Mathematische Annalen

Symmetric elastic knots

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Abstract

Minimizing the bending energy within knot classes leads to the concept of elastic knots which has been initiated by von der Mosel (Asymptot Anal 18(1–2):49–65, 1998). Motivated by numerical experiments in Bartels and Reiter (Math Comput 90(330):1499–1526, 2021) we prescribe dihedral symmetry and establish existence of dihedrally symmetric elastic knots for knot classes admitting this type of symmetry. Among other results we prove that the dihedral elastic trefoil is the union of two circles that form a (planar) figure-eight. We also discuss some generalizations and limitations regarding other symmetries and knot classes.

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Communicated by Giga.

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1 Introduction

The study of elastic knots was initiated by Gerlach et al. in [24]. Inspired by toy models of springy knotted wires (see the images in [24, Figure 7]) the existence of energy minimizing knotted configurations γ_ϑ has been established in any prescribed tame[1] knot class $\mathcal{K}$ [24, Theorem 2.1]. The total energy considered,

$$E_\vartheta := E + \vartheta \mathcal{R}, \quad \text{for } \vartheta > 0, \tag{1.1}$$

consists of the classic Euler–Bernoulli bending energy

$$E(\gamma) := \int_\gamma \kappa^2 \, ds \tag{1.2}$$

as the leading order term, together with a small multiple of a repulsive potential $\mathcal{R}$ to avoid self-intersections. In order to analyse the approximative shape of the minimizing knots γ_ϑ for small ϑ the authors study the limit $\vartheta \to 0$. It is shown that the minimizers γ_ϑ converge in C^1 to closed curves γ_0 that minimize the bending energy

$$E(\gamma_0) \le E(\beta) \quad \text{for all } \beta \in \mathscr{C}(\mathcal{K}),$$

where

$$\mathscr{C}(\mathcal{K}) := \{\gamma \in W^{2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3) : \mathscr{L}(\gamma) = 1, |\gamma'| > 0, [\gamma] = \mathcal{K}\}. \tag{1.3}$$

Here, $\mathbb{R}/\mathbb{Z}$ denotes the periodic interval of unit length. These limiting curves γ_0 are called *elastic knots for $\mathcal{K}$* according to [24, Definition 2.3], although they are not embedded unless $\mathcal{K}$ is trivial; see [24, Proposition 3.1]. One of the central results is the complete classification of elastic knots for all torus knot classes $\mathcal{T}(2, b)$ for odd $b \in \mathbb{Z}\backslash\{1, -1\}$.

Theorem ([24, Corollary 6.5(i)]) *The elastic torus knot γ_0 for $\mathcal{T}(2, b)$ for any odd $b \in \mathbb{Z}\backslash\{1, -1\}$ is the doubly covered circle.*

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[1] A knot class is called *tame* if it contains a polygonal representative [14, Definition 1.3], or equivalently, if and only if it contains a continuously differentiable representative [17, App. I].
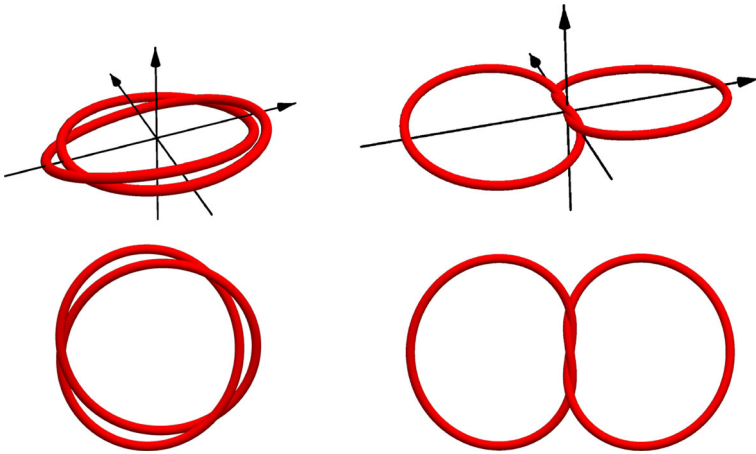
Fig. 1 Final states of the numerical gradient flow of Bartels et al. [7] for the total energy E_ϑ for small $\vartheta > 0$ in the case of the trefoil knot class. Left: The doubly covered circle is the usual limit configuration as predicted by [24]. Right: There are a few initial configurations that move towards an almost flat torus knot with dihedral symmetry under the gradient flow

This result is confirmed by mechanical experiments with thin elastic knotted wires, as well as by various numerical simulations performed by different groups of researchers as documented in [24, Introduction p. 94]. Also the more recent work on the corresponding numerical gradient flow of Bartels et al. [6,7] provides strong numerical evidence for the doubly covered circle as the only possible elastic knot for $\mathcal{T}(2, b)$; see the left of Fig. 1.

Symmetric configurations

Sometimes, however, this numerical gradient flow produces a different limiting configuration exhibiting a dihedral symmetry as depicted on the right of Fig. 1. This indicates the presence of a dihedrally symmetric critical point[2] of the total energy E_ϑ.

It is the purpose of the present paper to analytically support these infrequent but reproducible numerical observations. Namely, we use the principle of symmetric criticality of Palais [33] to prove the existence of symmetric critical points for the constrained variational problem

$$\text{Minimize the total energy } E_\vartheta \text{ on the set } \mathscr{C}(\mathcal{K}). \qquad (\text{P}_\vartheta)$$

Palais's principle, however, requires energy functionals of class C^1. To meet this precondition (and to avoid the technical issues connected with an alternative nonsmooth

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[2] This symmetric knot might be a saddle point, since there exist—as reported in [6]—symmetry breaking perturbations with smaller energy. A more systematic numerical investigation is under way to produce more evidence of the nature of this critical point.

variant [15] of this principle) we replace the nonsmooth ropelength functional used in [24] by a suitable power of the *tangent-point energy* TP_q. It is a self-avoiding energy defined on absolutely continuous regular closed curves $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ as

$$\mathcal{R}(\gamma) := TP_q^{\frac{1}{q-2}}(\gamma) := \left(\int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \frac{1}{r_{\mathrm{tp}}^q(\gamma(s), \gamma(t))} \left| \gamma'(s) \right| \left| \gamma'(t) \right| \, ds \, dt \right)^{\frac{1}{q-2}} \quad (1.4)$$

where we restrict to exponents $q \in (2, 4]$. Here, $r_{\mathrm{tp}}(\gamma(s), \gamma(t))$ stands for the radius of the unique circle through the points $\gamma(s)$ and $\gamma(t)$ that is tangent to γ at $\gamma(s)$. This energy was first suggested by Buck and Orloff [13] in the case $q = 2$ and for general $q > 2$ by Gonzalez and Maddocks in [27, p. 4773], and it was investigated analytically in detail in [38], [9], and [10]. Note that the Sobolev space $W^{2,2}$ continuously embeds into the fractional Sobolev space $W^{2-(1/q),q}$ for $q \in (2, 4]$, which provides the exact regularity framework to guarantee a finite and continuously differentiable tangent point energy [10, Remark 3.1], [42], so that the total energy E_ϑ is continuously differentiable on the open subset $W_{\mathrm{ir}}^{2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ of injective regular closed curves of class $W^{2,2}$. Consequently, Palais's principle of symmetric criticality is applicable. Furthermore, a suitably discretized version of TP_q for $q \in (2, 4]$ was used for $\mathcal{R}$ in the numerical gradient flow in [7] and [6].

Existence results

Theorem 1.1 (Existence of symmetric critical knots) *Given a knot class $\mathcal{K}$, assume that there is at least one knot with dihedral symmetry contained in $\mathscr{C}(\mathcal{K})$. Then for every $\vartheta > 0$ there exists an arclength parametrized knot Γ_ϑ of knot type $\mathcal{K}$ with dihedral symmetry that is critical for the constrained minimization problem (P_ϑ). More precisely, we have*

$$DE_\vartheta(\Gamma_\vartheta)h + \lambda D\mathscr{L}(\Gamma_\vartheta)h = 0 \quad \text{for all } h \in W^{2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3), \quad (1.5)$$

where $\lambda := E_\vartheta(\Gamma_\vartheta)$.

Here, the term *dihedral symmetry*[3] or, synonymously, D_2-*symmetry* refers to the action of the classic dihedral group D_2 (see Table 1) on parametrized space curves by rotating the curve's image by an angle of π about any of the three coordinate axes combined with an appropriate (dihedral) transformation of the periodic domain $\mathbb{R}/\mathbb{Z}$ of the curve. All this is made precise in Sect. 3; see the examples in Figs. 3 and 4 for a preliminary impression of D_2-symmetric curves. In particular, Fig. 4 depicts a dihedrally symmetric torus knot of class $\mathcal{T}(2, 5)$ constructed in Example 3.10 with a method which works for any odd $b \in \mathbb{Z}\backslash\{1, -1\}$, so that the torus knot classes $\mathcal{T}(2, b)$ satisfy the hypothesis of Theorem 1.1.

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[3] There are different definitions of a *symmetry group* for knots in the literature, cf. [1,14,28] and references therein. In the present paper we consider Euclidean symmetries of an actual space curve.

Corollary 1.2 (Existence of symmetric critical torus knots) *Let $b \in \mathbb{Z}\setminus\{-1, 1\}$ be odd. Then for every $\vartheta > 0$ there exists an arclength parametrized torus knot $\Gamma_\vartheta \in \mathscr{C}(\mathcal{T}(2, b))$ with dihedral symmetry, which is critical for the constrained minimization problem (P_ϑ) for $\mathcal{K} = \mathcal{T}(2, b)$.*

Apart from the D_2-symmetry these existence results contain no information about the actual shape of the critical knots Γ_ϑ. In fact, there is a large variety of possible shapes of dihedrally symmetric curves. In Lemma 3.5 we provide a general mechanism how to construct space curves with dihedral symmetry from just one arc satisfying rather mild conditions on its endpoints. To obtain more specific information on the shape of the symmetric critical points Γ_ϑ obtained in Theorem 1.1 and Corollary 1.2 it seems hard to exploit the variational equation (1.5) because of the complicated differential DTP_q of the non-local tangent-point energy TP_q as part of the total energy E_ϑ. Following the idea in [24] we study the limit $\vartheta \to 0$ instead, to obtain limit configurations whose shape can then be analyzed more easily to yield the approximative shape of the D_2-symmetric critical knots Γ_ϑ for small $\vartheta > 0$.

Theorem 1.3 (Existence of symmetric elastic knots) *Let $\mathcal{K}$ be a fixed knot class that contains a D_2-symmetric representative in $\mathscr{C}(\mathcal{K})$, and consider a sequence $\vartheta_j \to 0$ and the corresponding (P_{ϑ_j})-critical D_2-symmetric knots Γ_{ϑ_j} obtained in Theorem 1.1. Then there exists an arclength parametrized curve $\Gamma_0 \in W^{2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ with dihedral symmetry, and a subsequence $(\Gamma_{\vartheta_{j_k}})_k \subset (\Gamma_{\vartheta_j})_j$ such that the $\Gamma_{\vartheta_{j_k}}$ converge weakly in $W^{2,2}$ and strongly in C^1 to Γ_0 as $k \to \infty$. Moreover,*

$$E(\Gamma_0) \le E(\beta) \quad \text{for all } D_2\text{-symmetric } \beta \in \mathscr{C}(\mathcal{K}). \tag{1.6}$$

Definition 1.4 *(Symmetric elastic knots)* Any such curve Γ_0 obtained in Theorem 1.3 is called a *dihedral (or D_2-) elastic knot for $\mathcal{K}$.*

Shapes of symmetric elastic knots

The unknot class and the torus knot class $\mathcal{T}(2, b)$ for any odd $b \in \mathbb{Z}\setminus\{1, -1\}$ satisfy the hypothesis of Theorem 1.3; see Examples 3.8 and 3.10 . Consequently, there are D_2-elastic knots for the unknot class and for $\mathcal{T}(2, b)$, and we can determine their shapes, which also turn out to *characterize* these knot classes.

Theorem 1.5 (The D_2-elastic unknot) *Up to reparametrization and isometry, the only D_2-elastic unknot is the once covered circle of length one. Moreover, if a D_2-elastic knot for some knot class $\mathcal{K}$ is the once covered circle, then $\mathcal{K}$ is the unknot class. Only the unknot class $\mathcal{K}$ satisfies*

$$\inf_{\substack{\beta \in \mathscr{C}(\mathcal{K}) \\ \beta \text{ is } D_2\text{-symmetric}}} E(\beta) = \inf_{\substack{\beta \in W^{2,2} \\ \mathscr{L}(\beta)=1}} E(\beta) = (2\pi)^2. \tag{1.7}$$

If we have specific information about the infimal bending energy on non-trivial knots with dihedral symmetry, then we can identify the shape of the corresponding

D_2-elastic knot. To make this more precise, recall that the natural lower bound for the total curvature $\mathrm{TC}(\gamma) := \int_\gamma \kappa \, ds$ is 2π by virtue of Fenchel's theorem [22]. Applying Hölder's inequality it transfers to the natural lower bound $(2\pi)^2$ for the bending energy of closed curves of length one. Therefore, (1.7) is in fact equivalent to $\inf \{E(\beta)|\beta \in \mathscr{C}(\mathcal{K}), \beta \text{ is } D_2\text{-symmetric}\} \le (2\pi)^2$. In case of non-trivial knots, the total curvature is bounded below by 4π according to the famous result of Fáry and Milnor (see [21,30]) which gives rise to the lower bound $(4\pi)^2$ for the bending energy for any non-trivial knot class $\mathcal{K}$.

Knot classes $\mathcal{K}$ for which the infimal bending energy *equals* the natural lower bound $(4\pi)^2$, i.e., for which

$$\inf_{\substack{\beta \in \mathscr{C}(\mathcal{K}) \\ \beta \text{ is } D_2\text{-symmetric}}} E(\beta) = (4\pi)^2 = \inf_{\mathscr{C}(\mathcal{K})} E(\cdot) \tag{1.8}$$

are of particular interest, since they provide the variational problem with a high degree of rigidity; see Theorem 4.5.

