Singular orbits and Baker domains

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Abstract

We show that there is a transcendental meromorphic function with an invariant Baker domain U such that every singular value of f is a super-attracting periodic point. This answers a question of Bergweiler from 1993. We also show that U can be chosen to contain arbitrarily large round annuli, centred at zero, of definite modulus. This answers a question of Mihaljević and the author from 2013, and complements recent work of Barański et al concerning this question.

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*I long for the Person from Porlock
To bring my thoughts to an end,
I am becoming impatient to see him
I think of him as a friend.*

Stevie Smith

1 Introduction

Let $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a rational function of degree at least 2. The *Fatou set* $F(f)$ consists of those points $z \in \mathbb{C}$ near which the iterates of f form a normal family in the sense of Montel. In other words, these are the points at which the dynamics generated by f is stable under small perturbations. A connected component of $F(f)$ is called a *Fatou component*; such a component is *invariant* if $f(U) \subset U$. An invariant Fatou component may be an immediate basin of attraction of an attracting or parabolic fixed point, or a simply or doubly connected domain on which f is conjugate to an irrational rotation.

It was shown by Fatou [8, §30–31] that there is a close relationship between invariant Fatou components and the critical values of f . Indeed, every attracting or parabolic

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basin must contain a critical value, and the boundary of any rotation domain is contained in the *postcritical set* $\mathcal{P}(f)$, i.e., the closure of the union of all critical orbits of f . See Lemmas 8.5, 15.7 and Theorems 10.15, 11.17 of [10].

These relationships carry over to the case of a transcendental meromorphic function $f: \mathbb{C} \rightarrow \hat{\mathbb{C}}$, with the set of critical values replaced by the set $\text{sing}(f^{-1})$ of critical or asymptotic values of f , and the postcritical set by the *postsingular set* defined analogously. In this setting, there is another possible type of invariant Fatou component: An *invariant Baker domain*, in which the iterates converge to the essential singularity at ∞ . Bergweiler [3, Question 4] asked whether there is a relation between $\text{sing}(f^{-1})$ and the boundary of such a Baker domain. He also asked a more precise version of this question [3, Question 5]:

Question 1.1 Is it possible that a meromorphic function f has Baker domains if the forward orbit of z is bounded for all $z \in \text{sing}(f^{-1})$?

We give an affirmative answer.

Theorem 1.2 (Baker domains and super-attracting points) *There is a transcendental meromorphic function $f: \mathbb{C} \rightarrow \hat{\mathbb{C}}$ with a Baker domain such that every point of $\text{sing}(f^{-1})$ is a super-attracting periodic point of period 2.*

Regarding [3, Question 4], Bergweiler [4, Theorem 3] obtained the following answer when f is a transcendental *entire* function: If U contains no critical or asymptotic values of f , then there exists a sequence $(p_n)_{n=0}^\infty$ of postsingular points $p_n \in \mathcal{P}(f)$ of f such that

- (a) $|p_n| \rightarrow \infty$;
- (b) $\frac{\text{dist}(p_n, U)}{|p_n|} \rightarrow 0$; and
- (c) $\lim_{n \rightarrow \infty} |p_{n+1}|/|p_n| = 1$.

On the other hand [4, Theorem 1], there is an entire function with a Baker domain U such that $\text{dist}(\mathcal{P}(f), \partial U) > 0$, where dist denotes Euclidean distance.

For Baker domains of general transcendental *meromorphic* functions, it was shown in [11, Theorem 1.5] that an analogue of [4, Theorem 3] holds, without the hypothesis that U contains no critical or asymptotic values, where condition (c) is replaced by the weaker

$$\limsup_{n \rightarrow \infty} \frac{|p_{n+1}|}{|p_n|} < \infty. \quad (1.1)$$

The article [11] also poses the following question.

Question 1.3 ([11, Remark on p. 1603]) Can (1.1) be replaced by (c)?

In [7, Theorem A], it is shown that the answer is positive if $\mathbb{C} \setminus U$ has an unbounded connected component. Here we show that the answer to Question 1.3 is negative in general.

