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Harmonic extension through conical surfaces

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Abstract
This paper establishes extension results for harmonic functions which vanish on a conical surface. These are based on a detailed analysis of expansions for the Green function of an infinite cone.

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1 Introduction

The Schwarz reflection principle gives a simple formula for extending a harmonic function h on a domain $\omega \subset \mathbb{R}^N$ through a relatively open subset E of $\partial\omega$ on which h vanishes, provided E lies in a hyperplane. A corresponding reflection formula holds when E lies in a sphere. When $N \geq 3$ and N is odd, Ebenfelt and Khavinson [6] (cf. Chapter 12 of [16]) have shown that a point to point reflection law can only hold when the containing real analytic surface is either a hyperplane or a sphere. Thus more sophisticated methods are needed for extending a harmonic function which vanishes on any other type of set E.

This is the background to the following problem, which was posed by Dima Khavinson at various international conferences: if h is harmonic on an infinite cylinder and vanishes on the boundary, does it extend harmonically to all of $\mathbb{R}^N$? Of course, in the planar case, where h is harmonic on an infinite strip, the answer is readily seen to be positive by repeated application of the Schwarz reflection principle. In higher dimensions the problem was eventually also shown to have an affirmative answer [7] by analysis of the Green function of the cylinder. Subsequently, the authors investigated extension properties of harmonic functions on an annular cylinder

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$\{x' \in \mathbb{R}^{N-1} : a < \|x'\| < b\} \times \mathbb{R}$ that vanish on either one or both of the cylindrical boundary components (see [8,10,11]). The domain reflection results that emerged were noteworthy, given that reflection formulae for the harmonic functions themselves fail to exist. This raises the following general question.

Problem 1 For a domain ω in $\mathbb{R}^N$ and a subset E of $\partial\omega$ identify a larger domain ω_E such that each harmonic function on ω which vanishes continuously on E has a harmonic extension to ω_E.

Naturally we should assume that E is contained in a real-analytic surface, but the question is interesting even in the particular case where E is contained in the zero set of a polynomial. The cylindrical case corresponds to the polynomial $(x', x_N) \mapsto \|x'\|^2 - 1$. The next most natural case to consider is a cone. The analogue of Khavinson's question above would then be: if h is harmonic on an infinite cone and vanishes on the boundary, does h extend harmonically to all of $\mathbb{R}^N$, except for the negative axis of the cone? Again, in the planar case, such extension follows by repeated application of the Schwarz reflection principle.

A typical point of $\mathbb{R}^N$ $(N \geq 3)$ will be denoted by $x = (x', x_N)$, where $x' \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$, and we will write $\theta_x = \cos^{-1}(x_N / \|x\|)$ when $x \neq 0$. Let $0 < \theta_* < \pi$. We will show that harmonic functions h on the infinite cone

$$\Omega = \Omega(\theta_*) = \{x \in \mathbb{R}^N \backslash \{0\} : \theta_x < \theta_*\}$$

that vanish on $\partial\Omega$ have an extension to the set

$$\Omega(\pi) = \{x \in \mathbb{R}^N \backslash \{0\} : \theta_x < \pi\} = \mathbb{R}^N \backslash (\{0'\} \times (-\infty, 0]).$$

In fact, it is unnecessary to require that h vanishes at 0.

Theorem 1 *Let $0 < \theta_* < \pi$. If h is a harmonic function on $\Omega(\theta_*)$ that continuously vanishes on $\partial\Omega(\theta_*) \backslash \{0\}$, then h has a harmonic extension to $\Omega(\pi)$.*

The proof of Theorem 1 is technically more challenging than the corresponding result for the cylinder. However, it also yields tools applicable to reflection results for functions that are harmonic on a domain of the form

$$\Omega(\theta_0, \theta_*) = \{x \in \mathbb{R}^N \backslash \{0\} : \theta_0 < \theta_x < \theta_*\}$$

and vanish on $\partial\Omega(\theta_*)$. Strikingly, a dichotomy emerges between the cases where $\theta_* \leq \pi/2$ and $\theta_* > \pi/2$, as we will now see.

Theorem 2 *Let $0 \leq \theta_0 < \theta_* \leq \pi/2$. If h is a harmonic function on the domain $\Omega(\theta_0, \theta_*)$ that continuously vanishes on $\partial\Omega(\theta_*) \backslash \{0\}$, then h has a harmonic extension to the domain $\{x \in \mathbb{R}^N \backslash \{0\} : \theta_0 < \theta_x < 2\theta_* - \theta_0\}$.*

Theorem 3 *Let $0 \leq \theta_0 < \theta_* < \pi$, where $\theta_* > \pi/2$. If h is a harmonic function on the domain $\Omega(\theta_0, \theta_*)$ that continuously vanishes on $\partial\Omega(\theta_*) \backslash \{0\}$, then h has a harmonic* *extension to the domain*

$$\left\{ x \in \mathbb{R}^N \backslash \{0\} : \theta_0 < \theta_x \text{ and } \tan \frac{\theta_x}{2} \tan \frac{\theta_0}{2} < \left(\tan \frac{\theta_*}{2} \right)^2 \right\}.$$

The conditions arising in Theorems 2 and 3, and their sharpness, will be discussed in Sect. 2. Theorems 1–3 answer particular cases of Questions 4 and 5 in [9]. Theorem 3 also has the following immediate corollary.

Corollary 4 *Let $\pi/2 \le \theta_* < \pi$, and suppose that h is a harmonic function on the domain $\Omega(0, \theta_*)$ that continuously vanishes on $\partial\Omega(\theta_*)\backslash\{0\}$. Then h has a harmonic extension to the domain $\left(\mathbb{R}^{N-1}\backslash\{0'\}\right) \times \mathbb{R}$.*

We now have a reasonably complete set of harmonic extension results for conical surfaces to complement those known for cylinders. Our hope is that these will suggest further steps towards addressing the broader question in Problem 1.

The extension of harmonic functions through conical surfaces is obviously related to extension properties of the Green function for a cone, and harmonic functions on conical domains are naturally related to Legendre functions. The plan of the paper is thus as follows. In Sect. 3 we assemble and develop some relevant material concerning Legendre functions. This is subsequently used, in conjunction with contour integration, to establish an expansion of the fundamental function that is adapted to the geometry of cones, and then two different expansions for the Green function of the cone $\Omega(\theta_*)$. The first of these latter expansions is used to establish the second and also has later application. The second yields extension properties of the Green function that are used in proving Theorem 1. Theorems 2 and 3 rely on both Theorem 1 and further extension properties of the Green function. These latter properties are established using bounds for ratios of conical functions that may be of independent interest.

2 Sharpness of results

The domain of extension in Theorem 2 is formed by angular reflection. This is natural, since in the planar analogue of the result the function h is harmonic in an angle and would extend to an angle of twice the aperture by Schwarz reflection. The sharpness of this result in higher dimensions is demonstrated by the following example.

Example 1 Let $N = 4$ and $0 < \theta_0 < \theta_* < \pi$, where $2\theta_* - \theta_0 < \pi$, and define the planar angle

$$\omega(\theta) = \{(s, t) \in \mathbb{R}^2 : s > 0 \text{ and } t > \|(s, t)\| \cos\theta\} \quad (0 < \theta < \pi).$$

Further, let u be the Green potential in $\omega(\theta_*)$ of a dense sum of point masses on the half line $\partial\omega(\theta_0) \cap \omega(\theta_*)$, and extend u to $\omega(2\theta_* - \theta_0)$ by the Schwarz reflection principle. The function

$$(x', x_4) \mapsto \left\|x'\right\|^{-1} u(\left\|x'\right\|, x_4)$$

is, by computation of the Laplacian, harmonic on $\Omega(2\theta_* - \theta_0)\backslash\overline{\Omega(\theta_0)}$, and it vanishes on $\partial\Omega(\theta_*)\backslash\{0\}$. It cannot be extended as a harmonic function because it is unbounded near every point of $\partial\Omega(\theta_0) \cup \partial\Omega(2\theta_* - \theta_0)$.

Surprisingly, however, when $\theta_* > \pi/2$ and $2\theta_* - \theta_0 < \pi$ the above example no longer gives a sharp bound for how far h can be extended. To compare the domains of extension in Theorems 2 and 3 we note that, if $0 \leq \theta_0 < \theta_* < \theta_x < \pi$, where $\theta_* > \pi/2$, then

$$\theta_0 + \theta_x \leq \pi \Longrightarrow \tan\frac{\theta_x}{2}\tan\frac{\theta_0}{2} < \left(\tan\frac{\theta_*}{2}\right)^2,$$

$$\tan\frac{\theta_x}{2}\tan\frac{\theta_0}{2} \leq \left(\tan\frac{\theta_*}{2}\right)^2 \Longrightarrow \theta_0 + \theta_x < 2\theta_*, \tag{1}$$

where the latter inequality follows from the observation that

$$\begin{aligned}\frac{\tan(\theta_x/2)}{\tan(\theta_*/2)}\frac{\tan(\theta_0/2)}{\tan(\theta_*/2)} &= \exp\left(\int_{\theta_*}^{\theta_x}\csc\theta\, d\theta - \int_{\theta_0}^{\theta_*}\csc\theta\, d\theta\right)\\ &\geq \exp\left(\int_{2\theta_*-\theta_x}^{\theta_*}\csc\theta\, d\theta - \int_{\theta_0}^{\theta_*}\csc\theta\, d\theta\right) = \frac{\tan(\theta_0/2)}{\tan\left(\theta_* - \theta_x/2\right)}.\end{aligned}$$

(By csc we mean $1/\sin$.)

The sharpness of Theorem 3 is shown by the next example.

Example 2 Let $N = 3$ and $0 < \theta_0 < \theta_* < \pi$, let $y = (\sin\theta_0, 0, \cos\theta_0)$ and $w_\theta = (\sin\theta, 0, \cos\theta)$, and let S denote the unit sphere in $\mathbb{R}^3$. The Green function $\mathbf{G}_{\theta_*}$ for the Laplace-Beltrami operator on $S \cap \Omega(\theta_*)$ satisfies

$$\mathbf{G}_{\theta_*}(w_\theta, y) = \log\frac{\left|\tan^2(\theta_*/2) - \tan(\theta/2)\tan(\theta_0/2)\right|}{\left|\tan(\theta/2) - \tan(\theta_0/2)\right|\tan(\theta_*/2)}.$$

(See, for example, formula (13) in [12].) Hence the function defined by $h(x) = \mathbf{G}_{\theta_*}(x/\|x\|, y)$, which satisfies the hypotheses of Theorem 3, has a singularity at w_θ if $\tan(\theta/2)\tan(\theta_0/2) = (\tan(\theta_*/2))^2$.

Let T denote the stereographic projection that maps a typical point x of $S\backslash\{(0', -1)\}$ to the point where the line through $(0', -1)$ and x meets the plane $\mathbb{R}^2 \times \{1\}$. Then any point of $S \cap \partial\Omega(\theta)$ is mapped by T to a point of the form $(y', 1)$, where $\|y'\| = 2\tan(\theta/2)$. Hence, in Theorem 3, the intersection of the enlarged domain with S is mapped by T to an annulus, of which the outer boundary circle is the image of the inner boundary circle under inversion in $T(S \cap \partial\Omega(\theta_*))$.

3 Preparatory material

The ultraspherical (or Gegenbauer) polynomials $C_n^{(\lambda)}$, where $\lambda > 0$ and $n = 0, 1, 2, ...$, are defined by the equation

$$(1 - 2t\xi + \xi^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(t)\xi^n \quad (\xi \in (-1, 1), t \in [-1, 1]) \tag{2}$$

(see (4.7.23) in Szegö [21], where the notation $P_n^{(\lambda)}$ is used instead). They satisfy the differential equation

$$(1 - t^2) f''(t) - (2\lambda + 1)t f'(t) + n(n + 2\lambda) f(t) = 0 \tag{3}$$

and clearly

$$C_n^{(\lambda)}(-t) = (-1)^n C_n^{(\lambda)}(t) \tag{4}$$

(see (4.7.4) and (4.7.5) in [21]). We will also need the fact that

$$\left| C_n^{(\lambda)}(t) \right| \le C_n^{(\lambda)}(1) = \binom{n + 2\lambda - 1}{n} \quad (|t| \le 1) \tag{5}$$

(see Lemma 6(i) of [7]).

The Legendre (or Ferrers) functions of the first and second kinds, P_ν^μ and Q_ν^μ, respectively, are defined on the interval $(-1, 1)$ by equations (14.3.1) and (14.3.2) of [19]. (That source uses Roman type, P_ν^μ and Q_ν^μ, to distinguish functions defined on $(-1, 1)$ from functions on $(1, \infty)$.) They satisfy the equation

$$(1 - t^2) f''(t) - 2t f'(t) + \left(\nu(\nu + 1) - \frac{\mu^2}{1 - t^2} \right) f(t) = 0 \quad (-1 < t < 1) \tag{6}$$

(see (14.2.2) in [19]). We collect below some properties of these functions.

Lemma 5 (i) *The ultraspherical polynomials are connected to the Legendre functions by the formula*

$$C_n^{\left(\frac{N-2}{2}\right)}(t) = \frac{2^{\frac{N-3}{2}} \Gamma\left(\frac{N-1}{2}\right) \Gamma(n + N - 2)}{(1 - t^2)^{\frac{N-3}{4}} \Gamma(N - 2)\Gamma(n + 1)} P_{n+\frac{N-3}{2}}^{\frac{3-N}{2}}(t) \quad (|t| < 1, n = 0, 1, \ldots).$$

(ii) *If* $\mu \in \mathbb{R}$ *and* $p \in \mathbb{Z}$, *then* $P_{\mu+p}^{-\mu}(t) = (-1)^p P_{\mu+p}^{-\mu}(-t)$.