In Sect. 5 we analyze a sequence of specific D_2-symmetric $(2, b)$-torus knots like in Fig. 4 to show that all torus knot classes $\mathcal{T}(2, b)$ for odd integers $b \in \setminus\{-1, 1\}$ satisfy condition (1.8). It actually turns out that there are no other knot classes satisfying (1.8). This leads to the following central *characterization of D_2-elastic $(2, b)$-torus knots.*

Theorem 1.6 *(D_2-elastic $(2, b)$-torus knots) The following statements hold up to isometry and reparametrization.*

(i) *The unique D_2-elastic $(2, b)$-torus knot for any odd $b \in \mathbb{Z}\backslash\{1, -1\}$ is the tangential pair of co-planar circles with exactly one point in common, denoted by tpc_π. Any sequence of D_2-symmetric E_ϑ-critical $(2, b)$-torus knots Γ_ϑ converges* strongly *in $W^{2,2}$ to tpc_π as $\vartheta \to 0$.*

(ii) *If a D_2-elastic knot for some knot class $\mathcal{K}$ is tpc_π then $\mathcal{K} = \mathcal{T}(2, b)$ for some odd $b \in \mathbb{Z}\backslash\{1, -1\}$.*

(iii) *If any non-trivial knot class $\mathcal{K}$ satisfies*

$$\inf_{\substack{\beta \in \mathscr{C}(\mathcal{K}) \\ \beta \text{ is } D_2\text{-symmetric}}} E(\beta) \le (4\pi)^2 \tag{1.8*}$$

then $\mathcal{K} = \mathcal{T}(2, b)$ for some odd $b \in \mathbb{Z}\backslash\{1, -1\}$.

Note that (1.8) and (1.8*) are in fact equivalent due to the Fáry–Milnor theorem.

The one-parameter family of *tangential pairs of circles* tpc_φ for $\varphi \in [0, \pi]$ was introduced in [24]; see Fig. 2. It consists of (isometric images of) pairs of circles each with radius $1/(4\pi)$ that intersect each other tangentially in at least one point. The parameter φ describes the angle between the two planes spanned by the two circles. Only for $\varphi = 0$ and $\varphi = \pi$ the two planes coincide, and the tangential pair of co-planar circles addressed in the rigidity result, Theorem 4.5, is (an isometric image of) tpc_π; see Example 3.9. Part (i) of Theorem 1.6 improves the weak $W^{2,2}$-subconvergence of D_2-symmetric E_ϑ-critical knots Γ_ϑ, established in Theorem 1.3 for general knot classes $\mathcal{K}$, now to the *strong* convergence of *every* sequence of D_2-symmetric E_ϑ-critical

Fig. 2 The family of tangential pairs of circles tpc_φ parametrized by the opening angle φ between the two planes containing the circles
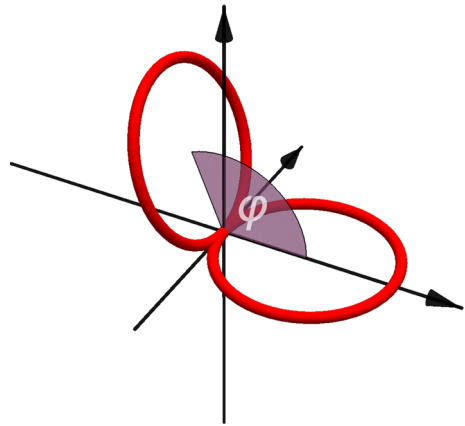
points Γ_ϑ to tpc_π for all torus knot classes $\mathcal{T}(2, b)$. Therefore, the limit curve tpc_π describes the approximate shape of the Γ_ϑ for small ϑ, thus supporting the sometimes experimentally observed final configurations of the numerical gradient flow of Bartels et al. [7]; see Fig. 1 (right). However, as pointed out before, it seems unlikely that these D_2-symmetric critical points Γ_ϑ are local minimizers of E_ϑ. To clarify this, one would need to analyze the second variation of the total energy containing the complicated non-local terms of the tangent-point energy TP_q.

Outline

The paper is structured as follows. In Sect. 2.1 we briefly review the basics of the principle of symmetric criticality along the lines of the presentation in [25] and [26, Section 2], from which we adopted our approach to apply symmetric criticality to knotted space curves. The relevant facts about the tangent-point energy are presented in Sect. 2.2.

In Sect. 3 we discuss the group action in detail, show how to construct and characterize dihedrally symmetric space curves with increasing regularity (Definition 3.3 and Proposition 3.4). Of particular importance in our context are dihedrally symmetric circles and planar tangential pairs of circles whose exact location in space is determined in Corollary 3.11. We also prove in Lemma 3.12 that reparametrization to arclength does not destroy the symmetry and provide a sharp a priori estimate on the size of D_2-symmetric curves; see Lemma 3.13. We finally identify the suitable Banach manifold of $W^{2,2}$-regular knots in Lemma 3.14, on which the group D_2 acts in a sufficiently regular way required by Palais's symmetric criticality principle (Lemma 3.15).

Section 4 is devoted to proving the existence of the D_2-symmetric critical points as stated in Theorem 1.1, by first minimizing a rescaled total energy on symmetric knots (Theorem 4.1). By symmetric criticality these minimizers turn out to be critical points among all knots in the given knot class (Corollary 4.2) satisfying the desired Euler–Lagrange equation (1.5) as shown in Corollary 4.3. Moreover, the existence of D_2-symmetric elastic knots in any given tame knot class, i.e., the proof of Theorem

1.3, is established. The remainder of Sects. 4 and 5 deal with the shape of symmetric elastic knots, that is, the proofs of Theorems 1.5 and 1.6, the latter with the help of a general rigidity result (Theorem 4.5) for all knot classes satisfying (1.8) and an explicit convergence proof of the torus knots of Example 3.10 towards tpc_π carried out in Lemma 5.1.

In Sect. 6 we briefly touch on higher regularity of the symmetric critical knots obtained in Theorem 1.1, as well as on the question whether symmetric elastic knots are embedded. The concept of symmetric knots also applies to other symmetry classes than D_2. We give a brief outlook on the general case in that section.

The simulations shown in Figs. 1 and 5 have been carried out using the algorithm described in [6] which bases on an earlier work by Bartels [4]. We also refer to the app KNOTevolve [5].

2 Preliminaries

2.1 Principle of symmetric criticality

Let us briefly recall the notion of a group action on a Banach manifold modeled over a Banach space to describe symmetry in a mathematically rigorous way, cf. [33, pp. 19–20, 26].

Definition 2.1 For $k \in \mathbb{N}$ let $\mathscr{M}$ be a C^k-Banach manifold modeled over a Banach space $\mathscr{B}$, and suppose that $(G, \circ)$ is a group.

(i) If there is a mapping τ assigning to each $(g, x) \in G \times \mathscr{M}$ a point $\tau_g(x) \in \mathscr{M}$ such that

$$\tau_{g \circ h}(x) = \tau_g(\tau_h(x)) \quad \text{for all} \ \ g, h \in G, \ x \in \mathscr{M}, \tag{2.1}$$

then the group G is said to *act on* $\mathscr{M}$, and τ is called a *representation of G on $\mathscr{M}$*.

(ii) If for each $g \in G$ the mapping $\tau_g : \mathscr{M} \to \mathscr{M}$ is a C^k-diffeomorphism, then $\mathscr{M}$ is called a *G-manifold (of class C^k)*. For an infinite Lie group G one additionally requires that the representation τ is of class C^k on $G \times \mathscr{M}$. If $\mathscr{M}$ is itself a Banach space and τ_g is linear then $\mathscr{M}$ is said to be a *G-space*.

(iii) For a G-manifold $\mathscr{M}$ the *G-symmetric subset* $\Sigma \subset \mathscr{M}$ is defined as

$$\Sigma := \{x \in \mathscr{M} : \tau_g(x) = x \quad \text{for all} \ g \in G\}. \tag{2.2}$$

(iv) A function $F : \mathscr{M} \to \mathbb{R}$ is called *G-invariant* if and only if

$$F(\tau_g(x)) = F(x) \quad \text{for all} \ \ g \in G, \ x \in \mathscr{M}. \tag{2.3}$$

Palais proved the following *symmetric criticality principle.*

Theorem ([33, Theorem 5.4]) *Suppose G is a compact Lie group and $\mathscr{M}$ a G-manifold of class C^1 over the Banach space $\mathscr{B}$ with the non-empty G-symmetric subset $\Sigma \subset \mathscr{M}$,*

and let $F : \mathcal{M} \to \mathbb{R}$ be a G-invariant function of class C^1. Then Σ is a C^1-submanifold of $\mathcal{M}$, and $x \in \Sigma$ is a critical point of F if and only if x is critical for the restricted functional $F|_\Sigma : \Sigma \to \mathbb{R}$. That is, if $D(F|_\Sigma)(x)v = 0$ for all $v \in T_x\Sigma$, then also $DF(x)w = 0$ for all $w \in T_x\mathcal{M}$.

As every finite group is a Lie group, cf., e.g., [16, p. 48, Example 5], we infer the following result that we apply in Sect. 4 to obtain dihedrally symmetric critical knots for the total energy E_ϑ.

Corollary 2.2 (Symmetric criticality for finite groups) *If G is a finite group and $\mathcal{M}$ a G-manifold of class C^1 with non-empty G-symmetric subset $\Sigma \subset \mathcal{M}$, and if $F \in C^1(\mathcal{M})$ is G-invariant, then any critical point of $F|_\Sigma$ is also a critical point of F.*

Our choice of a Banach manifold will simply be an open subset Ω of a Banach space $\mathcal{B}$. This allows us to identify the differential of the energy $F : \Omega \to \mathbb{R}$ with the classic Fréchet-differential $DF(x) : T_x\Omega \simeq \mathcal{B} \to T_{F(x)}\mathbb{R} \simeq \mathbb{R}$, which can be computed by means of the first variation

$$DF(x)h = \delta F(x, h) := \lim_{\epsilon \to 0} \epsilon^{-1}\big[F(x + \epsilon h) - F(x)\big] \quad \text{for } h \in \mathcal{B}.$$

2.2 Tangent-point energy

As mentioned in the introduction, Gonzalez and Maddocks suggested in [27, p. 4773] to consider the tangent-point energy (1.4) as a candidate for a valuable knot energy. This was confirmed in the work of P. Strzelecki and the third author [38] starting at a rather low level of regularity with just rectifiable curves. In fact, arclength parametrizations $\Gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ of rectifiable curves with finite tangent-point energy $\mathrm{TP}_q(\Gamma) < \infty$ for $q \geq 2$ are either injective or they are multiple coverings of one-dimensional manifolds [38, Theorem 1.1]. In addition, such curves Γ are of class $C^{1,1-(2/q)}$ if $q > 2$; see [38, Theorem 1.3].

In the present context, however, dealing with the bending energy E we start at the already higher regularity level of closed $W^{2,2}$-curves which—according to the Morrey–Sobolev embedding theorem—are automatically of class $C^{1,1/2}$. Consequently, it suffices to review Blatt's regularity results [9,10] on C^1-curves with finite tangent-point energy.[4] One of the central results [9,10, Theorem 1.1] characterizes finite energy among embedded curves by fractional Sobolev regularity $W^{2-(1/q),q}$. Here, we only need one part of that statement explicitly, in fact, in a slightly sharpened version for not necessarily arclength parametrized curves established in [26, Theorem 3.2 (ii)].[5]

Theorem 2.3 *Let $q \in (2, \infty)$ and suppose that $\gamma \in W^{2-(1/q),q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ is injective and satisfies $|\gamma'| > 0$ on $\mathbb{R}/\mathbb{Z}$. Then $\mathrm{TP}_q(\gamma) < \infty$.*

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[4] Notice that the two-parameter family of energies $\mathrm{TP}^{(p,q)}$ considered in [10] contains the tangent-point energy, more precisely $\mathrm{TP}_q = 2^q \mathrm{TP}^{(2q,q)}$.

[5] For the proof of [26, Lemma A.1], which is used to establish [26, Theorem 3.2 (ii)], see the updated arXiv version.

The other part of Blatt's characterization (or [38, Theorem 1.3] for that matter) can be used to quantify the degree of embeddedness for arclength parametrized curves $\Gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ by means of the *bi-Lipschitz constant*

$$\mathrm{BiLip}(\Gamma) := \inf_{\substack{s,t \in \mathbb{R}/\mathbb{Z} \\ s \neq t}} \frac{|\Gamma(s) - \Gamma(t)|}{|s - t|_{\mathbb{R}/\mathbb{Z}}}.$$

Lemma 2.4 *(Bi-Lipschitz estimate for finite* TP*-energy [10, Proposition 2.7]) For any $q > 2$ and $T > 0$ there is a constant $C = C(q,T) > 0$ such that any arclength parametrized and injective curve $\Gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ with $\mathrm{TP}_q(\Gamma) \leq T$ satisfies*

$$\mathrm{BiLip}(\Gamma) \geq C.$$

So far, we have reported on the effects that finite tangent-point energy has on the curve. Let us conclude this short review with continuity and regularity properties of the energy itself.

Theorem 2.5 (Regularity of the tangent-point energy) *Let $q > 2$. The tangent-point energy TP_q is sequentially lower semicontinuous with respect to C^1-convergence. Moreover, TP_q is continuously differentiable on regular embedded closed curves of fractional Sobolev regularity $W^{2-(1/q),q}$.*

Proof Lower semicontinuity of the tangent-point energy was shown in [37, p. 1513], whereas continuous differentiability was verified in [42] using the first variation formula in [10, Theorem 1.4] and the line of arguments used for the corresponding regularity statement for integral Menger curvature in [11, Theorem 3]. □

2.3 Isotopy stability

Several times in the proofs we will rely on the fact that knot classes are stable with respect to C^1-perturbations. Variants of the following statement can be found in [8, 18,19,34].

