



Time periodic motion of temperature driven compressible fluids

Eduard Feireisl¹ · Piotr Gwiazda² · Agnieszka Świerczewska-Gwiazda³

Received: 11 April 2022 / Revised: 7 September 2022 / Accepted: 3 October 2022 /

Published online: 25 October 2022

© The Author(s) 2022

Abstract

We consider the Navier–Stokes–Fourier system describing the motion of a compressible viscous fluid in a container with impermeable boundary subject to time periodic heating and under the action of a time periodic potential force. We show the existence of a time periodic weak solution for arbitrarily large physically admissible data.

Contents

1	Introduction	1604
2	Main result	1606
2.1	Constitutive theory	1606
2.2	Weak solutions	1608
2.3	Main result	1610
3	Approximate problem	1610

The work of E.F. was partially supported by the Czech Sciences Foundation (GAČR), Grant Agreement 21-02411S. The Institute of Mathematics of the Academy of Sciences of the Czech Republic is supported by RVO:67985840. This work is partially supported by the Simons Foundation Award No 663281 granted to the Institute of Mathematics of the Polish Academy of Sciences for the years 2021–2023. The work of A. Ś-G. and P.G. was partially supported by National Science Centre (Poland), agreement no 2021/43/B/ST1/02851.

✉ Agnieszka Świerczewska-Gwiazda
aswiercz@mimuw.edu.pl

Eduard Feireisl
feireisl@math.cas.cz

Piotr Gwiazda
pgwiazda@mimuw.edu.pl

¹ Institute of Mathematics of the Academy of Sciences of the Czech Republic,
Žitná 25, 115 67 Praha 1, Czech Republic

² Institute of Mathematics of Polish Academy of Sciences, Śniadeckich 8, 00-956 Warsaw, Poland

³ Institute of Applied Mathematics and Mechanics, University of Warsaw,
Banacha 2, 02-097 Warsaw, Poland

3.1 Existence of approximate solutions 1612
 3.2 Approximate ballistic energy balance 1613
 4 Uniform bounds 1613
 4.1 Mass conservation 1613
 4.2 Energy estimates 1613
 4.3 Pressure estimates 1617
 4.4 Uniform bounds for $\varepsilon \rightarrow 0$ 1621
 5 Convergence 1625
 6 Concluding remarks 1625
 References 1626

1 Introduction

There are numerous examples of turbulent fluid motion excited by changes of the boundary temperature, among which is the well studied problem of Rayleigh–Bénard convection, see e.g. Davidson [8]. Motivated by similar problems in astrophysics of gaseous stars, we consider a general compressible viscous possibly rotating fluid, occupying a bounded domain $\Omega \subset R^d$, $d = 2, 3$, driven by periodic changes of boundary temperature. The relevant system of field equations for the standard variables: the mass density $\varrho = \varrho(t, x)$, the velocity $\mathbf{u} = \mathbf{u}(t, x)$, and the (absolute) temperature $\vartheta = \vartheta(t, x)$ reads:

$$\begin{aligned} \partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) &= 0, & (1.1) \\ \partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \varrho(\boldsymbol{\omega} \times \mathbf{u}) + \nabla_x p(\varrho, \vartheta) &= \operatorname{div}_x \mathbb{S} + \varrho \nabla_x G, & (1.2) \\ \partial_t(\varrho e(\varrho, \vartheta)) + \operatorname{div}_x(\varrho e(\varrho, \vartheta) \mathbf{u}) + \nabla_x \mathbf{q} &= \mathbb{S} : \mathbb{D}_x \mathbf{u} - p(\varrho, \vartheta) \operatorname{div}_x \mathbf{u}, & (1.3) \end{aligned}$$

where \mathbb{S} is the viscous stress given by Newton’s rheological law

$$\mathbb{S}(\vartheta, \mathbb{D}_x \mathbf{u}) = \mu(\vartheta) \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) + \eta(\vartheta) \operatorname{div}_x \mathbf{u} \mathbb{I}, \tag{1.4}$$

and \mathbf{q} is the heat flux given by Fourier’s law

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta) \nabla_x \vartheta. \tag{1.5}$$

The momentum equation is augmented by the Coriolis force with the rotation constant vector $\boldsymbol{\omega}$, the associated centrifugal force as well as the gravitation and other possible inertial time-periodic forces are regrouped in the potential G . The fluid occupies a bounded smooth domain $\Omega \subset R^d$, $d = 2, 3$ endowed with the Dirichlet boundary conditions

$$\begin{aligned} \mathbf{u}|_{\partial \Omega} &= 0, \\ \vartheta|_{\partial \Omega} &= \vartheta_B. \end{aligned} \tag{1.6}$$

The functions $\vartheta_B = \vartheta_B(t, x)$ and $G = G(t, x)$ are smooth and T -periodic in the time variable,

$$\begin{aligned}\vartheta_B(t + T, x) &= \vartheta_B(t, x), \\ G(t + T, x) &= G(t, x).\end{aligned}\tag{1.7}$$

Hereafter, the problem (1.1)–(1.6) is referred to as *Navier–Stokes–Fourier system*.

Our goal is to show the existence of a time-periodic solution to problem (1.1)–(1.7). There is a substantial number of references, where such a result is proven under some smallness and smoothness assumption on the data. Valli and Zajaczkowski [24, 25] observe that the distance of two smooth global in time solutions decays in time for the system close to a stable equilibrium, and, as a by product, they deduce the existence of a time periodic solution. Similar ideas have been followed by many authors, see Březina and Kagei [4], [5], Jin and Yang [17], Kagei and Oomachi [18], Kagei and Tsuda [19], Tsuda [23] to name only a few. Turbulent fluid flows given by large forces out of equilibrium are mostly considered in the framework of *weak* solutions. Based on the mathematical theory of compressible fluids developed by Lions [20, 21], the existence of large time periodic solutions for the simplified isentropic system was proved in [9] for the isentropic pressure–density equation of state $p(\varrho) = a\varrho^\gamma$, $\gamma \geq \frac{9}{5}$. The later development of the theory in [12] enabled to extend the result to the case $\gamma > \frac{5}{3}$, see Cai and Tan [6].

The situation is more delicate for the complete fluid systems including thermal effects. As a direct consequence of the Second law of thermodynamics, the existence of (forced) time periodic solutions is ruled out for problems with purely conservative boundary conditions, see [13]. In [10], the heat flux was controlled by means of a Robin type boundary condition

$$\mathbf{q} \cdot \mathbf{n} = d(\vartheta - \Theta_0) \text{ on } \partial\Omega,\tag{1.8}$$

with a given “mean” temperature Θ_0 . Accordingly, the internal energy is transferred out of the fluid domain in the high temperature regime and the time periodic motion is possible, see [10, Theorem 1]. Our goal is to show a similar result for the Dirichlet boundary conditions (1.6). Note that the problem is much more delicate than in [10] as the heat flux through the boundary is a priori not controlled. Additional novelty is that the function ϑ_B in (3.1) is time dependent whereas its counterpart Θ_0 in [10], cf. (1.8), depends only on x . Finally we note that the presence of the Coriolis force in the momentum equation, though physically relevant in some situations, does not represent any extra analytical difficulties.

Our approach is based on several rather new ideas that appeared only recently in the mathematical theory of open fluid systems.

- The concept of *weak solution* for the Navier–Stokes–Fourier system based on a combination of the entropy inequality and the ballistic energy balance developed in [7].
- *Uniform bounds* and large time asymptotics of the weak solutions in the spirit of [14].

- An *approximation scheme* based on a penalization of the Dirichlet boundary conditions via (1.8).

The concept of weak solution developed in the monograph [11] and used in [10] is based on the total energy balance as an integral part of the definition of weak solution to the Navier–Stokes–Fourier system. This approach applies solely to problems with *conservative* boundary conditions, where the energy flux vanishes on the boundary of the physical space or it is at least controlled as in (1.8). The problems with inhomogeneous *Dirichlet* boundary conditions require an alternative approach developed in [7], where the energy is replaced by the *ballistic energy*, for which the boundary flux is again controllable. This approach has been used recently in [14], where the existence of bounded absorbing sets and asymptotic compactness of bounded trajectories was established.

The constitutive restrictions imposed on the equations of state as well as the transport coefficients are the same as in the existence theory [7]. In particular, the general equation of state of real monoatomic gases proposed in [11, Chapters 1,2] is included. From this point of view, the result is apparently better than in the isentropic case studied in [9], and later revisited by Cai and Tan [6], where the condition $\gamma > \frac{5}{3}$ is needed. The price to pay is the potential form of the driving force $\mathbf{f} = \nabla_x G$ that, however, includes the physically relevant centrifugal as well as gravitational forces.

The paper is organized as follows. In Sect. 2, we introduce the basic hypotheses concerning the constitutive relations and state the main result. In Sect. 3, we introduce an approximation scheme inspired by [10]. Section 4 is the heart of the paper. Here we establish the necessary uniform bounds to perform the limit in the sequence of approximate solutions. Finally, in Sect. 5, we obtain the desired solution as a limit of the approximate sequence.

