Mathematische Annalen (2024) 388:1091–1115
https://doi.org/10.1007/s00208-022-02540-3

Mathematische Annalen

Affine fractional L^p Sobolev inequalities

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Received: 6 October 2022 / Revised: 5 December 2022 / Accepted: 7 December 2022 /
Published online: 24 December 2022

Abstract

Sharp affine fractional L^p Sobolev inequalities for functions on $\mathbb{R}^n$ are established. The new inequalities are stronger than (and directly imply) the sharp fractional L^p Sobolev inequalities. They are fractional versions of the affine L^p Sobolev inequalities of Lutwak, Yang, and Zhang. In addition, affine fractional asymmetric L^p Sobolev inequalities are established.

Mathematics Subject Classification 46E35 (35R11, 52A40)

1 Introduction

Sharp fractional L^2 Sobolev inequalities are receiving increasing attention in the last decades. They are central in the study of solutions of equations involving the fractional Laplace operator $(-\Delta)^{1/2}$ which arises naturally in many non-local problems such as the stationary form of reaction-diffusion equations [9], the Signorini problem (and its equivalent formulation as the thin obstacle problem) [2], and the Dirichlet-to-Neumann operator of harmonic functions in the half-space [29]. Also, the general operators $(-\Delta)^s$ for $s \in (0, 1)$ arise in stochastic theory, associated with symmetric Levy processes (see [29] and the references therein).

Let $0 < s < 1$ and $1 \le p < n/s$. The fractional L^p Sobolev inequalities state that

$$\|f\|^p_{\frac{np}{n-ps}} \le \sigma_{n,p,s} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+ps}} \,\mathrm{d}x\,\mathrm{d}y \tag{1}$$

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for $f \in W^{s,p}(\mathbb{R}^n)$, the fractional L^p Sobolev space of functions $f \in L^p(\mathbb{R}^n)$ with finite right side in (1) (see, for example, [27]). In general, the optimal constants $\sigma_{n,p,s}$ and extremal functions are not known (see [7] for a conjecture). Equality is always attained in (1). For $p = 1$, the extremal functions of (1) are multiples of indicator functions of balls, and the constants are explicitly known. The only further known case is $p = 2$, where the constants $\sigma_{n,2,s}$ can be obtained by duality from Lieb's sharp Hardy–Littlewood–Sobolev inequalities [18] (see, for example, [10]). The asymptotic behavior of $\sigma_{n,p,s}$ as $s \to 1^-$ was studied in [5]. Almgren and Lieb [1] and Frank and Seiringer [12] showed that the extremal functions of (1) are radially symmetric and of constant sign.

By a result of Bourgain, Brezis, and Mironescu [4],

$$\lim_{s\to 1^-} p(1-s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y = \alpha_{n,p}\int_{\mathbb{R}^n}|\nabla f(x)|^p\,\mathrm{d}x$$

for $f \in W^{1,p}(\mathbb{R}^n)$, the Sobolev space of L^p functions f with weak L^p gradient ∇f, where

$$\alpha_{n,p} = \int_{\mathbb{S}^{n-1}} |\langle \xi, \eta\rangle|^p\,\mathrm{d}\xi \tag{2}$$

for any $\eta \in \mathbb{S}^{n-1}$. Here, integration on the unit sphere $\mathbb{S}^{n-1}$ is with respect to the $(n-1)$-dimensional Hausdorff measure, ω_n is the volume of the n-dimensional unit ball and $\langle\cdot,\cdot\rangle$ is the inner product on $\mathbb{R}^n$. For $p = 1$ and $p = 2$, this allows to deduce the sharp L^p Sobolev inequalities from (1) by calculating the limit of $\sigma_{n,p,s}/(1-s)$ as $s \to 1^-$.

Zhang [32] and Lutwak, Yang, and Zhang [24] obtained the following sharp affine L^p Sobolev inequality that is significantly stronger than the classical L^p Sobolev inequality:

$$\|f\|^p_{\frac{np}{n-p}} \le \sigma_{n,p}\frac{n\omega_n^{\frac{n+p}{n}}}{\alpha_{n,p}}|\Pi_p^* f|^{-\frac{p}{n}} \le \sigma_{n,p}\int_{\mathbb{R}^n}|\nabla f(x)|^p\,\mathrm{d}x \tag{3}$$

for $f \in W^{1,p}(\mathbb{R}^n)$ and $1 < p < n$, where the inequality between the first and third terms is the classical L^p Sobolev inequality and the optimal constants $\sigma_{n,p}$ were determined by Aubin [3] and Talenti [30]. We have rewritten the explicit constant for the first inequality from [24] using (2). Here $\Pi_p^* f$ is the L^p polar projection body of f, a convex body associated to f that was introduced with different notation in [24] (see Sect. 2.5), and $|\cdot|$ is the n-dimensional Lebesgue measure.

The main aim of this paper is to establish affine fractional L^p Sobolev inequalities that are stronger than the Euclidean fractional L^p Sobolev inequalities from (1) and are fractional counterparts of (3). The case $p = 1$ was studied in [16], so from now on, let $p > 1$.

Theorem 1 *Let* $0 < s < 1$ *and* $1 < p < n/s$. *For* $f \in W^{s,p}(\mathbb{R}^n)$,

$$\begin{aligned}\|f\|^p_{\frac{np}{n-ps}} &\le \sigma_{n,p,s} n \omega_n^{\frac{n+ps}{n}} \left(\frac{1}{n} \int_{\mathbb{S}^{n-1}} \left(\int_0^\infty t^{ps-1} \int_{\mathbb{R}^n} |f(x+t\xi) - f(x)|^p \, \mathrm{d}x \, \mathrm{d}t \right)^{-\frac{n}{ps}} \mathrm{d}\xi \right)^{-\frac{ps}{n}} \\ &\le \sigma_{n,p,s} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+ps}} \, \mathrm{d}x \, \mathrm{d}y.\end{aligned}$$

There is equality in the first inequality if and only if $f = h_{s,p} \circ \phi$ *for some* $\phi \in \mathrm{GL}(n)$, *where* $h_{s,p}$ *is an extremal function of* (1). *There is equality in the second inequality if* f *is radially symmetric.*

In order to prove Theorem 1, we introduce the s-fractional L^p polar projection body $\Pi_p^{*,s} f$ associated to f, defined as the star-shaped set whose gauge function for $\xi \in \mathbb{S}^{n-1}$ is

$$\|\xi\|^{ps}_{\Pi_p^{*,s} f} = \int_0^\infty t^{-ps-1} \int_{\mathbb{R}^n} |f(x+t\xi) - f(x)|^p \, \mathrm{d}x \, \mathrm{d}t$$

(see Section 3 for details). The affine fractional Sobolev inequality can now be written as

$$\|f\|^p_{\frac{np}{n-ps}} \le \sigma_{n,p,s} n \omega_n^{\frac{n+ps}{n}} |\Pi_p^{*,s} f|^{-\frac{ps}{n}}. \tag{4}$$

Since both sides of (4) are invariant under translations of f, and for volume-preserving linear transformations $\phi : \mathbb{R}^n \to \mathbb{R}^n$,

$$\Pi_p^{*,s} (f \circ \phi^{-1}) = \phi \Pi_p^{*,s} f,$$

it follows that (4) is an affine inequality. In Theorem 10, we will show that

$$\lim_{s \to 1^-} p(1-s) |\Pi_p^{*,s} f|^{-\frac{ps}{n}} = |\Pi_p^* f|^{-\frac{p}{n}},$$

which establishes the connection to the L^p polar projection bodies introduced by Lutwak, Yang and Zhang [24].

In Sect. 4, we introduce fractional asymmetric L^p polar projection bodies as fractional counterparts of the asymmetric L^p polar projection bodies of Haberl and Schuster [14], which in turn are functional versions of the asymmetric L^p polar projection bodies of convex bodies introduced in [19]. We obtain affine fractional asymmetric L^p Sobolev inequalities for non-negative functions that are stronger than the inequalities for the symmetric fractional L^p polar projection bodies.

In the proofs of the main results, we use anisotropic fractional Sobolev norms, which were introduced in [20, 21] and depend on a star-shaped set $K \subset \mathbb{R}^n$. In Sect. 10, we discuss which choice of K (with given volume) gives the minimal fractional Sobolev norm and connect it to the corresponding quest for an optimal L^p Sobolev norm solved by Lutwak, Yang, and Zhang [25].

2 Preliminaries

We collect results on function spaces, Schwarz symmetrization, star-shaped sets, anisotropic Sobolev norms, and L^p polar projection bodies that will be used in the following.

2.1 Function spaces

For $p \geq 1$ and measurable $f : \mathbb{R}^n \to \mathbb{R}$, let

$$\|f\|_p = \Big(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x\Big)^{1/p}.$$

We set $\{f \geq t\} = \{x \in \mathbb{R}^n : f(x) \geq t\}$ for $t \in \mathbb{R}$ and use similar notation for level sets, etc. We say that f is non-zero, if $\{f \neq 0\}$ has positive measure, and we identify functions that are equal up to a set of measure zero. For $p \geq 1$, let

$$L^p(\mathbb{R}^n) = \Big\{f : \mathbb{R}^n \to \mathbb{R} : f \text{ is measurable, } \|f\|_p < \infty\Big\}.$$

Here and below, when we use measurability and related notions, we refer to the n-dimensional Lebesgue measure on $\mathbb{R}^n$.

