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Convergent star products on cotangent bundles of Lie groups

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Abstract

For a connected real Lie group G we consider the canonical standard-ordered star product arising from the canonical global symbol calculus based on the half-commutator connection of G. This star product trivially converges on polynomial functions on T^*G thanks to its homogeneity. We define a nuclear Fréchet algebra of certain analytic functions on T^*G, for which the standard-ordered star product is shown to be a well-defined continuous multiplication, depending holomorphically on the deformation parameter $\hbar$. This nuclear Fréchet algebra is realized as the completed (projective) tensor product of a nuclear Fréchet algebra of entire functions on G with an appropriate nuclear Fréchet algebra of functions on $\mathfrak{g}^*$. The passage to the Weyl-ordered star product, i.e. the Gutt star product on T^*G, is shown to preserve this function space, yielding the continuity of the Gutt star product with holomorphic dependence on $\hbar$.

Contents

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1 Introduction

Formal deformation quantization as introduced in [1] is one of the very successful quantization schemes for Hamiltonian mechanical systems. The basic idea is to deform the commutative algebra of smooth functions $\mathscr{C}^\infty(M)$ on a Poisson manifold M into a noncommutative algebra $\mathscr{C}^\infty(M)[[\hbar]]$ by introducing a formal star product $\star$: this is an associative product bilinear over the formal power series in $\hbar$ such that the zeroth order in $\hbar$ is the undeformed pointwise product of functions and the first order commutator equals the Poisson bracket. Additional requirements are that each order of $\star$ consists of bidifferential operators on M.

The existence of such formal star products was first shown on symplectic manifolds [13, 18] and then by Kontsevich for the general case of Poisson manifolds [30]. While these results provide spectacular successes with many further developments and applications, for honest physical applications one has to overcome the formal power series formulation: the deformation parameter $\hbar$ is to be interpreted as Planck's constant. Thus one is interested in *strict* versions of deformation quantization.

One scenario to obtain reasonable definitions and results for strictness is to use C^*-algebraic deformations instead of formal deformations. This has been introduced by Rieffel, see in particular [45, 46], and used by many others in the sequel, see e.g. [3–6, 34]. The basic ingredient is to use (oscillatory) integral formulas for the star product, which then admit good enough estimates to arrive at constructions of C^*-norms. While giving strong results, the main difficulty with these approaches is that unfortunately there is no general construction of star products via oscillatory integrals available.

Thus a different approach was proposed, namely to use the formal star products and investigate their convergence directly. It turns out that in various classes of examples the following strategy is successful: first one needs to understand the example well enough to find a small subalgebra of functions for which the star product converges for some trivial reasons. In the examples considered so far, the star products simply terminate after finitely many terms on e.g. polynomial functions on a vector space. Here no general results are available and one is restricted to classes of examples. In a second step one then tries to establish a locally convex topology for the small subalgebra in such a way that the star product becomes continuous. Again, also in this step no general results are available, but examples show promising cases. Having succeeded, a completion of the small subalgebra then gives a hopefully large and interesting locally convex algebra, typically a Fréchet algebra, which then can be investigated further.

In finite dimensions this program might not seem more promising than the previous ones, as it also lacks general existence theorems. However, different types of examples can be covered, yielding e.g. analogs of unbounded operator algebras. Moreover, infinite-dimensional examples are very well possible, where oscillatory integrals definitely are no longer available. Thus this approach can be seen as complementing the previous strict deformation quantizations by new and different examples. A detailed overview on these ideas can be found in the review [56], the original results are in [2, 16, 17, 32, 47, 48, 51, 55].

Of indisputable interest for geometric mechanics are the cotangent bundles with their canonical Poisson structure. Their quantization is known to be strongly related to various symbol calculi for pseudo-differential operators. In fact, the asymptotic expansion of the corresponding integrals yield star products when interpreted correctly. Important for us is the other direction: one can directly construct (global) star products on cotangent bundles T^*Q out of a covariant derivative on the configuration space Q, see e.g. [7–9, 19, 41–43]. One of their crucial features is the homogeneity with respect to the Euler vector field, which causes the functions $\mathrm{Pol}(T^*Q)$ polynomial in the fibers to form a subalgebra, on which the star product trivially converges. Thus we have found a good starting point for the above program.

A particular case of cotangent bundles is obtained for a Lie group G as configuration space. This highly symmetric situation admits a distinguished covariant derivative, the half-commutator connection, which is entirely Lie-theoretic. The corresponding (standard-ordered) star product $\star_{\mathrm{std}}$ has been introduced already in [26] and was further investigated in [8]. Using a left-invariant volume form on G one then can pass to a Weyl ordered star product, as well. The left-invariant polynomial functions $\mathrm{Pol}^\bullet(T^*G)^G \cong \mathrm{S}^\bullet(\mathfrak{g})$ are in linear bijection to the symmetric algebra over the Lie algebra. The star product $\star_{\mathrm{std}}$ restricts and yields the Gutt star product on $\mathrm{S}^\bullet(\mathfrak{g})$, thereby quantizing the linear Poisson structure on $\mathfrak{g}^*$. For this star product, the above convergence program has been carried through in [17] by establishing a nuclear locally convex topology on $\mathrm{S}^\bullet_{R'}(\mathfrak{g})$ such that $\star_{\mathrm{std}}$ becomes continuous. Here $R' \geq 1$ is a parameter. The completion is explicitly given by certain real-analytic functions on the vector space $\mathfrak{g}^*$ with controlled growth at infinity and becomes largest for the limiting case $R' = 1$.

Using the trivialization $T^*G \cong G \times \mathfrak{g}^*$ we arrive at the first main result of this paper: We define a subspace $\mathscr{E}_R(G)$ of real-analytic functions on a connected Lie group G together with a suitable nuclear Fréchet topology, depending on a parameter $R \geq 0$ in such a way that $\mathscr{E}_R(G) \otimes \mathrm{S}^\bullet_{R'}(\mathfrak{g})$ becomes a subalgebra of $\mathrm{Pol}(T^*G)$ for which the star product $\star_{\mathrm{std}}$ is continuous. Here the tensor product is equipped with the projective topology. The completion $\widehat{\mathrm{Pol}}_{R,R'}(T^*G)$ is a nuclear Fréchet algebra with largest completion for $R = 0$ and $R' = 1$. The assumption to have a connected Lie group is convenient as then real-analytic functions are determined by their Taylor expansion at the unit element.

While the precise size of $\mathscr{E}_R(G)$ and $\mathrm{Pol}_{R,R'}(T^*G)$ is not easy to grasp, the representative functions on G always belong to $\mathscr{E}_R(G)$ as soon as $0 \leq R < 1$, thus guaranteeing a nontrivial algebra of functions. Moreover, the continuity properties of $\star_{\mathrm{std}}$ immediately imply the continuity of the standard-ordered quantization. This results in a symbol calculus for differential operators on G with coefficient functions in $\mathscr{E}_R(G)$ acting continuously on $\mathscr{E}_R(G)$.

The second result is that the star product of two functions in the completion $\mathrm{Pol}_{R,R'}(T^*G)$ depends holomorphically on $\hbar \in \mathbb{C}$. This way, the star product becomes a convergent series in $\hbar$ as wanted. As a consequence, also the standard-ordered quantization is holomorphic in $\hbar$ and yields not only differential operators but certain pseudo-differential operators for which the composition is holomorphic in $\hbar$.

Finally, the passage from the standard-ordered star product to the physically more appealing Weyl-ordered star product is compatible with the above topologies: the equivalence transformation preserves $\mathrm{Pol}_{R,R'}(T^*G)$, is continuous, and depends holomorphically on $\hbar$ itself. Thus the Weyl-ordered star product inherits all the nice properties of $\star_{\mathrm{std}}$ with the additional feature that the complex conjugation is now a *-involution and the corresponding Weyl-ordered quantization is a *-representation.

While these results yield another large class of examples for the aforementioned program to construct convergent star products, we also mention the following list of further questions and possible continuations:

- Having a *-algebra we can ask for its normalized positive functionals, i.e. its states. Here one first question is whether each classical state, i.e. a positive Borel measure on T^*G, can be deformed into a state of the Weyl-ordered star product algebra? Ideally, this can be accomplished in a way with good dependence on $\hbar$. Note that, unlike in [2, 32], such a deformation is expected to be necessary. In the case of formal star products this is known to be possible in general [10].
- The standard-ordered or Weyl-ordered representation gives now certain pseudo-differential operators which can be studied by means of the symbols in $\mathrm{Pol}_{R,R'}(T^*G)$. The strong analytic framework should help to establish functional-analytic properties like self-adjointness in the same spirit as this was done in [32, 51].
- Since the star product $\star_{\mathrm{std}}$ has all needed symmetry properties this raises the question whether we can construct further classes of examples of convergent star products by means of phase space reduction starting with T^*G. In view of the examples [32, 47] one expects a more complicated dependence on $\hbar$ after reduction with singularities reflecting the geometry of the reduced phase spaces.

The paper is organized as follows: in Sect. 2 we recall the basic construction of $\star_{\mathrm{std}}$ and $\star_{\mathrm{Weyl}}$ on T^*G and establish formulas which allow for efficient estimations. Section 3 contains the construction of the topology on $\mathrm{S}^\bullet(\mathfrak{g})$. We recall some of the basic properties of the resulting algebra. Section 4 is at the heart of the paper. We define the entire functions on G by means of their Lie-Taylor coefficients and study first properties of the resulting space $\mathscr{E}_R(G)$. In particular, we show that representative functions belong to $\mathscr{E}_R(G)$ for $0 \leq R < 1$. In Sect. 5 we combine the entire functions $\mathscr{E}_R(G)$ on G with the polynomials $\mathrm{S}^\bullet_{R'}(\mathfrak{g})$ to the observable algebra $\mathrm{Pol}_{R,R'}(T^*G)$, whose completion will then be studied in the final Sect. 6. Here the continuity of the star products is established. In two appendices we recall the general construction of star products on cotangent bundles and explain some combinatorial aspects of the Leibniz rule.

2 Star products on T^*G

In this section, we specialize the constructions of a global symbol calculus and the corresponding star products on cotangent bundles [7–9] to the cotangent bundle of a Lie group G, see also Appendix A for a brief introduction to the general situation. The main idea is that having a *global* frame of left invariant vector fields simplifies many of the formulas and allows us to use previously local formulas now globally. Essentially, all formulas we present are known from [26] as well as [8], but given in slightly different form, making it necessary to adapt them to our needs.

2.1 The standard-ordered star product on T^*G

Let G be an n-dimensional Lie group with Lie algebra $\mathfrak{g} = T_{\mathrm{e}}G$. We write $X_\xi \in \Gamma^\infty(TG)$ for the left invariant vector field with $X_\xi(\mathrm{e}) = \xi$ and $\theta^\alpha \in \Gamma^\infty(T^*G)$ for the left invariant one-form with $\theta^\alpha(\mathrm{e}) = \alpha$, where $\xi \in \mathfrak{g}$ and $\alpha \in \mathfrak{g}^*$. Then the natural pairing $\theta^\alpha(X_\xi) = \alpha(\xi) \in \mathscr{C}^\infty(G)$ yields a constant function on G.

After once and for all choosing a basis $(e_1, \dots, e_n)$ of the Lie algebra $\mathfrak{g}$ with corresponding dual basis $(e^1, \dots, e^n)$ of $\mathfrak{g}^*$, we write shorthand

$$X_i = X_{e_i} \quad \text{and} \quad \theta^i = \theta^{e_i} \tag{2.1}$$

for $i = 1, \dots, n$ in the sequel. Following [8], see also Appendix A, to construct a standard-ordered star product on the cotangent bundle T^*G, we have to specify a torsion-free covariant derivative on G first. The perhaps first surprising observation is that the most natural covariant derivative, the half-commutator connection ∇ on G, is *not* the Levi-Civita connection for a Riemannian metric in general. It would be the Levi-Civita connection of a biinvariant pseudo Riemannian metric. However, a positive definite one might not exist at all. Since we have a trivial tangent bundle, it suffices to specify ∇ on left invariant vector fields. One sets

$$\nabla_{X_\xi} X_\eta = \frac{1}{2} X_{[\xi,\eta]}, \tag{2.2}$$

which is torsion-free, as taking left invariant vector fields is a Lie algebra morphism by the very definition of the Lie algebra. This then induces covariant derivatives on the various tensor bundles and their complexifications, as usual. This is the only covariant derivative we shall use in the sequel, wherefore we stick to the simple notation ∇.

The covariant derivatives of the global frames $X_1, \dots, X_n$ and $\theta^1, \dots, \theta^n$ are thus given by the structure constants $c_{ij}^k = e^k([e_i, e_j])$ of the Lie algebra $\mathfrak{g}$, i.e. we have

$$\nabla_{X_i} X_j = \frac{1}{2} c_{ij}^k X_k \quad \text{and} \quad \nabla_{X_i} \theta^k = -\frac{1}{2} c_{ij}^k \theta^j. \tag{2.3}$$

Here and in the following we shall use Einstein's summation convention. The antisymmetry of the structure constants now gives the following result for the powers of the symmetrized covariant derivative from (A.3):

Lemma 2.1 *Let G be a Lie group.*

(i) *Let $\alpha \in \Gamma^\infty(\mathrm{S}^k T^*_{\mathbb{C}} G)$. Its symmetrized covariant derivative is given by the global formula*

$$\mathrm{D}\alpha = \theta^i \vee \left(\nabla_{X_i} \alpha\right), \tag{2.4}$$

where $\vee$ denotes the symmetric tensor product as usual.

(ii) *For the kth power of D acting on a function $\psi \in \mathscr{C}^\infty(G)$ we have the global formula*

$$\mathrm{D}^k \psi = \left(\mathscr{L}_{X_{i_1}} \cdots \mathscr{L}_{X_{i_k}} \psi\right) \cdot \theta^{i_1} \vee \cdots \vee \theta^{i_k}. \tag{2.5}$$

Proof We have already noted (i) in (A.5) with the crucial feature that now we have a global frame. The second statement is a straightforward induction based on (2.4) and (2.3). □

Since we have a global frame for TG, we can use it to identify the invariant polynomial functions on T^*G with the complexified symmetric algebra over the Lie algebra $\mathfrak{g}$:

Lemma 2.2 *Let G be a Lie group.*

(i) *We have the canonical isomorphism*

$$\mathrm{S}^\bullet_{\mathbb{C}}(\mathfrak{g}) \cong \Gamma^\infty\left(\mathrm{S}^\bullet_{\mathbb{C}} TG\right)^G \xrightarrow{\mathcal{J}} \mathrm{Pol}^\bullet(T^*G)^G \tag{2.6}$$

between the symmetric algebra of the Lie algebra and the invariant polynomials on T^*G.

(ii) *We have the isomorphisms*

$$\mathscr{C}^\infty(G) \otimes \mathrm{S}^\bullet(\mathfrak{g}) \cong \Gamma^\infty\left(\mathrm{S}^\bullet_{\mathbb{C}}(TG)\right) \cong \mathrm{Pol}(T^*G) \tag{2.7}$$

of graded algebras induced by the pullback π^* with the cotangent bundle projection π.

Here $\mathcal{J}$ is the canonical algebra isomorphism (A.1). This factorization will be used extensively in the sequel. Note that we do not have to complexify the symmetric algebra in (2.7), but doing so would not change the resulting algebra. We will switch between these points of view, whenever it is convenient to do so in the sequel.

Using this observation and Lemma 2.1, we get the following surprisingly simple formula for the standard-ordered quantization map:

Proposition 2.3 (Standard-ordered quantization map)
Let G be a Lie group.

(i) *The standard-ordered quantization map on invariant polynomial functions is globally given by*

$$\varrho_{\mathrm{std}}\big(\mathcal{J}(X_{\xi_1} \vee \cdots \vee X_{\xi_k})\big) = \left(\frac{\hbar}{i}\right)^k \frac{1}{k!} \sum_{\sigma \in S_k} \mathscr{L}_{X_{\xi_{\sigma(1)}}} \cdots \mathscr{L}_{X_{\xi_{\sigma(k)}}} \tag{2.8}$$

for $\xi_1, \ldots, \xi_k \in \mathfrak{g}$.

(ii) *It provides an isomorphism*

$$\varrho_{\mathrm{std}} \colon \mathrm{S}^{(\bullet)}_{\mathbb{C}}(\mathfrak{g}) \longrightarrow \mathrm{DiffOp}^{(\bullet)}(G)^G \tag{2.9}$$

between the complexification of the symmetric algebra over the Lie algebra and the invariant differential operators on G, both viewed as filtered vector spaces.

(iii) *For* $\phi \in \mathscr{C}^\infty(G)$ *and* $\xi_1, \ldots, \xi_k \in \mathfrak{g}$ *one has*

$$\varrho_{\mathrm{std}}\big(\pi^*(\phi)\mathcal{J}(X_{\xi_1} \vee \cdots \vee X_{\xi_k})\big) = \phi \cdot \varrho_{\mathrm{std}}\big(\mathcal{J}(X_{\xi_1} \vee \cdots \vee X_{\xi_k})\big), \tag{2.10}$$

i.e. the smooth function ϕ *acts as a multiplication operator.*

Proof The first part was obtained in [8, Prop. 11]. For (ii) we note that $\varrho_{\mathrm{std}}(\mathcal{J}(X_{i_1} \vee \cdots \vee X_{i_k}))$ is clearly an invariant differential operator. Conversely, if $D \in \mathrm{DiffOp}^k(G)^G$ is invariant, it has an invariant leading symbol $\sigma_k(D) \in \Gamma^\infty(\mathrm{S}^k_{\mathbb{C}} TG)^G \cong \mathrm{S}^k_{\mathbb{C}}(\mathfrak{g})$. Quantizing this symbol via ϱ_{std} gives an invariant differential operator with the same leading symbol, thus $D - \varrho_{\mathrm{std}}(\sigma_k(D))$ is of strictly lower order. A simple induction on k then proves the isomorphism (2.9), since we already know that ϱ_{std} is injective. The last statement is clear, as ϱ_{std} is left $\mathscr{C}^\infty(G)$-linear in general. □

As the standard-ordered quantization map is the quantization map for the standard-ordered star product $\star_{\mathrm{std}}$ in the sense of (A.9), the strategy is now to use (2.8) to derive a formula for the standard-ordered star product suitable for estimation.

Thanks to Lemma 2.2, we can compute the star products for $\mathscr{C}^\infty(G) \otimes \mathrm{S}^\bullet(\mathfrak{g})$ directly. The star product of two functions from $\mathscr{C}^\infty(G)$ is the commutative pointwise product, a feature which holds for all cotangent bundles and not only for T^*G. The next combination we are interested in are two elements of $\mathrm{S}^\bullet_{\mathbb{C}}(\mathfrak{g})$. Since the covariant derivative ∇ we use to construct $\star_{\mathrm{std}}$ is left invariant, their star product is an invariant polynomial, i.e. an element of $\mathrm{S}^\bullet_{\mathbb{C}}(\mathfrak{g})$ again. From [8, Lem. 10] we infer that $\star_{\mathrm{std}}$ coincides with the Gutt star product [26] on $\mathrm{S}^\bullet_{\mathbb{C}}(\mathfrak{g})$, which is obtained from the linear Poincaré–Birkhoff–Witt isomorphism

$$\mathrm{S}^\bullet_{\mathbb{C}}(\mathfrak{g}) \cong \mathrm{U}_{\mathbb{C}}(\mathfrak{g}) \tag{2.11}$$

to the universal enveloping algebra via symmetrization. Incorporating the correct powers of the formal parameter into the definition then yields the star product $\star_{\mathfrak{g}}$ for $\mathrm{S}^\bullet_{\mathbb{C}}(\mathfrak{g})$, where we follow the sign conventions from [17]. Finally, we have to take care of the mixed products: the property of a standard ordered star product immediately gives $(\phi \otimes 1) \star_{\mathrm{std}} (\psi \otimes \xi) = (\phi\psi) \otimes \xi$ for all $\phi, \psi \in \mathscr{C}^\infty(G)$ and $\xi \in \mathrm{S}^\bullet(\mathfrak{g})$. Thus it is the opposite order which needs to be computed. We summarize the result from [8, Prop. 11] in the following proposition, adapting it to our present notation:

Proposition 2.4 *Let G be a Lie group.*

(i) *Functions act trivially from the left, i.e. we have*

$$\big(\phi \otimes 1\big) \star_{\mathrm{std}} \big(\psi \otimes \xi\big) = (\phi \cdot \psi) \otimes \xi \tag{2.12}$$

for all $\phi, \psi \in \mathscr{C}^\infty(G)$ and $\xi \in \mathrm{S}^\bullet_{\mathbb{C}}(\mathfrak{g})$.

(ii) *Products of invariant polynomials are given by the Lie algebra star product $\star_{\mathfrak{g}}$, i.e. we have*

$$\big(1 \otimes \xi\big) \star_{\mathrm{std}} \big(1 \otimes \eta\big) = 1 \otimes \big(\xi \star_{\mathfrak{g}} \eta\big) \tag{2.13}$$

for $\xi, \eta \in \mathrm{S}^\bullet_{\mathbb{C}}(\mathfrak{g})$.

(iii) *The remaining combination of interest is*

$$\begin{aligned} &\big(1 \otimes \xi_1 \vee \cdots \vee \xi_k\big) \star_{\mathrm{std}} \big(\phi \otimes 1\big) \\ &\quad = \sum_{p=0}^{k} \left(\frac{\hbar}{i}\right)^p \frac{1}{p!\,(k-p)!} \sum_{\sigma \in S_k} \Big(\mathscr{L}_{X_{\xi_{\sigma(1)}}} \cdots \mathscr{L}_{X_{\xi_{\sigma(p)}}} \phi\Big) \otimes \xi_{\sigma(p+1)} \vee \cdots \vee \xi_{\sigma(k)}, \end{aligned} \tag{2.14}$$

where $\xi_1, \ldots, \xi_k \in \mathfrak{g}$ and $\phi \in \mathscr{C}^\infty(G)$.

(iv) *In general, one has*

$$\begin{aligned} &(\phi \otimes \xi_1 \vee \cdots \vee \xi_k) \star_{\mathrm{std}} (\psi \otimes \eta) \\ &\quad = (\phi \otimes 1) \star_{\mathrm{std}} (\mathbb{1} \otimes \xi_1 \vee \cdots \vee \xi_k) \star_{\mathrm{std}} (\psi \otimes 1) \star_{\mathrm{std}} (\mathbb{1} \otimes \eta) \qquad (2.15) \\ &\quad = \sum_{p=0}^{k} \left(\frac{\hbar}{i}\right)^p \frac{\phi}{p!\,(k-p)!} \sum_{\sigma \in S_k} \mathscr{L}_{X_{\xi_{\sigma(1)}}} \cdots \mathscr{L}_{X_{\xi_{\sigma(p)}}} \psi \otimes \big(\xi_{\sigma(p+1)} \vee \cdots \vee \xi_{\sigma(k)}\big) \star_{\mathfrak{g}} \eta \qquad (2.16) \end{aligned}$$

for $\phi, \psi \in \mathscr{C}^\infty(G)$, $\xi_1, \ldots, \xi_k \in \mathfrak{g}$ and $\eta \in \mathrm{S}^\bullet(\mathfrak{g})$.