2 Main result

Before stating the main result, we recall the form of the constitutive equations proposed in [11, Chapters 1,2]. To comply with the Second law of thermodynamics, we postulate the existence of *entropy* s , related to the internal energy e and the pressure p through Gibbs' equation

$$\vartheta Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta)D\left(\frac{1}{\varrho}\right). \quad (2.1)$$

2.1 Constitutive theory

Similarly to [11, Chapters 1,2] we consider the pressure equation of state in the form

$$p(\varrho, \vartheta) = p_m(\varrho, \vartheta) + p_{\text{rad}}(\vartheta),$$

where p_m is the pressure of a general *monoatomic* gas related to the internal energy through

$$p_m(\varrho, \vartheta) = \frac{2}{3}\varrho e_m(\varrho, \vartheta), \quad (2.2)$$

augmented by the radiation pressure

$$p_{\text{rad}}(\vartheta) = \frac{a}{3}\vartheta^4, \quad a > 0.$$

Similarly, the internal energy reads

$$e(\varrho, \vartheta) = e_m(\varrho, \vartheta) + e_{\text{rad}}(\varrho, \vartheta), \quad e_{\text{rad}}(\varrho, \vartheta) = \frac{a}{\varrho}\vartheta^4.$$

Now, Gibbs' equ. (2.1) gives rise to a specific form of p_m ,

$$p_m(\varrho, \vartheta) = \vartheta^{\frac{5}{2}} P\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right)$$

for a certain $P \in C^1[0, \infty)$. Consequently,

$$p(\varrho, \vartheta) = \vartheta^{\frac{5}{2}} P\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right) + \frac{a}{3}\vartheta^4, \quad e(\varrho, \vartheta) = \frac{3}{2} \frac{\vartheta^{\frac{5}{2}}}{\varrho} P\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right) + \frac{a}{\varrho}\vartheta^4, \quad a > 0. \quad (2.3)$$

In addition, we suppose

$$P(0) = 0, \quad P'(Z) > 0 \text{ for } Z \geq 0, \quad 0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} \leq c \text{ for } Z > 0, \quad (2.4)$$

that may be seen as a direct consequence of *hypothesis of thermodynamic stability*, see [11, Chapter 1], and Bechtel et al. [1]. It follows that the function $Z \mapsto P(Z)/Z^{\frac{5}{3}}$ is decreasing, and we suppose

$$\lim_{Z \rightarrow \infty} \frac{P(Z)}{Z^{\frac{5}{3}}} = p_\infty > 0. \quad (2.5)$$

The associated entropy takes the form

$$s(\varrho, \vartheta) = \mathcal{S}\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right) + \frac{4a}{3} \frac{\vartheta^3}{\varrho}, \quad (2.6)$$

where

$$S'(Z) = -\frac{3}{2} \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2} < 0. \quad (2.7)$$

Finally, we impose the Third law of thermodynamics, see e.g. Belgiorno [2, 3], requiring the total entropy to vanish as soon as the absolute temperature approaches zero,

$$\lim_{Z \rightarrow \infty} S(Z) = 0. \quad (2.8)$$

It is easy to check that (2.4)–(2.8) imply

$$0 \leq \varrho \mathcal{S} \left(\frac{\varrho}{\vartheta^{\frac{3}{2}}} \right) \leq c \left(1 + \varrho \log^+(\varrho) + \varrho \log^+(\vartheta) \right). \quad (2.9)$$

As for the transport coefficients, we suppose that they are continuously differentiable functions of the absolute temperature satisfying

$$\begin{aligned} 0 < \underline{\mu}(1 + \vartheta) &\leq \mu(\vartheta), \quad |\mu'(\vartheta)| \leq \bar{\mu}, \\ 0 &\leq \eta(\vartheta) \leq \bar{\eta}(1 + \vartheta), \\ 0 < \underline{\kappa}(1 + \vartheta^\beta) &\leq \kappa(\vartheta) \leq \bar{\kappa}(1 + \vartheta^\beta), \end{aligned} \quad (2.10)$$

where, in accordance with the existence theory developed in [7], we require

$$\beta > 6. \quad (2.11)$$

2.2 Weak solutions

It is convenient to identify the time periodic functions (distributions) with objects defined on a periodic “flat sphere”

$$S_T = [0, T]_{|[0, T]}.$$

We are ready to introduce the concept of *time periodic solution* to the Navier–Stokes–Fourier system (1.1)–(1.7).

Definition 2.1 (weak solution) We say that a trio $(\varrho, \vartheta, \mathbf{u})$ is a weak time–periodic solution to the problem (1.1)–(1.7) if the following holds:

- **Regularity class:**

$$\begin{aligned} \varrho &\in C_{\text{weak}}(S_T; L^\gamma(\Omega)) \text{ for } \gamma = \frac{5}{3}, \\ \mathbf{u} &\in L^2(S_T; W_0^{1,2}(\Omega; R^d)), \varrho \mathbf{u} \in C_{\text{weak}}(S_T, L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d)), \\ \vartheta^{\beta/2}, \log(\vartheta) &\in L^2(S_T; W^{1,2}(\Omega)), \\ (\vartheta - \vartheta_B) &\in L^2(S_T; W_0^{1,2}(\Omega)). \end{aligned} \tag{2.12}$$

- **Equation of continuity:**

$$\int_{S_T} \int_{\Omega} [\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi] \, dx \, dt = 0, \tag{2.13}$$

$$\int_{S_T} \int_{\Omega} [b(\varrho) \partial_t \varphi + b(\varrho) \mathbf{u} \cdot \nabla_x \varphi + (b(\varrho) - b'(\varrho) \varrho) \operatorname{div}_x \mathbf{u} \varphi] \, dx \, dt = 0 \tag{2.14}$$

for any $\varphi \in C^1(S_T \times \overline{\Omega})$, and any $b \in C^1(R), b' \in C_c(R)$.

- **Momentum equation:**

$$\begin{aligned} &\int_{S_T} \int_{\Omega} [\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi} - \varrho (\boldsymbol{\omega} \times \mathbf{u}) \cdot \boldsymbol{\varphi} + p \operatorname{div}_x \boldsymbol{\varphi}] \, dx \, dt \\ &= \int_{S_T} \int_{\Omega} [\mathbb{S} : \nabla_x \boldsymbol{\varphi} - \varrho \nabla_x G \cdot \boldsymbol{\varphi}] \, dx \, dt \end{aligned} \tag{2.15}$$

for any $\boldsymbol{\varphi} \in C_c^1(S_T \times \Omega; R^d)$.

- **Entropy inequality:**

$$\begin{aligned} &-\int_{S_T} \int_{\Omega} [\varrho s \partial_t \varphi + \varrho s \mathbf{u} \cdot \nabla_x \varphi + \frac{\mathbf{q}}{\vartheta} \cdot \nabla_x \varphi] \, dx \, dt \\ &\geq \int_{S_T} \int_{\Omega} \frac{\varphi}{\vartheta} \left[\mathbb{S} : \mathbb{D}_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right] \, dx \, dt \end{aligned} \tag{2.16}$$

for any $\varphi \in C_c^1(S_T \times \Omega), \varphi \geq 0$;

- **Ballistic energy balance:**

$$\begin{aligned} &-\int_{S_T} \partial_t \psi \int_{\Omega} \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e - \tilde{\vartheta} \varrho s \right] \, dx \, dt + \int_{S_T} \psi \int_{\Omega} \frac{\tilde{\vartheta}}{\vartheta} \left[\mathbb{S} : \mathbb{D}_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right] \, dx \, dt \\ &\leq \int_{S_T} \psi \int_{\Omega} \left[\varrho \mathbf{u} \cdot \nabla_x G - \varrho s \mathbf{u} \cdot \nabla_x \tilde{\vartheta} - \frac{\mathbf{q}}{\vartheta} \cdot \nabla_x \tilde{\vartheta} \right] \, dx \, dt \end{aligned} \tag{2.17}$$

for any $\psi \in C^1(S_T), \psi \geq 0$, and any $\tilde{\vartheta} \in C^1(S_T \times \overline{\Omega})$,

$$\tilde{\vartheta} > 0, \tilde{\vartheta}|_{\partial\Omega} = \vartheta_B. \tag{2.18}$$

The weak time–periodic solutions are therefore the weak solutions in the sense of [7] that are T -periodic in the time variable. The instantaneous values of the *conservative variables* $\varrho(\tau, \cdot)$, $(\varrho\mathbf{u})(\tau, \cdot)$ are well defined as well as the right and left-hand limits of the total entropy $S = \varrho s(\varrho, \vartheta)$,

$$\begin{aligned} \langle S(\tau-, \cdot); \phi \rangle &\equiv \lim_{\delta \rightarrow 0+} \frac{1}{\delta} \int_{\tau-\delta}^{\tau} \int_{\Omega} \varrho s(t, \cdot) \phi \, dx \, dt, \\ \langle S(\tau+, \cdot); \phi \rangle &\equiv \lim_{\delta \rightarrow 0+} \frac{1}{\delta} \int_{\tau}^{\tau+\delta} \int_{\Omega} \varrho s(t, \cdot) \phi \, dx \, dt. \end{aligned}$$

2.3 Main result

Having collected the necessary preliminary material we are ready to state our main result.

Theorem 2.2 (existence of time periodic solutions) *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded domain of class $C^{2+\nu}$. Suppose that the pressure p , the internal energy e , the entropy s , as well as the transport coefficients μ, η , and κ satisfy the hypotheses (2.2)–(2.11). Finally, let the data $G \in W^{1,\infty}(S_T \times \Omega)$, $\vartheta_B \in C^3(S_T \times \mathbb{R}^d)$ be time periodic as stated in (1.7), and*

$$\inf_{S_T \times \Omega} \vartheta_B = \underline{\vartheta} > 0.$$

Then for any M_0 there exists at least one time periodic solution $(\varrho, \vartheta, \mathbf{u})$ of the problem (1.1)–(1.7) in the sense specified in Definition 2.1 satisfying

$$\int_{\Omega} \varrho(t, \cdot) \, dx = M_0 \text{ for any } t \in S_T.$$

Remark 2.3 In the hypotheses of Theorem 2.2, we assume that $\vartheta_B|_{\partial\Omega}$ is a restriction of a (smooth) function defined on the whole space \mathbb{R}^d .

The rest of the paper is devoted to the proof of Theorem 2.2.

3 Approximate problem

The most efficient way of constructing suitable approximate solutions seems adapting the result of [10] to the present setting. Specifically, the approximation scheme is based on penalization of the Dirichlet boundary condition for the temperature via the Robin boundary conditions

$$\mathbf{q} \cdot \mathbf{n} = \frac{1}{\varepsilon} |\vartheta - \vartheta_B|^k (\vartheta - \vartheta_B), \quad k \geq 0, \text{ on } \partial\Omega, \tag{3.1}$$

where $\varepsilon > 0$ is a small parameter.

