z)dz$, where $\Pi(0)$ is generated by $c, d, \mathrm{e}^{c}, \mathrm{e}^{-c}$ and

$$c = x+y, \quad d = \frac{2}{3}\alpha_1 + \frac{4}{3}\alpha_2 - \frac{2k+3}{3}x - \frac{2k+9}{3}y.$$

Thus the embedding (11.1) commutes with $\int \mathrm{e}^{x}(z)dz$ and we have

$$x = -\frac{\widetilde{\alpha}_1 + 2\widetilde{\alpha}_2}{3} - \frac{2k+3}{6}c + \frac{1}{2}d$$

so that

$$\mathrm{e}^{x} = \mathrm{e}^{p} \otimes \mathrm{e}^{-\frac{2k+3}{6}c+\frac{1}{2}d}, \quad p = -\frac{\widetilde{\alpha}_1 + 2\widetilde{\alpha}_2}{3}.$$

Since generators $T(z)$, $W(z)$ of $\mathcal{Z}^k$ in [6] can be written by

$$
\begin{aligned}
T &= \frac{1}{k+3}\left((k+2)\partial(\widetilde{\alpha}_1+\widetilde{\alpha}_2)+\frac{1}{3}:(\widetilde{\alpha}_1^2+\widetilde{\alpha}_1\widetilde{\alpha}_2+\widetilde{\alpha}_2^2):\right),\\
W &= -\frac{(k+2)^2}{6}\partial^2(\widetilde{\alpha}_1-\widetilde{\alpha}_2)-\frac{k+2}{6}:((\partial\widetilde{\alpha}_1)(2\widetilde{\alpha}_1+\widetilde{\alpha}_2)-(\widetilde{\alpha}_1+2\widetilde{\alpha}_2)\partial\widetilde{\alpha}_2):\\
&\quad -\frac{1}{27}:(2\widetilde{\alpha}_1^3+3\widetilde{\alpha}_1^2\widetilde{\alpha}_2-3\widetilde{\alpha}_1\widetilde{\alpha}_2^2-2\widetilde{\alpha}_2^3):,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
T(z)\mathrm{e}^p(w) &\sim \frac{(4k+9)\mathrm{e}^p(w)}{3(z-w)^2}-\frac{:(\widetilde{\alpha}_1+2\widetilde{\alpha}_2)\mathrm{e}^p:(w)}{3(z-w)},\\
W(z)\mathrm{e}^p(w) &\sim -\frac{(k+3)(4k+9)(5k+12)\mathrm{e}^p(w)}{27(z-w)^3}\\
&\quad -\frac{(k+3)(5k+12):(\widetilde{\alpha}_1+2\widetilde{\alpha}_2)\mathrm{e}^p:(w)}{18(z-w)^2}\\
&\quad +\frac{(k+3):(\widetilde{\alpha}^2-2\widetilde{\alpha}_1\widetilde{\alpha}_1\widetilde{\alpha}_2-2\widetilde{\alpha}_2^2+(4k+9)\partial\widetilde{\alpha}_1+2(k+3)\partial\widetilde{\alpha}_2)\mathrm{e}^p:(w)}{9(z-w)}.
\end{aligned}
$$

Therefore

$$
T_0\mathrm{e}^p=\frac{4k+9}{3}\mathrm{e}^p,\quad W_0\mathrm{e}^p=-\frac{(k+3)(4k+9)(5k+12)}{27}\mathrm{e}^p.
$$

The assertion now follows by taking $v_0=e^p$ and $\widetilde{L}=\mathcal{Z}^k.e^p$. □

Let $i=-\frac{2k+3}{6}c+\frac{d}{2}$. Then $e^x=e^p\otimes e^i$. Then $\widetilde{L}=\mathcal{Z}^k.e^p$ is an highest weight $\mathcal{Z}^k$-module with highest weight $(\frac{4k+9}{3},-\frac{(4k+9)(k+3)(5k+12)}{27})$ from Proposition 11.1. We denote its irreducible quotient by $L_{11;12}$. By Proposition 11.1, we have that there exist the intertwining operator $\widetilde{\mathcal{Y}}(\cdot,z)$ of the type

$$
\binom{\widetilde{L}\otimes\Pi(0)e^i}{\widetilde{L}\otimes\Pi(0)e^i\quad \mathcal{Z}^k\otimes\Pi(0)},
$$

such that

- $e^x(z)=\widetilde{\mathcal{Y}}(e^x,z)$,
- e_0^x is a $\mathcal{W}^k$-homomorphism and $e_0^x|\mathcal{W}^k=0$.

