Finiteness and infiniteness results for Torelli groups of (hyper-)Kähler manifolds

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Abstract

The Torelli group $\mathcal{T}(X)$ of a closed smooth manifold X is the subgroup of the mapping class group $\pi_0(\text{Diff}^+(X))$ consisting of elements which act trivially on the integral cohomology of X . In this note we give counterexamples to Theorem 3.4 by Verbitsky (Duke Math J 162(15):2929–2986, 2013) which states that the Torelli group of simply connected Kähler manifolds of complex dimension ≥ 3 is finite. This is done by constructing under some mild conditions homomorphisms $J : \mathcal{T}(X) \rightarrow H^3(X; \mathbb{Q})$ and showing that for certain Kähler manifolds this map is non-trivial. We also give a counterexample to Theorem 3.5 (iv) in (Verbitsky in Duke Math J 162(15):2929–2986, 2013) where Verbitsky claims that the Torelli group of hyperkähler manifolds are finite. These examples are detected by the action of diffeomorphisms on $\pi_4(X)$. Finally we confirm the finiteness result for the special case of the hyperkähler manifold $K^{[2]}$.

1 Introduction

Let X be a closed connected oriented manifold. The **mapping class group** $\mathcal{M}(X)$ is the group of connected components of the orientation preserving self diffeomor-

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phisms of X . The subgroup consisting of diffeomorphisms which act trivially on the integral cohomology ring is called the **Torelli group** of X , denote by $\mathcal{T}(X)$. In this note we consider two classes of manifolds, the first is characterized by the following assumptions:

Assumption (*) The manifold X has trivial first rational homology and the first Pontrjagin class (considered in rational cohomology) $p_1(X) \in H^4(X; \mathbb{Q})$ is a linear combination of products of 2-dimensional classes, i.e. there exist classes $z_i \in H^2(X; \mathbb{Q})$ and rational numbers a_{ij} such that $p_1(X) = \sum a_{ij} z_i \cup z_j$. We denote the set consisting of the classes z_i and of the rational numbers a_{ij} by D , like definition data.

We construct a homomorphism

$$J_D : \mathcal{T}(X) \rightarrow H^3(X; \mathbb{Q})$$

which we consider as a sort of Johnson homomorphism since it is landing in an abelian group and constructed via the mapping torus. The construction is as follows. Consider the mapping torus X_f of f . Since f acts trivially on the cohomology groups the Wang sequence looks like:

$$0 \rightarrow H^{k-1}(X; \mathbb{Q}) \xrightarrow{\delta} H^k(X_f; \mathbb{Q}) \xrightarrow{i^*} H^k(X; \mathbb{Q}) \rightarrow 0$$

Since $H^1(X; \mathbb{Q}) = 0$ this implies that the inclusion induces an isomorphism $H^2(X_f; \mathbb{Q}) \rightarrow H^2(X; \mathbb{Q})$ and so for each class $z \in H^2(X; \mathbb{Q})$ there is a unique class $\bar{z} \in H^2(X_f; \mathbb{Q})$ restricting to z . Next we look at $p_1(X_f) - \sum a_{ij} \bar{z}_i \cup \bar{z}_j \in H^4(X_f; \mathbb{Q})$. By construction its restriction to $H^4(X; \mathbb{Q})$ is zero. Thus there exists a unique class denoted by $J_D(f) \in H^3(X; \mathbb{Q})$ mapping to this class in the Wang sequence.

Proposition 1.1 *Let X fulfill the Assumptions (*), then the map*

$$J_D : \mathcal{T}(X) \rightarrow H^3(X; \mathbb{Q})$$

is a homomorphism.

Next we are looking for Kähler manifolds where this invariant is non-trivial. For this we prove a purely topological result.

Theorem 1.2 *Let X be a simply connected 6-manifold with $H_2(X; \mathbb{Z}) \cong \mathbb{Z}$ and nontrivial cohomology product $H^2(X; \mathbb{Z}) \times H^2(X; \mathbb{Z}) \rightarrow H^4(X; \mathbb{Z})$. Then the Assumptions (*) are fulfilled and the invariant is independent of any choices and is denoted by J . The image of J is a lattice in $H^3(X; \mathbb{Q})$, i.e. a finitely generated abelian group of same rank as the dimension of $H^3(X; \mathbb{Q})$.*

Considering hypersurfaces $X(d)$ of degree $d > 1$ in $\mathbb{C}P^4$ we obtain complex three-dimensional Kähler manifolds with infinite Torelli group, since $d > 1$ implies that $H^3(X(d); \mathbb{Q}) \neq 0$. Taking the product with copies of $\mathbb{C}P^1$ one obtains examples in all

complex dimensions ≥ 3 . Namely if $f \in \mathcal{T}(X(d))$ then by definition of J_D we have for $f \times \text{id} : X(d) \times (\mathbb{CP}^1)^r \rightarrow X(d) \times (\mathbb{CP}^1)^r$ and the obvious choice of D

$$J_D(f \times \text{id}) = J(f)$$

where we identify $H^3(X(d); \mathbb{Q})$ with $H^3(X(d) \times (\mathbb{CP}^1)^r; \mathbb{Q})$ under the projection map.