(iii) *If* $\mu \in \mathbb{R}$, *then*

$$(1 - t^2) \frac{d P_\nu^{-\mu}}{dt}(t) = (\nu + 1)t P_\nu^{-\mu}(t) - (\mu + \nu + 1) P_{\nu+1}^{-\mu}(t).$$

(iv) *If* $-1 < t < 1$, *then*

$$(1-t^2)\left(P_\nu^{-\mu}(t)\frac{dQ_\nu^{-\mu}}{dt}(t) - Q_\nu^{-\mu}(t)\frac{dP_\nu^{-\mu}}{dt}(t)\right) = \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}, \tag{7}$$

$$(1-t^2)\left(P_\nu^{-\mu}(t)\frac{d}{dt}P_\nu^{-\mu}(-t) - P_\nu^{-\mu}(-t)\frac{d}{dt}P_\nu^{-\mu}(t)\right) = \frac{2}{\Gamma(\nu+\mu+1)\Gamma(\mu-\nu)}. \tag{8}$$

(v) *[Mehler–Dirichlet formula] If* $\mu \geq 0$, $0 < \theta < \pi$ *and* $\nu \in \mathbb{C}$, *then*

$$P_\nu^{-\mu}(\cos\theta) = \frac{\sqrt{2}}{\sqrt{\pi}(\sin\theta)^\mu \Gamma(\mu+\frac{1}{2})}\int_0^\theta \cos\left(\left(\nu+\frac{1}{2}\right)t\right)(\cos t - \cos\theta)^{\mu-\frac{1}{2}}\,dt. \tag{9}$$

In particular, $P_{-\nu-1}^{-\mu} = P_\nu^{-\mu}$.

(vi) *If* $\mu \geq 0$ *and* $0 < \theta < \pi$, *then the function* $\nu \mapsto P_\nu^{-\mu}(\cos\theta)$ *has infinitely many zeros, all of which are real and simple. The positive zeros form an increasing sequence* (ν_m) *which satisfies* $\nu_m > \mu + m - 1$. *All the remaining zeros are of the form* $\{-\nu - 1 : \nu$ *is a positive zero*$\}$.

(vii) *If* $\nu \in \mathbb{C}$, *then*

$$\left|P_\nu^{-\mu}(\cos\theta)\right| \leq 2^{3/2}\sqrt{\pi}\left(\frac{\sin\theta}{1+\cos\theta}\right)^\mu \frac{e^{|\mathrm{Im}\nu|\theta}}{\Gamma(\mu+\frac{1}{2})} \quad \left(0 \leq \theta \leq \frac{\pi}{2}\right),$$

$$\left|P_\nu^{-\mu}(\cos\theta)\right| \leq 2^{3/2}\sqrt{\pi}\left(\frac{1-\cos\theta}{\sin\theta}\right)^{\max\{\mu,\frac{1}{2}\}} \frac{e^{|\mathrm{Im}\nu|\theta}}{\Gamma(\mu+\frac{1}{2})} \quad \left(\frac{\pi}{2} < \theta < \pi\right).$$

(viii) *If* $\nu \geq \mu \geq 0$ *and* $-1 < t < 1$, *then*

$$\left\{P_\nu^{-\mu}(t)\right\}^2 + \left\{\frac{2}{\pi}Q_\nu^{-\mu}(t)\right\}^2 \leq \frac{4^\mu}{\pi\left(1-t^2\right)^{\max\{\mu,\frac{1}{2}\}}}\left\{\frac{\Gamma\left(\frac{\nu+\mu+1}{2}\right)\Gamma(\nu-\mu+1)}{\Gamma\left(\frac{\nu-\mu}{2}+1\right)\Gamma(\nu+\mu+1)}\right\}^2.$$

(ix) *If* $\nu \geq \mu \geq 0$ *and* $-1 < t < 1$, *then*

$$\left|\frac{d}{dt}\frac{P_\nu^{-\mu}(t)}{(1-t^2)^{\mu/2}}\right| \leq \frac{2^{\mu+1}}{\sqrt{\pi}\left(1-t^2\right)^{\max\{\mu+1,\mu/2+5/4\}}}\frac{\Gamma(\nu-\mu+1)\Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\Gamma(\nu+\mu+1)\Gamma\left(\frac{\nu-\mu+1}{2}\right)}.$$

(x) *If* $-1 < t < 1$ *and* $\mu \geq 0$, *then*

$$2\nu(\nu+1)\int_t^1 \tau\left\{P_\nu^{-\mu}(\tau)\right\}^2 d\tau = \left((1-t^2)\frac{dP_\nu^{-\mu}}{dt}\right)^2$$

$$+\left\{P_\nu^{-\mu}(t)\right\}^2\left\{\nu(\nu+1)(1-t^2)-\mu^2\right\}. \tag{10}$$

Proof (i)–(v). See (14.3.21), (14.9.10), (14.10.4), (14.2.4), (14.2.3) and (14.12.1) of [19].

(vi) It is shown in [17] (cf. Section 238 of [14]) that the function $\nu \mapsto P_\nu^{-\mu}(\cos\theta)$ has infinitely many zeros, all of which are real. The argument in [18] shows that these zeros are simple and $\nu_m > \mu + m - 1$. (These results are given for the case where $\mu > 0$, but the arguments extend easily to cover also the case where $\mu = 0$.) The final assertion of (vi) is a consequence of (v), since $P_{-\nu-1}^{-\mu} = P_\nu^{-\mu}$ and it follows from (9) that $P_\nu^{-\mu} \neq 0$ when $-1 \leq \nu \leq 0$.

(vii) To see that this follows from (v) we note that

$$\left|\cos\left(\left(\nu+\tfrac{1}{2}\right)t\right)\right| = \tfrac{1}{2}\left|e^{i\left(\nu+\frac{1}{2}\right)t} + e^{-i\left(\nu+\frac{1}{2}\right)t}\right| \leq e^{|\mathrm{Im}\,\nu|t},$$

$$(\cos t - \cos\theta)^{\mu-\frac{1}{2}} \leq (1-\cos\theta)^{\mu-\frac{1}{2}} \quad \left(0 \leq t \leq \theta,\ \mu \geq \tfrac{1}{2}\right),$$

and

$$\begin{aligned}\int_0^\theta (\cos t - \cos\theta)^{\mu-\frac{1}{2}}\,dt &= \int_0^\theta \left(2\sin\frac{t+\theta}{2}\sin\frac{\theta-t}{2}\right)^{\mu-\frac{1}{2}} dt \\ &\leq \frac{2^{\mu-\frac{1}{2}}}{(\min\{\sin(\theta/2),\sin\theta\})^{\frac{1}{2}-\mu}}\int_0^\theta \left(\frac{\theta-t}{\pi}\right)^{\mu-\frac{1}{2}} dt \\ &\leq \frac{2\theta^{\mu+\frac{1}{2}}\pi^{\frac{1}{2}-\mu}}{(\sin\theta)^{\frac{1}{2}-\mu}} \quad \left(0 \leq \mu < \frac{1}{2}\right),\end{aligned}$$

since $\sin\phi \geq 2\phi/\pi$ on $(0,\pi/2)$, and sin is concave and satisfies $\sin(\phi/2) \geq (\sin\phi)/2$ on $(0,\pi)$. If $0 \leq \theta \leq \pi/2$, the desired estimate now follows on noting that $1-\cos\theta = \left(\sin^2\theta\right)/(1+\cos\theta)$. If $\pi/2 < \theta < \pi$, we instead note that $\min\{\sin(\theta/2),\sin\theta\} \geq (\sin\theta)/\sqrt{2}$.

(viii) When $\mu \geq \frac{1}{2}$ this follows on combining equations (5) and (19) in Durand [4], and when $0 \leq \mu < \frac{1}{2}$ we instead use (5) and (23) there.

(ix) This follows on combining the first two lines of (29) with (5) in [4].

(x) This is equivalent to formula (5.3) in [15]. We recall the short proof here for completeness. Let $F(t)$ denote the right hand side of (10). Then, by (6),

$$\begin{aligned}F'(t) &= 2(1-t^2)\frac{dP_\nu^{-\mu}}{dt}\left\{(1-t^2)\frac{d^2P_\nu^{-\mu}}{dt^2} - 2t\frac{dP_\nu^{-\mu}}{dt}\right\} \\ &\quad +2P_\nu^{-\mu}(\tau)\frac{dP_\nu^{-\mu}}{dt}\left\{\nu(\nu+1)(1-t^2)-\mu^2\right\} - 2\nu(\nu+1)t\left\{P_\nu^{-\mu}(t)\right\}^2 \\ &= -2\nu(\nu+1)t\left\{P_\nu^{-\mu}(t)\right\}^2.\end{aligned}$$

We see from (iii) and (vii) that $F(t) \to 0$ as $t \to 1-$, so the result follows. □

If $x', y' \in \mathbb{R}^{N-1}$, then we define $\phi_{x',y'} \in [0, \pi]$ by the equation

$$\cos\phi_{x',y'} = \frac{\langle x', y'\rangle}{\|x'\|\,\|y'\|}$$

whenever the denominator is non-zero. We also recall that $\cos\theta_x = x_N/\|x\|$. The following result shows how some of the above functions relate to harmonicity.

Proposition 6 *Let $w \in \mathbb{C}$, $y' \in \mathbb{R}^{N-1}\backslash\{0'\}$ and $k \in \mathbb{N}\cup\{0\}$. Then the function*

$$h(x) = (\sin\theta_x)^{\frac{3-N}{2}} \frac{e^{w\log\|x\|}}{\|x\|^{\frac{N-2}{2}}} P_{w-\frac{1}{2}}^{\frac{3-N}{2}-k}(\cos\theta_x) C_k^{\left(\frac{N-3}{2}\right)}(\cos\phi_{x',y'}) \quad (N \geq 4),$$

$$h(x) = \frac{e^{w\log\|x\|}}{\|x\|^{\frac{1}{2}}} P_{w-\frac{1}{2}}^{-k}(\cos\theta_x)\cos(k\phi_{x',y'}) \quad (N = 3)$$

is harmonic on $\Omega(\pi)$ when suitably interpreted on the positive x_N-axis.

Proof We will give the details when $N \geq 4$ and leave the adjustments required when $N = 3$ to the reader. Let $r = \|x\|$, $\theta = \theta_x$ and $\phi = \phi_{x',y'}$. Then

$$\Delta h = \frac{\partial^2 h}{\partial r^2} + \frac{N-1}{r}\frac{\partial h}{\partial r} + \frac{1}{r^2}\left(\Lambda_1 + \frac{\Lambda_2}{(\sin\theta)^2}\right)h,$$

where

$$\Lambda_1 = \frac{1}{(\sin\theta)^{N-2}}\frac{\partial}{\partial\theta}\left\{(\sin\theta)^{N-2}\frac{\partial}{\partial\theta}\right\}, \quad \Lambda_2 = \frac{1}{(\sin\phi)^{N-3}}\frac{\partial}{\partial\phi}\left\{(\sin\phi)^{N-3}\frac{\partial}{\partial\phi}\right\}.$$

Since

$$\frac{\partial^2 h}{\partial r^2} + \frac{N-1}{r}\frac{\partial h}{\partial r} = \frac{h}{r^2}\left\{w^2 - \left(\frac{N-2}{2}\right)^2\right\},$$

it is enough to show that

$$\left(\Lambda_1 + \frac{\Lambda_2}{(\sin\theta)^2} + w^2 - \left(\frac{N-2}{2}\right)^2\right)\left\{\frac{f(\cos\theta)}{(\sin\theta)^{\frac{N-3}{2}}} C_k^{\left(\frac{N-3}{2}\right)}(\cos\phi)\right\} = 0,$$

where $f(t) = P_{w-\frac{1}{2}}^{\frac{3-N}{2}-k}(t)$.

Now

$$\frac{d}{d\phi}\left\{\sin^{N-3}\phi\frac{d}{d\phi}\left(C_k^{\left(\frac{N-3}{2}\right)}(\cos\phi)\right)\right\} = \sin^{N-1}\phi\frac{d^2 C_k^{\left(\frac{N-3}{2}\right)}}{d\phi^2}(\cos\phi)$$

$$-(N-2)\sin^{N-3}\phi\cos\phi\frac{dC_k^{\left(\frac{N-3}{2}\right)}}{d\phi}(\cos\phi),$$

so (3) yields

$$\Lambda_2\left(C_k^{\left(\frac{N-3}{2}\right)}(\cos\phi)\right)=-k(k+N-3)C_k^{\left(\frac{N-3}{2}\right)}(\cos\phi).$$

Thus it remains to check that

$$\left(\Lambda_1-\frac{k(k+N-3)}{\sin^2\theta}\right)\frac{f(\cos\theta)}{(\sin\theta)^{\frac{N-3}{2}}}=\left(\left(\frac{N-2}{2}\right)^2-w^2\right)\frac{f(\cos\theta)}{(\sin\theta)^{\frac{N-3}{2}}}. \quad (11)$$

Next,

$$(\sin\theta)^{N-2}\frac{d}{d\theta}\frac{f(\cos\theta)}{(\sin\theta)^{\frac{N-3}{2}}}=\frac{3-N}{2}(\sin\theta)^{\frac{N-3}{2}}\cos\theta f(\cos\theta)-(\sin\theta)^{\frac{N+1}{2}}f'(\cos\theta).$$

Thus

$$\begin{aligned}&(\sin\theta)^{\frac{1-N}{2}}\frac{d}{d\theta}\left((\sin\theta)^{N-2}\frac{d}{d\theta}\left\{\frac{f(\cos\theta)}{(\sin\theta)^{\frac{N-3}{2}}}\right\}\right)\\&=f(\cos\theta)\left\{\frac{N-3}{2}-\left(\frac{N-3}{2}\right)^2\cot^2\theta\right\}+\left\{\sin^2\theta f''(\cos\theta)-2\cos\theta f'(\cos\theta)\right\}\\&=f(\cos\theta)\left\{\left(\frac{N-3}{2}\right)^2+\frac{N-3}{2}-w^2+\frac{1}{4}\right\}+\frac{f(\cos\theta)}{\sin^2\theta}k(k+N-3),\end{aligned}$$

by (6), and (11) follows.

