The nilpotent cone in the Mukai system of rank two and genus two

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Abstract

We study the nilpotent cone in the Mukai system of rank two and genus two. We compute the degrees and multiplicities of its irreducible components and describe their cohomology classes.

1 Introduction

Let (S, H) be a polarized K3 surface of genus g and fix two coprime integers $n \geq 1$ and s . The moduli space $M = M_H(v)$ of H -Gieseker stable coherent sheaves with Mukai vector $v = (0, nH, s)$ is a smooth Hyperkähler variety of dimension $2(n^2(g-1)+1)$. A point in M corresponds to a stable sheaf $\mathcal{E}$ on S such that $\mathcal{E}$ is pure of dimension one with support in the linear system $|nH|$. Taking the (Fitting) support defines a Lagrangian fibration

$$f: M \longrightarrow |nH| \cong \mathbb{P}^{n^2(g-1)+1}, [\mathcal{E}] \mapsto \text{Supp}(\mathcal{E})$$

known as the *Mukai system* [5,22]. Over a general point in $|nH|$ which corresponds to a smooth curve $D \subset S$ the fibers of f are abelian varieties isomorphic to $\text{Pic}^\delta(D)$, where $\delta = s - n^2(1-g)$. So, M can also be viewed as a relative compactified Jacobian associated to the universal curve $\mathcal{C} \rightarrow |nH|$.

The Mukai system is of special interest because of its relation to the classical and widely studied *Hitchin system*, see [14] for a survey. Let C be a smooth curve of

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genus g . A Higgs bundle on C is a pair $(\mathcal{E}, \phi)$ consisting of a vector bundle $\mathcal{E}$ on C and a morphism $\phi: \mathcal{E} \rightarrow \mathcal{E} \otimes \omega_C$, called Higgs field. The moduli space $M_{\text{Higgs}}(n, d)$ of stable Higgs bundles of rank n and degree d is a smooth and quasi-projective symplectic variety. Sending $(\mathcal{E}, \phi)$ to the coefficients of its characteristic polynomial $\chi(\phi)$ defines a proper Lagrangian fibration

$$\chi: M_{\text{Higgs}}(n, d) \longrightarrow \bigoplus_{i=1}^n H^0(C, \omega_C^i).$$

It is equivariant with respect to the $\mathbb{C}^*$ -action that is given by scaling the Higgs field on $M_{\text{Higgs}}(n, d)$ and by multiplication with t^i in the corresponding summand on the base. As a corollary the topology of $M_{\text{Higgs}}(n, d)$ is controlled by the fiber over the origin. This fiber

$$N := \chi^{-1}(0) = \{(\mathcal{E}, \phi) \in M_{\text{Higgs}}(n, d) \mid \phi \text{ is nilpotent}\}$$

is called the *nilpotent cone*. In the late '80s Beauville, Narasimhan, and Ramanan discovered a beautiful interpretation of the space of Higgs bundles [4]. They showed that a Higgs bundle $(\mathcal{E}, \phi)$ with characteristic polynomial s corresponds to a pure sheaf of rank one on a so called spectral curve $C_s \subset T^*C$ inside the cotangent bundle of C . The curve C_s is defined in terms of $s = \chi(\phi)$ and is linearly equivalent to nC , the n -th order thickening of the zero section $C \subset T^*C$. This idea was taken up by Donagi, Ein, and Lazarsfeld in [9]: The space $M_{\text{Higgs}}(n, d)$ appears as a moduli space of stable sheaves on T^*C that are supported on curves in the linear system $|nC|$. Consequently, $M_{\text{Higgs}}(n, d)$ has a natural compactification $\overline{M}_{\text{Higgs}}(n, d)$ given by a moduli space of sheaves on the projective surface $S_0 = \mathbb{P}(\omega_C \oplus \mathcal{O}_C)$ with respect to the polarization $H_0 = \mathcal{O}_{S_0}(C)$. The Hitchin map extends to $\overline{M}_{\text{Higgs}}(n, d) \rightarrow |nH_0| \cong \mathbb{P}(\bigoplus_{i=0}^n H^0(\omega_C^i))$ and is nothing but the support map; the nilpotent cone is the fiber over the point $nC \in |nH_0|$. However, $\overline{M}_{\text{Higgs}}(n, d)$ cannot admit a symplectic structure as it is covered by rational curves. At this point the Mukai system enters the picture. If S is a K3 surface that contains the curve C as a hyperplane section, one can degenerate (S, H) to (S_0, H_0) and consequently the Mukai system $M_H(v) \rightarrow |nH|$ with $v = (0, nH, d + n(1 - g))$ degenerates to the compactified Hitchin system [9, §1]. From our perspective, this is a powerful approach to studying the Hitchin system. For instance, in a recent paper [7], de Cataldo, Maulik and Shen prove the P=W conjecture for $g = 2$ by means of the corresponding specialization map on cohomology.

In this note, we study the geometry of the nilpotent cone in the Mukai system, which is defined in parallel to the Hitchin system

$$N_C := f^{-1}(nC),$$

for some curve $C \in |H|$. Alternatively, one could say that we study the most singular fiber type, see (3.1). We will fix the invariants $n = 2$ and $g = 2$ and the Mukai vector $v = (0, 2H, -1)$. In this case and if C is irreducible, the nilpotent cone has two

irreducible components

$$(N_C)_{\text{red}} = N_0 \cup N_1,$$

where the first component is isomorphic to the moduli space $M_C(2, 1)$ of stable vector bundles of rank two and degree one on C and the second component is the closure of $N_C \setminus N_0$. Both components are Lagrangian subvarieties of $M = M_H(v)$. If C is smooth, then N_0 is smooth and the singularities of N_1 are contained in $N_0 \cap N_1$ (each understood with their reduced structure). However, both components occur with multiplicities.

Our first result is the computation of the multiplicities of the components as well as their degrees. Here, the degree is meant with respect to a naturally defined distinguished ample class $u_1 \in H^2(M, \mathbb{Z})$, see Definition 4.7.

Theorem 1.1 *Let $C \in |H|$ be an irreducible curve. The degrees of the two components of the nilpotent cone N_C are given by*

$$\deg_{u_1} N_0 = 5 \cdot 2^9 \text{ and } \deg_{u_1} N_1 = 5^2 \cdot 2^{11}$$

and their multiplicities are

$$\text{mult}_{N_C} N_0 = 2^3 \text{ and } \text{mult}_{N_C} N_1 = 2.$$

Moreover, any fiber F of the Mukai system has degree $5 \cdot 3 \cdot 2^{13}$.

As the smooth locus of every component with its reduced structure deforms from the Mukai to the Hitchin system, the multiplicities and degrees must coincide. Here, indeed, the same multiplicities can be found in [23, Propositions 34 and 35] and [15, Proposition 6], whereas, up to our knowledge, the degrees have not been determined in the literature. In our case, the degrees determine the multiplicities.

Our second result is a description of the classes $[N_0]$ and $[N_1] \in H^{10}(M, \mathbb{Z})$. The projective moduli spaces of stable sheaves on K3 surfaces are known to be deformation equivalent to Hilbert schemes of points. In our case, M is actually birational to $S^{[5]}$ [6, Lemma 3.2.7]. In particular, there is an isomorphism $H^*(M, \mathbb{Z}) \cong H^*(S^{[5]}, \mathbb{Z})$. The cohomology ring of $S^{[5]}$ is well understood, e.g. [19, §4] and the references therein. Recall that for any Hyperkähler variety X of dimension $2n$ there is an embedding $S^i H^2(X, \mathbb{Q}) \hookrightarrow H^{2i}(X, \mathbb{Q})$ for all $i \leq n$ [24, Theorem 1.7].

Theorem 1.2 *The classes $[N_0]$ and $[N_1] \in H^{10}(M, \mathbb{Q})$ are linearly independent and span a totally isotropic subspace of $H^{10}(M, \mathbb{Q})$ with respect to the intersection pairing. They are given by*

$$[N_0] = \frac{1}{48}[F] + \beta \text{ and } [N_1] = \frac{5}{12}[F] - 4\beta,$$

where $[F]$ is the class of a general fiber of the Mukai system and $0 \neq \beta \in (S^5 H^2(M, \mathbb{Q}))^\perp$ satisfies $\beta^2 = 0$. As $\deg_{u_1} \beta = 0$, the class β is not effective.

Outline

In Sect. 2 we introduce the Mukai system. In Sect. 3 we reduce the study of N_C to the case of a smooth curve C . We describe the irreducible components of the nilpotent cone following [9, §3], where it is shown that any point $[\mathcal{E}] \in N_C \setminus N_0$ fits into an extension of the form

$$0 \rightarrow \mathcal{L}(x) \otimes \omega_C^{-1} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \rightarrow 0,$$

where $\mathcal{L} \in \text{Pic}^1(C)$ is a line bundle and $x \in C$ a point. We specify a space $W \rightarrow \text{Pic}^1(C) \times C$ parameterizing such extensions, and a compactification $\overline{W}$ of W that comes with a birational map $v: \overline{W} \rightarrow N_1$. In the Hitchin case, this idea originates from [23].

In Sect. 4 we prove Theorem 1.1. The proof relies on the functorial properties of the definition of u_1 via the determinant line bundle construction, see Sect. 4.1. It allows us to relate $u_1|_F$ and $u_1|_{N_0}$ with the (generalized) theta divisor on $F = f^{-1}(D) \cong \text{Pic}^3(D)$ for $D \in |nC|$ smooth and $M_C(2, 1)$, respectively, see Propositions 4.8 and 4.10. For $[N_1]$ the degree computation is achieved by determining $v^*u_1 \in H^2(\overline{W}, \mathbb{Z})$. Finally, the multiplicities are inferred from knowing the degrees. The last Sect. 5 is devoted to the proof of Theorem 1.2. It uses our previous results.

Notation

All schemes are of finite type over $k = \mathbb{C}$. In the entire paper, S is a K3 surface polarized by a primitive, ample class $H \in \text{NS}(S)$ with $H^2 = 2g - 2$.