Thus we obtain the Corollary contradicting Verbitsky's theorem:

Corollary 1.3 *For all $n \geq 3$ there are Kähler manifolds of complex dimension n with infinite Torelli group, actually containing elements of infinite order. In particular taking for example the quintic in $\mathbb{CP}^4$ we obtain Calabi–Yau manifolds with infinite Torelli group.*

Remark 1.4 For simply connected 6-manifolds X with $H_2(X; \mathbb{Z}) \cong \mathbb{Z}$ we compute in [7] the full mapping class group as well as the Torelli group and give generators of these groups. In particular we prove that an element f in the Torelli group has finite order if and only if $J(f) = 0$.

Remark 1.5 In general the invariant J_D depends on the choice of the data. So, in general there are many homomorphisms depending on the choice of the data. The simplest example is $X = S^2 \times S^4 \# S^3 \times S^3$, where for $z = 0$ and z the generator of $H^2(X; \mathbb{Z})$ the difference of the J -invariants with these data is non-trivial, as shown in [7].

Now we come to the second class of manifolds which is characterized by the following assumptions:

Assumption ()** The manifolds X are simply connected and eight-dimensional, whose Betti numbers satisfy one of the following conditions

- (1) $b_2 > 2, b_3 \geq 1, b_4 > b_2(b_2 + 1)/2$;
- (2) $b_2 = 2, b_3 > 1, b_4 > b_2(b_2 + 1)/2$;
- (3) $b_2 = 2, b_3 = 1, b_4 > b_2(b_2 + 1)/2$ and $p_1(X) = 0 \in H^4(X; \mathbb{Q})$.

Theorem 1.6 *If X satisfies the assumptions (**), then the Torelli group $\mathcal{T}(X)$ is infinite.*

We apply this result to find a counterexample to Theorem 3.5 (iv) in [13]. For this we consider the complex four-dimensional hyperkähler manifold $K^2(T)$. For the construction and the following information we refer to [11] (where it is called K_2). The manifold is simply connected and the Betti numbers are: $b_2(K^2(T)) = 7$, $b_3(K^2(T)) = 8$ and $b_4(K^2(T)) = 108$. Thus our theorem implies:

Corollary 1.7 *The Torelli group of the hyperkähler manifold $K^2(T)$ is not finite.*

This contradicts Verbitsky's Theorem 3.5, iv ([13]).

There is another complex four-dimensional hyperkähler manifold denoted by $K^{[2]}$, where K is a K_3 -surface (for the construction see again for example [11]). In this case we show that the Torelli group is finite. This follows from:

Theorem 1.8 *Let X be a simply-connected closed 8-manifold satisfying the following conditions*

- (1) the cup product induces an isomorphism $\cup: \text{Sym}^2(H^2(X; \mathbb{Q})) \rightarrow H^4(X; \mathbb{Q})$.
- (2) $H^3(X; \mathbb{Q}) = 0$,
- (3) $p_1(X) \neq 0 \in H^4(X; \mathbb{Q})$.

Then $\mathcal{T}(X)$ is finite.

Corollary 1.9 *The Torelli group of $K^{[2]}$, where K is a K_3 -surface, is finite.*

Proof By [11] the Betti numbers are $b_2 = 23$, $b_3 = 0$ and $b_4 = 276$, which is the same as the dimension of the second symmetric product of $\mathbb{Q}^{23}$. Since the second symmetric power of the second cohomology of a hyperkähler manifold always injects into the 4-th cohomology [2], this implies (1). That $p_1(K^{[2]})$ is non-zero follows from [1].

Remark 1.10 This implies that Verbitsky's Theorem 3.5 is true for $K^{[2]}$ and so his argument for his main result about the moduli space of hyperkähler manifolds holds for $K^{[2]}$. In the general case the finiteness statement is used at several places in the paper so that the corresponding statements are either wrong or open. This is summarized in an erratum [14].

Looijenga [9] has defined a different moduli space, called the separated moduli space, and gives a complete proof of the global Torelli Theorem for this space. He further proves that this moduli space is equal to the one considered by Verbitsky (giving a different proof that the relation defining the moduli space is transitive). Thus the global Torelli Theorem by Verbitsky holds, but with a different proof.