Theorem 1.4 (Annuli of definite modulus) *Let $\rho > 0$. Then the function in Theorem 1.2 can be chosen such that there is a sequence $R_j \rightarrow \infty$ with*

$$\{z \in \mathbb{C} : R_j < |z| < \rho R_j\} \subset U \quad \text{for all } j. \quad (1.2)$$

The construction in our paper is inspired by those of Rippon and Stallard [12, Theorem 1.2] and Barański et al. [6, Theorem C]. Both of these construct meromorphic functions with multiply-connected Baker domains, obtained from an affine map by inserting poles at a sequence of points tending to infinity. We shall use quasiconformal surgery instead of explicit formulae, which allows us to obtain the required control of the singular orbits of f .

Notation

$\mathbb{C}$, $\hat{\mathbb{C}}$ and $\mathbb{D}$ denote the complex plane, Riemann sphere and unit disc, respectively. The Euclidean disc of radius r around $z \in \mathbb{C}$ is denoted by $D(z, r)$.

2 Quasiconformal surgery

The function in Theorem 1.2 is constructed by a “cut and paste” quasiconformal surgery (see [5, Chapter 7]), starting from the linear map

$$f_0(z) := \mu z, \quad (2.1)$$

where $\mu > 1$. Consider the closed and connected set

$$U_0 := \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\} \cup \bigcup_{j \geq 0} \left\{ z \in \mathbb{C} : \mu^j \leq |z| \leq \mu^{j+\frac{1}{2}} \right\},$$

which is forward-invariant under f_0 . Also let D_0 be a closed disc

$$D_0 = \overline{D(\xi_0, r_0)} \subset \{z \in \mathbb{C} : \operatorname{Re} z < 0 \text{ and } \sqrt{\mu} < |z| < \mu\} \subset \mathbb{C} \setminus U_0;$$

If we define

$$D_j := f_0^j(D_0) = \overline{D(\mu^j \cdot \xi_0, \mu^j \cdot r_0)} =: \overline{D(\xi_j, r_j)},$$

then all D_j are disjoint from U_0 , and from each other.

We construct our functions from f_0 by a quasiconformal surgery that inserts poles in the discs D_j . The following makes this procedure precise.

Proposition 2.1 (Quasiconformal surgery of f_0) *Let $\mu > 1$, and let f_0 , U_0 and $(D_j)_{j=0}^\infty$ be defined as above. Let $K > 1$, and let*

$$h_j : D_j \rightarrow \hat{\mathbb{C}}$$

be a sequence of K -quasiregular functions such that each h_j extends continuously to ∂D_j with $h_j = f_0$ on ∂D_j . Then

$$F: \mathbb{C} \rightarrow \hat{\mathbb{C}}; \quad z \mapsto \begin{cases} h_j(z) & \text{if } z \in D_j \\ f_0(z) & \text{if } z \notin \bigcup_j D_j. \end{cases} \quad (2.2)$$

is K -quasiregular, with no finite asymptotic values, and every critical value of F is a critical value of some h_j .

Suppose furthermore that, for infinitely many j , the map h_j is not a homeomorphism $h_j: D_j \rightarrow D_{j+1}$, and that for every j there is an open set W_j with the following properties.

- (a) The dilatation of h_j is supported on W_j .
- (b) F is conformal on $F^n(W_j)$ for all $n \geq 1$.

Then there is a quasiconformal homeomorphism $\psi: \mathbb{C} \rightarrow \mathbb{C}$ with $\psi(0) = 0$ such that

$$f := \psi \circ F \circ \psi^{-1} \quad (2.3)$$

is a transcendental meromorphic function. Moreover, ψ is conformal on U_0 , and $\psi(U_0)$ is contained in an invariant Baker domain U of f .

Proof That F is K -quasiregular follows from Royden's glueing lemma [2, Lemma 2], or the quasiconformal removability of quasicarcs [5, Theorem 1.19]. Any curve to infinity must intersect U_0 , and therefore F is unbounded on any such curve. In particular, F has no finite asymptotic values. As $F = f_0$ on U_0 , every critical value of F must come from one of the h_j .