Lemma 2.6 (Ambient isotopy is open in C^1) *For any embedded $\gamma \in C^1(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$ there is an $\varepsilon > 0$ such that any $\tilde{\gamma} \in C^1(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$ with $\|\tilde{\gamma} - \gamma\|_{C^1} < \varepsilon$ is also embedded and belongs to the same knot class as γ.*

3 Group action on parametrized curves

In order to describe the dihedral symmetry of parametrized closed curves $\gamma : \mathbb{R}/\ell\mathbb{Z} \to \mathbb{R}^3$ we use two different representations of the dihedral group $D_2 := \{d_0 \equiv e, d_1, d_2, d_3\}$ with the multiplication table depicted in Table 1, where e denotes the identity element. Namely, in view of the symmetry of the curves' images we consider the subgroup $\{\mathrm{Id}_{\mathbb{R}^3} \equiv R_0, R_1, R_2, R_3\} \subset SO(3)$ containing the rotations R_i about

Table 1 The multiplication table of the dihedral group D_2

	e	d_1	d_2	d_3
e	e	d_1	d_2	d_3
d_1	d_1	e	d_3	d_2
d_2	d_2	d_3	e	d_1
d_3	d_3	d_2	d_1	e

the coordinate axes $\mathbb{R}\mathbf{e}_i$ for $i = 1, 2, 3$, with rotational angle π, that is, written as matrices with respect to the standard coordinate basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \subset \mathbb{R}^3$,

$$R_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad R_2 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad R_3 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.1)$$

To take into account the curves' parametrizations, we use in addition the mappings $\psi_i^\ell : \mathbb{R}/\ell\mathbb{Z} \to \mathbb{R}/\ell\mathbb{Z}$ on the periodic domain $\mathbb{R}/\ell\mathbb{Z}$, defined as

$$\psi_i^\ell(t) := \begin{cases} t \pmod{\ell} & \text{for } i = 0, \\ -t + \frac{\ell}{2} \pmod{\ell} & \text{for } i = 1, \\ t - \frac{\ell}{2} \pmod{\ell} & \text{for } i = 2, \\ -t + \ell \pmod{\ell} & \text{for } i = 3. \end{cases} \quad (3.2)$$

It is easy to check that for all mutually distinct $i, j, k \in \{1, 2, 3\}$ the following identities hold.

$$R_i \circ R_i = \mathrm{Id}_{\mathbb{R}^3} \quad \text{and} \quad \psi_i^\ell \circ \psi_i^\ell = \mathrm{Id}_{\mathbb{R}/\ell\mathbb{Z}}, \quad (3.3)$$

$$R_i \circ R_j = R_k \quad \text{and} \quad \psi_i^\ell \circ \psi_j^\ell = \psi_k^\ell, \quad (3.4)$$

$$R_i|_{\mathbb{R}\mathbf{e}_i} = \mathrm{Id}_{\mathbb{R}\mathbf{e}_i} \quad \text{and} \quad R_i|_{\mathbb{R}\mathbf{e}_k} = R_j|_{\mathbb{R}\mathbf{e}_k} = -\mathrm{Id}|_{\mathbb{R}\mathbf{e}_k}. \quad (3.5)$$

Now we define how D_2 acts on the Banach space $C^0(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ of continuously parametrized closed curves (equipped with the norm $\|\cdot\|_{C^0}$).

Definition 3.1 Let $\tau^\ell : D_2 \times C^0(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3) \to C^0(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$, mapping $(d_i, \gamma) \mapsto \tau_{d_i}^\ell(\gamma)$ for $d_i \in D_2$, $i = 0, 1, 2, 3$, and $\gamma \in C^0(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$, be given by

$$\tau_{d_i}^\ell(\gamma)(t) := R_i \circ \gamma\big(\psi_i^\ell(t)\big) \quad \text{for } t \in \mathbb{R}/\ell\mathbb{Z}, \ i = 0, 1, 2, 3. \quad (3.6)$$

Lemma 3.2 ($C^0(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$ *is a smooth D_2-space*) *The mapping τ^ℓ acts on $C^0(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$, and under this action $C^0(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$ becomes a smooth D_2-space.*

Proof It is obvious that $\tau_{d_i}^\ell(\gamma) \in C^0(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$ for each $i = 0, 1, 2, 3$, and $\gamma \in C^0(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$, since any rotation in the image and affine linear transformation of the periodic domain neither changes the C^0-regularity nor the ℓ-periodicity.

According to the multiplication table of the group D_2 (see Table 1) and by the properties (3.3)–(3.5) we have

$$\tau^\ell_{d_i \circ d_i}(\gamma)(t) \overset{\text{Table 1}}{=} \tau^\ell_e(\gamma)(t) \overset{(3.6),(3.2)}{=} \gamma(t) \text{ for all } t \in \mathbb{R}/\ell\mathbb{Z}, \ i = 0, 1, 2, 3.$$

On the other hand, again by (3.6) and (3.3),

$$\tau^\ell_{d_i}\big(\tau^\ell_{d_i}(\gamma)\big)(t) \overset{(3.6)}{=} R_i \circ \big(\tau^\ell_{d_i}(\gamma)(\cdot)\big)(\psi^\ell_i(t))$$
$$\overset{(3.6)}{=} R_i \circ R_i \circ \gamma(\psi^\ell_i(\psi^\ell_i(t))) \overset{(3.3)}{=} \gamma(t) \text{ for all } t \in \mathbb{R}/\ell\mathbb{Z}, \ i = 0, 1, 2, 3,$$

which proves the homomorphism property (2.1) for identical group elements in D_2. For $i \neq j = 0$ there is nothing to prove since $j = 0$ corresponds to the identity elements in the respective representations of D_2. For $i \neq j, i, j \in \{1, 2, 3\}$ we use the multiplication rules in Table 1 and the definition (3.6) to find, on the one hand,

$$\tau^\ell_{d_i \circ d_j}(\gamma)(t) \overset{\text{Table 1}}{=} \tau^\ell_{d_k}(\gamma) \overset{(3.6)}{=} R_k \circ \gamma(\psi^\ell_k(t)), \tag{3.7}$$

whereas (3.6), as well as (3.4) lead to

$$\tau^\ell_{d_i}\big(\tau^\ell_{d_j}(\gamma)\big)(t) \overset{(3.6)}{=} \tau^\ell_{d_i}\big(R_j \circ \gamma(\psi^\ell_j(\cdot))\big)(t)$$
$$\overset{(3.6)}{=} R_i \circ R_j \circ \gamma(\psi^\ell_j(\psi^\ell_i(t))) \overset{(3.4)}{=} R_k \circ \gamma(\psi^\ell_k(t)),$$

which equals the expression in (3.7). We have shown so far that τ is indeed a representation of D_2 on $C^0(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$. Since $\tau^\ell_{d_i}(\sigma\gamma + \eta)(t) = R_i \circ (\sigma\gamma + \eta)(\psi^\ell_i(t)) = \sigma R_i \circ \gamma(\psi^\ell_i(t)) + R_i \circ \eta(\psi^\ell_i(t)) = \sigma\tau^\ell_{d_i}(\gamma) + \tau^\ell_{d_i}(\eta)$ for all $\gamma, \eta \in C^0(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$ and $\sigma \in \mathbb{R}$, one finds that $\tau^\ell_{d_i} : C^0(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3) \to C^0(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$ is linear for all $i = 0, 1, 2, 3$, so that the Banach space $C^0(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$ is indeed a D_2-space in the sense of Definition 2.1, part (ii).   □

Symmetric curves are of particular interest here, which are defined as follows.

Definition 3.3 (Dihedrally symmetric curves) A curve is called *dihedrally symmetric* or *D_2-symmetric* if it belongs to the *D_2-symmetric set*

$$\Sigma^\ell := \{\gamma \in C^0(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3) : 0 < \mathcal{L}(\gamma) < \infty, \ \tau^\ell_{d_i}(\gamma) = \gamma \text{ for all } d_i \in D_2\}. \tag{3.8}$$

Now we provide a method to systematically construct examples of D_2-symmetric curves (of finite length). For that we glue copies of an open or closed arc $\alpha : [0, \ell/4] \to \mathbb{R}^3$ of finite length together to obtain a mapping $g : [0, \ell) \to \mathbb{R}^3$ as

$$g(t) := \begin{cases} \alpha(t) & \text{for } t \in [0, \ell/4), \\ R_1 \circ \alpha(\psi^\ell_1(t)) & \text{for } t \in [\ell/4, \ell/2), \\ R_2 \circ \alpha(\psi^\ell_2(t)) & \text{for } t \in [\ell/2, 3\ell/4), \\ R_3 \circ \alpha(\psi^\ell_3(t)) & \text{for } t \in [3\ell/4, \ell), \end{cases} \tag{3.9}$$

and investigate first under which circumstances this glueing process leads to a closed curve with a certain regularity. Notice that $\psi_i^\ell([i\ell/4, (i+1)\ell/4)) = (0, \ell/4]$ for $i = 1, 3$, and $\psi_2^\ell([\ell/2, 3\ell/4)) = [0, \ell/4)$ by definition of the ψ_j^ℓ in (3.2), so that g in (3.9) is well-defined. It turns out that this construction does not only produce examples of D_2-symmetric curves but also *characterizes* this symmetry[6].

Proposition 3.4 (i) *$\gamma \in \Sigma^\ell$ if and only if there exists an arc $\alpha \in C^0([0, \ell/4], \mathbb{R}^3)$ of positive and finite length satisfying*

$$\alpha(0) \in \mathbb{R}\mathbf{e_3} \quad and \quad \alpha(\ell/4) \in \mathbb{R}\mathbf{e_1}, \tag{3.10}$$

such that γ coincides with the ℓ-periodic extension of g defined in (3.9). In particular, the points $\gamma(0)$ and $\gamma(\ell/2)$ are contained in $\mathbb{R}\mathbf{e_3}$, whereas $\gamma(\ell/4)$ and $\gamma(3\ell/4)$ are contained in $\mathbb{R}\mathbf{e_1}$.

(ii) *$\gamma \in \Sigma^\ell \cap W^{1,p}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$ for some $p \in [1, \infty]$ if and only if $\gamma = g$, where α is of class $W^{1,p}$ and satisfies (3.10).*

(iii) *$\gamma \in \Sigma^\ell \cap C^1(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$ if and only if $\gamma = g$ in the sense of (i), where $\alpha \in C^1([0, \ell/4], \mathbb{R}^3)$ satisfies (3.10) and*

$$\alpha'(0) \in \text{span}\{\mathbf{e_1}, \mathbf{e_2}\} \quad and \quad \alpha'(\ell/4) \in \text{span}\{\mathbf{e_2}, \mathbf{e_3}\}. \tag{3.11}$$

Moreover, $\gamma \in \Sigma^\ell \cap W^{2,p}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$ for some $p \in [1, \infty]$ if and only if, in addition to the above properties, α is of class $W^{2,p}$.

(iv) *Let S_i be the reflection in the coordinate plane $\mathbf{e}_i^\perp$ for $i = 1, 2, 3$. Then, $S_i(\Sigma^\ell) \subset \Sigma^\ell$, $S_i(\Sigma^\ell \cap C^1) \subset \Sigma^\ell \cap C^1$, and $S_i(\Sigma^\ell \cap W^{k,p}) \subset \Sigma^\ell \cap W^{k,p}$ for $p \in [1, \infty]$, $k = 1, 2$.*

The proof of this proposition will follow from the following partial results.

Lemma 3.5 (Glueing produces closed curves) *Suppose $\alpha \in C^0([0, \ell/4], \mathbb{R}^3)$ has length $\mathscr{L}(\alpha) \in (0, \infty)$, then the mapping g defined according to (3.9) has length $\mathscr{L}(g) = 4\mathscr{L}(\alpha)$. Moreover, g is closed and continuous, i.e., of class $C^0(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$ if and only if (3.10). Finally, if α is continuously differentiable, the curve g is of class $C^1(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$ if and only if in addition to (3.10) the tangents of α satisfy (3.11).*

Proof Since α is continuous one has $\mathscr{L}_{[0,\ell/4)}(g) = \mathscr{L}(\alpha)$ (see, e.g., [31, VIII, Section 5, Theorem 1, p. 223]), and because the rotated images of α have the same length as α, the statement about $\mathscr{L}(g)$ is immediate.

The ℓ-periodic extension of the piecewise defined curve g is continuous if and only if the following four identities hold true:

$$g(0) = \lim_{t \nearrow \ell} R_3 \circ \alpha(-t + \ell), \quad g(\ell/4) = \lim_{t \nearrow \ell/4} \alpha(t), \tag{3.12}$$

$$g(\ell/2) = \lim_{t \nearrow \ell/2} R_1 \circ \alpha(-t + \ell/2), \quad g(3\ell/4) = \lim_{t \nearrow 3\ell/4} R_2 \circ \alpha(t - \ell/2), \tag{3.13}$$

---

[6] Fixing the rotational axes with (3.1) and the corresponding parameter transformations in (3.2) enforces a rigidity on the class of D_2-symmetric curves, which is reflected in the statements of Proposition 3.4 and Corollary 3.11 below.

where we have already plugged in the definition (3.9) in the respective limits on the right-hand sides. Using the continuity of α on the right-hand side we obtain the conditions $\alpha(0) = R_3 \circ \alpha(0)$ and $\alpha(\ell/4) = R_1 \circ \alpha(\ell/4)$ that are equivalent to (3.12), and with the help of (3.3) and (3.4) exactly the same conditions equivalent to (3.13). From (3.5) we infer $\ker(\mathrm{Id} - R_i) = \mathbb{R}\mathbf{e}_i$ for $i = 1, 2, 3$, which implies that these identities on the endpoints $\alpha(0)$ and $\alpha(\ell/4)$ are equivalent to (3.10).

Provided that α is continuosly differentiable, C^1-regularity of g is equivalent to the pointwise conditions (3.12) and (3.13) in combination with the tangential conditions

$$g'(0) = \lim_{t \nearrow \ell} R_3 \circ \big(-\alpha'(-t+\ell) \big), \quad g'(\ell/4) = \lim_{t \nearrow \ell/4} \alpha'(t), \tag{3.14}$$

$$g'(\ell/2) = \lim_{t \nearrow \ell/2} R_1 \circ \big(-\alpha'(-t+\ell/2) \big), \quad g'(3\ell/4) = \lim_{t \nearrow 3\ell/4} R_2 \circ \alpha'(t-\ell/2), \tag{3.15}$$

where the minus signs in the respective left equations are a consequence of the chain rule. Now by continuity of α' we obtain from (3.14) the equivalent conditions $\alpha'(0) = -R_3 \circ \alpha'(0)$ and $\alpha'(\ell/4) = -R_1 \circ \alpha'(\ell/4)$. From $\ker(\mathrm{Id} + R_i) = \mathrm{span}\{\mathbf{e}_j, \mathbf{e}_k\}$ we deduce that they are equivalent to (3.11). Exploiting (3.15) again with the help of (3.3) and (3.4) leads to the same conditions on $\alpha'(0)$ and $\alpha'(\ell/4)$.   □

From the Morrey–Sobolev embedding in one dimension together with well-known glueing properties for Sobolev functions [2, E3.6 & E3.7] one readily obtains the following corollary.