Proof The presented formulae are obtained from [8, Prop. 11] after the suitable identification of polynomial functions with elements in $\mathscr{C}^\infty(G) \otimes \mathrm{S}^\bullet_{\mathbb{C}}(\mathfrak{g})$. We list them here, since directly working with the symmetric algebra will be easier for continuity estimates down the line. □

2.2 Weyl ordering and the Neumaier operator

This completes our algebraic considerations for the standard-ordered star product. In a next step, we turn towards other ordering prescriptions. More precisely, we are going to simplify the general formulas for the Neumaier operator (A.15) by utilizing the Lie-theoretic situation. As for the standard-ordered star product, having a global frame allows us to obtain considerably more explicit formulas, see again [8, Sect. 8].

The main idea is that we use the half-commutator connection to lift the left-invariant global frame $X_1, \ldots, X_n$ to vector fields

$$Y_i = X_i^{\mathrm{hor}} \in \Gamma^\infty\big(T(T^*G)\big) \tag{2.17}$$

on the cotangent bundle. Together with the vertical lifts of the global frame $\theta^1, \ldots, \theta^n$, denoted by

$$Z^i = (\theta^i)^{\mathrm{ver}} \in \Gamma^\infty\big(T(T^*G)\big), \tag{2.18}$$

one thus obtains a global frame $Y_1, \ldots, Y_n, Z^1, \ldots Z^n$ for the tangent bundle of T^*G. Having a global frame it is of course advantageous to express differential operators like the Laplacian Δ_0 from (A.14) of the pseudo Riemannian metric g_0 by iterated Lie derivatives with respect to the frame vector fields instead of covariant derivatives.

Since we also need the choice of a volume density μ on G to construct the Weyl ordering, we use the left-invariant volume form $\mu = \theta^1 \wedge \cdots \wedge \theta^n$. The required one-form α with $\nabla_X \mu = \alpha(X)\mu$ is then given by $\alpha(X_\xi) = -\frac{1}{2}\mathrm{tr}(\mathrm{ad}_\xi)$ for $\xi \in \mathfrak{g}$ and hence

$$\alpha = \frac{1}{2} c_{ik}^i \theta^k \in \Gamma^\infty(T^*G). \tag{2.19}$$

Note that in general $\alpha \neq 0$ unless the Lie algebra is unimodular. The vertical lift of α gives $\alpha^{\mathrm{ver}} = \frac{1}{2} c_{ik}^i Z^k$. For the operator N we need the combination $\Delta_0 + \mathscr{L}_{\alpha^{\mathrm{ver}}}$, wherefore we compute the action of this operator on factorizing tensors explicitly:

Proposition 2.5 *For $\phi \in \mathscr{C}^\infty(G)$ and $\xi_1, \ldots, \xi_k \in \mathfrak{g}$ one has*

$$\big(\Delta_0 + \mathscr{L}_{\alpha^{\mathrm{ver}}}\big)(\phi \otimes \xi_1 \vee \cdots \vee \xi_k) = \sum_{\ell=1}^{k} \mathscr{L}_{X_{\xi_\ell}} \phi \otimes \xi_1 \vee \overset{\xi_\ell}{\cdots} \vee \xi_k + 2\mathscr{L}_{\alpha^{\mathrm{ver}}}, \tag{2.20}$$

where we identify elements of $\mathscr{C}^\infty(G) \otimes \mathrm{S}(\mathfrak{g})$ with polynomial functions $\mathrm{Pol}(T^*G)$ as before.

Proof In general, the covariant divergence $\mathrm{div}_\nabla \colon \Gamma^\infty(\mathrm{S}^k TQ) \longrightarrow \Gamma^\infty(\mathrm{S}^{k-1} TQ)$ on an arbitrary manifold Q with covariant derivative ∇ is given by the local formula

$$\mathrm{div}_\nabla\Big|_U = \mathrm{i_s}(\mathrm{e}^i)\nabla_{\mathrm{e}_i}, \tag{$*$}$$

where $\mathrm{e}_1, \ldots, \mathrm{e}_n \in \Gamma^\infty(TU)$ is a local frame on an open subset $U \subseteq M$ with corresponding dual frame $\mathrm{e}^1, \ldots, \mathrm{e}^n \in \Gamma^\infty(T^*U)$ and $\mathrm{i_s}(\,\cdot\,)$ denotes the symmetric insertion derivation. As one easily verifies, this provides a global definition independent of the local frame. Directly from the definition one infers the Leibniz rule

$$\mathrm{div}_\nabla(\phi X) = \mathrm{i_s}(\mathrm{d}\phi) X + \phi \,\mathrm{div}_\nabla(X) \tag{$**$}$$

for all $\phi \in \mathscr{C}^\infty(Q)$ and $X \in \Gamma^\infty(\mathrm{S}^k TQ)$. On polynomial functions $\mathcal{J}(X) \in \mathrm{Pol}^k(T^*Q)$ with $X \in \Gamma^\infty(\mathrm{S}^k TQ)$ the Lie derivatives with respect to horizontal and vertical lifts act like

$$\mathscr{L}_{Y^{\mathrm{hor}}} \mathcal{J}(X) = \mathcal{J}(\nabla_Y X) \quad \text{and} \quad \mathscr{L}_{\beta^{\mathrm{ver}}} \mathcal{J}(X) = \mathcal{J}(\mathrm{i_s}(\beta) X),$$

where $Y \in \Gamma^\infty(TQ)$ and $\beta \in \Gamma^\infty(T^*Q)$. Since $\mathcal{J}$ is an algebra homomorphism this can be easily checked on generators. Using the local expression (A.14) for the Laplacian with respect to the g_0 as well as the local formulas for the horizontal and vertical lifts, one verifies

$$\Delta_0 \circ \mathcal{J} = \mathcal{J} \circ \mathrm{div}_\nabla,$$

see also [8, Eq. (111)]. Now we focus on the Lie group case. Here we first notice that $\mathrm{div}_\nabla(X_\xi) = -\frac{1}{2} \mathrm{tr}\, \mathrm{ad}_\xi$ for all $\xi \in \mathfrak{g}$. Note that (∗) becomes a global formula once we use the global frame $X_1, \ldots, X_n$. From the antisymmetry of the structure constants we get the divergence of higher polynomials as

$$\mathrm{div}_\nabla(X_{\xi_1} \vee \cdots \vee X_{\xi_k}) = -\tfrac{1}{2} \mathrm{i_s}(\mathrm{tr}\,\mathrm{ad})(X_{\xi_1} \vee \cdots \vee X_{\xi_k}).$$

Together with the Leibniz rule (∗∗) we arrive at the explicit formula

$$\mathrm{div}_\nabla(\phi X_{\xi_1} \vee \cdots \vee X_{\xi_k}) = \sum_{\ell=1}^{k} \mathscr{L}_{X_{\xi_\ell}} \phi \cdot X_{\xi_1} \vee \overset{\ell}{\cdots} \vee X_{\xi_k} - \tfrac{1}{2} \phi \mathrm{i_s}(\mathrm{tr}\,\mathrm{ad})(X_{\xi_1} \vee \cdots \vee X_{\xi_k}).$$

Applying the algebra isomorphism $\mathcal{J}$ turns the divergence into the Laplacian and the insertion of the modular one-form into the Lie derivative in direction of the vertical lift of α, finally proving (2.20). □

From this explicit description of the Laplacian Δ_0 we see that it might be advantageous to focus on the combination

$$\Delta = \Delta_0 - \mathscr{L}_{\alpha^{\mathrm{ver}}} \tag{2.21}$$

acting on polynomial functions as

$$\Delta(\phi \otimes \xi_1 \vee \cdots \vee \xi_k) = \sum_{\ell=1}^{k} \mathscr{L}_{X_{\xi_\ell}} \phi \otimes \xi_1 \vee \overset{\xi_\ell}{\cdots} \vee \xi_k. \tag{2.22}$$

The vertical Lie derivative $\mathscr{L}_{\alpha^{\mathrm{ver}}}$ is now easily shown to commute with both operators Δ_0 and Δ. Thus the Neumaier operator $\mathcal{N}_\kappa$ factorizes

$$\mathcal{N}_\kappa = \exp\left(-\mathrm{i}\kappa\hbar(\Delta_0 + \mathscr{L}_{\alpha^{\mathrm{ver}}})\right) = \exp\left(-\mathrm{i}\kappa\hbar\Delta\right) \circ \exp\left(2\mathrm{i}\kappa\hbar \mathscr{L}_{\alpha^{\mathrm{ver}}}\right). \tag{2.23}$$

As observed in [8, Lem. 11], the second factor $\exp(2i\kappa\hbar\mathscr{L}_{\alpha^{\mathrm{ver}}})$ is an automorphism of $\star_{\mathrm{std}}$ for all κ. Thus the simpler operator

$$N_\kappa = \exp\left(-i\kappa\hbar\Delta\right) \tag{2.24}$$

with Δ as in (2.22) is still an equivalence transformation from $\star_{\mathrm{std}}$ to $\star_\kappa$ for all $\kappa \in \mathbb{R}$. The reason to use N_κ instead of $\mathcal{N}_\kappa$ is that the simpler formulas are easier to estimate later on.

Corollary 2.6 *The Laplacian is given by the mixed Poisson bracket, i.e. we have*

$$\Delta(\phi \otimes \xi_1 \vee \cdots \vee \xi_k) = \left\{\mathbb{1} \otimes \xi_1 \vee \cdots \vee \xi_k, \phi \otimes 1\right\}. \tag{2.25}$$

Proof Indeed, the Poisson bracket of a function $\phi \in \mathscr{C}^\infty(G)$ with left-invariant vector fields can be directly obtained from Proposition 2.5 and the first order commutator of $\star_{\mathrm{std}}$ as in Proposition 2.4. Note that such a formula is only possible since we can factorize elements in $\mathrm{Pol}(T^*G)$ into $\mathscr{C}^\infty(G)$ and $\mathrm{S}^\bullet(\mathfrak{g})$. □

Corollary 2.7 *Let $\ell \leq k$. The powers of Δ act as*

$$\left(\Delta\right)^\ell(\phi \otimes \xi_1 \vee \cdots \vee \xi_k) = \frac{1}{(k-\ell)!}\sum_{\sigma \in S_k} \mathscr{L}_{X_{\xi_{\sigma(1)}}} \cdots \mathscr{L}_{X_{\xi_{\sigma(\ell)}}}\phi \otimes \xi_{\sigma(\ell+1)} \vee \cdots \vee \xi_{\sigma(k)}. \tag{2.26}$$

For $\ell > k$ the result is zero for degree reasons.

To show the continuity of the κ-Neumaier operators later on, we require a more explicit formula. Remarkably, (2.25) exponentiates very nicely: it turns out that the square N^2 of the Neumaier operator is given by the mixed star product from Theorem 2.4:

Proposition 2.8 *Let G be a Lie group. For $\kappa = \frac{1}{2}$, the square of the Neumaier operator $N = N_{\frac{1}{2}}$ is given by*

$$N^2(\phi \otimes \xi_1 \vee \cdots \vee \xi_k) = (\mathbb{1} \otimes \xi_1 \vee \cdots \vee \xi_k) \star_{\mathrm{std}} (\phi \otimes 1) \tag{2.27}$$

for $\phi \in \mathscr{C}^\infty(G)$ and $\xi_1, \ldots, \xi_k \in \mathfrak{g}$.

Proof Taking another look at (2.14) and (2.26) confirms our claim. □

Notably, incorporating $\kappa \neq 1$ is not that easy here. Down the line, the trick will thus be to absorb it into the $\hbar$ dependence, at which point we can employ Proposition 2.8 again.

3 The R'-topologies on the symmetric algebra

In view of the factorization $\mathrm{Pol}(T^*G) \cong \mathscr{C}^\infty(G) \otimes \mathrm{S}^\bullet(\mathfrak{g})$, we want to define a suitable locally convex topology on the symmetric algebra $\mathrm{S}^\bullet(\mathfrak{g})$ in such a way that the star product $\star_{\mathfrak{g}}$ is continuous. This has been accomplished and studied in detail in [17]. We briefly recall the construction based on the earlier work [55] and recollect some of the crucial features.

Let V be a locally convex space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We fix a parameter $R' \in \mathbb{R}$. Then for a seminorm p on V and a weight $c \geq 0$ one defines the seminorm $\mathrm{p}_{R',c}$

$$\mathrm{p}_{R',c} \colon \mathrm{S}^\bullet(V) \longrightarrow \mathbb{R}_0^+, \quad \mathrm{p}_{R',c} = \sum_{k=0}^{\infty} k!^{R'} \, c^k \, \mathrm{p}^k, \tag{3.1}$$

where p^k denotes the k-fold projective tensor power of the seminorm p acting on $\mathrm{S}^k(V) \subseteq \mathrm{T}^\bullet(V)$. By convention, $\mathrm{S}^0(V) = \mathbb{K} = \mathrm{T}^0(V)$ and p^0 is the absolute value on $\mathbb{K}$.

Let $\mathcal{P}$ be a defining system of seminorms for V. The locally convex topology on $\mathrm{S}^\bullet(V)$ induced by the set of seminorms $\big(\mathrm{p}_{R',c}\big)_{\mathrm{p}\in\mathcal{P},\, c\geq 0}$ is called the R'-topology. We write $\mathrm{S}^\bullet_{R'}(V)$ for $\mathrm{S}^\bullet(V)$ equipped with the R'-topology. It is independent of the chosen defining system of seminorms and thus intrinsic to V. However, it depends on R' in a very sensitive way. We note the obvious inequalities

$$\mathrm{p}_{R',c} \leq \mathrm{p}_{S',c} \quad \text{and} \quad \mathrm{p}_{R',c} \leq \mathrm{p}_{R',d} \quad \text{and} \quad \mathrm{p}_{R',c} \leq \mathrm{q}_{R',c}, \tag{3.2}$$

whenever $R' \leq S'$, $0 \leq c \leq d$ and $\mathrm{p} \leq \mathrm{q}$. This implies that the inclusion (in fact equality)

$$\mathrm{S}^\bullet_{S'}(V) \subseteq \mathrm{S}^\bullet_{R'}(V) \tag{3.3}$$

is continuous. Thus it extends to a continuous inclusion for the completions.

In [55], all continuous seminorms were chosen. In this case, there is no need for the parameter $c \geq 0$ as with p also $c\mathrm{p}$ is continuous. Nevertheless, a smaller collection is sometimes convenient: Let $V = \mathfrak{g}$ be the Lie algebra of a Lie group G. In the sequel we will always choose a basis of $\mathfrak{g}$ and equip it with the corresponding ℓ^1-topology. This then induces an R'-topology on $\mathrm{S}^\bullet(\mathfrak{g})$ via $\mathcal{P} = \{\|\cdot\|_1\}$, i.e. the system consists of a single norm. Note, however, that the topology for $\mathrm{S}^\bullet_{R'}(\mathfrak{g})$ is not at all normable. In general, we also note

$$\big(\lambda \cdot \mathrm{p}\big)_{R',c} = \mathrm{p}_{R',\lambda c} \tag{3.4}$$

for all $c \geq 0$, $\lambda \geq 0$ and $R' \in \mathbb{R}$. Another consequence of having all polynomial weights at our disposal is that instead of the ℓ^1-like seminorms (3.1) we could have used the ℓ^∞-like seminorms

$$\mathrm{p}_{R',c,\infty} \colon \mathrm{S}^\bullet(V) \longrightarrow \mathbb{R}_0^+, \quad \mathrm{p}_{R',c,\infty} = \sup_{k\in\mathbb{N}_0} k!^{R'} \, c^k \, \mathrm{p}^k(v_k) \tag{3.5}$$

with $c \geq 0$. The mutual estimates between them show that the resulting locally convex topology stays the same, see [55, Lem. 3.4]. We collect a few less obvious properties of the R'-topology from [55]:

Proposition 3.1 *Let V be a locally convex vector space with defining system of seminorms $\mathcal{P}$ and $R' \in \mathbb{R}$.*

(i) *Let $k \geq 0$. The subspace topology induced by the inclusion $\mathrm{S}^k(V) \subset \mathrm{S}^\bullet(V)$ is the projective tensor power topology and the inclusion is continuous.*
(ii) *The R'-topology is coarser than the locally convex direct sum topology.*
(iii) *The R'-topology is finer than the subspace topology induced by the Cartesian product topology.*
(iv) *The locally convex space $\mathrm{S}^\bullet_{R'}(V)$ is Hausdorff iff V is Hausdorff.*
(v) *The locally convex space $\mathrm{S}^\bullet_{R'}(V)$ is first countable iff V is first countable.*
(vi) *Let $R' \geq 0$. The locally convex space $\mathrm{S}^\bullet_{R'}(V)$ is nuclear iff V is nuclear.*
(vii) *The completion $\hat{\mathrm{S}}^\bullet_{R'}(V)$ of $\mathrm{S}^\bullet_{R'}(V)$ is explicitly given by*

$$\hat{\mathrm{S}}^\bullet_{R'}(V) = \left\{ v \in \prod_{k=0}^{\infty} \hat{\mathrm{S}}^k(V) \;\middle|\; p_{R',c}(v) < \infty \text{ for all } c \geq 0 \text{ and } p \in \mathcal{P} \right\}, \tag{3.6}$$

where $\hat{\mathrm{S}}^k(V)$ denotes the completion of $\mathrm{S}^k(V)$ with respect to the projective tensor product topology.
(viii) *Let $R' \geq 0$ and V be nuclear, Hausdorff as well as first countable. Then the completion $\hat{\mathrm{S}}^\bullet_{R'}(V)$ is nuclear Fréchet, Montel, separable, and reflexive.*

Proof All statements except for the last one have been obtained in [55]. The earlier parts of our theorem guarantee that nuclearity, the Hausdorff property and first countability get inherited by $\mathrm{S}^\bullet_{R'}(V)$. Moreover, everything passes to the completion $\hat{\mathrm{S}}^\bullet_{R'}(V)$, as well (see [54, (50.3)] for the nuclearity). By [54, Prop. 50.2] every nuclear Fréchet space is Montel. Nuclear Montel spaces are separable by [29, Section 11.6, Thm. 2] and by [54, Cor. 36.9] every Montel space is reflexive. □

From [55] we also get the following continuity statements for the symmetric tensor product:

Proposition 3.2 *Let V be a locally convex vector space, $c \geq 0$ and $R' \in \mathbb{R}$.*

(i) *The symmetric tensor product*

$$\vee \colon \mathrm{S}^\bullet_{R'}(V) \times \mathrm{S}^\bullet_{R'}(V) \longrightarrow \mathrm{S}^\bullet_{R'}(V) \tag{3.7}$$

is continuous. More precisely, for $R' \geq 0$ we have

$$p_{R',c}(v \vee w) \leq p_{R',2^{R'}c}(v) \cdot p_{R',2^{R'}c}(w) \tag{3.8}$$

for all $v, w \in \mathrm{S}^\bullet(V)$ and the seminorms $p_{R',c}$ are submultiplicative for $R' \leq 0$.

(ii) *Let $R' \geq 0$. For $\varphi \in V'$ the evaluation functionals*

$$\delta_\varphi \colon \mathrm{S}^\bullet_{R'}(V) \longrightarrow \mathbb{C}, \quad \delta_\varphi(v) = \sum_{k=0}^{\infty} \varphi^k(v_k) \tag{3.9}$$

are continuous algebra characters.

(iii) *Let $R' < 0$ and $\varphi \in V^*$ with continuous δ_φ as in (3.9). Then we have $\varphi = 0$.*

Proof The only new statement is (iii): let $R' < 0$ and $\varphi \in V^*$ with continuous δ_φ, i.e. we find a continuous seminorm p on V and $c \geq 0$ such that we have

$$\left| \sum_{k=0}^{\infty} \varphi^k(v_k) \right| = \left|\delta_\varphi(v)\right| \leq \mathrm{p}_{R',c}(v) = \sum_{k=0}^{\infty} k!^{R'} \, c^k \, \mathrm{p}^k(v_k) \tag{$*$}$$

for all $v \in V$. Assume that we had $\varphi \neq 0$. Then there exists a $v \in V$ with $\varphi(v) \neq 0$. Consider now $v^{\otimes n} \in \mathrm{T}^n(V)$ for all $n \in \mathbb{N}$. Then ($*$) implies

$$|\varphi(v)|^n = \left| \sum_{k=0}^{\infty} \varphi^k(v^{\otimes n}) \right| \leq \mathrm{p}_{R',c}\big(v^{\otimes n}\big) = n!^{R'} \cdot c^n \cdot \mathrm{p}^n\big(v^{\otimes n}\big) = n!^{R'} \cdot c^n \cdot \mathrm{p}(v)^n$$

for all $n \in \mathbb{N}$. However, this inequality is absurd: factorial growth in n can not be estimated by the *fixed* base $\left(\frac{c \cdot \mathrm{p}(v)}{|\varphi(v)|}\right)^{-(R'^{-1})}$ to the power of n. $\square$

4 The *R*-entire functions

The purpose of this section is to introduce and study Fréchet subalgebras $\mathscr{E}_R(G) \subseteq \mathscr{C}^\infty(G)$ depending on another parameter $R \in \mathbb{R}$. These Fréchet algebras will ultimately serve as the other tensor factors in the observable algebra for our strict deformation. While in the critical borderline case $R = 0$ the algebra $\mathscr{E}_0(G)$ can be seen as a Lie–theoretic descendent of the algebra of all holomorphic entire functions, the algebras $\mathscr{E}_R(G)$ for $R > 0$ share many properties with the classical and well studied Fréchet algebras of entire holomorphic functions of *finite order and minimal type*.

In this section G denotes always a real Lie group with corresponding Lie algebra $\mathfrak{g}$ of dimension $n \in \mathbb{N}$. We furthermore assume that the Lie group G is *connected*. As it is standard, we denote for an open set $U \subseteq \mathbb{C}^n$ the set of all holomorphic functions $F \colon U \longrightarrow \mathbb{C}$ by $\mathcal{H}(U)$.

4.1 Lie-Taylor series of smooth functions on a Lie group

Taking another look at (2.8) and (3.6), we anticipate certain power series of Lie derivatives $\mathscr{L}_{X_\xi}$ for $\xi \in \mathfrak{g}$ to make an appearance, since the completion $\hat{\mathrm{S}}^\bullet_{R'}(\mathfrak{g})$ of $\mathrm{S}^\bullet_{R'}(\mathfrak{g})$ contains already non-trivial (though not all) entire functions, when interpreting the elements of the completion as maps on $\mathfrak{g}^*$. Thus we need to find a space of functions on G on which all elements of $\hat{\mathrm{S}}^{\bullet}_{R'}(\mathfrak{g})$ act and which is preserved by this action.