Using the skew symmetry of the intertwining operators (cf. [53]) we get that

$$
I\binom{\widetilde{L}\otimes\Pi(0)e^i}{\widetilde{L}\otimes\Pi(0)e^i\quad \mathcal{Z}^k\otimes\Pi(0)}\cong I\binom{\widetilde{L}\otimes\Pi(0)e^i}{\mathcal{Z}^k\otimes\Pi(0)\quad \widetilde{L}\otimes\Pi(0)e^i},
$$

which implies that

$$
e^x(z)v=e^{zD}Y_{\widetilde{L}\otimes\Pi(0)e^i}(v,-z)(e^p\otimes e^i),
$$

where $Y_{\widetilde{L}\otimes\Pi(0)e^i}(\cdot,z)$ is the vertex operator that defines the module structure on $\widetilde{L}\otimes\Pi(0)e^i$. For $v\in\mathcal{W}^k\hookrightarrow\mathcal{Z}^k\otimes\Pi(0)$, we get

$$\int\left(e^x(z)v\right)dz=\int\left(e^{-zD}Y_{\widetilde{L}\otimes\Pi(0)e^i}(v,-z)(e^p\otimes e^i)\right)dz=0. \tag{11.2}$$

Assume that $L_{11;12}$ is an (irreducible) $\mathcal{Z}_k$-module. Then we have

$$\dim I\binom{L_{11;12}\otimes\Pi(0)e^i}{L_{11;12}\otimes\Pi(0)e^i \quad \mathcal{Z}_k\otimes\Pi(0)}=\dim I\binom{L_{11;12}\otimes\Pi(0)e^i}{\mathcal{Z}_k\otimes\Pi(0) \quad L_{11;12}\otimes\Pi(0)e^i}=1.$$

This implies that there is a unique, up to a scalar factor, intertwining operator $\mathcal{Y}(\cdot,z)$ of type $\binom{L_{11;12}\otimes\Pi(0)e^i}{L_{11;12}\otimes\Pi(0)e^i \quad \mathcal{Z}_k\otimes\Pi(0)}$ which is obtained as a transpose of the vertex operator $Y_{L_{11;12}\otimes\Pi(0)e^i}(\cdot,z)$ defining the unique structure of a module on $L_{11;12}\otimes\Pi(0)e^i$.

As a consequence, we get:

$$\mathcal{Y}(v[11;12]\otimes e^i,z)v=e^{zD}Y_{L_{11;12}\otimes\Pi(0)e^i}(v,-z)(v[11;12]\otimes e^i),$$

where $Y_{L_{11;12}\otimes\Pi(0)e^i}(\cdot,z)$ is the vertex operator that defines the $\mathcal{Z}_k\otimes\Pi(0)$-module structure on $L_{11;12}\otimes\Pi(0)e^i$.

Since $L_{11;12}\otimes\Pi(0)e^i$ is a simple quotient of $\widetilde{L}\otimes\Pi(0)e^i$, the relation (11.2) implies that for $v\in\mathcal{W}_k$,

$$\begin{aligned}&\int\left(\mathcal{Y}(v[11;12]\otimes e^i,z)v\right)dz\\&\quad=\int\left(e^{zD}Y_{L_{11;12}\otimes\Pi(0)e^i}(v,-z)(v[11;12]\otimes e^i)\right)dz=0.\end{aligned} \tag{11.3}$$

In this way we have proved the following proposition.

Proposition 11.2 *Assume that $L_{11;12}$ is a $\mathcal{Z}_k$-module. Let $v[11;12]$ denotes the highest weight vector of $L_{11;12}$. Let*

$$Q=\int\mathcal{Y}(v[11;12]\otimes e^i,z)dz:\mathcal{Z}_k\otimes\Pi(0)\to L_{11;12}\otimes\Pi(0).e^i.$$

Then Q is an non-trivial screening operator which commutes with the action of $\mathcal{W}_k$ and we have

$$Qe^{-c}=v[11;12]\otimes e^{i-c}.$$

For the construction of logarithmic modules we shall use methods from [3] and [11].