Now suppose that the additional conditions hold. Since h_j maps ∂D_j to ∂D_{j+1} in one-to-one fashion, if h_j is not a homeomorphism, then h_j must have at least one pole in D_j . As this happens for infinitely many j , we see that F has infinitely many poles. Let $\omega_0 := F_0^*(0)$ be the complex dilatation of F . By (b), the restriction of ω_0 to

$$W := \bigcup_{j,n \geq 0} F^n(W_j)$$

is forward-invariant under F . As F is meromorphic outside of W , we may extend $\omega_0|_W$ by pull-back to the grand orbit of W . Extending the resulting differential by the standard complex structure, we obtain an F -invariant measurable Beltrami differential ω on all of $\hat{\mathbb{C}}$. Observe that $\omega \equiv 0$ on U_0 .

By the measurable Riemann mapping theorem, there is a quasiconformal homeomorphism $\psi: \mathbb{C} \rightarrow \hat{\mathbb{C}}$ that solves the Beltrami equation for ω . Then ψ is conformal on U_0 . Moreover, the map f defined by (2.3) is meromorphic; it is transcendental since F has infinitely many poles. Let $z \in \psi(U_0)$; say $z = \psi(\zeta)$ with $\zeta \in U_0$. Then

$$\begin{aligned} f_0(z) &= \psi(F(\psi^{-1}(z))) = \psi(F(\zeta)) \in \psi(U_0), \quad \text{and} \\ f_0^n(z) &= \psi(F^n(\psi^{-1}(z))) = \psi(F^n(\zeta)) = \psi(f_0^n(\zeta)) \rightarrow \infty. \end{aligned}$$

So $\psi(U_0)$ is contained in a Baker domain of f , as claimed. $\square$

Proposition 2.2 (Large annuli in U) *For every $\rho > 1$, there is $\mu > 1$ with the following property. Let $f_0(z) = \mu \cdot z$, and let f be obtained from f_0 as in Proposition 2.1. Then f satisfies (1.2) for a sequence $R_j \rightarrow \infty$.*

Proof Recall that U_0 contains arbitrarily large round annuli, each of modulus at least $M := (\log \mu)/2$. If $M \geq \Lambda(\rho)$, where $\Lambda(\rho)$ is the modulus of the Teichmüller annulus [1, §4–11], then any annulus that separates 0 from ∞ and has modulus at least M contains a round annulus of modulus $\log \rho$ centred at zero. Hence we can take $\mu = e^{2\Lambda(\rho)}$. $\square$

3 Proof of Theorem 1.2

To prove Theorem 1.2, we construct a suitable sequence of functions h_j to use in Proposition 2.1. The idea is that each h_j has critical values in D_{j+1} , which are then mapped back to the original critical points by h_{j+1} . We begin with the following, where f_0 is again given by (2.1).

Proposition 3.1 *Let $\Delta = D(\zeta, r) \subset \mathbb{C}$ be a round disc, and let $K > 1$, $0 < \eta < r$, and $0 < \vartheta < \mu \cdot \eta$. For every $a \in \mathbb{C} \setminus f_0(\Delta)$, there is a K -quasiregular map*

$$g: \Delta \rightarrow \hat{\mathbb{C}}$$

such that

- (1) g extends continuously to $\partial\Delta$, where it agrees with f_0 .
- (2) $g(\zeta) = a$.
- (3) g has exactly two critical points, c_1 and c_2 , with $g(c_1) \neq g(c_2)$.
- (4) Each critical point c_j satisfies $g(c_j) \in D(f_0(\zeta), \vartheta)$.
- (5) g is meromorphic on $D(\zeta, \eta)$.
- (6) g is injective (and hence quasiconformal) on $\Delta \setminus D(\zeta, \eta)$. Furthermore, g satisfies $\mu r > |g(z) - f_0(\zeta)| > \vartheta$ for $r > |z - \zeta| \geq \eta$.

Proof Set

$$\alpha := \frac{a}{r\mu} - \frac{\zeta}{r} \neq 0.$$

Let $\varepsilon > 0$, and consider the map

$$\varphi = \varphi_{\alpha, \varepsilon}: \mathbb{D} \rightarrow \hat{\mathbb{C}}; \quad z \mapsto z + \frac{\varepsilon \alpha}{\alpha z + \varepsilon} = z + \frac{\varepsilon}{z - p},$$

where $p = -\varepsilon/\alpha$. Then $\varphi(0) = \alpha$ and $\varphi(p) = \infty$. By direct calculation, φ has two simple critical points at $p \pm \sqrt{\varepsilon}$, with critical values at $p \pm 2\sqrt{\varepsilon}$.

