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On the Gibbons' conjecture for equations involving the *p*-Laplacian

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Abstract

In this paper we prove the validity of Gibbons' conjecture for the quasilinear elliptic equation $-\Delta_p u = f(u)$ on $\mathbb{R}^N$. The result holds true for $(2N+2)/(N+2) < p < 2$ and for a very general class of nonlinearity f.

Mathematics Subject Classification 35J62, 35J92, 35B06, 35B50, 35B51

1 Introduction

In this work we are concerned with the study of qualitative properties of weak solutions of class C^1 to the quasilinear elliptic equation

$$-\Delta_p u = f(u) \quad \text{in } \mathbb{R}^N, \tag{$\mathcal{P}$}$$

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where we denote a generic point of $\mathbb{R}^N$ by (x', y) with $x' = (x_1, x_2, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$ and $y = x_N \in \mathbb{R}$, $p > 1$ and $N > 1$. The nonlinear function f will be assumed to satisfy the following assumptions :

$$(h_f) \qquad \begin{cases} f \in C^1([-1,1]), \quad f(-1) = f(1) = 0, \\ f'_+(-1) < 0, \quad f'_-(1) < 0, \\ \mathcal{N}_f := \{t \in [-1,1] \mid f(t) = 0\} \text{ is a finite set.} \end{cases}$$

A very special case covered by our assumptions is the well-known semilinear Allen–Cahn equation

$$-\Delta u = u(1-u^2) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

for which the following conjecture has been stated

Gibbons' conjecture [5] *Assume $N > 1$ and consider a bounded solution $u \in C^2(\mathbb{R}^N)$ of (1.1) such that*

$$\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1,$$

uniformly with respect to x'. Then, is it true that

$$u(x) = \tanh\left(\frac{x_N - \alpha}{\sqrt{2}}\right),$$

for some $\alpha \in \mathbb{R}$?

Gibbons' conjecture was proven independently and with different methods by [2,3,10,11] (see also [12] for further results in the semilinear scalar case and [17] for a result on related semilinear elliptic systems. The case of a uniformly elliptic fully nonlinear operator and the one of the fractional laplacian are treated in [18] (see also [19]).

Here we study Gibbons' conjecture for the quasilinear equation ($\mathcal{P}$). To the best of our knowledge, there are no general results in this framework. This lack of results is mainly due to the fact that, unlike the semilinear case, when working with the singular operator $-\Delta_p(\cdot)$, both the weak and the strong comparison principles might fail. This (possible) failure being caused either by the presence of critical points or by the fact that the nonlinearity f changes sign. Those difficulties are even more magnified by the fact that we are facing a problem on an unbounded domain, the entire euclidean space $\mathbb{R}^N$. Also, in the pure quasilinear case, $p \neq 2$, we cannot exploit the usual arguments and tricks related to the linearity of the Laplace operator. Despite all those problems and difficulties, we are able to study and solve the quasilinear version of Gibbons' conjecture by making use of the celebrated moving planes method which goes back to the papers of Alexandrov [1] and Serrin [25] (see also [4,20]).

Our main result is the following

Theorem 1.1 *Assume $N > 1$, $(2N+2)/(N+2) < p < 2$ and let $u \in C^1(\mathbb{R}^N)$ be a weak solution of ($\mathcal{P}$), such that*

$$|u| \leq 1 \quad on \quad \mathbb{R}^N$$

and

$$\lim_{y\to+\infty} u(x', y) = 1 \quad and \quad \lim_{y\to-\infty} u(x', y) = -1, \tag{1.2}$$

uniformly with respect to $x' \in \mathbb{R}^{N-1}$. *If* f *fulfills* (h_f), *then* u *depends only on* y *and*

$$\partial_y u > 0 \quad in \quad \mathbb{R}^N. \tag{1.3}$$

To get our main result, we first prove a new weak comparison principle for quasilinear equations in half-spaces and then we exploit it to start the moving plane procedure at infinity in the y-direction. Then, by a delicate analysis based on the use of the techniques developed in [7,8] and [14–16], the translation invariance of the considered problem and the method introduced in [10], we obtain the monotonicity of the solution in all the directions of the upper hemi-sphere $\mathbb{S}^{N-1}_+ := \{\nu \in \mathbb{S}^{N-1}_+ \mid (\nu, e_N)\}$. This result, in turn, will provide the desired one-dimensional symmetry result as well as the strict monotonicity.