The above calculation is not valid when $\theta_x=0$, or when $\phi_{x',y'}\in\{0,\pi\}$. In the latter case we can use the continuity of $C_k^{\left(\frac{N-3}{2}\right)}$ to see that the set

$$\{x\in\left(\mathbb{R}^{N-1}\backslash\{0\}\right)\times\mathbb{R}:\phi_{x',y'}=0\text{ or }\pi\}$$

is a removable singularity for the harmonic function h, by Corollary 5.2.3 of [1]. A similar argument, combined with Lemma 5(vii), shows that the positive x_N-axis is also removable for h. □

Corollary 7 *Let* $\nu>0$, $y'\in\mathbb{R}^{N-1}\backslash\{0\}$ *and* $k\in\mathbb{N}\cup\{0\}$. *Then any function of the form*

$$x\mapsto\frac{A\|x\|^{\nu}+B\|x\|^{2-N-\nu}}{(\sin\theta_x)^{\frac{N-3}{2}}}P_{\nu+\frac{N-3}{2}}^{\frac{3-N}{2}-k}(\cos\theta_x)C_k^{\left(\frac{N-3}{2}\right)}(\cos\phi_{x',y'})\quad(N\geq 4),$$

$$x \mapsto \left(A\,\|x\|^{\nu} + B\,\|x\|^{-\nu-1}\right) P_{\nu}^{-k}(\cos\theta_x)\cos(k\phi_{x',y'}) \quad (N=3),$$

where $A, B \in \mathbb{R}$, *is harmonic on* $\Omega(\pi)$ *when suitably interpreted on the positive* x_N*-axis.*

Proof We put $w = \pm\left(\nu + \frac{N-2}{2}\right)$ in the proposition and use the fact that $P_{\lambda}^{-\mu} = P_{-\lambda-1}^{-\mu}$, by Lemma 5(v). □

Corollary 8 *Let* $\lambda, c \in \mathbb{R}$, $y \in \mathbb{R}^N \backslash \{0\}$ *and* $k \in \mathbb{N} \cup \{0\}$. *Then the function*

$$x \mapsto \frac{\cos(\lambda \log\|x\| + c)}{\|x\|^{\frac{N-2}{2}} (\sin\theta_x)^{\frac{N-3}{2}}} P_{-\frac{1}{2}+i\lambda}^{\frac{3-N}{2}-k}(\cos\theta_x) C_k^{\left(\frac{N-3}{2}\right)}(\cos\phi_{x',y'}) \quad (N \geq 4),$$

$$x \mapsto \frac{\cos(\lambda \log\|x\| + c)}{\|x\|^{\frac{1}{2}}} P_{-\frac{1}{2}+i\lambda}^{-k}(\cos\theta_x)\cos(k\phi_{x',y'}) \quad (N=3)$$

is harmonic on $\Omega(\pi)$ *when suitably interpreted on the positive* x_N*-axis.*

Proof We put $w = i\lambda$ in the proposition, take real and imaginary parts of h, and expand $\cos(\lambda \log\|x\| + c)$ using the addition formula. □

Functions of the form $P_{-\frac{1}{2}+i\lambda}^{-\mu}$ are known as conical (or Mehler) functions. We record below some of their further properties for future reference.

Lemma 9 (i) $P_{-\frac{1}{2}+i\lambda}^{-\mu} > 0$ *and* $P_{-\frac{1}{2}+i\lambda}^{-\mu} = P_{-\frac{1}{2}-i\lambda}^{-\mu}$ *on* $(-1, 1)$.
(ii) *If* $\mu \geq 0$, *then the function* $\theta \mapsto P_{-\frac{1}{2}+i\lambda}^{-\mu}(-\cos\theta)/P_{-\frac{1}{2}+i\lambda}^{-\mu}(\cos\theta)$ *is decreasing on* $(0, \pi)$.
(iii) *If* $\mu \geq 0$, *then the function* $\theta \mapsto P_{-\frac{1}{2}+i\lambda}^{-\mu}(\cos\theta)$ *is increasing on* $(0, \pi)$.

Proof (i) This is clear from the Mehler–Dirichlet formula (9).

(ii) Since $\Gamma(\overline{z}) = \overline{\Gamma(z)}$, it follows from (8) that the function $t \mapsto P_{-\frac{1}{2}+i\lambda}^{-\mu}(-t)/P_{-\frac{1}{2}+i\lambda}^{-\mu}(t)$ is increasing on $(-1, 1)$.

(iii) By definition,

$$P_{\nu}^{-\mu}(t) = \frac{1}{\Gamma(1+\mu)}\left(\frac{1-t}{1+t}\right)^{\mu/2} {}_2F_1\left(\nu+1, -\nu; 1+\mu; \frac{1-t}{2}\right),$$

so

$$P_{-\frac{1}{2}+i\lambda}^{-\mu}(\cos\theta) = \frac{1}{\Gamma(1+\mu)}\left(\frac{1-\cos\theta}{1+\cos\theta}\right)^{\mu/2} {}_2F_1\left(\frac{1}{2}+i\lambda, \frac{1}{2}-i\lambda; 1+\mu; \frac{1-\cos\theta}{2}\right).$$

Since the coefficients in the expansion

$${}_2F_1\left(\frac{1}{2}+i\lambda, \frac{1}{2}-i\lambda; 1+\mu; s\right) = 1 + \frac{\left|\frac{1}{2}+i\lambda\right|^2}{(1+\mu)\,1!}s + \frac{\left|\frac{1}{2}+i\lambda\right|^2\left|\frac{3}{2}+i\lambda\right|^2}{(1+\mu)(2+\mu)\,2!}s^2 + \cdots$$

are all positive, we now see that the function $\theta \mapsto P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta)$ is the product of two positive increasing functions on $(0,\pi)$. $\square$

In the next three sections we will adapt an argument outlined on pp.69–72 of Dougall [3] for $\mathbb{R}^3$ to establish expansions of the Green function for $\Omega(\theta_*)$ in all dimensions.

4 An expansion of the fundamental function

If $x, y \in \mathbb{R}^N$, then we define $\gamma_{x,y} \in [0,\pi]$ by the equation

$$\cos\gamma_{x,y} = \frac{\langle x, y\rangle}{\|x\|\,\|y\|}$$

whenever the denominator is non-zero. Since $\langle x, y\rangle = \left\langle x', y'\right\rangle + x_N y_N$ and $\left\|x'\right\| = \|x\|\sin\theta_x$, it follows that

$$\cos\gamma_{x,y} = \cos\theta_x\cos\theta_y + \sin\theta_x\sin\theta_y\cos\phi_{x',y'}.$$

It will be convenient to define

$$R^{-\mu}_\nu(t) = \Gamma(\nu+\mu+1)\Gamma(\mu-\nu)P^{-\mu}_\nu(-t). \tag{12}$$

We recall from p. 1938 of [5] (cf. equation (80) in [13]) an addition formula for $P^{-\mu}_\nu$, namely

$$\frac{P^{-\mu}_\nu(\cos\gamma_{x,y})}{(\sin\gamma_{x,y})^\mu} = \frac{2^\mu\Gamma(\mu)}{(\sin\theta_x\sin\theta_y)^\mu}\sum_{k=0}^\infty (k+\mu)\,\frac{\Gamma(k+\nu+\mu+1)\Gamma(k+\mu-\nu)}{\Gamma(\nu+\mu+1)\Gamma(\mu-\nu)} \times P^{-\mu-k}_\nu(\cos\theta_x)P^{-\mu-k}_\nu(\cos\theta_y)(-1)^k C^{(\mu)}_k\left(\cos\phi_{x',y'}\right) \tag{13}$$

when $\theta_x+\theta_y < \pi$. (The restriction in [5] that $\phi_{x',y'} < \pi$ may be removed by dominated convergence, in the light of (5) and the asymptotic behaviour of $P^{-\mu}_\nu$ for large μ, as described in (14.15.1) of [19].) Since

$$\cos\gamma_{-x,y} = -\cos\gamma_{x,y}, \quad \sin\gamma_{-x,y} = \sin\gamma_{x,y},$$

and analogous formulae hold for θ_{-x} and $\phi_{-x',y'}$, we can replace x by $-x$ in (13), and use (4) and (12) to obtain

$$\frac{R^{-\mu}_\nu(\cos\gamma_{x,y})}{(\sin\gamma_{x,y})^\mu} = \frac{2^\mu\Gamma(\mu)}{(\sin\theta_x\sin\theta_y)^\mu}\sum_{k=0}^\infty (k+\mu)\,R^{-\mu-k}_\nu(\cos\theta_x)P^{-\mu-k}_\nu(\cos\theta_y)C^{(\mu)}_k\left(\cos\phi_{x',y'}\right) \tag{14}$$

when $\theta_{-x}+\theta_y<\pi$, that is, when $\theta_y<\theta_x$. When $\mu=0$ the appropriate analogue of (13) may be found by combining equations (14.18.1) and (14.9.3) of [19]. This leads to the formula

$$R_\nu^0(\cos\gamma_{x,y})=\sum_{k=0}^{\infty}{}' R_\nu^{-k}(\cos\theta_x)P_\nu^{-k}(\cos\theta_y)\cos\left(k\phi_{x',y'}\right), \tag{15}$$

where

$$\sum_{k=0}^{\infty}{}' g(k)=g(0)+2\{g(1)+g(2)+\cdots\}.$$

Equations (14) and (15) are valid when $\gamma_{x,y},\theta_x,\theta_y\in(0,\pi)$ and $\theta_y<\theta_x$.

Let

$$a_N=\frac{2^{\frac{N-3}{2}}\Gamma\left(\frac{N-1}{2}\right)}{\Gamma(N-2)}\quad(N\geq 3),$$

and suppose that $\|y\|<\|x\|$ and $0<\gamma_{x,y}<\pi$. Then (2) and parts (i), (ii) of Lemma 5 show that

$$\begin{aligned}\|x-y\|^{2-N}&=\|x\|^{2-N}\left(1-2\frac{\langle x,y\rangle}{\|x\|^2}+\left(\frac{\|y\|}{\|x\|}\right)^2\right)^{\frac{2-N}{2}}\\&=\|x\|^{2-N}\sum_{n=0}^{\infty}\left(\frac{\|y\|}{\|x\|}\right)^n C_n^{\left(\frac{N-2}{2}\right)}\left(\cos\gamma_{x,y}\right)\\&=a_N\frac{\|x\|^{2-N}}{(\sin\gamma_{x,y})^{\frac{N-3}{2}}}\sum_{n=0}^{\infty}\left(\frac{\|y\|}{\|x\|}\right)^n\frac{\Gamma(n+N-2)}{\Gamma(n+1)}P_{n+\frac{N-3}{2}}^{\frac{3-N}{2}}(\cos\gamma_{x,y})\\&=a_N\frac{(\|x\|\,\|y\|)^{\frac{2-N}{2}}}{(\sin\gamma_{x,y})^{\frac{N-3}{2}}}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\|y\|}{\|x\|}\right)^{n+\frac{N-2}{2}}\frac{\Gamma(n+N-2)}{\Gamma(n+1)}P_{n+\frac{N-3}{2}}^{\frac{3-N}{2}}(-\cos\gamma_{x,y}).\end{aligned}$$

Hence

$$\|x-y\|^{2-N}=a_N\frac{(\|x\|\,\|y\|)^{\frac{2-N}{2}}}{(\sin\gamma_{x,y})^{\frac{N-3}{2}}}\sum_{n=0}^{\infty}(-1)^n f(n), \tag{16}$$

where

$$f(z)=e^{\left(z+\frac{N-2}{2}\right)\log\left(\frac{\|y\|}{\|x\|}\right)}\frac{\Gamma(z+N-2)}{\Gamma(z+1)}P_{z+\frac{N-3}{2}}^{\frac{3-N}{2}}(-\cos\gamma_{x,y}).$$

For any $\kappa\in\mathbb{N}$ let $c(\kappa)$ denote the contour around the boundary of the rectangle

$$\left\{z\in\mathbb{C}:\frac{2-N}{2}<\mathrm{Re}z<\kappa+\frac{1}{2}\text{ and }|\mathrm{Im}z|<\kappa\right\}, \tag{17}$$

oriented anticlockwise. The function $z \mapsto P^{\frac{3-N}{2}}_{z+\frac{N-3}{2}}(-\cos\gamma_{x,y})$ is entire (see (14.3.1) and §15.2(ii) of [19]). Thus the residue theorem yields

$$\frac{1}{2\pi i}\int_{c(\kappa)} \frac{f(z)}{\cos\left(\pi\left(z+\frac{1}{2}\right)\right)}dz = \frac{-1}{\pi}\sum_{n=0}^{\kappa}(-1)^n f(n), \tag{18}$$

since the singularities of the integrand in $\mathbb{Z}\cap\left[\frac{2-N}{2},0\right)$ are removable. By Lemma 5(vii) the above integrand is bounded in modulus by

$$C(N,\gamma_{x,y})\kappa^{N-3}e^{-\kappa\gamma_{x,y}}$$

on the top and bottom sides of the contour, and by

$$C(N,\gamma_{x,y})\kappa^{N-3}e^{\kappa\log\frac{\|y\|}{\|x\|}}$$

on the right hand side. Since we can parametrize the reverse path $-c(\kappa)$ on the left hand side of the rectangle as $(2-N)/2+i\lambda$ $(-\kappa\le\lambda\le\kappa)$, we can let $\kappa\to\infty$ in (18) to see that

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{\exp\left(i\lambda\log\frac{\|y\|}{\|x\|}\right)}{\cos\left(\pi\left(i\lambda+\frac{3-N}{2}\right)\right)}\frac{\Gamma\left(i\lambda+\frac{N-2}{2}\right)}{\Gamma\left(i\lambda+\frac{4-N}{2}\right)}P^{\frac{3-N}{2}}_{-\frac{1}{2}+i\lambda}(-\cos\gamma_{x,y})d\lambda = \sum_{n=0}^{\infty}(-1)^n f(n).$$