Corollary 3.6 [Glueing Sobolev arcs] *If $\alpha \in W^{1,p}((0, \ell/4), \mathbb{R}^3)$ for any $p \in [1, \infty]$, then g defined in (3.9) is a closed curve of class $W^{1,p}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$ if and only if the continuous representative of α satisfies (3.10). If $\alpha \in W^{2,p}((0, \ell/4), \mathbb{R}^3)$, then g is a closed curve of class $W^{2,p}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$ if and only if the C^1-representative of α satisfies (3.10) and (3.11).*

Now we are in the position to prove that the constructed curve g in (3.9) is also D_2-symmetric.

Lemma 3.7 (D_2-symmetric curves) *If $\alpha \in C^0([0, \ell/4], \mathbb{R}^3)$ has finite and positive length and satisfies (3.10), then the curve g defined in (3.9) is D_2-symmetric, that is, $g \in \Sigma^\ell$. If α is of class $W^{1,p}$ for some $p \in [1, \infty]$, then so is the D_2-symmetric curve g, and if $\alpha \in C^1([0, \ell/4], \mathbb{R}^3)$ or if α is of class $W^{2,p}$, and satisfies (3.11) in addition, then g is a D_2-symmetric closed curve of class C^1, or $W^{2,p}$, respectively.*

Proof The regularity statements follow from Lemma 3.5 and Corollary 3.6, so it suffices to prove D_2-symmetry, for which we merely need to show that

$$\tau_{d_i}^\ell(g) = g \quad \text{for } i = 1, 2, 3. \tag{3.16}$$

We only treat the case $i = 1$ in full detail, the cases $i = 2, 3$ are very similar.

For $t \in [0, \ell/4)$ we have $\psi_1^\ell(t) \in (\ell/4, \ell/2]$, and therefore by definition of g

$$\tau_{d_1}^\ell(g)(t) \overset{(3.6)}{=} R_1 \circ g(\psi_1^\ell(t)) \overset{(3.9)}{=} R_1 \circ R_1 \circ \alpha(\psi_1^\ell \circ \psi_1^\ell(t)) \overset{(3.3)}{=} \alpha(t) = g(t).$$

Notice that we have also used the continuity of g established in Lemma 3.5 to treat the parameter $t = 0$, since $\psi_1^\ell(0) = \ell/2$, so that

$$g(\psi_1^\ell(0)) = g(\ell/2) = \lim_{t \nearrow \ell/2} g(t) \stackrel{(3.9)}{=} \lim_{t \nearrow \ell/2} R_1 \circ \alpha(\psi_1^\ell(t)) = R_1 \circ \alpha(0).$$

For $t \in [\ell/4, \ell/2)$ one has $\psi_1^\ell(t) \in (0, \ell/4]$, so that

$$\tau_{d_1}^\ell(g)(t) \stackrel{(3.6)}{=} R_1 \circ g(\psi_1^\ell(t)) \stackrel{(3.9)}{=} R_1 \circ \alpha(\psi_1^\ell(t)) \stackrel{(3.9)}{=} g(t),$$

where again, we have used the continuity of g to treat $t = \ell/4$ by means of $g(\psi_1^\ell(\ell/4)) = g(\ell/4) = \lim_{t \nearrow \ell/4} g(t)$ similarly as above.

For $t \in [\ell/2, 3\ell/4)$ we have $\psi_1^\ell(t) \in (-\ell/4, 0] \equiv (3\ell/4, \ell] \pmod{\ell}$, so that

$$\begin{aligned}
\tau_{d_1}^\ell(g)(t) &\stackrel{(3.6)}{=} R_1 \circ g(\psi_1^\ell(t)) \stackrel{(3.9)}{=} R_1 \circ R_3 \circ \alpha(\psi_3^\ell \circ \psi_1^\ell(t)) \\
&\stackrel{(3.4)}{=} R_2 \circ \alpha(\psi_2^\ell(t)) \stackrel{(3.9)}{=} g(t),
\end{aligned}$$

where the parameter $t = \ell/2$ was treated as before via $g(\psi_1^\ell(\ell/2)) = g(0) = \lim_{t \nearrow \ell} g(t)$. Finally, for $t \in [3\ell/4, \ell)$, one has $\psi_1^\ell(t) \in (-\ell/2, -\ell/4] \equiv (\ell/2, 3\ell/4] \pmod{\ell}$, so that

$$\begin{aligned}
\tau_{d_1}^\ell(g)(t) &\stackrel{(3.6)}{=} R_1 \circ g(\psi_1^\ell(t)) \stackrel{(3.9)}{=} R_1 \circ R_2 \circ \alpha(\psi_2^\ell \circ \psi_1^\ell(t)) \\
&\stackrel{(3.4)}{=} R_3 \circ \alpha(\psi_3^\ell(t)) \stackrel{(3.9)}{=} g(t),
\end{aligned}$$

the parameter $t = 3\ell/4$ treated by continuity of g via $g(\psi_1^\ell(3\ell/4)) = g(-\ell/4) = g(3\ell/4) = \lim_{t \nearrow 3\ell/4} g(t)$ by ℓ-periodicity of g.

Thus, we have finished the detailed argument for $i = 1$. The case $i = 3$ is analogous, and $i = 2$ is somewhat simpler, since there is no inversion in the domain which saves us the additional continuity argument to treat the respective boundary parameters. $\qquad\square$

Now we can present the

Proof of Proposition 3.4 For part (i) assume that $\gamma \in \Sigma^\ell$ so $\gamma = R_i \circ \gamma(\psi_i^\ell(\cdot))$ for $i = 0, 1, 2, 3$. Let $\alpha := \gamma|_{[0,\ell/4]}$ and define g according to (3.9). Then

$$\begin{aligned}
g(t) &= R_j \circ \alpha(\psi_j^\ell(t)) = R_j \circ \gamma(\psi_j^\ell(t)) = \gamma(t) \\
&\text{for } t \in [j\ell/4, (j+1)\ell/4), \ j = 0, 1, 2, 3.
\end{aligned}$$

The other implication in part (i) and also parts (ii) and (iii) follow from Lemmata 3.5 and 3.7, and Corollary 3.6. Finally, part (iv) follows from the characterizations of Σ^ℓ, $\Sigma^\ell \cap C^1$ and $\Sigma^\ell \cap W^{k,p}$ for $k = 1, 2$ in terms of the generating arc α, established in the previous parts (i)–(iii). These characterizations can be combined with the fact
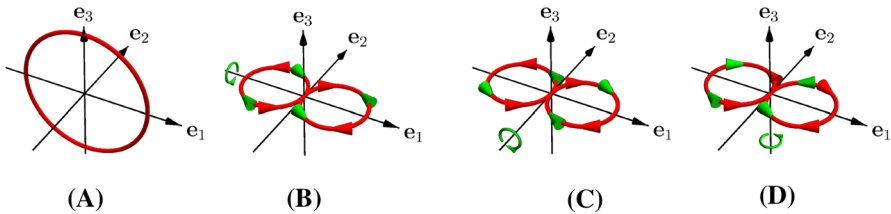
Fig. 3 D_2-symmetric curves generated by (3.9). **A** The once covered circle in the $\mathbf{e_1}$-$\mathbf{e_3}$-plane from Example 3.8. **B–D** A tangential pair of co-planar circles exhibiting symmetries with respect to the coordinate axes as described in Example 3.9. 180° rotations about the coordinate axes $\mathbb{R}\mathbf{e_i}$ indicated in green are accompanied by parameter transformations ψ_i^ℓ that reverse orientation for $i = 1, 3$ in **B** and **D**, while preserving it for $i = 2$ in **C**; see red and green arrows along the curves

that the conditions (3.10) and (3.11) on α are invariant under the reflections S_i for $i = 1, 2, 3$.   □

We use the glueing mechanism (3.9) to construct a few explicit examples of dihedrally symmetric curves parametrized on $\mathbb{R}/\ell\mathbb{Z}$.

Example 3.8 As a first generating arc $\alpha_1 \in C^\infty([0, \ell/4], \mathbb{R}^3)$ we choose a quadrant (i.e., one quarter of a circle) in the $\mathbf{e_1}$-$\mathbf{e_3}$-plane with arclength $\ell/4$, that is,

$$\alpha_1(t) := \frac{\ell}{2\pi} \begin{pmatrix} \sin(2\pi t/\ell) \\ 0 \\ \cos(2\pi t/\ell) \end{pmatrix} \qquad \text{for } t \in [0, \ell/4], \tag{3.17}$$

so that $\alpha := \alpha_1$ satisfies the conditions (3.10), (3.11), and the regularity assumptions of Lemma 3.5, Corollary 3.6, and Lemma 3.7. According to these results the curve $g \equiv g_1$ defined in (3.9) for this particular choice of $\alpha = \alpha_1$ is a $C^{1,1}$-closed and D_2-symmetric curve, that is, $g_1 \in \Sigma^\ell \cap C^{1,1}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$. It is easy to check that g_1 is the once covered circle whose parametrization equals (3.17) if one extends the domain of the latter to all of $[0, \ell]$, so g_1 is actually C^∞ on $\mathbb{R}/\ell\mathbb{Z}$; see Fig. 3A.

Example 3.9 We construct a dihedrally symmetric tangential pair of co-planar circles with exactly one self-intersection point within the $\mathbf{e_1}$-$\mathbf{e_2}$-plane; see Fig. 3B–D. For that we take as a generating arc $\alpha_2 \in C^\infty([0, \ell/4], \mathbb{R}^3)$ a semicircle of radius $\ell/(4\pi)$. To be more precise, we set

$$\alpha_2(t) := \frac{\ell}{4\pi} \begin{pmatrix} 1 - \cos(4\pi t/\ell) \\ \sin(4\pi t/\ell) \\ 0 \end{pmatrix} \qquad \text{for } t \in [0, \ell/4], \tag{3.18}$$

and easily check that conditions (3.10) and (3.11) as well as the regularity assumptions of Lemma 3.5, Corollary 3.6, and Lemma 3.7 are satisfied for $\alpha := \alpha_2$. Glueing according to (3.9) yields the D_2-symmetric tangential pair of co-planar circles with parametrization $g \equiv g_2 \in \Sigma^\ell \cap C^{1,1}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$, which in the case $\ell = 1$ we also denote by tpc_π. This curve has the same trace as the corresponding curve in [24, Formula (3.2) for $\varphi := \pi$], only with reversed orientation.

Fig. 4 Dihedrally symmetric torus knots of class $\mathcal{T}(2, 5)$ constructed in Example 3.10 for $\epsilon = 0.03$ (left) and for $\epsilon = 0.003$ (right)

In a similar manner, we may construct a tpc_π-curve in the $\mathbf{e_2}$-$\mathbf{e_3}$-plane by letting

$$\alpha_2(t) := \frac{\ell}{4\pi} \begin{pmatrix} 0 \\ \sin(4\pi t/\ell) \\ 1 + \cos(4\pi t/\ell) \end{pmatrix} \qquad \text{for } t \in [0, \ell/4]. \tag{3.19}$$

We will show in Corollary 3.11 that (3.18) and (3.19) are the *only two options* for the generating arc to construct D_2-symmetric tangential pairs of circles tpc_π.

Our third example produces dihedrally symmetric torus knots of class $\mathcal{T}(2, b)$ for any odd $b \in \mathbb{Z}\backslash\{1, -1\}$; see Fig. 4.

Example 3.10 The generating curve α_3 consists of a helical part h, which after the glueing forms together with its rotated copies a rational tangle that determines the knot class [1, Section 2.3], and a piece σ of a stadium curve, which after the glueing closes the tangle to form the knot; see Fig. 4. For the precise formulas, which we are going to take up again in Sect. 5 to compute the infimal bending energy on D_2-symmetric torus knots, it suffices to consider a fixed odd integer $b \geq 3$. Indeed, the reflection of a D_2-symmetric representative of the torus knot class $\mathcal{T}(2, b)$ in a coordinate plane $\mathbf{e_i}$ for $i = 1, 2, 3$, produces a representative of $\mathcal{T}(2, -b)$ that still possesses the D_2-symmetry according to part (iv) of Proposition 3.4. We also fix two parameters $\epsilon, \varrho > 0$, and define the helical part $h = h^\epsilon$ of the generating arc as

$$h^\epsilon(t) := \begin{pmatrix} \varrho(-1)^{(b-1)/2} \sin\phi_\epsilon(t) \\ t \\ \varrho \cos\phi_\epsilon(t) \end{pmatrix} \qquad \text{for } t \in [0, \infty), \tag{3.20}$$

where $\phi_\epsilon(t) := \pi \cdot \phi(t/\epsilon)$ for the piecewise smooth parameter transformation $\phi \in C^{1,1}([0, \infty))$ given by

$$\phi(t) := \begin{cases} t & \text{for } t \in \left[0, \frac{b-1}{2}\right], \\ \frac{b}{2} - \frac{1}{2}\left(t - \frac{b+1}{2}\right)^2 & \text{for } t \in \left[\frac{b-1}{2}, \frac{b+1}{2}\right], \\ \frac{b}{2} & \text{for } t \geq \frac{b+1}{2}. \end{cases} \tag{3.21}$$

Note that

$$h^\varepsilon\left(\frac{(b+1)\varepsilon}{2}\right) = \begin{pmatrix} \varrho(-1)^{(b-1)/2}\sin\frac{b\pi}{2} \\ \frac{(b+1)\varepsilon}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \varrho \\ \frac{(b+1)\varepsilon}{2} \\ 0 \end{pmatrix}. \tag{3.22}$$