Consider now extended vertex algebra

$$\mathcal{V}_k=\mathcal{Z}_k\otimes\Pi(0)\oplus L_{11;12}\otimes\Pi(0)e^i.$$

Let $\mathcal{M}$ be any $\mathcal{V}_k$-module. Define

$$(\widetilde{\mathcal{M}}, \widetilde{Y}_{\mathcal{M}}(\cdot, z)) := (\mathcal{M}, Y_{\mathcal{M}}(\Delta(s, z)\cdot, z)),$$

where

$$s = v[11; 12] \otimes e^i,$$
$$\Delta(s, z) = z^{s_0} \exp\left(\sum_{n=1}^{\infty} \frac{s(n)}{-n}(-z)^{-n}\right).$$

Since

$$\mathcal{W}_k \subset \mathrm{Ker}(Q : \mathcal{V}_k \to \mathcal{V}_k),$$

we conclude that $\widetilde{Y}_{\mathcal{M}}(\cdot, z)$ defines on $\widetilde{\mathcal{M}}$ the structure of a $\mathcal{W}_k$-module. The action of $\widetilde{L}^{BP}(0)$ on $\widetilde{\mathcal{M}}$ is

$$\widetilde{L}^{BP}(0) = L^{BP}(0) + Q.$$

Proposition 11.3 *Assume that $Z_{11;12}$ is a $\mathcal{Z}_k$-module and $2k + 3 \notin \mathbb{Z}_{\geq 0}$.*

- *Assume that $\mathcal{M}$ is a $L^{BP}(0)$ semi-simple $\mathcal{V}$-module such that Q acts non-trivially on $\mathcal{M}$. Then $\widetilde{\mathcal{M}}$ is a logarithmic $\mathcal{W}_k$-module.*
- *$\widetilde{\mathcal{V}_k}$ is a logarithmic $\mathcal{W}_k$-module of $\widetilde{L}^{BP}(0)$ nilpotent rank two.*

Remark 11.4 Let $k = -3 + \frac{u}{v}$ be an admissible level, that is u, v are coprime positive integers and $u \geq 3$. Then $Z_{11;12}$ is a $\mathcal{Z}_k$-module if $v > 3$.

12 Logarithmic $L_k(\mathfrak{sl}_3)$-modules

First we recall that the $\beta\gamma$ vertex operator algebra $\mathcal{S}(1)$ has the logarithmic (projective) modules $\mathcal{P}_s$, $s \in \mathbb{Z}$, of $L^{\mathcal{S}(1)}(0)$-nilpotent rank two (cf. [15, 26]). We get:

Proposition 12.1 *Assume $k \notin \mathbb{Z}_{\geq 0}$ and assume that N is irreducible, highest weight or relaxed $\mathcal{W}_k$-module. Then for each $r, s \in \mathbb{Z}$, $\lambda \in \mathbb{C}$:*

$$N \otimes \mathcal{P}_s \otimes \Pi(0).e^{\frac{r}{2}d_2 + \lambda c_2}$$

is an logarithmic $L_k(\mathfrak{sl}_3)$-module of $L_{sug}(0)$-nilpotent rank two.

Proof Since $L_k(\mathfrak{sl}_3) \hookrightarrow \mathcal{W}_k \otimes \mathcal{S}(1) \otimes \Pi(0)$, we get that $N \otimes \mathcal{P}_s \otimes \Pi(0).e^{\frac{r}{2}d_2 + \lambda c_2}$ is a $L_k(\mathfrak{sl}_3)$-module.

Using (3.4) we get that

$$L_{sug} \mapsto L^{\mathrm{BP}} + \frac{1}{2}\partial J + L^{\mathcal{S}(1)} + L^{\Pi(0)},$$

where L^{BP}, $L^{\mathcal{S}(1)}$, $L^{\Pi(0)}$ are commuting Virasoro vectors on $\mathcal{W}_k$, $\mathcal{S}(1)$, $\Pi(0)$, respectively. Since $J(0)$, $L^{\mathrm{BP}}(0)$, $L^{\Pi(0)}$ act semisimply, and $L^{\mathcal{S}(1)}(0)$ has nilpotent rank two, we conclude that $L_{sug}(0)$ has nilpotent rank two. The proof follows. □

We can now consider the logarithmic $\mathcal{W}_k$-modules from Sect. 11. For simplicity we shall consider only logarithmic module $\widetilde{\mathcal{V}}_k$.

Proposition 12.2 *Assume that $k = -3 + \frac{u}{v}$ be an admissible level such that is u, v are coprime positive integers and $v \geq 4$.*

Then for each $r, s \in \mathbb{Z}$, $\lambda \in \mathbb{C}$:

$$\widetilde{\mathcal{V}}_k \otimes \mathcal{P}_s \otimes \Pi(0).e^{\frac{r}{2}d_2+\lambda c_2}$$

is a logarithmic $L_k(\mathfrak{sl}_3)$-module of $L_{sug}(0)$-nilpotent rank three.