Formalizing this idea, it turns out that the functions we are looking for are exactly the "entire vectors" for suitably chosen seminorms on $\mathscr{C}^\infty(G)$ and the lifted Lie algebra representation

$$\mathscr{L}\colon \mathrm{U}_{\mathbb{C}}(\mathfrak{g}) \longrightarrow \mathrm{End}\big(\mathscr{C}^\infty(G)\big), \quad \mathscr{L}(\xi_1 \cdots \xi_k) = \mathscr{L}_{X_{\xi_1}} \cdots \mathscr{L}_{X_{\xi_k}}. \tag{4.1}$$

Here $\mathrm{U}(\mathfrak{g})$ denotes the universal enveloping algebra of the Lie algebra $\mathfrak{g}$, as before.

To make the notion of an "entire vector" precise, we introduce some more notation. Let $\mathbb{N}_n = \{1, \ldots, n\}$ and $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}_n^k$ be an ordered k-tuple. For a fixed basis $\mathcal{B} = (e_1, \ldots, e_n)$ of the Lie algebra $\mathfrak{g}$, we then write

$$\mathscr{L}_{X_\alpha} = \mathscr{L}_{X_{\alpha_1}} \cdots \mathscr{L}_{X_{\alpha_k}}, \tag{4.2}$$

where we once again use the convention (2.1). Finally, we also use the shorthand notation

$$\underline{z}_\alpha = z_{\alpha_1} \cdots z_{\alpha_k} \quad \text{for} \quad \underline{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n. \tag{4.3}$$

Definition 4.1 (*Lie-Taylor series and majorants*)
Let $\phi \in \mathscr{C}^\infty(G)$ be a smooth function and let $\mathcal{B} = (e_1, \ldots, e_n)$ be a basis of the Lie algebra $\mathfrak{g}$.

(i) We call the formal series

$$\mathrm{T}_\phi\colon G \longrightarrow \mathbb{C}[\![\underline{z}]\!], \quad \mathrm{T}_\phi(\underline{z}; g) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\alpha \in \mathbb{N}_n^k} \big(\mathscr{L}_{X_\alpha}\phi\big)(g) \cdot \underline{z}_\alpha \tag{4.4}$$

the Lie-Taylor series of ϕ at the point $g \in G$ (w.r.t. the basis $\mathcal{B}$).

(ii) Using the coefficients

$$c_k(\phi) = \frac{1}{k!} \sum_{\alpha \in \mathbb{N}_n^k} \big|\big(\mathscr{L}_{X_\alpha}\phi\big)(\mathrm{e})\big| \tag{4.5}$$

we define the Lie-Taylor majorant of ϕ (w.r.t. the basis $\mathcal{B}$) as

$$\mathrm{F}_\phi(z) = \sum_{k=0}^{\infty} c_k(\phi) \cdot z^k \in \mathbb{C}[\![z]\!]. \tag{4.6}$$

Remark 4.2 (Lie-Taylor majorants)
Let $\phi \in \mathscr{C}^\infty(G)$ be a smooth function.

(i) The Lie-Taylor majorant $\mathrm{F}_\phi(\|\underline{z}\|_\infty)$ is a majorant of $\mathrm{T}_\phi(\underline{z}; e)$, i.e.

$$\left|\mathrm{T}_\phi(\underline{z}; \mathrm{e})\right| \leq \mathrm{F}_\phi\left(\|\underline{z}\|_\infty\right) \tag{4.7}$$

for all $\underline{z} \in \mathbb{C}^n$. In particular, if $\mathrm{F}_\phi \in \mathcal{H}(\mathrm{B}_r(0))$, i.e. F_ϕ is holomorphic on the open disk $\mathrm{B}_r(0) = \{z \in \mathbb{C} \mid |z| < r\}$, then $\mathrm{T}_\phi(\,\cdot\,; \mathrm{e})$ is holomorphic on the polydisk $\mathrm{B}_r(0)^n \subseteq \mathbb{C}^n$.

(ii) The coefficients $c_k(\phi)$ and hence F_ϕ depend on the choice of the basis $\mathcal{B}$, so one should tend to write $c_{k,\mathcal{B}}(\phi)$ instead. However, if $\mathcal{B}'$ is another basis of $\mathfrak{g}$, then it is easy to see that there is a constant $M = M(\mathcal{B}, \mathcal{B}') > 0$ such that

$$c_{k,\mathcal{B}'}(\phi) \leq M^k \cdot c_{k,\mathcal{B}}(\phi) \tag{4.8}$$

for all $k \in \mathbb{N}_0$ and $\phi \in \mathscr{C}^\infty(G)$. In particular, if $\mathrm{F}_\phi \in \mathcal{H}(\mathbb{C})$ w.r.t. some basis of $\mathfrak{g}$, then $\mathrm{F}_\phi \in \mathcal{H}(\mathbb{C})$ w.r.t. any basis. The upshot is that whenever we are dealing with *entire* Lie-Taylor majorants or only care about analyticity with no specific radius of convergence, we can safely ignore the specific choice of the basis of $\mathfrak{g}$.

The following simple observations will prove very useful:

Proposition 4.3 (Leibniz and chain rule)
Let $\phi \in \mathscr{C}^\infty(G)$ and $z \in \mathbb{C}$.

(i) *Let $\psi \in \mathscr{C}^\infty(G)$ be another smooth function. We have the Leibniz inequality*

$$\left|\mathrm{F}_{\phi\cdot\psi}(z)\right| \leq \mathrm{F}_\phi\left(|z|\right) \cdot \mathrm{F}_\psi\left(|z|\right). \tag{4.9}$$

(ii) *Let $\Phi\colon G \longrightarrow H$ be a morphism of Lie groups. Then*

$$c_k\left(\Phi^*\phi\right) \leq (Dn)^k \cdot c_k(\phi) \tag{4.10}$$

$$\text{and} \quad \left|\mathrm{F}_{\Phi^*\phi}(z)\right| \leq \mathrm{F}_\phi\left(Dn \cdot |z|\right), \tag{4.11}$$

where D is the matrix supnorm of the matrix representation of the tangent map $T_e\Phi\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ in the bases used for the construction of the Taylor majorants.

Proof The Leibniz rule (i) is an easy consequence of the noncommutative higher Leibniz rule (B.1), the Cauchy product formula and the triangle inequality. For (ii) recall that for $\xi \in \mathfrak{g}$, the left invariant vector fields X_ξ^G and $X_{T_{\mathrm{e}_G}\Phi\xi}^H$ are Φ-related. This implies for the corresponding Lie derivatives

$$\mathscr{L}_{X_\xi^G}\left(\Phi^*\phi\right) = \Phi^*\left(\mathscr{L}_{X^H_{T_\mathrm{e}\Phi\xi}}\phi\right).$$

As in the formulation, we set

$$D = \max_{i,j=1,\ldots,n}\left|(T_\mathrm{e}\Phi)_i^j\right|,$$

where we take matrix representation $(T_e\Phi)_i^j = d_i^j$ of $T_e\Phi\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ with respect to the chosen bases. Thus we obtain for $k \in \mathbb{N}_0$

$$
\begin{aligned}
c_k\big(\Phi^*\phi\big) &= \frac{1}{k!} \sum_{\alpha\in\mathbb{N}_n^k} \Big|\big(\mathscr{L}^G_{X_\alpha}\Phi^*\phi\big)(\mathrm{e})\Big| \\
&= \frac{1}{k!} \sum_{\alpha\in\mathbb{N}_n^k} \bigg|\Phi^*\Big(\mathscr{L}_{X^H_{T_e\Phi(e_{\alpha_1})}} \cdots \mathscr{L}_{X^H_{T_e\Phi(e_{\alpha_k})}}\phi\Big)\Big|_{\mathrm{e}}\bigg| \\
&= \frac{1}{k!} \sum_{\alpha\in\mathbb{N}_n^k} \bigg|d^{j_1}_{\alpha_1}\cdots d^{j_k}_{\alpha_k}\Big(\mathscr{L}_{X^H_{j_1}} \cdots \mathscr{L}_{X^H_{j_k}}\phi\Big)\Big|_{\Phi(\mathrm{e})}\bigg| \\
&\le \frac{(Dn)^k}{k!} \sum_{\beta\in\mathbb{N}_n^k} \Big|\big(\mathscr{L}^H_{X_\beta}\phi\big)(\mathrm{e})\Big| \\
&= (Dn)^k c_k(\phi),
\end{aligned}
$$

where we wrote $\beta = (j_1, \ldots, j_k)$. This implies (4.11) at once. □

To understand the representation (4.1) it is essential to estimate the Lie-Taylor majorant of Lie-derivatives $\mathscr{L}_{X_\xi}\phi$ in terms of the formal "complex" derivative

$$
\mathrm{F}'_\phi(z) = \sum_{k=0}^{\infty}(k+1)c_{k+1}(\phi)\cdot z^k \tag{4.12}
$$

of the Lie-Taylor majorant F_ϕ of ϕ. Such an estimate is provided by the following result. Here and in what follows we slightly abuse notation and denote by $\|\xi\|_\infty$ the supnorm of the coordinate vector $(\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ of $\xi \in \mathfrak{g}$ w.r.t. the basis $\mathcal{B}$.

Proposition 4.4 *Let $\xi \in \mathfrak{g}$, $k \in \mathbb{N}_0$, $\phi \in \mathscr{C}^\infty(G)$ and $z \in \mathbb{C}$. We have the estimates*

$$
c_k\big(\mathscr{L}_{X_\xi}\phi\big) \le \|\xi\|_\infty \cdot (k+1)c_{k+1}(\phi) \tag{4.13}
$$

and

$$
\Big|\mathrm{F}_{\mathscr{L}_{X_\xi}\phi}(z)\Big| \le \|\xi\|_\infty \cdot \mathrm{F}'_\phi\big(|z|\big). \tag{4.14}
$$

Proof The estimate (4.13) is just

$$
\begin{aligned}
c_k\big(\mathscr{L}_{X_\xi}\phi\big) &= \frac{1}{k!} \sum_{\alpha\in\mathbb{N}_n^k} \big|\big(\mathscr{L}_{X_\alpha}\mathscr{L}_{X_\xi}\phi\big)(\mathrm{e})\big| \\
&\le \frac{\|\xi\|_\infty}{k!} \sum_{\alpha\in\mathbb{N}_n^{k+1}} \big|\big(\mathscr{L}_{X_\alpha}\phi\big)(\mathrm{e})\big| \le \|\xi\|_\infty \cdot (k+1)c_{k+1}(\phi),
\end{aligned}
$$

from which (4.14) follows at once via (4.12). □

We now shift our focus to the set $\mathscr{C}^\omega(G)$ of *(real) analytic* functions on G. Here we can say more and warrant our terminology from Definition 4.1. For a more conceptual understanding, we recall the classical concept of Lie-Taylor series, which can be found e.g. in the beginning of [27, Sec. 2.1.4] for Lie groups and [21, (1.48)] for arbitrary analytic manifolds:

Proposition 4.5 (Lie-Taylor)
Let M be an analytic manifold, $\phi \in \mathscr{C}^\omega(M)$ and $X \in \Gamma^\omega(M)$ an analytic vector field with corresponding flow Φ.

(i) *Given $p \in M$, there exists a parameter $r > 0$ such that the Lie-Taylor formula*

$$\phi\big(\Phi(t, p)\big) = \big(\exp(t\mathscr{L}_X)\phi\big)(p) \tag{4.15}$$

holds, whenever $|t| < r$.

(ii) *The series (4.15) is $\mathscr{C}^\infty\big(B_r(0)\big)$-convergent in the parameter t.*

(iii) *Given a Lie algebra element $\xi \in \mathfrak{g}$, we get a well-defined exponential operator*

$$\exp\big(\mathscr{L}_{X_\xi}\big)\colon \mathscr{C}^\omega(G) \longrightarrow \mathscr{C}^\omega(G). \tag{4.16}$$

Proof We first compute how powers of $\mathscr{L}_X$ act on ϕ. To this end, we rewrite

$$\mathscr{L}_X\phi\Big|_{\Phi(t,p)} = \frac{\mathrm{d}}{\mathrm{d}s}\Phi\big(s, \Phi(t, p)\big)^*\phi\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\Phi(s+t, p)^*\phi\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}t}\phi\big(\Phi(t, p)\big),$$

which we can now iterate easily. This yields

$$\mathscr{L}_X^k\phi\Big|_{\Phi(t,p)} = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\phi\big(\Phi(t, p)\big) \tag{4.17}$$

for all $k \in \mathbb{N}_0$. Note that these considerations also work for smooth functions. The case $X(p) = 0$ is special, so we deal with it first: then $\mathscr{L}_X\phi = X(p)\phi = 0$ and thus the right hand side of (4.19) reduces to the constant term, which is just $\phi(p)$. From the differential equation it is moreover clear that $\Phi(t, p) \equiv p$ in this case, so the left hand side matches. Thus the interesting case $X(p) \neq 0$ remains. Here we finally use the analyticity to obtain a chart (U, x) of M centered at p such that the function

$$\psi\colon x(U) \longrightarrow \mathbb{K}, \quad \psi = \phi \circ x$$

is given by its power series around 0 on all of $x(U) \subseteq \mathbb{K}^n$ as well as $x^{-1}(te_1) = \Phi(t, p)$ for t with $te_1 \in x(U)$. The latter condition is achievable by [21, Lem. 1.9.2], which yields an analytic chart if we go through its construction starting with an analytic chart as well as using an analytic vector field: having a chart with $X\Big|_U = \partial_1$ then gives the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t, p) = \frac{\partial}{\partial x^1}\big(\Phi(t, p)\big),$$

which applied to a function on U gets solved by $\Phi(t, p) = x^{-1}(te_1)$. Note that the left hand side acts by pullback here. Thus by uniqueness of solutions our condition indeed holds. Let now $(e_1, \dots, e_n)$ be a basis of $\mathbb{K}^n$. As $x(U)$ is open, we find an $r > 0$ with $\mathrm{B}_r(0) \subseteq x(U)$ for some auxiliary norm on $\mathbb{K}^n$. For $|t| < r$ we have the simple Taylor expansion

$$
\begin{aligned}
\psi(te_1) &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \big(\partial_1^k \psi\big)(0) \\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{\mathrm{d}^k}{\mathrm{d}s^k} \phi\big(x(se_i)\big)\Big|_{s=0} \\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{\mathrm{d}^k}{\mathrm{d}s^k} \phi\big(\Phi(s, p)\big)\Big|_{s=0} \\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathscr{L}_X^k \phi\Big|_{\Phi(0,p)} \\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathscr{L}_X^k \phi\Big|_{p},
\end{aligned}
$$

which is exactly (4.15). This calculation gives also the uniform convergence statement: the kth partial sum on the right hand side of (4.15) exactly corresponds to the kth partial sum of the classical Taylor series. As the Taylor series is $\mathscr{C}^\infty$-convergent in the interior of the polydisk of convergence, so is (4.15) in t. The statement (iii) follows immediately from reading (4.15) backwards. $\square$

Taking $M = G$ as a Lie group and $X = X_\xi$ as a left invariant vector field for some $\xi \in \mathfrak{g}$ gives the flow $\Phi(t, g) = g\exp(t\xi)$, which notably also yields a suitable chart centered at g, as we just used in the proof. The Lie-Taylor series then has the form

$$
\big(\phi \circ r_{\exp(t\xi)}\big)(g) = \phi\big(g\exp(t\xi)\big) = \big(\exp(t\mathscr{L}_{X_\xi})\phi\big)(g) \tag{4.18}
$$

for all $g \in G$ and sufficiently small $t \in \mathbb{R}$. It moreover coincides with the Taylor series of $\phi \circ \ell_g \circ \exp$ on the ray through 0 in direction of ξ. Taking now $\xi = x^j e_j$ as a basis decomposition yields the following:

Corollary 4.6 (Lie-Taylor series on G) *Let $\phi \in \mathscr{C}^\omega(G)$ and $g \in G$. Then there is a radius $r > 0$ such that*

$$
\phi\big(g\exp(x^j e_j)\big) = \mathrm{T}_\phi(\underline{x}; g) \tag{4.19}
$$

for all $\underline{x} = (x^1, \dots, x^n) \in \mathbb{R}^n$ with $\|\underline{x}\|_\infty < r$. The right hand side of (4.19) is $\mathscr{C}^\infty$-convergent, whenever it converges at all. In particular, if the Lie-Taylor majorant F_ϕ of ϕ is an entire holomorphic function, then

$$
\mathbb{R}^n \longrightarrow \mathbb{C}, \quad \underline{x} \mapsto \phi\big(g\exp(x^j e_j)\big) \tag{4.20}
$$

has a holomorphic extension to $\mathbb{C}^n$*, which is provided by the Lie-Taylor series* $\mathrm{T}_\phi(\,\cdot\,; g)$ *of* ϕ *at* g*.*

The next result has a similar flavor as Proposition 4.4 and roughly ensures that the concept of Lie-Taylor majorants is also compatible with the exponentiated action (4.16) of G on $\mathscr{C}^\infty(G)$ by pullbacks with right multiplications $r_g(h) = hg$.

Proposition 4.7 *Let* $\xi \in \mathfrak{g}$*,* $\phi \in \mathscr{C}^\omega(G)$ *and* $z \in \mathbb{C}$*. Then*

$$\left|\mathrm{F}_{\phi \circ r_{\exp(\xi)}}(z)\right| \leq \mathrm{F}_\phi\left(|z| + \|\xi\|_\infty\right). \tag{4.21}$$

Proof Using (4.18) and applying (4.13) repeatedly yields

$$c_k\big(\phi \circ r_{\exp\xi}\big) = c_k\big(\exp(\mathscr{L}_{X_\xi})\phi\big) \leq \sum_{\ell=0}^{\infty} \frac{1}{\ell!} c_k\big(\mathscr{L}_{X_\xi}^\ell \phi\big) \leq \sum_{\ell=0}^{\infty} \frac{(k+\ell)!}{k!\,\ell!} c_{k+\ell}(\phi)\|\xi\|_\infty^\ell.$$

Consequently,

$$\begin{aligned}
\left|\mathrm{F}_{\phi \circ r_{\exp\xi}}(z)\right| &\leq \sum_{k=0}^{\infty} c_k\big(\phi \circ r_{\exp\xi}\big) \cdot |z|^k \\
&\leq \sum_{\ell=0}^{\infty} \frac{\|\xi\|_\infty^\ell}{\ell!} \sum_{k=0}^{\infty} \frac{(k+\ell)!}{k!} c_{k+\ell}(\phi) \cdot |z|^k \\
&= \sum_{\ell=0}^{\infty} \frac{\|\xi\|_\infty^\ell}{\ell!} \mathrm{F}_\phi^{(\ell)}\left(|z|\right) \\
&\overset{(\star)}{=} \mathrm{F}_\phi\left(|z| + \|\xi\|_\infty\right),
\end{aligned}$$

provided that $\mathrm{F}_\phi(|z| + \|\xi\|_\infty) < \infty$. In fact, in this case F_ϕ is holomorphic on the disk $\mathrm{B}_{|z|+\|\xi\|_\infty}(0)$, so we can expand F_ϕ as a Taylor series around $|z|$ in the disk $\mathrm{B}_{\|\xi\|_\infty}(|z|)$, at least. This proves $(\star)$ with $\|\xi\|_\infty$ replaced by $r\|\xi\|_\infty$ for any $0 < r < 1$. Letting $r \to 1$ gives $(\star)$. If $\mathrm{F}_\phi\left(|z| + \|\xi\|_\infty\right) = \infty$, then the estimate is certainly true. $\square$

Intuitively, (4.21) lets us estimate Taylor expansions of analytic functions with perturbed expansion point on the group by perturbing the expansion point on the Lie algebra.

Lemma 4.8 (Inversion invariance of Lie-Taylor majorants)
Let $\mathrm{inv}\colon G \longrightarrow G$ *denote group inversion on* G*. Then* $\mathrm{F}_{\phi\circ \mathrm{inv}} = \mathrm{F}_\phi$ *for all* $\phi \in \mathscr{C}^\infty(G)$*.*

Proof This is immediate from

$$\mathscr{L}_{X_\xi}(\phi \circ \mathrm{inv})\Big|_{\mathrm{e}} = \frac{\mathrm{d}}{\mathrm{d}t}\big(\phi \circ \mathrm{inv}\big)(\mathrm{e} \cdot \exp(t\xi))\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\phi\big(\exp(-t\xi)\big)\Big|_{t=0} = -\mathscr{L}_{X_\xi}\phi\Big|_{\mathrm{e}}$$

for $\xi \in \mathfrak{g}$ by definition of the Lie derivative. $\square$

Corollary 4.9 *There is a locally constant function*

$$\gamma\colon G \longrightarrow [0,\infty)$$

such that for any $\phi \in \mathscr{C}^\omega(G)$ *and any* $\underline{z} \in \mathbb{C}^n$ *the Lie-Taylor majorant* $\mathrm{F}_\phi(\|\underline{z}\|_\infty + \gamma(g))$ *is a majorant for the Lie-Taylor series* $\mathrm{T}_\phi(\underline{z}; g)$ *of* ϕ *at* g *evaluated at* $\underline{z} \in \mathbb{C}^n$ *and hence*

$$\left|\mathrm{T}_\phi(\underline{z}; g)\right| \leq \mathrm{F}_\phi\left(\|\underline{z}\|_\infty + \gamma(g)\right). \tag{4.22}$$

Proof We denote left multiplication with g by ℓ_g, as usual. The left invariance of X_ξ gives $\mathscr{L}_{X_\xi} \circ \ell_g^* = \ell_g^* \circ \mathscr{L}_{X_\xi}$. Applying this to a function ϕ and evaluating at $e \in G$ gives

$$\mathrm{T}_\phi(\underline{z}; g) = \mathrm{T}_{\phi\circ\ell_g}(\underline{z}; \mathrm{e}). \tag{$*$}$$

Now $\phi \circ \ell_g = \phi \circ \mathrm{inv} \circ r_{g^{-1}} \circ \mathrm{inv}$, so Proposition 4.8 implies

$$\mathrm{F}_{\phi\circ\ell_g}(z) = \mathrm{F}_{\phi\circ\mathrm{inv}\circ r_{g^{-1}}}(z).$$

We now choose $\xi_1, \ldots, \xi_m \in \mathfrak{g}$ with $\|\xi_j\|_\infty \leq 1$ and $g^{-1} = \exp(\xi_1)\cdots\exp(\xi_m)$. By openness of the Lie exponential the integer m can be chosen in a locally constant manner, i.e. there is an open neighbourhood U of g^{-1} s.t. each $h \in U$ can be written as a product of m exponentials. Then applying Proposition 4.7 m-times and once again Proposition 4.8 yields

$$\left|\mathrm{F}_{\phi\circ\ell_g}(z)\right| = \left|\mathrm{F}_{\phi\circ\mathrm{inv}\circ r_{g^{-1}}}(z)\right| \leq \mathrm{F}_{\phi\circ\mathrm{inv}}\left(|z| + m\right) = \mathrm{F}_\phi\left(|z| + m\right).$$

Combining this with ($*$) and (4.7) completes the proof. ☐

4.2 Entire functions on *G*

In this subsection we first focus on the case $R = 0$ and introduce the pendant $\mathscr{E}_0(G)$ of the optimal case from the strict deformation [17, Prop. 3.2, (ii)], i.e. $\mathrm{S}^\bullet_{R'}(\mathfrak{g})$ with $R' = 1$. In a second step, we then introduce the algebras $\mathscr{E}_R(G)$ for $R > 0$. There are several reasons for this two–step approach. Firstly, our methods in the case $R = 0$ seem completely natural and do not call for any specific motivation. Secondly, it puts us in a position to reintroduce the classical notion of an entire vector for the lifted Lie algebra representation (4.1). While our approach to this notion is novel, the locally convex space we are about to consider is not. We will make this and its history precise in Remark 4.16. Thirdly, the construction for the case $R = 0$ provides a solid motivation for the cases $R > 0$, since it makes clear that we need to identify locally convex algebras of entire functions with controlled growth, whose topology is finer than that of locally uniform convergence, but which still are invariant with respect to differentiation and translation in the argument.