The paper is organized as follows: In Sect. 2 we recall the definition of weak solution of $(\mathcal{P})$, as well as some results about the strong maximum principle and the comparison principles for nonlinear equations involving the p-Laplace operator. In Sect. 3 we prove a new weak comparison principle in half-spaces. In Sect. 4 we prove the monotonicity of the solution in the y-direction, exploiting the moving plane procedure. In Sect. 5 we prove the one-dimensional symmetry and the strict monotonicity of the solution.

2 Strong maximum principles and strong comparison principles for quasilinear elliptic equations

The aim of this section is to recall some results about the strong comparison principles and the strong maximum principles for quasilinear elliptic equations that will be used several times in the proof of our main theorem. To this end we first recall the definition of weak solution for the quasilinear equation $-\Delta_p u = f(u)$.

Definition 2.1 Let Ω be an open set of $\mathbb{R}^N$, $N \geq 1$. We say that $u \in C^1(\Omega)$ is a *weak subsolution* to

$$-\Delta_p u = f(u) \quad \text{in} \quad \Omega \tag{2.1}$$

if

$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla\varphi)\, dx \leq \int_\Omega f(u)\varphi\, dx \qquad \forall \varphi \in C^\infty_c(\Omega),\ \varphi \geq 0. \tag{2.2}$$

Similarly, we say that $u \in C^1(\Omega)$ is a *weak supersolution* to (2.1) if

$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla\varphi)\, dx \geq \int_\Omega f(u)\varphi\, dx \qquad \forall \varphi \in C^\infty_c(\Omega),\ \varphi \geq 0. \tag{2.3}$$

Finally, we say that $u \in C^1(\Omega)$ is a weak *solution* of equation (2.1), if (2.2) and (2.3) hold. Sometimes for brevity, we shall use the term 'solution' to indicate a weak solution to the considered problem.

The first result that we are going to present is the classical strong maximum principle due to Vazquez [28] (see also the book of Pucci and Serrin [22])

Theorem 2.2 (Strong Maximum Principle and Höpf's Lemma, [22,28]) *Let $u \in C^1(\Omega)$ be a non-negative weak solution to*

$$-\Delta_p u + cu^q = g \geq 0 \quad in \quad \Omega$$

with $1 < p < +\infty$, $q \geq p-1$, $c \geq 0$ and $g \in L^\infty_{loc}(\Omega)$. If $u \not\equiv 0$, then $u > 0$ in Ω. Moreover for any point $x_0 \in \partial\Omega$ where the interior sphere condition is satisfied, and such that $u \in C^1(\Omega) \cup \{x_0\}$ and $u(x_0) = 0$ we have that $\partial_\nu u > 0$ for any inward directional derivative (this means that if y approaches x_0 in a ball $B \subseteq \Omega$ that has x_0 on its boundary, then $\lim_{y\to x_0} \frac{u(y)-u(x_0)}{|y-x_0|} > 0$).

It is very simple to guess that in the quasilinear case, maximum and comparison principles are not equivalent; for this reason we need also to recall the classical version of the strong comparison principle for quasilinear elliptic equations

Theorem 2.3 (Classical Strong Comparison Principle, [6,22])
Let $u, v \in C^1(\Omega)$ be two solutions to

$$-\Delta_p w = f(w) \quad in \quad \Omega \tag{2.4}$$

such that $u \leq v$ in Ω, with $1 < p < +\infty$ and let $\mathcal{Z} = \{x \in \Omega \mid |\nabla u(x)| + |\nabla v(x)| = 0\}$. If $x_0 \in \Omega \setminus \mathcal{Z}$ and $u(x_0) = v(x_0)$, then $u = v$ in the connected component of $\Omega \setminus \mathcal{Z}$ containing x_0.