The portion $\sigma = \sigma^\epsilon$ of a stadium curve of class $C^{1,1}$ consists of a semicircle of radius $r = r(\epsilon)$ satisfying

$$(b+1)\epsilon + \pi r = \ell/4, \tag{3.23}$$

and a (short) straight segment attached in a C^1-manner to the semicircle. The precise definition is

$$\sigma^\epsilon(t) := \begin{cases} \begin{pmatrix} \varrho + r - r\cos\left(\frac{1}{r}(t-(b+1)\epsilon/2)\right) \\ (b+1)\epsilon/2 + r\sin\left(\frac{1}{r}(t-(b+1)\epsilon/2)\right) \\ 0 \end{pmatrix} & \text{for } t \in \left[\frac{(b+1)\epsilon}{2}, \frac{\ell}{4} - \frac{(b+1)\epsilon}{2}\right], \\ \\ \begin{pmatrix} \varrho + 2r \\ (\ell/4) - t \\ 0 \end{pmatrix} & \text{for } t \in \left[\frac{\ell}{4} - \frac{(b+1)\epsilon}{2}, \frac{\ell}{4}\right]. \end{cases} \tag{3.24}$$

The generating arc α_3^ϵ for fixed $\epsilon > 0$ satisfying (3.23) is now defined as

$$\alpha_3^\epsilon(t) := \begin{cases} h^\epsilon(t) & \text{for } t \in \left[0, \frac{b+1}{2}\cdot\epsilon\right], \\ \sigma^\epsilon(t) & \text{for } t \in \left(\frac{b+1}{2}\cdot\epsilon, \frac{\ell}{4}\right], \end{cases} \tag{3.25}$$

and one easily checks with the help of (3.23) and (3.22) that α_3^ϵ itself is of class $C^{1,1}$ on $[0, \ell/4]$. Moreover, $\alpha_3^\epsilon(0) = h^\epsilon(0) = (0, 0, \varrho)^T \in \mathbb{R}\mathbf{e_3}$ and $\alpha_3^\epsilon(\ell/4) = (\varrho + 2r, 0, 0)^T \in \mathbb{R}\mathbf{e_1}$ as required in (3.10) of Lemma 3.5. Finally, $(\alpha_3^\epsilon)'(0) = (\varrho(-1)^{(b-1)/2}\pi/\epsilon, 1, 0)^T \in \mathrm{span}\{\mathbf{e_1}, \mathbf{e_2}\}$ and $(\alpha_3^\epsilon)'(\ell/4) = (0, -1, 0)^T \in \mathrm{span}\{\mathbf{e_2}, \mathbf{e_3}\}$, so that also (3.11) is satisfied. Therefore, all assumptions of Lemma 3.5, Corollary 3.6, and Lemma 3.7 are satisfied, so that the glueing (3.9) yields a D_2-symmetric torus knot $g_3^\epsilon \in \Sigma^\ell \cap C^{1,1}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$ representing the knot class $\mathcal{T}(2, b)$.

We establish in Section 5 the strong $W^{2,2}$-convergence of these torus knots g_3^ϵ for $\ell = 1$ to the tangential pair of co-planar circles constructed in Example 3.9, i.e., to tpc_π as $\epsilon \to 0$.

In light of the above examples, it is in order to ask about the location of the dihedrally symmetric curves appearing in the main theorems, namely the round circle and the tangential pair of circles tpc_π.

Corollary 3.11 (D_2-symmetric circle and tpc_π) *Up to reparametrization there are a unique D_2-symmetric circle and precisely two D_2-symmetric tpc_π-curves (which are of course isometric).*

(i) *Any D_2-symmetric circle $c : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ is centered at the origin and contained in the plane perpendicular to $\mathbf{e_2}$ with initial point $c(0) \in \mathbb{R}\mathbf{e_3}$.*

(ii) *A D_2-symmetric $\mathrm{tpc}_\pi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ is located in one of the two coordinate planes $\mathbf{e_1^\perp}$ and $\mathbf{e_3^\perp}$. The self-intersection point is at the origin and its tangent is parallel to $\mathbf{e_2}$ in both cases.*

Proof (i) The center M of any circle $C \subset \mathbb{R}^3$, which – as a set – is D_2-symmetric, is the origin, since if not, we could find $i \in \{1, 2, 3\}$, such that $R_i M \neq M$ and hence $R_i C$ is a circle of the same length as C, but with a center $R_i M$ different from M. Therefore, the sets $R_i C$ and C differ which contradicts the dihedral symmetry of C. Consequently, any injective arclength parametrization $c : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ of the once covered circle with D_2-symmetry satisfies the antipodal relation $c(t) = -c(t + 1/2)$ for all $t \in \mathbb{R}/\mathbb{Z}$. If there is one such parametrization c with D_2-symmetry then its image $c(\mathbb{R}/\mathbb{Z})$ must be contained in span$\{\mathbf{e_1}, \mathbf{e_3}\}$ since, by 1-periodicity,

$$c(t) \stackrel{(3.6)}{=} R_2 \circ c(t - 1/2) = R_2 \circ c(t + 1/2) = -R_2 \circ c(t) \qquad (3.26)$$

for all $t \in \mathbb{R}/\mathbb{Z}$. Moreover, we infer $c(0) \in \mathbb{R}\mathbf{e_3}$ from (3.10) in Proposition 3.4. The existence of such a parametrization was established in Example 3.8, cf. (3.17).

(ii) For tpc_π we can argue in a similar way to see that the self-intersection point is at the origin. By Lemma 3.12 below we may assume that tpc_π is an arclength parametrized curve $\mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$. Let the unit tangent of tpc_π at the origin be denoted by $\nu \in \mathbb{S}^2$. Due to symmetry, $\mathrm{tpc}_\pi(t + \frac{1}{2})$ is just the image of a 180-degree rotation R_ν of $\mathrm{tpc}_\pi(t)$ about ν. Thus we obtain

$$R_\nu \circ \mathrm{tpc}_\pi(t) = \mathrm{tpc}_\pi(t + \tfrac{1}{2}) \stackrel{(3.6)}{=} R_2 \circ \mathrm{tpc}_\pi(t) \qquad \text{for all } t \in \mathbb{R}/\mathbb{Z}.$$

This implies that the matrix product $R_\nu^{-1} R_2$ is the identity on the hyperplane which contains (the image of) tpc_π. As it belongs to $SO(3)$, it must even be $\mathrm{Id}_{\mathbb{R}^3}$, in particular $\nu = \mathbf{e_2}$.

From part (i) of Proposition 3.4 we infer that the image of tpc_π must contain points in $\mathbb{R}\mathbf{e_k} \setminus \{0\}$ for $k = 1$ or $k = 3$. Together with $\nu = \mathbf{e_2}$ we conclude that the planar curve tpc_π belongs either to $\mathbf{e_1^\perp}$ or $\mathbf{e_3^\perp}$. Both configurations are realized by Example 3.9, cf. (3.18) and (3.19). □

According to the following result one can reparametrize D_2-symmetric curves without affecting the symmetry. This, as well as the subsequent uniform a priori bound on the size of dihedrally symmetric curves, turns out to be useful ingredients in the existence proofs of Sect. 4.

Lemma 3.12 (Symmetry of arclength parametrization) *If $\gamma \in \Sigma^\ell$ has length $\mathscr{L}(\gamma) = L$, then its arclength parametrization Γ is also D_2-symmetric, that is, Γ is contained in the D_2-symmetric set Σ^L defined as in (3.8).*

Proof Differentiating the symmetry relation $\gamma = \tau_{d_i}^\ell(\gamma)$ on $\mathbb{R}/\ell\mathbb{Z}$ we obtain by (3.6) and (3.2)

$$\left|\gamma'(t)\right| = \left|\tau_{d_i}^\ell(\gamma)'(t)\right| \overset{(3.6)}{=} \left|R_i \circ \gamma'(\psi_i^\ell(t))(\psi_i^\ell)'(t)\right| \overset{(3.2)}{=} \left|\gamma'(\psi_i^\ell(t))\right| \quad (3.27)$$

for all $t \in \mathbb{R}/\ell\mathbb{Z}$ and $i = 0, 1, 2, 3$. This identity can be used to compute the arclength parameter

$$s\left(\psi_i^\ell(t)\right) = \mathscr{L}\left(\gamma|_{[0,\psi_i^\ell(t)]}\right) = \int_0^{\psi_i^\ell(t)} \left|\gamma'(\tau)\right| d\tau \overset{(3.27)}{=} \int_0^{\psi_i^\ell(t)} \left|\gamma'(\psi_i^\ell(\tau))\right| d\tau$$

$$= \int_{\psi_i^\ell(0)}^t \left|\gamma'(z)\right| \frac{1}{(\psi_i^\ell)'(\psi_i^\ell(z))} dz = \left(\text{sign}(\psi_i^\ell)'\right)\left[s(t) - s(\psi_i^\ell(0))\right],$$

(3.28)

where we changed variables to $z := \psi_i^\ell(\tau)$ with $z(0) = \psi_i^\ell(0)$, $z(\psi_i^\ell(t)) = \psi_i^\ell \circ \psi_i^\ell(t) = t$ by virtue of (3.3). Notice also that $(\psi_i^\ell)'(\cdot) = (-1)^i$ for $i = 0, 1, 2, 3$. Using (3.28) with $i = 2$ and $t = \ell$ we infer $2s(\frac{\ell}{2}) = s(\ell) = L$. Now it is easy to check that

$$\left(\text{sign}(\psi_i^\ell)'\right)\left[s(t) - s(\psi_i^\ell(0))\right] = \psi_i^L(s(t)) \quad \text{for all } t \in \mathbb{R}/\ell\mathbb{Z}, \ i = 0, 1, 2, 3, \quad (3.29)$$

where ψ_i^L is the transformation defined in (3.2) only with ℓ replaced by L. In other words, D_2 acts on the domain $\mathbb{R}/L\mathbb{Z}$ of the arclength parametrization Γ via the transformations ψ_i^L, $i = 0, 1, 2, 3$. Combining (3.28) with (3.29) we arrive at

$$s\left(\psi_i^\ell(t)\right) = \psi_i^L(s(t)) \quad \text{for all } t \in \mathbb{R}/\ell\mathbb{Z}, \ i = 0, 1, 2, 3, \quad (3.30)$$

so that the symmetry of γ leads to

$$\Gamma(s(t)) = \gamma(t) \overset{(3.6)}{=} \tau_{d_i}^\ell(\gamma)(t) \overset{(3.6)}{=} R_i \circ \gamma(\psi_i^\ell(t)) = R_i \circ \Gamma(s(\psi_i^\ell(t)))$$

$$\overset{(3.30)}{=} R_i \circ \Gamma\left(\psi_i^L(s(t))\right) \overset{(3.6)}{=} \tau_{d_i}^L(\Gamma)(s(t)) \quad (3.31)$$

for all $t \in \mathbb{R}/\ell\mathbb{Z}$, $i = 0, 1, 2, 3$, which establishes the symmetry of Γ. $\qquad\square$

Lemma 3.13 (Optimal L^∞-bound) *A closed curve $\gamma \in C^0(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$ of length $\Lambda \in (0, \infty)$ whose image has dihedral symmetry is contained in the closure of the ball $B_{\Lambda/4}(0)$.*

Proof Assume to the contrary that there is a point $(x_1, x_2, x_3) = x := \gamma(s)$ such that $|x| > \Lambda/4$. We may assume without loss of generality that $|x_1| \geq |x_2| \geq |x_3|$. The symmetry assumption means

$$R_1 \circ \gamma(\mathbb{R}/\ell\mathbb{Z}) = R_2 \circ \gamma(\mathbb{R}/\ell\mathbb{Z}) = R_3 \circ \gamma(\mathbb{R}/\ell\mathbb{Z}) = \gamma(\mathbb{R}/\ell\mathbb{Z}), \quad (3.32)$$

so that we have $R_i(x) \in \gamma(\mathbb{R}/\ell\mathbb{Z})$ for $i = 1, 2, 3$, and some permutation of the four points $A := x$, $B := R_1(x)$, $C := R_2(x)$, $D := R_3(x)$ forms a polygon inscribed in γ. By direct computation we infer

$$S := |A - B| = |C - D| = 2\sqrt{x_2^2 + x_3^2},$$

$$M := |A - C| = |B - D| = 2\sqrt{x_1^2 + x_3^2},$$

$$L := |A - D| = |B - C| = 2\sqrt{x_1^2 + x_2^2}, \tag{3.33}$$

with $0 \leq S \leq M \leq L$ according to our assumption on the coordinates of x. Of all the possible choices of permutations of the points A, B, C, D, the two closed polygons $P_1 := ABDCA$ and $P_2 := ACDBA$ have the shortest length $\mathscr{L}(P_1) = \mathscr{L}(P_2) = 2S + 2M$, which by means of (3.33) leads to the contradictive inequality

$$\Lambda \geq 2S + 2M \overset{(3.33)}{=} 4\left(\sqrt{x_2^2 + x_3^2} + \sqrt{x_1^2 + x_3^2}\right) \geq 4|x| > \Lambda.$$

$\square$

This L^∞ bound is optimal, since one can think of a sequence of ellipses of length L, all centered at the origin and contained in a fixed coordinate plane, converging to a straight segment of length $L/2$ on one coordinate axis. All such ellipses are contained in the ball of radius $L/4$ centered at the origin (cf. [32]).

Recall from the introduction the set $W_{\mathrm{ir}}^{2,2}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$ of closed, regular and embedded $W^{2,2}$-curves, each of which represents a tame knot class. Thus, for a given knot class $\mathcal{K}$ we introduce the subset

$$\Omega_{\mathcal{K}}^\ell := \{\gamma \in W_{\mathrm{ir}}^{2,2}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3) : [\gamma] = \mathcal{K}\}, \tag{3.34}$$

which – according to the Morrey–Sobolev embedding $W^{2,2} \hookrightarrow C^1$ – is the empty set unless $\mathcal{K}$ is tame; see Footnote 1. First we observe that $\Omega_{\mathcal{K}}^\ell$ is a Banach manifold.