Definition 4.10 (*Entire functions on G*) An analytic function $\phi \in \mathscr{C}^\omega(G)$ is called an entire function on G if its Lie-Taylor majorant $\mathrm{F}_\phi \in \mathcal{H}(\mathbb{C})$ is entire. By

$$\mathscr{E}_0(G) = \left\{\phi \in \mathscr{C}^\omega(G) \;\middle|\; \mathrm{F}_\phi \in \mathcal{H}(\mathbb{C})\right\} \tag{4.23}$$

we denote the set of all entire functions on G.

In particular, each element of $\mathscr{E}_0(G)$ is analytic by definition, so it does have a local Lie-Taylor series representation by Corollary 4.6. Hence by connectedness of G it follows that

$$\phi = 0 \quad \Longleftrightarrow \quad \mathrm{F}_\phi = 0. \tag{4.24}$$

Recall that the $\mathbb{C}$-vector space $\mathcal{H}(\mathbb{C})$ carries a canonical topology, namely the compact-open topology (or topology of locally uniform convergence). This locally convex topology can be induced by the family of norms

$$\|F\|_{0,c} = \max_{|z|\leq c} |F(z)|, \tag{4.25}$$

and is metrizable in a translation-invariant manner by

$$d(F, G) = d(F - G, 0) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\|F - G\|_{0,j}}{1 + \|F - G\|_{0,j}}$$

for $F, G \in \mathcal{H}(\mathbb{C})$. It is well-known that $(\mathcal{H}(\mathbb{C}), d)$ is then a multiplicatively convex (commutative) nuclear Fréchet algebra w.r.t. pointwise multiplication. We are thus naturally led to define a metric d_0 on the vector space $\mathscr{E}_0(G)$ by

$$d_0(\phi, \psi) = d(\mathrm{F}_{\phi-\psi}, 0) \tag{4.26}$$

for $\phi, \psi \in \mathscr{E}_0(G)$. An associated family of seminorms is given by

$$\begin{aligned} \mathrm{q}_{0,c}(\phi) &= \|\mathrm{F}_\phi\|_{0,c} = \max_{|z|\leq c} \left|\mathrm{F}_\phi(z)\right| = \mathrm{F}_\phi(c) \\ &= \sum_{k=0}^{\infty} c_k(\phi)\, c^k = \sum_{k=0}^{\infty} \frac{c^k}{k!} \sum_{\alpha\in\mathbb{N}_n^k} \left|\left(\mathscr{L}_{X_\alpha}\phi\right)(\mathrm{e})\right| \end{aligned} \tag{4.27}$$

with $c > 0$. Note that (4.24) ensures that each $\mathrm{q}_{0,c}\colon \mathscr{E}_0(G) \longrightarrow \mathbb{R}$ is in fact a *norm* on $\mathscr{E}_0(G)$. By the Leibniz inequality from Proposition 4.3, (i), the norms (4.27) are sub-multiplicative. In particular, $(\mathscr{E}_0(G), d_0)$ is a locally multiplicatively convex algebra w.r.t. to pointwise multiplication.

We now introduce the family of subspaces $\mathscr{E}_R(G)$, $R > 0$, of the algebra $\mathscr{E}_0(G)$ of all entire functions on G, which will serve as the other tensor factors in the observable algebra. The idea is to restrict the Lie-Taylor majorants $\mathrm{F}_\phi \in \mathcal{H}(\mathbb{C})$ to holomorphic entire functions of fixed *finite order and minimal type*. The intimate reason for this choice is that these functions and thus their Taylor coefficients do have controlled growth and they form a particularly well-studied subalgebra of $\mathscr{E}_0(G)$ with the added feature of being invariant w.r.t. to *differentiation* and *translation*, i.e. both the Lie algebra and the Lie group representations. This puts us in a position to make use of Propositions 4.4 and 4.7 and will directly lead us to faithful analogues of those algebras on the Lie group G. Recall that for any $\rho > 0$ a function $F \in \mathcal{H}(\mathbb{C})$ is said to have *finite order* $\leq \rho$ *and minimal type* if

$$\sup_{z\in\mathbb{C}} |F(z)| \exp\big(- \varepsilon\, |z|^{\rho} \big) < \infty \tag{4.28}$$

for every $\epsilon > 0$. We denote the set of all such functions by $\mathcal{H}_{\rho}(\mathbb{C})$. Note that for $\rho \leq 1$ we are speaking of entire functions of exponential type zero.

Definition 4.11 (*R-entire functions*) Let $R > 0$. A function $\phi \in \mathscr{C}^{\omega}(G)$ is called an R-entire function if $\mathrm{F}_{\phi} \in \mathcal{H}(\mathbb{C})$ has finite order $\leq \frac{1}{R}$ and minimal type. We denote the set of all R-entire functions by $\mathscr{E}_R(G)$.

Unwrapping the definition, we thus have

$$\phi \in \mathscr{E}_R(G) \quad\iff\quad \forall_{\varepsilon>0} \quad \|\phi\|_{R,\varepsilon} = \sup_{z\in\mathbb{C}} \big|\mathrm{F}_{\phi}(z)\big| \exp\big(- \varepsilon\, |z|^{1/R} \big) < \infty. \tag{4.29}$$

Remark 4.12 (i) Clearly, each of the sets $\mathscr{E}_R(G)$ is a unital subalgebra of $\mathscr{E}_0(G)$ and we have the inclusions

$$\mathscr{E}_R(G) \subseteq \mathscr{E}_S(G), \tag{4.30}$$

whenever $S \leq R$.

(ii) Let $\phi \in \mathscr{E}_R(G)$ for some $R > 0$. Then the Lie-Taylor series $\mathrm{T}_{\phi}(\,\cdot\,; \mathrm{e})$ of ϕ is an entire holomorphic function on $\mathbb{C}^n$ of order $\leq \frac{1}{R}$ and minimal type. We denote the set of such functions by $\mathcal{H}_{1/R}(\mathbb{C})$ and equip it with the family of norms (4.29). For a general treatment we refer e.g. to the textbook [35].

It is well-known (see e.g. [36, Prop. 4.2]) that $\mathcal{H}_{\rho}(\mathbb{C})$ equipped with the family of norms (4.28) is a nuclear Fréchet space. It follows at once from (4.29) that for $\varepsilon > 0$

$$\|\phi \cdot \psi\|_{R,\varepsilon} \leq \|\phi\|_{R,\varepsilon/2} \cdot \|\psi\|_{R,\varepsilon/2}, \tag{4.31}$$

so $(\mathscr{E}_R(G), d_R)$ is a locally convex commutative algebra w.r.t. pointwise multiplication. Note, however, that $\mathscr{E}_R(G)$ is *not* multiplicatively convex as soon as $R > 0$. In fact, it can be easily shown that $\mathscr{E}_R(G)$ has no entire holomorphic functional calculus.

It will turn out convenient to introduce an equivalent family of (semi)norms on $\mathscr{E}_R(G)$. As in the classical case of entire holomorphic functions of finite order this is achieved by relating the order of an analytic function $\phi \in \mathscr{C}^{\omega}(G)$ to the growth of its Taylor coefficients:

Definition 4.13 (*R-Lie-Taylor majorant*) Let $R \geq 0$ and $\phi \in \mathscr{C}^\infty(G)$. Then we call

$$\mathrm{F}_{R,\phi}(z) = \sum_{k=0}^{\infty} k!^R c_k(\phi) \cdot z^k = \sum_{k=0}^{\infty} k!^{R-1} \sum_{\alpha \in \mathbb{N}_n^k} \left|\left(\mathscr{L}_{X_\alpha}\phi\right)(\mathrm{e})\right| z^k \tag{4.32}$$

the R-Lie-Taylor majorant of ϕ and define for any $c \geq 0$

$$\mathrm{q}_{R,c}(\phi) = \mathrm{F}_{R,\phi}(c) = \sum_{k=0}^{\infty} k!^R \cdot c_k(\phi) \cdot c^k = \sum_{k=0}^{\infty} k!^{R-1} \sum_{\alpha \in \mathbb{N}_n^k} \left|\left(\mathscr{L}_{X_\alpha}\phi\right)(\mathrm{e})\right| c^k. \tag{4.33}$$

The following elementary result tells us that for a function $\phi \in \mathscr{C}^\omega(G)$ membership in $\mathscr{E}_R(G)$ can be checked using the R-Lie-Taylor majorant $\mathrm{F}_{R,\phi}$ and the seminorms $\mathrm{q}_{R,c}$:

$$\phi \in \mathscr{E}_R(G) \quad \Longleftrightarrow \quad \mathrm{F}_{R,\phi} \in \mathcal{H}(\mathbb{C}) \quad \Longleftrightarrow \quad \forall_{c>0}\ \mathrm{q}_{R,c}(\phi) < \infty, \tag{4.34}$$

and also that

$$d_R(\phi_n, \phi) \to 0 \quad \Longleftrightarrow \quad \mathrm{F}_{R,\phi_n-\phi} \to 0 \text{ in } \mathcal{H}(\mathbb{C}). \tag{4.35}$$

Remark 4.14 It seems (and perhaps is) over the top to introduce two different notions, the seminorms $\mathrm{q}_{R,c}$ and the R-Lie-Taylor majorant $\mathrm{F}_{R,\phi}$, for essentially the same object, namely $\mathrm{F}_{R,\phi}(c) = \mathrm{q}_{R,c}(\phi)$. However, it emphasizes that $\mathrm{q}_{R,c}(\phi)$ simply *is* a *holomorphic* function of one *complex* variable c, and this point of view brings along some useful tools such as the Cauchy integral formula.

Proposition 4.15 *Let $\phi, \psi \in \mathscr{C}^\infty(G)$. Then*

(i) *for any $R > 0$ and $\varepsilon > 0$,*

$$\|\phi\|_{R,\varepsilon} \leq q_{R,(R/\varepsilon)^R}(\phi); \tag{4.36}$$

(ii) *for any $R > 0$, $c > 0$ and $\varepsilon > 0$ such that $c \cdot \left(\frac{e\varepsilon}{R}\right)^R < 1$,*

$$q_{R,c}(\phi) \leq \frac{\|\phi\|_{R,\varepsilon}}{1 - c \cdot (\frac{e\varepsilon}{R})^R}; \tag{4.37}$$

(iii) *for any $R \geq 0$ and $c > 0$*

$$q_{R,c}(\phi \cdot \psi) \leq q_{R,2^Rc}(\phi) \cdot q_{R,2^Rc}(\psi). \tag{4.38}$$

Moreover, naive extension of (4.33) *to $R < 0$ yields submultiplicative seminorms $q_{R,c}$.*

(iv) *for any* $R \geq 0$, $\xi \in \mathfrak{g}$ *and* $z \in \mathbb{C}$

$$\left| \mathrm{F}_{R, \phi \circ r_{\exp \xi}}(z) \right| \leq \mathrm{F}_{R,\phi} \left(|z| + \|\xi\|_\infty \right) \tag{4.39}$$

and

$$\left| \mathrm{F}_{R, \mathscr{L}_{X_\xi} \phi}(z) \right| \leq \|\xi\|_\infty \cdot \mathrm{F}'_{R,\phi} \left(|z| \right). \tag{4.40}$$

Proof The proofs of (i) and (ii) are standard and will not be given here. For (iii) we recall $c_k(\phi \cdot \psi) \leq c_k(\phi) \cdot c_k(\psi)$ by the Leibniz inequality in Proposition 4.3, (i). Hence

$$\begin{aligned}
\mathrm{q}_{R,c}(\phi \cdot \psi) &\leq \sum_{k=0}^{\infty} k!^R \sum_{j=0}^{k} c_j(\phi) c^j c_{k-j}(\psi) c^{k-j} \\
&= \sum_{k=0}^{\infty} \binom{k}{j}^R \sum_{j=0}^{k} j!^R c_j(\phi) c^j \cdot (k-j)!^R c_{k-j}(\psi) c^{k-j} \\
&\leq \sum_{k=0}^{\infty} 2^{kR} \sum_{j=0}^{k} j!^R c_j(\phi) c^j \, (k-j)!^R c_{k-j}(\psi) c^{k-j} \\
&= \mathrm{q}_{R,2^R c}(\phi) \cdot \mathrm{q}_{R,2^R c}(\psi),
\end{aligned}$$

using the rather crude estimate $\binom{k}{j} \leq 2^k$ resp. the Cauchy product formula in the last two steps. Going through our computation for $R < 0$, one can be even cruder and estimate the binomial to the power of R by 1. The proof of (iv) is identical to the ones of Propositions 4.4 and 4.7 and will not be repeated here. The reader will notice that it *does* rely on $R \geq 0$, though. □

We are thus naturally led to equip the vector space $\mathscr{E}_R(G)$ with the family of seminorms (4.29) and the corresponding metric

$$d_R(\phi, \psi) = d_0\big(\mathrm{F}_{R,\phi-\psi}, 0\big) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\mathrm{q}_{R,j}(\phi - \psi)}{1 + \mathrm{q}_{R,c}(\phi - \psi)}. \tag{4.41}$$

Notably, the inclusions (4.30) are then continuous. Before we take a closer look at the somewhat deeper properties of the locally convex algebras $\mathscr{E}_R(G)$, we take a detour to relate our considerations to classical notions from infinite-dimensional representation theory.

Remark 4.16 (Representation theory)
In the literature, [23, Sec. 2] was the first to consider the Fréchet space of entire vectors for Lie algebra representations on Banach spaces B induced by strongly continuous Lie group representations $\pi\colon G \longrightarrow L(B)$. Recall that a vector $v \in B$ is

called $\mathscr{C}^k$-vector for π if the maps

$$\pi_v \colon G \longrightarrow B, \quad \pi_v(g) = \pi(g)v \tag{4.42}$$

are $\mathscr{C}^k$-functions with values in B, where $k \in \mathbb{N}_0 \cup \{\infty, \omega\}$. By sequential completeness of the Banach space B one can indeed define differentiability by means of differential quotients. One writes $\mathscr{C}^k(\pi) \subseteq B$ for the set of all $\mathscr{C}^k$-vectors for the representation π. The assumed strong continuity of the representation π then just means that $\mathscr{C}(\pi) = \mathscr{C}^0(G) = B$. Note that doing things in this pointwise fashion corresponds to considering all limits in the weak topology of $L(B)$. In this context, the natural question is about density of the spaces $\mathscr{C}^k(G)$ as subspaces of $\mathscr{C}(G)$. For a quite nice, albeit dated discussion, we refer to [39]. Some more modern discourses can be found in the textbooks [49, Chap. 10] and [53, Appendix D].

In fact, it is often necessary to go beyond the Banach space scenario and include more general locally convex spaces like e.g. Fréchet spaces as representation spaces as well.

Passing to the infinitesimal situation, we differentiate and obtain a Lie algebra representation $T\pi$ of not necessarily continuous linear operators $T\pi\xi$, each defined on some subspace of the representation space. By the classical [22] they share a common dense invariant domain, the so-called Gårding space $\mathscr{G}(\pi) \subseteq \mathscr{C}^\infty(\pi)$. For Fréchet spaces, the seminal work [14] proved the equality $\mathscr{G}(\pi) = \mathscr{C}^\infty(\pi)$.

Analytic vectors are more complicated. By [40, Sec. 3] a smooth vector $v \in \mathscr{C}^\infty(\pi)$ is analytic if and only if the formal exponential $\exp(T\pi\xi)v$ converges for all ξ in some neighbourhood of $0 \in \mathfrak{g}$. Reference [23] turned this into a seminorm condition and obtained families of Fréchet spaces $\mathscr{H}_t(\pi)$ this way: demanding convergence for $\xi \in \mathrm{B}_t(0) \subseteq \mathfrak{g}$ prescribes a uniform radius of convergence for $\exp(T\pi\xi)v$. Of particular interest are then the union over $t = \frac{1}{n}$ and the intersection over $t = n$ with $n \in \mathbb{Z}$: the former endows the space of analytic vectors with the structure of a locally convex inductive limit, the latter is Fréchet again as the countable intersection of Fréchet spaces. For obvious reasons, [23] calls this intersection entire vectors and so shall we. Note that there is no need to start with a Lie group representation at all: all notions make sense for Lie algebra representations from the start. The natural question is then, when such a representation can be integrated. Some answers can be found in [20, 38] and the much more recent developments [11, 49, 50].

The action we are interested in is the translation action of the group G by means of pullbacks

$$\ell^*_\bullet, r^*_\bullet \colon G \longrightarrow L\big(\mathscr{C}(G)\big) \tag{4.43}$$

by left and right multiplications, which is surprisingly ill-behaved. Note that the space of continuous functions $\mathscr{C}(G)$ is *not* Banach with respect to its usual topology of locally uniform convergence in general, but a Fréchet space. It is straightforward to generalize the notions of $\mathscr{C}^k$-vectors of a group representation into this more general setting. These more general cases were studied e.g. in [33, 37]. However, most of the useful techniques break down, unless the set $\pi(G)$ is an equicontinuous family of operators. Most importantly, the utilization of (Riemann or Bochner) integral methods require this assumption. Notably, the translation actions (4.43) are not equicontinuous at all: there is no compact subset of G that contains all compact subsets, unless the group is compact itself. Remarkably, we will be able to reproduce most of the pleasant results for our particular situation regardless.

Our Taylor formula (4.18) implies that each $\phi \in \mathscr{E}_0(G)$ is an entire vector with respect to the representation $r_\bullet^*$. Or differently put, the lifted Lie algebra representation (4.1) exponentiates to the Lie group representation $r_\bullet^*$. Our Definition 4.10 is thus the appropriate infinitesimal version of entire vectors in our locally convex situation. This matches with a straightforward reformulation of the textbook definitions [49, Def. 10.3.1 and 10.3.2] in the following sense: We consider the lifted Lie algebra representation (4.1) and equip $\mathscr{C}(G)$ with the continuous seminorm $|\delta_{\mathrm{e}}|$ given by the absolute value of the Dirac functional at the group unit

$$\delta_{\mathrm{e}} \colon \mathscr{C}(G) \longrightarrow \mathbb{C}, \quad \delta_{\mathrm{e}}(\phi) = \phi(\mathrm{e}). \tag{4.44}$$

While this is not the natural topology of $\mathscr{C}(G)$ at all, this is the useful choice for estimation and to generate examples later on. By the upcoming Theorem 4.17, (vi), this choice leads to the same locally convex space $\mathscr{E}_0(G)$ as *the natural one* a posteriori. This also identifies $\mathscr{E}_0(G)$ as the space of entire vectors for the group representations (4.43) in the sense of [23].

With this incomplete discussion in mind, we can show the following:

Theorem 4.17 (Representation theory) *Let G be a connected Lie group and let $R \geq 0$.*

(i) *Group inversion $inv \colon G \longrightarrow G$ induces an isometry of $(\mathscr{E}_R(G), d_R)$, that is, $\phi \circ inv \in \mathscr{E}_R(G)$ and*

$$d_R(\phi \circ inv, \psi \circ inv) = d_R(\phi, \psi) \tag{4.45}$$

for all $\phi, \psi \in \mathscr{E}_R(G)$.

(ii) *Pullbacks with left and right translations yield representations*

$$\ell_\bullet^*, r_\bullet^* \colon G \longrightarrow L\big(\mathscr{E}_R(G)\big) \tag{4.46}$$

by continuous linear maps.

(iii) *The space $\mathscr{E}_R(G)$ of entire functions is invariant under the lifted Lie algebra representation (4.1) by continuous maps. More precisely, we have the estimate*

$$q_{R,c}\big(\mathscr{L}_{X_\xi}\phi\big) \leq q_{R,c+1}(\phi) \cdot \|\xi\|_\infty \tag{4.47}$$

for $c \geq 0$, $\xi \in \mathfrak{g}$ and $\phi \in \mathscr{E}_R(G)$.

(iv) *The Lie-Taylor series $\mathrm{T}_\phi(\,\cdot\,; \underline{z})$ is absolutely convergent in $\mathscr{E}_R(G)$ for every $\underline{z} \in \mathbb{C}^n$. Thus every entire function $\phi \in \mathscr{E}_R(G)$ is an entire vector for the translation representations (4.46), which are in particular strongly continuous.*

(v) *The $\mathscr{E}_R(G)$-topology is finer than the $\mathscr{C}^\infty$-topology. In particular, the evaluation functionals*

$$\delta_{g,\alpha}\colon \mathscr{E}_R(G) \longrightarrow \mathbb{C}, \quad \delta_g(\phi) = \big(\mathscr{L}_{X_\alpha}\phi\big)(g) \tag{4.48}$$

are continuous linear maps for $g \in G$ and $\alpha \in \mathbb{N}_n^k$.

(vi) *The alternative seminorms*

$$r_{R,c}(\phi) = \sum_{k=0}^{\infty} k!^R \frac{c^k}{k!} \sum_{\alpha\in\mathbb{N}_n^k} \sup_{g\in K} \big|\big(\mathscr{L}_{X_\alpha}\phi\big)(g)\big| \tag{4.49}$$

with $c \geq 0$ and compact $K \subseteq G$ are well-defined on $\mathscr{E}_R(G)$ and constitute a defining system for the $\mathscr{E}_R$-topology. Thus $\mathscr{E}_0(G)$ is the space of entire vectors for the representation (4.43) if we equip $\mathscr{C}(G)$ with its canonical topology.