For $0 < s < 1$ and $p \geq 1$, we define the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ as

$$W^{s,p}(\mathbb{R}^n) = \Big\{f \in L^p(\mathbb{R}^n) : \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+ps}} \,\mathrm{d}x \,\mathrm{d}y < \infty\Big\}.$$

For $p \geq 1$, we set

$$W^{1,p}(\mathbb{R}^n) = \big\{f \in L^p(\mathbb{R}^n) : |\nabla f| \in L^p(\mathbb{R}^n)\big\},$$

where ∇f is the weak gradient of f.

2.2 Symmetrization

For a set $E \subset \mathbb{R}^n$, the indicator function 1_E is defined by $1_E(x) = 1$ for $x \in E$ and $1_E(x) = 0$ otherwise. Let $E \subseteq \mathbb{R}^n$ be a Borel set of finite measure. The Schwarz symmetral of E, denoted by $E^\star$, is the closed centered Euclidean ball with the same volume as E.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a non-negative measurable function with super-level sets $\{f \geq t\}$ of finite measure. The layer cake formula states that

$$f(x) = \int_0^\infty 1_{\{f \geq t\}}(x) \,\mathrm{d}t$$

for almost every $x \in \mathbb{R}^n$ and allows us to recover the function from its super-level sets. The Schwarz symmetral of f, denoted by $f^\star$, is defined by

$$f^\star(x) = \int_0^\infty 1_{\{f \geq t\}^\star}(x)\,\mathrm{d}t$$

for $x \in \mathbb{R}^n$. Hence, $f^\star$ is determined by the properties of being radially symmetric, decreasing, and having super-level sets of the same measure as those of f. Note that $f^\star$ is also called the symmetric decreasing rearrangement of f.

The proofs of our results make use of the Riesz rearrangement inequality, which is stated in full generality, for example, in [6].

Theorem 2 (Riesz's rearrangement inequality) *For $f, g, k : \mathbb{R}^n \to \mathbb{R}$ non-negative, measurable functions with super-level sets of finite measure,*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)k(x-y)g(y)\,\mathrm{d}x\,\mathrm{d}y \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^\star(x)k^\star(x-y)g^\star(y)\,\mathrm{d}x\,\mathrm{d}y.$$

We will use the characterization of equality cases of the Riesz rearrangement inequality due to Burchard [8].

Theorem 3 (Burchard) *Let A, B and C be sets of finite positive measure in $\mathbb{R}^n$ and denote by α, β and γ the radii of their Schwarz symmetrals $A^\star, B^\star$ and $C^\star$. For $|\alpha - \beta| < \gamma < \alpha + \beta$, there is equality in*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_A(y)1_B(x-y)1_C(x)\,\mathrm{d}x\,\mathrm{d}y \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_{A^\star}(y)1_{B^\star}(x-y)1_{C^\star}(x)\,\mathrm{d}x\,\mathrm{d}y$$

if and only if, up to sets of measure zero,

$$A = a + \alpha D,\ B = b + \beta D,\ C = c + \gamma D,$$

where D is a centered ellipsoid, and a, b and $c = a + b$ are vectors in $\mathbb{R}^n$.

2.3 Star-shaped sets and star bodies

A set $K \subseteq \mathbb{R}^n$ is star-shaped (with respect to the origin) if the interval $[0, x] \subset K$ for every $x \in K$. The gauge function $\|\cdot\|_K : \mathbb{R}^n \to [0, \infty]$ of a star-shaped set is defined as

$$\|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\},$$

and the radial function $\rho_K : \mathbb{R}^n \setminus \{0\} \to [0, \infty]$ as

$$\rho_K(x) = \|x\|_K^{-1} = \sup\{\lambda \geq 0 : \lambda x \in K\}.$$

The n-dimensional Lebesgue measure or volume of a star-shaped set K in $\mathbb{R}^n$ with measurable radial function is given by

$$|K| = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_K(\xi)^n \, \mathrm{d}\xi.$$

We call a star-shaped set $K \subset \mathbb{R}^n$ a star body if its radial function is strictly positive and continuous in $\mathbb{R}^n \setminus \{0\}$. On the set of star bodies, the q-radial sum for $q \neq 0$ of $K, L \subset \mathbb{R}^n$ is defined by

$$\rho^q(K \tilde{+}_q L, \xi) = \rho^q(K, \xi) + \rho^q(L, \xi)$$

for $\xi \in \mathbb{S}^{n-1}$ (cf. [28, Section 9.3]). The dual Brunn–Minkowski inequality (cf. [28, (9.41)]) states that for star bodies $K, L \subset \mathbb{R}^n$ and $q > 0$,

$$|K \tilde{+}_{-q} L|^{-q/n} \geq |K|^{-q/n} + |L|^{-q/n}, \tag{5}$$

with equality precisely if K and L are dilates, that is, there is $\lambda > 0$ such that $K = \lambda L$.

Let $\alpha \in \mathbb{R} \backslash \{0, n\}$. For star-shaped sets $K, L \subseteq \mathbb{R}^n$ with measurable radial functions, the dual mixed volume is defined as

$$\tilde{V}_\alpha(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_K(\xi)^{n-\alpha} \rho_L(\xi)^\alpha \, \mathrm{d}\xi.$$

Note that

$$\tilde{V}_\alpha(K, K) = |K|$$

and that

$$\tilde{V}_\alpha(K, L_1 \tilde{+}_\alpha L_2) = \tilde{V}_\alpha(K, L_1) + \tilde{V}_\alpha(K, L_2)$$

for star-shaped sets $K, L_1, L_2 \subseteq \mathbb{R}^n$ with measurable radial functions.

For star-shaped sets $K, L \subseteq \mathbb{R}^n$ of finite volume and $0 < \alpha < n$, the dual mixed volume inequality states that

$$\tilde{V}_\alpha(K, L) \leq |K|^{(n-\alpha)/n} |L|^{\alpha/n}. \tag{6}$$

Equality holds if and only if K and L are dilates, where we say that star-shaped sets K and L are dilates if $\rho_K = \lambda \, \rho_L$ almost everywhere on $\mathbb{S}^{n-1}$ for some $\lambda > 0$. The definition of dual mixed volume for star bodies is due to Lutwak [22], where also the dual mixed volume inequality is derived from Hölder's inequality (also see [28, Section 9.3] or [13, B.29]).

2.4 Anisotropic fractional Sobolev norms

Let $0 < s < 1$ and $p \geq 1$. For $K \subset \mathbb{R}^n$ a star body and $f \in W^{s,p}(\mathbb{R}^n)$, the anisotropic fractional L^p Sobolev norm of f with respect to K is

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{\|x - y\|_K^{n+ps}} \, \mathrm{d}x \, \mathrm{d}y. \tag{7}$$

It was introduced in [21] for K a convex body (also, see [20]). For $K = B^n$, the Euclidean unit ball, we obtain the classical s-fractional L^p Sobolev norm of f. The limit as $s \to 1^-$ was determined in [4] in the Euclidean case and in [21] in the anisotropic case. We will also consider the following asymmetric versions of (7),

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))_+^p}{\|x - y\|_K^{n+ps}} \, \mathrm{d}x \, \mathrm{d}y, \quad \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))_-^p}{\|x - y\|_K^{n+ps}} \, \mathrm{d}x \, \mathrm{d}y,$$

where $a_+ = \max\{a, 0\}$ and $a_- = \max\{-a, 0\}$ for $a \in \mathbb{R}$. The limits as $s \to 1^-$ were determined in [26].

2.5 L^p polar projection bodies

For $p \geq 1$ and $f \in W^{1,p}(\mathbb{R}^n)$, the L^p polar projection body is defined as the star body with gauge function given by

$$\|\xi\|_{\Pi_p^* f}^p = \int_{\mathbb{R}^n} |\langle \nabla f(x), \xi \rangle|^p \, \mathrm{d}x$$

for $\xi \in \mathbb{S}^{n-1}$, were $\langle \cdot, \cdot \rangle$ denotes the inner product. It is the polar body of a convex body. The definition is due to Lutwak, Yang, and Zhang [24]. For a convex body $K \subset \mathbb{R}^n$, they defined the L^p polar projection body (with a different normalization) in [23] by

$$\|\xi\|_{\Pi_p^* K}^p = \int_{\mathbb{S}^{n-1}} |\langle \xi, \eta \rangle|^p \, \mathrm{d}S_p(K, \eta), \tag{8}$$

where $S_p(K, \cdot)$ is the L^p surface area measure of K (for the definition of L^p surface area measures, see, for example, [28, Section 9.1]).

Asymmetric L^p polar projection bodies of convex bodies were introduced in [19]. For $f \in W^{1,p}(\mathbb{R}^n)$, the asymmetric L^p polar projection bodies of f are defined as the star bodies with gauge function given by

$$\|\xi\|_{\Pi_{p,\pm}^* f}^p = \int_{\mathbb{R}^n} \langle \nabla f(x), \xi \rangle_\pm^p \, \mathrm{d}x$$

for $\xi \in \mathbb{S}^{n-1}$.

3 Fractional L^p polar projection bodies

Let $0 < s < 1$ and $1 < p < n/s$. For a measurable function $f : \mathbb{R}^n \to \mathbb{R}$, define the s-fractional L^p polar projection body $\Pi_p^{*,s} f$ as the star-shaped set given by the gauge function

$$\|\xi\|_{\Pi_p^{*,s} f}^{ps} = \int_0^\infty t^{-ps-1} \int_{\mathbb{R}^n} |f(x+t\xi) - f(x)|^p \, \mathrm{d}x \, \mathrm{d}t \tag{9}$$

for $\xi \in \mathbb{R}^n$. Note that $\|\cdot\|_{\Pi_p^{*,s} f}$ is a one-homogeneous function on $\mathbb{R}^n$.