Lemma 3.14 ($\Omega_{\mathcal{K}}^\ell$ is a Banach manifold) *For any fixed tame knot class $\mathcal{K}$ the set $\Omega_{\mathcal{K}}^\ell$ is a non-empty open subset of the Banach space $W^{2,2}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$, hence a Banach manifold.*

Proof By the Morrey–Sobolev embedding result any curve $\gamma \in \Omega_{\mathcal{K}}^\ell$ is a regular C^1-knot representing the knot class $\mathcal{K}$, so that according to Lemma 2.6 the curve γ possesses a neighbourhood $\mathcal{U} \subset C^1(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$ such that any curve $\xi \in \mathcal{U}$ is regular and of the same knot type $\mathcal{K}$. Again by means of the Morrey–Sobolev embedding theorem we can choose the radius δ of the ball $B_\delta(\gamma) \subset W^{2,2}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$ so small that $B_\delta(\gamma) \subset \mathcal{U}$, which proves the claim. $\square$

Restricting the group action (3.6) to $\Omega_{\mathcal{K}}^\ell$ yields a smooth D_2-manifold.

Lemma 3.15 ($\Omega_{\mathcal{K}}^\ell$ is D_2-manifold) *The mapping τ^ℓ defined in (3.6) acts on the Banach manifold $\Omega_{\mathcal{K}}^\ell$, and under this action $\Omega_{\mathcal{K}}^\ell$ becomes a smooth D_2-manifold.*

Proof It is easy to see that $\tau_{d_i}^{\ell}(\gamma)$ is contained in $\Omega_{\mathcal{K}}^{\ell}$ for $i = 0, 1, 2, 3$, and $\gamma \in \Omega_{\mathcal{K}}^{\ell}$, since any rotation in the image and affine linear transformation of the periodic domain does not change the $W^{2,2}$-regularity and injectivity on $[0, \ell)$. Moreover, the knot class $\mathcal{K}$ is preserved as well, and

$$\left| \tau_{d_i}^{\ell}(\gamma)'(t) \right| = \left| R_i \circ \gamma'(\psi_i^{\ell}(t)) \left(\psi_i^{\ell} \right)'(t) \right| = \left| \gamma'(\psi_i^{\ell}(t)) \right| > 0 \quad \text{for all } t \in \mathbb{R}/\ell\mathbb{Z}.$$

The algebraic property (2.1) as well as the linearity of $\tau_{d_i}^{\ell} : \Omega_{\mathcal{K}}^{\ell} \to \Omega_{\mathcal{K}}^{\ell}$ for $i = 0, 1, 2, 3$ was verified in the proof of Lemma 3.2. Indeed, the linearity of $\tau_{d_i}^{\ell}$ leads to the differential

$$(d\tau_{d_i}^{\ell})_\gamma \eta = \tau_{d_i}^{\ell}(\eta) \quad \text{for all } \gamma \in \Omega_{\mathcal{K}}^{\ell}, \ \eta \in T_\gamma \Omega_{\mathcal{K}}^{\ell} \simeq W^{2,2}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$$

and $i = 0, 1, 2, 3$. Therefore, $\Omega_{\mathcal{K}}^{\ell}$ is a smooth D_2-manifold, since $\tau_{d_i}^{\ell} : \Omega_{\mathcal{K}}^{\ell} \to \Omega_{\mathcal{K}}^{\ell}$ is a diffeomorphism with (smooth) inverse $\left(\tau_{d_i}^{\ell} \right)^{-1} := \tau_{d_i}^{\ell}$ for each $i = 0, 1, 2, 3$ by means of the properties (3.3). $\qquad \square$

4 Existence theory under the D_2-symmetry constraint

Throughout this section we set $\ell = 1$. Instead of the total energy $E_\vartheta = E + \vartheta \mathrm{TP}_q^{1/(q-2)}$ which is positively (-1)-homogeneous, i.e., $E_\vartheta(r\gamma) = r^{-1} E_\vartheta(\gamma)$ for all $\gamma \in W^{2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ and $r > 0$, we consider the scale-invariant version

$$S_\vartheta(\gamma) := \mathscr{L}(\gamma) \cdot E_\vartheta(\gamma) = \mathscr{L}(\gamma) \cdot \left(E(\gamma) + \vartheta \mathrm{TP}_q^{\frac{1}{q-2}}(\gamma) \right). \tag{4.1}$$

We first minimize this scale-invariant total energy on the class of $W^{2,2}$-knots with dihedral symmetry, that is, we minimize S_ϑ on the D_2-symmetric subset

$$\Sigma_{\mathcal{K}} := \Sigma^1 \cap \Omega_{\mathcal{K}}^1, \tag{4.2}$$

where Σ^{ℓ} for general $\ell > 0$ was defined in (3.8) and $\Omega_{\mathcal{K}}^{\ell}$ in (3.34).

Theorem 4.1 (Symmetric minimizers of total scaled energy) *Assume that $\Sigma_{\mathcal{K}} \neq \emptyset$ for a given knot class $\mathcal{K}$. Then for any $\vartheta > 0$ there exists an arclength parametrized knot $\Gamma_\vartheta \in \Sigma_{\mathcal{K}}$ with length $\mathscr{L}(\Gamma_\vartheta) = 1$, such that*

$$S_\vartheta(\Gamma_\vartheta) = \inf_{\Sigma_{\mathcal{K}}} S_\vartheta(\cdot). \tag{4.3}$$

Before proving this crucial existence result, let us draw some immediate conclusions that also lead to the proofs of Theorems 1.1 and 1.3 stated in the introduction.

Corollary 4.2 (Symmetric S_ϑ-critical points) *Any symmetric locally minimizing knot* $\gamma \in \Sigma_\mathcal{K}$ *of* $S_\vartheta|_{\Sigma_\mathcal{K}}$ *is* S_ϑ-*critical, that is,*

$$DS_\vartheta(\gamma)h = 0 \quad \text{for all} \ \ h \in W^{2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3). \tag{4.4}$$

Proof Small D_2-symmetric variations of a locally minimizing knot $\gamma \in \Sigma_\mathcal{K}$ remain regular and in the same knot class $\mathcal{K}$, so that γ is $(S_\vartheta|_{\Sigma_\mathcal{K}})$-critical, that is, $D(S_\vartheta|_{\Sigma_\mathcal{K}})(\gamma)h = 0$ for all $h \in T_\gamma \Sigma_\mathcal{K}$. Using the definition of the group action (3.6), (3.2) one easily checks that S_ϑ is a D_2-invariant energy. Indeed, both E and TP_q are invariant under Euclidian transformations and reparametrization; here we even do not change the speed due to $|(\psi_i^\ell)'(\cdot)| = 1$ for all $i = 0, 1, 2, 3$. Furthermore, S_ϑ is of class C^1 by means of Theorem 2.5.

Moreover, $\Sigma_\mathcal{K}$ is the non-empty D_2-symmetric subset of the smooth D_2-manifold $\Omega_\mathcal{K}^1$, cf. Lemma 3.15, so that we can apply the version of Palais's principle of symmetric criticality stated in Corollary 2.2. $\qquad\square$

Criticality of S_ϑ is directly related to criticality for the constrained variational problem (lP_ϑ) for the original total energy E_ϑ.

Corollary 4.3 (Euler–Lagrange equation) *Any arclength parametrized critical point* $\Gamma \in W^{2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ *of* S_ϑ *satisfies*

$$DE_\vartheta(\Gamma) + \lambda \cdot D\mathscr{L}(\Gamma) = 0 \tag{4.5}$$

for the Lagrange multiplier $\lambda := E_\vartheta(\Gamma)$. *Moreover, any arclength parametrized critical point for the variational problem* (P_ϑ) *satisfies the same variational equation* (4.5).

Proof The Euler–Lagrange equation (4.5) is a direct consequence of (4.4) via the product rule and because $\mathscr{L}(\Gamma) = 1$. The variational equation for the constrained variational problem (P_ϑ) is

$$DE_\vartheta(\Gamma) + \mu \cdot D\mathscr{L}(\Gamma) = 0 \tag{4.6}$$

for some Lagrange multiplier $\mu \in \mathbb{R}$, since small variations $\Gamma + \epsilon h$ for $h \in W^{2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ remain regular and in the same knot class $\mathcal{K}$ by Morrey's compact embedding $W^{2,2}$ into C^1. Testing (4.6) with Γ itself we can use the fact that $|\Gamma'| \equiv 1$ to find

$$0 = DE_\vartheta(\Gamma)\Gamma + \mu \int_{\mathbb{R}/\mathbb{Z}} |\Gamma'(\tau)|^2 \, d\tau = DE_\vartheta(\Gamma)\Gamma + \mu,$$

so that $\mu = -DE_\vartheta(\Gamma)\Gamma = E_\vartheta(\Gamma)$ by the positive (-1)-homogeneity of E_ϑ. $\qquad\square$

Before proving Theorem 4.1 itself, we turn to another immediate application, namely the existence of symmetric critical knots for the total energy E_ϑ as stated in Theorem 1.1, and the existence of symmetric elastic knots; see Theorem 1.3.

Proof of Theorem 1.1 Since by assumption there is at least one D_2-symmetric knot contained in $\mathscr{C}(\mathcal{K})$ one has $\Sigma_{\mathcal{K}} \neq \emptyset$ so that Theorem 4.1 is applicable. The S_ϑ-minimizing knots $\Gamma_\vartheta \in \Sigma_{\mathcal{K}}$ obtained in that theorem have length one, so $\Gamma_\vartheta \in \mathscr{C}(\mathcal{K})$. They are S_ϑ-critical according to Corollary 4.2. Moreover, $\left|\Gamma'_\vartheta\right| \equiv 1$ on $\mathbb{R}/\mathbb{Z}$ so that Corollary 4.3 implies that the Euler–Lagrange equation (4.5) holds true, which is the variational equation (1.5) stated in Theorem 1.1 with the exact same Lagrange multiplier.   □

Proof of Theorem 1.3 For any $\vartheta > 0$ we find by virtue of Theorem 4.1 an arclength parametrized knot $\Gamma_\vartheta \in \Sigma_{\mathcal{K}}$ (of length $\mathscr{L}(\Gamma_\vartheta) = 1$ hence $\Gamma_\vartheta \in \mathscr{C}(\mathcal{K})$) such that

$$E_\vartheta(\Gamma_\vartheta) \overset{(4.1)}{=} S_\vartheta(\Gamma_\vartheta) \overset{(4.3)}{\leq} S_\vartheta(\beta) = E_\vartheta(\beta)$$

for all D_2-symmetric $\beta \in \mathscr{C}(\mathcal{K})$, since such β are contained in $\Sigma_{\mathcal{K}}$. By definition of the total energy E_ϑ we infer a uniform bound on the bending energies $E(\Gamma_\vartheta)$,

$$E(\Gamma_\vartheta) \leq E_\vartheta(\Gamma_\vartheta) \leq E_\vartheta(\beta) \leq E(\beta) + \mathrm{TP}_q^{\frac{1}{q-2}}(\beta) < \infty \quad \text{for all } \vartheta \in (0, 1] \ (4.7)$$

for all D_2-symmetric curves $\beta \in \mathscr{C}(\mathcal{K})$. Notice that the right-hand side is finite by virtue of Theorem 2.3. Together with the uniform L^∞-bound $\|\Gamma_\vartheta\|_{L^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)} \leq 1/4$, which follows from Lemma 3.13 since $\mathscr{L}(\Gamma_\vartheta) = 1$, we obtain the uniform bound

$$\|\Gamma_\vartheta\|_{W^{2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)} \leq C < \infty \quad \text{for all } \vartheta \in (0, 1]. \tag{4.8}$$

Hence, for any given sequence $\vartheta_j \to 0$ there exists a subsequence $(\vartheta_{j_k})_k \subset (\vartheta_j)_j$ such that the corresponding symmetric minimizing knots $\Gamma_{\vartheta_{j_k}}$ converge weakly in $W^{2,2}$ and strongly in C^1 to a limiting curve Γ_0 as $k \to \infty$. Therefore Γ_0 satisfies $|\Gamma'_0| \equiv 1$ on $\mathbb{R}/\mathbb{Z}$, $\mathscr{L}(\Gamma_0) = 1$, the dihedral symmetry relation $\tau^1_{d_i}(\Gamma_0) = \Gamma_0$ for $i = 0, 1, 2, 3$. By the lower semicontinuity of the bending energy E and by means of (4.7),

$$E(\Gamma_0) \leq \liminf_{k \to \infty} E(\Gamma_{\vartheta_{j_k}}) \leq \liminf_{k \to \infty} E_{\vartheta_{j_k}}(\Gamma_{\vartheta_{j_k}}) \overset{(4.7)}{\leq} \liminf_{k \to \infty} E_{\vartheta_{j_k}}(\beta) = E(\beta)$$

for all D_2-symmetric curves $\beta \in \mathscr{C}(\mathcal{K})$, which is the minimizing property (1.6).   □

Proof of Theorem 4.1 Since $\Sigma_{\mathcal{K}}$ was assumed to be non-empty, we have $\inf_{\Sigma_{\mathcal{K}}} S_\vartheta \in [(2\pi)^2, \infty)$, where we used that the Sobolev space $W^{2,2}$ continuously embeds[7] into the fractional Sobolev space $W^{2-(1/q),q}$ for $q \in (2, 4]$ so that the tangent-point energy of a regular embedded $W^{2,2}$-curve is finite according to Theorem 2.3. Hence there exists a minimal sequence $(\gamma_j)_j \subset \Sigma_{\mathcal{K}}$ with $\lim_{j \to \infty} S_\vartheta(\gamma_j) = \inf_{\Sigma_{\mathcal{K}}} S_\vartheta$. Due to the scale-invariance of S_ϑ we may assume that $\mathscr{L}(\gamma_j) = 1$ for all $j \in \mathbb{N}$, and we can

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[7] This follows, e.g., by the characterization of Sobolev spaces of real positive smoothness in terms of Triebel–Lizorkin and Besov spaces [35, Prop. 2.1.2] applied to an embedding with constant differential dimension [35, Rem. 2.2.3/2].