We would like to thank Daniel Huybrechts for bringing Verbitsky's paper [13] to our attention when we told him about our paper [7] where we give a complete computation of the mapping class group of certain 6-manifolds. Since counterexamples to Verbitsky's theorem might be of separate interest we wrote this note. We have sent this note to several people and received a paper by Richard Hain [4] confirming our result that Theorem 3.4 in [13] is incorrect by showing that the Torelli group of certain complex three-dimensional Kähler manifolds M have an abelian quotient of infinite order. These examples are different from our examples, they are detected by the induced map $\pi_3(M) \otimes \mathbb{Q} \rightarrow \pi_3(M) \otimes \mathbb{Q}$, whereas for complex three-dimensional complete intersections $\pi_3(X) \otimes \mathbb{Q} \cong H_3(X; \mathbb{Q})$, and so the Torelli groups acts trivial on $\pi_3(X) \otimes \mathbb{Q}$. We would like to thank the referee for his suggestions to improve the paper.

2 Proofs

Proof of Proposition 1.1 To prove that J_D is a homomorphism we first note that if X is a fibre in X_f then we have an exact sequence with rational coefficients

$$0 \rightarrow H^4(X_f, X) \rightarrow H^4(X_f) \rightarrow H^4(X) \rightarrow 0$$

and the term on the left is by suspension isomorphism isomorphic to $H^3(X)$ and under this isomorphism $J_D(f)$ corresponds to the class in $H^4(X_f, X)$ which restricts to $p_1(X_f) - \sum a_{ij} \bar{z}_i \cup \bar{z}_j$.

Now we construct X_{fg} from X_f by cutting $I \times X$ along $1/2 \times X$ and reglueing it via g from the left to the right. Then we consider the two fibres over $1/4$ and $3/4$ and denote them by X_1 and X_2 . We consider the restriction map (with rational coefficients)

$$H^4(X_{fg}, X_1 \cup X_2) \rightarrow H^4(X_{fg}, X_1)$$

We have an isomorphism

$$H^4(X_f, X) \oplus H^4(X_g, X) \xrightarrow{\cong} H^4(X_{fg}, X_1 \cup X_2)$$

If we identify $H^4(X_{fg}, X_1)$ with $H^3(X)$, the homomorphism

$$H^4(X_f, X) \oplus H^4(X_g, X) \xrightarrow{\cong} H^4(X_{fg}, X_1 \cup X_2) \rightarrow H^4(X_{fg}, X_1)$$

corresponds to the homomorphism $H^3(X) \oplus H^3(X) \rightarrow H^3(X)$ given by the sum, and the element (to distinguish the $\bar{z}_i$'s corresponding to X_f , X_g and X_{fg} we name them as $i_f^{-1}(z_i)$, $i_g^{-1}(z_i)$ and $i_{fg}^{-1}(z_i)$ respectively)

$$(p_1(X_f) - \sum a_{ij} i_f^{-1}(z_i) \cup i_f^{-1}(z_j), p_1(X_g) - \sum a_{ij} i_g^{-1}(z_i) \cup i_g^{-1}(z_j))$$

is mapped to $p_1(X_{fg}) - \sum a_{ij} i_{fg}^{-1}(z_i) \cup i_{fg}^{-1}(z_j)$. This implies that $J_D(fg) = J_D(f) + J_D(g)$.

Next we consider the 6-manifolds in Theorem 1.2. Let $z \in H^2(X)$ be a generator, by Poincaré duality $H^4(X) \cong \mathbb{Z}$ and thus z^2 equals to d times a generator of $H^4(X)$. The condition that the cohomology product $H^2(X) \times H^2(X) \rightarrow H^4(X)$ is non-trivial is equivalent to saying that d is nonzero. It's shown in [5, 15] that for every $d \neq 0$ there exist such X . Furthermore there is a connected sum decomposition $X = N \sharp_g(S^3 \times S^3)$, where N has the same homology as $\mathbb{CP}^3$.

We introduce a construction of diffeomorphisms, which is a generalization of Dehn twists. Let $S^3 \times D^3 \subset X$, $\alpha \in \pi_3(SO(4))$, choose a smooth map $\varphi: (D^3, \partial) \rightarrow (SO(4), I)$ representing α , such that a neighborhood of ∂D^3 is mapped to the identity. Define $f: X \rightarrow X$ by

$$f(u) = \begin{cases} (\varphi(y) \cdot x, y) & \text{if } u = (x, y) \in S^3 \times D^3 \\ u & \text{otherwise} \end{cases}$$

We call f a Dehn twist in $S^3 \times D^3$ with parameter α .

Proof of Theorem 1.2 The main ingredients of Theorem 1.2 already appeared in [6]. The proof of a slightly different situation will be given in [7]. For convenience of the reader we repeat the proof.

By the assumptions the rational Pontrjagin class $p_1(X) \in H^4(X; \mathbb{Q})$ is a multiple of z^2 , say, $p_1(X) = l \cdot z^2$, $l \in \mathbb{Q}$. Therefore our invariant $J(f)$ is the pre-image of $p_1(X_f) - l \cdot \bar{z}^2$ in $H^3(X)$.