Remark 12.3 Let us summarize our result on logarithmic modules.

- $k \in \mathbb{Z}_{\geq 0}$, $L_k(\mathfrak{sl}_3)$ is rational and there are no logarithmic modules.
- $k \in \frac{3}{2} + \mathbb{Z}_{\geq 0}$, $L_k(\mathfrak{sl}_3)$ is admissible, and there exist logarithmic modules of rank two constructed using $\beta\gamma$ projective modules $\mathcal{P}_s$. It is expected that there exist logarithmic modules of higher rank.
- If k is admissible satisfying condition of Proposition 12.2. Then there exist logarithmic modules of rank three. They are constructed using $\mathcal{W}_k$-modules of type $\widetilde{\mathcal{V}}_k$ and $\beta\gamma$ modules $\mathcal{P}_s$.
- If k is generic, there exist logarithmic modules of order three.

We hope to study these logarithmic modules in more details in our forthcoming publications, in particular we will investigate the construction of logarithmic modules of possibly higher rank. The eventual goal is to find logarithmic modules that are projective.

13 An affine analog of the Feigin–Tipunin algebra

In [2] and [4] we study logarithmic vertex algebras $\mathcal{V}^{(p)}$ which are obtained as a subalgebra of

$$\mathcal{A}^{(p)} \otimes \Pi(0)^{1/2},$$

where $\mathcal{A}^{(p)}$ is the doublet vertex algebra associated to $(1, p)$ Virasoro minimal models (cf. [12]). This vertex algebra has the natural $\mathfrak{sl}_2$-action which is used in [4] to construct a rigid braided tensor category KL_k associated to $\mathfrak{sl}_2$ for levels $k = -2 + 1/p$.

The doublet algebra $\mathcal{A}^{(p)}$ is a simple current extension of the triplet vertex algebra $\mathcal{W}(p)$ (cf. [10]). The Feigin–Tipunin algebra $\mathcal{W}_{\mathfrak{g}}(p)$ [51, 76] is a higher rank generalization of the triplet vertex algebra, where $\mathfrak{g}$ is a Lie algebra of type A, D, E. We conjecture:

Conjecture 13.1 *Let $k = -h^\vee + 1/p$.*

(1) *There exist a logarithmic vertex algebra $\mathcal{V}_{\mathfrak{g}}^{(p)}$ which is a semi-simple as $L_k(\mathfrak{g}) \times \mathfrak{g}$-module.*
(2) *KL_k is a semi-simple, rigid braided tensor category.*

In the case $\mathfrak{g} = \mathfrak{sl}_3$ we can construct such logarithmic vertex algebra.

Our realization of $L_k(\mathfrak{sl}_3)$ together with the free field realization from [6] imply that for $2k + 3 \notin \mathbb{Z}_{\geq 0}$:

$$L_k(\mathfrak{sl}_3) \hookrightarrow \mathcal{Z}_k \otimes \mathcal{S} \otimes \Pi(0)^{\otimes 2}.$$

We can transfer "classical" logarithmic VOAs to logarithmic affine VOAs:

Let $W_{A_2}(p)$ be the Feigin–Tipunin algebra of central charge $(1, p)$ models for $\mathcal{Z}_k$ (cf. [2, 51, 76]). Then

$$\mathcal{V}_{A_2}^{(p)} = (W_{A_2}(p) \otimes \mathcal{S} \otimes \Pi(0)^{1/2} \otimes \Pi(0)^{1/3})^{int_{\mathfrak{sl}_3}}$$

is a logarithmic VOA associated to $L_k(\mathfrak{sl}_3)$ for $k = -3 + 1/p$.

By using the semi-simplicity of $W_{A_2}(p)$ as $\mathcal{Z}_k$-modules from [76, 77] (see also [13] for the proof of the semi-simplicity in the case $p = 2$), the results from [6] and our new results from Sect. 7 one can show that

- $\mathcal{V}_{A_2}^{(p)}$ is a semi-simple $L_k(\mathfrak{sl}_3)$-module in KL_k for $k = -3 + 1/p$.

In our future work we shall study properties of this logarithmic vertex algebra and some approaches to Conjecture 13.1. The main remaining step in proving the conjecture is to prove the semi-simplicity of KL_k.

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