2 The Mukai system

In this section, we give a brief recollection on moduli spaces of sheaves on K3 surfaces and define the Mukai system. First recall that the Mukai vector induces an isomorphism

$$v: K(S)_{\text{num}} \xrightarrow{\sim} H_{\text{alg}}^*(S, \mathbb{Z}) = H^0(S, \mathbb{Z}) \oplus \text{NS}(S) \oplus H^4(S, \mathbb{Z}).$$

It is given by

$$v(\mathcal{E}) := \text{ch}(\mathcal{E})\sqrt{\text{td}(S)} = (\text{rk}(\mathcal{E}), c_1(\mathcal{E}), \chi(\mathcal{E}) - \text{rk}(\mathcal{E})).$$

We write $M_H(v)$ for the moduli space of pure, H -Gieseker stable sheaves on S with Mukai vector v . If v is primitive and positive and H is v -generic then $M_H(v)$ is an irreducible holomorphic symplectic manifold of dimension $\langle v, v \rangle + 2$, which is deformation equivalent to the Hilbert scheme of $\frac{1}{2}\langle v, v \rangle + 1$ points on S [18, Theorem 10.3.1]. Here, $\langle \cdot, \cdot \rangle$ is the Mukai pairing given by

$$\langle (r, c, s), (r', c', s') \rangle = cc' - rs' - r's.$$

Consider the Mukai vector

$$v := (0, nH, s) \in H_{\text{alg}}^*(S, \mathbb{Z}),$$

and assume that v is primitive. A pure sheaf $\mathcal{F}$ of Mukai vector v has one-dimensional support, first Chern class nH and Euler characteristic s . In particular, $\mathcal{F}$ admits a length one resolution $0 \rightarrow \mathcal{V} \xrightarrow{f} \tilde{\mathcal{V}} \rightarrow \mathcal{F}$ by two vector bundles of the same rank r [16, §1.1]. We define the (Fitting) support of $\mathcal{F}$ to be

$$\text{Supp}(\mathcal{F}) := V(\det f) \subset S$$

the vanishing scheme of the induced morphism $\det f = \wedge^r f: \wedge^r \mathcal{V} \rightarrow \wedge^r \tilde{\mathcal{V}}$, for any resolution $0 \rightarrow \mathcal{V} \xrightarrow{f} \tilde{\mathcal{V}}$ of $\mathcal{F}$ as above. This definition is well-defined, i.e. independent of the chosen resolution [13, Definition 20.4].

Example 2.1 Let $i: C \hookrightarrow S$ be an integral curve and $\mathcal{E}$ a vector bundle of rank n on C . Then

$$\text{Supp}(i_*\mathcal{E}) = nC$$

is the n -th order thickening of C in S .

By definition, $\text{Supp}(\mathcal{F})$ is linearly equivalent to $c_1(\mathcal{F})$ and $\text{Supp}(\mathcal{F})$ contains the usual support defined by the annihilator of $\mathcal{F}$. Moreover, the reduced locus $\text{Supp}(\mathcal{F})_{\text{red}}$ is the set-theoretic support of $\mathcal{F}$. The advantage of the above definition is, that it behaves well in families and thus induces a morphism [20, §2.2]

$$f: M_H(v) \longrightarrow |nH| \cong \mathbb{P}^{\tilde{g}}, \quad [\mathcal{E}] \mapsto \text{Supp}(\mathcal{E}).$$

Here, $\tilde{g} = n^2(g-1) + 1$. Moreover, $M_H(v)$ is irreducible holomorphic symplectic of dimension $n^2H^2 + 2 = 2\tilde{g}$ and hence, by Matsushita's result [21, Corollary 1] this morphism is a Lagrangian fibration (for an explicit proof see [9, Lemma 1.3]), called the *Mukai system* (of rank n and genus g).

3 The nilpotent cone for $n = 2$ and $g = 2$

We now specialize to the case that $n = 2$ and $s = 3 - 2g$ with $g = 2$, i.e. we fix the Mukai vector

$$v = (0, 2H, -1).$$

In particular, a stable vector bundle of rank two and degree one on a smooth curve $C \in |H|$ defines a point in $M := M_H(v)$. We have $\dim M = 8g - 6 = 10$ and M is birational to the Hilbert scheme $S^{[5]}$ of five points on S .

Taking (Fitting) supports defines a Lagrangian fibration

$$f: M \longrightarrow |2H| \cong \mathbb{P}^5.$$

We have a natural morphism $|H| \times |H| \rightarrow |2H|$. We define $\Sigma \subset |2H|$ as its image and $\Delta \subset \Sigma$ as the image of the diagonal. Then $\Sigma \cong \text{Sym}^2 |H|$ and $\Delta \cong |H|$. If every curve in $|H|$ is irreducible (e.g. if $\text{Pic}(S) = \mathbb{Z} \cdot H$) then Δ and Σ are exactly the loci of non-reduced and non-integral curves, respectively. And in this case we can distinguish three cases following [6, Proposition 3.7.1]:

$$f^{-1}(x) \begin{cases} \text{is reduced and irreducible} & \text{if } x \in |2H| \setminus \Sigma \\ \text{is reduced and has two irreducible components} & \text{if } x \in \Sigma \setminus \Delta \\ \text{has two irreducible components with multiplicities} & \text{if } x \in \Delta. \end{cases} \quad (3.1)$$

In the general case, the list is still valid for the geometric generic point in the respective subvariety. However, over points that correspond to curves with more irreducible components, one also finds more irreducible components in the fiber [6, Proof of Lemma 3.3.2].

We will study fibers of the third type, namely

$$N_C := f^{-1}(2C),$$

where $C \in |H|$ is irreducible. In analogy with the Hitchin system, we call N_C *nilpotent cone*.

For the rest of the paper, we fix a smooth curve $C \in |H|$ and write N instead of N_C . We will now identify the irreducible components of N following the ideas of [9].

3.1 Pointwise description of the nilpotent cone $N = N_C$

Let $[\mathcal{E}] \in N$ and consider its restriction $\mathcal{E}|_C$ to C . There are two cases, either $\mathcal{E}|_C$ is a stable rank two vector bundle on C or $\mathcal{E}|_C$ has rank one. By dimension reasons, the sheaves of the first kind contribute an irreducible component N_0 of N isomorphic to the moduli space $M_C(2, 1)$ of stable rank two and degree one vector bundles on C . In the second case, $\mathcal{E}|_C \cong \mathcal{L} \oplus \mathcal{O}_D$, where the first factor $\mathcal{L} := \mathcal{E}|_C / \text{torsion}$ is a line bundle on C and $D \subset C$ is an effective divisor. We set

$$E_1 := N \setminus N_0$$

with reduced structure.

Lemma 3.1 *Let $[\mathcal{E}] \in E_1$ and write $\mathcal{E}|_C = \mathcal{L} \oplus \mathcal{O}_D$. There is a short exact sequence of $\mathcal{O}_S$ -modules*

$$0 \rightarrow i_*(\mathcal{L}(D) \otimes \omega_C^{-1}) \rightarrow \mathcal{E} \rightarrow i_*\mathcal{L} \rightarrow 0. \quad (3.2)$$

Moreover, $k := \deg \mathcal{L} = 1$ and $d := \deg D = 2g - 2k - 1 = 1$.

Proof Noting that ω_C^{-1} is the conormal bundle of C in S , it is straightforward to obtain the sequence (3.2). Let us prove the numerical restrictions. From (3.2) we have

$$1 + 2(1 - g) = \chi(\mathcal{E}) = \chi(\mathcal{L}(D) \otimes \omega_C^{-1}) + \chi(\mathcal{L}) = 2k + d - (2g - 2) + 2(1 - g).$$

Thus $d = 2g - 2k - 1$ and we find $k \leq g - 1$. On the other hand, $\mathcal{E}$ is stable and therefore the reduced Hilbert polynomials [16, Definition 1.2.3] of $\mathcal{E}$ and $\mathcal{L}$ satisfy $p(\mathcal{E}, t) < p(\mathcal{L}, t)$, which amounts to

$$\frac{1}{2}(1 + 2(1 - g)) < k + 1 - g$$

or equivalently $k \geq 1$. $\square$

Remark 3.2 For $n = 2$ and arbitrary genus g , one has $\deg \mathcal{L} \in \{1, \dots, g - 1\}$ and a decomposition into locally closed subsets $N_{\text{red}} = N_0 \sqcup E_1 \sqcup \dots \sqcup E_{g-1}$ corresponding to the degree of $\mathcal{L}$. In fact, N_0 and the closures of E_k are the irreducible components of N .

We conclude that every point in E_1 defines a class in $\text{Ext}_S^1(i_*\mathcal{L}, i_*(\mathcal{L}(x) \otimes \omega_C^{-1}))$ for some point $x \in C$ and some line bundle $\mathcal{L} \in \text{Pic}^1(C)$. Conversely, an extension class in $\text{Ext}_S^1(i_*\mathcal{L}, i_*(\mathcal{L}(x) \otimes \omega_C^{-1}))$ defines a point in E_1 if and only if its middle term is stable and has the point x as support of its torsion part when restricted to C , i.e. if it is not pushed forward from C . It turns out that all such extensions are stable.

Lemma 3.3 Consider a coherent sheaf $\mathcal{E}$ on S that is given as an extension

$$0 \rightarrow i_*\mathcal{L} \rightarrow \mathcal{E} \rightarrow i_*\mathcal{L}' \rightarrow 0,$$

where $\mathcal{L}'$ and $\mathcal{L}$ are line bundles on C of degree k and $1 - k$, respectively, with $k \geq 1$. Moreover, assume that $\mathcal{E}$ itself does not admit the structure of an $\mathcal{O}_C$ -module. Then $\mathcal{E}$ is H -Gieseker stable.