Let $K \subset \mathbb{R}^n$ be a star body. The following simple calculation turns out to be useful. For $f \in W^{s,p}(\mathbb{R}^n)$,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{\|x-y\|_K^{n+ps}} \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(y+z)-f(y)|^p}{\|z\|_K^{n+ps}} \, \mathrm{d}z \, \mathrm{d}y \\
&= \int_{\mathbb{S}^{n-1}} \int_0^\infty \|t\xi\|_K^{-n-ps} \int_{\mathbb{R}^n} |f(y+t\xi) - f(y)|^p \, t^{n-1} \, \mathrm{d}y \, \mathrm{d}t \, \mathrm{d}\xi \\
&= \int_{\mathbb{S}^{n-1}} \int_0^\infty \|\xi\|_K^{-n-ps} t^{-ps-n} \, \|f(\cdot + t\xi) - f\|_p^p \, t^{n-1} \, \mathrm{d}t \, \mathrm{d}\xi \\
&= \int_{\mathbb{S}^{n-1}} \rho_K(\xi)^{n+ps} \int_0^\infty t^{-ps-1} \, \|f(\cdot + t\xi) - f\|_p^p \, \mathrm{d}t \, \mathrm{d}\xi \\
&= \int_{\mathbb{S}^{n-1}} \rho_K(\xi)^{n+ps} \rho_{\Pi_p^{*,s} f}(\xi)^{-ps} d\xi.
\end{aligned}$$

Hence,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{\|x-y\|_K^{n+ps}} \, \mathrm{d}x \, \mathrm{d}y = n \, \tilde{V}_{-ps}(K, \Pi_p^{*,s} f) \tag{10}$$

in this case.

Next, we establish basic properties of fractional L^p polar projection bodies.

Proposition 4 *For non-zero $f \in W^{s,p}(\mathbb{R}^n)$, the set $\Pi_p^{*,s} f$ is an origin-symmetric star body with the origin in its interior. Moreover, there is $c > 0$ depending only on f and p such that $\Pi_p^{*,s} f \subseteq c\, B^n$ for every $s \in (0, 1)$.*

Proof First, note that since for $\xi \in \mathbb{R}^n$ and $t > 0$,

$$\int_{\mathbb{R}^n} |f(x-t\xi) - f(x)|^p \, \mathrm{d}x = \int_{\mathbb{R}^n} |f(x) - f(x+t\xi)|^p \, \mathrm{d}x,$$

the set $\Pi_p^{*,s} f$ is origin-symmetric.

Next, we show that $\Pi_p^{*,s} f$ is bounded. We take $r > 1$ large enough so that $\|f\|_{L^p(rB^n)} \geq \frac{2}{3}\|f\|_p$ and easily see that for $t > 2r$,

$$\|f(\cdot + t\xi) - f(\cdot)\|_p \geq \|f(\cdot + t\xi) - f(\cdot)\|_{L^p(rB^n - t\xi)}$$

$$
\begin{aligned}
&= \|f(\cdot) - f(\cdot - t\xi)\|_{L^p(rB^n)} \\
&\geq \|f\|_{L^p(rB^n)} - \|f(\cdot - t\xi)\|_{L^p(rB^n)} \\
&\geq \frac{2}{3}\|f\|_p - \frac{1}{3}\|f\|_p.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\int_0^\infty t^{-ps-1} \int_{\mathbb{R}^n} \left|f(x+t\xi) - f(x)\right|^p \mathrm{d}x\,\mathrm{d}t &\geq \frac{\|f\|_p^p}{3^p} \int_r^\infty t^{-ps-1}\,\mathrm{d}t \\
&\geq \frac{\|f\|_p^p}{3^p}\frac{r^{-ps}}{ps} \geq c,
\end{aligned}
$$

which implies that $\Pi_p^{*,s} f \subseteq c\,B^n$ for $c > 0$ independent of s.

Now, we show that $\Pi_p^{*,s} f$ has the origin in its interior. First observe that for $\xi, \eta \in \mathbb{R}^n$, by the triangle inequality and a change of variables,

$$
\begin{aligned}
&\|\xi + \eta\|_{\Pi_p^{*,s} f}^{ps} \\
&\quad = \int_0^\infty t^{-ps-1} \|f(\cdot + t\xi + t\eta) - f(\cdot)\|_p^p\,\mathrm{d}t \\
&\quad \leq \int_0^\infty t^{-ps-1} \left(\|f(\cdot + t\xi + t\eta) - f(\cdot + t\xi)\|_p + \|f(\cdot + t\xi) - f(\cdot)\|_p\right)^p\,\mathrm{d}t \\
&\quad \leq \int_0^\infty t^{-ps-1} 2^{p-1} (\|f(\cdot + t\eta) - f(\cdot)\|_p^p + \|f(\cdot + t\xi) - f(\cdot)\|_p^p)\,\mathrm{d}t \\
&\quad = 2^{p-1}\|\xi\|_{\Pi_p^{*,s} f}^{ps} + 2^{p-1}\|\eta\|_{\Pi_p^{*,s} f}^{ps}.
\end{aligned}
\tag{11}
$$

Using the relation (10) with $K = B^n$, we get

$$
\int_{\mathbb{S}^{n-1}} \|\xi\|_{\Pi_p^{*,s} f}^{ps}\,\mathrm{d}\xi = \frac{1}{n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y,
$$

which is finite since $f \in W^{s,p}(\mathbb{R}^n)$. We choose $r > 0$ large enough so that the set $A = \{\xi \in \mathbb{S}^{n-1} : \|\xi\|_{\Pi_p^{*,s} f}^{s} < r\}$ has positive $(n-1)$-dimensional Hausdorff measure and contains a basis $\{\xi_1, \ldots, \xi_n\} \subseteq A$ of $\mathbb{R}^n$. Applying (if necessary) a linear transformation to $\Pi_p^{*,s} f$, we may assume without loss of generality that $\xi_i = e_i$ are the canonical basis vectors. For every $x \in \mathbb{R}^n$, writing $x = \sum x_i e_i$ and using (11), we get

$$
\|x\|_{\Pi_p^{*,s} f} \leq \left(2^{n(p-1)} \sum_{i=1}^n |x_i|^{ps} \|e_i\|_{\Pi_p^{*,s} f}^{ps}\right)^{\frac{1}{ps}} \leq d\,|x|,
\tag{12}
$$

where $d > 0$ is independent of x. This shows that $\Pi_p^{*,s} f$ has the origin as interior point.

Finally, we show that $\|\cdot\|_{\Pi_p^{*,s}f}$ is continuous. For $\xi, \eta \in \mathbb{R}^n$, by the triangle inequality and (12), we have

$$\begin{aligned}
&\|\xi+\eta\|_{\Pi_p^{*,s}f}^{ps}\\
&= \int_0^\infty t^{-1-ps}\|f(\cdot+t\xi+t\eta)-f(\cdot)\|_p^p \,\mathrm{d}t\\
&\le \int_0^\infty t^{-1-ps}\big(\|f(\cdot+t\eta)-f(\cdot)\|_p+\|f(\cdot+t\xi)-f(\cdot)\|_p\big)^p \,\mathrm{d}t\\
&\le \big(1+|\eta|^{\frac{s}{2}\frac{p}{p-1}}\big)^{p-1}\int_0^\infty t^{-1-ps}\left(\frac{\|f(\cdot+t\eta)-f(\cdot)\|_p^p}{|\eta|^{\frac{ps}{2}}}+\|f(\cdot+t\xi)-f(\cdot)\|_p^p\right)\mathrm{d}t\\
&= \big(1+|\eta|^{\frac{s}{2}\frac{p}{p-1}}\big)^{p-1}\big(|\eta|^{-\frac{ps}{2}}\|\eta\|_{\Pi_p^{*,s}f}^{ps}+\|\xi\|_{\Pi_p^{*,s}f}^{ps}\big)\\
&\le \big(1+|\eta|^{\frac{s}{2}\frac{p}{p-1}}\big)^{p-1}\big(d\,|\eta|^{\frac{ps}{2}}+\|\xi\|_{\Pi_p^{*,s}f}^{ps}\big),
\end{aligned}$$

where we used the inequality $a+b \le (1+r^{p/(p-1)})^{(p-1)/p}((r^{-1}a)^p+b^p)^{1/p}$ for $a, b, r > 0$, which is a consequence of Hölder's inequality.