For the proof of this result we suggest [6]. The main feature of Theorem 2.3 is that it holds far from the critical set. Now we present a result which holds true, under stronger assumptions, on the entire domain Ω.

Theorem 2.4 (Strong Comparison Principle, [7,23]) *Let $u, v \in C^1(\overline{\Omega})$ be two solutions to (2.4), where Ω is a bounded domain of $\mathbb{R}^N$ and $\frac{2N+2}{N+2} < p < +\infty$. Assume that at least one of the following two conditions (f_u),(f_v) holds:*

(f_u): *either*

$$f(u(x)) > 0 \quad in \quad \overline{\Omega} \tag{2.5}$$

or

$$f(u(x)) < 0 \quad in \quad \overline{\Omega}; \tag{2.6}$$

(f_v): *either*

$$f(v(x)) > 0 \quad in \quad \overline{\Omega} \tag{2.7}$$

or

$$f(v(x)) < 0 \quad in \quad \overline{\Omega}. \tag{2.8}$$

Suppose furthermore that

$$u \leq v \quad in \quad \Omega. \tag{2.9}$$

Then $u \equiv v$ *in* Ω *unless*

$$u < v \quad in \quad \Omega. \tag{2.10}$$

Proof The proof of this result follows by the same arguments in [7,15,23,24]. Note in fact that under the assumption ($\boldsymbol{f_u}$) or ($\boldsymbol{f_v}$), it follows that $|\nabla u|^{-1}$ or $|\nabla v|^{-1}$ has the summability properties exposed by Theorem 3.1 in [24]. Then the weighted Sobolev inequality is in force, see e.g. Theorem 8 in [15].

Now, it is sufficient to note that the Harnack comparison inequality given by Corollary 3.2 in [7] holds true, since the proof it is only based on the weighted Sobolev inequality.

Finally it is standard to see that the Strong Comparison Principle follows by the weak comparison Harnack inequality (that it is based on the Moser-iteration scheme [21,27]), see Theorem 1.4 in [7]. □

Let us now recall that the linearized operator at a fixed solution w of (2.4), $L_w(v, \varphi)$ (with weight $\rho = |\nabla w|^{p-2}$), is well defined in the set $\hat{\Omega} := \Omega \setminus \{x \in \Omega \mid |\nabla w| = 0\}$ for every smooth functions $v \in C^1(\hat{\Omega})$ and $\varphi \in C^1_c(\hat{\Omega})$ by

$$\begin{aligned} L_w(v, \varphi) \equiv \int_{\hat{\Omega}} & |\nabla w|^{p-2}(\nabla v, \nabla \varphi) + (p-2)|\nabla w|^{p-4}(\nabla w, \nabla v)(\nabla w, \nabla \varphi) \\ & - f'(w)v\varphi \, dx. \end{aligned} \tag{2.11}$$

Here below we recall two versions of the strong maximum principle for the linearized equation (2.15) that we shall use in our proofs. The first result holds far from the critical set:

Theorem 2.5 (Classical Strong Maximum Principle for the Linearized Operator, [22]) *Let* $u \in C^1(\overline{\Omega})$ *be a solution to problem* (2.4)*, with* $1 < p < +\infty$*. Let* $\eta \in \mathbb{S}^{N-1}$ *and let us assume that for any connected domain* $\Omega' \subset \Omega \setminus \{x \in \Omega \mid |\nabla u(x)| = 0\}$

$$\partial_\eta u \geq 0 \quad in \quad \Omega'. \tag{2.12}$$

Then $\partial_\eta u \equiv 0$ *in* Ω' *unless*

$$\partial_\eta u > 0 \quad in \quad \Omega'. \tag{2.13}$$

Next we recall a more general result which holds true on the entire domain Ω under the same assumptions of Theorem 2.4. To this end we have to define the linearized operator (2.11) at a fixed solution w of (2.4) in the whole domain Ω. We point out that under the hypothesis (2.16) or (2.17), thanks to [24, Theorem 3.1], the weight $\rho = |\nabla w|^{p-2} \in L^1(\Omega)$ if $p > (2N+2)/(N+2)$. Hence $L_w(v, \varphi)$, is well defined, for every v and φ in the weighted Sobolev space $H^{1,2}_\rho(\Omega)$ with[1] $\rho = |\nabla w|^{p-2}$ by

$$\begin{aligned} L_w(v, \varphi) \equiv \int_{\Omega} & |\nabla w|^{p-2}(\nabla v, \nabla \varphi) + (p-2)|\nabla w|^{p-4}(\nabla w, \nabla v)(\nabla w, \nabla \varphi) \\ & - f'(w)v\varphi \, dx. \end{aligned}$$