In the decomposition $X = N \sharp g(S^3 \times S^3)$ we number the standard embedded spheres

$$(S^3 \times \{pt\})_1 = S^3_1, (\{pt\} \times S^3)_1 = S^3_2, \dots, (S^3 \times \{pt\})_g = S^3_{2g-1}, (\{pt\} \times S^3)_g = S^3_{2g}.$$

They represent a symplectic basis $\{e_1, \dots, e_{2g}\}$ of $H_3(X)$.

Let $\alpha \in \pi_3(SO(3)) \cong \mathbb{Z}$ be a generator, ξ be the three-dimensional vector bundle over S^4 corresponding to α , $i_*(\alpha) \in \pi_3(SO(4))$ be the image induced by the inclusion $i: SO(3) \rightarrow SO(4)$. Then it's known that $p_1(\xi)$ equals 4 times a generator of $H^4(S^4)$ (see [10, Lemma 20.10]), and the four-dimensional vector bundle over S^4 corresponding to $i_*(\alpha)$ has trivial Euler class.

Now let $f_i: X \rightarrow X$ be the Dehn twist in $S^3_i \times D^3$ with parameter $i_*(\alpha)$. By the geometric construction, the action of f_i on homology is seen as follows: f_i acts trivially on all basis elements of the homology groups but one, namely the one e_j dual to e_i in the symplectic basis of $H_3(X)$, and $f_{i*}(e_j) = e_j + \chi(i_*(\alpha)) \cdot e_i$, where $\chi(i_*(\alpha))$ is the Euler characteristic of $i_*(\alpha)$. Since the Euler class of $i_*(\alpha)$ is trivial, f_i acts trivially on homology and hence is in $\mathcal{T}(X)$. We claim that $J(f_i)$ equals 4 times the Poincaré dual of e_i in $H^3(X)$. Notice that for the manifolds X under consideration, the homomorphism J can be defined using integral cohomology and is indeed a homomorphism from $\mathcal{T}(X)$ to $H^3(X)$. This implies that the image of J is a subgroup of $H^3(X)$ of the same rank, hence it is a lattice in $H^3(X; \mathbb{Q})$.

It is enough to show the claim for $i = 1$. By the adjunction formula this is equivalent to

$$\langle J(f_1), e_i \rangle = 0 \text{ for } i \neq 2 \text{ and } \langle J(f_1), e_2 \rangle = 4.$$

Since $f_1|_{S^3_1}$ is the identity, we have embeddings $S^3_i \times S^1 \subset X_{f_1}$. Let $\bar{e}_i = [S^3_i \times S^1] \in H_4(X_{f_1})$, then $\bar{e}_i$ is a pre-image of e_i in the short exact sequence

$$0 \rightarrow H_4(X) \rightarrow H_4(X_{f_1}) \xrightarrow{\partial} H_3(X) \rightarrow 0$$

There is a decomposition $X_{f_1} = (N \times S^1) \sharp_{S^1} (\sharp_g(S^3 \times S^3)_{f_1})$, where $\sharp_{S^1}$ denotes the fiber connected sum along S^1 . Then $H^4(X_{f_1}) = H^4(N) \oplus H^4(\sharp_g(S^3 \times S^3)_{f_1})$ and $\bar{z}^2 = (z^2, 0)$. This implies $\langle \bar{z}^2, \bar{e}_i \rangle = 0$. Therefore we have

$$\langle J(f_1), e_i \rangle = \langle J(f_1), \partial \bar{e}_i \rangle = \langle \delta J(f_1), \bar{e}_i \rangle = \langle p_1(X_{f_1}) - l \cdot \bar{z}^2, \bar{e}_i \rangle = \langle p_1(X_{f_1}), [S^3_i \times S^1] \rangle$$

Now notice that f_1 is the identity on a tubular neighborhood of S^3_i 's for $i \neq 2$, therefore the normal bundle of $S^3_i \times S^1$ in X_{f_1} is trivial, and we have

$$\langle p_1(X_{f_1}), [S^3_i \times S^1] \rangle = 0$$

for $i \neq 2$. Let

$$c: S^3 \times S^1 \rightarrow S^3 \times S^1 / S^3 \times \{pt\} = S^4 \vee S^1 \rightarrow S^4$$

be the quotient map, from the geometric construction it's easy to see that the normal bundle of $S^3_2 \times S^1$ in X_{f_1} equals $c^*\xi$. Therefore

$$\langle p_1(X_{f_1}), [S^3_2 \times S^1] \rangle = \langle c^*p_1(\xi), [S^3_2 \times S^1] \rangle = \langle p_1(\xi), [S^4] \rangle = 4.$$

The proofs of Theorems 1.6 and 1.8 are based on modified surgery theory [8]. We recall some basic definitions here. The normal k -type of a smooth oriented manifold X is a fibration $p : B \rightarrow BSO$, which is characterized by the assumptions that there is a lift of the normal Gauss map $\nu : X \rightarrow BO$ by a $(k+1)$ -equivalence $\bar{\nu} : X \rightarrow B$ and that the homotopy groups of the homotopy fibre F of p are trivial in degree $\geq k+1$ (for details see [8]).