(The convergence of this integral will become clear below.) Since

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)} \quad (z\notin\mathbb{Z}) \text{ and } \Gamma(\overline{z}) = \overline{\Gamma(z)}, \tag{19}$$

we see that

$$\begin{aligned}\frac{\Gamma\left(i\lambda+\frac{N-2}{2}\right)}{\cos\left(\pi\left(i\lambda+\frac{3-N}{2}\right)\right)\Gamma\left(i\lambda+\frac{4-N}{2}\right)} &= \frac{1}{\pi}\Gamma\left(-i\lambda+\frac{N-2}{2}\right)\Gamma\left(i\lambda+\frac{N-2}{2}\right)\\ &= \frac{1}{\pi}\left|\Gamma\left(i\lambda+\frac{N-2}{2}\right)\right|^2. \end{aligned}\tag{20}$$

Hence

$$\sum_{n=0}^{\infty}(-1)^n f(n) = \frac{1}{\pi}\int_0^{\infty}\cos\left(\lambda\log\frac{\|y\|}{\|x\|}\right)\left|\Gamma\left(i\lambda+\frac{N-2}{2}\right)\right|^2 P^{\frac{3-N}{2}}_{-\frac{1}{2}+i\lambda}(-\cos\gamma_{x,y})d\lambda,$$

because $P^{\frac{3-N}{2}}_{-\frac{1}{2}+i\lambda}(-\cos\gamma_{x,y})$ is real and symmetric in λ, by Lemma 9(i). Combining this with (16), we see that

$$\|x-y\|^{2-N} = \frac{a_N}{\pi}\frac{(\|x\|\,\|y\|)^{\frac{2-N}{2}}}{(\sin\gamma_{x,y})^{\frac{N-3}{2}}}\int_0^\infty \cos\left(\lambda\log\frac{\|y\|}{\|x\|}\right)\left|\Gamma\left(i\lambda+\frac{N-2}{2}\right)\right|^2 \times P^{\frac{3-N}{2}}_{-\frac{1}{2}+i\lambda}(-\cos\gamma_{x,y})d\lambda. \tag{21}$$

Noting that

$$\left|\Gamma\left(i\lambda+\frac{N-2}{2}\right)\right|^2 = 2\pi\lambda^{N-3}e^{-\pi\lambda}(1+o(1)) \quad (\lambda\to\infty), \tag{22}$$

by (5.11.9) in [19], we see from Lemma 5(vii) that the integral in (21) converges absolutely even when $\|y\| = \|x\|$. It follows from dominated convergence and symmetry that (21) is valid for any non-zero choices of $\|y\|$ and $\|x\|$, provided $\gamma_{x,y}\in(0,\pi)$. Since

$$R^{\frac{3-N}{2}}_{-\frac{1}{2}+i\lambda}(\cos\theta_x) = \left|\Gamma\left(i\lambda+\frac{N-2}{2}\right)\right|^2 P^{\frac{3-N}{2}}_{-\frac{1}{2}+i\lambda}(-\cos\theta_x), \tag{23}$$

by (12) and (20), we see from (21) that

$$\|x-y\|^{2-N} = \frac{a_N}{\pi}(\|x\|\,\|y\|)^{\frac{2-N}{2}}\int_0^\infty \cos\left(\lambda\log\frac{\|x\|}{\|y\|}\right)\frac{R^{\frac{3-N}{2}}_{-\frac{1}{2}+i\lambda}(\cos\gamma_{x,y})}{(\sin\gamma_{x,y})^{\frac{N-3}{2}}}d\lambda. \tag{24}$$

We now make the additional assumption that $0<\theta_y<\theta_x<\pi$, and deal first with the case where $N\geq 4$. We can combine (24) with (14) to see that

$$\|x-y\|^{2-N} = \frac{a_N}{\pi}\frac{2^{\frac{N-3}{2}}\Gamma\left(\frac{N-3}{2}\right)}{(\sin\theta_x\sin\theta_y)^{\frac{N-3}{2}}(\|x\|\,\|y\|)^{\frac{N-2}{2}}}\int_0^\infty\sum_{k=0}^\infty \cos\left(\lambda\log\frac{\|y\|}{\|x\|}\right) \times\left(k+\frac{N-3}{2}\right)R^{\frac{3-N}{2}-k}_{-\frac{1}{2}+i\lambda}(\cos\theta_x)P^{\frac{3-N}{2}-k}_{-\frac{1}{2}+i\lambda}(\cos\theta_y)C_k^{\left(\frac{N-3}{2}\right)}\left(\cos\phi_{x',y'}\right)d\lambda. \tag{25}$$

In view of the positivity of $P^{\frac{3-N}{2}-k}_{-\frac{1}{2}+i\lambda}$ (see Lemma 9(i)) and (5) the summand in (25) is bounded in absolute value by

$$\left(k+\frac{N-3}{2}\right)R^{\frac{3-N}{2}-k}_{-\frac{1}{2}+i\lambda}(\cos\theta_x)P^{\frac{3-N}{2}-k}_{-\frac{1}{2}+i\lambda}(\cos\theta_y)C_k^{\left(\frac{N-3}{2}\right)}(1).$$

In addition,

$$\begin{aligned}
&\frac{a_N}{\pi}\frac{2^{\frac{N-3}{2}}\Gamma\left(\frac{N-3}{2}\right)}{(\sin\theta_x\sin\theta_y)^{\frac{N-3}{2}}(\|x\|\,\|y\|)^{\frac{N-2}{2}}}\int_0^\infty\sum_{k=0}^\infty\left(k+\frac{N-3}{2}\right)\\
&\quad\times R^{\frac{3-N}{2}-k}_{-\frac{1}{2}+i\lambda}(\cos\theta_x)P^{\frac{3-N}{2}-k}_{-\frac{1}{2}+i\lambda}(\cos\theta_y)C_k^{\left(\frac{N-3}{2}\right)}(1)\\
&\quad=\left\|\left(\frac{\|y\|}{\|x\|}\right)^{1/2}\left(\frac{\|x'\|}{\|y'\|}y',x_N\right)-\left(\frac{\|x\|}{\|y\|}\right)^{1/2}y\right\|^{2-N}\\
&\quad=\left\{2\sqrt{\|x\|\,\|y\|}\sin\frac{\theta_x-\theta_y}{2}\right\}^{2-N}<\infty. \qquad (26)
\end{aligned}$$

Thus the integral in (25) still converges when we replace the summand by its absolute value. In particular, we can thus allow $\gamma_{x,y}$ to range over $(0,\pi]$, by dominated convergence.

When $N=3$ we instead combine (15) with (24) to see that

$$\begin{aligned}
\|x-y\|^{-1}&=\frac{1}{\pi\sqrt{\|x\|\,\|y\|}}\int_0^\infty{\sum_{k=0}^\infty}'\cos\left(\lambda\log\frac{\|y\|}{\|x\|}\right)\\
&\quad\times R^{-k}_{-\frac{1}{2}+i\lambda}(\cos\theta_x)P^{-k}_{-\frac{1}{2}+i\lambda}(\cos\theta_y)\cos\left(k\phi_{x',y'}\right)d\lambda. \qquad (27)
\end{aligned}$$

The analogue of (26) again holds, so the expansion in (27) has the same absolute convergence property.

We have established (25) and (27) for any $x,y\in\mathbb{R}^N\backslash\{0\}$ satisfying $0<\theta_y<\theta_x<\pi$. The integrals and summations are interchangeable, by Fubini's theorem.

5 An expansion for the Green function

We assume in this section that $x,y\in\mathbb{R}^N\backslash\{0\}$ and $\theta_x,\theta_y\in(0,\pi)$.

When $N\geq 4$, $y\in\Omega$ and $x\in\overline{\Omega}$ we define

$$\begin{aligned}
h_y(x)&=\frac{a_N}{\pi}\frac{2^{\frac{N-3}{2}}\Gamma\left(\frac{N-3}{2}\right)}{(\sin\theta_x\sin\theta_y)^{\frac{N-3}{2}}(\|x\|\,\|y\|)^{\frac{N-2}{2}}}\int_0^\infty\sum_{k=0}^\infty\cos\left(\lambda\log\frac{\|y\|}{\|x\|}\right)\\
&\quad\times\left(k+\frac{N-3}{2}\right)P^{\frac{3-N}{2}-k}_{-\frac{1}{2}+i\lambda}(\cos\theta_x)P^{\frac{3-N}{2}-k}_{-\frac{1}{2}+i\lambda}(\cos\theta_y)\frac{R^{\frac{3-N}{2}-k}_{-\frac{1}{2}+i\lambda}(\cos\theta_*)}{P^{\frac{3-N}{2}-k}_{-\frac{1}{2}+i\lambda}(\cos\theta_*)}\\
&\quad\times C_k^{\left(\frac{N-3}{2}\right)}\left(\cos\phi_{x',y'}\right)d\lambda. \qquad (28)
\end{aligned}$$

Since the function $\theta \to P^{\frac{3-N}{2}-k}_{-\frac{1}{2}+i\lambda}(\cos\theta)$ is positive and increasing on $(0,\pi)$, by Lemma 9, we see that

$$P^{\frac{3-N}{2}-k}_{-\frac{1}{2}+i\lambda}(\cos\theta_x)P^{\frac{3-N}{2}-k}_{-\frac{1}{2}+i\lambda}(\cos\theta_y)\frac{R^{\frac{3-N}{2}-k}_{-\frac{1}{2}+i\lambda}(\cos\theta_*)}{P^{\frac{3-N}{2}-k}_{-\frac{1}{2}+i\lambda}(\cos\theta_*)} \le P^{\frac{3-N}{2}-k}_{-\frac{1}{2}+i\lambda}(\cos\theta_y)R^{\frac{3-N}{2}-k}_{-\frac{1}{2}+i\lambda}(\cos\theta_*)$$

when $\theta_x \le \theta_*$. It now follows from (26), with $\theta_x = \theta_*$, and (5), that the right hand side of (28) is absolutely convergent, and from dominated convergence that h_y is continuous on $\overline{\Omega}$, when suitably interpreted at points where $\theta_x = 0$. Further, by Fubini's theorem and Corollary 8, the function h_y satisfies the volume mean value property in Ω, and so is harmonic there. It tends to 0 at infinity, by (26) again with $\theta_x = \theta_*$. Since $h_y(x) = \|x-y\|^{2-N}$ on $\partial\Omega$, by (25), it follows from the minimum principle that h_y is the greatest harmonic minorant of $\|\cdot - y\|^{2-N}$ on Ω. Hence, when $0 < \theta_y < \theta_x < \theta_*$, it follows from (25) and (28) that the Green function of Ω is given by

$$\begin{aligned} G_\Omega(x,y) &= \|x-y\|^{2-N} - h_y(x) \\ &= \frac{a_N}{\pi}\frac{2^{\frac{N-3}{2}}\Gamma\left(\frac{N-3}{2}\right)}{(\sin\theta_x \sin\theta_y)^{\frac{N-3}{2}}(\|x\|\,\|y\|)^{\frac{N-2}{2}}}\int_0^\infty \sum_{k=0}^{\infty}\cos\left(\lambda \log\frac{\|y\|}{\|x\|}\right) \\ &\quad \times \left(k+\frac{N-3}{2}\right) g_k(\lambda,\theta_x,\theta_y) C_k^{\left(\frac{N-3}{2}\right)}\left(\cos\phi_{x',y'}\right) d\lambda, \end{aligned} \tag{29}$$

where

$$g_k(\lambda,\theta_x,\theta_y) = \left\{\frac{R^{\frac{3-N}{2}-k}_{-\frac{1}{2}+i\lambda}(\cos\theta_x)}{P^{\frac{3-N}{2}-k}_{-\frac{1}{2}+i\lambda}(\cos\theta_x)} - \frac{R^{\frac{3-N}{2}-k}_{-\frac{1}{2}+i\lambda}(\cos\theta_*)}{P^{\frac{3-N}{2}-k}_{-\frac{1}{2}+i\lambda}(\cos\theta_*)}\right\} P^{\frac{3-N}{2}-k}_{-\frac{1}{2}+i\lambda}(\cos\theta_x)P^{\frac{3-N}{2}-k}_{-\frac{1}{2}+i\lambda}(\cos\theta_y).$$

The integration and summation can be interchanged in (29), by the absolute convergence of the expansions in (25) and (28). If $\theta_x < \theta_y$, then we replace $g_k(\lambda,\theta_x,\theta_y)$ by $g_k(\lambda,\theta_y,\theta_x)$ in (29), by the symmetry of the Green function.

When $N = 3$ analogous reasoning shows that

$$G_\Omega(x,y) = \frac{1}{\pi\sqrt{\|x\|\,\|y\|}}\int_0^\infty {\sum_{k=0}^{\infty}}' \cos\left(\lambda\log\frac{\|y\|}{\|x\|}\right) g_k(\lambda,\theta_x,\theta_y)\cos\left(k\phi_{x',y'}\right) d\lambda \tag{30}$$

when $\theta_y < \theta_x$. This was asserted long ago in p.71(1) of [3], though full details of the proof were not provided. (That paper used P_ν^μ to denote what today is called $P_\nu^{-\mu}$, as can seen from the definition on p.48.)