Now define $A: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ to be the affine map taking $\mathbb{D}$ to Δ ; i.e., $A(z) = r \cdot z + \zeta$. Define

$$g_\varepsilon: D(\zeta, \eta) \rightarrow \hat{\mathbb{C}}; \quad z \mapsto \mu \cdot A(\varphi_{\alpha, \varepsilon}(A^{-1}(z))).$$

For all sufficiently small ε , g_ε satisfies (3),(4) and (5). Furthermore,

$$g_\varepsilon(\zeta) = \mu \cdot (r\alpha + \zeta) = a,$$

so g also satisfies (2).

Set $\Delta_\eta := D(\zeta, \eta)$. As $\varepsilon \rightarrow 0$, $g_\varepsilon \rightarrow f_0$ uniformly on a neighbourhood of $\partial\Delta_\eta$. In particular, for sufficiently small ε , g_ε is conformal on a neighbourhood of $\partial\Delta_\eta$, and satisfies

$$\mu r > |g_\varepsilon(z) - f_0(\zeta)| > \vartheta$$

when $z \in \partial\Delta_\eta$. It follows that there is a quasiconformal homeomorphism G that maps the round annulus $\Delta \setminus \overline{\Delta_\eta}$ to the annulus bounded by $f_0(\partial\Delta)$ and $g_\varepsilon(\partial\Delta_\eta)$, and which agrees with f_0 on $\partial\Delta$ and with g_ε on $\partial\Delta_\eta$; see [9]. We define g to agree with g_ε on D_η and with G outside; then g satisfies (1)–(6).

Finally, we claim that the dilatation of G , and thus of g , tends to 1 as $\varepsilon \rightarrow 0$. Indeed, this follows readily from the construction in [9]. Alternatively, for small ε , we can define G directly by linear interpolation on each radius of $\Delta \setminus \overline{\Delta_\eta}$. An elementary calculation shows that the dilatation of G tends to 1. Hence, for sufficiently small ε , g is K -quasiconformal as claimed. $\square$

For the remainder of the paper, let D_j denote the discs introduced in Sect. 2.

Proposition 3.2 *For any $K > 1$, there is a sequence of K -quasiregular maps*

$$h_j: D_j \rightarrow \hat{\mathbb{C}}$$

with the following properties for all j .

- (a) *For every j , the map h_j has at least one critical point.*
- (b) *If c is a critical point of h_j , then $h_j(c) \in D_{j+1}$ and $h_{j+1}(h_j(c)) = c$.*

Furthermore, for each j there are open sets $V_j \subset W_j \subset D_j$ such that

- (c) *$D_j \setminus V_j$ is compact;*
- (d) *h_j is injective on W_j ;*
- (e) *$h_j(z) = f_0(z)$ for $z \in V_j$;*
- (f) *$h_j(W_j) \subset V_{j+1}$;*
- (g) *The dilatation of h_j is supported on W_j .*

Proof The sequence of functions is constructed inductively, in such a way as to ensure the following inductive hypotheses. For $j \geq 0$, let $\mathcal{C}_j \subset D_j$ be the set of critical points of h_j and $\Omega_{j+1} = h_j(\mathcal{C}_j) \subset \hat{\mathbb{C}}$ the set of critical values. The construction will ensure the following inductive hypotheses for $j \geq 0$:

- (A) $\Omega_{j+1} \subset D_{j+1}$.
- (B) $h_j: \mathcal{C}_j \rightarrow \Omega_{j+1}$ is bijective.
- (C) $h_j(W_j) \subset D_{j+1}$ and $\overline{h_j(W_j)} \cap \Omega_{j+1} = \emptyset$.

To anchor the recursive construction, we also set $\Omega_0 := \{\zeta_0\}$. This means that (A) holds also for $j = -1$.