To show Theorem 1.6 we will construct infinitely many mapping classes in $\mathcal{T}(X)$, whose actions on $\pi_4(X)$ are pairwise distinct. Let $p : B \rightarrow BSO$ be the normal 4-type of X , denote the homotopy fiber of p by F , and the kernel of the Hurewicz homomorphism $\pi_4(F) \rightarrow H_4(F)$ by $K\pi_4(F)$.

Lemma 2.1 *Under the assumptions (**) we have*

- (1) *the image of the Hurewicz map $\pi_4(B) \rightarrow H_4(B)$ has rank > 0 ;*
- (2) *the abelian group $K\pi_4(F)$ has rank > 0 .*

Proof We use Sullivan's rational homotopy theory [12] to show the assumptions (**) imply the conclusions in this lemma. For this we construct the minimal model from information of the cohomology ring.

Let $x_1, \dots, x_{b_2}$ be free generators of the minimal model in degree 2 with differential 0. Let $y_1, \dots, y_{b_3}$ be a part of the free generators in degree 3, which generate $H^3(X; \mathbb{Q})$. They have trivial differentials. The elements $x_i y_j$ generate a space of dimension $b_2 b_3$, on which the differential is trivial. Since by Poincaré duality the 5-th Betti number is b_3 , there must be at least $k = (b_2 - 1)b_3$ linearly independent indecomposable elements $z_1, \dots, z_k$ in degree 4, which by the differential are mapped injectively to a subspace of the vector space generated by the $x_i y_j$. In degree 4 the dimension of the subspace generated by $x_i x_j$ ($i \leq j$) is $b_2(b_2 + 1)/2$, whereas the 4-th Betti number is b_4 , there are additional (to the z_i) free generators $u_1, \dots, u_l$ (where $l \geq b_4 - b_2(b_2 + 1)/2 \geq 1$) with trivial differential. There might be more generators in degree 4 depending on the product $H^2(X; \mathbb{Q}) \otimes H^3(X; \mathbb{Q}) \rightarrow H^5(X; \mathbb{Q})$. But alone from this information the two conclusions in the lemma follow. Namely $\text{Hom}(\pi_4(X), \mathbb{Q})$ is isomorphic to the degree 4 subspace modulo the decomposable elements and so contains the subspace with basis $u_1, \dots, u_l, z_1, \dots, z_k$. The dual of the rational Hurewicz homomorphism maps u_i injectively. Dualizing we see that the dual of the u_i are mapped injectively showing (1) and the dual of the z_i are mapped to 0 in rational homology, so the rank of $K\pi_4(B) = \text{Ker}(\pi_4(B) \rightarrow H_4(B))$ is at least $k = (b_2 - 1)b_3$.

To analyze the rank of $K\pi_4(F)$, first note that $\pi_4(F) \otimes \mathbb{Q} \rightarrow \pi_4(B) \otimes \mathbb{Q}$ is an isomorphism if $p_1(X) = 0 \in H^4(X; \mathbb{Q})$ and is injective with a codimension 1 image if $p_1(X) \neq 0 \in H^4(X; \mathbb{Q})$. This follows from the long exact sequence of homotopy groups of the fibration $F \rightarrow B \rightarrow BSO$ and the facts that $\pi_5(BSO) = 0$, $\pi_4(BSO) \otimes \mathbb{Q} \cong \mathbb{Q}$ by the universal Pontrjagin class p_1 . The Leray–Serre spectral

sequence of the fibration shows that $H_4(F; \mathbb{Q}) \rightarrow H_4(B; \mathbb{Q})$ is injective. Now a little diagram chasing in the following commutative diagram

$$\begin{array}{ccccc} K\pi_4(F) \otimes \mathbb{Q} & \longrightarrow & \pi_4(F) \otimes \mathbb{Q} & \longrightarrow & H_4(F; \mathbb{Q}) \\ \downarrow & & \downarrow & & \downarrow \\ K\pi_4(B) \otimes \mathbb{Q} & \longrightarrow & \pi_4(B) \otimes \mathbb{Q} & \longrightarrow & H_4(B; \mathbb{Q}) \end{array}$$

shows that

$$K\pi_4(B) \otimes \mathbb{Q} \cap \text{Im}(\pi_4(F) \otimes \mathbb{Q} \rightarrow \pi_4(B) \otimes \mathbb{Q}) = K\pi_4(F) \otimes \mathbb{Q}.$$

This shows that under the assumptions (**) we have $\dim_{\mathbb{Q}} K\pi_4(F) \otimes \mathbb{Q} \geq 1$.