[1] We recall that, for $\rho \in L^1(\Omega)$, the space $H^{1,2}_\rho(\Omega)$ is defined as the completion of $C^1(\Omega)$ (or $C^\infty(\Omega)$) with the norm

$$\|v\|_{H^{1,2}_\rho} = \|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega,\rho)}, \tag{2.14}$$

We point out that the linearized operator is also well defined if $v \in L^2(\Omega)$, $|\nabla w|^{p-2}\nabla v \in L^2(\Omega)$ and $\varphi \in W^{1,2}(\Omega)$.

Finally $v \in H^{1,2}_\rho(\Omega)$ is a weak solution of the linearized operator if

$$L_w(v,\varphi) = 0 \qquad \forall \varphi \in H^{1,2}_{0,\rho}(\Omega). \tag{2.15}$$

For future use we recall that, as it follows by the regularity results in [8,23,24], the directional derivatives of the solution $\partial_\eta u$ ($\eta \in \mathbb{S}^{N-1}$) belong to the weighted Sobolev space $H^{1,2}_\rho(\Omega)$ and fulfill (2.15). Moreover (from [8,23,24]), it follows also that $|\nabla u|^{p-2}\nabla u \in W^{1,2}_{loc}(\Omega)$ so that if $\varphi \in W^{1,2}_c(\Omega)$ (with compact support), the linearized operator $L_w(\partial_\eta u, \varphi)$ is still well defined.

Theorem 2.6 (Strong Maximum Principle for the Linearized Operator, [7,23]) *Let $u \in C^1(\Omega)$ be a solution to problem (2.4), with $\frac{2N+2}{N+2} < p < +\infty$. Assume that either*

$$f(u(x)) > 0 \quad in \quad \overline{\Omega} \tag{2.16}$$

or

$$f(u(x)) < 0 \quad in \quad \overline{\Omega}. \tag{2.17}$$

If $\eta \in \mathbb{S}^{N-1}$ and $\partial_\eta u \geq 0$ in Ω, then either $\partial_\eta u \equiv 0$ in Ω or $\partial_\eta u > 0$ in Ω.

We conclude this section by the following

Remark 2.7 Here we want to point out the following two properties satisfied by any weak solution of class C^1 to ($\mathcal{P}$), with $1 < p < 2$, and such that $|u| \leq 1$ on $\mathbb{R}^N$. They will be used several times throughout the paper.

- Since $p < 2$, we have that : either $|u| < 1$ on $\mathbb{R}^N$ or $u \equiv \pm 1$ on $\mathbb{R}^N$. To see this, let us define the function $w = 1 - u$. It follows that w is a non-negative weak solution of class C^1 to
$$-\Delta_p w = g(w) \quad \text{in } \mathbb{R}^N,$$
with $g(w) := -f(1-w)$. Moreover $g \in C^1([0,2])$ (see assumptions (h_f) in the Introduction) and in particular it is a Lipschitz-continuous function on the interval $[0,2]$ with $g(1) = 0$. Since $p < 2$, it is therefore possible to find a constant $c > 0$ such that
$$-\Delta_p w + cw^{p-1} = g(w) + cw^{p-1} \geq 0 \qquad in \quad \mathbb{R}^N.$$
By Theorem 2.2, it follows that either $w \equiv 0$ in $\mathbb{R}^N$ or $w > 0$ in $\mathbb{R}^N$, that is, either $u \equiv 1$ on $\mathbb{R}^N$ or $u < 1$ on $\mathbb{R}^N$. Once $u < 1$ on $\mathbb{R}^N$ we can apply the same

Footnote 1 continued
where

$$\|\nabla v\|^2_{L^2(\Omega,\rho)} := \int_\Omega \rho(x)|\nabla v(x)|^2 dx.$$

Moreover, the space $H^{1,2}_{0,\rho}(\Omega)$ is consequently defined as the closure of $C^1_c(\Omega)$ (or $C^\infty_c(\Omega)$), w.r.t. the norm (2.14).

argument to the non-negative function $w = u + 1$ to get: either $u \equiv -1$ on $\mathbb{R}^N$ or $u > -1$ on $\mathbb{R}^N$. As claimed.