The approximate solutions $(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon)$ are defined similarly to Definition 2.1:

• **Regularity class:**

$$\begin{aligned} \varrho_\varepsilon &\in C_{\text{weak}}(S_T; L^\gamma(\Omega)) \text{ for } \gamma = \frac{5}{3}, \\ \mathbf{u}_\varepsilon &\in L^2(S_T; W_0^{1,2}(\Omega; R^d)), \varrho_\varepsilon \mathbf{u}_\varepsilon \in C_{\text{weak}}(S_T, L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d)) \\ \vartheta_\varepsilon^{\beta/2}, \log(\vartheta_\varepsilon) &\in L^2(S_T; W^{1,2}(\Omega)). \end{aligned} \tag{3.2}$$

• **Equation of continuity:**

$$\int_{S_T} \int_{\Omega} [\varrho_\varepsilon \partial_t \varphi + \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \varphi] \, dx \, dt = 0, \tag{3.3}$$

$$\int_{S_T} \int_{\Omega} [b(\varrho_\varepsilon) \partial_t \varphi + b(\varrho_\varepsilon) \mathbf{u}_\varepsilon \cdot \nabla_x \varphi + (b(\varrho_\varepsilon) - b'(\varrho_\varepsilon) \varrho_\varepsilon) \operatorname{div}_x \mathbf{u}_\varepsilon \varphi] \, dx \, dt = 0 \tag{3.4}$$

for any $\varphi \in C^1(S_T \times \overline{\Omega})$, and any $b \in C^1(R)$, $b' \in C_c(R)$.

• **Momentum equation:**

$$\begin{aligned} &\int_{S_T} \int_{\Omega} [\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \boldsymbol{\varphi} + \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \boldsymbol{\varphi} - \varrho_\varepsilon (\boldsymbol{\omega} \times \mathbf{u}_\varepsilon) \cdot \boldsymbol{\varphi} + p \operatorname{div}_x \boldsymbol{\varphi}] \, dx \, dt \\ &= \int_{S_T} \int_{\Omega} [\mathbb{S} : \nabla_x \boldsymbol{\varphi} - \varrho_\varepsilon \nabla_x G \cdot \boldsymbol{\varphi}] \, dx \, dt \end{aligned} \tag{3.5}$$

for any $\boldsymbol{\varphi} \in C_c^1(S_T \times \Omega; R^d)$.

• **Entropy inequality:**

$$\begin{aligned} &-\int_{S_T} \int_{\Omega} \left[\varrho_\varepsilon s \partial_t \varphi + \varrho_\varepsilon s \mathbf{u}_\varepsilon \cdot \nabla_x \varphi + \frac{\mathbf{q}}{\vartheta_\varepsilon} \cdot \nabla_x \varphi \right] \, dx \, dt \geq \int_{S_T} \int_{\Omega} \frac{\varphi}{\vartheta_\varepsilon} \left[\mathbb{S} : \mathbb{D}_x \mathbf{u}_\varepsilon - \frac{\mathbf{q} \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon} \right] \, dx \, dt \\ &+ \frac{1}{\varepsilon} \int_{S_T} \int_{\partial\Omega} \varphi \frac{|\vartheta_B - \vartheta_\varepsilon|^k (\vartheta_B - \vartheta_\varepsilon)}{\vartheta_\varepsilon} \, d\sigma_x \, dt \end{aligned} \tag{3.6}$$

for any $\varphi \in C^1(S_T \times \overline{\Omega})$, $\varphi \geq 0$.

• **Energy balance:**

$$\begin{aligned} &-\int_{S_T} \partial_t \psi \int_{\Omega} \left[\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \varrho_\varepsilon e \right] \, dx \, dt + \frac{1}{\varepsilon} \int_{S_T} \psi \int_{\partial\Omega} |\vartheta_\varepsilon - \vartheta_B|^k (\vartheta_\varepsilon - \vartheta_B) \, d\sigma_x \, dt \\ &= \int_{S_T} \psi \int_{\Omega} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x G \, dx \, dt \end{aligned} \tag{3.7}$$

for any $\psi \in C^1(S_T)$,

cf. [10, Section 2.2].

3.1 Existence of approximate solutions

Our aim is to use the existence result proved in [10, Theorem 1] to obtain the approximate solutions $(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon)_{\varepsilon>0}$. To perform this step some comments are in order. In comparison with [10], the present problem features the following new ingredients:

- The action of the Coriolis force in the momentum equation (3.5).
- The function ϑ_B in (3.1) is time dependent whereas its counterpart Θ_0 in [10] depends only on x .
- The exponent k in (3.1) equals zero in [10].

It is easy to check that the existence proof in [10] can be modified to accommodate the above changes as soon as suitable a priori bounds similar to those in [10, Section 2.4] are established. To see this, we start with the energy balance (3.7) with $\psi \equiv 1$ yielding

$$\begin{aligned} \frac{1}{\varepsilon} \int_{S_T} \int_{\partial\Omega} |\vartheta_\varepsilon - \vartheta_B|^k (\vartheta_\varepsilon - \vartheta_B) d\sigma_x \, dt &= \int_{S_T} \int_{\Omega} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x G \, dx \, dt = - \int_{S_T} \int_{\Omega} \varrho_\varepsilon \partial_t G \, dx \, dt \\ &\leq M_0 \|\partial_t G\|_{L^\infty(S_T \times \Omega)}, \text{ where } M_0 = \int_{\Omega} \varrho_\varepsilon \, dx. \end{aligned} \tag{3.8}$$

As $\vartheta_\varepsilon > 0$ a.a., (3.8) yields the bound

$$\|\vartheta_\varepsilon\|_{L^{k+1}(S_T \times \partial\Omega)} \lesssim 1 \tag{3.9}$$

in terms of the data and uniform for $\varepsilon \rightarrow 0$. Consequently, the entropy inequality (3.6) gives rise to the bound on the entropy production rate

$$\int_{S_T} \int_{\Omega} \frac{1}{\vartheta_\varepsilon} \left[\mathbb{S} : \mathbb{D}_x \mathbf{u}_\varepsilon - \frac{\mathbf{q} \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon} \right] \, dx \, dt \leq c(\varepsilon, G, \vartheta_B, M_0) \tag{3.10}$$

and the remaining estimates are obtained exactly as in [10, Section 2.4]. Note that the right-hand side of (3.10) may blow up for $\varepsilon \rightarrow 0$.

With the necessary a priori bounds at hand, we obtain a family of approximate solutions $(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon)_{\varepsilon>0}$ exactly as in [10].

Proposition 3.1 (Approximate solutions) *In addition to the hypotheses of Theorem 2.2, let*

$$\varepsilon > 0, \quad 6 < k + 1 = \beta, \quad M_0 > 0 \tag{3.11}$$

be given.

Then the approximate problem (3.2)–(3.7) admits a solution $(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon)$.

3.2 Approximate ballistic energy balance

Let $\tilde{\vartheta} \in C^1(S_T \times \overline{\Omega})$ satisfy (2.18). Choosing $\varphi(t, x) = \psi(t)\tilde{\vartheta}(t, x)$, where $\psi \in C^1(S_T)$, $\psi \geq 0$, as a test function in the approximate entropy inequality (3.6) and adding the resulting integral to the energy balance (3.7), we deduce

$$\begin{aligned}
 & - \int_{S_T} \partial_t \psi \int_{\Omega} \left[\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \varrho_\varepsilon e - \tilde{\vartheta} \varrho_\varepsilon s \right] dx dt + \int_{S_T} \psi \int_{\Omega} \frac{\tilde{\vartheta}}{\vartheta_\varepsilon} \left[\mathbb{S} : \mathbb{D}_x \mathbf{u}_\varepsilon - \frac{\mathbf{q} \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon} \right] dx dt \\
 & + \frac{1}{\varepsilon} \int_{S_T} \psi \int_{\partial\Omega} \frac{|\vartheta_\varepsilon - \vartheta_B|^{k+2}}{\vartheta_\varepsilon} d\sigma_x dt \\
 & \leq \int_{S_T} \psi \int_{\Omega} \left[\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x G - \varrho_\varepsilon s \mathbf{u}_\varepsilon \cdot \nabla_x \tilde{\vartheta} - \frac{\mathbf{q}}{\vartheta_\varepsilon} \cdot \nabla_x \tilde{\vartheta} - \partial_t \tilde{\vartheta} \varrho_\varepsilon s \right] dx dt. \tag{3.12}
 \end{aligned}$$

Inequality (3.12) is obviously a counter part of the ballistic energy balance (2.17) and will be used in the forthcoming part to deduce the necessary bounds on the family of approximate solutions.

4 Uniform bounds

In order to perform the limit $\varepsilon \rightarrow 0$ in the family of approximate solutions obtained in Proposition 3.1, we need uniform bounds independent of ε .

4.1 Mass conservation

Obviously, as the total mass of the fluid is conserved, we get

$$M_0 = \int_{\Omega} \varrho_\varepsilon(t, \cdot) dx \text{ for all } t \in S_T \Rightarrow \sup_{t \in S_T} \|\varrho_\varepsilon(t, \cdot)\|_{L^1(\Omega)} \lesssim 1. \tag{4.1}$$

4.2 Energy estimates

As both $\partial\Omega$ and the boundary data ϑ_B are smooth, we may suppose that

$$\Delta_x \vartheta_B(t, \cdot) = 0 \text{ in } \Omega \text{ for any } t \in S_T. \tag{4.2}$$

Choosing $\psi = 1$, $\tilde{\vartheta} = \vartheta_B$ in the ballistic energy inequality (3.12) we get

$$\begin{aligned}
 & \int_{S_T} \int_{\Omega} \frac{\vartheta_B}{\vartheta_\varepsilon} \left[\mathbb{S}(\mathbb{D}_x \mathbf{u}_\varepsilon) : \mathbb{D}_x \mathbf{u}_\varepsilon + \frac{\kappa(\vartheta_\varepsilon) |\nabla_x \vartheta_\varepsilon|^2}{\vartheta_\varepsilon} \right] dx dt + \frac{1}{\varepsilon} \int_{S_T} \int_{\partial\Omega} \frac{|\vartheta_\varepsilon - \vartheta_B|^{k+2}}{\vartheta_\varepsilon} d\sigma_x dt \\
 & \leq \int_{S_T} \int_{\Omega} \left[\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x G - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \mathbf{u}_\varepsilon \cdot \nabla_x \vartheta_B + \frac{\kappa(\vartheta_\varepsilon) \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x \vartheta_B - \partial_t \tilde{\vartheta} \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \right] dx dt. \tag{4.3}
 \end{aligned}$$