6 A second expansion for the Green function

We denote by $n_{k,m}$ the mth positive zero of the entire function $\nu \mapsto P_{\nu+\frac{N-3}{2}}^{\frac{3-N}{2}-k}(\cos\theta_*)$ and note from Lemma 5(vi) that

$$n_{k,m} > k + m - 1. \tag{31}$$

Suppose that $\|y\| < \|x\|$ and $\theta_x, \theta_y \in (0, \pi)$, and let

$$f(z) = e^{\left(z+\frac{N-2}{2}\right)\log\left(\frac{\|y\|}{\|x\|}\right)} \frac{P_{z+\frac{N-3}{2}}^{\frac{3-N}{2}-k}(\cos\theta_y)}{P_{z+\frac{N-3}{2}}^{\frac{3-N}{2}-k}(\cos\theta_*)}$$
$$\times \left\{ R_{z+\frac{N-3}{2}}^{\frac{3-N}{2}-k}(\cos\theta_x) P_{z+\frac{N-3}{2}}^{\frac{3-N}{2}-k}(\cos\theta_*) - R_{z+\frac{N-3}{2}}^{\frac{3-N}{2}-k}(\cos\theta_*) P_{z+\frac{N-3}{2}}^{\frac{3-N}{2}-k}(\cos\theta_x) \right\}.$$

We recall that $\Gamma(z)$ is holomorphic except for simple poles at the nonpositive integers, and that

$$\mathrm{Res}(\Gamma, -p) = \frac{(-1)^p}{p!} \quad (p = 0, 1, 2, \ldots).$$

Hence, by (12), the singularities of the function

$$z \mapsto R_{z+\frac{N-3}{2}}^{\frac{3-N}{2}-k}(\cos\theta) = \Gamma(z + N - 2 + k)\Gamma(k - z) P_{z+\frac{N-3}{2}}^{\frac{3-N}{2}-k}(-\cos\theta) \quad (\mathrm{Re} z > 2-N)$$

lie at the integers j satisfying $j \geq k$, and the residue at j is then

$$\frac{(-1)^{j-k}}{(j-k)!}(j + k + N - 3)! P_{j+\frac{N-3}{2}}^{\frac{3-N}{2}-k}(-\cos\theta).$$

The singularities of f at such points are thus removable, in view of Lemma 5(ii). The remaining singularities of f in $\{\mathrm{Re} z > 2 - N\}$ are simple poles at the points $(n_{k,m})_{m \geq 1}$.

We will apply the residue theorem to the contour integral of f around the boundary $d(\kappa)$ of the rectangle

$$\left\{ z \in \mathbb{C} : \frac{2-N}{2} < \mathrm{Re} z < \frac{\pi}{\theta_*}\left(\kappa + \frac{N-2}{4} + \frac{k}{2}\right) - \frac{1}{2} \text{ and } |\mathrm{Im} z| < \kappa \right\},$$

oriented anticlockwise, where $\kappa \in \mathbb{N}$. We recall from p.291 of [22] that, for fixed $\mu \geq 0$ and $\gamma, \delta \in (0, \pi)$,

$$P_\nu^{-\mu}(\cos\gamma) = \frac{\sqrt{2}\Gamma(\nu+1)}{\sqrt{\nu\pi \sin\gamma}\Gamma(\nu+\mu+1)} \left\{ \begin{array}{l} \left(1 + O\left(\frac{1}{\nu}\right)\right) \cos\left(\left(\nu + \frac{1}{2}\right)\gamma - \frac{\mu\pi}{2} - \frac{\pi}{4}\right) \\ +O\left(\frac{1}{\nu}\right) \sin\left(\left(\nu + \frac{1}{2}\right)\gamma - \frac{\mu\pi}{2} - \frac{\pi}{4}\right) \end{array} \right\}$$

as $|\nu| \to \infty$ in the set $\{|\mathrm{Arg}(\nu)| \leq \pi - \delta\}$, whence

$$\begin{aligned} &R_\nu^{-\mu}(\cos\theta_x)P_\nu^{-\mu}(\cos\theta_*) - R_\nu^{-\mu}(\cos\theta_*)P_\nu^{-\mu}(\cos\theta_x) \\ &= \frac{2}{\nu\sqrt{\sin\theta_x \sin\theta_*}}\left\{\left(1 + O\left(\frac{1}{\nu}\right)\right)\sin\left(\left(\nu + \frac{1}{2}\right)(\theta_* - \theta_x)\right)\right\} \end{aligned}$$

as $|\nu| \to \infty$ in the set $\{|\mathrm{Arg}(\nu)| \leq \pi - \delta, \mathrm{dist}(\nu - \mu, \mathbb{N}) > \varepsilon\}$ for any $\varepsilon > 0$, by (12), (19) and Stirling's formula. It follows that, for large κ,

$$|f(z)| \leq \frac{C(\theta_x, \theta_y, \theta_*)}{\kappa} e^{\kappa(\theta_y - \theta_x) + (\mathrm{Re} z + \frac{N-2}{2})\log\left(\frac{\|y\|}{\|x\|}\right)}$$

on the top and bottom sides of $d(\kappa)$, and that

$$|f(z)| \leq \frac{C(\theta_x, \theta_y, \theta_*)}{\kappa} e^{\kappa\frac{\pi}{\theta_*}\log\left(\frac{\|y\|}{\|x\|}\right) + (\theta_y - \theta_x)|\mathrm{Im} z|}$$

on the right hand side of $d(\kappa)$. If we temporarily assume that $\theta_y < \theta_x$, then we can apply the residue theorem and let $\kappa \to \infty$ to see that

$$\begin{aligned} &\frac{1}{\pi}\int_0^\infty \cos\left(\lambda \log\frac{\|y\|}{\|x\|}\right) g_k(\lambda, \theta_x, \theta_y) d\lambda \\ &= \sum_{m=1}^{\infty} e^{\left(n_{k,m} + \frac{N-2}{2}\right)\log\left(\frac{\|y\|}{\|x\|}\right)} P_{n_{k,m}+\frac{N-3}{2}}^{\frac{3-N}{2}-k}(\cos\theta_y) P_{n_{k,m}+\frac{N-3}{2}}^{\frac{3-N}{2}-k}(\cos\theta_x) \\ &\quad \times \frac{R_{n_{k,m}+\frac{N-3}{2}}^{\frac{3-N}{2}-k}(\cos\theta_*)}{\left.\frac{\partial}{\partial\nu} P_{\nu+\frac{N-3}{2}}^{\frac{3-N}{2}-k}(\cos\theta_*)\right|_{\nu = n_{k,m}}}. \end{aligned} \tag{32}$$

For any $\mu \geq 0$, $\nu > 0$ and $\tau_0 \in (-1, 1)$ satisfying $P_\nu^{-\mu}(\tau_0) = 0$, we know from §11(I) of [2] (cf. §7 of [17]; the result is stated for the case where $\mu > 0$, but remains valid also when $\mu = 0$) that

$$\int_{\tau_0}^1 \left\{P_\nu^{-\mu}(\tau)\right\}^2 d\tau = -\frac{(1 - \tau_0^2)}{2\nu + 1}\frac{\partial}{\partial\tau_0}P_\nu^{-\mu}(\tau_0)\frac{\partial}{\partial\nu}P_\nu^{-\mu}(\tau_0),$$

and from (8) that

$$-(1 - \tau_0^2)P_\nu^{-\mu}(-\tau_0)\frac{\partial}{\partial\tau_0}P_\nu^{-\mu}(\tau_0) = \frac{2}{\Gamma(\mu + \nu + 1)\Gamma(\mu - \nu)}.$$

Hence, by (12),

$$\int_{\tau_0}^1 \left\{P_\nu^{-\mu}(\tau)\right\}^2 d\tau = \frac{2}{2\nu + 1}\frac{\frac{\partial}{\partial\nu}P_\nu^{-\mu}(\tau_0)}{R_\nu^{-\mu}(\tau_0)}.$$

When $N \geq 4$ we then see from (29), an interchange of summation and integration, and (32), that

$$G_\Omega(x, y) = \frac{a_N 2^{\frac{N-3}{2}} \Gamma\left(\frac{N-3}{2}\right)}{(\sin\theta_x \sin\theta_y)^{\frac{N-3}{2}} (\|x\|\,\|y\|)^{\frac{N-2}{2}}} \sum_{k=0}^{\infty} \left(k + \frac{N-3}{2}\right) C_k^{\left(\frac{N-3}{2}\right)} \left(\cos\phi_{x',y'}\right)$$
$$\times \sum_{m=1}^{\infty} \left(\frac{\|y\|}{\|x\|}\right)^{n_{k,m}+\frac{N-2}{2}} \frac{P^{\frac{3-N}{2}-k}_{n_{k,m}+\frac{N-3}{2}}(\cos\theta_y) P^{\frac{3-N}{2}-k}_{n_{k,m}+\frac{N-3}{2}}(\cos\theta_x)}{\left(n_{k,m} + \frac{N-2}{2}\right) \int_{\cos\theta_*}^{1} \left\{P^{\frac{3-N}{2}-k}_{n_{k,m}+\frac{N-3}{2}}(\tau)\right\}^2 d\tau}, \tag{33}$$

and when $N = 3$ we use (30) in place of (29) to see that

$$G_\Omega(x, y) = \frac{1}{\sqrt{\|x\|\,\|y\|}} \sum_{k=0}^{\infty}{}' \cos\left(k\phi_{x',y'}\right)$$
$$\times \sum_{m=1}^{\infty} \left(\frac{\|y\|}{\|x\|}\right)^{n_{k,m}+\frac{1}{2}} \frac{P^{-k}_{n_{k,m}}(\cos\theta_y) P^{-k}_{n_{k,m}}(\cos\theta_x)}{\left(n_{k,m} + \frac{1}{2}\right) \int_{\cos\theta_*}^{1} \left\{P^{-k}_{n_{k,m}}(\tau)\right\}^2 d\tau}. \tag{34}$$

We temporarily assumed above that $\theta_y < \theta_x$. If $\theta_x < \theta_y$, then we define $x^* = (\|x\| / \|y\|)y$ and $y^* = (\|y\| / \|x\|)x$. We then observe that $G_\Omega(x^*, y^*) = G_\Omega(y, x) = G_\Omega(x, y)$, by (29) (or (30)) and the symmetry of the Green function, to arrive at (33) (or (34)) again. Our earlier assumption that θ_x, θ_y are non-zero can be dropped provided the formulae are suitably interpreted. Thus these formulae hold when $\theta_x \neq \theta_y$ and $\|y\| < \|x\|$. The corresponding formulae when $\|x\| < \|y\|$ are obtained by interchanging x and y in (33) and (34).

7 Extending the Green function of the cone

In preparation for the main result of this section we note the following lemma.

Lemma 10 *If $\nu \geq \mu \geq 0$, $-1 < t_0 < 1$ and $P_\nu^{-\mu}(t_0) = 0$, then*

$$\int_{t_0}^{1} \left\{P_\nu^{-\mu}(\tau)\right\}^2 d\tau \geq \frac{(1 - t_0^2)^{\max\{\mu, \frac{1}{2}\}}}{2^{2\mu-1}\pi(\nu + \frac{1}{2})^2} \left\{\frac{\Gamma\left(\frac{\nu-\mu}{2} + 1\right)}{\Gamma\left(\frac{\nu+\mu+1}{2}\right)}\right\}^2.$$

Proof It follows from parts (x), (iv) and then (viii) of Lemma 5 that

$$
\begin{aligned}
2\left(\nu+\frac{1}{2}\right)^2\int_{t_0}^{1}\left\{P_\nu^{-\mu}(\tau)\right\}^2 d\tau &\geq 2\nu(\nu+1)\int_{t_0}^{1}\tau\left\{P_\nu^{-\mu}(\tau)\right\}^2 d\tau \\
&= \left((1-t_0^2)\frac{dP_\nu^{-\mu}}{dt}(t_0)\right)^2 \\
&= \left\{\frac{1}{Q_\nu^{-\mu}(t_0)}\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\right\}^2 \\
&\geq \frac{(1-t_0^2)^{\max\{\mu,\frac{1}{2}\}}}{4^{\mu-1}\pi}\left\{\frac{\Gamma\left(\frac{\nu-\mu}{2}+1\right)}{\Gamma\left(\frac{\nu+\mu+1}{2}\right)}\right\}^2.
\end{aligned}
$$

□

Theorem 11 *Let $y \in \Omega$ and $a > 1$, and define*

$$
\omega_{y,a}^{(1)} = \left\{x \in \Omega(\pi) : \|x\| \sin\theta_x > \frac{a\,\|y\|}{(\min\{\sin(\theta_*/2), \sin\theta_*\})^3}\right\},
$$
$$
\omega_{y,a}^{(2)} = \left\{x \in \Omega(\pi) : \|y\| \sin\theta_x > \frac{a\,\|x\|}{(\min\{\sin(\theta_*/2), \sin\theta_*\})^3}\right\}.
$$

Then the formulae in (33) and (34) converge absolutely and uniformly to a harmonic function on $\omega_{y,a}^{(1)}$, and when x and y are interchanged they converge absolutely and uniformly to a harmonic function on $\omega_{y,a}^{(2)}$. In particular, $G_\Omega(\cdot, y)$ has a harmonic extension $\widetilde{G}_\Omega(\cdot, y)$ to the set $(\Omega\backslash\{y\}) \cup \omega_{y,a}^{(1)} \cup \omega_{y,a}^{(2)}$. Further,

$$
\left|\widetilde{G}_\Omega(x, y)\right| \leq \frac{C(N, a, \theta_*)\left(\theta_* - \theta_y\right)}{(\|x\|\,\|y\|)^{\frac{N-2}{2}}(\sin\theta_x)^{N-3}} \quad (x \in \omega_{y,a}^{(1)} \cup \omega_{y,a}^{(2)}). \tag{35}
$$

Proof Suppose first that $N \geq 4$ and $\|x\| > a\,\|y\|$. We assume, without loss of generality, that $1 < a \leq 2$, and define

$$
a_j = 1 + \frac{j}{4}(a-1) \quad (j = 1, 2, 3).
$$

By (31) we see that

$$
n_{k,m} + \frac{N-3}{2} > \frac{N-3}{2} + k,
$$

which will allow us to apply Lemma 10 and some results from Lemma 5.