- By classical regularity results [9,26] and since $\|f(u)\|_{L^\infty(\mathbb{R}^N)} \le \|f\|_{L^\infty([-1,1])}$ (here we have used the assumption $\|u\|_{L^\infty(\mathbb{R}^N)} \le 1$),
we deduce that : given $R \in (0, 1)$ there exist $\alpha \in (0, 1)$ and $C > 0$, depending only on N, p, R and $\|f\|_{L^\infty([-1,1])}$, so that

$$\|\nabla u\|_{L^\infty(\mathbb{R}^N)} \le C, \tag{2.18}$$

$$|\nabla u(x) - \nabla u(y)| \le C \left(\frac{|x-y|}{R}\right)^\alpha, \tag{2.19}$$

for every $x_0 \in \mathbb{R}^N$ and any $x, y \in B_R(x_0)$. In particular, $u \in C^{1,\alpha}(\mathbb{R}^N)$ and

$$\|u\|_{C^{1,\alpha}(\mathbb{R}^N)} \le C_0,$$

for some constant $C_0 > 0$ depending only on N, p and $\|f\|_{L^\infty([-1,1])}$. Indeed, taking $R = 1/2$, we get $\|\nabla u\|_{L^\infty(\mathbb{R}^N)} \le C(\frac{1}{2})$ by (2.18). Therefore, if $|x-y| < 1$, we have $|u(x) - u(y)| \le C(\frac{1}{2})|x-y| \le C(\frac{1}{2})|x-y|^\alpha$ while, if $|x-y| \ge 1$, we clearly have $|u(x) - u(y)| \le 2\|u\|_{L^\infty(\mathbb{R}^N)} \le 2\|u\|_{L^\infty(\mathbb{R}^N)}|x-y|^\alpha \le 2|x-y|^\alpha$. Hence the α-Hölder seminorm $[u]_{0,\alpha,\mathbb{R}^N} \le \max\{2, C(\frac{1}{2})\}$. Furthermore, if $|x - y| < 1$, then $x, y \in B_{\frac{1}{2}}(x_0)$, where x_0 is the middle point of the segment joining x and y. We can then apply (2.19) to get $|\nabla u(x) - \nabla u(y)| \le 2^\alpha C(\frac{1}{2})|x-y|^\alpha \le 2C(\frac{1}{2})|x-y|^\alpha$. Finally, if $|x-y| \ge 1$, then $|\nabla u(x) - \nabla u(y)| \le 2\|\nabla u\|_{L^\infty(\mathbb{R}^N)} \le 2C(\frac{1}{2})|x-y|^\alpha$. Hence $[\nabla u]_{0,\alpha,\mathbb{R}^N} \le 2C(\frac{1}{2})$. The desired claims follow.

3 Preliminary results

In this section we shall denote by Σ any (affine) open half-space of $\mathbb{R}^N$ of the form

$$\Sigma := \mathbb{R}^{N-1} \times (a, b),$$

where either $a = -\infty$ and $b \in \mathbb{R}$, or $a \in \mathbb{R}$ and $b = +\infty$.