Proof We have to prove $p(\mathcal{E}, t) < p(\mathcal{M}, t)$ or, equivalently, $\frac{\chi(\mathcal{E})}{c_1(\mathcal{E}) \cdot H} < \frac{\chi(\mathcal{M})}{c_1(\mathcal{M}) \cdot H}$ for every surjection $\mathcal{E} \twoheadrightarrow \mathcal{M}$. We can assume that $\text{Supp}(\mathcal{M}) = C$ and $\mathcal{M} = i_*\mathcal{M}'$, where $\mathcal{M}'$ is a line bundle on C . Then because $\mathcal{E}|_C \cong \mathcal{L}' \oplus \mathcal{T}$ for some torsion sheaf $\mathcal{T}$, we find

$$\text{Hom}_{\mathcal{O}_S}(\mathcal{E}, i_*\mathcal{M}') \cong \text{Hom}_{\mathcal{O}_C}(\mathcal{E}|_C, \mathcal{M}') \cong \text{Hom}_{\mathcal{O}_C}(\mathcal{L}', \mathcal{M}')$$

and thus $i_*\mathcal{L}' \xrightarrow{\sim} \mathcal{M}$.

Corollary 3.4 The closed points of E_1 are in bijection with the following set

$$\bigsqcup_{\substack{\mathcal{L} \in \text{Pic}^1(C) \\ x \in C}} \mathbb{P}(\text{Ext}_S^1(i_*\mathcal{L}, i_*(\mathcal{L}(x) \otimes \omega_C^{-1}))) \setminus \mathbb{P}(\text{Ext}_C^1(\mathcal{L}, \mathcal{L}(x) \otimes \omega_C^{-1})),$$

i.e. with extension classes $[v] \in \mathbb{P}(\mathrm{Ext}_S^1(i_*\mathcal{L}, i_*(\mathcal{L}(x) \otimes \omega_C^{-1})))$ such that v is not pushed forward from C . Here, $\mathcal{L}$ varies over all line bundles on C with $\deg \mathcal{L} = 1$, and x varies over all points in C . The bijection is established by Lemma 3.1.

In Proposition 3.5 below, we will see that there is a short exact sequence

$$0 \rightarrow \mathrm{Ext}_C^1(\mathcal{L}, \mathcal{L}(x) \otimes \omega_C^{-1}) \rightarrow \mathrm{Ext}_S^1(i_*\mathcal{L}, i_*(\mathcal{L}(x) \otimes \omega_C^{-1})) \xrightarrow{\rho_{\mathcal{L},x}} H^0(C, \mathcal{O}_C(x)) \rightarrow 0,$$

where $\rho_{\mathcal{L},x}$ has the following interpretation modulo a scalar factor. If $\mathcal{E}$ is the middle term of a representing sequence of $v \in \mathrm{Ext}_S^1(i_*\mathcal{L}, i_*(\mathcal{L}(x) \otimes \omega_C^{-1}))$, then

$$\mathcal{E}|_C \cong \mathcal{L} \oplus \mathcal{O}_{V(\rho_{\mathcal{L},x}(v))}.$$

Hence, another way to phrase Corollary 3.4 is by fixing for every $x \in C$ a defining section $s_x \in H^0(C, \mathcal{O}_C(x))$ as follows. Let $\Delta \hookrightarrow C \times C$ be the diagonal, yielding a section $s_\Delta \in H^0(C \times C, \mathcal{O}(\Delta))$. For every $x \in C$, we set $s_x = s_\Delta|_{\{x\} \times C}$. Then we can write

$$\text{points of } E_1 \xleftarrow{1:1} \bigsqcup_{\substack{\mathcal{L} \in \mathrm{Pic}^1(C) \\ x \in C}} \{v \in \mathrm{Ext}_S^1(i_*\mathcal{L}, i_*(\mathcal{L}(x) \otimes \omega_C^{-1})) \mid \rho_{\mathcal{L},x}(v) = s_x\}. \quad (3.3)$$

3.2 Extension spaces

So far, we have given a pointwise description of the nilpotent cone. Next, we will identify its irreducible components and their scheme structures. This subsection is a technical parenthesis in this direction. The reader may like to skip it.

Let S be a smooth projective surface and $i: C \hookrightarrow S$ a smooth curve with normal bundle $\mathcal{N}_{C/S} \cong \mathcal{O}_C(C)$. Let T be any scheme and let $\mathcal{F}$ and $\mathcal{F}'$ be two vector bundles on $T \times C$ considered as families of vector bundles on C . Denote by $\pi: T \times S \rightarrow T$ and $\pi_C: T \times C \rightarrow T$ the projections. For a morphism $f: X \rightarrow Y$, we write $\mathcal{E}xt f$ instead of $Rf_* R\mathcal{H}om$.

Proposition 3.5 *There is a short exact sequence of $\mathcal{O}_T$ -modules*

$$0 \rightarrow \mathcal{E}xt_{\pi_C}^1(\mathcal{F}', \mathcal{F}) \rightarrow \mathcal{E}xt_{\pi}^1((\mathrm{id} \times i)_*\mathcal{F}', (\mathrm{id} \times i)_*\mathcal{F}) \xrightarrow{\rho} \mathcal{E}xt_{\pi_C}^0(\mathcal{F}' \boxtimes \mathcal{O}_C(-C), \mathcal{F}) \rightarrow 0, \quad (3.4)$$

as well as for every $t \in T$ a short exact sequence of vector spaces

$$0 \rightarrow \mathrm{Ext}_C^1(\mathcal{F}'_t, \mathcal{F}_t) \xrightarrow{\xi} \mathrm{Ext}_S^1(i_*\mathcal{F}'_t, i_*\mathcal{F}_t) \xrightarrow{\rho_t} \mathrm{Ext}_C^0(\mathcal{F}'_t \otimes \mathcal{O}_C(-C), \mathcal{F}_t) \rightarrow 0. \quad (3.5)$$

Note that the fibers of (3.4) must, in general, not coincide with (3.5), see Lemma 3.6.

Proof Apply $R\pi_{C*} R\mathcal{H}om(-, \mathcal{F})$ or $R\mathcal{H}om(-, \mathcal{F}_t)$, to the exact triangle

$$\mathcal{F}' \boxtimes \mathcal{O}_C(-C)[1] \rightarrow L(\mathrm{id} \times i)^*(\mathrm{id} \times i)_*\mathcal{F}' \rightarrow \mathcal{F}' \xrightarrow{[1]}$$

in $D^b(T \times C)$ (see [17, Corollary 11.4]) or its counterpart in $D^b(C)$, respectively, and consider the induced cohomology sequence. $\square$

We can explicitly describe the morphism ρ_t in the sequence (3.5). Represent $v \in \text{Ext}_S^1(i_*\mathcal{F}'_t, i_*\mathcal{F}_t)$ by $0 \rightarrow i_*\mathcal{F}_t \rightarrow \mathcal{E} \rightarrow i_*\mathcal{F}'_t \rightarrow 0$. Restriction to C yields

$$\cdots \rightarrow \mathcal{F}'_t \otimes \mathcal{O}_C(-C) \xrightarrow{\delta(v)} \mathcal{F}_t \rightarrow \mathcal{E}|_C \rightarrow \mathcal{F}'_t \rightarrow 0,$$

where we inserted

$\text{Tor}_1^{\mathcal{O}_S}(i_*\mathcal{F}'_t, i_*\mathcal{O}_C) \cong \mathcal{F}'_t \otimes_{\mathcal{O}_C} \text{Tor}_1^{\mathcal{O}_S}(i_*\mathcal{O}_C, i_*\mathcal{O}_C) = \mathcal{F}'_t \otimes \mathcal{O}_C(-C)$. This gives a well-defined, linear map

$$\delta: \text{Ext}_S^1(i_*\mathcal{F}'_t, i_*\mathcal{F}_t) \rightarrow \text{Ext}_C^0(\mathcal{F}'_t \otimes \mathcal{O}_C(-C), \mathcal{F}_t).$$

As $\text{im } \xi = \ker \delta$, it follows by dimension reasons, that δ has to be surjective. So, $\rho_t = \delta$ up to post-composition with an isomorphism of $\text{Ext}_C^0(\mathcal{F}'_t \otimes \mathcal{O}_C(-C), \mathcal{F}_t)$.

Lemma 3.6 *For every $t \in T$ there is a commutative diagram of short exact sequences*

$$\begin{array}{ccccc} \mathcal{E}xt_{\pi_C}^1(\mathcal{F}', \mathcal{F})(t) & \longrightarrow & \mathcal{E}xt_{\pi}^1((\text{id} \times i)_*\mathcal{F}', (\text{id} \times i)_*\mathcal{F})(t) & \xrightarrow{\rho(t)} & \mathcal{E}xt_{\pi_C}^0(\mathcal{F}' \boxtimes \mathcal{O}_C(-C), \mathcal{F})(t) \\ \downarrow \cong & & \downarrow & & \downarrow \\ \text{Ext}_C^1(\mathcal{F}'_t, \mathcal{F}_t) & \longrightarrow & \text{Ext}_S^1(i_*\mathcal{F}'_t, i_*\mathcal{F}_t) & \xrightarrow{\rho_t} & \text{Ext}_C^0(\mathcal{F}'_t \otimes \mathcal{O}_C(-C), \mathcal{F}_t), \end{array}$$

where the first vertical arrow is an isomorphism. If $\text{Ext}_C^0(\mathcal{F}'_t \otimes \mathcal{O}_C(-C), \mathcal{F}_t)$ has constant dimension for all $t \in T$ all vertical arrows are isomorphisms.

Proof The vertical morphisms are the usual functorial base change morphisms. The lower line is (3.5) and hence also exact on the left. The first vertical arrow is an isomorphism because $\text{Ext}_C^2(\mathcal{F}'_t, \mathcal{F}_t) = 0$. Consequently, also the upper line is exact on the left.