Now suppose that $j \geq 0$ is such that h_i has been constructed for $i < j$, satisfying the inductive hypotheses. To define h_j , let $\omega_j^1, \dots, \omega_j^m$ be the elements of Ω_j . Choose pairwise disjoint small discs $\Delta_j^i = D(\omega_j^i, \rho_j^i) \subseteq D_j$, for $j = 1, \dots, m$; this is possible by (A). If $j > 0$, then by (C) we may also ensure that the closures of the Δ_j^i do not intersect $\overline{h_{j-1}(W_{j-1})}$. Set

$$V_j := D_j \setminus \bigcup_{i=1}^m \overline{\Delta_j^i} \supset h_{j-1}(W_{j-1}). \quad (3.1)$$

For each $i = 1, \dots, m$, apply Proposition 3.1 to $\Delta = \Delta_j^i$, where $a = c_{j-1}^i \in \mathcal{C}_{j-1}$ is such that

$$h_{j-1}(c_{j-1}^i) = \omega_j^i, \quad (3.2)$$

if $j > 0$, and $c_{-1}^1 = 0$ otherwise. We may choose ϑ and $\eta = \eta_j^i$ arbitrary, subject to the conditions given in the proposition. Let g_j^i denote the maps obtained. We define

$$h_j(z) := \begin{cases} f_0(z) & \text{if } z \in V_j; \\ g_j^i(z) & \text{if } z \in \Delta_j^i \end{cases} \quad \text{and} \quad (3.3)$$

$$W_j := D_j \setminus \bigcup_{i=1}^m \overline{D(\omega_j^i, \eta_j^i)} \supset V_j. \quad (3.4)$$

We claim that h_j satisfies (A)–(C). For the remainder of the proof, references to (1)–(6) shall mean the corresponding properties of the g_j^i established in Proposition 3.1.

The critical points and critical values of h_j are exactly those of the maps g_j^i . By (4), the critical values of g_j^i are in $f_0(\Delta_j^i) \subset f_0(D_j) = D_{j+1}$, establishing (A). Furthermore, by (3), g_j^i has exactly two different critical points, with different critical values belonging to $f_0(\Delta_j^i)$. Since different Δ_j^i are pairwise disjoint, and f_0 is injective, claim (B) follows. Finally, (C) is an immediate consequence of (4) and (6).

This completes the construction of the h_j . By (1) and Royden's glueing lemma (or the quasiconformal removability of quasiarcs), the map h_j is indeed a K -quasiregular map. It remains to establish (a)–(g).

- (a) Each h_j^i has exactly two critical values, and $\#\Omega_0 = 1$. It follows inductively that $\#\Omega_j = 2^j$ for all j , and in particular each h_j has at least one critical point.
- (b) If $c \in \mathcal{C}_j$ is a critical point of h_j , then we must have $c = c_j^i$ for some i , by (3.2) and (B). So by (2),

$$h_{j+1}(h_j(c)) = h_{j+1}(h_j(c_j^i)) = g_{j+1}^i(\omega_{j+1}^i) = c_j^i = c.$$

- (c) By definition of V_j in (3.1), $D_j \setminus V_j$ is compact.

- (d) $h_j|_{W_j}$ is injective by (3.4) and (6) .
- (e) By (3.3), $h_j = f_0$ on V_j .
- (f) By (3.1), $V_{j+1} \supset h_j(W_j)$.
- (g) By (5), the dilatation of h_j is supported on W_j . □

Proof of Theorems 1.2 and 1.4 Choose μ as in Proposition 2.2, and let $(h_j)_{j=0}^\infty$ be as in Proposition 3.2. Let F be the quasiregular map defined by (2.2). Then $F^2(c) = c$ for every critical point c of F .

The dilatation of h_j is supported on W_j , and h_j is conformal on $\overline{V_j}$. Moreover, $F^n(W_j) \subset V_{n+j}$ for all $n \geq 1$ and $j \geq 0$, so W_j satisfies the conditions of Proposition 2.1. Let f be the meromorphic function (2.3); then f has a Baker domain U and satisfies $f^2(c) = c$ for every critical point of f . This completes the Proof of Theorem 1.2.

Furthermore, by Proposition 2.2, the function also satisfies (1.2) for a sequence $R_j \rightarrow \infty$, proving Theorem 1.4. □

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