We obtain

$$\|\xi+\eta\|_{\Pi_p^{*,s}f}^{ps} \le \big(1+|\eta|^{\frac{s}{2}\frac{p}{p-1}}\big)^{p-1}\big(d\,|\eta|^{\frac{ps}{2}}+\|\xi\|_{\Pi_p^{*,s}f}^{ps}\big). \tag{13}$$

Applying inequality (13) to the vectors $\xi+\eta$ and $-\eta$, we get

$$\|\xi\|_{\Pi_p^{*,s}f}^{ps} = \|\xi+\eta-\eta\|_{\Pi_p^{*,s}f}^{ps} \le \big(1+|-\eta|^{\frac{s}{2}\frac{p}{p-1}}\big)^{p-1}\big(d\,|-\eta|^{\frac{ps}{2}}+\|\xi+\eta\|_{\Pi_p^{*,s}f}^{ps}\big),$$

which implies

$$\|\xi+\eta\|_{\Pi_p^{*,s}f}^{ps} \ge \big(1+|\eta|^{\frac{s}{2}\frac{p}{p-1}}\big)^{p-1}\|\xi\|_{\Pi_p^{*,s}f}^{ps} - d\,|\eta|^{\frac{ps}{2}}. \tag{14}$$

The continuity of $\|\cdot\|_{\Pi_p^{*,s}f}$ now follows from (13) and (14). □

4 Fractional asymmetric L^p polar projection bodies

Let $0 < s < 1$ and $1 < p < n/s$. For a measurable function $f : \mathbb{R}^n \to \mathbb{R}$, define the asymmetric s-fractional L^p polar projection bodies $\Pi_{p,+}^{*,s}f$ and $\Pi_{p,-}^{*,s}f$ as the star-shaped sets given by the gauge functions

$$\|\xi\|_{\Pi_{p,\pm}^{*,s}f}^{ps} = \int_0^\infty t^{-ps-1}\int_{\mathbb{R}^n}(f(x+t\xi)-f(x))_\pm^p \,\mathrm{d}x\,\mathrm{d}t$$

for $\xi \in \mathbb{R}^n$. We have $\Pi_{p,-}^{*,s}f = \Pi_{p,+}^{*,s}(-f) = -\Pi_{p,+}^{*,s}f$ and state our results just for $\Pi_{p,+}^{*,s}f$. Note that, as in the symmetric case, $\|\cdot\|_{\Pi_{p,+}^{*,s}f}^{ps}$ is a one-homogeneous function on $\mathbb{R}^n$. Also, note that

$$\|\xi\|_{\Pi_p^{*,s} f}^{ps} = \|\xi\|_{\Pi_{p,+}^{*,s} f}^{ps} + \|\xi\|_{\Pi_{p,-}^{*,s} f}^{ps} \tag{15}$$

for $\xi \in \mathbb{R}^n$.

Let $K \subset \mathbb{R}^n$ be a star body and $f \in W^{s,p}(\mathbb{R}^n)$. As in (10), we obtain that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))_+^p}{\|x-y\|_K^{n+ps}} \,\mathrm{d}x\,\mathrm{d}y = n\,\tilde{V}_{-ps}(K, \Pi_{p,+}^{*,s} f). \tag{16}$$

In the following proposition, we derive the basic properties of fractional asymmetric L^p polar projection bodies.

Proposition 5 *For non-zero $f \in W^{s,p}(\mathbb{R}^n)$, the set $\Pi_{p,+}^{*,s} f$ is a star body with the origin in its interior. Moreover, there is $c > 0$ depending only on f and p such that $\Pi_{p,+}^{*,s} f \subseteq c\,B^n$ for every $s \in (0,1)$.*

Proof Since the functions $(a)_+^p$ and $(a)_-^p$ are convex, the inequalities $(a+b)_+^p \ge (a)_+^p + p(a)_+^{p-1} b$ and $(a+b)_-^p \ge (a)_-^p + p(a)_-^{p-1} b$ hold for $a, b \in \mathbb{R}$.

If $\int_{\mathbb{R}^n} (f(x))_+^p \,\mathrm{d}x > 0$, take $\varepsilon > 0$ so small that $\varepsilon + p\varepsilon^{1/p} \|f\|_p^{p-1} \le \frac{1}{2} \int_{\mathbb{R}^n} (f(x))_+^p \,\mathrm{d}x$, and take $r > 0$ so large that $\int_{\mathbb{R}^n \setminus rB^n} |f(x)|^p \,\mathrm{d}x < \varepsilon$. For $z \in \mathbb{R}^n \setminus 2rB^n$, we obtain by Hölder's inequality that

$$\begin{aligned}
&\int_{rB^n} (f(x) - f(x+z))_+^p \,\mathrm{d}x \\
&\quad \ge \int_{rB^n} (f(x))_+^p - p\,(f(x))_+^{p-1} f(x+z) \,\mathrm{d}x \\
&\quad \ge \int_{rB^n} (f(x))_+^p \,\mathrm{d}x - p\Big(\int_{rB^n} (f(x))_+^p \,\mathrm{d}x\Big)^{\frac{p-1}{p}} \Big(\int_{rB^n} |f(x+z)|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \\
&\quad \ge \int_{rB^n} (f(x))_+^p \,\mathrm{d}x - p\Big(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x\Big)^{\frac{p-1}{p}} \Big(\int_{\mathbb{R}^n \setminus rB^n} |f(x)|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \\
&\quad \ge \int_{\mathbb{R}^n} (f(x))_+^p \,\mathrm{d}x - \varepsilon - p\,\|f\|_p^{p-1} \varepsilon^{\frac{1}{p}} \\
&\quad \ge \frac{1}{2} \int_{\mathbb{R}^n} (f(x))_+^p \,\mathrm{d}x.
\end{aligned}$$

In case $\int_{\mathbb{R}^n} (f(x))_+^p \,\mathrm{d}x = 0$ the previous inequality holds trivially for any $r > 0$.

By an analogous calculation and eventually increasing the value of r, we obtain that

$$\begin{aligned}
\int_{rB^n - z} (f(x) - f(x+z))_+^p \,\mathrm{d}x &= \int_{rB^n} (f(x) - f(x-z))_-^p \,\mathrm{d}x \\
&\ge \frac{1}{2} \int_{\mathbb{R}^n} (f(x))_-^p \,\mathrm{d}x.
\end{aligned}$$

It follows that $\int_{\mathbb{R}^n}(f(x)-f(x+z))_+^p\,\mathrm{d}x \ge \frac{1}{2}\|f\|_p^p$ for every $z\in\mathbb{R}^n\setminus 2rB^n$ with $r>0$ depending only on f. Finally,

$$\begin{aligned}\|\xi\|^{ps}_{\Pi^{*,s}_{p,+}f} &\ge \int_{2r}^{\infty} t^{-1-ps}\int_{\mathbb{R}^n}(f(x)-f(x+z))_+^p\,\mathrm{d}x\,\mathrm{d}t\\ &\ge \int_{2r}^{\infty} t^{-1-ps}\,\mathrm{d}t\,\frac{1}{2}\int_{\mathbb{R}^n}|f(x)|^p\,\mathrm{d}x\\ &\ge \frac{(2r)^{-ps}}{ps}\frac{1}{2}\int_{\mathbb{R}^n}|f(x)|^p\,\mathrm{d}x\\ &\ge \frac{(2r)^{-p}}{2p}\|f\|_p^p.\end{aligned}$$

Note that $\Pi^{*,s}_p f\subset \Pi^{*,s}_{p,+}f$. Hence, it follows from Proposition 4 that $\Pi^{*,s}_{p,+}f$ contains the origin in its interior, that is, there is $d>0$ such that

$$\|x\|_{\Pi^{*,s}_{p,+}f}\le d\,|x| \tag{17}$$

for every $x\in\mathbb{R}^n$.

Finally, we show that $\|\cdot\|_{\Pi^{*,s}_{p,+}f}$ is continuous. Observe that the inequality $(a+b)_+^p\le(a_++b_+)^p$ holds for any $a,b\in\mathbb{R}$. Hence, for $\xi,\eta\in\mathbb{R}^n$, we obtain that

$$\begin{aligned}&\int_{\mathbb{R}^n}(f(x+t\xi+t\eta)-f(x))_+^p\,\mathrm{d}x\\ &=\int_{\mathbb{R}^n}(f(x+t\xi+t\eta)-f(x+t\xi)+f(x+t\xi)-f(x))_+^p\,\mathrm{d}x\\ &\le\int_{\mathbb{R}^n}\big((f(x+t\xi+t\eta)-f(x+t\xi))_++(f(x+t\xi)-f(x))_+\big)^p\,\mathrm{d}x\\ &\le\int_{\mathbb{R}^n}(1+|\eta|^{\frac{s}{2}\frac{p}{(p-1)}})^{p-1}\Big(\frac{(f(x+t\xi+t\eta)-f(x+t\xi))_+^p}{|\eta|^{\frac{ps}{2}}}+(f(x+t\xi)-f(x))_+^p\Big)\,\mathrm{d}x\\ &\le(1+|\eta|^{\frac{s}{2}\frac{p}{(p-1)}})^{p-1}\Big(\frac{\|(f(\cdot+t\eta)-f(\cdot))_+\|_p^p}{|\eta|^{\frac{ps}{2}}}+\|(f(\cdot+t\xi)-f(\cdot))_+\|_p^p\Big),\end{aligned}$$

where we used the inequality $a+b\le(1+r^{p/(p-1)})^{(p-1)/p}((r^{-1}a)^p+b^p)^{1/p}$ for $a,b,r>0$, which is a consequence of Hölder's inequality. Thus, integrating and using (17), we obtain

$$\|\xi+\eta\|^{ps}_{\Pi^{*,s}_{p,+}f}\le(1+|\eta|^{\frac{s}{2}\frac{p}{p-1}})^{p-1}(d\,|\eta|^{\frac{ps}{2}}+\|\xi\|^{ps}_{\Pi^{*,s}_{p,+}f}). \tag{18}$$

Applying inequality (18) to the vectors $\xi+\eta$ and $-\eta$, we get

$$\|\xi\|^{ps}_{\Pi^{*,s}_{p,+}f}=\|\xi+\eta-\eta\|^{ps}_{\Pi^{*,s}_{p,+}f}\le(1+|-\eta|^{\frac{s}{2}\frac{p}{p-1}})^{p-1}(d\,|-\eta|^{\frac{ps}{2}}+\|\xi+\eta\|^{ps}_{\Pi^{*,s}_{p,+}f}),$$

which implies

$$\|\xi+\eta\|^{ps}_{\Pi^{*,s}_{p,+}f} \geq (1+|\eta|^{\frac{s}{2}\frac{p}{p-1}})^{-(p-1)}\|\xi\|^{ps}_{\Pi^{*,s}_{p,+}f} - d\,|\eta|^{\frac{ps}{2}}. \tag{19}$$

The continuity of $\|\cdot\|_{\Pi^{*,s}_{p,+}f}$ now follows from (18) and (19). □

5 The limit of fractional L^p polar projection bodies

We establish the limiting behavior of s-fractional L^p polar projection bodies for $1 < p < n/s$ as $s \to 1^-$ in the symmetric and asymmetric case. For $p = 1$, a corresponding result was proved in [16].