3.3 Irreducible components of N

In this section, we show that E_1 is irreducible and has the same dimension as N . Therefore its closure

$$N_1 := \overline{E}_1 \subset N_{\text{red}},$$

with reduced structure is an irreducible component of N . For the proof, we need some more notation. Let $\mathcal{P}_1$ be a Poincaré line bundle on $\text{Pic}^1(C) \times C$ and $\Delta \subset C \times C$ the diagonal. Set $T := \text{Pic}^1(C) \times C$ and on T define the following sheaves

$$\mathcal{V} := R^1 p_{12*}(p_{23}^* \mathcal{O}(\Delta) \otimes p_3^* \omega_C^{-1}),$$

$$\mathcal{W} := R^1 p_{12*} R \mathcal{H}om((\text{id} \times i)_* p_{13}^* \mathcal{P}_1, (\text{id} \times i)_*(p_{13}^* \mathcal{P}_1 \otimes p_{23}^* \mathcal{O}(\Delta) \otimes p_3^* \omega_C^{-1})) \text{ and}$$

$$\mathcal{U} := p_{12*} p_{23}^* \mathcal{O}(\Delta),$$

where p_{ij} are the appropriate projections from $\mathrm{Pic}^1(C) \times C \times C$. Considering the fiber dimensions, we see that $\mathcal{V}$ and $\mathcal{U}$ are vector bundles of rank 2 and 1, respectively. In fact, s_Δ induces an isomorphism $s_\Delta: \mathcal{O}_T \xrightarrow{\sim} \mathcal{U} = p_{12*}p_{23}^*\mathcal{O}(\Delta)$. Moreover, by Proposition 3.5 they fit into a short exact sequence

$$0 \rightarrow \mathcal{V} \rightarrow \mathcal{W} \xrightarrow{\rho} \mathcal{O}_T \rightarrow 0. \quad (3.6)$$

Consequently, also $\mathcal{W}$ is a vector bundle and ρ induces a map of geometric vector bundles

$$\rho: \underline{\mathrm{Spec}}_T(\mathrm{Sym}^\bullet \mathcal{W}^\vee) \rightarrow T \times \mathbb{A}^1.$$

We set

$$W := \rho^{-1}(T \times \{1\})$$

with the projection $\tau: W \rightarrow T$. We retain some immediate consequences of the construction.

- (i) W is a principal homogeneous space under $\underline{\mathrm{Spec}}_T(\mathrm{Sym}^\bullet \mathcal{V}^\vee)$. In particular, it is an affine bundle over T .
- (ii) Let $t = (\mathcal{L}, x) \in T$. Then by Lemma 3.6 we have

$$W_t = \tau^{-1}(t) \cong \mathbb{P}(\mathrm{Ext}_S^1(i_*\mathcal{L}, i_*(\mathcal{L}(x) \otimes \omega_C^{-1}))) \setminus \mathbb{P}(\mathrm{Ext}_C^1(\mathcal{L}, \mathcal{L}(x) \otimes \omega_C^{-1})).$$

- (iii) $\dim W = 5$.
- (vi) W is compactified by the projective bundle $\overline{W} := \mathbb{P}(\mathcal{W})$ with boundary isomorphic to $\mathbb{P}(\mathcal{V})$, i.e.

$$\overline{W} = W \cup \mathbb{P}(\mathcal{V}).$$

Remark 3.7 Actually, $\mathcal{V} \cong p_2^*(\omega_C \oplus \omega_C)$ and hence $\mathbb{P}(\mathcal{V}) \cong \mathbb{P}^1 \times \mathrm{Pic}^1(C) \times C$.

Next, we relate E_1 and $\overline{W}$. Recall that $N_1 := \overline{E}_1 \subset N_{\mathrm{red}}$. We keep all the notations from the previous section, and

$$\begin{array}{ccccc} \overline{W} \times C & \hookrightarrow & \overline{W} \times S & \xrightarrow{\pi'} & \overline{W} := \mathbb{P}(\mathcal{W}) \\ \downarrow \tau_C & & \downarrow \tau_S & & \downarrow \tau \\ T \times C & \hookrightarrow & T \times S & \xrightarrow{\pi} & T. \end{array}$$

Proposition 3.8 *There exists a ‘universal’ extension represented by*

$$0 \rightarrow \tau_S^*(\mathrm{id} \times i)_*(\mathcal{P}_1 \boxtimes \mathcal{O}(\Delta) \boxtimes \omega_C^{-1}) \boxtimes \mathcal{O}_\tau(1) \rightarrow \mathcal{G}_{\mathrm{univ}} \rightarrow \tau_S^*(\mathrm{id} \times i)_*p_{13}^*\mathcal{P}_1 \rightarrow 0,$$

such that $\mathcal{G}_{\text{univ}} \in \text{Coh}(\overline{W} \times S)$ defines a birational morphism

$$v: \overline{W} \longrightarrow N_1.$$

In particular, $N_{\text{red}} = N_0 \cup N_1$ is a decomposition into irreducible components.

Proof We set $\mathcal{F} := \mathcal{P}_1 \boxtimes \mathcal{O}(\Delta) \boxtimes \omega_C^{-1}$ and $\mathcal{F}' := p_{13}^* \mathcal{P}_1$. We are looking for a ‘universal’ extension, i.e. for

$$v_{\text{univ}} \in \text{Ext}_{\overline{W} \times S}^1(\tau_S^*(\text{id} \times i)_* \mathcal{F}', \tau_S^*(\text{id} \times i)_* \mathcal{F} \otimes \pi'^* \mathcal{O}_\tau(1)),$$

such that for $w \in W \subset \overline{W}$ the restriction of v_{univ} to $\{w\} \times S$ is the extension corresponding to $w \in W_{\tau(w)} \subset \text{Ext}_S^1(i_* \mathcal{F}'_{\tau(w)}, i_* \mathcal{F}_{\tau(w)})$.

By definition, $\mathcal{W} = R^1 \pi_* R \mathcal{H}om((\text{id} \times i)_* \mathcal{F}', (\text{id} \times i)_* \mathcal{F})$. Hence, there is a base change map

$$\tau^* \mathcal{W} \rightarrow R^1 \pi'_* L \tau_S^* R \mathcal{H}om((\text{id} \times i)_* \mathcal{F}', (\text{id} \times i)_* \mathcal{F}).$$

We get

$$\begin{aligned} H^0(\overline{W}, \tau^* \mathcal{W} \otimes \mathcal{O}_\tau(1)) &\rightarrow H^0(\overline{W}, R^1 \pi'_* L \tau_S^* R \mathcal{H}om((\text{id} \times i)_* \mathcal{F}', (\text{id} \times i)_* \mathcal{F}) \otimes \mathcal{O}_\tau(1)) \\ &\xleftarrow{\sim} H^1(\overline{W} \times S, L \tau_S^* R \mathcal{H}om((\text{id} \times i)_* \mathcal{F}', (\text{id} \times i)_* \mathcal{F}) \otimes \pi'^* \mathcal{O}_\tau(1)) \\ &= \text{Ext}_{\overline{W} \times S}^1(\tau_S^*(\text{id} \times i)_* \mathcal{F}', \tau_S^*(\text{id} \times i)_* \mathcal{F} \otimes \pi'^* \mathcal{O}_\tau(1)), \end{aligned} \quad (3.7)$$

where the indicated isomorphism comes from the Leray spectral sequence. It is an isomorphism, because

$$\begin{aligned} &R^0 \pi'_* L \tau_S^* R \mathcal{H}om((\text{id} \times i)_* \mathcal{F}', (\text{id} \times i)_* \mathcal{F}) \\ &\cong R^0 \pi'_* L \tau_C^* R \mathcal{H}om(L(\text{id} \times i)^*(\text{id} \times i)_* \mathcal{F}', \mathcal{F}) = 0, \end{aligned}$$

where $\pi_C: T \times C \rightarrow T$. The last equality follows from the long exact sequence

$$\begin{aligned} \cdots \rightarrow 0 &= R^0 \pi'_* L \tau_C^* R \mathcal{H}om(\mathcal{F}', \mathcal{F}) \rightarrow R^0 \pi'_* L \tau_C^* R \mathcal{H}om(L(\text{id} \times i)^*(\text{id} \times i)_* \mathcal{F}', \mathcal{F}) \\ &\rightarrow 0 = R^0 \pi'_* L \tau_C^* R \mathcal{H}om(\mathcal{F}' \boxtimes \mathcal{O}_C(-C)[1], \mathcal{F}) \rightarrow \cdots \end{aligned}$$

Finally, we consider the universal surjection as an element in $H^0(\overline{W}, \tau^* \mathcal{W} \otimes \mathcal{O}_\tau(1))$ and take its image under (3.7). This produces the desired extension.

By construction, $\mathcal{G}_{\text{univ}} \in \text{Coh}(\overline{W} \times S)$ defines a morphism $v: \overline{W} \rightarrow N_1 \subset M$ which restricts to a bijection $W \rightarrow E_1$ (see Corollary 3.4 and (3.3)). By degree reasons an extension on C of the form $0 \rightarrow \mathcal{L} \rightarrow \mathcal{E} \rightarrow \mathcal{L}' \rightarrow 0$, where $\deg \mathcal{L}' = 1$ and $\deg \mathcal{L} = 0$ is stable or split. However, the split extensions do not occur in $\mathbb{P}(\mathcal{V})$. Hence, v is everywhere defined. Moreover, the boundary $\overline{W} \setminus W = \mathbb{P}(\mathcal{V})$ maps to $N_1 \setminus E_1 = N_0 \cap N_1$. $\square$

Remark 3.9 One can show that $\nu: W \rightarrow E_1$ is actually an isomorphism of schemes. Moreover, $\nu: \overline{W} \rightarrow N_1$ is finite and hence a normalization map. Its tangent map is analyzed in [11, Proposition 7.5] and provides a characterization of the singularities of N_1 .

4 Proof of Theorem 1.1

We will now prove Theorem 1.1.

Theorem *Let $C \in |H|$ be an irreducible curve. The degrees of the two components of the nilpotent cone $N_C = N_0 \cup N_1$ are given by*

$$\deg_{u_1} N_0 = 5 \cdot 2^9 \text{ and } \deg_{u_1} N_1 = 5^2 \cdot 2^{11}$$

and their multiplicities are

$$\text{mult}_{N_C} N_0 = 2^3 \text{ and } \text{mult}_{N_C} N_1 = 2.$$

Moreover, any fiber F of the Mukai system has degree $5 \cdot 3 \cdot 2^{13}$.