By Lemma 5(viii),

$$
\left|P_{n_{k,m}+\frac{N-3}{2}}^{\frac{3-N}{2}-k}(\cos\theta_x)\right| \leq \frac{2^{k+\frac{N-3}{2}}}{\sqrt{\pi}(\sin\theta_x)^{k+\frac{N-3}{2}}}\frac{\Gamma\left(\frac{n_{k,m}+k+N-2}{2}\right)\Gamma(n_{k,m}-k+1)}{\Gamma\left(\frac{n_{k,m}-k}{2}+1\right)\Gamma(n_{k,m}+k+N-2)} \tag{36}
$$

and, by Lemma 10,

$$I_{k,m}^2 \geq \frac{(\sin\theta_*)^{2k+N-3}}{2^{2k+N-4}\pi\left(n_{k,m}+\frac{N-2}{2}\right)^2}\left\{\frac{\Gamma\left(\frac{n_{k,m}-k}{2}+1\right)}{\Gamma\left(\frac{n_{k,m}+k+N-2}{2}\right)}\right\}^2, \tag{37}$$

where

$$I_{k,m}=\left(\int_{\cos\theta_*}^1\left\{P^{\frac{3-N}{2}-k}_{n_{k,m}+\frac{N-3}{2}}(\tau)\right\}^2 d\tau\right)^{1/2}.$$

Using the Legendre duplication formula,

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)=2^{1-2z}\sqrt{\pi}\,\Gamma(2z), \tag{38}$$

we see that

$$\left\{\frac{\Gamma\left(\frac{n_{k,m}+k+N-2}{2}\right)}{\Gamma\left(\frac{n_{k,m}-k}{2}+1\right)}\right\}^2 \leq C(N)2^{-2k}\frac{\Gamma(n_{k,m}+k+N-2)}{\Gamma(n_{k,m}-k+1)}.$$

Thus, by (36) and (37),

$$\frac{\left|P^{\frac{3-N}{2}-k}_{n_{k,m}+\frac{N-3}{2}}(\cos\theta_x)\right|}{I_{k,m}} \leq \frac{C(N)\left(n_{k,m}+\frac{N-2}{2}\right)}{(\sin\theta_x\sin\theta_*)^{k+\frac{N-3}{2}}}. \tag{39}$$

When $\theta_*/2<\theta_y<\theta_*$ we combine Lemma 5(ix) with the mean value theorem and use the concavity of $\sin\theta$ on $(0,\pi)$ to see that

$$\left|\frac{P^{\frac{3-N}{2}-k}_{n_{k,m}+\frac{N-3}{2}}(\cos\theta_y)}{(\sin\theta_y)^{\frac{N-3}{2}+k}}\right| \leq C(N)\frac{\left(\cos\theta_y-\cos\theta_*\right)2^k}{(\min\{\sin(\theta_*/2),\sin\theta_*\})^{N-1+2k}} \times\frac{\Gamma(n_{k,m}-k+1)\Gamma\left(\frac{n_{k,m}+k+N-1}{2}\right)}{\Gamma(n_{k,m}+k+N-2)\Gamma\left(\frac{n_{k,m}-k+1}{2}\right)}.$$

Using (38) again we see that

$$\frac{\Gamma\left(\frac{n_{k,m}+k+N-1}{2}\right)\Gamma(n_{k,m}-k+1)}{\Gamma\left(\frac{n_{k,m}-k+1}{2}\right)\Gamma(n_{k,m}+k+N-2)}\frac{\Gamma\left(\frac{n_{k,m}+k+N-2}{2}\right)}{\Gamma\left(\frac{n_{k,m}-k}{2}+1\right)}=2^{3-N-2k},$$

so

$$\frac{\left|C_k^{(\frac{N-3}{2})}(\cos\phi_{x',y'})P_{n_{k,m}+\frac{N-3}{2}}^{\frac{3-N}{2}-k}(\cos\theta_y)\right|}{(\sin\theta_y)^{\frac{N-3}{2}}I_{k,m}} \leq C(N,\theta_*)\frac{(\theta_*-\theta_y)\left(n_{k,m}+\frac{N-2}{2}\right)C_k^{(\frac{N-3}{2})}(1)}{(\min\{\sin(\theta_*/2),\sin\theta_*\})^{2k}} \quad (\theta_*/2<\theta_y<\theta_*), \tag{40}$$

in view of (37) and (5).

We next consider the case where $0 \leq \theta_y \leq \theta_*/2$. Let

$$B_{y,a} = \left\{w \in \mathbb{R}^N : \|w-y\| < \|y\|\frac{a-1}{4}\sin\left(\frac{\theta_*}{2}\right)\right\},$$

whence

$$B_{y,a} \subset \Omega \cap \left\{w \in \mathbb{R}^N : |\|w\|-\|y\|| < \|y\|\frac{a-1}{4}\sin\left(\frac{\theta_*}{2}\right)\right\}.$$

If h is a harmonic function on Ω, then h^2 is subharmonic there, and so we can use the volume mean value inequality to see that

$$\begin{aligned}\{h(y)\}^2 &\leq \frac{C(N)}{\{\|y\|\sin(\theta_*/2)(a-1)/4\}^N}\int_{B_{y,a}}\{h(w)\}^2\,dw \\ &\leq \frac{C(N,a,\theta_*)}{\|y\|^N}\int_{\Omega\cap\{|\|w\|-\|y\||<\|y\|\sin(\theta_*/2)(a-1)/4\}}\{h(w)\}^2\,dw.\end{aligned}$$

By Corollary 7 we can apply this inequality to the harmonic function given by

$$h(w) = \frac{\|w\|^{n_{k,m}}}{(\sin\theta_w)^{\frac{N-3}{2}}}P_{n_{k,m}+\frac{N-3}{2}}^{\frac{3-N}{2}-k}(\cos\theta_w)C_k^{\left(\frac{N-3}{2}\right)}(\cos\phi_{x',w'})$$

(interpreted, as usual, in the limiting sense on $\{0\}^{N-1}\times(0,\infty)$) to see that

$$\frac{\|y\|^{n_{k,m}}}{(\sin\theta_y)^{\frac{N-3}{2}}}\left|C_k^{\left(\frac{N-3}{2}\right)}(\cos\phi_{x',y'})P_{n_{k,m}+\frac{N-3}{2}}^{\frac{3-N}{2}-k}(\cos\theta_y)\right| \leq C(N,a,\theta_*)\,(a_1\|y\|)^{n_{k,m}} \times C_k^{\left(\frac{N-3}{2}\right)}(1)I_{k,m},$$

whence

$$\frac{\left|C_k^{(\frac{N-3}{2})}(\cos\phi_{x',y'})P_{n_{k,m}+\frac{N-3}{2}}^{\frac{3-N}{2}-k}(\cos\theta_y)\right|}{(\sin\theta_y)^{\frac{N-3}{2}}I_{k,m}} \leq C(N,a,\theta_*)a_1^{n_{k,m}}C_k^{\left(\frac{N-3}{2}\right)}(1) \quad (0\leq\theta_y\leq\theta_*/2). \tag{41}$$

Since the sets

$$\left\{t^2\left(\frac{a_1}{a_2}\right)^t : t \geq 0\right\} \quad \text{and} \quad \left\{\left(\frac{a_2}{a_3}\right)^k \binom{k+N-4}{k} : k \in \mathbb{N}\right\}$$

are bounded above by a constant $C(a, N)$, we can use (39)–(41), (5) and (31), to see that

$$\begin{aligned}
&\sum_{m=1}^{\infty}\left(k+\frac{N-3}{2}\right)\left(\frac{\|y\|}{\|x\|}\right)^{n_{k,m}+\frac{N-2}{2}} \frac{\left|P^{\frac{3-N}{2}-k}_{n_{k,m}+\frac{N-3}{2}}(\cos\theta_y) P^{\frac{3-N}{2}-k}_{n_{k,m}+\frac{N-3}{2}}(\cos\theta_x)\right|}{(\sin\theta_y)^{\frac{N-3}{2}}\left(n_{k,m}+\frac{N-2}{2}\right) I^2_{k,m}} \\
&\quad\times\left|C_k^{\left(\frac{N-3}{2}\right)}(\cos\phi_{x',y'})\right| \\
&\leq \frac{C(N,a,\theta_*)C_k^{\left(\frac{N-3}{2}\right)}(1)\left(\theta_*-\theta_y\right)}{(\sin\theta_x)^{k+\frac{N-3}{2}}(\min\{\sin(\theta_*/2),\sin\theta_*\})^{3k}} \sum_{m=1}^{\infty}\left(n_{k,m}+\frac{N-2}{2}\right)^2\left(\frac{a_1\|y\|}{\|x\|}\right)^{n_{k,m}+\frac{N-2}{2}} \\
&\leq \frac{C(N,a,\theta_*)\left(\theta_*-\theta_y\right)}{(\sin\theta_x)^{k+\frac{N-3}{2}}(\min\{\sin(\theta_*/2),\sin\theta_*\})^{3k}}\binom{k+N-4}{k}\sum_{m=1}^{\infty}\left(\frac{a_2\|y\|}{\|x\|}\right)^{n_{k,m}+\frac{N-2}{2}} \\
&\leq \frac{C(N,a,\theta_*)\left(\theta_*-\theta_y\right)}{(\sin\theta_x)^{k+\frac{N-3}{2}}(\min\{\sin(\theta_*/2),\sin\theta_*\})^{3k}}\binom{k+N-4}{k}\left(\frac{a_2\|y\|}{\|x\|}\right)^{k} \\
&\leq \frac{C(N,a,\theta_*)\left(\theta_*-\theta_y\right)}{(\sin\theta_x)^{k+\frac{N-3}{2}}(\min\{\sin(\theta_*/2),\sin\theta_*\})^{3k}}\left(\frac{a_3\|y\|}{\|x\|}\right)^{k} \\
&\leq \frac{C(N,a,\theta_*)\left(\theta_*-\theta_y\right)}{(\sin\theta_x)^{\frac{N-3}{2}}}\left(\frac{a_3}{a}\right)^k \qquad (x\in\omega^{(1)}_{y,a}).
\end{aligned}$$

It follows that the expression for $G_\Omega(x, y)$ in (33) converges absolutely to a harmonic function in $\omega^{(1)}_{y,a}$ and satisfies the estimate (35) there.

For the set $\omega^{(2)}_{y,a}$ we interchange x and y in (33) and argue similarly.

Analogous reasoning applies when $N = 3$. □

8 Proof of Theorem 1

We will adapt the approach taken in Theorem 19 of [10]. Theorem 1 follows from the result below on letting $c \to \infty$. We define

$$A(c) = \{x \in \mathbb{R}^N : c^{-1} < \|x\| < c\} \quad (c > 1).$$

Theorem 12 *Let $c > 1$ and let h be a harmonic function on the set $\Omega\cap A(c)$ which continuously vanishes on $\partial\Omega \cap A(c)$. Then h has a harmonic extension to the intersection of the sets*

$$\left\{x \in A(c) : c^{-1} < \|x\| \sin\theta_x\,(\min\{\sin\theta_*, \sin(\theta_*/2)\})^3\right\}$$

and

$$\left\{x \in A(c) : \|x\| < c \sin\theta_x \left(\min\{\sin\theta_*, \sin(\theta_*/2)\}\right)^3\right\}.$$

Proof Let $1 < c'' < c' < c$. On $\Omega \cap A(c')$ we can write h as the difference, $h_1 - h_2$, of two positive harmonic functions that vanish on $\partial\Omega \cap A(c')$. (Each of these is a Dirichlet solution with non-negative boundary data.) Next, let h_i^* $(i = 1, 2)$ be defined as h_i on $\Omega \cap \overline{A(c'')}$, as 0 on $\partial\Omega$ and also on $\Omega \backslash A(c')$, and extended to Ω by solving the Dirichlet problem in $\Omega \cap \left[A(c') \backslash \overline{A(c'')}\right]$. Then h_i^* is subharmonic on $\Omega \backslash \overline{A(c'')}$ and superharmonic on $\Omega \cap A(c')$, and continuously vanishes on $\partial\Omega$. By the Riesz decomposition theorem (Theorem 4.4.1 of [1]) and standard estimates of the Green function (cf. Theorems 4.2.4 and 4.2.5 of [1]) we can represent h_i^* as a Green potential $G_\Omega \Lambda_i$, where Λ_i is a signed measure on $\Omega \cap \left[\partial A(c') \cup \partial A(c'')\right]$ satisfying

$$\int (\theta_* - \theta_y)\, |d\Lambda_i|\,(y) < \infty.$$

(More precisely, the Riesz decomposition theorem shows that $h_i^* - G_\Omega \Lambda_i$ is harmonic on Ω, and the representation then follows from the fact that h_i^* and $G_\Omega \Lambda_i$ both vanish at the boundary.)

Let $a > 1$. It follows from Theorem 11 that the formula

$$\widetilde{h}(x) = \int_{\Omega \cap \left[\partial A(c') \cup \partial A(c'')\right]} \widetilde{G}_\Omega(x, y) d(\Lambda_1 - \Lambda_2)(y)$$

defines a harmonic extension of h from $\Omega \cap \overline{A(c'')}$ to the intersection of the sets

$$\left\{x \in A(c'') : \frac{a}{c''} < \|x\| \sin\theta_x \left(\min\{\sin\theta_*, \sin(\theta_*/2)\}\right)^3\right\} \tag{42}$$

and

$$\left\{x \in A(c'') : \|x\| < \frac{c''}{a} \sin\theta_x \left(\min\{\sin\theta_*, \sin(\theta_*/2)\}\right)^3\right\}. \tag{43}$$

Since c'' may be arbitrarily close to c, and a may be arbitrarily close to 1, the result follows. □

9 Bounds for ratios of conical functions

Several authors have considered bounds on ratios of modified Bessel functions: see, for example, [20] and the references provided there. In this section we establish corresponding bounds on ratios of conical functions in preparation for the proofs of Theorems 2 and 3. We begin with two elementary lemmas concerning Riccati equations.

Lemma 13 *Let h, α β and γ be differentiable functions on an interval (a, b) such that*

$$h'(t) = \alpha(t)\{h(t)\}^2 + \beta(t)h(t) + \gamma(t). \tag{44}$$

If $\beta' h > 0$, $\alpha' \geq 0$, $\gamma' \geq 0$ and $\liminf_{t\to a+} h'(t) > 0$, then $h' > 0$ on (a, b).