Let $0 < s < 1$ and $1 < p < n/s$. Set $p' = p/(p-1)$. We say that $f_k \to f$ weakly in $L^p(\mathbb{R}^n)$ if

$$\int_{\mathbb{R}^n} f_k(x)g(x)\,\mathrm{d}x \to \int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x$$

for every $g \in L^{p'}(\mathbb{R}^n)$ as $k \to \infty$. Set $B_{p',+} = \{g \in L^{p'}(\mathbb{R}^n) : g \geq 0, \|g\|_{p'} \leq 1\}$.

We require the following lemmas.

Lemma 6 *The following statements hold.*

(1) *For $f \in L^p(\mathbb{R}^n)$,*

$$\|f_+\|_p = \sup_{g\in B_{p',+}} \int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x.$$

(2) *Let $f_k, f \in L^p(\mathbb{R}^n)$. If $f_k \to f$ weakly in $L^p(\mathbb{R}^n)$ as $k \to \infty$, then*

$$\liminf_{k\to\infty} \|(f_k)_+\|_p \geq \|f_+\|_p.$$

(3) *Assume f_k is a bounded sequence in $L^p(\mathbb{R}^n)$. If*

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} f_k(x)g(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x$$

for every g in a dense subset $D \subseteq L^{p'}(\mathbb{R}^n)$, then $f_k \to f$ weakly in $L^p(\mathbb{R}^n)$ as $k \to \infty$.

Proof First, we prove (1). Let $g \in B_{p',+}$ and write $f = f_+ - f_-$. Since f_- and g are non-negative, it follows from Hölder's inequality that

$$\int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x \leq \int_{\mathbb{R}^n} f_+(x)g(x)\,\mathrm{d}x \leq \|f_+\|_p.$$

For the opposite inequality, take $g = \|f_+\|_p^{-p/p'} f_+^{p/p'}$ and note that $g \in B_{p',+}$ and

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x &= \|f_+\|_p^{-\frac{p}{p'}} \int_{\mathbb{R}^n} f(x)f_+(x)^{\frac{p}{p'}}\,\mathrm{d}x \le \|f_+\|_p^{-\frac{p}{p'}} \int_{\mathbb{R}^n} f_+(x)^p\,\mathrm{d}x\\ &= \|f_+\|_p.\end{aligned}$$

Next, we prove (2). Fix k_0 and $g_0 \in B_{p',+}$. By (1), we have

$$\int_{\mathbb{R}^n} f_{k_0}(x)g_0(x)\,\mathrm{d}x \le \sup_{g\in B_{p',+}} \int_{\mathbb{R}^n} f_{k_0}(x)g(x)\,\mathrm{d}x = \|(f_{k_0})_+\|_p.$$

Since this inequality holds for every k_0,

$$\int_{\mathbb{R}^n} f(x)g_0(x)\,\mathrm{d}x = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k(x)g_0(x)\,\mathrm{d}x \le \liminf_{k\to\infty} \|(f_k)_+\|_p.$$

Thus, by (1),

$$\|f_+\|_p = \sup_{g\in B_{p',+}} \int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x \le \liminf_{k\to\infty} \|(f_k)_+\|_p.$$

Finally, we prove (3). Take $c \ge \max\{\|f_k\|_p, \|f\|_p\}$. Let $\varepsilon > 0$ and $g \in L^{p'}(\mathbb{R}^n)$. Take $h \in D$ such that $\|g - h\|_{p'} < \varepsilon/(2c)$. Then

$$\begin{aligned}&\Big|\int_{\mathbb{R}^n} f_k(x)g(x)\,\mathrm{d}x - \int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x\Big|\\ &\le \Big|\int_{\mathbb{R}^n} f_k(x)(g(x)-h(x))\,\mathrm{d}x\Big| + \Big|\int_{\mathbb{R}^n} f_k(x)h(x)\,\mathrm{d}x - \int_{\mathbb{R}^n} f(x)h(x)\,\mathrm{d}x\Big|\\ &\quad + \Big|\int_{\mathbb{R}^n} f(x)(g(x)-h(x))\,\mathrm{d}x\Big|\\ &\le c\varepsilon/(2c) + \Big|\int_{\mathbb{R}^n} f_k(x)h(x)\,\mathrm{d}x - \int_{\mathbb{R}^n} f(x)h(x)\,\mathrm{d}x\Big| + c\varepsilon/(2c)\end{aligned}$$

and the statement follows. □

Lemma 7 *For $f \in W^{1,p}(\mathbb{R}^n)$ and fixed $\xi \in \mathbb{S}^{n-1}$,*

$$\lim_{t\to 0} \Big\|\Big(\frac{f(\cdot + t\xi) - f(\cdot)}{t}\Big)_+\Big\|_p^p = \int_{\mathbb{R}^n} \langle \nabla f(x), \xi\rangle_+^p\,\mathrm{d}x.$$

Proof Let $g : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with compact support. Write div_x for the divergence taken with respect to the variable x. Using integration by parts, we obtain for $\xi \in \mathbb{S}^{n-1}$ and $t > 0$,

$$\begin{aligned}\int_{\mathbb{R}^n} g(x)\frac{f(x+t\xi)-f(x)}{t}\,\mathrm{d}x &= \int_{\mathbb{R}^n} f(x)\frac{g(x-t\xi)-g(x)}{t}\,\mathrm{d}x\\ &= -\int_{\mathbb{R}^n} f(x)\int_0^1 \langle \nabla g(x-rt\xi),\xi\rangle\,\mathrm{d}r\,\mathrm{d}x\\ &= -\int_{\mathbb{R}^n} f(x)\mathrm{div}_x\Big(\int_0^1 g(x-rt\xi)\,\mathrm{d}r\,\xi\Big)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^n}\Big(\int_0^1 g(x-rt\xi)dr\Big)\langle\nabla f(x),\xi\rangle\,\mathrm{d}x.\end{aligned}$$

By Minkowski's integral inequality $\|\int_0^1 g(\cdot - rt\xi)\,\mathrm{d}r\|_{p'} \le \|g\|_{p'}$, and we deduce

$$\Big\|\frac{f(\cdot+t\xi)-f(\cdot)}{t}\Big\|_p \le \|\langle\nabla f(\cdot),\xi\rangle\|_p < \infty.$$

Hence, $\frac{f(\cdot+t\xi)-f(\cdot)}{t}$ is uniformly bounded in $L^p(\mathbb{R}^n)$ on $(0,\infty)$.

By Lemma 6(3),

$$\lim_{t\to 0}\int_{\mathbb{R}^n} g(x)\frac{f(x+t\xi)-f(x)}{t}\,\mathrm{d}x = \int_{\mathbb{R}^n} g(x)\langle\nabla f(x),\xi\rangle\,\mathrm{d}x$$

for every $g\in L^{p'}(\mathbb{R}^n)$. Hence, $\frac{f(\cdot+t\xi)-f(\cdot)}{t}$ converges weakly to $\langle\nabla f(\cdot),\xi\rangle$ as $t\to 0$.

By Lemma 6(2),

$$\liminf_{t\to 0}\Big\|\Big(\frac{f(\cdot+t\xi)-f(\cdot)}{t}\Big)_+\Big\|_p \ge \|\langle\nabla f(\cdot),\xi\rangle_+\|_p.$$

For the opposite inequality, we recall that for any $g\in B_{p',+}$, the function $x\mapsto \int_0^1 g(x-rt\xi)\,\mathrm{d}r$ is in $B_{p',+}$ as well. Hence,

$$\begin{aligned}\int_{\mathbb{R}^n} g(x)\frac{f(x+t\xi)-f(x)}{t}\,\mathrm{d}x &= \int_{\mathbb{R}^n}\Big(\int_0^1 g(x-rt\xi)\,\mathrm{d}r\Big)\langle\nabla f(x),\xi\rangle\,\mathrm{d}x\\ &\le \|\langle\nabla f(x),\xi\rangle_+\|_p.\end{aligned}$$

Again by Lemma 6(1),

$$\Big\|\Big(\frac{f(\cdot+t\xi)-f(\cdot)}{t}\Big)_+\Big\|_p \le \|\langle\nabla f(\cdot),\xi\rangle_+\|_p$$

for each $t>0$. □

The following result is Lemma 4 in [16].

Lemma 8 *If $\varphi:[0,\infty)\to[0,\infty)$ be a measurable function with $\lim_{t\to 0^+}\varphi(t)=\varphi(0)$ and such that $\int_0^\infty t^{-s_0}\varphi(t)\,\mathrm{d}t<\infty$ for some $s_0\in(0,1)$, then*

$$\lim_{s\to 1^-}(1-s)\int_0^\infty t^{-s}\varphi(t)\,\mathrm{d}t=\varphi(0).$$

We are now able to prove the main result of this section.

Theorem 9 *Let $f\in W^{1,p}(\mathbb{R}^n)$. For $\xi\in\mathbb{S}^{n-1}$,*

$$\lim_{s\to 1^-}(p(1-s))^{\frac{1}{p}}\|\xi\|_{\Pi^{*,s}_{p,+}f}=\|\xi\|_{\Pi^{*}_{p,+}f}.$$

Moreover,

$$\lim_{s\to 1^-}p(1-s)|\Pi^{*,s}_{p,+}f|^{-\frac{ps}{n}}=|\Pi^{*}_{p,+}f|^{-\frac{p}{n}},$$

and

$$\lim_{s\to 1^-}p(1-s)\tilde{V}_{-ps}(K,\Pi^{*,s}_{p,+}f)=\tilde{V}_{-p}(K,\Pi^{*}_{p,+}f)$$

for every star body $K\subset\mathbb{R}^n$.