reparametrize to arclength to obtain a minimal sequence Γ_j with $\left|\Gamma'_j\right| = 1$ for all j, and with

$$\lim_{j\to\infty} E_\vartheta(\Gamma_j) = \lim_{j\to\infty} S_\vartheta(\Gamma_j) = \inf_{\Sigma_{\mathcal{K}}} S_\vartheta(\cdot). \tag{4.9}$$

Note that the first equation holds since we have $\mathscr{L}(\Gamma_j) = 1$ for all $j \in \mathbb{N}$. Moreover, $\Gamma_j \in \Sigma_{\mathcal{K}}$ for all j due to Lemma 3.12 for $\ell = L = 1$, and therefore, $\|\Gamma_j\|_{L^\infty} \le 1/4$ for all $j \in \mathbb{N}$ by virtue of Lemma 3.13. Now (4.9) implies that

$$\int_{\mathbb{R}/\mathbb{Z}} \left|\Gamma''_j(s)\right|^2 ds = E(\Gamma_j) \le E_\vartheta(\Gamma_j) \le \inf_{\Sigma_{\mathcal{K}}} S_\vartheta(\cdot) + 1 < \infty \quad \text{for all } j \gg 1,$$

which together with the uniform L^∞-bound and with $\left|\Gamma'_j\right| \equiv 1$ for all j, yields a uniform bound on the full $W^{2,2}$-norm of the Γ_j for $j \gg 1$. Consequently, there exists a subsequence $(\Gamma_{j_k})_k \subset (\Gamma_j)_j$ converging weakly in $W^{2,2}$ and strongly in C^1 to a limit curve $\Gamma_\vartheta \in W^{2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ as $k \to \infty$. The C^1-convergence implies that $\left|\Gamma'_\vartheta\right| \equiv 1$, and that $\mathscr{L}(\Gamma_\vartheta) = 1$. Moreover, taking the limit $k \to \infty$ in the symmetry relation

$$\tau^1_{d_i}(\Gamma_{j_k})(t) = \Gamma_{j_k}(t) \quad \text{for all } t \in \mathbb{R}/\mathbb{Z}$$

we find $\tau^1_{d_i}(\Gamma_\vartheta) = \Gamma_\vartheta$. To prove that Γ_ϑ is contained in $\Sigma_{\mathcal{K}}$ it suffices to show that Γ_ϑ is embedded since then $[\Gamma_\vartheta] = \mathcal{K}$ by Lemma 2.6. The uniform $W^{2,2}$-bound on the Γ_{j_k} implies by the Morrey–Sobolev embedding also a uniform bound on the $W^{2-(1/q),q}$-norms of the Γ_{j_k}. This in turn yields a uniform positive lower bound B on the the bi-Lipschitz constants $\mathrm{BiLip}(\Gamma_{j_k})$ according to Lemma 2.4. Passing to the limit $k \to \infty$ in the corresponding inequality

$$\left|\Gamma_{j_k}(s) - \Gamma_{j_k}(t)\right| \ge B\,|s - t|_{\mathbb{R}/\mathbb{Z}} \quad \text{for all } s, t \in \mathbb{R}/\mathbb{Z}$$

one obtains from the C^1-convergence $\Gamma_{j_k} \to \Gamma_\vartheta$

$$|\Gamma_\vartheta(s) - \Gamma_\vartheta(t)| \ge B\,|s - t|_{\mathbb{R}/\mathbb{Z}} \quad \text{for all } s, t \in \mathbb{R}/\mathbb{Z}.$$

By Theorem 2.5 the tangent-point energy is lower-semicontinuous with respect to the strong C^1-convergence, and therefore also the total scaled energy S_ϑ with respect to the combined weak $W^{2,2}$- and strong C^1-convergence, which implies by virtue of the fact that $\mathscr{L}(\Gamma_\vartheta) = 1$,

$$\inf_{\Sigma_{\mathcal{K}}} S_\vartheta \le S_\vartheta(\Gamma_\vartheta) = E_\vartheta(\Gamma_\vartheta) \le \liminf_{k\to\infty} S_\vartheta(\Gamma_{j_k}) = \inf_{\Sigma_{\mathcal{K}}} S_\vartheta(\cdot).$$

$\square$

We can now identify the shape of the D_2-elastic unknot as the once covered circle contained in the $\mathbf{e_1}$-$\mathbf{e_3}$-plane with starting point on the $\mathbf{e_3}$-axis. This information is

even more concrete than stated in Theorem 1.5, because of our choice of rotational axes in (3.6) representing the dihedral group D_2 on $\mathbb{R}^3$ and of the dihedral parameter transformations of the domain $\mathbb{R}/\mathbb{Z}$ described in (3.2).

Proof of Theorem 1.5 The once covered circle of length one uniquely minimizes the bending energy E in $\mathscr{C}(\mathcal{K})$ where $\mathcal{K}$ is the unknot class according to the stability result by Langer and Singer [29]. But it also uniquely minimizes the tangent-point energy TP_q by the two uniqueness proofs of Volkmann and Blatt; see [39, Cor. 5.12]. Therefore the once covered circle of length one and all its isometric images also uniquely minimize the total energy E_ϑ within $\mathscr{C}(\mathcal{K})$ for any $\vartheta > 0$. The dihedral symmetry forces the E_ϑ-minimizing circles to lie in the $\mathbf{e_1}$-$\mathbf{e_3}$-plane, with initial point contained in $\mathbb{R}\mathbf{e_3}$; see Corollary 3.11. These properties transfer via C^1-convergence of E_{ϑ_j}-minimizers as $\vartheta_j \to 0$ to the elastic unknot.

Now assume that a D_2-elastic knot for some knot class $\mathcal{K}$ is the once covered circle, which represents the unknot class. According to Lemma 2.6 there is an entire C^1-neighborhood of the once covered circle that only consists of unknots. But by definition of elastic knots this once covered circle is the C^1-limit of E_{ϑ_j}-minimizing knots $\Gamma_{\vartheta_j}, \vartheta_j \to 0$, all representing the knot class $\mathcal{K}$. This implies that $\mathcal{K}$ is the unknot.

Notice finally, for the proof of (1.7), that Fenchel's lower bound of 2π for the total curvature of any closed curve combined with Hölder's inequality implies that $(2\pi)^2$ is the infimal bending energy for the trivial knot class. This value is attained by the once covered circle. Now (1.7) immediately follows from the fact that according to Corollary 3.11 there are round D_2-symmetric circles.   □

Now we turn to non-trivial knot classes satisfying assumption (1.8) on the infimal bending energy. Here we need [24, Theorem A.1] where the Fáry–Milnor theorem on the lower bound for total curvature of non-trivially knotted curves has been extended to the C^1-closure of knots. We restate it for the convenience of the reader.

Theorem 4.4 (Fáry–Milnor extension) *Let $\mathcal{K}$ be a non-trivial (tame) knot class and suppose γ belongs to the C^1-closure of $\mathscr{C}(\mathcal{K})$. Then $\mathrm{TC}(\gamma) \geq 4\pi$.*

This result permits to prove the following rigidity result, which is the essential ingredient for the proof of Theorem 1.6 presented in Section 5.

Theorem 4.5 (Rigidity & strong convergence) *If a knot class $\mathcal{K}$ satisfies (1.8) then any D_2-elastic knot Γ_0 for $\mathcal{K}$ is (up to reparametrization) the tangential pair of co-planar circles with exactly one point in common described in Corollary 3.11. In addition, any subsequence of D_2-symmetric E_ϑ-minimizers $\Gamma_\vartheta \in \mathscr{C}(\mathcal{K})$ converges strongly in $W^{2,2}$ to (an isometric image of) Γ_0 as $\vartheta \to 0$.*

Proof From Theorem 1.5 we infer that $\mathcal{K}$ is nontrivial. Applying the extended Fáry–Milnor Theorem 4.4 to any D_2-elastic knot Γ_0 which according to Theorem 1.3 lies in the C^1-closure of $\mathscr{C}(\mathcal{K})$, we estimate by means of Hölder's inequality

$$(4\pi)^2 \overset{\text{Thm. } 4.4}{\leq} \left(\int_{\Gamma_0} \kappa_{\Gamma_0} \, ds \right)^2 \leq E(\Gamma_0) \overset{(1.6)}{\leq} \inf_{\substack{\beta \in \mathscr{C}(\mathcal{K}) \\ \beta \, D_2\text{-symmetric}}} E(\beta) \overset{(1.8)}{=} (4\pi)^2.$$

(4.10)

Consequently, we have equality everywhere, in particular equality in Hölder's inequality, which implies a constant integrand $\kappa_{\Gamma_0} = 4\pi$ a.e. on $\mathbb{R}/\mathbb{Z}$.

Next we prove that Γ_0 has at least one double point. Indeed, otherwise, by Lemma 2.6 the curve Γ_0 would be contained in $\mathscr{C}(\mathcal{K})$ since Γ_0 is the strong C^1-limit of the E_{ϑ_j}-minimizers $\Gamma_{\vartheta_j} \in \mathscr{C}(\mathcal{K})$ as $\vartheta_j \to 0$. Combining the minimizing property (1.6) of Γ_0 with our assumption (1.8), which also implies that $(4\pi)^2 = \inf_{\mathscr{C}(\mathcal{K})} E$, we find that Γ_0 is an embedded minimizer of the bending energy within $\mathscr{C}(\mathcal{K})$, hence a stable critical point of E. Applying the stability result of Langer and Singer [29], we find Γ_0 to be the once covered circle representing the unknot class, which contradicts the fact that $\mathcal{K}$ is nontrivial. So, we have shown that Γ_0 is not injective.

According to [24, Cor. 3.4] the elastic D_2-symmetric knot Γ_0 belongs, up to isometry and reparametrization, to the one-parameter family of tangentially intersecting circles tpc_φ for $\varphi \in [0, \pi]$ explicitly given in [24, Formula (3.2)]. One easily checks that the only possible candidates that may respect the D_2-symmetry are the doubly covered circle tpc_0 and the tangential pair of co-planar circles tpc_π with only one touching point.

But (any isometric image of) tpc_0 is not only 1-periodic but also 1/2-periodic, so that the symmetry assumption $\tau_{d_2}^1(\mathrm{tpc}_0) = \mathrm{tpc}_0$ would lead to

$$\mathrm{tpc}_0(t) = \mathrm{tpc}_0(t+1/2) = \tau_{d_2}^1(\mathrm{tpc}_0)(t+1/2) \overset{(3.6)}{=} R_2 \circ \mathrm{tpc}_0\big(\psi_2^1(t+1/2)\big)$$
$$\overset{(3.2)}{=} R_2 \circ \mathrm{tpc}_0(t) \ \text{ for all } \ t \in \mathbb{R}/\mathbb{Z}, \tag{4.11}$$

so that $\mathrm{tpc}_0(t) \in \mathbb{R}\mathbf{e_2}$ for all $t \in \mathbb{R}/\mathbb{Z}$, which is a contradiction.

So, (up to isometry) the only remaining option is $\Gamma_0 = \mathrm{tpc}_\pi$, and that this curve indeed has the D_2-symmetry has been verified in Example 3.9.

It remains to establish strong convergence. Now that we have identified the weak $W^{2,2}$-limit of the E_{ϑ_j}-minimizers for any sequence $\vartheta_j \to 0$, we can use our assumption (1.8) to find for any given $\delta > 0$ a D_2-symmetric curve $\beta \in \mathscr{C}(\mathcal{K})$ such that $E(\beta) \leq (4\pi)^2 + \delta$. The minimizing property of the Γ_ϑ yields

$$(4\pi)^2 \leq E(\Gamma_\vartheta) \leq E_\vartheta(\Gamma_\vartheta) \leq E_\vartheta(\beta) \leq (4\pi)^2 + \delta + \vartheta \mathrm{TP}_q(\beta)^{\frac{1}{q-2}} < \infty,$$

where we used the classic Fáry-Milnor theorem for the first inequality. Taking the limit $\vartheta \to 0$ gives

$$(4\pi)^2 \leq \liminf_{\vartheta \to 0} E(\Gamma_\vartheta) \leq \limsup_{\vartheta \to 0} E(\Gamma_\vartheta) \leq (4\pi)^2 + \delta \ \text{ for all } \ \delta > 0.$$

Therefore, $\lim_{\vartheta \to 0} E(\Gamma_\vartheta) = (4\pi)^2 = E(\Gamma_0)$, since $\kappa_{\Gamma_0} = 4\pi$ a.e. on $\mathbb{R}/\mathbb{Z}$. This, together with the C^1-convergence of the Γ_ϑ to the same limit (up to permutation of the axes and reparametrization) leads to convergence in the $W^{2,2}$-norm. Combining this with the weak convergence to the now unique weak limit Γ_0 (up to permutation of the axes and reparametrization) gives finally strong convergence in $W^{2,2}$ by the subsequence principle. □

5 Infimal bending energy on torus knots with dihedral symmetry

We now investigate the convergence properties of the D_2-symmetric torus knots g_3^ϵ introduced in Example 3.10 fixing $\ell := 1$ and $\varrho := \epsilon^2$.