By virtue of hypothesis (2.10) and Korn’s inequality, we obtain

$$\|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega; R^d)}^2 \lesssim \int_\Omega \frac{\vartheta_B}{\vartheta_\varepsilon} \mathbb{S}(\mathbb{D}_x \mathbf{u}_\varepsilon) : \mathbb{D}_x \mathbf{u}_\varepsilon \, dx.$$

Moreover, again by virtue of (2.10),

$$\int_\Omega \left[|\nabla_x \vartheta_\varepsilon^{\frac{\beta}{2}}|^2 + |\nabla_x \log(\vartheta_\varepsilon)|^2 \right] dx \lesssim \int_\Omega \frac{\vartheta_B}{\vartheta_\varepsilon} \frac{\kappa(\vartheta_\varepsilon) |\nabla_x \vartheta_\varepsilon|^2}{\vartheta_\varepsilon} dx.$$

By Poincarè inequality (see e.g. Theorem 4.4.6 in [27]) we obtain that

$$\begin{aligned} \int_\Omega |\vartheta_\varepsilon^{\frac{\beta}{2}}|^2 dx &\lesssim \int_{\partial\Omega} |\vartheta_\varepsilon^{\frac{\beta}{2}}|^2 d\sigma_x + \int_\Omega \left| \vartheta_\varepsilon^{\frac{\beta}{2}} - \int_{\partial\Omega} \vartheta_\varepsilon^{\frac{\beta}{2}} d\sigma_x \right|^2 dx \lesssim \int_{\partial\Omega} |\vartheta_\varepsilon^{\frac{\beta}{2}}|^2 d\sigma_x \\ &+ \int_\Omega |\nabla_x \vartheta_\varepsilon^{\frac{\beta}{2}}|^2 dx, \end{aligned}$$

as well as

$$\int_\Omega |\log(\vartheta_\varepsilon)|^2 dx \lesssim \int_{\partial\Omega} |\log(\vartheta_\varepsilon)|^2 d\sigma_x + \int_\Omega |\nabla_x \log(\vartheta_\varepsilon)|^2 dx.$$

Collecting the last three inequalities, hypothesis (3.11) and estimating the boundary terms

$$|\vartheta_\varepsilon^{\frac{\beta}{2}}|^2 + |\log(\vartheta_\varepsilon)|^2 \lesssim \frac{|\vartheta_\varepsilon - \vartheta_B|^{k+2}}{\vartheta_\varepsilon} \lesssim \frac{1}{2\varepsilon} \frac{|\vartheta_\varepsilon - \vartheta_B|^{k+2}}{\vartheta_\varepsilon}$$

gives

$$\begin{aligned} &\left\| \vartheta_\varepsilon^{\frac{\beta}{2}} \right\|_{W^{1,2}(\Omega)}^2 + \|\log(\vartheta_\varepsilon)\|_{W^{1,2}(\Omega)}^2 \\ &\lesssim \left(1 + \frac{1}{2\varepsilon} \int_{\partial\Omega} \frac{|\vartheta_\varepsilon - \vartheta_B|^{k+2}}{\vartheta_\varepsilon} d\sigma_x + \int_\Omega \frac{\vartheta_B}{\vartheta_\varepsilon} \frac{\kappa(\vartheta_\varepsilon) |\nabla_x \vartheta_\varepsilon|^2}{\vartheta_\varepsilon} dx \right). \end{aligned}$$

Gathering the previous observations, we may infer that

$$\begin{aligned} &\int_{S_T} \left[\|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega; R^d)}^2 + \left\| \vartheta_\varepsilon^{\frac{\beta}{2}} \right\|_{W^{1,2}(\Omega)}^2 + \|\log(\vartheta_\varepsilon)\|_{W^{1,2}(\Omega)}^2 \right] dt + \frac{1}{\varepsilon} \int_{S_T} \int_{\partial\Omega} \frac{|\vartheta_\varepsilon - \vartheta_B|^{\beta+1}}{\vartheta_\varepsilon} d\sigma_x dt \\ &\lesssim \left(1 + \left| \int_{S_T} \int_\Omega \left[\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x G - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \mathbf{u}_\varepsilon \cdot \nabla_x \vartheta_B + \frac{\kappa(\vartheta_\varepsilon) \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x \vartheta_B - \partial_t \tilde{\varrho}_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \right] dx dt \right) \right). \end{aligned} \tag{4.4}$$

Now, as $\varrho_\varepsilon, \mathbf{u}_\varepsilon$ solve the equation of continuity (2.13),

$$\int_{S_T} \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x G \, dx \, dt = - \int_{S_T} \varrho_\varepsilon \partial_t G \, dt \leq c(M_0, G).$$

In addition, denoting

$$\mathcal{K}(\vartheta) = \int_1^\vartheta \frac{\kappa(z)}{z} \, dz,$$

we obtain, by virtue of (4.2),

$$\int_\Omega \frac{\kappa(\vartheta_\varepsilon) \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x \vartheta_B \, dx = \int_\Omega \nabla_x \mathcal{K}(\vartheta_\varepsilon) \cdot \nabla_x \vartheta_B \, dx = \int_{\partial\Omega} \mathcal{K}(\vartheta_\varepsilon) \nabla_x \vartheta_B \cdot \mathbf{n} \, d\sigma_x.$$

Consequently, as κ satisfies hypothesis (2.10), we conclude

$$\begin{aligned} \left| \int_\Omega \frac{\kappa(\vartheta_\varepsilon) \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x \vartheta_B \, dx \right| &= \left| \int_{\partial\Omega} \mathcal{K}(\vartheta_\varepsilon) \nabla_x \vartheta_B \cdot \mathbf{n} \, d\sigma_x \right| \\ &\lesssim \left(1 + \int_{\partial\Omega} \frac{|\vartheta_\varepsilon - \vartheta_B|^{\beta+1}}{\vartheta_\varepsilon} \, d\sigma_x \right). \end{aligned}$$

Thus inequality (4.4) reduces to

$$\begin{aligned} &\int_{S_T} \left[\|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega; R^d)}^2 + \left\| \vartheta_\varepsilon^{\frac{\beta}{2}} \right\|_{W^{1,2}(\Omega)}^2 + \|\log(\vartheta_\varepsilon)\|_{W^{1,2}(\Omega)}^2 \right] \, dt \\ &+ \frac{1}{\varepsilon} \int_{S_T} \int_{\partial\Omega} \frac{|\vartheta_\varepsilon - \vartheta_B|^{\beta+1}}{\vartheta_\varepsilon} \, d\sigma_x \, dt \\ &\lesssim \left(1 + \int_{S_T} \int_\Omega \left(|\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \mathbf{u}_\varepsilon \cdot \nabla_x \vartheta_B| + |\partial_t \tilde{\vartheta} \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)| \right) \, dx \, dt \right). \end{aligned} \tag{4.5}$$

In accordance with hypothesis (2.6), we decompose the entropy as

$$\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) = \varrho_\varepsilon \mathcal{S} \left(\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}} \right) + \frac{4a}{3} \vartheta_\varepsilon^3.$$

Consequently, the radiation component may be handled as

$$\int_\Omega |\vartheta_\varepsilon^3 \mathbf{u}_\varepsilon \cdot \nabla_x \vartheta_B| \, dx \leq \delta \|\mathbf{u}_\varepsilon\|_{L^2(\Omega; R^d)}^2 + c(\delta, \vartheta_B) \int_\Omega \vartheta_\varepsilon^6 \, dx$$

for any $\delta > 0$. Consequently, as $\beta > 6$, this term can be absorbed by the left-hand side of (4.5) yielding

$$\begin{aligned} & \int_{S_T} \left[\|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega; R^d)}^2 + \left\| \vartheta_\varepsilon^{\frac{\beta}{2}} \right\|_{W^{1,2}(\Omega)}^2 + \|\log(\vartheta_\varepsilon)\|_{W^{1,2}(\Omega)}^2 \right] dt \\ & + \frac{1}{\varepsilon} \int_{S_T} \int_{\partial\Omega} \frac{|\vartheta_\varepsilon - \vartheta_B|^{\beta+1}}{\vartheta_\varepsilon} d\sigma_x dt \\ & \lesssim \left(1 + \int_{S_T} \int_{\Omega} \left| \varrho_\varepsilon \mathcal{S} \left(\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}} \right) \mathbf{u}_\varepsilon \cdot \nabla_x \vartheta_B \right| + \left| \varrho_\varepsilon \mathcal{S} \left(\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}} \right) \partial_t \tilde{\vartheta} \right| dx dt \right). \end{aligned} \tag{4.6}$$

Finally, following the arguments of [14, Section 4.4], we make use of the Third law of thermodynamics enforced through hypothesis (2.8). Specifically, if

$$\frac{\varrho}{\vartheta^{\frac{3}{2}}} < r \text{ meaning } \varrho < r\vartheta^{\frac{3}{2}},$$

we get, by virtue of (2.9),

$$0 \leq \varrho \mathcal{S} \left(\frac{\varrho}{\vartheta^{\frac{3}{2}}} \right) \lesssim \left(1 + r\vartheta^{\frac{3}{2}} \left[\log^+(r\vartheta^{\frac{3}{2}}) + \log^+(\vartheta) \right] \right). \tag{4.7}$$

Consequently, we deduce from (4.6),

$$\begin{aligned} & \int_{S_T} \left[\|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega; R^d)}^2 + \left\| \vartheta_\varepsilon^{\frac{\beta}{2}} \right\|_{W^{1,2}(\Omega)}^2 + \|\log(\vartheta_\varepsilon)\|_{W^{1,2}(\Omega)}^2 \right] dt \\ & + \frac{1}{\varepsilon} \int_{S_T} \int_{\partial\Omega} \frac{|\vartheta_\varepsilon - \vartheta_B|^{\beta+1}}{\vartheta_\varepsilon} d\sigma_x dt \\ & \lesssim \left(\Lambda(r) + \int_{S_T} \int_{\Omega} \mathbb{1}_{\left\{ \frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}} \geq r \right\}} \left| \varrho_\varepsilon \mathcal{S} \left(\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}} \right) \mathbf{u}_\varepsilon \cdot \nabla_x \vartheta_B \right| dx dt \right. \\ & \left. + \int_{S_T} \int_{\Omega} \mathbb{1}_{\left\{ \frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}} \geq r \right\}} \left| \varrho_\varepsilon \mathcal{S} \left(\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}} \right) \partial_t \tilde{\vartheta} \right| dx dt \right), \end{aligned} \tag{4.8}$$

where $\Lambda(r) \rightarrow \infty$ as $r \rightarrow \infty$.