Proof Define $\varphi:[0,\infty)\to[0,\infty)$ by

$$\varphi(t)=\Big\|\Big(\frac{f(\cdot+t\xi)-f(\cdot)}{t}\Big)_+\Big\|_p^p,$$

and note that $\varphi(t)\le\Big(\frac{2\|f\|_p}{t}\Big)^p$ for $t>0$. By Lemma 8 and Lemma 7,

$$\lim_{s\to 1^-}p(1-s)\int_0^\infty t^{p(1-s)-1}\Big\|\Big(\frac{f(\cdot+t\xi)-f(\cdot)}{t}\Big)_+\Big\|_p^p\,\mathrm{d}t=\int_{\mathbb{R}^n}\langle\nabla f(x),\xi\rangle_+^p\,\mathrm{d}x.$$

By Proposition 4, we can use the dominated convergence theorem to obtain

$$\begin{aligned}
&\lim_{s\to 1^-}n\,|(p(1-s))^{-\frac{1}{ps}}\Pi^{*,s}_{p,+}f|\\
&\quad=\lim_{s\to 1^-}\int_{\mathbb{S}^{n-1}}\Big(p(1-s)\int_0^\infty t^{p(1-s)-1}\Big\|\Big(\frac{f(\cdot+t\xi)-f(\cdot)}{t}\Big)_+\Big\|_p^p\,\mathrm{d}t\Big)^{-\frac{n}{ps}}\mathrm{d}\xi\\
&\quad=\int_{\mathbb{S}^{n-1}}\Big(\int_{\mathbb{R}^n}\langle\nabla f(x),\xi\rangle_+^p\,\mathrm{d}x\Big)^{-\frac{n}{p}}\mathrm{d}\xi\\
&\quad=n\,|\Pi^{*}_{p,+}f|,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{s\to 1^-}np(1-s)\tilde{V}_{-ps}(K,\Pi^{*,s}_{p,+}f)&=\lim_{s\to 1^-}p(1-s)\int_{\mathbb{S}^{n-1}}\|\xi\|_K^{n+ps}\,\|\xi\|^{ps}_{\Pi^{*,s}_{p,+}f}\,\mathrm{d}\xi\\
&=\int_{\mathbb{S}^{n-1}}\|\xi\|_K^{n}\,\|\xi\|^{p}_{\Pi^{*}_{p,+}f}\,\mathrm{d}\xi\\
&=n\,\tilde{V}_{-p}(K,\Pi^{*}_{p,+}f),
\end{aligned}$$

which completes the proof of the theorem. □

The following result is an immediate consequence of Theorem 9 and (15).

Theorem 10 *Let $f \in W^{1,p}(\mathbb{R}^n)$. For $\xi \in \mathbb{S}^{n-1}$,*

$$\lim_{s\to 1^-} (p(1-s))^{\frac{1}{p}} \|\xi\|_{\Pi_p^{*,s} f} = \|\xi\|_{\Pi_p^* f}.$$

Moreover,

$$\lim_{s\to 1^-} p(1-s)|\Pi_p^{*,s} f|^{-\frac{ps}{n}} = |\Pi_p^* f|^{-\frac{p}{n}},$$

and

$$\lim_{s\to 1^-} p(1-s)\tilde{V}_{-ps}(K, \Pi_p^{*,s} f) = \tilde{V}_{-p}(K, \Pi_p^* f) \tag{20}$$

for every star body $K \subset \mathbb{R}^n$.

6 Anisotropic fractional Pólya–Szegő inequalities

We will establish anisotropic Pólya–Szegő inequalities for fractional L^p Sobolev norms and their asymmetric counterparts.

Theorem 11 *If $f \in L^p(\mathbb{R}^n)$ is non-negative and $K \subset \mathbb{R}^n$ a star body, then*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f(x)-f(y))_+^p}{\|x-y\|_K^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y \geq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f^\star(x)-f^\star(y))_+^p}{\|x-y\|_{K^\star}^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y. \tag{21}$$

Equality holds for non-zero $f \in W^{s,p}(\mathbb{R}^n)$ if and only if K is a centered ellipsoid and f is a translate of $f^\star \circ \phi$ for some $\phi \in \mathrm{SL}(n)$.

Proof Writing

$$\|z\|_K^{-n-ps} = \int_0^\infty k_t(z)\,\mathrm{d}t$$

where $k_t(z) = 1_{t^{-1/(n+ps)}K}(z)$, we obtain

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f(x)-f(y))_+^p}{\|x-y\|_K^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y = \int_0^\infty \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (f(x)-f(y))_+^p k_t(x-y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

Note that

$$(f(x)-f(y))_+^p = p\int_0^\infty (f(x)-r)_+^{p-1} 1_{\{f<r\}}(y)\,\mathrm{d}r.$$

Hence, for $t > 0$, it follows from Fubini's theorem that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (f(x)-f(y))_+^p\, k_t(x-y)\,\mathrm{d}x\,\mathrm{d}y$$

$$= p \int_0^\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (f(x) - r)_+^{p-1} k_t(x - y) 1_{\{f<r\}}(y) \, dx \, dy \, dr$$
$$= p \int_0^\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (f(x) - r)_+^{p-1} k_t(x - y)(1 - 1_{\{f \ge r\}}(y)) \, dx \, dy \, dr.$$

Let $r, t > 0$. Note that $\int_{\mathbb{R}^n} (f(x) - r)_+^{p-1} \, dx < \infty$ and that

$$\begin{aligned}&\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (f(x) - r)_+^{p-1} k_t(x - y)(1 - 1_{\{f \ge r\}}(y)) \, dx \, dy \\ &\quad = p \, \|k_t\|_1 \int_{\mathbb{R}^n} (f(x) - r)_+^{p-1} \, dx \\ &\quad - p \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (f(x) - r)_+^{p-1} k_t(x - y) 1_{\{f \ge r\}}(y) \, dx \, dy.\end{aligned}$$

The first term is finite since $\{f > r\}$ has finite measure, $f \in L^{\frac{np}{n-ps}}(\mathbb{R}^n)$ and $\frac{np}{n-ps} > p-1$. Clearly, the first term is invariant under Schwarz symmetrization. For the second term, by the Riesz rearrangement inequality, Theorem 2, we have

$$\begin{aligned}&\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (f(x) - r)_+^{p-1} k_t(x - y) 1_{\{f \ge r\}}(y) \, dx \, dy \\ &\quad \le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (f^\star(x) - r)_+^{p-1} k_t^\star(x - y) 1_{\{f^\star \ge r\}}(y) \, dx \, dy\end{aligned}$$

for $r, t > 0$. Note that

$$(f(x) - r)_+^{p-1} = (p - 1) \int_0^\infty (\tilde{r} - r)_+^{p-2} 1_{\{f \ge \tilde{r}\}}(x) \, d\tilde{r},$$

and that the corresponding equation holds for $f^\star$. Hence, if there is equality in (21), then, for $(\tilde{r}, r, t) \in (0, \infty)^3 \backslash M$ with $|M| = 0$, we have

$$\begin{aligned}&\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{\{f \ge \tilde{r}\}}(x) 1_{t^{-1/(n+ps)} K}(x - y) 1_{\{f \ge r\}}(y) \, dx \, dy \\ &\quad = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{\{f^\star \ge \tilde{r}\}}(x) 1_{t^{-1/(n+ps)} K^\star}(x - y) 1_{\{f^\star \ge r\}}(y) \, dx \, dy.\end{aligned}$$

For almost every $(\tilde{r}, r) \in (0, \infty)^2$, we have $(\tilde{r}, r, t) \in (0, \infty)^3 \backslash M$ for almost every $t > 0$. For such $(\tilde{r}, r)$ with $\tilde{r} \le r$ and $t > 0$ sufficiently large, the assumptions of Theorem 3 are fulfilled and therefore there are a centered ellipsoid D and $a, b \in \mathbb{R}^n$ (depending on $(\tilde{r}, r, t)$) such that

$$\{f \ge \tilde{r}\} = a + \alpha D, \quad t^{-1/(n+ps)} K = b + \beta D, \quad \{f \ge r\} = c + \gamma D$$

where $c = a + b$. Since $K = t^{1/(n+ps)} b + (|K|/|D|)^{1/n} D$, the centered ellipsoid D does not depend on $(\tilde{r}, r, t)$ and also a, c do not depend on t. It follows that $b = 0$

and that K is a multiple of D. Hence, $a = c$ is a constant vector, which concludes the proof. □

The following result is a variation of [17, Theorem 3.1].

Theorem 12 *If $f \in L^p(\mathbb{R}^n)$ is non-negative and $K \subset \mathbb{R}^n$ a star body, then*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{\|x-y\|_K^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y \ge \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f^\star(x)-f^\star(y)|^p}{\|x-y\|_{K^\star}^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y.$$

Equality holds for non-zero $f \in W^{s,p}(\mathbb{R}^n)$ if and only if K is a centered ellipsoid and f is a translate of $f^\star \circ \phi$ for some $\phi \in \mathrm{SL}(n)$.

Proof Since

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f(x)-f(y))_-^p}{\|x-y\|_K^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f(x)-f(y))_+^p}{\|x-y\|_{-K}^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y,$$

the result follows from Theorem 11 for K and $-K$. □

7 Affine fractional Pólya–Szegő inequalities

We establish affine Pólya–Szegő inequalities for fractional asymmetric and symmetric L^p polar projection bodies.