Let $Z^4(B; K\pi_4(F))$ be the group of 4-cocycles with coefficients in $K\pi_4(F)$, $\text{Aut}(B)$ be the group of fiber homotopy classes of fiber homotopy equivalences of $p: B \rightarrow BSO$. We will first define a map

$$\Phi: Z^4(B; K\pi_4(F)) \rightarrow \text{Aut}(B)$$

as follows. (Here we remark that Φ is only a map between sets, and depends on various choices.)

Let $B^{(4)}$ be the 4-skeleton of B , $e_1^4, \dots, e_m^4$ be the 4-cells, then the 4 dimensional cellular chain group $C_4(B)$ is a free abelian group generated by e_i^4 ($i = 1, \dots, m$). Let $q: B^{(4)} \rightarrow B^{(4)} \vee \bigvee_{i=1}^m S_i^4$ be the pinch map, where we pinch each 4-cell e_i^4 to $e_i^4 \vee S_i^4$. Let $\alpha: C_4(B) \rightarrow K\pi_4(F)$ be a cocycle. For each $\alpha(e_i^4) \in K\pi_4(F)$, by abuse of notation, we view it as a map $S_i^4 \rightarrow B$. Let $i: B^{(4)} \rightarrow B$ be the inclusion, define $h: B^{(4)} \rightarrow B$ to be the composition

$$h = (i \vee \bigvee_{i=1}^m \alpha(e_i^4)) \circ q: B^{(4)} \rightarrow B^{(4)} \vee \bigvee_{i=1}^m S_i^4 \rightarrow B.$$

By the construction h is compatible with the fiber projection p , i. e. the following diagram commutes up to homotopy

$$\begin{array}{ccc} B^{(4)} & \xrightarrow{h} & B \\ \downarrow i & & \downarrow p \\ B & \xrightarrow{p} & BSO \end{array}$$

Now we show that we can extend h to a map on the 5-skeleton of B compatible with p . Let e^5 be a 5-cell, with attaching map $f: S^4 \rightarrow B^{(4)}$. By Blakers-Massey theorem we have an isomorphism

$$\pi_4(B^{(4)} \vee \bigvee_{i=1}^m S_i^4) \cong \pi_4(B^{(4)}) \oplus \bigoplus_{i=1}^m \pi_4(S_i^4),$$

under this isomorphism the image of $[q \circ f]$ is clearly $([f], \partial e^5)$. Since α is a cocycle, $\langle \alpha, \partial e^5 \rangle = \langle \delta \alpha, e^5 \rangle = 0$, therefore h extends to a map $h: B^{(5)} \rightarrow B$, compatible with the fiber projection p . (The extension may not be unique, we choose one extension.) Since $\pi_n(F) = 0$ for $n \geq 5$, we can extend h further to a fiber map $h: B \rightarrow B$.

Lemma 2.2 *h is a fiber homotopy equivalence. h_* is the identity on $H_i(B)$ for $i \leq 4$. h^* is the identity on $H^i(B)$ for $i \leq 3$. h_* is the identity on $H_8(B; \mathbb{Q})$.*

Proof It's clear from the construction that h_* is the identity on $H_i(B)$ for $i \leq 3$ and h^* is the identity on $H^i(B)$ for $i \leq 3$. On the chain level h_* is

$$h_*: C_4(B) \rightarrow C_4(B), \quad e_i^4 \mapsto e_i^4 + \alpha(e_i^4)$$

where $\alpha(e_i^4)$ is a boundary by definition. Therefore h_* is the identity on $H_4(B)$. By the Hurewicz theorem we see that $h_*: \pi_i(B) \rightarrow \pi_i(B)$ is an isomorphism for $i \leq 3$ and is a surjection for $i = 4$. Notice that $\pi_4(B)$ is a finitely generated abelian group, a surjective endomorphism of a finitely generated abelian group must also be injective, therefore h_* is an isomorphism on $\pi_4(B)$. Since $p: B \rightarrow BSO$ is 4-coconnected, h induces isomorphisms on $\pi_i(B)$ for all i . Therefore h is a fiber homotopy equivalence.

Since $\pi_n(F) = 0$ for $n \geq 5$, by rational homotopy theory, there are no indecomposable cohomology classes in $H^n(F; \mathbb{Q})$ for $n \geq 5$. Therefore elements in $H^8(F; \mathbb{Q})$ are linear combinations of products of cohomology classes of degree ≤ 4 . Since h_* is the identity on $H_i(F)$ for $i \leq 4$, h^* is the identity on $H^i(F; \mathbb{Q})$ for $i \leq 4$ and hence also the identity on $H^8(F; \mathbb{Q})$. Thus h_* is the identity on $H_8(F; \mathbb{Q})$. By the Leray–Serre spectral sequence of the fibration $F \rightarrow B \rightarrow BSO$, h_* is the identity on $H_8(B; \mathbb{Q})$.