We also recall some known inequalities which will be used in this section. For any $\eta, \eta' \in \mathbb{R}^N$ with $|\eta| + |\eta'| > 0$ there exists positive constants C_1, C_2, C_3 depending only on p such that

$$[\eta - \eta'] \ge C_1(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|^2,$$

$$||\eta|^{p-2}\eta - |\eta'|^{p-2}\eta'| \le C_2(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|, \tag{3.1}$$

$$||\eta|^{p-2}\eta - |\eta'|^{p-2}\eta'| \le C_3|\eta - \eta'|^{p-1} \quad \text{if} \quad 1 < p \le 2.$$

The first result that we need is a weak comparison principle between a subsolution and a supersolution to ($\mathcal{P}$) ordered on the boundary of some open half-space Σ of $\mathbb{R}^N$. We prove the following

Proposition 3.1 *Assume $N > 1$, $p > 1$ and $f \in C^1(\mathbb{R})$. Let $u, v \in C^{1,\alpha}_{loc}(\overline{\Sigma})$ such that $|\nabla u|, |\nabla v| \in L^\infty(\Sigma)$ and*

$$\begin{cases} -\Delta_p u \le f(u) & \text{in } \Sigma, \\ -\Delta_p v \ge f(v) & \text{in } \Sigma, \\ u \le v & \text{on } \partial\Sigma, \end{cases} \tag{3.2}$$

where Σ is the open half-space $\mathbb{R}^{N-1} \times (-\infty, b)$. Moreover, let us assume that there are $\delta > 0$, sufficiently small, and $L > 0$ such that

$$f'(t) < -L \quad \text{in} \quad [-1, -1+\delta], \tag{3.3}$$

$$-1 \le u \le -1+\delta \quad \text{in} \quad \Sigma. \tag{3.4}$$

Then

$$u \le v \quad \text{in } \Sigma. \tag{3.5}$$

The same result is true if $\Sigma = \mathbb{R}^{N-1} \times (a, +\infty)$ and (3.3) and (3.4) are replaced by

$$f'(t) < -L \quad \text{in} \quad [1-\delta, 1] \quad \text{and} \quad 1-\delta \le v \le 1 \quad \text{in} \quad \Sigma.$$

Proof We prove the result when (3.3) and (3.4) are in force. The other case is similar.

We distinguish two cases:

Case 1 $1 < p < 2$. We set

$$\psi := w^\alpha \varphi_R^{\alpha+1}, \tag{3.6}$$

where $\alpha > 1$, $R > 0$ large, $w := (u - v)^+$ and φ_R is a standard cutoff function such that $0 \le \varphi_R \le 1$ on $\mathbb{R}^N$, $\varphi_R = 1$ in B_R, $\varphi_R = 0$ outside B_{2R}, with $|\nabla \varphi_R| \le 2/R$ in $B_{2R} \setminus B_R$. Let us define $\mathcal{C}(2R) := \Sigma \cap B_{2R} \cap \text{supp}(\omega)$. First of all we notice that $\psi \in W_0^{1,p}(\mathcal{C}(2R))$. By density arguments we can take ψ as test function in (2.2) and (2.3), so that, subtracting we obtain

$$\begin{aligned} &\alpha \int_{\mathcal{C}(2R)} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla w) w^{\alpha-1} \varphi_R^{\alpha+1}\, dx \\ &\le -(\alpha+1) \int_{\mathcal{C}(2R)} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla \varphi_R) w^\alpha \varphi_R^{\alpha+1}\, dx \\ &\quad + \int_{\mathcal{C}(2R)} [f(u) - f(v)] w^\alpha \varphi_R^{\alpha+1}\, dx\,. \end{aligned} \tag{3.7}$$

From (3.7), using (3.1) and noticing that f is decreasing in $[-1, -1+\delta]$, we obtain

$$\begin{aligned}
&\alpha C_1 \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 w^{\alpha-1} \varphi_R^{\alpha+1}\, dx \\
&\leq \alpha \int_{\mathcal{C}(2R)} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla w) w^{\alpha-1} \varphi_R^{\alpha+1}\, dx \\
&\leq -(\alpha+1) \int_{\mathcal{C}(2R)} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla \varphi_R) w^{\alpha} \varphi_R^{\alpha}\, dx \\
&\quad + \int_{\mathcal{C}(2R)} f'(\xi)(u-v)^+ w^{\alpha} \varphi_R^{\alpha+1}\, dx \\
&\leq (\alpha+1) C_3 \int_{\mathcal{C}(2R)} |\nabla w|^{p-1} |\nabla \varphi_R| w^{\alpha} \varphi_R^{\alpha}\, dx - L \int_{\mathcal{C}(2R)} (u-v)^+ w^{\alpha} \varphi_R^{\alpha+1}\, dx,
\end{aligned} \tag{3.8}$$