All degrees will be computed with respect to a naturally defined distinguished ample class $u_1 \in H^2(M, \mathbb{Z})$, which we construct in Sect. 4.1. We set

$$d_i = \deg_{u_1}(N_i) := \int_M [N_i] u_1^5$$

for $i = 0, 1$, where by abuse of notation $[N_i] \in H^{10}(M, \mathbb{Z})$ is the Poincaré dual of the fundamental homology class $[N_i] \in H_{10}(M, \mathbb{Z})$.

The multiplicity is defined as follows. Let η_i be the generic point of N_i . Then

$$m_i = \text{mult}_N N_i := \text{lg}_{\mathcal{O}_{N, \eta_i}} \mathcal{O}_{N_i, \eta_i} = \text{lg}_{\mathcal{O}_{N_i, \eta_i}} \mathcal{O}_{N_i, \eta_i}.$$

In particular, we have an equality $[F] = m_0[N_0] + m_1[N_1] \in H^{10}(M, \mathbb{Z})$ for any fiber F . Consequently, inserting $m_0 = 2^3$ and $m_1 = 2$, we find

$$\deg_{u_1}(F) = 5 \cdot 2^{12} + 5^2 \cdot 2^{12} = 5 \cdot 3 \cdot 2^{13}.$$

as stated in the theorem. Luckily, it turns out that the multiplicities are small in comparison with the degrees so that it is possible to determine the multiplicities from the knowledge of the degrees but not vice versa.

Proof of the multiplicities knowing all the degrees Let $F \subset M$ be a smooth fiber. Then, we have $\deg F = m_0 d_0 + m_1 d_1$ and hence

$$5 \cdot 3 \cdot 2^{13} = m_0 \cdot 5 \cdot 2^9 + m_1 \cdot 5^2 \cdot 2^{11}.$$

The only possible solutions are $(m_0, m_1) = (28, 1)$ or $(m_0, m_1) = (8, 2)$. However, by [8, Proposition 4.11]

$$\dim T_{[\mathcal{E}]}N = \dim \operatorname{Ext}_{2C}^1(\mathcal{E}, \mathcal{E}) = \dim N + 1 \text{ for all } [\mathcal{E}] \in E_1.$$

Hence, N_1 is not reduced and the first solution is ruled out.

Remark 4.1 We will prove Theorem 1.1 for a fixed smooth curve $C \in |H|$, which implies the case of an irreducible and possibly singular curve by a deformation argument as follows. According to the careful analysis in [6, Section 3.7, in particular Propositions 3.7.23 & 3.7.19] the above description of the irreducible components of $f^{-1}(2C)$ is valid for every irreducible curve $C \in |H|$. Hence, if one deforms from a smooth to a singular, irreducible curve in $|H|$, the irreducible components of the fiber with their reduced structure deform as well. Consequently, degrees and multiplicities remain constant.

4.1 Construction of the ample class u_1

We use the determinant line bundle construction [16, Lemma 8.1.2] in order to produce an ample class on the moduli space M .

Let X and T be two projective varieties and assume that X is smooth. Let $p: T \times X \rightarrow T$ and $q: T \times X \rightarrow X$ denote the two projections. For any $\mathcal{W} \in \operatorname{Coh}(X \times T)$ flat over T , we define $\lambda_{\mathcal{W}}: K(X)_{\text{num}} \rightarrow H^2(T, \mathbb{Z})$ to be the following composition

$$K(X)_{\text{num}} \xrightarrow{q^*} K^0(T \times X)_{\text{num}} \xrightarrow{[\mathcal{W}]} K^0(T \times X)_{\text{num}} \xrightarrow{Rp_*} K^0(T)_{\text{num}} \xrightarrow{\det} \operatorname{NS}(T) \subset H^2(T, \mathbb{Z}).$$

We will take advantage of the functorial properties of this definition. These are

- (i) $f^* \lambda_{\mathcal{W}} = \lambda_{(f \times \operatorname{id})^* \mathcal{W}}$ for any morphism $f: T' \rightarrow T$ and
- (ii) $\lambda_{(\operatorname{id} \times i)_* \mathcal{W}}(x) = \lambda_{\mathcal{W}}(Li^* x)$ for all $x \in K(X)_{\text{num}}$ if $i: Y \hookrightarrow X$ is the inclusion of a closed, smooth subscheme and $\mathcal{W} \in \operatorname{Coh}(T \times Y)$.

The construction is especially interesting if $X = M_T(c)$ is a fine moduli space, that parametrizes coherent sheaves of class c on T . Let $\mathcal{E}_{\text{univ}}$ be a universal sheaf on $M_T(c) \times T$, then

$$\lambda_{\mathcal{E}_{\text{univ}} \otimes p^* \mathcal{M}}(x) = \lambda_{\mathcal{E}_{\text{univ}}}(x) + \chi(c \cdot x) c_1(\mathcal{M})$$

for all $\mathcal{M} \in \operatorname{Pic}(M_T(c))$. Hence,

$$\lambda_{M_T(c)} := \lambda_{\mathcal{E}_{\text{univ}}} : c^{\perp, \chi} \longrightarrow \operatorname{NS}(M_T(c))$$

is well-defined and does not depend on the choice of universal sheaf. Here,

$$c^{\perp, \chi} = \{x \in K(T)_{\text{num}} \mid \chi(x \cdot c) = 0\}.$$

Example 4.2 Let C be a smooth curve of any genus $g \geq 0$. Then

$$(\mathrm{rk}, \deg): K(C)_{\mathrm{num}} \xrightarrow{\sim} \mathbb{Z} \oplus \mathbb{Z}.$$

Fix $n \geq 1$ and $d \in \mathbb{Z}$ coprime and let $c = (n, d) \in K(C)_{\mathrm{num}}$. Then $M_C(c) = M_C(n, d)$ is the moduli space of stable vector bundles of rank n and degree d on C and we find $c^{\perp, \chi} = \langle (-n, d + n(1 - g)) \rangle$. The generalized Theta divisor can be defined by

$$\Theta_{M_C(n, d)} := \lambda_{M_C(n, d)}(-n, d + n(1 - g)),$$

see [12, Théorème D]. A special case is $M_C(1, k) = \mathrm{Pic}^k(C)$. In this case, we find $c^{\perp, \chi} = \langle (-1, k + 1 - g) \rangle$ and

$$\Theta_k := \lambda_{\mathrm{Pic}^k(C)}(-1, k + 1 - g)$$

is the class of the canonical Theta divisor in $\mathrm{Pic}^k(C)$.

Remark 4.3 Denote by $\mathrm{SM}_C(n, d)$ the moduli space of vector bundles with fixed determinant, i.e. a fiber of $\det: M_C(n, d) \rightarrow \mathrm{Pic}(C)$ and by $\Theta_{\mathrm{SM}_C(n, d)}$ the restriction of $\Theta_{M_C(n, d)}$ to $\mathrm{SM}_C(n, d)$. Taking the tensor product defines an étale map

$$h: \mathrm{SM}_C(n, d) \times \mathrm{Pic}^0(C) \longrightarrow M_C(n, d)$$

of degree n^{2g} . Using [10, Corollary 6], we find the following relation if (n, d) are coprime

$$h^* \Theta_{M_C(n, d)} = p_1^* \Theta_{\mathrm{SM}_C(n, d)} + n^2 p_2^* \Theta_0. \quad (4.1)$$

Lemma 4.4 Let C be a smooth curve of genus g and $\mathcal{P}$ a Poincaré line bundle on $\mathrm{Pic}^k(C) \times C$. Then

$$\lambda_{\mathcal{P}}: K(C)_{\mathrm{num}} \rightarrow H^2(\mathrm{Pic}^k(C), \mathbb{Z})$$

is given by

$$(r, d) \mapsto (d + (k + 1 - g)r)\mu - r\Theta_k,$$

where $p_1^* \mu = c_1^{2,0}(\mathcal{P}) \in H^2(\mathrm{Pic}^k(C) \times C, \mathbb{Z})$ is the $(2, 0)$ Künneth component of $c_1(\mathcal{P})$.

By tensoring with a suitable line bundle on $\mathrm{Pic}^k(C)$, we can assume $c_1^{2,0}(\mathcal{P}) = 0$.

Proof Let us abbreviate $\mathrm{Pic}^k(C)$ to Pic^k . We decompose

$$c_1(\mathcal{P}) = c^{2,0} + c^{1,1} + c^{0,2}$$

into its Künneth components and write $c^{2,0} = p^* \mu$ for some $\mu \in H^2(\mathrm{Pic}^k, \mathbb{Z})$. Then by [1, VIII §2] the class $\gamma = c^{1,1}$ satisfies $\gamma^2 = -2\rho p^* \Theta_k$. Moreover, by

definition, $c^{0,2} = k\rho$, where ρ is the pullback of the class of a point on C . Together, $c_1(\mathcal{P}) = p^*\mu + \gamma + k\rho$ and

$$\mathrm{ch}(\mathcal{P}) = 1 + p^*\mu + \gamma + k\rho + \rho p^*(k\mu - \Theta_k).$$

Now, let $x = (r, d) \in K(C)_{\mathrm{num}}$. The Grothendieck–Riemann–Roch theorem gives

$$\begin{aligned} \mathrm{ch}(Rp_*(\mathcal{P} \otimes q^*x)) &= p_*(\mathrm{ch}(\mathcal{P} \otimes q^*x) \mathrm{td}(\mathrm{Pic}^k \times C)) = p_*(\mathrm{ch}(\mathcal{P}) \mathrm{ch}(q^*x) q^* \mathrm{td}(C)) \\ &= p_*(\mathrm{ch}(\mathcal{P})(r + ((1 - g)r + d)\rho)) \\ &= kr + (1 - g)r + d + (kr + (1 - g)r + d)\mu - r\Theta_k. \end{aligned}$$

In particular, $\lambda_{\mathcal{P}}(x) = c_1(Rp_*(\mathcal{P} \otimes q^*x)) = ((k + 1 - g)r + d)\mu - r\Theta_k$. $\square$

We come back to our original situation, i.e. (S, H) is a polarized K3 surface of genus 2 and $M = M_H(v)$ parametrizes H -stable sheaves with fixed Mukai vector $v = (0, 2H, -1)$ or equivalently, with Chern character $v_{\mathrm{ch}} = (0, 2H, -1)$. In this setting λ_M induces an isomorphism, [16, Theorem 6.2.15]

$$\lambda_M: v_{\mathrm{ch}}^{\perp, \chi} \xrightarrow{\sim} \mathrm{NS}(M). \quad (4.2)$$

As v and v_{ch} coincide, we will notationally not distinguish between them anymore. We find

$$v^{\perp, \chi} = \{(2c \cdot H, c, s) \mid c \in \mathrm{NS}(S), s \in \mathbb{Z}\}.$$

Warning 4.5 In this setting, one usually wants to consider the morphism λ_M in terms of the Mukai vector and the Mukai pairing instead of the Chern character and the intersection product, i.e. one considers the composition

$$v^{\perp, \langle \cdot, \cdot \rangle} \xrightarrow{\sim} v_{\mathrm{ch}}^{\perp, \chi} \xrightarrow{\lambda_M} \mathrm{NS}(M_H(v)),$$

which identifies the Mukai pairing on the left hand side with the Beauville–Bogomolov form on the right hand side. Here $v_{\mathrm{ch}} \cdot \sqrt{\mathrm{td}(S)} = v$. Explicitly, if $v = (r, c, s)$, then $v_{\mathrm{ch}} = (r, c, s - r)$ and

$$\langle (r', c', s'), (r, c, s) \rangle = \chi((-r', c', -s' - r') \cdot (r, c, s - r)).$$

Thus the first arrow is given by $(r', c', s') \mapsto (-r', c', -s' - r')$.