Now, again by hypothesis (2.8),

$$0 \leq \mathbb{1}_{\left\{ \frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}} \geq r \right\}} \mathcal{S} \left(\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}} \right) \leq \mathcal{S}(r) \rightarrow 0 \text{ as } r \rightarrow \infty.$$

In an analogous way we treat the term $\int_{S_T} \int_{\Omega} \partial_t \tilde{\vartheta} \varrho_\varepsilon s \, dx \, dt$. Going back to (4.8) we conclude

$$\begin{aligned} & \int_{S_T} \left[\|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega; R^d)}^2 + \left\| \vartheta_\varepsilon^{\frac{\beta}{2}} \right\|_{W^{1,2}(\Omega)}^2 + \|\log(\vartheta_\varepsilon)\|_{W^{1,2}(\Omega)}^2 \right] dt \\ & + \frac{1}{\varepsilon} \int_{S_T} \int_{\partial\Omega} \frac{|\vartheta_\varepsilon - \vartheta_B|^{\beta+1}}{\vartheta_\varepsilon} \, d\sigma_x \, dt \\ & \lesssim \left(\Lambda(r) + \mathcal{S}(r) \int_{S_T} \int_{\Omega} (|\varrho_\varepsilon \mathbf{u}_\varepsilon| + \varrho_\varepsilon) \, dx \, dt \right), \\ & \Lambda(r) \rightarrow \infty, \mathcal{S}(r) \rightarrow 0 \text{ as } r \rightarrow \infty. \end{aligned} \tag{4.9}$$

4.3 Pressure estimates

To close the estimates we have to control the density in terms of the integrals on the right-hand side of (4.9). To this end, we use the nowadays standard pressure estimates obtained via Bogovskii operator. Specifically, we use the quantity

$$\varphi(t, x) = \mathcal{B} \left[\varrho_\varepsilon^\omega - \frac{1}{|\Omega|} \int_{\Omega} \varrho_\varepsilon^\omega \, dx \right], \omega > 0,$$

as a test function in the momentum equation (3.5). Here \mathcal{B} denotes the operator enjoying the following properties:

- $\mathcal{B} : L_0^q(\Omega) \equiv \left\{ v \in L^q(\Omega) \mid \int_{\Omega} v \, dx = 0 \right\} \rightarrow W_0^{1,q}(\Omega; R^d), 1 < q < \infty;$ (4.10)

- $\operatorname{div}_x \mathcal{B}[v] = v;$

- if $v = \operatorname{div}_x \mathbf{g}$, with $\mathbf{g} \in L^q(\Omega; R^d)$, $\operatorname{div}_x \mathbf{g} \in L^r(\Omega)$, $\mathbf{g} \cdot \mathbf{n}|_{\partial\Omega} = 0$, then

$$\|\mathcal{B}[\operatorname{div}_x \mathbf{g}]\|_{L^r(\Omega; R^d)} \lesssim \|\mathbf{g}\|_{L^r(\Omega; R^d)}, \tag{4.11}$$

see Galdi [15, Chapter 3] or Geißert et al. [16]. Boundedness of the operator \mathcal{B} stated in (4.10), (4.11) will be systematically used in the estimates below.

After a straightforward manipulation (see e.g. [9]), we obtain

$$\begin{aligned}
 & \int_{S_T} \int_{\Omega} p(\varrho_\varepsilon, \vartheta_\varepsilon) \varrho_\varepsilon^\omega \, dx \, dt \\
 &= \int_{S_T} \frac{1}{|\Omega|} \left(\int_{\Omega} \varrho_\varepsilon^\omega \, dx \right) \left(\int_{\Omega} p(\varrho_\varepsilon, \vartheta_\varepsilon) \, dx \right) \, dt \\
 &\quad - \int_{S_T} \int_{\Omega} \varrho_\varepsilon (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \mathcal{B} \left[\varrho_\varepsilon^\omega - \frac{1}{|\Omega|} \int_{\Omega} \varrho_\varepsilon^\omega \, dx \right] \, dx \, dt \\
 &\quad + \int_{S_T} \int_{\Omega} \varrho_\varepsilon (\boldsymbol{\omega} \times \mathbf{u}_\varepsilon) \cdot \mathcal{B} \left[\varrho_\varepsilon^\omega - \frac{1}{|\Omega|} \int_{\Omega} \varrho_\varepsilon^\omega \, dx \right] \, dx \, dt \\
 &\quad + \int_{S_T} \int_{\Omega} \mathbb{S}(\vartheta_\varepsilon, \mathbb{D}_x \mathbf{u}_\varepsilon) : \nabla_x \mathcal{B} \left[\varrho_\varepsilon^\omega - \frac{1}{|\Omega|} \int_{\Omega} \varrho_\varepsilon^\omega \, dx \right] \, dx \, dt \\
 &\quad - \int_{S_T} \int_{\Omega} \varrho_\varepsilon \nabla_x G \cdot \mathcal{B} \left[\varrho_\varepsilon^\omega - \frac{1}{|\Omega|} \int_{\Omega} \varrho_\varepsilon^\omega \, dx \right] \, dx \, dt \\
 &\quad + \int_{S_T} \int_{\Omega} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B}[\operatorname{div}_x(\varrho_\varepsilon^\omega \mathbf{u}_\varepsilon)] \, dx \, dt \\
 &\quad + (\omega - 1) \int_{S_T} \int_{\Omega} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B} \left[\varrho_\varepsilon^\omega \operatorname{div}_x \mathbf{u}_\varepsilon - \frac{1}{|\Omega|} \int_{\Omega} \varrho_\varepsilon^\omega \operatorname{div}_x \mathbf{u}_\varepsilon \, dx \right] \, dx \, dt. \tag{4.12}
 \end{aligned}$$

Since the total mass M_0 is constant, the smoothing properties of \mathcal{B} yield

$$\left\| \mathcal{B} \left[\varrho_\varepsilon^\omega - \frac{1}{|\Omega|} \int_{\Omega} \varrho_\varepsilon^\omega \, dx \right] \right\|_{L^\infty(S_T \times \Omega; R^d)} \leq c(M_0) \text{ as soon as } \omega < \frac{1}{d}.$$

Moreover, in accordance with hypotheses (2.3)–(2.5),

$$\varrho^{\frac{5}{3}} + \vartheta^4 \lesssim p(\varrho, \vartheta) \lesssim \varrho^{\frac{5}{3}} + \vartheta^4 + 1.$$

In view of these facts, inequality (4.12) gives rise to

$$\begin{aligned}
 & \int_{S_T} \int_{\Omega} \varrho_\varepsilon^{\frac{5}{3} + \omega} \, dx \, dt \leq c(M_0) \left(1 + \int_{S_T} \int_{\Omega} \vartheta_\varepsilon^4 \, dx \, dt \right. \\
 &\quad - \int_{S_T} \int_{\Omega} \varrho_\varepsilon (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \mathcal{B} \left[\varrho_\varepsilon^\omega - \frac{1}{|\Omega|} \int_{\Omega} \varrho_\varepsilon^\omega \, dx \right] \, dx \, dt \\
 &\quad + \int_{S_T} \int_{\Omega} \varrho_\varepsilon (\boldsymbol{\omega} \times \mathbf{u}_\varepsilon) \cdot \mathcal{B} \left[\varrho_\varepsilon^\omega - \frac{1}{|\Omega|} \int_{\Omega} \varrho_\varepsilon^\omega \, dx \right] \, dx \, dt \\
 &\quad \left. + \int_{S_T} \int_{\Omega} \mathbb{S}(\vartheta_\varepsilon, \mathbb{D}_x \mathbf{u}_\varepsilon) : \nabla_x \mathcal{B} \left[\varrho_\varepsilon^\omega - \frac{1}{|\Omega|} \int_{\Omega} \varrho_\varepsilon^\omega \, dx \right] \, dx \, dt \right)
 \end{aligned}$$

$$\begin{aligned}
 &+ \int_{S_T} \int_{\Omega} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B}[\operatorname{div}_x(\varrho_\varepsilon^\omega \mathbf{u}_\varepsilon)] \, dx \, dt \\
 &+ (\omega - 1) \int_{S_T} \int_{\Omega} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B} \left[\varrho_\varepsilon^\omega \operatorname{div}_x \mathbf{u}_\varepsilon - \frac{1}{|\Omega|} \int_{\Omega} \varrho_\varepsilon^\omega \operatorname{div}_x \mathbf{u}_\varepsilon \, dx \right] \, dx \, dt.
 \end{aligned}
 \tag{4.13}$$