Theorem 13 *If $f \in W^{s,p}(\mathbb{R}^n)$ is non-negative, then*

$$|\Pi_{p,+}^{*,s} f|^{-ps/n} \ge |\Pi_{p,+}^{*,s} f^\star|^{-ps/n}. \tag{22}$$

Equality holds if and only if f is a translate of $f^\star \circ \phi$ for some $\phi \in \mathrm{SL}(n)$.

Proof By Theorem 11, (16) and the dual mixed volume inequality, we obtain for $K \subset \mathbb{R}^n$ a star body that

$$\begin{aligned}\tilde{V}_{-ps}(K, \Pi_{p,+}^{*,s} f) &\ge \tilde{V}_{-ps}(K^\star, \Pi_{p,+}^{*,s} f^\star)\\ &\ge |K^\star|^{(n+ps)/n} |\Pi_{p,+}^{*,s} f^\star|^{-ps/n}\\ &= |K|^{(n+ps)/n} |\Pi_{p,+}^{*,s} f^\star|^{-ps/n}.\end{aligned}$$

Setting $K = \Pi_{p,+}^{*,s} f$, we see that

$$|\Pi_{p,+}^{*,s} f| = \tilde{V}_{-ps}(\Pi_{p,+}^{*,s} f, \Pi_{p,+}^{*,s} f) \ge |\Pi_{p,+}^{*,s} f|^{(n+ps)/n} |\Pi_{p,+}^{*,s} f^\star|^{-ps/n},$$

which completes the proof of the inequality. By Theorem 11, there is equality in (7) if and only if f is a translate of $f^\star \circ \phi$ for some $\phi \in \mathrm{SL}(n)$. □

The following result is obtained in the same way as Theorem 13 by replacing Theorem 11 with Theorem 12.

Theorem 14 *If* $f \in L^p(\mathbb{R}^n)$ *is non-negative, then*

$$|\Pi_p^{*,s} f|^{-ps/n} \geq |\Pi_p^{*,s} f^\star|^{-ps/n}.$$

Equality holds for $f \in W^{s,p}(\mathbb{R}^n)$ *if and only if* f *is a translate of* $f^\star \circ \phi$ *for some* $\phi \in \mathrm{SL}(n)$.

We remark that by Theorem 10 we obtain from Theorem 14 in the limit as $s \to 1^-$ that

$$|\Pi_p^* f|^{-p/n} \geq |\Pi_p^* f^\star|^{-p/n},$$

which is equivalent to the Pólya–Szegő inequality for L^p projection bodies by Cianchi, Lutwak, Yang, and Zhang [11, Theorem 2.1]. Similarly, by Theorem 9 we obtain from Theorem 13 in the limit as $s \to 1^-$ that

$$|\Pi_{p,+}^* f|^{-p/n} \geq |\Pi_{p,+}^* f^\star|^{-p/n},$$

which is equivalent to the Pólya–Szegő inequality for asymmetric L^p projection bodies by Haberl, Schuster and Xiao [15, Theorem 1].

8 Affine fractional asymmetric L^p Sobolev inequalities

We establish the following affine fractional asymmetric L^p Sobolev inequalities and show that they are stronger than Theorem 1.

Theorem 15 *Let* $0 < s < 1$ *and* $1 < p < n/s$. *For non-negative* $f \in W^{s,p}(\mathbb{R}^n)$,

$$\|f\|^p_{\frac{np}{n-ps}} \leq 2\,\sigma_{n,p,s} n\omega_n^{\frac{n+ps}{n}} |\Pi_{p,+}^{*,s} f|^{-\frac{ps}{n}} \leq 2\sigma_{n,p,s} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f(x)-f(y))_+^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y.$$

There is equality in the first inequality if and only if $f = h_{s,p} \circ \phi$ *for some* $\phi \in \mathrm{GL}(n)$ *where* $h_{s,p}$ *is an extremal function of* (1). *There is equality in the second inequality if* f *is radially symmetric.*

Proof By Theorem 13,

$$|\Pi_{p,+}^{*,s} f|^{-ps/n} \geq |\Pi_{p,+}^{*,s} f^\star|^{-ps/n},$$

with equality if f is a translate of $f^\star \circ \phi$ for some $\phi \in \mathrm{SL}(n)$. Since $f^\star$ is radially symmetric, $\Pi_{p,+}^{*,s} f^\star = \Pi_{p,-}^{*,s} f^\star$ is a ball. Hence, it follows from (16) that

$$2n\omega_n^{\frac{n+ps}{n}} |\Pi_{p,+}^{*,s} f^\star|^{-\frac{ps}{n}} = 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f^\star(x)-f^\star(y))_+^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f^\star(x) - f^\star(y)|^p}{|x-y|^{n+ps}} \,\mathrm{d}x\,\mathrm{d}y.$$

The fractional Sobolev inequality (1) shows that

$$\sigma_{n,p,s}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f^\star(x) - f^\star(y)|^p}{|x-y|^{n+ps}} \,\mathrm{d}x\,\mathrm{d}y \geq \|f^\star\|^p_{\frac{np}{n-ps}}.$$

Combining these inequalities and their equality cases, we complete the proof of the first inequality of the theorem.

For the second inequality, we set $K = B^n$ in (16) and apply the dual mixed volume inequality (6) to obtain

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f(x) - f(y))_+^p}{|x-y|^{n+ps}} \,\mathrm{d}x\,\mathrm{d}y = n\tilde{V}_{-ps}(B^n, \Pi^{*,s}_{p,+} f) \geq n\omega_n^{\frac{n+ps}{n}} |\Pi^{*,s}_{p,+} f|^{-\frac{ps}{n}}.$$

There is equality precisely if $\Pi^{*,s}_{p,+} f$ is a ball, which is the case for radially symmetric functions. □

Note that it follows from the definition of fractional symmetric and asymmetric L^p polar projection bodies that

$$\Pi^{*,s}_p f = \Pi^{*,s}_{p,+} f \,\tilde{+}_{-ps}\, \Pi^{*,s}_{p,-} f.$$

We use the dual Brunn–Minkowski inequality (5) and obtain that

$$|\Pi^{*,s}_p f|^{-\frac{ps}{n}} \geq |\Pi^{*,s}_{p,+} f|^{-\frac{ps}{n}} + |\Pi^{*,s}_{p,-} f|^{-\frac{ps}{n}},$$

with equality precisely if the star bodies $\Pi^{*,s}_{p,+} f$ and $\Pi^{*,s}_{p,-} f$ are dilates. Thus, it follows that for non-negative f, Theorem 15 implies Theorem 1, and it is, in general, substantially stronger than Theorem 1. Of course, they coincide for even functions.

9 Affine fractional L^p Sobolev inequalities: Proof of Theorem 1

For non-negative f, the first inequality in Theorem 1 follows from Theorem 15, as mentioned before. For general f and $x, y \in \mathbb{R}^n$, we use

$$|f(x) - f(y)| \geq \big||f(x)| - |f(y)|\big|,$$

where equality holds if and only if $f(x)$ and $f(y)$ are both non-negative or non-positive. We obtain

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y \geq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\big||f(x)| - |f(y)|\big|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y,$$

with equality if and only if f has constant sign for almost every $x, y \in \mathbb{R}^n$. Using the result for $|f|$, we obtain the first inequality of the theorem and its equality case.

For the second inequality, we set $K = B^n$ in (10) and apply the dual mixed volume inequality (6) as in the proof of Theorem 15.

10 Optimal fractional L^p Sobolev bodies

The following important question was asked by Lutwak, Yang and Zhang [25] for a given $f \in W^{1,p}(\mathbb{R}^n)$ and $1 \le p < n$: For which origin-symmetric convex bodies $K \subset \mathbb{R}^n$ is

$$\inf \left\{ \int_{\mathbb{R}^n} \|\nabla f(x)\|_{K^*}^p \,\mathrm{d}x : K \text{ origin-symmetric convex body}, |K| = \omega_n \right\} \tag{23}$$

attained? An optimal L^p Sobolev body of f is a convex body where the infimum is attained.

Lutwak, Yang and Zhang [25] showed that the infimum in (23) is attained (up to normalization) at the unique origin-symmetric convex body $\langle f \rangle_p$ in $\mathbb{R}^n$ such that

$$\int_{\mathbb{S}^{n-1}} g(\xi) \,\mathrm{d}S_p(\langle f \rangle_p, \xi) = \int_{\mathbb{R}^n} g(\nabla f(x)) \,\mathrm{d}x \tag{24}$$

for every even $g \in C(\mathbb{R}^n)$ that is positively homogeneous of degree p, where $S_p(K, \cdot)$ is the L_p surface area measure of K. Setting $g = \|\cdot\|_{K^*}$, they obtain from the L^p Minkowski inequality that

$$\frac{1}{n} \int_{\mathbb{R}^n} \|\nabla f(x)\|_{K^*}^p \,\mathrm{d}x = V_p(\langle f \rangle_p, K) \ge |\langle f \rangle_p|^{(n-p)/n} |K|^{p/n}, \tag{25}$$

with equality precisely if K and $\langle f \rangle_p$ are homothetic (see [28, Section 9.1] for the definition of the L_p mixed volume $V_p(\cdot, \cdot)$ and the L^p Minkowski inequality). Hence, they obtain from their solution to their functional version (24) of the L^p Minkowski problem that $\langle f \rangle_p$ is the optimal L^p Sobolev body associated to f. Tuo Wang [31] obtained corresponding results for $f \in BV(\mathbb{R}^n)$ and $p = 1$.