Proof of Theorem 1.6 By Lemma 2.1 the image of $\pi_4(B) \rightarrow H_4(B)$ has rank $k > 0$. Let $x_1, \dots, x_k$ be homology classes in the image which generate a free abelian subgroup of rank k in $H_4(B)$, $y_i \in \pi_4(B)$ be a pre-image of x_i , represented by a map $S^4 \rightarrow B$. Then we may take $B^{(4)}$ of the form

$$B^{(4)} = X \vee \bigvee_{i=1}^k S_i^4$$

where X is a 4-complex, and $[S_i^4] = y_i$ for $i = 1, \dots, k$.

Let $\varphi \in \text{Hom}(H_4(B), K\pi_4(F))$ be a homomorphism, since there are surjective homomorphisms

$$Z^4(B, K\pi_4(F)) \rightarrow H^4(B, K\pi_4(F)) \rightarrow \text{Hom}(H_4(B), K\pi_4(F))$$

we may pick a pre-image of φ , say $\alpha \in Z^4(B, K\pi_4(F))$. Let $h: B \rightarrow B$ be the image of α under Φ . Consider the action of h on $\pi_4(B)$, especially the image of y_i under h_* : let $\iota: S^4 \rightarrow B$ be the inclusion of S_i^4 , clearly the homotopy class of the composition

$$h \circ \iota: S^4 \rightarrow B^{(4)} \rightarrow B^{(4)} \vee \bigvee_{i=1}^m S_i^4 \rightarrow B$$

equals $y_i + \varphi(x_i) \in \pi_4(B)$.

By Lemma 2.1 there are infinitely many $\varphi \in \text{Hom}(H_4(B), K\pi_4(F))$ such that $\{\varphi(x_i) \mid i = 1, \dots, k\}$ are pairwise distinct. Therefore we have constructed infinitely many $h \in \text{Aut}(B)$ with properties in Lemma 2.2.

Let $\Omega_8(B, p)$ be the 8 dimensional B -bordism group. It's isomorphic to the eight-dimensional stable homotopy group of the corresponding Thom spectrum. By the Atiyah–Hirzebruch spectral sequence there is an isomorphism

$$\Omega_8(B, p) \otimes \mathbb{Q} \rightarrow H_8(B; \mathbb{Q})$$

where the image of a bordism class $[N, f] \in \Omega_8(B, p)$ is the image of the fundamental class $f_*[N] \in H_8(B; \mathbb{Q})$.

Now we fix a normal 4-smoothing $\bar{v}: X \rightarrow B$. Consider the normal 4-smoothing $\bar{v} \circ h$, where $h \in \text{Aut}(B)$ is constructed as above. Since h_* is the identity on $H_8(B; \mathbb{Q})$, we have

$$[X, \bar{v}] = [X, h \circ \bar{v}] \in \Omega_8(B, p) \otimes \mathbb{Q}$$

Therefore there are infinitely many h (indexed by $i \in \mathbb{N}$) such that

$$[X, h_i \circ \bar{v}] = [X, h_j \circ \bar{v}] \in \Omega_8(B, p).$$

Let W_i be a B -bordism between $[X, h_0 \circ \bar{v}]$ and $[X, h_i \circ \bar{v}]$, by the theory of modified surgery, the obstruction to changing W by surgery to an h -cobordism is in $L_9(\mathbb{Z}) = 0$. Therefore there is a diffeomorphism $f_i: X \rightarrow X$ such that $h_i \circ \bar{v} \circ f_i \simeq h_0 \circ \bar{v}$. Since $\bar{v}$ is a 5-equivalence, we see that f_{i*} is the identity on $H_i(X)$ for $i \leq 4$, and f_i^* is the identity on $H^i(X)$ for $i \leq 3$. Therefore by Poincaré duality, f_{i*} is the identity on $H_*(X)$ and hence $f \in \mathcal{T}(X)$. But f_{i*} on $\pi_4(X)$ are pairwise distinct. Therefore we have constructed infinitely many elements in $\mathcal{T}(X)$.

Proof of Theorem 1.8 Let $p: B \rightarrow BSO$ be the normal 4-type of X , F be the homotopy fiber of p . Fix a normal 4-smoothing $\bar{v}: X \rightarrow B$.

We claim that $\pi_3(X)$ and $\pi_4(X)$ are finite. For this we consider the minimal model of X . It has generators $x_1, \dots, x_r$ in degree 2, where r is the second Betti number. By assumption $x_i \cup x_j$ for $i \leq j$ is a basis of $H^4(X; \mathbb{Q})$. This and the fact that $H^3(X; \mathbb{Q}) = 0$ implies that there are no generators in degree 3 and hence $\pi_3(X)$ is finite. This implies that there are no decomposable elements in degree 5. Thus the differential on elements of degree 4 is zero. But then there are no indecomposable elements in degree 4, since they would produce indecomposable cohomology classes. Thus $\pi_4(X)$ is finite. Finally we conclude that also $\pi_3(F)$ and $\pi_4(F)$ are finite. This follows from the homotopy sequence of the fibration $F \rightarrow B \rightarrow BSO$ since $\pi_i(X) \cong \pi_i(B)$ for $i \leq 4$ and $p_1(X) \neq 0$ implying that $\pi_4(B) \rightarrow \pi_4(BSO)$ is non-trivial.