Proof Let

$$t_0 = \sup\{t \in (a, b) : h' > 0 \text{ on } (a, t)\}.$$

Then $t_0 > a$, by hypothesis. If $t_0 < b$, then $h'(t_0) = 0$ and so

$$h''(t_0) = \alpha'(t_0)\{h(t_0)\}^2 + \beta'(t_0)h(t_0) + \gamma'(t_0) > 0.$$

This yields a contradiction, since $h' > h'(t_0)$ on (a, t_0). Thus $t_0 = b$ as claimed. □

Lemma 14 *Suppose that*

$$h'(t) = -A(t)\{h(t) - B(t)\}\{h(t) + C(t)\} \quad (t \in (a, b)),$$

where h, A, B and C are all positive.

(i) *If $h' > 0$ on (a, b), then $0 < h < B$.*
(ii) *If $h' < 0$ on (a, b), then $0 < B < h$.*

Proof Since $h + C > 0$ and $A > 0$, we see that h' and $h - B$ have opposite signs. □

Proposition 15 *Let $0 < \theta_1 < \theta_2 < \pi$ and $\mu, \lambda \in \mathbb{R}$. Then*

$$\frac{P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_2)}{P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_1)} = \exp\left(\int_{\theta_1}^{\theta_2}\left\{\mu\cot\theta + \left(\lambda^2 + \left(\mu + \frac{1}{2}\right)^2\right)h_\mu(\theta)\right\}d\theta\right), \tag{45}$$

where

$$h_\mu(\theta) = \frac{P^{-\mu-1}_{-\frac{1}{2}+i\lambda}(\cos\theta)}{P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta)} \quad (0 < \theta < \pi).$$

Proof We note from (14.10.2) in [19] that

$$\sqrt{1-t^2}P_\nu^{1-\mu}(t) - (\nu + \mu + 1)P^{-\mu}_{\nu+1}(t) + (\nu - \mu + 1)tP_\nu^{-\mu}(t) = 0,$$

and combine this with Lemma 5(iii) to see that

$$(1-t^2)\frac{dP_\nu^{-\mu}}{dt} = -\sqrt{1-t^2}P_\nu^{1-\mu}(t) + \mu t P_\nu^{-\mu}(t). \tag{46}$$

We also know from (14.10.1) in [19] that

$$P_\nu^{1-\mu}(t) - 2\mu \frac{t}{\sqrt{1-t^2}} P_\nu^{-\mu}(t) + (\nu+\mu+1)(\nu-\mu) P_\nu^{-\mu-1}(t) = 0, \qquad (47)$$

and combine this with (46) to see that

$$(1-t^2)\frac{dP_\nu^{-\mu}}{dt} = -\mu t P_\nu^{-\mu}(t) + (\nu+\mu+1)(\nu-\mu)\sqrt{1-t^2}\,P_\nu^{-\mu-1}(t). \qquad (48)$$

Hence

$$\frac{1}{P^{-\mu}_{-\frac{1}{2}+i\lambda}(t)} \frac{dP^{-\mu}_{-\frac{1}{2}+i\lambda}}{dt} = -\frac{\mu t}{1-t^2} - \frac{\lambda^2 + (\mu+\frac{1}{2})^2}{\sqrt{1-t^2}} \frac{P^{-\mu-1}_{-\frac{1}{2}+i\lambda}(t)}{P^{-\mu}_{-\frac{1}{2}+i\lambda}(t)},$$

and so

$$\log \frac{P^{-\mu}_{-\frac{1}{2}+i\lambda}(t_2)}{P^{-\mu}_{-\frac{1}{2}+i\lambda}(t_1)} = -\int_{t_1}^{t_2} \left\{ \frac{\mu t}{\sqrt{1-t^2}} + \left(\lambda^2 + \left(\mu + \frac{1}{2}\right)^2\right) \frac{P^{-\mu-1}_{-\frac{1}{2}+i\lambda}(t)}{P^{-\mu}_{-\frac{1}{2}+i\lambda}(t)} \right\} \frac{dt}{\sqrt{1-t^2}}.$$

Equation (45) follows on substituting $t = \cos\theta$. □

Theorem 16 *If $\lambda \in \mathbb{R}$ and $\mu > -\frac{1}{2}$, then*

$$f_1(\theta) \le h_\mu(\theta) \le f_2(\theta) \quad (0 < \theta < \pi),$$

where h_μ is as in Proposition 15,

$$f_1(\theta) = \frac{1}{\sqrt{\lambda^2 + \left\{\left(\mu+\frac{3}{2}\right)\csc\theta\right\}^2} + \left(\mu+\frac{1}{2}\right)\cot\theta}$$

and

$$f_2(\theta) = \frac{1}{\sqrt{\lambda^2 + \left\{\left(\mu+\frac{1}{2}\right)\csc\theta\right\}^2} + \left(\mu+\frac{1}{2}\right)\cot\theta}.$$

Proof Let $F_\mu(\theta) = P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta)$. We note from (46) and (48) that

$$F'_{\mu+1}(\theta) = F_\mu(\theta) - (\mu+1)(\cot\theta)\,F_{\mu+1}(\theta)$$

and

$$F'_\mu(\theta) = \left\{\lambda^2 + \left(\mu+\frac{1}{2}\right)^2\right\} F_{\mu+1}(\theta) + \mu(\cot\theta)\,F_\mu(\theta).$$

Since $h_\mu = F_{\mu+1}/F_\mu$ we now see that

$$h'_\mu(\theta) = 1 - (2\mu+1)\,(\cot\theta)\,h_\mu(\theta) - \left\{\lambda^2 + \left(\mu+\frac{1}{2}\right)^2\right\}\{h_\mu(\theta)\}^2. \tag{49}$$

Further,

$$\Gamma(1+\mu)F_\mu(\theta)\left(\frac{2}{1-\cos\theta}\right)^{\mu/2} \to 1 \quad (\theta\to 0+),$$

by (14.8.1) of [19], so it follows from (49) that

$$\begin{aligned}\lim_{\theta\to 0+} h'_\mu(\theta) &= 1-(2\mu+1)\lim_{\theta\to 0+}\frac{F_{\mu+1}(\theta)}{(\sin\theta)\,F_\mu(\theta)} - 0\\ &= 1-\frac{2\mu+1}{\mu+1}\lim_{\theta\to 0+}\sqrt{\frac{1-\cos\theta}{2\sin^2\theta}} = \frac{1}{2(\mu+1)} > 0.\end{aligned} \tag{50}$$

The derivative of the function $\theta \mapsto -(2\mu+1)\cot\theta$ is positive, because $\mu > -\frac{1}{2}$. Since also $h_\mu > 0$, we can apply Lemma 13 to Eq. (49) to conclude that $h'_\mu > 0$ on $(0,\pi)$.

It follows from Lemma 14 that $h_\mu(\theta)$ is bounded above by the positive root of the equation

$$1-(2\mu+1)\,(\cot\theta)\,t - \left\{\lambda^2 + \left(\mu+\frac{1}{2}\right)^2\right\}t^2 = 0,$$

namely,

$$\frac{\sqrt{\lambda^2 + \left\{\left(\mu+\frac{1}{2}\right)\csc\theta\right\}^2} - \left(\mu+\frac{1}{2}\right)\cot\theta}{\lambda^2+\left(\mu+\frac{1}{2}\right)^2}, \tag{51}$$

which equals $f_2(\theta)$. Further, from (47),

$$\begin{aligned}\frac{F_{\mu-1}(\theta)}{F_\mu(\theta)} &= 2\mu\cot\theta + \left\{\lambda^2+\left(\mu+\frac{1}{2}\right)^2\right\}\frac{F_{\mu+1}(\theta)}{F_\mu(\theta)}\\ &\le 2\mu\cot\theta + \left\{\lambda^2+\left(\mu+\frac{1}{2}\right)^2\right\}f_2(\theta)\\ &= \sqrt{\lambda^2+\left\{\left(\mu+\frac{1}{2}\right)\csc\theta\right\}^2} + \left(\mu-\frac{1}{2}\right)\cot\theta,\end{aligned}$$

whence $h_\mu(\theta) \ge f_1(\theta)$. □

10 Proofs of Theorems 2 and 3

Proposition 17 *Let* $y \in \Omega$ *and* $\delta > 0$. *If* $\theta_* \leq \pi/2$, *then* $G_\Omega(\cdot, y)$ *has a harmonic extension* $\overline{G}_\Omega(y, \cdot)$ *to the set*

$$\left\{x \in \mathbb{R}^N \backslash \{0, y\} : \theta_x < 2\theta_* - \theta_y\right\},$$

and $\overline{G}_\Omega(\cdot, \cdot)$ *is bounded on the set*

$$\omega_{1,\delta} = \{(x, y) : \|x\| > \delta,\ \ \|y\| > \delta,\ \delta < \theta_y \leq \theta_*, \theta_* \leq \theta_x \leq 2\theta_* - \theta_y - \delta\}.$$

Proof We will give the argument when $N \geq 4$. Only slight adjustments are required when $N = 3$. It is enough, by Corollary 8, to show that the expansion (29) (or, indeed, the expansion (28)) converges absolutely and uniformly when $x, y \in \omega_{1,\delta}$. Let $\mu = (N-3)/2 + k$ and $\Lambda = \lambda^2 + \left(\mu + \frac{1}{2}\right)^2$.

By Lemma 9(ii),

$$\begin{aligned}
\left|g_k(\lambda, \theta_x, \theta_y)\right| &\leq \frac{R^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_*)}{P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_*)} P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_x) P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_y) \\
&= R^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_*) P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos(\theta_y + \theta_x - \theta_*)) Q \qquad (52)
\end{aligned}$$

when $(x, y) \in \omega_{1,\delta}$, where

$$Q = \frac{P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_x)}{P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_*)} \frac{P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_y)}{P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos(\theta_y + \theta_x - \theta_*))}.$$

By Theorem 16 and the formula (51) for $f_2(\theta)$,

$$h_\mu(\theta) \leq f_2(\theta) = \frac{1}{\Lambda}\left\{H(\theta) - \left(\mu + \frac{1}{2}\right)\cot\theta\right\}, \qquad (53)$$

where

$$H(\theta) = \sqrt{\lambda^2 + \left\{\left(\mu + \frac{1}{2}\right)\csc\theta\right\}^2}, \qquad (54)$$

so

$$\mu\cot\theta + \Lambda h_\mu(\theta) \leq H(\theta) - \frac{1}{2}\cot\theta.$$

Hence, by Proposition 15,

$$\frac{P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_x)}{P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_*)} = \exp\left(\int_{\theta_*}^{\theta_x}\left\{\mu\cot\theta + \Lambda h_\mu(\theta)\right\}d\theta\right)$$
$$\leq \exp\left(\int_{\theta_*}^{\theta_x}\left\{H(\theta) - \frac{1}{2}\cot\theta\right\}d\theta\right). \tag{55}$$

We claim that

$$\int_{\theta_*}^{\theta_x} H(\theta)d\theta \leq \int_{\theta_y}^{\theta_y+\theta_x-\theta_*} H(\theta)d\theta.$$

If $\theta_x \leq \pi/2$, this is clear from the monotonicity of H on $(0, \pi/2]$ and the fact that $\theta_y < \theta_*$. If $\theta_x > \pi/2$, we use the symmetry of H about $\pi/2$ as well as the above monotonicity to see that

$$\int_{\theta_*}^{\theta_x} H(\theta)d\theta = \int_{\theta_*}^{\pi/2} H(\theta)d\theta + \int_{\pi-\theta_x}^{\pi/2} H(\theta)d\theta$$
$$\leq \int_{\pi/2-(\theta_x-\theta_*)}^{\pi/2} H(\theta)d\theta \leq \int_{\theta_y}^{\theta_y+\theta_x-\theta_*} H(\theta)d\theta,$$

because $\theta_y + \theta_x - \theta_* \leq \theta_* \leq \pi/2$.

Since also $\theta_x < 2\theta_* - \theta_y \leq \pi - \theta_y$, and so $|\cot| \leq \cot\theta_y$ on (θ_*, θ_x), we now see from (55) that

$$\frac{P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_x)}{P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_*)} \leq \exp\left(\int_{\theta_y}^{\theta_y+\theta_x-\theta_*}\left\{H(\theta) + \frac{1}{2}\cot\theta_y\right\}d\theta\right)$$
$$\leq \exp\left(\int_{\theta_y}^{\theta_y+\theta_x-\theta_*}\left\{\mu\cot\theta + \Lambda f_2(\theta) + \cot\theta_y\right\}d\theta\right),$$

by the equality in (53). Proposition 15 and Theorem 16 also show that

$$\frac{P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_y)}{P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos(\theta_y+\theta_x-\theta_*))} = \exp\left(-\int_{\theta_y}^{\theta_y+\theta_x-\theta_*}\left\{\mu\cot\theta + \Lambda h_\mu(\theta)\right\}d\theta\right)$$
$$\leq \exp\left(-\int_{\theta_y}^{\theta_y+\theta_x-\theta_*}\left\{\mu\cot\theta + \Lambda f_1(\theta)\right\}d\theta\right).$$

Hence

$$Q \leq \exp\left(\int_{\theta_y}^{\theta_y+\theta_x-\theta_*} \left\{\Lambda\left(f_2(\theta)-f_1(\theta)\right)+\cot\theta_y\right\} d\theta\right). \tag{56}$$

Now

$$\begin{aligned}\frac{f_2(\theta)-f_1(\theta)}{f_1(\theta)f_2(\theta)} &= \sqrt{\lambda^2+\left\{\left(\mu+\frac{3}{2}\right)\csc\theta\right\}^2}-\sqrt{\lambda^2+\left\{\left(\mu+\frac{1}{2}\right)\csc\theta\right\}^2}\\ &= \frac{2(\mu+1)\csc^2\theta}{\sqrt{\lambda^2+\left\{\left(\mu+\frac{3}{2}\right)\csc\theta\right\}^2}+\sqrt{\lambda^2+\left\{\left(\mu+\frac{1}{2}\right)\csc\theta\right\}^2}}\\ &\leq \csc\theta, \end{aligned} \tag{57}$$

and

$$f_1(\theta) \leq f_2(\theta) \leq \Lambda^{-1/2} \quad (0<\theta\leq\pi/2),$$

so

$$\Lambda\left\{f_2(\theta)-f_1(\theta)\right\} \leq \Lambda\,(\csc\theta)\,f_1(\theta)f_2(\theta) \leq \csc\theta \quad (0<\theta\leq\pi/2).$$