Proof The statement (i) is just Lemma 4.8 again. For (ii) we consider $g = \exp(\xi_1)\cdots\exp(\xi_m)$ with $\xi_1, \ldots, \xi_m \in \mathfrak{g}$ s.t. $\|\xi_j\|_\infty \leq 1$. Applying Proposition 4.15, (iv), m-times we obtain

$$\big|\mathrm{F}_{R,\phi\circ r_g}(z)\big| \leq \mathrm{F}_{R,\phi}\big(|z| + m\big)$$

for $z \in \mathbb{C}$. This proves

$$\mathrm{q}_{R,c}(\phi \circ r_g) \leq \mathrm{q}_{R,c+m}(\phi)$$

for $c \geq 0$. The corresponding property of left multiplication by g follows now from (i) by once again noting that $\phi \circ \ell_g = \phi \circ \mathrm{inv} \circ r_{g^{-1}} \circ \mathrm{inv}$. Thus the translations r_g and ℓ_g are continuous selfmaps of $\mathscr{E}_R(G)$. For (iii) the Cauchy integral formula yields

$$\mathrm{F}'_{R,\phi}(c) = \frac{1}{2\pi}\left|\int_{\partial\mathrm{B}_r(c)} \frac{\mathrm{F}_{R,\phi}(w)}{(w-c)^2}\,\mathrm{d}w\right| \leq r^{-1}\,\mathrm{F}_{R,\phi}(c+r) = r^{-1}\cdot \mathrm{q}_{R,c+r}(\phi) \tag{$*$}$$

for every $r > 0$. Taking $r = 1$, Proposition 4.15, (iv), shows that

$$\mathrm{q}_{R,c}\big(\mathscr{L}_{X_\xi}\phi\big) = \mathrm{F}_{R,\mathscr{L}_{X_\xi}\phi}(c) \leq \|\xi\|_\infty \cdot \mathrm{F}'_{R,\phi}(c) \leq \|\xi\|_\infty \cdot \mathrm{q}_{R,c+1}(\phi).$$

This completes (iii), wherefore asking for (iv) makes sense at all: each of the $\mathscr{L}_{X_\alpha}\phi$ is an R-entire function itself. The idea is now to choose $r_k = \frac{1}{k}$ in ($*$), which gives due to $\|e_j\|_\infty = 1$ for $N > 0$

$$\sum_{k=N}^{\infty} \frac{1}{k!} \sum_{\alpha\in\mathbb{N}_n^k} \mathrm{q}_{R,c}\big(\mathscr{L}_{X_\alpha}\phi\big) \leq \sum_{k=N}^{\infty} \frac{1}{k!} \sum_{\alpha\in\mathbb{N}_n^k} k\cdot \mathrm{q}_{R,c+1}(\phi) = \mathrm{q}_{R,c+1}(\phi) \sum_{k=N}^{\infty} \frac{k\cdot n^k}{k!} \longrightarrow 0.$$

Thus the Taylor series $\mathrm{T}(\,\cdot\,, \underline{z})$ is indeed absolutely convergent in $\mathscr{E}_R(G)$. In other words, each of the maps

$$G \ni g \mapsto r_g^* \phi \in \mathscr{E}_R(G)$$

is analytic by (4.18). The statement about left translations follows the usual way. As analytic maps are continuous, this also gives the strong continuity of (4.46). We now turn to (v) and notice that it suffices to handle the case $R = 0$. Let $K \subseteq G$ be a compact set. By a covering argument it is easy to see that there is a positive integer m (depending only on K) with the property that for any $g \in K$ there are $\xi_1, \ldots, \xi_m \in \mathrm{B}_1(0)^{\mathrm{cl}} \subseteq \mathfrak{g}$ such that $g = \exp(\xi_1) \cdots \exp(\xi_m)$. This implies for any $\phi \in \mathscr{E}_0(G)$

$$|\phi(g)| = \left|\big(\phi \circ r_g\big)(\mathrm{e})\right| = \left|\mathrm{F}_{\phi \circ r_g}(0)\right| = \left|\mathrm{F}_{\phi \circ r_{\exp(\xi_1) \cdots \exp(\xi_m)}}(0)\right|.$$

Applying Proposition 4.7 m-times we obtain

$$|\phi(g)| \leq \left|\mathrm{F}_\phi\left(\|\xi_1\|_\infty + \cdots + \|\xi_m\|_\infty\right)\right| \leq \mathrm{q}_{0,m}(\phi) \tag{$**$}$$

for all $g \in K$, yielding the claim. Keeping this compact subset $K \subseteq G$, we turn towards (vi). Note that $\mathrm{r}_{0,c,\{\mathrm{e}\}} = \mathrm{q}_{0,c}$, wherefore the topology induced by the seminorms (4.49) is certainly finer than the $\mathscr{E}_0$-topology. Using ($*$) with $r_k = \frac{1}{k}$ again and what we have just shown gives on the other hand

$$\mathrm{r}_{R,c}(\phi) \leq \sum_{k=0}^{\infty} k!^{R-1} c^k \sum_{\alpha \in \mathbb{N}_n^k} \mathrm{q}_{R,m}\big(\mathscr{L}_{X_\alpha}\phi\big) \leq \sum_{k=0}^{\infty} k!^{R-1} (cn)^k \cdot \max\{k, 1\} \cdot \mathrm{q}_{R,m+1}(\phi),$$

wherefore both topologies do indeed coincide. Here we have used $\mathrm{q}_{R,c} \leq \mathrm{q}_{R,c+1}$. □

The following theorem summarizes the main properties of the locally convex algebras $(\mathscr{E}_R(G), d_R)$ and shows that they share many of the pleasant properties of the Fréchet algebras $\mathcal{H}_{1/R}(\mathbb{C})$ of all entire holomorphic functions (of finite order and minimal type).

Theorem 4.18 (Properties of $\mathscr{E}_R(G)$)
Let G be a connected Lie group and let $R \geq 0$. Then the locally convex algebra $(\mathscr{E}_R(G), d_R)$ is

(i) *a Fréchet algebra;*
(ii) *nuclear;*
(iii) *a Montel space: Every bounded and closed set in $\mathscr{E}_0(G)$ is compact;*
(iv) *separable and reflexive.*

Proof We tackle (i) first. Let (ϕ_j) be a Cauchy sequence in $(\mathscr{E}_R(G), d_R)$. By Theorem 4.17, (v), there is a $\phi \in \mathscr{C}^\infty(G)$ such that $\phi_j \to \phi$ converges in $\mathscr{C}^\infty(G)$. We need to show that $\phi \in \mathscr{E}_R(G)$, i.e. ϕ is analytic as well as $\mathrm{q}_{R,c}(\phi) < \infty$ for all $c \geq 0$, and $\phi_j \to \phi$ in $(\mathscr{E}_R(G), d_R)$. We proceed in several steps.

(1) For each $g \in G$ and each $k \in \mathbb{N}_0$ the kth order homogeneous Taylor polynomial of ϕ_j at g converges to the kth order homogeneous Taylor polynomial of ϕ at g, that is,

$$\begin{aligned}\mathrm{T}_{k,\phi_j}(\underline{z}; g) &= \frac{1}{k!} \sum_{\alpha \in \mathbb{N}_n^k} \big(\mathscr{L}_{X_\alpha} \phi_j\big)(g) \cdot \underline{z}^\alpha \quad \overset{j \to \infty}{\longrightarrow} \quad \frac{1}{k!} \sum_{\alpha \in \mathbb{N}_n^k} \big(\mathscr{L}_{X_\alpha} \phi\big)(g) \cdot \underline{z}^\alpha \\ &= \mathrm{T}_{k,\phi}(\underline{z}; g)\end{aligned}$$

locally uniformly w.r.t. $\underline{z} \in \mathbb{C}^n$. This follows from $\phi_j \to \phi$ in $\mathscr{C}^\infty(G)$ by continuity of the linear operators

$$\mathscr{C}^\infty(G) \ni \psi \mapsto \mathrm{T}_{k,\psi}(\,\cdot\,; g) \in \mathcal{H}(\mathbb{C}^n).$$

(2) Fix $g \in G$. Then $\mathrm{T}_{\phi_j}(\,\cdot\,; g) \to \mathrm{T}_\phi(\,\cdot\,; g)$ locally uniformly on $\mathbb{C}^n$. In order to see this, we note that as a Cauchy sequence, (ϕ_j) is in particular bounded in $(\mathscr{E}_0(G), d_0)$. With other words, $(\mathrm{F}_{\phi_j}) \subseteq \mathcal{H}(\mathbb{C})$ is locally bounded on $\mathbb{C}$. By Corollary 4.9 we can deduce that $(\mathrm{T}_{\phi_j}(\,\cdot\,; g))$ is locally bounded in $\mathcal{H}(\mathbb{C}^n)$. Since

$$\mathrm{T}_{\phi_j}(\underline{z}; g) = \sum_{k=0}^\infty \mathrm{T}_{k,\phi_j}(\underline{z}; g) \quad \text{and} \quad \mathrm{T}_\phi(\underline{z}; g) = \sum_{k=0}^\infty \mathrm{T}_{k,\phi}(\underline{z}; g),$$

we see from (1) and a standard Montel-type normal family argument that $\mathrm{T}_{\phi_j}(\,\cdot\,; g) \to \mathrm{T}_\phi(\,\cdot\,; g)$ locally uniformly on $\mathbb{C}^n$.

(3) In view of Corollary 4.6, we also have

$$\mathrm{T}_{\phi_j}(\underline{x}; g) = \phi_j\big(g \exp(x^\ell e_\ell)\big) \to \phi\big(g \exp(x^\ell e_\ell)\big)$$

locally uniformly for $(x^1, \ldots, x^n) \in \mathbb{R}^n$. It follows that

$$\mathrm{T}_\phi(\underline{x}; g) = \phi\big(g \exp(x^\ell e_\ell)\big)$$

for each $g \in G$ and $\underline{x} \in \mathbb{R}^n$. In particular, $\phi \in \mathscr{C}^\omega(G)$, since $\underline{x} \mapsto g \exp(x^\ell e_\ell)$ is an analytic diffeomorphism around $0 \in \mathbb{C}^n$.

(4) Next, we show $\mathrm{q}_{R,c}(\phi) < \infty$ for all $c \geq 0$. By Theorem 4.17, (iii) we have $\mathscr{L}_{X_\alpha} \phi_j \to \mathscr{L}_{X_\alpha} \phi$ for each k-tuple $\alpha \in \mathbb{N}_n^k$. Hence we have convergence of the k-th Taylor coefficients $c_k(\phi_j) \to c_k(\phi)$. Using the local boundedness of $(\mathrm{F}_{R,\phi_j}(\,\cdot\,))$ in $\mathcal{H}(\mathbb{C})$ it easily follows that (F_{R,ϕ_j}) converges in $\mathcal{H}(\mathbb{C})$ to $\mathrm{F}_{R,\phi}$. In particular, $\mathrm{F}_{R,\phi} \in \mathcal{H}(\mathbb{C})$, so $\mathrm{q}_{R,c}(\phi) = \mathrm{F}_{R,\phi}(c) < \infty$ for each $c \geq 0$.

(5) Finally, we prove $\phi_j \to \phi$ in $(\mathscr{E}_R(G), d_R)$. An equivalent statement is that $\mathrm{F}_{R,\phi_j - \phi}(c) \to 0$ for each $c \geq 0$. By

$$\big|\mathrm{F}_{R,\phi_j - \phi}(z)\big| \leq \mathrm{F}_{R,\phi_j}\left(|z|\right) + \mathrm{F}_{R,\phi}\left(|z|\right)$$

the sequence $(\mathrm{F}_{R,\phi_j-\phi}) \subseteq \mathcal{H}(\mathbb{C})$ is locally bounded. Hence the convergence of each of the k-Taylor coefficients $c_k(\phi_j-\phi)$ to 0 and a simple Montel-type argument imply $\mathrm{F}_{R,\phi_j-\phi} \to 0$ even locally uniformly on $\mathbb{C}$.

For (ii) it is convenient to identify $\mathscr{E}_R(G)$ with a Köthe space Λ_R in the following manner: write $\mathbb{N}_n^\infty = \bigcup_{k=0}^\infty \mathbb{N}_n^k$ and define a Köthe matrix by

$$a_{\alpha c} = (k!)^R \cdot \frac{c^k}{k!}$$

for $\alpha \in \mathbb{N}_n^k$, $k \in \mathbb{N}_0$ and $c \in \mathbb{N}$. Note that $a_{\alpha c} \leq a_{\alpha c'}$, whenever $c \leq c'$. We identify each $\phi \in \mathscr{E}_R(G)$ with the sequence (ϕ_α) defined by

$$\phi_\alpha = \big(\mathscr{L}_{X_\alpha}\phi\big)(\mathrm{e}).$$

Note that $c_k(\phi) = \frac{1}{k!}\sum_{\alpha\in\mathbb{N}_n^k} |\phi_\alpha|$ from (4.5). This yields an injective isometry

$$\mathscr{E}_R(G) \longrightarrow \Lambda_R,$$

as the net (ϕ_α) contains even *more* information than just the Lie-Taylor coefficients at e. We use the Grothendieck–Pietsch Theorem as it can be found in [44, Thm. 6.1.2] to check nuclearity of Λ_R. As moreover any subspace of a nuclear space is nuclear, see [44, Prop. 5.1.1] or [54, (50.3)], the claim will follow. Thus let $c \in \mathbb{N}$. We have to find a $c' \in \mathbb{N}$ such that the series

$$\sum_{\alpha\in\mathbb{N}_n^\infty} \frac{a_{\alpha c}}{a_{\alpha c'}} = \sum_{k=0}^\infty \sum_{\alpha\in\mathbb{N}_n^k} \frac{c^k k!}{c'^k k!} = \sum_{k=0}^\infty n^k \frac{c^k}{c'^k}$$

converges. Taking $c' = 2cn$ does the job. By positivity of all the numbers, all of our considerations are independent of the enumeration we choose for the index set $\mathbb{N}_n^\infty$, wherefore we do not make this choice at all. Thus the nuclearity of $\mathscr{E}_R(G)$ follows. By [54, Prop. 50.2] every nuclear Fréchet space is Montel, which gives (iii). Notably, this can also be shown directly by using the classical Montel Theorem for the Taylor majorants. Nuclear Montel spaces are separable by [29, Section 11.6, Thm. 2] and by [54, Cor. 36.9] every Montel space is reflexive. □

Rounding out this section, we discuss the simple but surprisingly far reaching example of the circle group $\mathbb{S}^1$, which is closely related to $\mathrm{GL}_1(\mathbb{C}) \cong \mathbb{C}^\times$ through the notion of universal complexification:

Example 4.19 (Circle group) Let $G = \mathbb{S}^1$, which is connected, but not simply connected. Its universal complexification $\big(\mathbb{S}^1_\mathbb{C}, \eta\big)$ in the sense of [28, Def. 15.1.2.] is given by

$$\mathbb{S}^1_\mathbb{C} = \mathbb{C}^\times = \mathbb{C}\setminus\{0\} \quad \text{and} \quad \eta\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1_\mathbb{C}, \quad \eta = \mathrm{id}_{\mathbb{C}^\times}\big|_{\mathbb{S}^1}, \tag{4.50}$$

i.e. we embed $\mathbb{S}^1$ as the unit circle into $\mathbb{C}\backslash\{0\}$. This induces a complex structure, which coincides with the one given by exponential charts due to the local existence of holomorphic logarithms. Given a morphism of complex Lie groups $\Phi\colon \mathbb{S}^1 \longrightarrow H$, we set

$$\Phi_{\mathbb{C}}\colon \mathbb{C}^{\times} \longrightarrow H, \quad \Phi_{\mathbb{C}}\big(r \cdot \mathrm{e}^{2\pi \mathrm{i} t}\big) = r \cdot \Phi(\mathrm{e}^{2\pi \mathrm{i} t}), \tag{4.51}$$

which is a holomorphic group morphism and clearly fulfils $\Phi = \Phi_{\mathbb{C}} \circ \eta$. Notably, the universal complexification $\mathbb{C}^{\times}$ is not compact, even though the circle group $\mathbb{S}^1$ was, but their fundamental groups are clearly isomorphic. Analogously, one obtains the universal complexification for higher tori as products of $\mathbb{C}^{\times}$. Notice that the group morphism given by the reflection on the unit circle

$$\sigma\colon \mathbb{C}^{\times} \longrightarrow \mathbb{C}^{\times}, \quad \sigma\big(r \cdot e^{2\pi \mathrm{i} t}\big) = \frac{1}{r} \cdot e^{2\pi \mathrm{i} t} \tag{4.52}$$

is antiholomorphic and fixes $\eta(\mathbb{S}^1) = \mathbb{S}^1$. Thus σ is the unique antiholomorphic complex conjugation on $\mathbb{S}^1_{\mathbb{C}}$, whose existence is guaranteed by [28, Thm. 15.1.4, (iv)].

Let $f\colon \mathbb{C}^{\times} \longrightarrow \mathbb{C}$ be holomorphic. As $\exp(\mathbb{C}) = \mathbb{C}\backslash\{0\} = \mathbb{C}^{\times}$, we can form the composition $f \circ \exp\colon \mathbb{C} \longrightarrow \mathbb{C}$, which is holomorphic as the composition of holomorphic maps and thus entire in the classical sense. By commutativity of $\mathbb{S}^1_{\mathbb{C}}$, the Lie-Taylor majorant F_f is entire if and only if the Taylor series of $f \circ \exp$ converges absolutely. Thus holomorphic functions on $\mathbb{C}^{\times}$ are automatically entire in our sense. Consequently, we have shown

$$\mathscr{E}_0\big(\mathbb{S}^1_{\mathbb{C}}\big) \cap \mathcal{H}(\mathbb{S}^1_{\mathbb{C}}) = \mathcal{H}\big(\mathbb{S}^1_{\mathbb{C}}\big) \tag{4.53}$$

as locally convex algebras in view of Theorem 4.17, (v): having a convergent Taylor series gives uniform estimates for the function values at once. Moreover, this restricts nicely to the real situation, i.e. we have

$$\mathscr{E}_0\big(\mathbb{S}^1\big) = \eta^* \mathscr{E}_0\big(\mathbb{S}^1_{\mathbb{C}}\big) = \eta^* \mathcal{H}\big(\mathbb{S}^1_{\mathbb{C}}\big). \tag{4.54}$$

This can be seen as follows: given $\phi \in \mathscr{E}_0(\mathbb{S}^1)$, the composition $\phi \circ \exp\colon \mathbb{C} \longrightarrow \mathbb{C}$ is entire and 2π-periodic (as the Lie exponential is $\exp(z) = \mathrm{e}^{\mathrm{i}z}$ here). Applying [52, Thm. 3.10.1] to $a = b = n \in \mathbb{N}$ with the obvious rescaling yields a $\mathcal{H}$-convergent Laurent expansion

$$\phi\big(\mathrm{e}^{2\pi \mathrm{i} z}\big) = \phi \circ \exp\Big|_{2\pi z} = \sum_{k=-\infty}^{\infty} a_k \mathrm{e}^{2\pi \mathrm{i} k z} \tag{4.55}$$

for $z \in \mathbb{C}$. This is the holomorphic extension of ϕ to $\mathbb{C}^{\times}$ we were looking for. Rephrasing the convergence of (4.55), we find two entire functions $F, G \in \mathcal{H}(\mathbb{C})$ with

$$\phi(z) = F(z) + G(\tfrac{1}{z}) = F(z) + G(\overline{z}) \tag{4.56}$$

for all $z \in \mathbb{S}^1$. If now $\phi \in \mathscr{E}_R(G)$ for $R > 0$, we still get the Laurent expansion (4.55). It is even convergent in the $\mathcal{H}_{1/R}$-topology, see again Proposition 4.15. Thus (4.56) yields

$$\mathscr{E}_R(\mathbb{S}^1) = \eta^* \mathcal{H}_{1/R}\big(\mathbb{C}^\times\big) \tag{4.57}$$

as locally convex algebras for $R > 0$.

This example suggests to use the universal complexification of G also in general to understand the algebra $\mathscr{E}_R(G)$.

Remark 4.20 (Negative R) As already suggested in Proposition 4.15, (iii), one can in principle consider arbitrary $R \in \mathbb{R}$ by using the series of seminorms in (4.33). However, the approach we have taken to deal with $R \geq 0$ *ceases* to work here: the Lie-Taylor majorants need no longer be entire. This already happens in the abelian case $G = (\mathbb{R}, +) = \mathfrak{g}$. Consider the geometric series

$$g : \mathbb{R} \longrightarrow \mathbb{C}, \quad g(x) = \frac{1}{1 + \mathrm{i}x}, \tag{4.58}$$

whose Taylor series converges if and only if $|x| < 1$. The underlying problem here is of course the singularity at $x = \mathrm{i}$, which is hidden in the universal complexification $\mathbb{C}$ of $\mathbb{R}$. Nevertheless, we have

$$\mathrm{q}_{R,c}(g) = \sum_{k=0}^{\infty} k!^R \cdot c^k < \infty \tag{4.59}$$

for all $c > 0$ and $R < 0$. Consequently, we know at lot less about the nature of such functions: in fact, all of the upcoming results simply work for *arbitrary* $R \in \mathbb{R}$ and are somewhat algebraic in nature. Nevertheless, we shall admit $R < 0$ and consider the R-entire functions $\mathscr{E}_R(G)$ also in this extended sense in the sequel.

By the already mentioned Proposition 4.15, (iii), the resulting additional vector spaces are locally multiplicatively convex. Also the continuous inclusions (4.30), the inversion invariance and the continuity of Lie derivatives from Theorem 4.17, (i) and (iii) remain correct. Indeed, we have the obvious inequalities

$$\mathrm{q}_{R,c} \leq \mathrm{q}_{S,c} \quad \text{and} \quad \mathrm{q}_{R,c} \leq \mathrm{q}_{R,d}, \tag{4.60}$$

whenever $-\infty < R \leq S < \infty$ and $0 \leq c \leq d$. First countability and the Hausdorff property are also clearly still intact. Beyond these trivial observations, our methods break down immediately. For instance, we can no longer infer anything from (4.21) if the right hand side is infinite.

4.3 Representative functions

At this point of our discussion it is not at all clear, whether there are examples of entire functions on a given Lie group beyond the constant ones. To remedy this, we first note the following compatibility of the entire functions with pullbacks by group morphisms:

Proposition 4.21 *Let G and H be Lie groups and $R \in \mathbb{R}$. Let $\Phi\colon G \longrightarrow H$ be a Lie group morphism. The pullback with Φ is a morphism of locally convex algebras*

$$\Phi^*\colon \mathscr{E}_R(H) \longrightarrow \mathscr{E}_R(G). \tag{4.61}$$

More precisely, we have the estimate

$$q_{R,c}\big(\Phi^*\phi\big) \leq q_{R,cnD}(\phi) \tag{4.62}$$

for $\phi \in \mathscr{E}_R(G)$ and $c \geq 0$, where D is the matrix supnorm of the matrix representation of the tangent map $T_e\Phi\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ in the bases used for the construction of the seminorms of $\mathscr{E}_R(G)$ and $\mathscr{E}_R(H)$, respectively.

Proof Note that the pullback Φ^* with Φ is an algebra morphism

$$\Phi^*\colon \mathscr{C}^\omega(H) \longrightarrow \mathscr{C}^\omega(G),$$

i.e. its restriction to $\mathscr{E}_R(H)$ is an algebra morphism with values in $\mathscr{C}^\omega(G)$. Consequently, this is just a rephrasing of the chain rule Proposition 4.3, (ii). Note that the additional weight from $R \neq 0$ does not interfere with the argument, as it shows up on both sides of the estimate. □

This way, we obtain a contravariant functor $\mathscr{E}_R(\,\cdot\,)$ from the category of connected real Lie groups to the category of commutative Fréchet algebras. Thus our construction of the entire functions fits nicely into the otherwise functorial framework of the R'-topologies and the deformation quantization itself.