The following steps will be performed for $d = 3$. Obviously even better estimates can be obtained if $d = 2$. First,

$$\begin{aligned}
 &\left| \int_{S_T} \int_{\Omega} \varrho_\varepsilon (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \mathcal{B} \left[\varrho_\varepsilon^\omega - \frac{1}{|\Omega|} \int_{\Omega} \varrho_\varepsilon^\omega \, dx \right] \, dx \, dt \right| \\
 &\lesssim \int_{S_T} \|\varrho_\varepsilon\|_{L^\gamma(\Omega)} \|\mathbf{u}_\varepsilon\|_{L^6(\Omega; \mathbb{R}^3)}^2 \|\varrho_\varepsilon^\omega\|_{L^q(\Omega)} \, dt \\
 &\lesssim \sup_{t \in S_T} \|\varrho_\varepsilon\|_{L^\gamma(\Omega)} \int_{S_T} \|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega; \mathbb{R}^3)}^2 \, dt \sup_{t \in S_T} \|\varrho_\varepsilon^\omega\|_{L^q(\Omega)} \, dt,
 \end{aligned}$$

where

$$q = \frac{3\gamma}{2\gamma - 3} > 1 \text{ provided } \gamma > \frac{3}{2}.$$

Fixing

$$\gamma = \frac{5}{3}, \quad \omega = \frac{3\gamma}{2\gamma - 3} = \frac{1}{15}
 \tag{4.14}$$

and using the fact that the total mass M_0 is conserved, we get

$$\begin{aligned}
 &\left| \int_{S_T} \int_{\Omega} \varrho_\varepsilon (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \mathcal{B} \left[\varrho_\varepsilon^\omega - \frac{1}{|\Omega|} \int_{\Omega} \varrho_\varepsilon^\omega \, dx \right] \, dx \, dt \right| \\
 &\leq c(M_0) \sup_{t \in S_T} \|\varrho_\varepsilon\|_{L^\gamma(\Omega)} \int_{S_T} \|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega; \mathbb{R}^3)}^2 \, dt.
 \end{aligned}$$

Seeing that the integral containing the Coriolis force can be controlled in a similar way we may rewrite (4.13) in the form

$$\begin{aligned}
 &\int_{S_T} \int_{\Omega} \varrho_\varepsilon^{\frac{5}{3} + \omega} \, dx \, dt \leq c(M_0) \left(1 + \int_{S_T} \int_{\Omega} \vartheta_\varepsilon^4 \, dx \, dt \right) \\
 &+ \sup_{t \in S_T} \|\varrho_\varepsilon\|_{L^\gamma(\Omega)} \int_{S_T} \|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega; \mathbb{R}^3)}^2 \, dt \\
 &+ \int_{S_T} \int_{\Omega} \mathbb{S}(\vartheta_\varepsilon, \mathbb{D}_x \mathbf{u}_\varepsilon) : \nabla_x \mathcal{B} \left[\varrho_\varepsilon^\omega - \frac{1}{|\Omega|} \int_{\Omega} \varrho_\varepsilon^\omega \, dx \right] \, dx \, dt
 \end{aligned}$$

$$\begin{aligned}
 &+ \int_{S_T} \int_{\Omega} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B}[\operatorname{div}_x(\varrho_\varepsilon^\omega \mathbf{u}_\varepsilon)] \, dx \, dt \\
 &+ (\omega - 1) \int_{S_T} \int_{\Omega} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B} \left[\varrho_\varepsilon^\omega \operatorname{div}_x \mathbf{u}_\varepsilon - \frac{1}{|\Omega|} \int_{\Omega} \varrho_\varepsilon^\omega \operatorname{div}_x \mathbf{u}_\varepsilon \, dx \right] \, dx \, dt.
 \end{aligned}
 \tag{4.15}$$

In a similar way, we get

$$\begin{aligned}
 &\left| \int_{S_T} \int_{\Omega} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B}[\operatorname{div}_x(\varrho_\varepsilon^\omega \mathbf{u}_\varepsilon)] \, dx \, dt \right| \\
 &\lesssim \int_{S_T} \|\varrho_\varepsilon\|_{L^\gamma(\Omega)} \|\mathbf{u}_\varepsilon\|_{L^6(\Omega; \mathbb{R}^3)} \|\varrho_\varepsilon^\omega \mathbf{u}_\varepsilon\|_{L^q(\Omega; \mathbb{R}^3)} \, dt,
 \end{aligned}$$

where

$$\frac{1}{\gamma} + \frac{1}{6} + \frac{1}{q} = 1.$$

In addition,

$$\|\varrho_\varepsilon^\omega \mathbf{u}_\varepsilon\|_{L^q(\Omega; \mathbb{R}^3)} \leq \|\mathbf{u}_\varepsilon\|_{L^6(\Omega; \mathbb{R}^3)} \|\varrho_\varepsilon^\omega\|_{L^p(\Omega)}, \text{ where } \frac{1}{q} = \frac{1}{6} + \frac{1}{p};$$

whence

$$\left| \int_{\tau}^{\tau+1} \int_{\Omega} \varrho \mathbf{u} \cdot \mathcal{B}[\operatorname{div}_x(\varrho^\omega \mathbf{u})] \, dx \, dt \right| \leq c(M) \sup_{t \in (\tau, \tau+1)} \|\varrho\|_{L^{\frac{5}{3}}(\Omega)} \int_{\tau}^{\tau+1} \|\mathbf{u}\|_{W_0^{1,2}(\Omega; \mathbb{R}^3)}^2 \, dt$$

as soon as (4.14) holds.

Finally,

$$\begin{aligned}
 &\left| \int_{S_T} \int_{\Omega} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B} \left[\varrho_\varepsilon^\omega \operatorname{div}_x \mathbf{u}_\varepsilon - \frac{1}{|\Omega|} \int_{\Omega} \varrho_\varepsilon^\omega \operatorname{div}_x \mathbf{u}_\varepsilon \, dx \right] \, dx \, dt \right| \\
 &\leq \int_{S_T} \|\varrho_\varepsilon\|_{L^\gamma(\Omega)} \|\mathbf{u}_\varepsilon\|_{L^6(\Omega; \mathbb{R}^3)} \left\| \mathcal{B} \left[\varrho_\varepsilon^\omega \operatorname{div}_x \mathbf{u}_\varepsilon - \frac{1}{|\Omega|} \int_{\Omega} \varrho_\varepsilon^\omega \operatorname{div}_x \mathbf{u}_\varepsilon \, dx \right] \right\|_{L^q(\Omega; \mathbb{R}^3)} \, dt,
 \end{aligned}$$

where

$$\frac{1}{\gamma} + \frac{1}{6} + \frac{1}{q} = 1.$$

Here,

$$\left\| \mathcal{B} \left[\varrho_\varepsilon^\omega \operatorname{div}_x \mathbf{u}_\varepsilon - \frac{1}{|\Omega|} \int_{\Omega} \varrho_\varepsilon^\omega \operatorname{div}_x \mathbf{u}_\varepsilon \, dx \right] \right\|_{L^q(\Omega; \mathbb{R}^3)} \lesssim \|\varrho_\varepsilon^\omega \operatorname{div}_x \mathbf{u}_\varepsilon\|_{L^r(\Omega; \mathbb{R}^3)}, \quad q = \frac{3r}{3-r},$$

and

$$\|\varrho_\varepsilon^\omega \operatorname{div}_x \mathbf{u}_\varepsilon\|_{L^r(\Omega; R^3)} \leq \|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega; R^3)} \|\varrho_\varepsilon^\omega\|_{L^p(\Omega)}, \text{ with } \frac{1}{2} + \frac{1}{p} = \frac{1}{r}.$$

Thus using (4.14) we may infer that

$$\begin{aligned} & \left| \int_{S_T} \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B} \left[\varrho_\varepsilon^\omega \operatorname{div}_x \mathbf{u}_\varepsilon - \frac{1}{|\Omega|} \int_\Omega \varrho_\varepsilon^\omega \operatorname{div}_x \mathbf{u}_\varepsilon \, dx \right] \, dx \, dt \right| \\ & \leq c(M_0) \sup_{t \in S_T} \|\varrho_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)} \int_{S_T} \|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega; R^3)}^2 \, dt. \end{aligned}$$

Going back to (4.15) and summarizing the previous estimates we conclude

$$\begin{aligned} & \int_{S_T} \int_\Omega \varrho_\varepsilon^{\frac{5}{3}+\omega} \, dx \, dt \leq c(M_0) \left(1 + \int_{S_T} \int_\Omega \vartheta_\varepsilon^4 \, dx \, dt \right. \\ & \quad + \sup_{t \in S_T} \|\varrho_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)} \int_{S_T} \|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega; R^3)}^2 \, dt \\ & \quad \left. + \int_{S_T} \int_\Omega \mathbb{S}(\vartheta_\varepsilon, \mathbb{D}_x \mathbf{u}_\varepsilon) : \nabla_x \mathcal{B} \left[\varrho_\varepsilon^\omega - \frac{1}{|\Omega|} \int_\Omega \varrho_\varepsilon^\omega \, dx \right] \, dx \, dt \right), \text{ where } \omega = \frac{1}{15}. \end{aligned} \tag{4.16}$$

The last step is estimating

$$\begin{aligned} & \int_\Omega \mathbb{S}(\vartheta_\varepsilon, \mathbb{D}_x \mathbf{u}_\varepsilon) : \nabla_x \mathcal{B} \left[\varrho_\varepsilon^\omega - \frac{1}{|\Omega|} \int_\Omega \varrho_\varepsilon^\omega \, dx \right] \, dx \\ & \leq (1 + \|\vartheta_\varepsilon\|_{L^4(\Omega)}) \|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega; R^3)} \left\| \nabla_x \mathcal{B} \left[\varrho_\varepsilon^\omega - \frac{1}{|\Omega|} \int_\Omega \varrho_\varepsilon^\omega \, dx \right] \right\|_{L^4(\Omega; R^3)} \\ & \leq c(M) (1 + \|\vartheta_\varepsilon\|_{L^4(\Omega)}) \|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega; R^3)}. \end{aligned}$$

We therefore conclude the pressure estimates:

$$\begin{aligned} & \int_{S_T} \int_\Omega \varrho_\varepsilon^{\frac{5}{3}+\omega} \, dx \, dt \leq c(M_0) \left[1 + \int_{S_T} \int_\Omega \vartheta_\varepsilon^4 \, dx \, dt \right. \\ & \quad \left. + \left(1 + \sup_{t \in S_T} \|\varrho_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)} \right) \int_{S_T} \|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega; R^3)}^2 \, dt \right], \omega = \frac{1}{15}. \end{aligned} \tag{4.17}$$