Definition 4.6 For all $s \in \mathbb{Z}$ we define

$$l_s := \lambda_M((-4, -H, s)) \in H^2(M, \mathbb{Z}).$$

The value of s does not have any relevance for our computations. However, with the results of [3], it can be proven that l_s is ample for $s \gg 0$ and one can even compute the precise boundary of the ample cone.

Definition 4.7 For everything what follows, we fix $s_0 \gg 0$ such that l_{s_0} is ample and set

$$u_1 := l_{s_0}.$$

4.2 Degree of a general fiber

We compute the degree of a general fiber.

Proposition 4.8 Let $D \in |2H|$ be a smooth curve and let $F := f^{-1}(D)$ be the corresponding fiber. Let $u = \lambda_M(x)$ with $x = (2c.H, c, s) \in v^{\perp, X}$. Then

$$u|_F = -2c.H \cdot \Theta_3,$$

where $\Theta_3 \in H^2(\text{Pic}^3(D), \mathbb{Z})$ is the class of the Theta divisor. In particular, we have

$$\deg_{u_1} F = 5! \cdot 2^{10}.$$

Proof Let $i: D \hookrightarrow S$ be the inclusion. The inclusion $\text{Pic}^3(D) \cong F \hookrightarrow M$ is defined by $(\text{id} \times i)_* \mathcal{P}_3$, where $\mathcal{P}_3$ is a Poincaré line bundle on $\text{Pic}^3(D) \times D$. Hence,

$$u|_{\text{Pic}^3(D)} = \lambda_{(\text{id} \times i)_* \mathcal{P}_3}(x) = \lambda_{\mathcal{P}_3}(Li^*x).$$

Now, $Li^*: K(S)_{\text{num}} \rightarrow K(D)_{\text{num}} \cong \mathbb{Z}^{\oplus 2}$ maps (r, c, s) to $(r, c.D)$ and thus Li^*x to $2c.H \cdot (1, 1)$, whereas by definition $\theta_3 = \lambda_{\mathcal{P}_3}(-1, -1)$. Finally,

$$\deg_{u_1} F = \int_{\text{Pic}^3(D)} (4\Theta_3)^5 = 2^{10} \cdot 5!.$$

Remark 4.9 One can also prove the above result using the Beauville–Bogomolov form $(\cdot, \cdot)_{BB}$ on $H^2(M, \mathbb{Z})$. Let $u_0 = f^*c_1(\mathcal{O}(1)) \in H^2(M, \mathbb{Z})$. Then $[F] = u_0^5 \in H^{10}(M, \mathbb{Z})$ and

$$\deg_{u_1}(F) = \int_M u_0^5 u_1^5 = 5! \cdot (u_0, u_1)_{BB}^5,$$

where we use that $(u_0, u_0)_{BB} = 0$ and that M is birational to $S^{[5]}$ in order to determine the correct Fujiki constant. One verifies that $u_0 = \lambda_M((0, 0, 1))$ [25, Lem 4.4] whereas, by definition, $u_1 = \lambda_M(-4, -H, s_0)$ with $s_0 \gg 0$. After correct identification (cf. Warning 4.5), one has

$$(\lambda_M(r, c, s), \lambda_M(r', c', s'))_{BB} = \langle (r, c, s), (r', c', s') \rangle + 2rr'.$$

This gives $(u_0, u_1)_{BB} = 4$.

4.3 Degree of the vector bundle component N_0

Next, we deal with the component N_0 , which is isomorphic to $M_C(2, 1)$.

Proposition 4.10 *Let $x = \lambda_M(u)$ with $u = (2c.H, c, s) \in v^{\perp, \chi}$. Then*

$$x|_{N_0} = -c.H\Theta,$$

where $\Theta \in H^2(N_0, \mathbb{Z})$ is the the generalized Theta divisor. In particular,

$$u_1|_{N_0} = 2\Theta,$$

and given $x_i = \lambda_M(2c_i.H, c_i, s_i)$ for $i = 1, \dots, 5$, we find

$$\int_M x_1 \dots x_5 [N_0] = - \prod_{i=1}^5 c_i.H \int_{N_0} \Theta^5 = -5 \cdot 2^4 \prod_{i=1}^5 c_i.H.$$

Hence, $\deg_{u_1} N_0 = 5 \cdot 2^9$.

Proof Let $i: C \hookrightarrow S$ be the inclusion. The inclusion $N_0 \hookrightarrow M$ is defined by $(\text{id} \times i)_* \mathcal{E}_{\text{univ}}$, where $\mathcal{E}_{\text{univ}}$ is the universal vector bundle on $N_0 \times C$. Hence,

$$x|_{N_0} = \lambda_{(\text{id} \times i)_* \mathcal{E}_{\text{univ}}}(u) = \lambda_{N_0}(Li^*u).$$

Now, $Li^*: K(S)_{\text{num}} \rightarrow K(C)_{\text{num}} \cong \mathbb{Z}^{\oplus 2}$ maps (r, c, s) to $(r, c.H)$. In particular, $Li^*u = c.H(2, 1)$, whereas by definition $\theta = \lambda_{N_0}(-2, -1)$.

Next, we compute $\int_{N_0} \Theta^5$ by pulling back along $h: \text{SM}_C(2, 1) \times \text{Pic}^0(C) \rightarrow N_0$ from Remark 4.3.

$$\begin{aligned} \int_{M_C(2,1)} \Theta^5 &\stackrel{(4.1)}{=} \frac{1}{2^4} \int_{\text{SM}_C(2,1) \times \text{Pic}^0(C)} (p_1^* \Theta_{\text{SM}} + 4p_2^* \Theta_0)^5 \\ &= \frac{1}{2^4} \binom{5}{3} \int_{\text{SM}_C(2,1)} \Theta_{\text{SM}}^3 \int_{\text{Pic}^0(C)} (4\Theta_0)^2 = 5 \cdot 2^4. \end{aligned}$$

The value $\int_{\text{SM}_C(2,1)} \Theta_{\text{SM}}^3 = 4$ is given by the leading term of the Verlinde formula [27].

Remark 4.11 The general formula is

$$\int_{M_C(n,d)} \Theta^{\dim M_C(n,d)} = \dim M_C(n, d)! (2^{2g-2} - 2) \frac{(-1)^g 2^{2g-2} B_{2g-2}}{(2g-2)!},$$

where B_i is the i -th Bernoulli number. The second Bernoulli number is $B_2 = \frac{1}{6}$

Remark 4.12 In the general case, where $[v = (0, nH, s)$ and $u_1 = \lambda_M(-n(2g - 2), sH, *)$ with $s = n + d(1 - g)$,] we find

$$u_1|_F = n(2g - 2)\Theta_\delta \text{ and } u_1|_{N_0} = (2g - 2)\Theta. \text{ Thus}$$

$$\begin{aligned} \deg_{u_1} F &= (n(2g - 2))^{\dim N} \cdot \dim N! \\ \text{and } \deg_{u_1} N_0 &= (2g - 2)^{\dim N} \int_{M_C(n, d)} \Theta^{\dim M_C(n, d)}. \end{aligned}$$

Here, $\dim N = n^2(2g - 2) + 2$.

4.4 Degree of the other component N_1

We complete the proof of Theorem 1.1 by dealing with the remaining component N_1 . Recall from Proposition 3.8 that there is a birational map $v: \overline{W} \rightarrow N_1$, where $\tau: \overline{W} = \mathbb{P}(\mathcal{W}) \rightarrow T = \text{Pic}^1(C) \times C$.

Proposition 4.13 *Let $x_i = \lambda_M(u_i)$ with $u_i = (2c_i.H, c_i, s_i) \in v^{\perp, \chi}$ for $i = 1, \dots, 5$. Then*

$$\int_M x_1 \dots x_5 [N_1] = \int_{\overline{W}} \prod_{i=1}^5 v^*(x_i|_{N_1}) = -5^2 \cdot 2^6 \prod_{i=1}^5 c_i.H. \quad (4.3)$$

In particular, $\deg_{u_1} N_1 = 5^2 \cdot 2^{11}$.

Note that the first equality in (4.3) is immediate, because $v: \overline{W} \rightarrow N_1$ is birational. For the proof of the proposition, we need to introduce some more notation. We abbreviate $\text{Pic}^1(C)$ to Pic^1 and in the following all cohomology groups have $\mathbb{Z}$ coefficients. We set

$$\zeta = c_1(\mathcal{O}_\tau(1)) \in H^2(\overline{W}) \text{ and write } \rho = p_2^*[\text{pt}] \in H^2(\text{Pic}^1 \times C)$$

for the pullback of the class of a point on C . If no confusion is likely, we suppress pullbacks from our notation, e.g. we will write $\Theta_1 \in H^2(\text{Pic}^1 \times C)$ and also $\Theta_1 \in H^2(\mathbb{P}(\mathcal{W}))$ instead of $p_1^*\Theta_1$ and $\tau^*p_1^*\Theta_1$, respectively. Moreover, we define

$$\pi := c_1(\mathcal{P}) - c_1^{2,0}(\mathcal{P}) \in H^2(\text{Pic}^1 \times C),$$

where $\mathcal{P}$ is a Poincaré line bundle. Note that π is independent of the choice of $\mathcal{P}$.