Generating examples of entire functions on one group thus lets us transport them to other groups in a continuous way. Once again, this does not guarantee the existence of interesting entire functions, yet. The idea is now that group representations on finite dimensional vector spaces are particularly nice group morphisms, to which we can associate special functions on G: the *representative functions* or *matrix coefficients*, see e.g. [15, Sect. 4.3]. Recall that a choice of generators is of the form

$$\pi_{ij}\colon G \longrightarrow \mathbb{C}, \quad \pi_{ij}(g) = \pi(g)_{ij}, \tag{4.63}$$

where π ranges over all continuous (and thus automatically analytic) finite dimensional representations of G. We are going to show by direct estimation that every representative function is R-entire for $R < 1$. To this end, we recall the following well-known Lemma:

Lemma 4.22 *Let G be a Lie group and $\phi \in \mathscr{C}^\omega(G)$ be a representative function and write $\langle\phi\rangle = \{\ell_g^*\phi \in \mathscr{C}^\omega(G) \mid g \in G\}$ for the orbit of ϕ under the left action of G on itself.*

(i) *The orbit* $\langle\phi\rangle$ *is finite dimensional and coincides with the orbit of* ϕ *under right translation, i.e. we have*

$$\langle\phi\rangle = \left\{ r_g^*\phi \in \mathscr{C}^\omega(G) \;\middle|\; g \in G \right\}. \tag{4.64}$$

(ii) *Let* $\xi \in \mathfrak{g}$. *The Lie derivative* $\mathscr{L}_{X_\xi}$ *is contained in the orbit* $\langle\phi\rangle$ *of* ϕ. *The same is true for right invariant vector fields.*

Theorem 4.23 (Representative functions) *Let* G *be a Lie group,* $R < 1$ *and* $\phi \in \mathscr{C}^\omega(G)$ *be a representative function. Then* $\phi \in \mathscr{E}_R(G)$. *More precisely, choosing an auxiliary norm* $\|\cdot\|$ *on* $\langle\phi\rangle$, *we have the estimate*

$$q_{R,c}(\phi) \le \Delta \cdot \|\phi\| \sum_{k=0}^{\infty} k!^{R-1}\,(c\Psi n)^k \tag{4.65}$$

for $c \ge 0$, *where* Ψ *is the maximum of the operator seminorms*

$$\Psi = \max_{i=1,\ldots,n}\ \max_{\psi\in\langle\phi\rangle, |\psi(e)|\le 1} \left|\left(\mathscr{L}_{X_i}\psi\right)(e)\right| \tag{4.66}$$

of the Lie derivatives in direction of left invariant vector fields on the orbit $\langle\phi\rangle$ *of* ϕ *and* Δ *is the operator norm of the Dirac functional at the group unit, i.e.*

$$\Delta = \max_{\psi\in B_1(0)^{cl}} |\psi(e)|\,. \tag{4.67}$$

Proof We equip the, by Lemma 4.22, (i), finite-dimensional orbit $\langle\phi\rangle$ with some auxiliary norm $\|\cdot\|$. Part (ii) of the same lemma implies that the "left invariant derivatives"

$$\mathscr{L}_{X_i} : \langle\phi\rangle \longrightarrow \langle\phi\rangle$$

are well-defined and thus continuous as linear maps on a finite-dimensional topological vector space. Consequently, the maximum of the operator norms

$$\Psi = \max_{i=1,\ldots,n}\ \max_{\psi\in \mathrm{B}_1(0)^{\mathrm{cl}}} \|\mathscr{L}_{X_i}\psi\|$$

is finite. For the very same reasons, the Dirac functional δ_{e} at the group unit is continuous and thus its operator norm

$$\Delta = \max_{\psi\in \mathrm{B}_1(0)^{\mathrm{cl}}} |\psi(\mathrm{e})|$$

is finite, as well. Putting both observations together, we obtain

$$\mathrm{q}_\alpha(\phi) = \left|\delta_{\mathrm{e}} \circ \mathscr{L}_{X_{\alpha_k}} \cdots \mathscr{L}_{X_{\alpha_1}}\phi\right| \le \Delta \cdot \Psi^k \cdot \|\phi\|$$

for any k-tuple $\alpha \in \{1, \ldots, n\}^k$. Plugging this into the full seminorms yields finally

$$\begin{aligned} \mathrm{q}_{R,c}(\phi) &= \sum_{k=0}^{\infty} k!^{R-1}\, c^k \sum_{\alpha \in \{1,\ldots,n\}^k} \mathrm{q}_\alpha(\phi) \\ &\leq \sum_{k=0}^{\infty} k!^{R-1}\, c^k \sum_{\alpha \in \{1,\ldots,n\}^k} \Psi^k \cdot \Delta \cdot \|\phi\| \\ &= \Delta \cdot \|\phi\| \sum_{k=0}^{\infty} k!^{R-1}\, (c\Psi n)^k, \end{aligned}$$

which converges for all $c \geq 0$ iff $R < 1$. □

Notably, choosing a basis of $\langle \phi \rangle$ allows to proceed in the spirit of Proposition 4.21 to derive a similar estimate based on the matrix supnorm of the Lie derivatives in direction of the left invariant vector fields instead. This is fairly cumbersome in terms of bookkeeping due to numerous indices, wherefore we chose the more abstract approach. For $R = 0$, we recognize the series in the estimate (4.65) as $\exp(c\Psi n)$. Note that the condition $R < 1$ is sharp:

Example 4.24 (Exponential representation) Consider the representation exp of the abelian Lie group $(\mathbb{R}, +)$ on $(\mathbb{R}^\times, \cdot)$. Indeed, we have

$$\ell_x^* \exp\Big|_t = \exp(x + t) = \exp(x) \cdot \exp\Big|_t \tag{4.68}$$

for $x, t \in \mathbb{R}$, confirming that exp is a representative function with one dimensional orbit. This matches with Lemma 4.22, (ii), as $\exp' = \exp$. This also implies

$$\mathrm{q}_{R,c}(\exp) = \sum_{k=0}^{\infty} k!^{R-1}\, c^k \cdot 1 \tag{4.69}$$

for $R \in \mathbb{R}$ and $c \geq 0$. The series in (4.69) converges for all $c \geq 0$ iff $R < 1$.

For a compact Lie group, we now know that the span of the matrix coefficients is already *dense* in $\mathscr{C}(G)$. This is the classical Peter-Weyl Theorem, see [15, Thm. (4.6.1)]. Thus the same is true for $\mathscr{E}_R(G)$, whenever $R < 1$. Here we also use that pointwise complex conjugation is an isometry of $\mathscr{E}_R(G)$. In the language of representation theory, see again the lengthy Remark 4.16, this means that the subspace of entire vectors is dense in the space of continuous vectors for either of the representations (4.1) or (4.43).

5 The R, R'-topologies on the observable algebra

Having studied both the R-entire functions $\mathscr{E}_R(G)$ and the symmetric algebra $\mathrm{S}^\bullet_{R'}(\mathfrak{g})$ with the R'-topology in isolation, we now projectively tensorize them together to the observable algebra of our strict deformation:

Definition 5.1 (*(R, R')-Topologies*) Let G be a Lie group and $R, R' \in \mathbb{R}$. We equip the tensor product

$$\mathrm{Pol}^\bullet_{R,R'}(T^*G) = \mathscr{E}_R(G) \otimes \mathrm{S}^\bullet_{R'}(\mathfrak{g}) \tag{5.1}$$

with the projective tensor product topology and call it the (R, R')-topology.

Due to $\mathscr{E}_R(G) \subseteq \mathscr{C}^\infty(G)$ and the decomposition (2.7) we have the inclusion $\mathrm{Pol}^\bullet_{R,R'}(T^*G) \subseteq \mathrm{Pol}(T^*G)$, explaining the notation.

As projective tensor products moreover inherit most of the desirable properties of their factors, we immediately obtain the following statements for $\mathrm{Pol}^\bullet_{R,R'}(T^*G)$ and the commutative pointwise multiplication:

Proposition 5.2 *Let G be a connected Lie group and $R, R' \in \mathbb{R}$.*

(i) *The projective tensor product topology turns $\mathrm{Pol}^\bullet_{R,R'}(T^*G)$ into a unital Hausdorff and first countable locally convex algebra.*
(ii) *Complex conjugation is a continuous involution on $\mathrm{Pol}^\bullet_{R,R'}(T^*G)$.*
(iii) *Let $R, R' \leq 0$. Then $\mathrm{Pol}^\bullet_{R,R'}(T^*G)$ is locally multiplicatively convex.*
(iv) *The completion $\widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*G)$ of $\mathrm{Pol}^\bullet_{R,R'}(T^*G)$ contains the completion of each factor and they are dense, i.e.*

$$\hat{\mathscr{E}}_R(G) \otimes \hat{\mathrm{S}}^\bullet_{R'}(\mathfrak{g}) \subseteq \widehat{\mathrm{Pol}}^\bullet_{R,R'}(G). \tag{5.2}$$

(v) *The completion $\widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*G)$ is a commutative Fréchet *-algebra.*
(vi) *Let $R \leq S$ and $R' \leq S'$. We have the continuous inclusions of locally convex algebras*

$$\mathrm{Pol}^\bullet_{S,R'}(T^*G) \subseteq \mathrm{Pol}^\bullet_{R,R'}(T^*G) \quad \text{and} \quad \mathrm{Pol}^\bullet_{R,S'}(T^*G) \subseteq \mathrm{Pol}^\bullet_{R,R'}(T^*G). \tag{5.3}$$

(vii) *Let $R, R' \geq 0$. The locally convex algebras $\mathrm{Pol}^\bullet_{R,R'}(T^*G)$ and $\widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*G)$ are nuclear.*
(viii) *Let $R, R' \geq 0$. The locally convex algebra $\widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*G)$ is Montel, reflexive and separable.*

Proof All statements are standard results about projective tensor products and have nothing to do with our particular example. For detailed treatments, see e.g. the textbooks [54, Chap. 43, 50], [31, §41] and [29, Chap. 15]. The continuity of the complex conjugation is clear as all our seminorms are invariant under complex conjugation. □

As a first consequence of the construction, we note that restricting to momentum zero, which geometrically is the map ι^*, and prolonging constantly in momentum direction, which is π^*, provide continuous maps:

Proposition 5.3 *Let $R, R' \geq 0$.*

(i) *The restriction to the zero section yields a continuous map*

$$\iota^* \colon \widehat{\mathrm{Pol}}^{\bullet}_{R,R'}(T^*G) \longrightarrow \mathscr{E}_R(G). \tag{5.4}$$

(ii) *The pullback*

$$\pi^* \colon \mathscr{E}_R(G) \longrightarrow \mathrm{Pol}^{\bullet}_{R,R'}(T^*G) \subseteq \widehat{\mathrm{Pol}}^{\bullet}_{R,R'}(T^*G) \tag{5.5}$$

is continuous.

Proof From the above factorization we have $\iota^* = \mathrm{id}_{\mathscr{E}_R(G)} \otimes \delta_0$ on the dense subalgebra $\mathrm{Pol}^{\bullet}_{R,R'}(T^*G)$ with the δ-functional

$$\delta_0 \colon \mathrm{S}^{\bullet}_{R'}(\mathfrak{g}) \longrightarrow \mathbb{C}.$$

This is a continuous functional for $R' \geq 0$ according to Proposition 3.2, (ii). The functoriality of the projective tensor product implies the continuity of ι^*, which then extends to the completion. The pullback is even simpler, we have

$$\pi^* \phi = \phi \otimes 1,$$

which is again continuous by general properties of the projective tensor product. □

As tensor products of continuous linear maps are continuous in the projective tensor product topology, we moreover obtain the following continuity of evaluations and symmetries:

Proposition 5.4 *Let $R, R' \in \mathbb{R}$.*

(i) *Assume $R, R' \geq 0$ and let $g \in G$, $\alpha \in \mathbb{N}_n^k$ and $\eta \in \mathfrak{g}^*$. The tensor product*

$$\delta_{g,\alpha} \otimes \delta_\eta \colon \mathrm{Pol}^{\bullet}_{R,R'}(T^*G) \longrightarrow \mathbb{C} \tag{5.6}$$

of the evaluation functionals (3.9) and (4.48) is continuous.

(ii) *Let $\Phi \colon G \longrightarrow H$ be a covering map of Lie groups. The pullback with the point transformation*

$$(T_*\Phi)^* \colon \mathrm{Pol}^{\bullet}_{R,R'}(T^*H) \longrightarrow \mathrm{Pol}^{\bullet}_{R,R'}(T^*G) \tag{5.7}$$

is well-defined and continuous.

Proof The first part is clear. For the second, recall that the canonical isomorphism (A.1) fulfils

$$(T_*\Phi)^* \mathcal{J}(X) = \mathcal{J}\big(\Phi^* X\big)$$

for $X \in \Gamma^\infty(\mathrm{S}^\bullet(T_{\mathbb{C}}H))$. In our polynomial factorization from (2.7), the point transformation is thus given by

$$\Phi^* \otimes \big((T_{\mathrm{e}}\Phi)^{-1} \otimes \cdots \otimes (T_{\mathrm{e}}\Phi)^{-1}\big).$$

With this formula, our claims are clear in view of Proposition 4.21 and the basis independence of the R'-topology. Note that the invertibility of the tangent map $T_{\mathrm{e}}\Phi$ is equivalent to the group morphism Φ being a covering map. $\square$

Recall that we may endow the cotangent bundle T^*G with a natural Lie group structure by choosing a trivialization. More precisely, this allows for the semidirect product structure $T^*G = G \ltimes_{\mathrm{Ad}^*} \mathfrak{g}^*$ coming from the coadjoint representation. The natural question is thus whether this group structure preserves our observable algebra $\widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*G) \subseteq \mathscr{C}^\infty(T^*G)$. As before, we denote the left and right multiplications with $(g, \eta) \in T^*G$ by $\ell_{(g,\eta)}$ and $r_{(g,\eta)}$, respectively.

Proposition 5.5 *Let $R, R' \geq 0$.*

(i) *The pullbacks with left multiplications on T^*G yield representations*

$$\ell^* \colon T^*G \longrightarrow L\big(\widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*G)\big) \tag{5.8}$$

by continuous linear maps and $\widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*G)$ consists of corresponding entire vectors.

(ii) *Assume furthermore $R < 1$. The pullbacks with right multiplications on T^*G yield representations*

$$r^* \colon T^*G \longrightarrow L\big(\widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*G)\big) \tag{5.9}$$

by continuous linear maps and $\widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*G)$ consists of corresponding entire vectors.

Proof Let $(g, \eta), (h, \chi) \in G \times \mathfrak{g}^*, \phi \in \mathscr{E}_R(G)$ and $\xi_1, \ldots, \xi_k \in \mathfrak{g}$. We note the explicit formulae

$$\begin{aligned}
&\ell^*_{(g,\eta)}\big(\phi \otimes \xi_1 \vee \cdots \vee \xi_k\big)\Big|_{(h,\chi)} \\
&\quad = \ell^*_g\phi \otimes \big(\eta(\xi_1)\cdots\eta(\xi_k)\cdot 1 + \mathrm{Ad}_{g^{-1}}\xi_1 \vee \cdots \vee \mathrm{Ad}_{g^{-1}}\xi_k\big)\Big|_{(h,\chi)}, \\
&r^*_{(h,\chi)}\big(\phi \otimes \xi_1 \vee \cdots \vee \xi_k\big)\Big|_{(g,\eta)} \\
&\quad = r^*_h\phi \otimes \xi_1 \vee \cdots \vee \xi_k\Big|_{(g,\eta)} + r^*_h\phi \cdot \chi\big(\mathrm{Ad}_{\mathrm{inv}(\,\cdot\,)}\xi_1\big)\cdots\chi\big(\mathrm{Ad}_{\mathrm{inv}(\,\cdot\,)}\xi_k\big) \otimes 1\Big|_{(g,\eta)}
\end{aligned}$$

for the pullbacks. From here, the continuity estimates can be handled by the same techniques we have employed throughout the paper, see in particular Theorem 4.17 and the upcoming Lemma 6.2. Notice that the maps

$$\Phi_\xi \colon G \ni g \mapsto \chi\big(\mathrm{Ad}_g\xi\big) \in \mathbb{C}$$

are nothing but representative functions, which we have studied in Theorem 4.23. This explains the additional requirement of $R < 1$ in (ii). □

After these abstract considerations, we derive a more explicit description of $\widehat{\mathrm{Pol}}^{\bullet}_{R,R'}(T^*G)$. A first observation is that for finite dimensional vector spaces V, we have an analogue of (A.1), implementing the isomorphism $\mathrm{S}^\bullet(V^*) \cong \mathrm{Pol}^\bullet(V)$ of graded vector spaces. We shall identify both without further comment in the sequel. This moreover gives $\mathrm{p}_{R',c} = \mathrm{q}_{R',c}$ for $R' \in \mathbb{R}$ and $c \geq 0$. Here the slightly different prefactors match, as differentiation produces another factorial. In particular, the subspace topology induced by $\mathrm{S}^\bullet_{R'}(V) \subseteq \mathscr{E}_R(V^*)$ is the $\mathrm{S}^\bullet_{R'}$-topology again. The idea is now that $\mathscr{E}_R(V^*)$ is the completion of $\mathrm{S}^\bullet_{R'}(V)$:

Lemma 5.6 *Let V be a finite dimensional vector space over $\mathbb{C}$ and $R' \geq 0$. Then we have*

$$\hat{\mathrm{S}}^\bullet_{R'}(V) \cong \mathscr{E}_{R'}(V^*) \tag{5.10}$$

with embedding given by the isomorphism $\mathfrak{J}$. For $R' < 0$ we have the inclusion $\mathscr{E}_{R'}(V^*) \subseteq \hat{\mathrm{S}}^\bullet_{R'}(V)$.

Proof Truncating the Taylor series of an entire function yields the desired polynomial approximation by elements of $\mathrm{S}^\bullet_{R'}(V)$ at once. Notably, this still works for negative R', but the completeness of $\mathscr{E}_{R'}(V^*)$ relies on $R' \geq 0$, see again Theorem 4.18, (i). □

Corollary 5.7 *Let $R' \geq 0$ and $R \in \mathbb{R}$.*

(i) *Every function $\chi \in \widehat{\mathrm{Pol}}^{\bullet}_{R,R'}(T^*G)$ is smooth.*
(ii) *Let $R \geq 0$. The (R, R')-topology is finer than the $\mathscr{C}^\infty$-topology.*

Proof We invoke the triviality of the bundle $T^*G \cong G \times \mathfrak{g}$ once more: every derivative on $\mathscr{C}^\infty(T^*G)$ factorizes into derivatives on G and $\mathfrak{g}$, which commute with each other. Using this, (ii) is immediate from Lemma 5.6 and Theorem 4.18, (v). The case $R < 0$ in (i) is trivial, as $\mathscr{E}_R(G) \subseteq \mathscr{C}^\omega(G)$ by its very definition. □

Using the vector space structure of $\mathfrak{g}$, we arrive now at the following explicit description of the completion $\widehat{\mathrm{Pol}}^{\bullet}_{R,R'}(T^*G)$:

Proposition 5.8 *Let $R \in \mathbb{R}$, $R' \geq 0$ and $\chi \in \widehat{\mathrm{Pol}}^{\bullet}_{R,R'}(T^*G)$, viewed as an element of $\mathscr{C}^\infty(T^*G)$. Then there is a unique absolutely convergent decomposition*

$$\chi(g, \eta) = \sum_{k=0}^{\infty} \sum_{\alpha \in \mathbb{N}_n^k} c_\alpha(g) \cdot \eta_1^{\alpha_1} \cdots \eta_n^{\alpha_n} \tag{5.11}$$

for $g \in G$ and $\eta \in \mathfrak{g}^*$, where each $c_\alpha \colon G \longrightarrow \mathbb{C}$ is R-entire and c_α is independent of the ordering of the entries of $\alpha \in \mathbb{N}_n^k$. Moreover, for every $g \in G$ and as a function of η, (5.11) is an element of $\mathscr{E}_{R'}(\mathfrak{g})$.

Proof By Corollary 5.7, we know that $\chi \in \mathscr{C}^\infty(T^*G)$. Invoking [54, Thm. 45.1], we find a summable sequence $(\phi_k) \subseteq \mathscr{E}_R(G)$ and another (not necessarily summable) sequence $(\psi_k) \subseteq \mathscr{E}_{R'}(\mathfrak{g})$ s.t.

$$\chi = \sum_{k=1}^{\infty} \phi_k \otimes \psi_k,$$

where the series converges absolutely in the projective tensor product topology. Given a $g \in G$ we use the product structure $T^*G \cong G \times \mathfrak{g}^*$ to define

$$c_\alpha(g) = \mathscr{L}_{X_\alpha^{\mathfrak{g}^*}} \chi(g, \cdot) = \sum_{k=1}^{\infty} \phi_k(g) \otimes \mathscr{L}_{X_\alpha^{\mathfrak{g}^*}} \psi_k$$

as the α-th Lie-Taylor coefficient of $\chi(g, \cdot)\colon \mathfrak{g}^* \longrightarrow \mathbb{C}$. Note that this way, the $c_\alpha(g)$ do indeed have the claimed symmetry property, as the Lie derivatives on $\mathfrak{g}$ are just partial derivatives corresponding to the basis we chose. By summability of (ϕ_k), we moreover have $c_\alpha \in \mathscr{E}_R(G)$. Interchanging the series, it is straightforward to check that the right hand side of (5.11) indeed converges absolutely to the function χ we started with. This also gives the remaining statement, as each $\psi_k \in \mathscr{E}_{R'}(\mathfrak{g})$. $\square$

Corollary 5.9 *Let $R \in \mathbb{R}$ and $R' \geq 0$. We have the inclusion $\widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*G) \subseteq \mathscr{C}^\omega(T^*G)$ of algebras.*

6 Continuity results

We begin our considerations on continuity by restating [17, Prop. 3.2, (ii), and Prop. 3.6] on the Lie algebra star product $\star_{\mathfrak{g}}$. By (2.15) this star product is the restriction of $\star_{\mathrm{std}}$ to the second tensor factors, i.e. polynomials in the momenta only. For convenience, we already specialize to the situation we are interested in, namely finite-dimensional Lie algebras instead of general asymptotic estimate algebras [17]:

Proposition 6.1 *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $R' \geq 1$. The Lie algebra star product*

$$\star_{\mathfrak{g}}\colon \mathrm{S}^\bullet_{R'}(\mathfrak{g}) \times \mathrm{S}^\bullet_{R'}(\mathfrak{g}) \longrightarrow \mathrm{S}^\bullet_{R'}(\mathfrak{g}) \tag{6.1}$$

is well-defined and continuous. More precisely, we have the estimate

$$p_{R',c'}\big(\xi \star_{\mathfrak{g}} \eta\big) \leq p_{R',\tilde{c}'}(\xi) \cdot p_{R',\tilde{c}'}(\eta) \tag{6.2}$$

for $\xi, \eta \in \mathrm{S}^\bullet_{R'}(\mathfrak{g})$, $c' \geq 1$ and $\tilde{c}' = 32(\hbar + 1)c'$. In particular, $\tilde{c}'$ is a continuous function of $\hbar$. Moreover, the map

$$\mathbb{C} \ni \hbar \mapsto \xi \star_{\mathfrak{g}} \eta \in \mathrm{S}^{\bullet}_{R'}(\mathfrak{g}) \tag{6.3}$$

is entire for all $\xi, \eta \in \mathrm{S}^{\bullet}_{R'}(\mathfrak{g})$.