Let $f \in \mathcal{T}(X)$ be a self-diffeomorphism, then $\bar{v} \circ f$ is also a normal 4-smoothing. Let $\mathcal{T}_0(X)$ be the subset of $\mathcal{T}(X)$ consisting of self-diffeomorphisms f such that $\bar{v} \circ f$ and $\bar{v}$ are homotopic as liftings of v .

Lemma 2.3 *Under our conditions $\mathcal{T}_0(X)$ is a finite index subgroup of $\mathcal{T}(X)$.*

Proof It suffices to show that there are finitely many homotopy classes of $\bar{\nu} \circ f$ over p , which in turn follows if there are finitely many lifts of the Gauss map ν over p . This follows by induction over the skeletons and the Puppe sequence and the fact that f induces the identity on $\pi_2(X) = H_2(X)$ from the fact that the homotopy groups $\pi_3(F)$ and $\pi_4(F)$ are finite and $\pi_i(F) = 0$ for $i \geq 5$.

We finish the proof of the theorem by showing that $\mathcal{T}_0(X)$ is finite. Given $f \in \mathcal{T}_0(X)$, choosing a homotopy $h: X \times [0, 1] \rightarrow B$ between $\bar{\nu}$ and $\bar{\nu} \circ f$ we obtain a normal B -structure $\varphi: X_f \rightarrow B$, where X_f is the mapping torus. This represents an element $[X_f, \varphi]$ in $\Omega_9(B, p)$. Again by the Atiyah–Hirzebruch spectral sequence we have $\Omega_9(B, p) \otimes \mathbb{Q} \cong H_9(B; \mathbb{Q})$.

Lemma 2.4 $H_9(B; \mathbb{Q}) = 0$.

Proof By the Leray–Serre spectral sequence of the fibration $F \rightarrow B \rightarrow BSO$, it suffices to show that $H^5(F; \mathbb{Q}) = 0 = H^9(F; \mathbb{Q})$. But since there are no indecomposable classes in $H^n(F; \mathbb{Q})$ for $n \geq 5$, it is enough to show that $H^3(F; \mathbb{Q}) = 0$. Again by the Leray–Serre spectral sequence there is an exact sequence

$$H_4(BSO; \mathbb{Q}) \xrightarrow{d} H_3(F; \mathbb{Q}) \rightarrow H_3(B; \mathbb{Q}) = 0$$

where d can be identified with the composition

$$H_4(BSO; \mathbb{Q}) = \pi_4(BSO) \otimes \mathbb{Q} \xrightarrow{\partial} \pi_3(F) \otimes \mathbb{Q} \rightarrow H_3(F; \mathbb{Q}).$$

If $p_1(X) \neq 0 \in H^4(X; \mathbb{Q})$, then $p_*: \pi_4(B) \otimes \mathbb{Q} \rightarrow \pi_4(BSO) \otimes \mathbb{Q}$ is surjective. This implies $H_3(F; \mathbb{Q}) = 0$.

Therefore the finitely generated abelian group $\Omega_9(B, p)$ is finite. There are finitely many $f_1, \dots, f_r \in \mathcal{T}_0(X)$, together with homotopies $h_1, \dots, h_r$ such that for any $f \in \mathcal{T}_0(X)$ and homotopy h , $[X_f, \varphi] = [X_{f_i}, \varphi_i]$ for some $1 \leq i \leq r$. This implies that there is a B -bordism W between $X \times [0, 1]$ and $X \times [0, 1]$, where the two boundary components are identified by f and f_i respectively. The modified surgery obstruction to changing W to a relative h -cobordism is $\theta(W) \in L_{10}(\mathbb{Z}) \cong \mathbb{Z}/2$. If $\theta(W) = 0$, there is a relative h -cobordism between $X \times [0, 1]$ and $X \times [0, 1]$. Such an h -cobordism gives rise to a diffeomorphism $F: X \times [0, 1] \rightarrow X \times [0, 1]$ which is a pseudo-isotopy between f and f_i . Therefore by Cerf's pseudo-isotopy theorem [3] f is isotopic to f_i . If $\theta(W) \neq 0$, then for any f' with this property, we may glue W and W' and get a B -bordism V between $X \times [0, 1]$ and $X \times [0, 1]$ where the two boundary components are identified by f and f' respectively. Now the surgery obstruction $\theta(V) = 0$, therefore f and f' are isotopic. This shows that there are at most $2r$ elements in $\mathcal{T}_0(X)$.

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