Let $0 < s < 1$ and $1 < p < n/s$. The results by Lutwak, Yang and Zhang [25] suggest the following question for a given $f \in W^{s,p}(\mathbb{R}^n)$: For which star bodies $L \subset \mathbb{R}^n$ is

$$\inf \left\{ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{\|x - y\|_L^{n+ps}} \,\mathrm{d}x \,\mathrm{d}y : L \text{ star body}, |L| = \omega_n \right\} \tag{26}$$

attained? An optimal s-fractional L^p Sobolev body of f is a star body where the infimum is attained.

By (10) and the dual mixed volume inequality (6),

$$\frac{1}{n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{\|x-y\|_L^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y=\tilde{V}_{-ps}(L,\Pi_p^{*,s}f)\geq |L|^{(n+ps)/n}|\Pi_p^{*,s}f|^{-(ps)/n},$$

and there is equality precisely if L is a dilate of $\Pi_p^{*,s}f$. Hence, $\Pi_p^{*,s}f$ is the unique optimal s-fractional L^p Sobolev body associated to f.

To understand how the solutions to (23) and (26) are related, we use the following result: For $f\in W^{1,p}(\mathbb{R}^n)$ and $L\subset\mathbb{R}^n$ a star body,

$$\lim_{s\to 1^-}p(1-s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{\|x-y\|_L^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y=\int_{\mathbb{R}^n}\|\nabla f(x)\|_{\mathrm{Z}_p^*L}\,\mathrm{d}x, \tag{27}$$

where the convex body Z_pK, defined for $\xi\in\mathbb{S}^{n-1}$ by

$$h_{\mathrm{Z}_pL}(\xi)^p=\int_{\mathbb{S}^{n-1}}|\langle\xi,\eta\rangle|^p\rho_L(\eta)^{n+p}\,\mathrm{d}\eta,$$

is a multiple of the L^p centroid body of L. This can be proved as in [21], where the corresponding result was established for a convex body L (with a different normalization of Z_pL). It also follows from Theorem 10. Indeed, by (10) and (20),

$$\lim_{s\to 1^-}p(1-s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{\|x-y\|_L^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y=\tilde{V}_{-p}(L,\Pi_p^*f).$$

Using that

$$\Pi_p^*f=\Pi_p^*\langle f\rangle_p \tag{28}$$

for $f\in W^{1,p}(\mathbb{R}^n)$, which follows from (24) by setting $g=|\langle\cdot,\eta\rangle|^p$ for $\eta\in\mathbb{S}^{n-1}$ and using (8) and (9) (cf. [25]), and that

$$V_p(K,\mathrm{Z}_pL)=\tilde{V}_{-p}(L,\Pi_p^*K) \tag{29}$$

for K a convex body and L a star body, a well-known relation that follows from Fubini's theorem, we now obtain (27) from the first equation in (25).

Using (27), we obtain from (26) in the limit as $s\to 1^-$ for a given $f\in W^{1,p}(\mathbb{R}^n)$, the following question: For which star bodies $L\subset\mathbb{R}^n$ is

$$\inf\left\{\int_{\mathbb{R}^n}\|\nabla f(x)\|_{\mathrm{Z}_p^*L}\,\mathrm{d}x : L \text{ star body}, |L|=\omega_n\right\} \tag{30}$$

attained? By (25) and the dual mixed volume inequality (6), we have

$$\frac{1}{n}\int_{\mathbb{R}^n}\|\nabla f(x)\|_{\mathrm{Z}_p^*L}^p\,\mathrm{d}x=V_p(\langle f\rangle_p,\mathrm{Z}_pL)=\tilde{V}_{-p}(L,\Pi_p^*f)\geq |L|^{(n+p)/n}|\Pi_p^*f|^{-p/n},$$

with equality precisely if L and $\Pi_p^* f$ are dilates, where we have used (28) and (29). From Theorem 10, we obtain that a suitably scaled sequence of optimal s-fractional Sobolev bodies converges to a multiple of the optimal body for (30) as $s \to 1^-$.

Acknowledgements J. Haddad was partially supported by CNPq, reference no. PQ 305559/2021-4 and by PAIDI 2020, reference no. PY20_00664. M. Ludwig was supported, in part, by the Austrian Science Fund (FWF): P 34446.

Funding Open access funding provided by Austrian Science Fund (FWF).

Declarations

Conflicts of interest On behalf of all authors, the corresponding author states that there is no conflict of interest. Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

References

1. Almgren, F.J., Jr., Lieb, E.H.: Symmetric decreasing rearrangement is sometimes continuous. J. Amer. Math. Soc. **2**, 683–773 (1989)
2. Athanasopoulos, I., Caffarelli, L.A.: Optimal regularity of lower-dimensional obstacle problems. J. Math. Sci. (N.Y.) **132**, 274–284 (2006)
3. Aubin, T.: Problèmes isopérimétriques et espaces de Sobolev. J. Differ. Geom. **11**, 573–598 (1976)
4. Bourgain, J., Brezis, H., Mironescu, P.: Another look at Sobolev spaces, In: Menaldi, J.L., Rofman, E., Sulem, A. (eds.). Optimal Control and Partial Differential Equations. A volume in honor of A. Bensoussans's 60th birthday, Amsterdam: IOS Press; Tokyo: Ohmsha, (2001)
5. Bourgain, J., Brezis, H., Mironescu, P.: Limiting embedding theorems for $W^{s,p}$ when $s \to 1$ and applications. J. Anal. Math. **87**, 77–101 (2002)
6. Brascamp, H.J., Lieb, E.H., Luttinger, J.M.: A general rearrangement inequality for multiple integrals. J. Funct. Anal. **17**, 227–237 (1974)
7. Brasco, L., Mosconi, S., Squassina, M.: Optimal decay of extremals for the fractional Sobolev inequality. Calc. Var. Partial Differ. Equ. **55**, 1–32 (2016)
8. Burchard, A.: Cases of equality in the Riesz rearrangement inequality. Ann. Math. **2**(143), 499–527 (1996)
9. Caffarelli, L., Mellet, A., Sire, Y.: Traveling waves for a boundary reaction-diffusion equation. Adv. Math. **230**, 433–457 (2012)
10. Carlen, E.: Duality and stability for functional inequalities. Ann. Fac. Sci. Toulouse Math. **6**(26), 319–350 (2017)
11. Cianchi, A., Lutwak, E., Yang, D., Zhang, G.: Affine Moser-Trudinger and Morrey-Sobolev inequalities. Calc. Var. Partial Differ. Equ. **36**, 419–436 (2009)
12. Frank, R., Seiringer, R.: Non-linear ground state representations and sharp Hardy inequalities. J. Funct. Anal. **255**, 3407–3430 (2008)
13. Gardner, R.: Geometric Tomography, Encyclopedia of Mathematics and its Applications, vol. 58, 2nd edn. Cambridge University Press, Cambridge (2006)

14. Haberl, C., Schuster, F.E.: Asymmetric affine L_p Sobolev inequalities. J. Funct. Anal. **257**, 641–658 (2009)
15. Haberl, C., Schuster, F.E., Xiao, J.: An asymmetric affine Pólya-Szegö principle. Math. Ann. **352**, 517–542 (2012)
16. Haddad, J., Ludwig, M.: Affine fractional Sobolev and isoperimetric inequalities, arXiv:2207.06375 (2022)
17. Kreuml, A.: The anisotropic fractional isoperimetric problem with respect to unconditional unit balls. Commun. Pure. Appl. Anal. **20**, 783–799 (2021)
18. Lieb, E.: Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. Ann. Math. **2**(118), 349–374 (1983)
19. Ludwig, M.: Minkowski valuations. Trans. Amer. Math. Soc. **357**, 191–4213 (2005)
20. Ludwig, M.: Anisotropic fractional perimeters. J. Differ. Geom. **96**, 77–93 (2014)
21. Ludwig, M.: Anisotropic fractional Sobolev norms. Adv. Math. **252**, 150–157 (2014)
22. Lutwak, E.: Dual mixed volumes. Pacific J. Math. **58**, 531–538 (1975)
23. Lutwak, E., Yang, D., Zhang, G.: L_p affine isoperimetric inequalities. J. Differ. Geom. **56**, 111–132 (2000)
24. Lutwak, E., Yang, D., Zhang, G.: Sharp affine L_p Sobolev inequalities. J. Differ. Geom. **62**, 17–38 (2002)
25. Lutwak, E., Yang, D., Zhang, G.: Optimal Sobolev norms and the L^p Minkowski problem. Int. Math. Res. Not., Art. ID 62987, 21 (2006)
26. Ma, D.: Asymmetric anisotropic fractional Sobolev norms. Arch. Math. (Basel) **103**, 167–175 (2014)
27. Maz'ya, V.: Sobolev Spaces with Applications to Elliptic Partial Differential Equations, augmented ed., Grundlehren der Mathematischen Wissenschaften, vol. 342, Springer, Heidelberg, (2011)
28. Schneider, R.: Convex Bodies: the Brunn-Minkowski Theory, Second expanded ed., Encyclopedia of Mathematics and its Applications, vol. 151, Cambridge University Press, Cambridge, (2014)
29. Silvestre, L.: Regularity of the obstacle problem for a fractional power of the Laplace operator. Comm. Pure Appl. Math. **60**, 67–112 (2007)
30. Talenti, G.: Best constant in Sobolev inequality. Ann. Mat. Pura Appl. **110**, 353–372 (1976)
31. Wang, T.: The affine Sobolev-Zhang inequality on $\mathrm{BV}(\mathbb{R}^n)$. Adv. Math. **230**, 2457–2473 (2012)
32. Zhang, G.: The affine Sobolev inequality. J. Differ. Geom. **53**, 183–202 (1999)