Another look at (2.15) reveals that there is only one other interesting type of product to consider: a polynomial in $\mathrm{S}(\mathfrak{g})$ on the left and a function on G on the right. The crucial idea is that the mixed product corresponds to a dual pairing in the spirit of [24, 25], where only the sum $R + R'$ of the parameters in $\mathrm{Pol}^{\bullet}_{R,R'}(T^*G)$ matters for continuity, but not their individual values. Note, however, that we do more than just pair: instead of applying the differential operator to the function, we commute the differential operators with the left multiplication, yielding numerous additional contributions. Nevertheless, this yields the following continuity result:

Lemma 6.2 *Let G be a Lie group and $R, R' \in \mathbb{R}$ with $R + R' \geq 1$. The restricted standard-ordered star product*

$$\star_{\mathrm{std}} : \big(\mathbb{1} \otimes \mathrm{S}^{\bullet}_{R'}(\mathfrak{g})\big) \times \big(\mathscr{E}_R(G) \otimes 1\big) \longrightarrow \mathrm{Pol}^{\bullet}_{R,R'}(T^*G) \tag{6.4}$$

is well-defined and continuous with respect to the R, R'-topology. More precisely, in each symmetric degree and for $c, c' \geq 1$ we have the estimate

$$\big(q_{R,c} \otimes p_{R',c'}\big)\big((\mathbb{1} \otimes \xi) \star_{\mathrm{std}} (\phi \otimes 1)\big) \leq 2 \cdot p_{R',d'}(\xi) \cdot q_{R,2c}(\phi) \tag{6.5}$$

for $\phi \in \mathscr{E}_R(G)$, $\xi \in \hat{\mathrm{S}}^{\bullet}_{R'}(\mathfrak{g})$ and $d' = \max\{2\hbar, c'\}$, which depends continuously on $\hbar$. In particular, the map

$$\mathbb{C} \ni \hbar \mapsto (\mathbb{1} \otimes \xi) \star_{\mathrm{std}} (\phi \otimes 1) \in \widehat{\mathrm{Pol}}^{\bullet}_{R,R'}(T^*G) \tag{6.6}$$

is entire for all $\phi \in \mathscr{E}_R(G)$ and $\xi \in \hat{\mathrm{S}}^{\bullet}_{R'}(\mathfrak{g})$.

Proof Let $\phi \in \mathscr{E}_R(G)$, $k \in \mathbb{N}_0$, $\big(e_1, \ldots, e_n\big)$ be a basis of $\mathfrak{g}$ corresponding to the ℓ^1-norm p we chose and $1 \leq i_1, \ldots, i_k \leq n$. Recall that by Proposition 2.4, (iii), we have the explicit formula

$$\begin{aligned}
&\big(\mathbb{1} \otimes e_{i_1} \vee \cdots \vee e_{i_k}\big) \star_{\mathrm{std}} \big(\phi \otimes 1\big) \\
&\quad = \sum_{p=0}^{k} \left(\frac{\hbar}{\mathrm{i}}\right)^{p} \frac{1}{p!\,(k-p)!} \sum_{\sigma \in S_k} \mathscr{L}_{X_{i_{\sigma(p)}}} \cdots \mathscr{L}_{X_{i_{\sigma(1)}}} \phi \otimes e_{\sigma(p+1)} \vee \cdots \vee e_{\sigma(k)}.
\end{aligned}$$

First note

$$\begin{aligned}
\sum_{\sigma \in S_k} \mathrm{q}_{\alpha}\Big(\mathscr{L}_{X_{i_{\sigma(p)}}} \cdots \mathscr{L}_{X_{i_{\sigma(1)}}} \phi\Big) &= \sum_{\sigma \in S_k} \Big|\big(\mathscr{L}_{X_{\alpha_\ell}} \cdots \mathscr{L}_{X_{\alpha_1}} \mathscr{L}_{X_{i_{\sigma(p)}}} \cdots \mathscr{L}_{X_{i_{\sigma(1)}}} \phi\big)(\mathrm{e})\Big| \\
&= \sum_{\sigma \in S_k} \mathrm{q}_{(i_{\sigma(1)}, \ldots, i_{\sigma(p)}, \alpha_1, \ldots, \alpha_\ell)}(\phi)
\end{aligned}$$

for $\alpha \in \{1, \ldots, n\}^{\ell}$. In the sequel we write $(i_{\sigma(1)}, \ldots, i_{\sigma(p)}, \alpha)$ for $(i_{\sigma(1)}, \ldots, i_{\sigma(p)}, \alpha_1, \ldots, \alpha_\ell)$ by slight abuse of notation. Note that we sum over S_k, but only use the first p values of the permutation. For the other factor we use [12, Lem. A.1], which essentially says that projective tensor products of ℓ^1-norms yield ℓ^1-norms associated to the product bases. We write p^k for the k-th projective tensor power of the ℓ^1-norm p. This gives

$$\mathrm{p}^{k-p}\big(e_{\sigma(p+1)} \vee \cdots \vee e_{\sigma(k)}\big) = 1 = \mathrm{p}^k\big(e_{i_1} \vee \cdots \vee e_{i_k}\big).$$

Here it is important that we use the ℓ^1-norm p with respect to the above basis, otherwise we would only get estimates instead of equalities. This implies

$$\begin{aligned}\mathrm{p}_{R',c'}\big(e_{\sigma(p+1)} \vee \cdots \vee e_{\sigma(k)}\big) &= (k-p)!^{R'}\, c'^{k-p} \\ &= \left(\frac{(k-p)!}{k!}\right)^{R'} c'^{-p}\, \mathrm{p}_{R',c'}\big(e_{i_1} \vee \cdots \vee e_{i_k}\big).\end{aligned}$$

Let now $R, R' \leq 1$ such that $R + R' \geq 1$ and $c, c' \geq 1$. Due to

$$\mathrm{q}_{R',c'} \leq \mathrm{q}_{R',\max\{c',2\hbar\}}$$

we may assume $c' \geq 2\hbar$ without loss of generality. Otherwise we just estimate c' by a yet another polynomial weight $\tilde{c} \geq 2\hbar$ in the very first step. With this in mind, we obtain

$$\begin{aligned}&\Big(\mathrm{q}_{R,c} \otimes \mathrm{p}_{R',c'}\Big)\big((\mathbb{1} \otimes e_{i_1} \vee \cdots \vee e_{i_k})\star_{\mathrm{std}} (\phi \otimes 1)\big) \\ &\quad\leq \sum_{p=0}^{k} \frac{\hbar^p}{p!\,(k-p)!} \sum_{\sigma \in S_k} \mathrm{q}_{R,c}\Big(\mathscr{L}_{X_{i_{\sigma(p)}}} \cdots \mathscr{L}_{X_{i_{\sigma(1)}}} \phi\Big) \\ &\qquad \cdot \mathrm{p}_{R',c'}\big(e_{\sigma(p+1)} \vee \cdots \vee e_{\sigma(k)}\big) \\ &\quad= \sum_{p=0}^{k} \frac{\hbar^p}{p!\,(k-p)!} \sum_{\sigma \in S_k} \sum_{\ell=0}^{\infty} \ell!^{R-1}\, c^{\ell} \sum_{\alpha \in \{1,\ldots,n\}^{\ell}} \mathrm{q}_{\alpha}\Big(\mathscr{L}_{X_{i_{\sigma(p)}}} \cdots \mathscr{L}_{X_{i_{\sigma(1)}}} \phi\Big) \\ &\qquad \cdot \mathrm{p}_{R',c'}\big(e_{\sigma(p+1)} \vee \cdots \vee e_{\sigma(k)}\big) \\ &\quad\leq \sum_{p=0}^{k} \frac{\hbar^p}{p!\,(k-p)!} k! \sum_{\ell=0}^{\infty} \ell!^{R-1}\, c^{\ell} \sum_{\beta \in \{1,\ldots,n\}^{\ell+p}} \mathrm{q}_{\beta}(\phi) \\ &\qquad \cdot \left(\frac{(k-p)!}{k!}\right)^{R'} c'^{-p}\, \mathrm{p}_{R',c'}\big(e_{i_1} \vee \cdots \vee e_{i_k}\big) \\ &\quad= \mathrm{p}_{R',c'}\big(e_{i_1} \vee \cdots \vee e_{i_k}\big) \sum_{p=0}^{k} \frac{\hbar^p\, k!^{1-R'}}{p!^R\, (k-p)!^{1-R'}\, c'^p}\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{m=p}^{\infty} p!^{R-1} \, (m-p)!^{R-1} \, c^{m-p} \sum_{\beta \in \{1,\ldots,n\}^m} \mathrm{q}_\beta(\phi) \\
&\overset{(*)}{\leq} \mathrm{p}_{R',c'}\big(e_{i_1} \vee \cdots \vee e_{i_k}\big) \sum_{p=0}^{k} \binom{k}{p}^{1-R'} \frac{1}{p!^{R'+R-1}} \frac{\hbar^p}{c'^p} \\
&\times \sum_{m=p}^{\infty} m!^{R-1} \, \big(2^{1-R} c\big)^m \sum_{\beta \in \{1,\ldots,n\}^m} \mathrm{q}_\beta(\phi) \\
&\overset{(*')}{\leq} \mathrm{p}_{R',c'}\big(e_{i_1} \vee \cdots \vee e_{i_k}\big) 2^{k(1-R')} \sum_{p=0}^{k} \frac{\hbar^p}{c'^p} \cdot \mathrm{q}_{R,2^{1-R}c}(\phi) \\
&\leq \mathrm{p}_{R',2^{1-R'}c'}\big(e_{i_1} \vee \cdots \vee e_{i_k}\big) \cdot \mathrm{q}_{R,2^{1-R}c}(\phi) \sum_{p=0}^{\infty} 2^{-p} \\
&\leq \mathrm{p}_{R',2c'}\big(e_{i_1} \vee \cdots \vee e_{i_k}\big) \cdot \mathrm{q}_{R,2c}(\phi) \cdot 2,
\end{aligned}
$$

where we have used $R \leq 1$ as well as $c \geq 1$ in $(*)$, then $R' \leq 1$ in $(*')$, and $c' \geq 2\hbar$ in the final estimate. Note again that we can make this assumption on c' without loss of generality by (3.2). This observation and the analogous statement (4.60) for $\mathscr{E}_R$ also gives the worse estimate (6.5) from what we have computed. If we have $R' \geq 1$, we estimate the binomial coefficient in the step $(*')$ by 1 instead, which once again implies (6.5). In the case that $R \geq 1$ we note $(m-p)!^{R-1} \leq m!^{R-1}$, yielding

$$
\begin{aligned}
\ldots &\overset{(*)}{\leq} \mathrm{p}_{R',c'}\big(e_{i_1} \vee \cdots \vee e_{i_k}\big) \sum_{p=0}^{k} \binom{k}{p}^{1-R'} \frac{1}{p!^{R'}} \frac{\hbar^p}{c'^p} \sum_{m=0}^{\infty} m!^{R-1} \, c^m \sum_{\beta \in \{1,\ldots,n\}^m} \mathrm{q}_\beta(\phi) \\
&= \mathrm{p}_{R',c'}\big(e_{i_1} \vee \cdots \vee e_{i_k}\big) \cdot \mathrm{q}_{R,c}(\phi) \cdot \sum_{p=0}^{k} \binom{k}{p}^{1-R'} \frac{1}{p!^{R'}} \frac{\hbar^p}{c'^p} \qquad (\dagger) \\
&= \mathrm{p}_{R',c'}\big(e_{i_1} \vee \cdots \vee e_{i_k}\big) \cdot \mathrm{q}_{R,c}(\phi) \cdot \sum_{p=0}^{k} \left(\frac{k!}{(k-p)!}\right)^{1-R'} \frac{1}{p!} \frac{\hbar^p}{c'^p}. \qquad (\ddagger)
\end{aligned}
$$

From here, (†) gives the case $R' \geq 1$ and (‡) the case $R' \leq 1$ in the same fashion as before. Note that passing to the series in p makes our estimate independent of the symmetric order k. Thus we have shown (6.5) on generators. Consider now an arbitrary function $P \in \hat{\mathrm{S}}^\bullet_{R'}(\mathfrak{g})$ in the left factor. Expanding P in the induced basis of $\mathrm{S}^\bullet(\mathfrak{g})$ corresponding to the basis of $\mathfrak{g}$ we chose earlier gives

$$
P = \sum_{k=0}^{\infty} \sum_{i_1 \leq \cdots \leq i_k = 1}^{n} a^{i_1 \cdots i_k} e_{i_1} \vee \cdots \vee e_{i_k} \in \mathrm{S}^\bullet_{R'}(\mathfrak{g}).
$$

By distributivity of the standard-ordered star product this now implies

$$\begin{aligned}
&\Big(\mathrm{q}_{R,c} \otimes \mathrm{p}_{R',c'}\Big)\big((\mathbb{1} \otimes P) \star_{\mathrm{std}} (\phi \otimes 1)\big) \\
&\quad \leq 2 \cdot \mathrm{q}_{R,2c}(\phi) \sum_{k=0}^{\infty} \sum_{i_1 \leq \cdots \leq i_k = 1}^{n} \Big|a^{i_1 \cdots i_k}\Big| \cdot \mathrm{p}_{R',2c'}\big(e_{i_1} \vee \cdots \vee e_{i_k}\big) \\
&\quad = 2 \cdot \mathrm{q}_{R,2c}(\phi) \cdot \mathrm{p}_{R',2c'}(P),
\end{aligned}$$

where we once again utilized [12, Lem. A.1] to first infer the "orthogonality"

$$\mathrm{p}^k \left(\sum_{i_1 \leq i_2 \leq \ldots \leq i_k} a^{i_1 \ldots i_k} e_{i_1} \vee \cdots \vee e_{i_k} \right) = \sum_{i_1 \leq i_2 \leq \ldots \leq i_k} \Big|a^{i_1 \ldots i_k}\Big| \, \mathrm{p}^k(e_{i_1} \vee \cdots \vee e_{i_k})$$

within a *fixed* symmetric degree. Thus we have shown that (6.5) also holds for arbitrary polynomial functions P. This finally implies the continuity of the standard ordered star product. For the holomorphy first note that our estimate works for $\hbar$ in a locally uniform and bounded way by continuity of the involved weights with respect to $\hbar$. Taking another look at the formula for the star product from (2.14) we see that the star product is an absolutely convergent power series in $\hbar$, i.e. the limit of polynomials in $\hbar$, which are holomorphic. By our estimate the corresponding sequence is Cauchy with respect to the locally uniform topology, i.e. it converges to some element in the completion and that element is vector-valued holomorphic, as well. □

We have gathered all the necessary ingredients to prove the continuity of the full star product. Notably, the sharp condition $R' \geq 1$ from Proposition 6.1 breaks the symmetry between R and R' from Lemma 6.2 and reduces the condition $R + R' \geq 1$ to $R \geq 0$. Investing moreover the continuity of the pointwise product on $\mathscr{E}_R(G)$ now yields our main result:

Theorem 6.3 (Continuity of $\star_{\mathrm{std}}$) *Let G be a Lie group, $R \geq 0$ and $R' \geq 1$. The full standard-ordered star product*

$$\star_{\mathrm{std}} : \mathrm{Pol}^{\bullet}_{R,R'}(T^*G) \times \mathrm{Pol}^{\bullet}_{R,R'}(T^*G) \longrightarrow \mathrm{Pol}^{\bullet}_{R,R'}(T^*G) \tag{6.7}$$

is well-defined and continuous, extending to a continuous product

$$\star_{\mathrm{std}} : \widehat{\mathrm{Pol}}^{\bullet}_{R,R'}(T^*G) \times \widehat{\mathrm{Pol}}^{\bullet}_{R,R'}(T^*G) \longrightarrow \widehat{\mathrm{Pol}}^{\bullet}_{R,R'}(T^*G). \tag{6.8}$$

More precisely, for $c, c' \geq 1$ there is a $d \geq 1$, which is continuous with respect to $\hbar$, such that

$$\big(q_{R,c} \otimes p_{R',c'}\big)(P \star_{\mathrm{std}} Q) \leq 2 \cdot \big(q_{R,d} \otimes p_{R',d}\big)(P) \cdot \big(q_{R,d} \otimes p_{R',d}\big)(Q) \tag{6.9}$$

holds for $P, Q \in \widehat{\mathrm{Pol}}^{\bullet}_{R,R'}(T^*G)$.

Proof We first consider factorizing functions. Let $\phi, \psi \in \mathscr{E}_R(G)$, $\eta \in \mathrm{S}^\bullet_{R'}(\mathfrak{g})$ as well as $\xi_1, \ldots, \xi_k \in \mathfrak{g}$. By Proposition 2.4, (vi), the full star product can be written as

$$
\begin{aligned}
&(\phi \otimes \xi_1 \vee \cdots \vee \xi_k) \star_{\mathrm{std}} (\psi \otimes \eta) \\
&\quad = \sum_{p=0}^{k} \left(\frac{\hbar}{\mathrm{i}}\right)^p \frac{\phi}{p!\,(k-p)!} \sum_{\sigma \in S_k} \mathscr{L}_{X_{\xi_{\sigma(1)}}} \cdots \mathscr{L}_{X_{\xi_{\sigma(p)}}} \psi \otimes (\xi_{\sigma(p+1)} \vee \cdots \vee \xi_{\sigma(k)}) \star_{\mathfrak{g}} \eta.
\end{aligned}
$$

Note that, compared to (2.14), we left multiply with the function ϕ in the first tensor factor and compose with the Lie algebra star product in the second one. Let $c, c' \geq 1$ and write $\tilde{c}' = 16(\hbar + 1)c'$. Using (6.2), (6.5) as well as the continuity estimate for pointwise products from (4.38) gives

$$
\begin{aligned}
&\Big(\mathrm{q}_{R,c} \otimes \mathrm{p}_{R',c'}\Big)\big((\phi \otimes \xi_1 \vee \cdots \vee \xi_k) \star_{\mathrm{std}} (\psi \otimes \eta)\big) \\
&\quad \leq \sum_{p=0}^{k} \frac{\hbar^p}{p!\,(k-p)!} \sum_{\sigma \in S_k} \mathrm{q}_{R,c}\Big(\phi \cdot \mathscr{L}_{X_{\xi_{\sigma(1)}}} \cdots \mathscr{L}_{X_{\xi_{\sigma(p)}}} \psi\Big) \mathrm{p}_{R',c'} \\
&\quad\quad \big(\xi_{\sigma(p+1)} \vee \cdots \vee \xi_{\sigma(k)} \star_{\mathfrak{g}} \eta\big) \\
&\quad \leq \sum_{p=0}^{k} \frac{\hbar^p \cdot \mathrm{q}_{R,2^R c}(\phi) \cdot \mathrm{p}_{R',\tilde{c}'}(\eta)}{p!\,(k-p)!} \\
&\quad\quad \sum_{\sigma \in S_k} \mathrm{q}_{R,2^R c}\Big(\mathscr{L}_{X_{\xi_{\sigma(1)}}} \cdots \mathscr{L}_{X_{\xi_{\sigma(p)}}} \psi\Big) \mathrm{p}_{R',\tilde{c}'}\big(\xi_{\sigma(p+1)} \vee \cdots \vee \xi_{\sigma(k)}\big).
\end{aligned}
$$

Thus what remains to be estimated is

$$
\sum_{p=0}^{k} \frac{\hbar^p}{p!\,(k-p)!} \sum_{\sigma \in S_k} \mathrm{q}_{R,2^R c}\Big(\mathscr{L}_{X_{\xi_{\sigma(1)}}} \cdots \mathscr{L}_{X_{\xi_{\sigma(p)}}} \psi\Big) \mathrm{p}_{R',\tilde{c}'}\big(\xi_{\sigma(p+1)} \vee \cdots \vee \xi_{\sigma(k)}\big),
$$

which is exactly what we obtained by applying the triangle inequality to the mixed product

$$
\Big(\mathrm{q}_{R,2^R c} \otimes \mathrm{p}_{R',\tilde{c}'}\Big)\big((\mathbb{1} \otimes \xi_1 \vee \cdots \vee \xi_k) \star_{\mathrm{std}} (\phi \otimes 1)\big).
$$

Thus we can utilize our estimate from Lemma 6.2 to obtain (6.9), taking d as the largest coefficient we obtain upon putting everything together, which is obviously continuous in $\hbar$ as a pointwise maximum of continuous functions. Then the bilinear version of the argument as in the end of Lemma 6.2 extends (6.9) to arbitrary polynomial functions, as all other factors were already in full generality. Finally, the usual infimum argument (see e.g. [54, Prop. 43.4]) for projective tensor products gives this estimate also for arbitrary mixed tensors, which implies continuity of the standard-ordered star product $\star_{\mathrm{std}}$ on the entire observable algebra $\mathrm{Pol}^\bullet_{R,R'}(T^*G)$. From here it extends to the completion by continuity, preserving the estimates (6.9). □

The second main statement is that the star product we obtained is a holomorphic deformation in the following sense:

Theorem 6.4 (Holomorphic dependence on $\hbar$) *Let G be a connected Lie group and let $R \geq 0$ and $R' \geq 1$. Then*

$$\mathbb{C} \ni \hbar \;\mapsto\; P \star_{\mathrm{std}} Q \in \widehat{\mathrm{Pol}}^{\bullet}_{R,R'}(T^*G) \tag{6.10}$$

is entire for all $P, Q \in \widehat{\mathrm{Pol}}^{\bullet}_{R,R'}(T^*G)$. Its Taylor series in $\hbar$ coincides with the formal star product in the sense that the $\hbar^k$-term is given by (2.16).

Proof As long as $P, Q \in \mathrm{Pol}^{\bullet}_{R,R'}(T^*G)$, their star product is a polynomial in $\hbar$, thus entire. For general elements P and Q in the completion, we can once again estimate locally uniformly in $\hbar$ and our explicit formula as well as [17, Lem. 2.8 with $z = -\mathrm{i}\hbar$] then imply that we have polynomial partial sums, i.e. vector-valued holomorphic functions. Together, this implies vector-valued holomorphy of the full star product for fixed factors. The second statement is clear for elements $P, Q \in \mathrm{Pol}^{\bullet}_{R,R'}(T^*G)$ and extends to the completion by virtue of Proposition 5.8. $\square$

A first application of this continuity result is the continuity of the standard-ordered quantization map:

Corollary 6.5 *Let G be a Lie group and $R \geq 0$ and $R' \geq 1$. The standard-ordered quantization map ϱ_{std} yields a continuous bilinear map*

$$\varrho_{\mathrm{std}} \colon \widehat{\mathrm{Pol}}^{\bullet}_{R,R'}(T^*G) \times \mathscr{E}_R(G) \ni (f, \phi) \;\mapsto\; \varrho_{\mathrm{std}}(f)\phi \in \mathscr{E}_R(G). \tag{6.11}$$

In particular, every operator $\varrho_{\mathrm{std}}(f)$ with $f \in \widehat{\mathrm{Pol}}^{\bullet}_{R,R'}(T^*G)$ is a continuous endomorphism of $\mathscr{E}_R(G)$.