Proof of Proposition 4.13 We will split the proof into the following three steps.

(i) Let $x = \lambda_M(2c.H, c, s)$. Then

$$v^*(x|_{N_1}) = \lambda_{\mathcal{G}_{\text{univ}}}(x) = c.H(-4\Theta_1 + 2\pi - 7\rho - \zeta) \in H^2(\overline{W}).$$

(ii) We have

$$(-4\Theta_1 + 2\pi - 7\rho - \zeta)^5 = -5^2 2^5 \zeta^2 \rho \Theta_1^2 \in H^{10}(\overline{W}).$$

(iii) The top cohomology group $H^{10}(\overline{W})$ generated by $\frac{1}{2}\zeta^2\rho\Theta_1^2$ and we have

$$\int_{\mathbb{P}(\mathcal{W})} \zeta^2 \rho \Theta_1^2 = 2.$$

□

Proof of (i) In Proposition 3.8, we defined the morphism $\nu: \overline{W} \rightarrow N_1$ by means of $\mathcal{G}_{\text{univ}} \in \text{Coh}(\overline{W} \times S)$, which sits in the (universal) extension

$$0 \rightarrow \tau_S^*(\text{id} \times i)_*(\mathcal{P}_1 \boxtimes \mathcal{O}(\Delta) \boxtimes \omega_C^{-1}) \boxtimes \mathcal{O}_\tau(1) \rightarrow \mathcal{G}_{\text{univ}} \rightarrow \tau_S^*(\text{id} \times i)_* p_{13}^* \mathcal{P}_1 \rightarrow 0,$$

where $\tau_S = \tau \times \text{id}_S: \overline{W} \times S \rightarrow \text{Pic}^1 \times C \times S$. So, by construction, we have

$$\begin{aligned} \lambda_{\mathcal{G}_{\text{univ}}}(x) &= \lambda_{\tau_S^*(\text{id} \times i)_*(\mathcal{P}_1 \boxtimes \mathcal{O}(\Delta) \boxtimes \omega_C^{-1}) \boxtimes \mathcal{O}_\tau(1)}(x) + \lambda_{\tau_S^*(\text{id} \times i)_* p_{13}^* \mathcal{P}_1}(x) \\ &= \lambda_{\tau_S^*(\text{id} \times i)_*(\mathcal{P}_1 \boxtimes \mathcal{O}(\Delta) \boxtimes \omega_C^{-1})}(x) + k(Li^*x) \cdot \zeta + \tau^* p_1^* \lambda_{\mathcal{P}_1}(Li^*x) \\ &= \tau^*(\lambda_{\mathcal{P}_1 \boxtimes \mathcal{O}(\Delta)}(Li^*x \cdot \omega^{-1}) + p_1^* \lambda_{\mathcal{P}_1}(Li^*x)) + k(Li^*x) \cdot \zeta, \end{aligned}$$

where

$$\omega = c_1(\omega_C) \text{ and } k(Li^*x) = \text{rk } Rp_*(\mathcal{P}_1 \boxtimes \mathcal{O}(\Delta) \boxtimes \omega_C^{-1} \boxtimes Li^*x) = \chi(Li^*x) = -c.H.$$

The term $\lambda_{\mathcal{P}_1 \boxtimes \mathcal{O}(\Delta)}(Li^*x \cdot \omega^{-1}) + p_1^* \lambda_{\mathcal{P}_1}(Li^*x)$, is determined in Lemmas 4.14 and 4.4. Note that each summand depends on the choice of a Poincaré line bundle, whereas the sum does not. Together,

$$\begin{aligned} \nu^*(x|_{N_1}) &= \tau^*(\lambda_{\mathcal{P}_1 \boxtimes \mathcal{O}(\Delta)}(Li^*x \cdot \omega^{-1}) + p_1^* \lambda_{\mathcal{P}_1}(Li^*x)) - c.H\zeta \\ &= c.H(p_1^*(\lambda_{\mathcal{P}_1}(2, -3) + \lambda_{\mathcal{P}_1}(2, 1)) + 2c_1(\mathcal{P}_1) - 7\rho - \zeta) \\ &= c.H(-4\Theta_1 + 2\pi - 7\rho - \zeta). \end{aligned}$$

Lemma 4.14 *Let $\mathcal{F} \in \text{Coh}(X \times C)$. Then*

$$\lambda_{\mathcal{F} \boxtimes \mathcal{O}(\Delta)}(x) = p_1^* \lambda_{\mathcal{F}}(x) + rc_1(\mathcal{F}) + c_0(\mathcal{F})(d - 2r)\rho$$

for all $x = (r, d) \in \mathbf{K}(C)_{\text{num}}$. In particular,

$$\lambda_{\mathcal{P}_1 \boxtimes \mathcal{O}(\Delta)}(Li^*x \cdot \omega^{-1}) = c.H(p_1^* \lambda_{\mathcal{P}}(2, -3) + 2c_1(\mathcal{P}_1) - 7\rho).$$

Proof We have

$$[\mathcal{F} \boxtimes \mathcal{O}(\Delta)] = [p_{13}^* \mathcal{F}] + [(\text{id} \times i_\Delta)_*(p_2^* \omega_C^{-1} \otimes \mathcal{F})] \in \mathbf{K}(X \times C \times C),$$

where $i_\Delta: C \rightarrow C \times C$ is the diagonal and thus

$$\lambda_{\mathcal{F} \boxtimes \mathcal{O}(\Delta)}(x) = \lambda_{p_{13}^* \mathcal{W}}(x) + \lambda_{(\text{id} \times i_\Delta)_*(p_2^* \omega_C^{-1} \otimes \mathcal{F})}(x) \in H^2(X \times C)$$

for all $x \in K(C)_{\text{num}}$. Now,

$$\begin{aligned}\lambda_{(\text{id} \times i_{\Delta})_* (p_2^* \omega^{-1} \cdot [\mathcal{F}])}(x) &= \det R p_{12*}((\text{id} \times i_{\Delta})_*(p_2^* \omega^{-1} \cdot [\mathcal{F}]) \cdot p_3^* x) \\ &= \det R(p_{12} \circ (\text{id} \times i_{\Delta}))_*([\mathcal{F}] \cdot p_2^*(\omega^{-1} \cdot x)) \\ &= \det([\mathcal{F}] \cdot p_2^*(\omega^{-1} \cdot x)) = r c_1(\mathcal{F}) + c_0(\mathcal{F})(d - r(2g - 2))\rho.\end{aligned}$$

□

To prove the remaining steps, we need to understand the cohomology ring $H^*(\overline{W})$.

Lemma 4.15 *We have*

$$H^*(\overline{W}) \cong H^*(\text{Pic}^1 \times C)[\zeta]/\zeta^3 + 4\rho\zeta^2.$$

In particular,

$$H^{10}(\overline{W}) = \zeta^2 \cdot H^6(\text{Pic}^1 \times C).$$

Proof By definition, $\overline{W} = \mathbb{P}(\mathcal{W})$. Hence,

$$H^*(\overline{W}) \cong H^*(\text{Pic}^1 \times C)[\zeta]/\zeta^3 + c_1(\mathcal{W})\zeta^2 + c_2(\mathcal{W})\zeta + c_3(\mathcal{W}).$$

We use the short exact sequence $0 \rightarrow \mathcal{V} \rightarrow \mathcal{W} \rightarrow \mathcal{O}_T \rightarrow 0$ from (3.6) to compute the Chern classes of $\mathcal{W}$. Note that

$$\mathcal{V} = R^1 p_{12*}(p_{23}^* \mathcal{O}(\Delta) \otimes p_3^* \omega_C^{-1}) \cong p_2^* R^1 p_{1*}(\mathcal{O}(\Delta) \otimes p_2^* \omega_C^{-1}).$$

So the Chern classes of $\mathcal{V}$ can be computed by the push forward along the first projection of the following the short exact sequence

$$0 \rightarrow p_2^* \omega_C^{-1} \rightarrow \mathcal{O}(\Delta) \boxtimes \omega_C^{-1} \rightarrow p_2^* \omega_C^{-2}|_{\Delta} \rightarrow 0.$$

We find

$$0 \longrightarrow \omega_C^{-2} \rightarrow \mathcal{O}_C \otimes H^1(C, \omega_C^{-1}) \rightarrow R^1 p_{1*}(\mathcal{O}(\Delta) \otimes p_2^* \omega_C^{-1}) \longrightarrow 0.$$

Hence, $c_1(\mathcal{W}) = 4\rho$ and $c_i(\mathcal{W}) = 0$ if $i \geq 2$. □

Proof of (ii) and (iii) We want to show that

$$(-4\Theta_1 + 2\pi - 7\rho - \zeta)^5 = -5^2 2^5 \zeta^2 \rho \Theta_1^2 \in H^{10}(\overline{W}).$$

We compute

$$\begin{aligned}(-4\Theta_1 + 2\pi - 7\rho - \zeta)^5 &= \binom{5}{3}(-\zeta^3)(-4\Theta_1 + 2\pi - 7\rho)^2 + \binom{5}{2}\zeta^2(-4\Theta_1 + 2\pi - 7\rho)^3 \\ &= 10 \cdot \zeta^2((4\rho(-4\Theta_1 + 2\pi - 7\rho)^2 + (-4\Theta_1 + 2\pi - 7\rho)^3).\end{aligned}$$

The result is a combination of π , θ and ρ , which are classes of type $(1, 1) + (0, 2)$, $(2, 0)$ and $(0, 2)$, respectively. Moreover, in the proof of Lemma 4.4 we computed $\pi = \rho + \gamma$ and $\pi^2 = \gamma^2 = -2\rho\Theta_1$. Hence, the only non-zero combinations are $\pi^2\Theta_1 = -2\rho\Theta_1^2 = -2\pi\Theta_1^2$. We find

$$\begin{aligned} & 10 \cdot \zeta^2((4\rho(-4\Theta_1 + 2\pi - 7\rho)^2 + (-4\Theta_1 + 2\pi - 7\rho)^3) \\ &= 10 \cdot \zeta^2(2^6\rho\Theta_1^2 + 3(-2^4\pi^2\Theta_1 + 2^5\pi\Theta_1^2 - 7 \cdot 2^4\rho\Theta_1^2) \\ &= 10(2^6 + 3(2^5 + 2^5 - 7 \cdot 2^4))\zeta^2\rho\Theta_1^2 = -5^2 2^5 \zeta^2 \rho \Theta_1^2. \end{aligned}$$

Finally, we want to show that $\int_{\overline{W}} \zeta^2 \rho \Theta^2 = 2$. Indeed,

$$\int_{\overline{W}} \zeta^2 \rho \Theta^2 = \tau_* \zeta^2 \int_{\text{Pic}^1} \Theta^2 \int_C \rho = 2.$$

□

This concludes the proof of the proposition.