Since $\theta_y+\theta_x-\theta_* \leq \pi/2$, we now see from (56) that

$$Q \leq \exp\left(\int_{\theta_y}^{\theta_y+\theta_x-\theta_*}(\csc\theta+\cot\theta_y)d\theta\right) \leq \exp\left(\pi\csc\delta\right)$$

when $(x,y)\in\omega_{1,\delta}$. It follows from (52) that

$$\left|g_k(\lambda,\theta_x,\theta_y)\right| \leq C(\delta)R_{-\frac{1}{2}+i\lambda}^{-\mu}(\cos\theta_*)P_{-\frac{1}{2}+i\lambda}^{-\mu}(\cos(\theta_y+\theta_x-\theta_*)).$$

Since, by (26),

$$\begin{aligned}&\frac{a_N}{\pi}\frac{2^{\frac{N-3}{2}}\Gamma\left(\frac{N-3}{2}\right)}{(\sin\theta_x\sin\theta_y)^{\frac{N-3}{2}}\left(\|x\|\,\|y\|\right)^{\frac{N-2}{2}}}\int_0^\infty\sum_{k=0}^\infty\left(k+\frac{N-3}{2}\right)\\ &\quad\times R_{-\frac{1}{2}+i\lambda}^{\frac{3-N}{2}-k}(\cos\theta_*)P_{-\frac{1}{2}+i\lambda}^{\frac{3-N}{2}-k}(\cos(\theta_y+\theta_x-\theta_*))C_k^{\left(\frac{N-3}{2}\right)}(1)\\ &\quad=\left(\frac{\sin\theta_*\sin(\theta_y+\theta_x-\theta_*)}{\sin\theta_x\sin\theta_y}\right)^{\frac{N-3}{2}}\left\{2\sqrt{\|x\|\,\|y\|}\sin\left(\theta_*-\frac{\theta_x+\theta_y}{2}\right)\right\}^{2-N}\\ &\quad\leq C(N,\delta),\end{aligned}$$

the proof is complete. □

Proposition 18 *Let $\theta_* > \pi/2$, $y \in \Omega$ and $0 < \delta < \min\{\theta_* - \theta_y, \theta_* - \pi/2\}$. Then $G_\Omega(\cdot, y)$ has a harmonic extension $\overline{G}_\Omega(y, \cdot)$ to the set*

$$\left\{x \in \mathbb{R}^N \backslash \{0\} : \theta_y < \theta_x \text{ and } \tan\frac{\theta_x}{2}\tan\frac{\theta_y}{2} < \left(\tan\frac{\theta_*}{2}\right)^2\right\},$$

and $\overline{G}_\Omega(y, \cdot)$ is bounded on the set

$$\omega_{2,\delta} = \left\{\begin{array}{c}(x, y) : \|x\| > \delta,\ \ \|y\| > \delta,\ \delta < \theta_y \le \theta_* \le \theta_x, \\ \tan(\theta_x/2)\tan(\theta_y/2) < \tan^2((\theta_* - \delta)/2)\end{array}\right\}.$$

Proof We modify the previous proof. Again we will assume, for simplicity, that $N \ge 4$. This time we note that

$$\begin{aligned}\left|g_k(\lambda, \theta_x, \theta_y)\right| &\le \frac{R^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_*)}{P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_*)} P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_x) P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_y) \\ &= R^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_*) P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos(\theta_* - \delta))T,\end{aligned} \tag{58}$$

where

$$T = \frac{P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_x)}{P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_*)} \frac{P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_y)}{P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos(\theta_* - \delta))}. \tag{59}$$

It follows from our choice of δ that

$$\theta_* - \delta > \pi/2, \tag{60}$$

and from (1) that

$$\theta_x - \theta_* < \theta_* - \theta_y - \delta \ \text{ when } (x, y) \in \omega_{2,\delta}. \tag{61}$$

Also, if $0 \le a < b$, then

$$\lambda \mapsto \sqrt{\lambda^2 + b} - \sqrt{\lambda^2 + a} \text{ is decreasing on } [0, \infty). \tag{62}$$

Let $H(\theta)$ be as in (54). Then

$$\int_{\theta_*}^{\theta_x} H(\theta)d\theta = \int_0^{\theta_x - \theta_*} \sqrt{\lambda^2 + \left\{\left(\mu + \frac{1}{2}\right)\csc(\vartheta + \theta_*)\right\}^2} d\vartheta \tag{63}$$

and

$$\int_{\theta_y}^{\theta_*-\delta} H(\theta)d\theta = \int_0^{\theta_*-\theta_y-\delta} \sqrt{\lambda^2 + \left\{\left(\mu+\frac{1}{2}\right)\csc(\theta_*-\delta-\vartheta)\right\}^2}\, d\vartheta$$
$$\geq \int_0^{\theta_x-\theta_*} \sqrt{\lambda^2 + \left\{\left(\mu+\frac{1}{2}\right)\csc(\theta_*-\delta-\vartheta)\right\}^2}\, d\vartheta$$
$$+\int_{\theta_x-\theta_*}^{\theta_*-\theta_y-\delta} \left(\mu+\frac{1}{2}\right)\csc(\theta_*-\delta-\vartheta)d\vartheta, \tag{64}$$

by (61). Also,

$$\sin(\vartheta+\theta_*) \leq \sin(\theta_*-\delta-\vartheta) \quad (0<\vartheta<\theta_x-\theta_*), \tag{65}$$

in view of (60). It follows from (63), (64), (62) and then (65) that

$$\int_{\theta_*}^{\theta_x} H(\theta)d\theta - \int_{\theta_y}^{\theta_*-\delta} H(\theta)d\theta \leq \int_0^{\theta_x-\theta_*} \left\{ \begin{array}{c} \sqrt{\lambda^2+\left\{\left(\mu+\frac{1}{2}\right)\csc(\vartheta+\theta_*)\right\}^2} \\ -\sqrt{\lambda^2+\left\{\left(\mu+\frac{1}{2}\right)\csc(\theta_*-\delta-\vartheta)\right\}^2} \end{array} \right\} d\vartheta$$
$$-\int_{\theta_x-\theta_*}^{\theta_*-\theta_y-\delta} \left(\mu+\frac{1}{2}\right)\csc(\theta_*-\delta-\vartheta)d\vartheta$$
$$\leq \int_0^{\theta_x-\theta_*} \left(\mu+\frac{1}{2}\right)\{\csc(\vartheta+\theta_*)-\csc(\theta_*-\delta-\vartheta)\}\, d\vartheta$$
$$-\int_{\theta_x-\theta_*}^{\theta_*-\theta_y-\delta} \left(\mu+\frac{1}{2}\right)\csc(\theta_*-\delta-\vartheta)d\vartheta$$
$$= \left(\mu+\frac{1}{2}\right)\left(\int_{\theta_*}^{\theta_x}\csc\theta\, d\theta - \int_{\theta_y}^{\theta_*-\delta}\csc\theta\, d\theta\right)$$
$$= \left(\mu+\frac{1}{2}\right)\log\left(\frac{\tan(\theta_x/2)\tan(\theta_y/2)}{\tan(\theta_*/2)\tan((\theta_*-\delta)/2)}\right) \leq 0.$$

Hence, by (55), (53) and the fact that log sin is a primitive for cot,

$$\frac{P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_x)}{P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_*)} \leq \exp\left(\int_{\theta_*}^{\theta_x}\left\{H(\theta)-\frac{1}{2}\cot\theta\right\}d\theta\right)$$
$$= \sqrt{\frac{\sin\theta_*}{\sin\theta_x}}\exp\left(\int_{\theta_*}^{\theta_x} H(\theta)d\theta\right)$$
$$\leq \sqrt{\frac{\sin\theta_*}{\sin\theta_x}}\exp\left(\int_{\theta_y}^{\theta_*-\delta} H(\theta)d\theta\right)$$
$$= \sqrt{\frac{\sin\theta_*}{\sin\theta_x}}\exp\left(\int_{\theta_y}^{\theta_*-\delta}\left\{\Lambda f_2(\theta)+\left(\mu+\frac{1}{2}\right)\cot\theta\right\}d\theta\right).$$

Since, by Proposition 15 and Theorem 16,

$$\frac{P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_y)}{P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos(\theta_*-\delta))} = \exp\left(-\int_{\theta_y}^{\theta_*-\delta}\left\{\mu\cot\theta+\Lambda h_\mu(\theta)\right\}d\theta\right)$$
$$\leq \exp\left(-\int_{\theta_y}^{\theta_*-\delta}\left\{\mu\cot\theta+\Lambda f_1(\theta)\right\}d\theta\right),$$

we now see from (59) that

$$T \leq \sqrt{\frac{\sin\theta_*}{\sin\theta_x}}\exp\left(\int_{\theta_y}^{\theta_*-\delta}\left\{\Lambda\left(f_2(\theta)-f_1(\theta)\right)+\frac{1}{2}\cot\theta\right\}d\theta\right)$$
$$= \sqrt{\frac{\sin\theta_*\sin(\theta_*-\delta)}{\sin\theta_x\sin\theta_y}}\exp\left(\int_{\theta_y}^{\theta_*-\delta}\Lambda\left(f_2(\theta)-f_1(\theta)\right)d\theta\right).$$

Now

$$f_2(\theta) = \frac{\csc\theta}{\Lambda}\left\{\sqrt{\lambda^2\sin^2\theta+\left(\mu+\frac{1}{2}\right)^2}-\left(\mu+\frac{1}{2}\right)\cos\theta\right\}$$
$$\leq \frac{2\csc\theta}{\Lambda}\left\{\sqrt{\lambda^2\sin^2\theta+\left(\mu+\frac{1}{2}\right)^2}\right\} \leq \frac{2\csc\theta}{\sqrt{\Lambda}},$$

so from (57) we have

$$\Lambda\left(f_2(\theta)-f_1(\theta)\right) \leq \Lambda(\csc\theta)f_1(\theta)f_2(\theta) \leq \Lambda(\csc\theta)\left\{f_2(\theta)\right\}^2 \leq 4\csc^3\theta.$$

Hence

$$T \leq \sqrt{\frac{\sin\theta_*\sin(\theta_*-\delta)}{\sin\theta_x\sin\theta_y}}\exp\left(4\int_{\theta_y}^{\theta_*-\delta}\csc^3\theta d\theta\right) \leq C(\theta_*,\delta)$$

when $(x, y) \in \omega_{2,\delta}$, since

$$\frac{1}{\sin\theta_x} \leq \frac{1-\cos\theta_x}{\sin\theta_x} = \tan\frac{\theta_x}{2} \leq \frac{\{\tan(\theta_*/2)\}^2}{\tan(\theta_y/2)}.$$

It follows from (58) that

$$\left|g_k(\lambda,\theta_x,\theta_y)\right| \leq C(\theta_*,\delta)R^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta_*)P^{-\mu}_{-\frac{1}{2}+i\lambda}(\cos(\theta_*-\delta)).$$

The argument is completed by observing from (26) that

$$\begin{aligned}&\frac{a_N}{\pi}\frac{2^{\frac{N-3}{2}}\Gamma\left(\frac{N-3}{2}\right)}{(\sin\theta_x\sin\theta_y)^{\frac{N-3}{2}}(\|x\|\,\|y\|)^{\frac{N-2}{2}}}\int_0^\infty\sum_{k=0}^\infty\left(k+\frac{N-3}{2}\right)\\&\times R_{-\frac{1}{2}+i\lambda}^{\frac{3-N}{2}-k}(\cos\theta_*)P_{-\frac{1}{2}+i\lambda}^{\frac{3-N}{2}-k}(\cos(\theta_*-\delta))C_k^{\left(\frac{N-3}{2}\right)}(1)\\&=\left(\frac{\sin\theta_*\sin(\theta_*-\delta)}{\sin\theta_x\sin\theta_y}\right)^{\frac{N-3}{2}}\left\{2\sqrt{\|x\|\,\|y\|}\sin\left(\frac{\delta}{2}\right)\right\}^{2-N}\\&\le C(N,\delta,\theta_*),\end{aligned}$$

when $(x,y)\in\omega_{2,\delta}$. □

Proof of Theorem 2 Let $\theta_0<\theta_-<\theta_+<\theta_*$ and $1<c''<c'$. As in the proof of Theorem 12, we can represent h in $\left[\Omega(\theta_*)\backslash\overline{\Omega(\theta_+)}\right]\cap A(c'')$ as the potential $G_\Omega\Lambda$ of a signed measure Λ on the union of the sets

$$\partial\left(A(c')\cap[\Omega(\theta_*)\backslash\overline{\Omega(\theta_-)}]\right)\cap\Omega(\theta_*)\text{ and }\partial\left(A(c'')\cap[\Omega(\theta_*)\backslash\overline{\Omega(\theta_+)}]\right)\cap\Omega(\theta_*).$$

Then $h=h_a+h_b$, where

$$h_a(x)=\int_{\Omega\backslash A(c'')}G_\Omega(x,y)d\Lambda(y)$$

and

$$h_b(x)=\int_{A(c'')\cap[\partial\Omega(\theta_-)\cup\partial\Omega(\theta_+)]}G_\Omega(x,y)d\Lambda(y).$$

It follows from Theorem 12 that h_a has a harmonic extension to the intersection of the sets (42) and (43), and from Proposition 17 that h_b has a harmonic extension to the set $\Omega(2\theta_*-\theta_+)\backslash\overline{\Omega(\theta_+)}$. The result now follows on letting $c''\to\infty$ and $\theta_+\to\theta_0+$. □

Proof of Theorem 3 We follow the above argument except that we use Proposition 18 to see that h_b has a harmonic extension to the set

$$\left\{x\in\mathbb{R}^N\backslash\{0\}:\theta_+<\theta_x\text{ and }\tan\frac{\theta_x}{2}\tan\frac{\theta_+}{2}<\left(\tan\frac{\theta_*}{2}\right)^2\right\}.$$

□

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