Proof According to (A.20) we have for $f \in \mathrm{Pol}^{\bullet}_{R,R'}(T^*G)$ and $\phi \in \mathscr{E}_R(G)$

$$\varrho_{\mathrm{std}}(f)\phi = \iota^*(f \star_{\mathrm{std}} \pi(\phi)),$$

which is a composition of the continuous linear maps π^* and ι^*, see Proposition 5.3, and the continuous bilinear star product. As usual, this extends the completion. $\square$

By invoking the semiclassical limit we immediately obtain the continuity of the Poisson bracket:

Corollary 6.6 *Let G be a Lie group, $R \geq 0$ and $R' \geq 1$. The Poisson bracket*

$$\{\,\cdot\,,\,\cdot\,\} \colon \mathrm{Pol}^{\bullet}_{R,R'}(T^*G) \times \mathrm{Pol}^{\bullet}_{R,R'}(T^*G) \longrightarrow \mathrm{Pol}^{\bullet}_{R,R'}(T^*G) \tag{6.12}$$

is well-defined and continuous. Moreover, the explicit formula (2.25) *extends to the completion $\widehat{\mathrm{Pol}}^{\bullet}_{R,R'}(T^*G)$ order by order.*

Of course, both Corollaries 6.5 and 6.6 can be shown by direct estimation and the explicit formulas (2.25) and (2.8), as well. Notably, this extends the statements to arbitrary values of R and R'. The underlying reason for this is that each of the mappings is an honest differential operator, i.e. only *finitely* many differentiations have to be estimated at once. Analogously, the same is true for the bidifferential operators $D_k \in \mathrm{DiffOp}(T^*G)$ given by

$$D_k(P, Q) = \frac{\mathrm{d}^k}{\mathrm{d}\hbar^k}\Big(P \star_{\mathrm{std}} Q\Big)\Big|_{\hbar=0} \tag{6.13}$$

for $P, Q \in \mathscr{C}^\infty(T^*G)$ and $k \in \mathbb{N}_0$.

After having established the continuity of the structure maps for classical mechanics and its standard-ordered quantization, we turn towards other ordering prescriptions obtained by means of the Neumaier operator. Instead of directly deriving continuity estimates for the considerably more complicated formulas, we show the continuity of the κ-Neumaier operators. From Proposition 2.8 we immediately get the continuity of N^2 and, ultimately, the continuity of N_κ for all $\kappa \in \mathbb{R}$:

Proposition 6.7 *Let G be a Lie group, $\kappa \in \mathbb{R}$ and $R, R' \in \mathbb{R}$ with $R + R' \geq 1$.*

(i) *The κ-Neumaier operator*

$$N_\kappa : \mathrm{Pol}^\bullet_{R,R'}(T^*G) \longrightarrow \mathrm{Pol}^\bullet_{R,R'}(T^*G) \tag{6.14}$$

is well-defined and continuous.

(ii) *The κ-Neumaier operator N_κ extends by continuity to*

$$N_\kappa : \widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*G) \longrightarrow \widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*G) \tag{6.15}$$

and its explicit formula extends to the completion $\widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*G)$ order by order.

(iii) *For all $P \in \widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*G)$ the map*

$$\mathbb{C} \ni \hbar \;\mapsto\; N_\kappa(P) \in \widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*G) \tag{6.16}$$

is entire.

(iv) *Let now in addition $R \geq 0$ and $R' \geq 1$. Then the κ-ordered star product extends to a continuous multiplication*

$$\star_\kappa : \widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*G) \times \widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*G) \longrightarrow \widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*G). \tag{6.17}$$

(v) *For $R \geq 0$ and $R' \geq 1$ the κ-ordered star product yields an entire function*

$$\mathbb{C} \ni \hbar \;\mapsto\; P \star_\kappa Q \in \widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*G) \tag{6.18}$$

for all $P, Q \in \widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*G)$.

Proof This is a somewhat immediate consequence of (2.27): first we get the continuity for $N^2 = N_2$ and all $\hbar$. Rescaling now $\hbar$ appropriately can be re-interpreted as a rescaling of $\kappa = 2$ in the continuity estimates for N_2. This gives the continuity for all κ. The second statement is then an immediate consequence of Proposition 5.8. The entirety of $\hbar \mapsto N_\kappa(P)$ now follows from the entirety of (6.6) and the formula (2.27). Next,

$$P \star_\kappa Q = N_{-\kappa}\big((N_\kappa P) \star_{\mathrm{std}} (N_\kappa Q)\big) \tag{$*$}$$

for $P, Q \in \mathrm{Pol}^\bullet_{R,R'}(T^*G)$ gives continuity of the κ-ordered star product $\star_\kappa$ as a composition of continuous maps. Being continuous, $\star_\kappa$ extends to the completion as usual. Finally, ($*$) implies entirety of the κ-ordered star products for fixed factors as a composition of entire functions. □

Corollary 6.8 *Let G be a connected Lie group and let $R \geq 0$ and $R' \geq 1$. Then the Weyl star product $\star_{\mathrm{Weyl}}$ is a continuous multiplication*

$$\star_{\mathrm{Weyl}} : \widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*G) \times \widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*G) \longrightarrow \widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*G) \tag{6.19}$$

with entire dependence on $\hbar$.

Proposition 6.9 *Let $\Phi : G \longrightarrow H$ be a covering map of Lie groups. Then pullback with the point transformation*

$$(T_*\Phi)^* : \widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*H) \longrightarrow \widehat{\mathrm{Pol}}^\bullet_{R,R'}(T^*G) \tag{6.20}$$

is a continuous homomorphism with respect to the κ-ordered star products on T^*G and T^*H, respectively.

Proof The fact that $(T_*\Phi)^*$ is a homomorphism holds in general since Φ preserves the half-commutator connection. The continuity was obtained in Proposition 5.4, (ii). □

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Appendix A: Star products on cotangent bundles

In this short appendix we recall the basic facts on star products on general cotangent bundles from [7–9, 41, 42] to put the construction on the cotangent bundle of a Lie group into the right perspective.

Let Q be a smooth manifold, the configuration space, and denote its cotangent bundle by the projection $\pi: T^*Q \longrightarrow Q$. For the zero section we will write $\iota: Q \longrightarrow T^*Q$. On a cotangent bundle (as on any vector bundle) we have smooth functions which are polynomial in the fiber directions. They will be denoted by $\mathrm{Pol}^\bullet(T^*Q) \subseteq \mathscr{C}^\infty(T^*Q)$, where we write $\mathrm{Pol}^k(T^*Q)$ for those, which are homogeneous polynomials of degree $k \in \mathbb{N}_0$. Recall that we always consider complex-valued functions $\mathscr{C}^\infty(T^*Q)$.

As any vector bundle, T^*Q has a particular vector field, the Euler vector field $\xi \in \Gamma^\infty(T(T^*Q))$, whose flow is given by $(t, \alpha_q) \mapsto \mathrm{e}^t \alpha_q$, where $t \in \mathbb{R}$ and $\alpha_q \in T^*_q Q$ for $q \in Q$. It can be used to characterize $\mathrm{Pol}^k(T^*Q)$ as the eigenfunctions of the Lie derivative $\mathscr{L}_\xi$ to the eigenvalue $k \in \mathbb{N}_0$ and no other eigenvalues occur. Note that the canonical map $\mathcal{J}: \Gamma^\infty(T_{\mathbb{C}}Q) \longrightarrow \mathrm{Pol}^1(T^*Q)$, sending a complex vector field $X \in \Gamma^\infty(T_{\mathbb{C}}Q)$ to the linear function defined by $(\mathcal{J}(X))(\alpha_q) = \alpha_q(X(q))$, extends to a graded unital algebra isomorphism

$$\mathcal{J}: \bigoplus_{k=0}^{\infty} \Gamma^\infty(\mathrm{S}^k_{\mathbb{C}} TQ) \longrightarrow \mathrm{Pol}^\bullet(T^*Q), \tag{A.1}$$

if we set $\mathcal{J}(u) = \pi^* u$ for $u \in \mathscr{C}^\infty(Q) = \Gamma^\infty(\mathrm{S}^0_{\mathbb{C}} TQ)$. Here $\mathrm{S}^k_{\mathbb{C}} TQ$ denotes the k-th complexified symmetric power of the tangent bundle TQ.

To establish a global symbol calculus for the algebra of differential operators $\mathrm{DiffOp}(Q)$ acting on $\mathscr{C}^\infty(Q)$, we choose a torsion-free covariant derivative ∇ on Q. We use the same symbol for all induced covariant derivatives on the various tensor bundles. The covariant derivative ∇ induces a symmetrized covariant derivative

$$\mathrm{D}: \Gamma^\infty(\mathrm{S}^k_{\mathbb{C}} T^*Q) \longrightarrow \Gamma^\infty(\mathrm{S}^{k+1}_{\mathbb{C}} T^*Q) \tag{A.2}$$

in such a way that for functions $u \in \mathscr{C}^\infty(Q)$ we have $\mathrm{D}u = \mathrm{d}u$ and for one-forms $\alpha \in \Gamma^\infty(T^*_{\mathbb{C}} Q)$ we have

$$(\mathrm{D}\alpha)(X, Y) = \nabla_X(\alpha(Y)) + \nabla_Y(\alpha(X)) - \alpha(\nabla_X Y) - \alpha(\nabla_Y X). \tag{A.3}$$

Then D is defined on higher symmetric forms by requiring a Leibniz rule with respect to the symmetric tensor product $\vee$, i.e. we have $\mathrm{D}(\alpha \vee \beta) = \mathrm{D}\alpha \vee \beta + \alpha \vee \mathrm{D}\beta$. In local coordinates (U, x) of Q this can then be written as

$$\mathrm{D} = \mathrm{d}x^i \vee \nabla_{\frac{\partial}{\partial x^i}}. \tag{A.4}$$

In fact, if $\mathrm{e}_1, \ldots, \mathrm{e}_n \in \Gamma^\infty(TU)$ is a local frame of TQ on an open subset $U \subseteq Q$ with dual local frame $\mathrm{e}^1, \ldots, \mathrm{e}^n \in \Gamma^\infty(T^*U)$ then we have

$$\mathrm{D} = \mathrm{e}^i \vee \nabla_{\mathrm{e}_i} \tag{A.5}$$

for sections on U. This (local) formula will play a crucial role whenever we have a *global* frame, i.e. on a parallelizable manifold.

There are now various ways to define the global symbol calculus with respect to ∇. Following [8] one defines the standard-ordered quantization map

$$\varrho_{\mathrm{std}} \colon \mathrm{Pol}^\bullet(T^*Q) \longrightarrow \mathrm{DiffOp}(Q) \tag{A.6}$$

by specifying the differential operators $\varrho_{\mathrm{std}}(\mathcal{J}(X))$ for all $X \in \Gamma^\infty(\mathrm{S}^k TQ)$ on functions $\psi \in \mathscr{C}^\infty(Q)$ as

$$\varrho_{\mathrm{std}}\big(\mathcal{J}(X)\big)\psi = \iota^*\left(\mathrm{i_s}(X)\mathrm{e}^{-\mathrm{i}\hbar\mathrm{D}}\psi\right), \tag{A.7}$$

where $\iota^* \colon \prod_{k=0}^\infty \Gamma^\infty(\mathrm{S}^k_{\mathbb{C}} T^*Q) \longrightarrow \mathscr{C}^\infty(Q)$ is the projection onto the symmetric degree $k = 0$ and $\mathrm{i_s}(\,\cdot\,)$ denotes the symmetric insertion map, which is defined as follows: for a vector field $X \in \Gamma^\infty(T_{\mathbb{C}}Q)$ it is the insertion into the first argument as usual. For higher degrees we require $\mathrm{i_s}(X \vee Y) = \mathrm{i_s}(X)\mathrm{i_s}(Y)$ to get the correct prefactors. For a function $X = u \in \mathscr{C}^\infty(Q)$ we set $\mathrm{i_s}(u) = u$ as multiplication operator. Finally, the formal exponential series of the iterated symmetrized covariant derivatives of ψ is interpreted as element in the Cartesian product over all symmetric degrees. Since $\mathcal{J}$ is an isomorphism, this indeed specifies ϱ_{std} on all polynomial functions $\mathrm{Pol}^\bullet(T^*Q)$ as wanted.

We note that (A.6) is a $\mathscr{C}^\infty(Q)$-linear isomorphism whenever $\hbar \neq 0$, where $\mathrm{Pol}^\bullet(T^*Q)$ is equipped with the canonical $\mathscr{C}^\infty(Q)$-module structure via π^* and $\mathrm{DiffOp}(Q)$ is considered as left $\mathscr{C}^\infty(Q)$-module as usual. Moreover, ϱ_{std} is compatible with the filtrations of the differential operator by the degree of differentiation and the filtration of $\mathrm{Pol}^\bullet(T^*Q)$ induced by the degree of the polynomials. Taking into account the $\hbar$-dependence gives the *homogeneity*

$$\left[\hbar\tfrac{\partial}{\partial\hbar}, \varrho_{\mathrm{std}}(f)\right] = \varrho_{\mathrm{std}}\left(\mathsf{H}f\right) \tag{A.8}$$

for all $f \in \mathrm{Pol}^\bullet(T^*Q)$ possibly depending on $\hbar$ as well, where $\mathsf{H} = \hbar\frac{\partial}{\partial\hbar} + \mathscr{L}_\xi$. From a physical point of view this means that ϱ_{std} is dimensionless.

The bijection (A.6) allows us to pull back the operator product to $\mathrm{Pol}^\bullet(T^*Q)$. This gives an associative product, the *standard-ordered star product* $\star_{\mathrm{std}}$, for $\mathrm{Pol}^\bullet(T^*Q)$ such that

$$f \star_{\mathrm{std}} g = \varrho_{\mathrm{std}}^{-1}\big(\varrho_{\mathrm{std}}(f)\varrho_{\mathrm{std}}(g)\big) \tag{A.9}$$

for $f, g \in \mathrm{Pol}^\bullet(T^*Q)$. The homogeneity properties shows that for $f, g \in \mathrm{Pol}^\bullet(T^*Q)$ the standard-ordered star product $f \star_{\mathrm{std}} g$ is a polynomial in $\hbar$ of degree at most the sum of the degree of f and g. More precisely,

$$\mathsf{H}(f \star_{\mathrm{std}} g) = \mathsf{H} f \star_{\mathrm{std}} g + f \star_{\mathrm{std}} \mathsf{H} g \tag{A.10}$$

for all $f, g \in \mathrm{Pol}^\bullet(T^*Q)$. Hence we have unique bilinear operators $C_r\colon \mathrm{Pol}^\bullet(T^*Q) \times \mathrm{Pol}^\bullet(T^*Q) \longrightarrow \mathrm{Pol}^\bullet(T^*Q)$ with

$$f \star_{\mathrm{std}} g = \sum_{r=0}^{\infty} \hbar^r C_r(f, g), \tag{A.11}$$

where each C_r changes the polynomial degree by $-r$. In particular, the sum is always finite as long as f and g are polynomial functions.

It is a not completely obvious fact that the operators C_r in $\star_{\mathrm{std}}$ are actually bidifferential operators and thus extend to a formal star product for $\mathscr{C}^\infty(T^*Q)[\![\hbar]\!]$. In fact, one way to show this is to identify $\star_{\mathrm{std}}$ with the Fedosov star product based on standard-ordering, see [8]. Note, however, that for functions in $\mathrm{Pol}^\bullet(T^*Q)$ the usual convergence problem of formal star products is absent since the series (A.11) terminates after finitely many contributions.

The standard-ordered symbol calculus has one serious flaw: it lacks compatibility with the *-involutions. For the differential operators $\mathrm{DiffOp}(Q)$ one has no intrinsic involution. However, fixing a smooth positive density $\mu \in \Gamma^\infty(|\Lambda^n| \, T^*Q)$ one induces an inner product for $\mathscr{C}_0^\infty(Q)$ by

$$\langle \phi, \psi \rangle_\mu = \int_Q \overline{\phi} \psi \mu, \tag{A.12}$$

where $\phi, \psi \in \mathscr{C}_0^\infty(Q)$. We fix μ once and for all to define the adjoint of a differential operator $D \in \mathrm{DiffOp}(Q)$ by requiring

$$\langle D^* \phi, \psi \rangle_\mu = \langle \phi, D \psi \rangle_\mu \tag{A.13}$$

for all $\phi, \psi \in \mathscr{C}_0^\infty(Q)$. A non-trivial global integration by parts then computes the adjoint D^*, explicitly using the standard-ordered symbol calculus, which we briefly recall:

Firstly, we define the one-form $\alpha \in \Gamma^\infty(T^*Q)$ by $\nabla_X \mu = \alpha(X)\mu$, thus measuring how μ is not covariantly constant with respect to the chosen covariant derivative ∇. In many cases one can achieve $\alpha = 0$, say for a Levi-Civita covariant derivative ∇ of a Riemannian metric g and the corresponding Riemannian volume density μ_g.

Secondly, and more importantly, we note that ∇ allows to horizontally lift tangent vectors $v_q \in T_q Q$ to tangent vectors $v_q^{\mathrm{hor}}\big|_{\alpha_q} \in T_{\alpha_q} T^*Q$. Canonically, we can lift one-forms $\beta_q \in T_q^* Q$ vertically to tangent vectors $\beta_q^{\mathrm{ver}}\big|_{\alpha_q} \in T_{\alpha_q} T^*Q$. This gives a splitting $T_{\alpha_q} T^*Q = \mathrm{Hor}_{\alpha_q} \oplus \mathrm{Ver}_{\alpha_q}$ for all $\alpha_q \in T_q^*Q$ with the additional property, specific for a cotangent bundle, that the horizontal and the vertical space are equipped with a natural pairing originating from the pairing of $T_q Q$ and $T_q^* Q$. Thus we obtain a pseudo Riemannian metric g_0 on T^*Q of split signature (n, n). This metric has a Laplace operator $\Delta_0 \in \mathrm{DiffOp}^2(T^*Q)$ for functions on T^*Q, which locally is given by

$$\Delta_0 = \frac{\partial^2}{\partial q^i \partial p_i} + p_r \pi^*(\Gamma^r_{ij}) \frac{\partial^2}{\partial p_i \partial p_j} + \pi^*(\Gamma^i_{ij}) \frac{\partial}{\partial p_j}, \tag{A.14}$$

where $(T^*U, (q, p))$ is a Darboux chart induced by a local chart (U, x) on Q and where Γ^r_{ij} are the Christoffel symbols of ∇ with respect to the chart (U, x).

Putting things together we can then consider the Neumaier operator

$$\mathcal{N} = \exp\big(-\tfrac{\mathrm{i}\hbar}{2}(\Delta_0 + \mathscr{L}_{\alpha^{\mathrm{ver}}})\big), \tag{A.15}$$

which is a well-defined endomorphism of $\mathrm{Pol}^\bullet(T^*Q)$, since both Δ_0 and the Lie derivative in direction of the vertical lift of α decrease the polynomial degree by one, thus making the exponential series terminate on polynomial functions. Using $\mathcal{N}$ one can write the integration by parts to compute the adjoint of a differential operator as

$$\varrho_{\mathrm{std}}(f)^* = \varrho_{\mathrm{std}}(\mathcal{N}^2 \overline{f}) \tag{A.16}$$

for all $f \in \mathrm{Pol}^\bullet(T^*Q)$, see [8, 9]. Since (A.6) is an isomorphism, this computes the adjoint of all differential operators explicitly, once we base their description on the standard-ordered symbol calculus ϱ_{std}.

One can then use $\mathcal{N}$ to pass from the standard-ordering to a Weyl ordering and, more generally, to a κ-ordering interpolating between the two. For $\kappa \in \mathbb{R}$ one defines a new ordering

$$\varrho_\kappa(f) = \varrho_{\mathrm{std}}(\mathcal{N}_\kappa f) \quad \text{where} \quad \mathcal{N}_\kappa = \exp(-\mathrm{i}\hbar\kappa(\Delta_0 + \mathscr{L}_{\alpha^{\mathrm{ver}}})) \tag{A.17}$$

together with a corresponding κ-ordered star product

$$f \star_\kappa g = \mathcal{N}_\kappa^{-1}\big(\mathcal{N}_\kappa(f) \star_{\mathrm{std}} \mathcal{N}_\kappa(g)\big) \tag{A.18}$$

for $f, g \in \mathrm{Pol}^\bullet(T^*Q)$, see [7, 41, 42]. The case $\kappa = \frac{1}{2}$ is then called the Weyl ordering ϱ_{Weyl} with the corresponding Weyl star product $\star_{\mathrm{Weyl}}$. For the Weyl star product one has

$$\varrho_{\mathrm{Weyl}}(f)^* = \varrho_{\mathrm{Weyl}}(\overline{f}) \quad \text{and} \quad \overline{f \star_{\mathrm{Weyl}} g} = \overline{g} \star_{\mathrm{Weyl}} \overline{f} \tag{A.19}$$

for all $f, g \in \mathrm{Pol}^\bullet(T^*Q)$.

Finally, we note the useful relation

$$\varrho_{\mathrm{std}}(f)\phi = \iota^*(f \star_{\mathrm{std}} \pi^*\phi) \tag{A.20}$$

for all $f \in \mathrm{Pol}(T^*Q)$ and $\phi \in \mathscr{C}^\infty(Q)$. This allows to reconstruct the standard-ordered representation from the standard-ordered star product.

Appendix B: Noncommutative higher Leibniz rule

The following well-known Leibniz rules are completely algebraic, wherefore we treat them as such.

Proposition 1 *Let R be a (not necessarily associative) ring with $\mathbb{Q} \subseteq R$, $D_1, \ldots, D_n \in \mathit{Der}(R)$ derivations and $a, b \in R$.*

(i) *We have the higher Leibniz rule*

$$D_n \cdots D_1(ab) = \sum_{p=0}^{n} \sum_{\sigma \in Sh(p,n-p)} \big(D_{\sigma(n)} \cdots D_{\sigma(p+1)} a\big)\big(D_{\sigma(p)} \cdots D_{\sigma(1)} b\big). \tag{B.1}$$

Here $Sh(p, n-p)$ denotes the set of $(p, n-p)$-shuffles, i.e. permutations $\sigma \in S_n$ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p) \quad \textit{and} \quad \sigma(p+1) < \sigma(p+2) < \cdots < \sigma(n). \tag{B.2}$$

(ii) *Symmetrizing, it furthermore holds that*

$$\sum_{\sigma \in S_n} D_{\sigma(n)} \cdots D_{\sigma(1)}(ab) = \sum_{\sigma \in S_n} \sum_{p=0}^{n} \binom{n}{p} \big(D_{\sigma(n)} \cdots D_{\sigma(p+1)} a\big)\big(D_{\sigma(p)} \cdots D_{\sigma(1)} b\big). \tag{B.3}$$

Proof Part (i) is a straightforward induction. Use that $\sigma \in \mathrm{Sh}(p, n-p)$ satisfies either $\sigma(p) = n$ or $\sigma(n) = n$ by (B.2). The statement (ii) is an easy consequence of $|\mathrm{Sh}(p, n-p)| = \binom{n}{p}$. □

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