5 Proof of Theorem 1.2

In this section, we prove Theorem 1.2.

Theorem *The classes $[N_0]$ and $[N_1] \in H^{10}(M, \mathbb{Q})$ are linearly independent and span a totally isotropic subspace of $H^{10}(M, \mathbb{Q})$ with respect to the intersection pairing. They are given by*

$$[N_0] = \frac{1}{48}[F] + \beta \text{ and } [N_1] = \frac{5}{12}[F] - 4\beta,$$

where $[F]$ is the class of a general fiber of the Mukai system and $0 \neq \beta \in (S^5 H^2(M, \mathbb{Q}))^\perp$ satisfies $\beta^2 = 0$. As $\deg_{u_1} \beta = 0$, the class β is not effective.

From now on, all cohomology groups have $\mathbb{Q}$ -coefficients.

Before coming to the proof, we want to point out, that the irreducible components over points in $\Sigma \setminus \Delta$ (see (3.1)) are of different cohomological nature. Let $D \in \Sigma \setminus \Delta$ be a reducible curve with two smooth components C_1 and C_2 meeting transversally. Then the two components N'_1 and N'_2 of $f^{-1}(D)$ contain an open sublocus parametrizing line bundles on D of bi-degree $(2, 1)$ and $(1, 2)$, respectively [6, Proposition 3.7.1 and Lemma 3.3.2]. The monodromy around $\Sigma \setminus \Delta$ exchanges C_1 and C_2 and consequently the classes of the irreducible components. We find

$$[N'_1] = [N'_2] = \frac{1}{2}[F].$$

In particular, the two components are linearly dependent. This is not true over Δ .

Proposition 5.1 *The classes $[N_0]$ and $[N_1] \in H^{10}(M)$ are linearly independent.*

The proof uses the following simple observation.

Lemma 5.2 *Let $M \rightarrow B$ be a Lagrangian fibration and F a smooth fiber. Then*

$$c_i(\mathcal{T}_M)|_F = 0 \text{ for all } i > 0.$$

Proof We have a short exact sequence $0 \rightarrow \mathcal{T}_F \rightarrow \mathcal{T}_M|_F \rightarrow \mathcal{N}_{F/M} \rightarrow 0$. Now, $F \subset M$ is Lagrangian and hence $\mathcal{N}_{F/M} \cong \Omega_F$. Moreover, F is an abelian variety and hence all its Chern classes of degree greater than zero are trivial.

Proof of Proposition 5.1 Assume that $[N_0]$ and $[N_1]$ are linearly dependent. Then there is some $\lambda \in \mathbb{Q}$ such that $[F] = \lambda[N_0]$, where $F \subset M$ is a smooth fiber. In particular, by the above lemma, any product of $[N_0]$ and the Chern classes of M vanishes. However, we will show that

$$\int_M c_2(\mathcal{T}_M) \cdot u_1^3 \cdot [N_0] \neq 0,$$

leading to the desired contradiction. We have $c(\mathcal{T}_M|_{N_0}) = c(\mathcal{T}_{N_0})c(\Omega_{N_0})$ and thus

$$c_2(\mathcal{T}_M)|_{N_0} = (2c_2 - c_1^2)(\mathcal{T}_{N_0}).$$

Moreover, our computation will use the following two inputs. Let $\alpha \in H^2(\mathrm{SM}_C(2, 1))$ be the degree two Künneth component of $(c_1^2 - c_2)(\mathcal{V}_{\mathrm{univ}})$ with $\mathcal{V}_{\mathrm{univ}}$ being a universal bundle on $\mathrm{SM}_C(2, 1) \times C$. It is known, e.g. [26, §5A], that

$$c_1(\mathcal{T}_{\mathrm{SM}_C(2,1)}) = 2\alpha, \quad c_2(\mathcal{T}_{\mathrm{SM}_C(2,1)}) = 3\alpha^2 \quad \text{and} \quad \int_{\mathrm{SM}_C(2,1)} \alpha^3 = 4.$$

Further, by [12, Théorème F] it is known that $\mathcal{O}_{\mathrm{SM}_C(2,1)}(-2\Theta) \cong \omega_{\mathrm{SM}_C(2,1)}$. Hence,

$$\Theta = -\frac{1}{2}c_1(\omega_{\mathrm{SM}_C(2,1)}) = \frac{1}{2}c_1(\mathcal{T}_{\mathrm{SM}_C(2,1)}).$$

This gives,

$$\begin{aligned} \int c_2(\mathcal{T}_M) u_1^3 [N_0] &\stackrel{4.10}{=} \int_{N_0} (2c_2 - c_1^2)(\mathcal{T}_{N_0}) \cdot (2\Theta)^3 \\ &= \frac{1}{2^4} \int_{\mathrm{SM}_C(2,1) \times \mathrm{Pic}^0} h^*((2c_2 - c_1^2)(\mathcal{T}_{N_0}) \cdot (2\Theta)^3) \\ &\stackrel{(4.1)}{=} \frac{2^3}{2^4} \int_{\mathrm{SM}_C(2,1) \times \mathrm{Pic}^0} p_1^*(2c_2 - c_1^2)(\mathcal{T}_{\mathrm{SM}}) \cdot (p_1^*\Theta_{\mathrm{SM}} + 4p_2^*\Theta_0)^3 \\ &= \frac{1}{2} \int_{\mathrm{SM}_C(2,1) \times \mathrm{Pic}^0} p_1^*(2c_2 - c_1^2)(\mathcal{T}_{\mathrm{SM}}) \cdot (3p_1^*\Theta_{\mathrm{SM}} \cdot 4^2 p_2^*\Theta_0^2) \\ &= 3 \cdot 2^3 \int_{\mathrm{SM}_C(2,1)} (2c_2 - c_1^2)(\mathcal{T}_{\mathrm{SM}}) \cdot \frac{1}{2} c_1(\mathcal{T}_{\mathrm{SM}}) \int_{\mathrm{Pic}^0} \Theta_0^2 \\ &= 3 \cdot 2^4 \int_{\mathrm{SM}_C(2,1)} (6\alpha^2 - 4\alpha^2) \alpha = 3 \cdot 2^7 \neq 0. \end{aligned}$$

Proof of Theorem 1.2 We set $V := S^5 H^2(M) \subset H^{10}(M)$ so that we have an orthogonal decomposition with respect to the cup product $H^{10}(M) = V \oplus V^\perp$. Accordingly, we write $[N_i] = \alpha_i + \beta_i$ with $\alpha_i \in V$ and $0 \neq \beta_i \in V^\perp$ for $i = 1, 2$. We claim that

$$20[N_0] - [N_1] \in V^\perp. \quad (5.1)$$

To see this, we decompose the second cohomology group into its transcendental and algebraic part, i.e. $H^2(M) = T(M) \oplus \text{NS}(M)$. Now, for $i = 1, 2$ consider

$$T(M) \rightarrow H^{12}(M), \quad \alpha \mapsto \alpha \cdot [N_i]. \quad (5.2)$$

As the symplectic form $\sigma \in T(M)$ vanishes on N_i , it follows by irreducibility of the Hodge structure $T(M)$ that the assignment (5.2) is trivial. Hence, it suffices to show that $20[N_0] - [N_1] \in (S^5 \text{NS}(M))^\perp$. By (4.2) any element in $S^5 \text{NS}(M)$ is of the form $x_1 x_2 \dots x_5$, where $x_i = \lambda_M(2c_i \cdot H, c_i, s_i)$. According to Propositions 4.10 and 4.13

$$\int [N_1] x_1 x_2 \dots x_5 = -5^2 2^6 \prod_{i=1}^5 c_i \cdot H = 20 \int [N_0] x_1 x_2 \dots x_5.$$

This proves (5.1).

Next, we write $[N_1] - 20[N_0] = \alpha_1 - 20\alpha_0 + \beta_1 - 20\beta_0 \in V^\perp$ and conclude $\alpha_1 = 20\alpha_0$. We set $\alpha = \alpha_0$. On the one hand, we have by Theorem 1.1

$$2^3[N_0] + 2[N_1] = [F] = u_0^5 \in V,$$

but also

$$u_0^5 = 48\alpha + 8\beta_0 + 2\beta_1.$$

This gives $48\alpha = u_0^5$ and $\beta_1 = -4\beta_0$. Setting $\beta = \beta_0$ gives the desired expression.

The last assertion follows from $[N_0]^2 = (\frac{1}{48}u_0^5 + \beta)^2 = \beta^2$ and

$$[N_0]^2 = \int_{N_0} c_5(\mathcal{N}_{N_0/M}) = \int_{N_0} c_5(\Omega_{N_0}) = -e(N_0),$$

which is known to vanish, see [2, §9]. Hence $\beta^2 = 0$, which implies $[N_1]^2 = 0$ and finally, as $[F]^2 = 0$, also $[N_0] \cdot [N_1] = 0$. $\square$

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