4.4 Uniform bounds for $\varepsilon \rightarrow 0$

As $\beta > 6$, we deduce from the inequalities (4.9), (4.17) that

$$\int_{S_T} \int_\Omega \vartheta_\varepsilon^4 \, dx \, dt \lesssim \left(1 + \int_{S_T} \|\vartheta_\varepsilon^{\frac{\beta}{2}}\|_{W^{1,2}(\Omega)}^2 \, dt \right) \lesssim \left(1 + \int_{S_T} \int_\Omega \varrho_\varepsilon |\mathbf{u}_\varepsilon| \, dx \, dt \right)$$

provided we fix $r = 1$ in (4.9). Furthermore,

$$\begin{aligned} \int_{S_T} \int_{\Omega} \varrho_\varepsilon |\mathbf{u}_\varepsilon| \, dx \, dt &\leq \frac{1}{2} \int_{S_T} \int_{\Omega} \varrho_\varepsilon \, dx \, dt + \frac{1}{2} \int_{S_T} \int_{\Omega} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 \, dx \, dt \\ &\leq \frac{1}{2} T M_0 + \frac{1}{2} \sup_{t \in S_T} \|\varrho_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)} \int_{S_T} \|\mathbf{u}_\varepsilon\|_{L^5(\Omega; R^3)}^2 \, dt \\ &\leq c(M_0) \left(1 + \sup_{t \in S_T} \|\varrho_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)} \int_{S_T} \|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega; R^3)}^2 \, dt \right). \end{aligned}$$

Consequently, inequality (4.17) reduces to

$$\begin{aligned} \int_{S_T} \int_{\Omega} \varrho_\varepsilon^{\frac{5}{3} + \omega} \, dx \, dt &\leq c(M_0) \left[1 + \left(1 + \sup_{t \in S_T} \|\varrho_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)} \right) \int_{S_T} \|\mathbf{u}_\varepsilon\|_{W^{1,2}(\Omega; R^3)}^2 \, dt \right], \\ \omega &= \frac{1}{15}. \end{aligned} \tag{4.18}$$

Next, going back to (4.9) we get

$$\int_{S_T} \|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega; R^3)}^2 \, dt \lesssim \left(\mathcal{S}(r) \int_{S_T} \int_{\Omega} (\varrho_\varepsilon |\mathbf{u}_\varepsilon| + \varrho_\varepsilon) \, dx \, dt + \Lambda(r) \right)$$

where, by means of the standard Sobolev embedding theorem,

$$\begin{aligned} \int_{\Omega} \varrho_\varepsilon |\mathbf{u}_\varepsilon| \, dx &\leq \|\sqrt{\varrho_\varepsilon}\|_{L^2(\Omega)} \|\sqrt{\varrho_\varepsilon}\|_{L^3(\Omega)} \|\mathbf{u}_\varepsilon\|_{L^6(\Omega; R^3)} \\ &\leq c(M_0) \|\sqrt{\varrho_\varepsilon}\|_{L^3(\Omega)} \|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega; R^3)}. \end{aligned}$$

Consequently,

$$\int_{S_T} \|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega; R^3)}^2 \, dt \lesssim \left(\mathcal{S}(r) \int_{S_T} \|\varrho_\varepsilon\|_{L^{\frac{3}{2}}(\Omega)} \, dt + \Lambda(r) \right). \tag{4.19}$$

Now, introducing the total energy of the system,

$$E(\varrho, \vartheta, \mathbf{u}) = \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta)$$

we first observe that

$$\sup_{t \in S_T} \int_{\Omega} E(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon) \, dx \lesssim \left(1 + \int_{S_T} \int_{\Omega} E(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon) \, dx \, dt \right). \tag{4.20}$$

The estimate (4.20) follows from the mean value theorem and the ballistic energy inequality (3.12). Indeed, in view of the uniform bounds established in Sect. 4.2, we

first deduce (4.20) for the ballistic energy

$$E(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon) - \vartheta_B \varrho_\varepsilon \mathcal{S}(\varrho_\varepsilon, \vartheta_\varepsilon),$$

and then use (2.9) to observe that the entropy part $\vartheta_B \varrho_\varepsilon \mathcal{S}(\varrho_\varepsilon, \vartheta_\varepsilon)$ is a lower order perturbation.

Now, we estimate the kinetic energy using (4.19),

$$\begin{aligned} & \int_{S_T} \int_{\Omega} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 \, dx \, dt \\ & \leq \sup_{t \in S_T} \|\varrho_\varepsilon\|_{L^{\frac{3}{2}}(\Omega)} \int_{S_T} \|\mathbf{u}_\varepsilon\|_{L^6(\Omega; \mathbb{R}^3)}^2 \, dt \\ & \leq c \sup_{t \in S_T} \|\varrho_\varepsilon\|_{L^{\frac{3}{2}}(\Omega)} \int_{\tau}^{\tau+1} \|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega; \mathbb{R}^3)}^2 \, dt \\ & \lesssim \Lambda(r) \sup_{t \in S_T} \|\varrho_\varepsilon\|_{L^{\frac{3}{2}}(\Omega)} + \mathcal{S}(r) \sup_{t \in S_T} \|\varrho_\varepsilon\|_{L^{\frac{3}{2}}(\Omega)} \int_{S_T} \|\varrho_\varepsilon\|_{L^{\frac{3}{2}}(\Omega)} \, dt. \end{aligned}$$

In addition, by interpolation,

$$\|\varrho_\varepsilon\|_{L^{\frac{3}{2}}(\Omega)} \leq \|\varrho_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)}^{\frac{5}{6}} \|\varrho_\varepsilon\|_{L^1(\Omega)}^{\frac{1}{6}}. \tag{4.21}$$

Consequently,

$$\begin{aligned} & \int_{S_T} \int_{\Omega} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 \, dx \, dt \leq \\ & \lesssim c(M_0) \Lambda(r) \sup_{t \in S_T} \|\varrho_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)}^{\frac{5}{6}} + \mathcal{S}(r) \sup_{t \in S_T} \|\varrho_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)}^{\frac{5}{6}} \int_{S_T} \|\varrho_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)}^{\frac{5}{6}} \, dt. \end{aligned} \tag{4.22}$$

Combining (4.18), (4.19), (4.21) we get

$$\begin{aligned} & \int_{S_T} \int_{\Omega} \varrho_\varepsilon^{\frac{5}{3}+\omega} \, dx \, dt \\ & \leq c(M_0) \left[1 + \left(1 + \sup_{t \in S_T} \|\varrho_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)} \right) \int_{S_T} \|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega; \mathbb{R}^3)}^2 \, dt \right] \\ & \leq c(M_0) \left[1 + \left(1 + \sup_{t \in S_T} \|\varrho_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)} \right) \left(\mathcal{S}(r) \int_{S_T} \|\varrho_\varepsilon\|_{L^{\frac{3}{2}}(\Omega)} \, dt + \Lambda(r) \right) \right] \\ & \leq c(M_0) \left[1 + \left(1 + \sup_{t \in S_T} \|\varrho_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)} \right) \left(\mathcal{S}(r) \int_{S_T} \|\varrho_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)}^{\frac{5}{6}} \, dt + \Lambda(r) \right) \right]. \end{aligned} \tag{4.23}$$

Interpolating L^1 and $L^{\frac{5}{3}+\omega}$ and using boundedness of the total mass we have

$$\int_{S_T} \int_{\Omega} \varrho_{\varepsilon}^{\frac{5}{3}} \, dx \, dt \leq c(M_0) \left(\int_{S_T} \int_{\Omega} \varrho_{\varepsilon}^{\frac{5}{3}+\omega} \, dx \, dt \right)^{\frac{10}{11}} \text{ provided } \omega = \frac{1}{15}. \tag{4.24}$$

Thus summing up (4.20)–(4.24) we may infer that

$$\begin{aligned} (4.20) &\Rightarrow \sup_{t \in S_T} \int_{\Omega} E(\varrho_{\varepsilon}, \vartheta_{\varepsilon}, \mathbf{u}_{\varepsilon}) \, dx \lesssim \left(1 + \int_{S_T} \int_{\Omega} E(\varrho_{\varepsilon}, \vartheta_{\varepsilon}, \mathbf{u}_{\varepsilon}) \, dx \, dt \right) \\ &\lesssim \left[1 + \int_{S_T} \int_{\Omega} \left(\|\mathbf{u}_{\varepsilon}\|_{W_0^{1,2}(\Omega; R^d)}^2 + \|\vartheta_{\varepsilon}^{\frac{\beta}{2}}\|_{W^{1,2}(\Omega)}^2 + \|\log(\vartheta_{\varepsilon})\|_{W^{1,2}(\Omega)}^2 \right) \, dx \, dt \right. \\ &\quad \left. + \int_{S_T} \int_{\Omega} \varrho_{\varepsilon} |\mathbf{u}_{\varepsilon}|^2 \, dx \, dt + \int_{S_T} \int_{\Omega} \varrho_{\varepsilon}^{\frac{5}{3}} \, dx \, dt \right] \\ (4.9) &\Rightarrow \lesssim \left[1 + \int_{S_T} \int_{\Omega} \varrho_{\varepsilon} |\mathbf{u}_{\varepsilon}|^2 \, dx \, dt + \int_{S_T} \int_{\Omega} \varrho_{\varepsilon}^{\frac{5}{3}} \, dx \, dt \right] \\ (4.22) &\Rightarrow \lesssim \left[1 + \int_{S_T} \int_{\Omega} \varrho_{\varepsilon}^{\frac{5}{3}} \, dx \, dt \right. \\ &\quad \left. + \Lambda(r) \left(\sup_{t \in S_T} \int_{\Omega} E(\varrho_{\varepsilon}, \vartheta_{\varepsilon}, \mathbf{u}_{\varepsilon}) \, dx \right)^{\frac{1}{2}} + \mathcal{S}(r) \left(\sup_{t \in S_T} \int_{\Omega} E(\varrho_{\varepsilon}, \vartheta_{\varepsilon}, \mathbf{u}_{\varepsilon}) \, dx \right) \right] \\ (4.24) &\Rightarrow \leq c(M_0) \left[1 + \left(\int_{S_T} \int_{\Omega} \varrho_{\varepsilon}^{\frac{5}{3}+\omega} \, dx \, dt \right)^{\frac{10}{11}} \right. \\ &\quad \left. + \Lambda(r) \left(\sup_{t \in S_T} \int_{\Omega} E(\varrho_{\varepsilon}, \vartheta_{\varepsilon}, \mathbf{u}_{\varepsilon}) \, dx \right)^{\frac{1}{2}} + \mathcal{S}(r) \left(\sup_{t \in S_T} \int_{\Omega} E(\varrho_{\varepsilon}, \vartheta_{\varepsilon}, \mathbf{u}_{\varepsilon}) \, dx \right) \right] \\ (4.23) &\Rightarrow \leq c(M_0) \left[\Lambda(r) + \mathcal{S}^{\frac{10}{11}}(r) \left(\sup_{t \in S_T} \int_{\Omega} E(\varrho_{\varepsilon}, \vartheta_{\varepsilon}, \mathbf{u}_{\varepsilon}) \, dx \right) \right. \\ &\quad \left. + \Lambda(r) \left(\sup_{t \in S_T} \int_{\Omega} E(\varrho_{\varepsilon}, \vartheta_{\varepsilon}, \mathbf{u}_{\varepsilon}) \, dx \right)^{\frac{1}{2}} + \mathcal{S}(r) \left(\sup_{t \in S_T} \int_{\Omega} E(\varrho_{\varepsilon}, \vartheta_{\varepsilon}, \mathbf{u}_{\varepsilon}) \, dx \right) \right]. \tag{4.25} \end{aligned}$$