Lemma 5.1 ($W^{2,2}$-convergence of D_2-symmetric torus knots) *Let $\ell = 1$, $\varrho = \epsilon^2$, and $b \in \mathbb{Z}\setminus\{1, -1\}$ be odd. Then the D_2-symmetric torus knots $g_3^\epsilon \in \Sigma_{\mathcal{T}(2,b)}$ constructed in Example 3.10 for $\ell = 1$ and $\varrho = \epsilon^2$ converge strongly in $W^{2,2}$ to the tangential pair of co-planar circles tpc_π (see Example 3.9) as $\epsilon \to 0$.*

Corollary 5.2 (Minimal bending energy for D_2-symmetric torus knots) *The torus knot classes $\mathcal{T}(2, b)$ for odd $b \in \mathbb{Z}\setminus\{1, -1\}$ are the only knot classes $\mathcal{K}$ that satisfy*

$$\inf_{\substack{\beta \in \mathscr{C}(\mathcal{K}) \\ \beta \ D_2\text{-symmetric}}} E(\beta) = (4\pi)^2 = \inf_{\mathscr{C}(\mathcal{K})} E(\cdot). \tag{1.8}$$

Proof The convergence of torus knots established in Lemma 5.1 together with an appropriate rescaling to unit length, i.e., the $W^{2,2}$-convergence of $g_3^\epsilon/\mathscr{L}(g_3^\epsilon) \in \Sigma_{\mathcal{T}(2,b)}$ to tpc_π, allows us to identify the infimal bending energy $(4\pi)^2 = E(\mathrm{tpc}_\pi)$ on the class of D_2-symmetric curves in $\mathscr{C}(\mathcal{T}(2, b))$, since by the Morrey–Sobolev embedding we have $g_3^\epsilon \to \mathrm{tpc}_\pi$ in C^1, and hence $\mathscr{L}(g_3^\epsilon) \to \mathscr{L}(\mathrm{tpc}_\pi) = 1$, and therefore also $g_3^\epsilon/\mathscr{L}(g_3^\epsilon) \to \mathrm{tpc}_\pi$ in $W^{2,2}$ as $\epsilon \to 0$. That the torus knot classes $\mathcal{T}(2, b)$ are the only possible knot classes to satisfy the right equality in (1.8) was proven in [24, Corollary 4.4]. $\qquad\qquad\square$

Proof of Lemma 5.1 For simplicity we restrict the explicit arguments to the case that $b \geq 3$. By symmetry it suffices to prove the $W^{2,2}$-convergence of the generating arcs $\alpha^\epsilon := \alpha_3^\epsilon$ of g_3^ϵ to the corresponding generating arc α_2 of tpc_π defined in Example 3.9. Since α_3^ϵ is piecewise defined (see (3.25)) we focus first on the interval $I_1(\epsilon) := [0, (b+1)\epsilon/2]$ where $\alpha_3^\epsilon = h^\epsilon$, and obtain by direct computation from the explicit expressions (3.20) and (3.21) for the helical part h^ϵ and the parameter transformation ϕ (for $\varrho = \epsilon^2$)

$$\left|(h^\epsilon)''(t)\right|^2 = \pi^4 \phi'^4(t/\epsilon) + \pi^2 \phi''^2(t/\epsilon) \quad \text{for all } t \in I_1(\epsilon), \tag{5.1}$$

which implies

$$\left\|(h^\epsilon)''\right\|_{L^2(I_1(\epsilon),\mathbb{R}^3)}^2 \leq \epsilon\pi^2 \int_0^{(b+1)/2} \left(\pi^2 \phi'^4(z) + 1\right) dz, \tag{5.2}$$

where we changed variables to $z := t/\epsilon$ and used that $\left|\phi''\right| \leq 1$ on $[0, (b+1)/2]$. Now, $\phi' = 1$ on $[0, (b-1)/2]$ whereas $\phi'(t) = -(t-(b+1)/2)$ for $t \in [(b-1)/2, (b+1)/2]$ according to (3.21) so that we obtain from (5.2)

$$\left\|(h^\epsilon)''\right\|_{L^2(I_1(\epsilon),\mathbb{R}^3)}^2 \leq \epsilon\pi^2 \left[\pi^2 \left(\frac{b-1}{2} + \frac{1}{5}\right) + \frac{b+1}{2}\right] < \epsilon\pi^4(b+1). \tag{5.3}$$

The prospective arclength parametrized limit curve tpc_π has constant curvature $\left|\mathrm{tpc}_\pi''\right| = 4\pi$ a.e., so that by virtue of (5.3)

$$\left\|(\alpha^\epsilon)'' - \alpha_2''\right\|_{L^2(I_1(\epsilon),\mathbb{R}^3)}^2 \overset{(5.3)}{\leq} 2\epsilon\pi^4(b+1) + (4\pi)^2(b+1)\epsilon. \qquad (5.4)$$

Now we consider the interval $I_2(\epsilon) := [(b+1)\epsilon/2, (1/4) - (b+1)\epsilon/2]$ and recall our condition (3.23) on the radius r of the stadium curve σ^ϵ, namely (now for $\ell = 1$)

$$(b+1)\epsilon + \pi r = 1/4. \qquad (5.5)$$

This implies that the auxiliary function $f(t,s) := \left|r^{-1}(t - (b+1)s/2) - 4\pi t\right|^2$ satisfies for any $\epsilon \in (0, (8(b+1))^{-1})$

$$0 \leq f(t,s) < 1024\pi^2(b+1)^2\epsilon^2 \quad \text{for all } (t,s) \in [0,1] \times [0,2\epsilon] \qquad (5.6)$$

since $\sqrt{f(t,s)} < 4\pi(b+1)\left|8\varepsilon t - s\right| \leq 4\pi(b+1)\max(8\varepsilon, 2\varepsilon) \leq 32\pi(b+1)\varepsilon$. Inequality (5.6) can be used to estimate

$$\left\|(\alpha^\epsilon)'' - \alpha_2''\right\|_{L^2(I_2(\epsilon),\mathbb{R}^3)}^2 \leq 2\int_{I_2(\epsilon)} \left|(1/r) - 4\pi\right|^2 dt$$
$$+ 2 \cdot (4\pi)^2 \int_{I_2(\epsilon)} \left|\begin{pmatrix} \cos((t - (b+1)\epsilon/2)/r) - \cos(4\pi t) \\ -\sin((t - (b+1)\epsilon/2)/r) + \sin(4\pi t) \\ 0 \end{pmatrix}\right|^2 dt.$$

The first integrand equals $f(1,0)$, and the second integrand can be estimated from above by $2f(t,\epsilon)$ for $t \in I_2(\epsilon)$, since both, cos and sin, have Lipschitz constant 1. Therefore, we can apply the auxiliary estimate (5.6) to arrive at

$$\left\|(\alpha^\epsilon)'' - \alpha_2''\right\|_{L^2(I_2(\epsilon),\mathbb{R}^3)}^2 < \frac{1}{2}(1 + 2(4\pi)^2)1024\pi^2(b+1)^2\epsilon^2, \qquad (5.7)$$

where we also used that $\mathscr{L}^1(I_2(\epsilon)) < 1/4$. Finally, on the interval $I_3(\epsilon) := [(1/4) - (b+1)\epsilon/2, 1/4]$ the stadium curve $\sigma|_{I_3(\epsilon)} = \alpha^\epsilon$ is a straight segment so that

$$\left\|(\alpha^\epsilon)'' - \alpha_2''\right\|_{L^2(I_3(\epsilon),\mathbb{R}^3)}^2 = \left\|\alpha_2''\right\|_{L^2(I_3(\epsilon),\mathbb{R}^3)}^2 = (4\pi)^2(b+1)\epsilon/2. \qquad (5.8)$$

Summarizing (5.4), (5.7), and (5.8) we obtain a constant $C_1 \geq 1$ independent of ϵ such that

$$\left\|(\alpha^\epsilon)'' - \alpha_2''\right\|_{L^2([0,1/4],\mathbb{R}^3)} \leq C_1\sqrt{\epsilon} \quad \text{for all } 0 < \epsilon < \frac{1}{8(b+1)}, \qquad (5.9)$$

which by means of Poincaré's inequality [20, Section 5.8.1] applied to γ' (satisfying $\int_{\mathbb{R}/\mathbb{Z}} \gamma'(\tau) d\tau = 0$ because γ is 1-periodic) implies that there is a constant $C_2 \geq 1$ such that

$$\left\| (\alpha^\epsilon)' - \alpha_2' \right\|_{W^{1,2}([0,1/4],\mathbb{R}^3)} \leq C_2 \sqrt{\epsilon} \quad \text{for all } 0 < \epsilon < \frac{1}{8(b+1)}. \tag{5.10}$$

To conclude the proof it therefore suffices to prove the uniform convergence of the α^ϵ to α_2 on $[0, 1/4]$ as $\epsilon \to 0$. We have for any $t \in [0, \frac{1}{4}]$

$$\left| \alpha^\epsilon(t) - \alpha_2(t) \right| = \left| \alpha^\epsilon(0) - \alpha_2(0) \right| + \left| \int_0^t \left((\alpha^\epsilon)' - \alpha_2' \right) \right|$$

$$\leq \varrho + \sqrt{t} \left\| (\alpha^\epsilon)' - \alpha_2' \right\|_{L^2} \leq \epsilon^2 + \tfrac{1}{2} C_2 \sqrt{\epsilon}.$$

Taking the supremum over $t \in [0, \frac{1}{4}]$ concludes the proof. $\qquad\square$

Proof of Theorem 1.6   (i) Corollary 5.2 implies that the torus knot classes $\mathcal{T}(2, b)$ for odd $b \in \mathbb{Z} \backslash \{1, -1\}$ satisfy condition (1.8), so that Theorem 4.5 is applicable.

(ii) If a D_2-symmetric elastic knot for some knot class $\mathcal{K}$ is tpc_φ then there is a sequence of curves $(\gamma_k)_{k \in \mathbb{N}} \subset \mathscr{C}(\mathcal{K})$ such that $\gamma_k \to \mathrm{tpc}_\pi$ with respect to the C^1-norm. For all γ_k sufficiently close to tpc_π with respect to the C^1-norm, we obtain some cumulative angle $\Delta\beta$ as described in [24, Prop. 4.2] which yields the existence of some odd integer b such that γ_k is a $(2, b)$-torus knot if $|b| \geq 3$ and unknotted if $b = \pm 1$. The latter is ruled out by Theorem 1.5.

(iii) This is an immediate consequence of Corollary 5.2. $\qquad\square$

6 Discussion and open problems

6.1 Higher regularity of D_2-symmetric E_ϑ-critical knots

The Euler–Lagrange operator of TP_q for $q > 2$ studied in [10] seems to be related to the q-Laplacian for which one cannot expect full regularity, at least in the non-fractional case. So it is open whether the arclength parametrized D_2-symmetric critical points of the constrained variational problem (P_ϑ) obtained in Theorem 1.1 are of class $C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$.

Choosing instead of TP_q the decoupled tangent-point functionals $\mathrm{TP}^{(p,2)}$ for $p \in (4, 5)$, cf. Footnote 4, whose domain is a Hilbert space, we can generalize the bootstrapping argument from [10] to obtain C^∞-regularity. We might even derive analyticity by extending the arguments given in [12,36,41].

6.2 Non-embeddedness of symmetric elastic knots

Similarly as in [24, Proposition 3.1] we expect that also symmetric elastic knots for non-trivial knot classes must have double points. According to the stability result of Langer and Singer [29], the only stable critical point of the bending energy E is the once covered circle. However, due to the fact that the symmetry constraint only permits to apply symmetry preserving variations in (1.6), we cannot immediately apply this

tool in the present work. Consequently, in contrast to the case of (not necessarily symmetric) elastic knots treated in [24], we are presently not able to show that

- every D_2-elastic knot for a non-trivial knot class $\mathcal{K}$ must have self-intersection points, and that
- inequality (1.6) is strict unless $\mathcal{K}$ is the unknot class.

It is an interesting question whether one may derive a weaker version of the stability result in [29], that is applicable in our situation, e.g., stating that the round circle would be the only local minimizer within the D_2-symmetric subclass Σ^1 introduced in Definition 3.3. In this case, one could argue as in [24] to conclude that any embedded minimizer of the bending energy within the set $\Sigma_\mathcal{K}$ of D_2-symmetric knots defined in (4.2) would in fact be a local minimizer within Σ^1. Hence, $\mathcal{K}$ would be the unknot contradicting the assumption of a non-trivial knot class $\mathcal{K}$. Thus, the D_2-symmetric minimizer of the bending energy could not be embedded, and the infimum of the bending energy could not be attained in $\Sigma_\mathcal{K}$.

6.3 Other knot classes and symmetries

In a similar manner as in Definition 1.4 we may define *G-elastic knots* for any symmetry group G. Most results from the Introduction, namely Theorems 1.1, 1.3, and 1.5 as well as Corollary 1.2, carry over to more general symmetry groups while Theorem 1.6 is restricted to D_2. We briefly speculate about some examples involving other knot classes or symmetry groups different from D_2.

General torus knots

Let $a, b \in \mathbb{N}$ be coprime with $2 \leq a < b$. We expect the elastic knot to be the a-times covered circle [24] which then would agree with the D_b-elastic knot.

Consequently, the D_a-elastic knot is likely to be the union of a circles of radius $1/(2\pi a)$ that tangentially meet in one common point. The angle between two consecutive circles amounts to $2\pi/a$, which is also observed experimentally by means of the numerical gradient flow of Bartels et al.; see Fig. 5A. An animation is available as an ancillary file at arXiv:2105.08558v2.

(A)   **(B)**   **(C)**
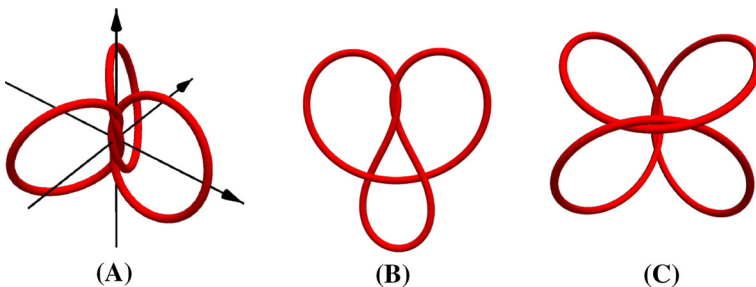
Fig. 5 Numerical approximations of candidates for **A** the D_3-elastic $(3, 4)$-torus knot, **B** the elastic figure-eight knot, **C** the D_2-elastic figure-eight knot

The figure-eight knot

Simulations that have been carried out earlier suggest that an elastic figure eight (4_1) is either planar [3] or spherical [23,24]; see Fig. 5B, C. Recent numerical experiments [6] support the former. So the planar configuration (B) could be a global minimizer within the figure-eight class whereas the spherical configuration (C) might merely be a local minimizer.

Assuming that this is true and that there are no further candidates, we may conjecture that the planar configuration is an elastic knot, while the spherical configuration is a D_2-elastic knot. In contrast to the latter, the former does not enjoy a D_2-symmetry. Using the theory developed above, we can only state that there exists a D_2-elastic figure-eight knot which may or may not coincide with the elastic figure-eight.

Availability of data and material Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The author declares that they have no conflict of interest.

Code availability Details on the code that has been used for the simulations in Sect. 6 can be found in [6].

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