As $\mathcal{S}(r) \rightarrow 0$ as $r \rightarrow \infty$, we fix $r > 0$ large enough to deduce from (4.25) the desired energy bound

$$\sup_{t \in S_T} \int_{\Omega} E(\varrho_{\varepsilon}, \vartheta_{\varepsilon}, \mathbf{u}_{\varepsilon}) \, dx \leq c(M_0). \tag{4.26}$$

5 Convergence

Our ultimate goal is to perform the limit in the sequence of approximate solutions $(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon)_{\varepsilon>0}$ to obtain the existence of the time-periodic solution claimed in Theorem 2.2. With the energy estimate (4.26) at hand, this is nowadays well understood routine matter. Indeed the test functions used in the entropy inequality (2.16) are compactly supported thus unaffected by the boundary integral in its approximate counterpart (3.6). Similarly, the approximate ballistic energy (3.12) is in fact stronger than (2.17) due to the penalization

$$\frac{1}{\varepsilon} \int_{S_T} \psi \int_{\partial\Omega} \frac{|\vartheta_\varepsilon - \vartheta_B|^{k+2}}{\vartheta_\varepsilon} d\sigma_x dt \lesssim 1, \quad \psi \geq 0. \tag{5.1}$$

In particular, for $\psi = 1$, the above inequality together with the (3.12) yield

$$\vartheta_\varepsilon \rightarrow \vartheta \text{ weakly in } L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^d))$$

with the limit trace $\vartheta|_{\partial\Omega} = \vartheta_B$ as required in Theorem 2.2.

Consequently, the proof of convergence is exactly the same as in the existence theory elaborated in [7] with the exception of the strong convergence of the density, the “initial” value of which is unspecified in the periodic setting. Fortunately, the compactness arguments based on Lions’ identity and boundedness of the oscillation defect measure can be modified to accommodate the time periodic setting exactly as in [10, Section 9.3]. Thus the proof of Theorem 2.2 can be completed.

6 Concluding remarks

In comparison with [10], the available a priori bounds do not allow to handle a general driving force $\varrho \mathbf{g}$ in the momentum equation. Although the potential case $\mathbf{g} = \nabla_x G$ is physically relevant, more general (non-potential) forces occur when the fluid is stirred up by the motion of the container. A detailed inspection of the arguments in Sect. 4.3 reveals that they could be considerably improved in the case $d = 2$ due to the Sobolev embedding $W^{1,2} \subset L^q$ for any finite q . Similar improvement may also be expected in the case the total mass M_0 is small, cf. Wang and Wang [26]. We therefore strongly conjecture that the present result can be extended to a general driving force \mathbf{g} provided

- either $d = 2$,
- or

$$M_0 = \int_{\Omega} \varrho \, dx$$

is small enough with respect to the amplitude of \mathbf{g} .

As potentiality of \mathbf{g} was also used in the estimate (3.8) crucial for boundedness of the approximate sequence, the proof of the above conjecture would require a different kind of approximation scheme.

Finally, let us discuss briefly the possibility of extending the results to non-smooth spatial domains. In view of the Sobolev space theory, notably various embedding theorems, one is tempted to say that everything works well for domains with Lipschitz boundary. Indeed we believe that such an extension is possible, however, there are some technical difficulties to overcome in the construction of the approximate solutions, see e.g. Poul [22].

Data Availability Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Bechtel, S.E., Rooney, F.J., Forest, M.G.: Connection between stability, convexity of internal energy, and the second law for compressible Newtonian fluids. *J. Appl. Mech.* **72**, 299–300 (2005)
2. Belgiorno, F.: Notes on the third law of thermodynamics. I. *J. Phys. A* **36**, 8165–8193 (2003)
3. Belgiorno, F.: Notes on the third law of thermodynamics. II. *J. Phys. A* **36**, 8195–8221 (2003)
4. Březina, J., Kagei, Y.: Decay properties of solutions to the linearized compressible Navier–Stokes equation around time-periodic parallel flow. *Math. Models Methods Appl. Sci.* **22**(7), 1250007 (2012). (53)
5. Březina, J., Kagei, Y.: Spectral properties of the linearized compressible Navier–Stokes equation around time-periodic parallel flow. *J. Differ. Equ.* **255**(6), 1132–1195 (2013)
6. Cai, H., Tan, Z.: Weak time-periodic solutions to the compressible Navier–Stokes equations. *Acta Math. Sci. Ser. B (Engl. Ed.)* **36**(2), 499–513 (2016)
7. Chaudhuri, N., Feireisl, E.: Navier–Stokes–Fourier system with Dirichlet boundary conditions. arxiv preprint No. [arXiv:2106.05315](https://arxiv.org/abs/2106.05315) (2021)
8. Davidson, P.A.: *Turbulence: An Introduction for Scientists and Engineers*. Oxford University Press, Oxford (2004)
9. Feireisl, E., Matušů-Nečasová, Š., Petzeltová, H., Straškraba, I.: On the motion of a viscous compressible flow driven by a time-periodic external flow. *Arch. Rational Mech. Anal.* **149**, 69–96 (1999)
10. Feireisl, E., Mucha, P., Novotný, A., Pokorný, M.: Time periodic solutions to the full Navier–Stokes–Fourier system. *Arch. Rational. Mech. Anal.* **204**, 745–786 (2012)
11. Feireisl, E., Novotný, A.: Singular limits in thermodynamics of viscous fluids. In: *Advances in Mathematical Fluid Mechanics*, 2nd edn. Birkhäuser/Springer, Cham (2017)
12. Feireisl, E., Novotný, A., Petzeltová, H.: On the existence of globally defined weak solutions to the Navier–Stokes equations of compressible isentropic fluids. *J. Math. Fluid Mech.* **3**, 358–392 (2001)
13. Feireisl, E., Petzeltová, H.: On the long-time behaviour of solutions to the Navier–Stokes–Fourier system with a time-dependent driving force. *J. Dyn. Differ. Equ.* **19**(3), 685–707 (2007)
14. Feireisl, E., Świerczewska-Gwiazda, A.: The Rayleigh–Bénard problem for compressible fluid flows. arxiv preprint No. [arXiv:2110.10198](https://arxiv.org/abs/2110.10198) (2021)

15. Galdi, G.P.: An Introduction to the Mathematical Theory of the Navier–Stokes Equations. I. Springer, New York (1994)
16. Geißert, M., Heck, H., Hieber, M.: On the equation $\operatorname{div} u = g$ and Bogovskii’s operator in Sobolev spaces of negative order. In: Partial Differential Equations and Functional Analysis, Volume 168 of Oper. Theory Adv. Appl., pp. 113–121. Birkhäuser, Basel (2006)
17. Jin, Ch., Yang, T.: Time periodic solution to the compressible Navier–Stokes equations in a periodic domain. *Acta Math. Sci. Ser. B (Engl. Ed.)* **36**(4), 1015–1029 (2016)
18. Kagei, Y., Oomachi, R.: Stability of time periodic solution of the Navier–Stokes equation on the half-space under oscillatory moving boundary condition. *J. Differ. Equ.* **261**(6), 3366–3413 (2016)
19. Kagei, Y., Tsuda, K.: Existence and stability of time periodic solution to the compressible Navier–Stokes equation for time periodic external force with symmetry. *J. Differ. Equ.* **258**(2), 399–444 (2015)
20. Lions, P.-L.: *Mathematical Topics in Fluid Dynamics, Vol. 1, Incompressible Models*. Oxford Science Publication, Oxford (1996)
21. Lions, P.-L.: *Mathematical Topics in Fluid Dynamics, Vol. 2, Compressible Models*. Oxford Science Publication, Oxford (1998)
22. Poul, L.: Existence of weak solutions to the Navier–Stokes–Fourier system on Lipschitz domains. In: *Discrete Contin. Dyn. Syst., Proceedings of the 6th AIMS International Conference*, suppl., pp. 834–843 (2007)
23. Tsuda, K.: Existence and stability of time periodic solution to the compressible Navier–Stokes–Korteweg system on \mathbb{R}^3 . *J. Math. Fluid Mech.* **18**(1), 157–185 (2016)
24. Valli, A.: Navier–Stokes equations for compressible fluids: global estimates and periodic solutions. In: *Nonlinear Functional Analysis and its Applications, Part 2 (Berkeley, Calif., 1983)*, Volume 45 of Proc. Sympos. Pure Math., pp. 467–476. Amer. Math. Soc., Providence (1986)
25. Valli, A., Zajackowski, M.: Navier–Stokes equations for compressible fluids: global existence and qualitative properties of the solutions in the general case. *Commun. Math. Phys.* **103**, 259–296 (1986)
26. Wang, X., Wang, W.: On global behavior of weak solutions to the Navier–Stokes equations of compressible fluid for $\gamma = 5/3$. *Bound. Value Probl.* **2015**, 176 (2015). (13)
27. Ziemer, W.: *Weakly Differentiable Functions*. Springer, Berlin (1989)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.