where ξ is some point that belongs to (v, u). Hence, recalling also that $|\nabla u|, |\nabla v| \in L^\infty(\Sigma)$, we deduce

$$\begin{aligned}
&\alpha C_1 \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 w^{\alpha-1} \varphi_R^{\alpha+1}\, dx \\
&\leq (\alpha+1) C_3 \int_{\mathcal{C}(2R)} |\nabla w|^{p-1} |\nabla \varphi_R| w^{\alpha} \varphi_R^{\alpha}\, dx - L \int_{\mathcal{C}(2R)} w^{\alpha+1} \varphi_R^{\alpha+1}\, dx \qquad (3.9)\\
&\leq (\alpha+1) C \int_{\mathcal{C}(2R)} |\nabla \varphi_R| w^{\alpha} \varphi_R^{\alpha}\, dx - L \int_{\mathcal{C}(2R)} w^{\alpha+1} \varphi_R^{\alpha+1}\, dx
\end{aligned}$$

where $C = C(p, \|\nabla u\|_{L^\infty(\Sigma)}, \|\nabla v\|_{L^\infty(\Sigma)})$. Exploiting the weighted Young inequality with exponents $\alpha+1$ and $(\alpha+1)/\alpha$ in (3.9), we obtain

$$\begin{aligned}
&\alpha C_1 \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 w^{\alpha-1} \varphi_R^{\alpha+1}\, dx \\
&\leq \frac{C}{\sigma^{\alpha+1}} \int_{\mathcal{C}(2R)} |\nabla \varphi_R|^{\alpha+1}\, dx + \alpha C \sigma^{\frac{\alpha+1}{\alpha}} \int_{\mathcal{C}(2R)} w^{\alpha+1} \varphi_R^{\alpha+1}\, dx \\
&\quad - L \int_{\mathcal{C}(2R)} w^{\alpha+1} \varphi_R^{\alpha+1}\, dx \\
&\leq \frac{C}{\sigma^{\alpha+1}} \int_{\mathcal{C}(2R)} |\nabla \varphi_R|^{\alpha+1}\, dx + \left(\alpha C \sigma^{\frac{\alpha+1}{\alpha}} - L\right) \int_{\mathcal{C}(2R)} w^{\alpha+1} \varphi_R^{\alpha+1}\, dx \\
&\leq \frac{2^{\alpha+1} C}{\sigma^{\alpha+1} R^{\alpha-(N-1)}} + \left(\alpha C \sigma^{\frac{\alpha+1}{\alpha}} - L\right) \int_{\mathcal{C}(2R)} w^{\alpha+1} \varphi_R^{\alpha+1}\, dx.
\end{aligned}$$

Now taking $\alpha > N-1$, if we choose $\sigma = \sigma(p, \alpha, L, N, \|\nabla u\|_{L^\infty(\Sigma)}, \|\nabla v\|_{L^\infty(\Sigma)}) > 0$ sufficiently small so that

$$\alpha C \sigma^{\frac{\alpha+1}{\alpha}} - L < 0,$$

we obtain

$$\int_{\mathcal{C}(R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 w^{\alpha-1}\, dx \leq \frac{\tilde{C}}{\alpha\sigma^{\alpha+1} R^{\alpha-(N-1)}}. \tag{3.10}$$

Passing to the limit in (3.10) for $R \to +\infty$, by Fatou's Lemma we have

$$\int_{\Sigma} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 w^{\alpha-1}\, dx \leq 0.$$

This implies that $u \leq v$ in Σ.
Case 2 $p \geq 2$. We set

$$\psi := w\varphi_R^2, \tag{3.11}$$

where $R > 0$, $w := (u-v)^+$ and φ_R is the standard cutoff function defined above. First of all we notice that $\psi \in W_0^{1,p}(B_{2R})$. Let us define $\mathcal{C}(2R) := \Sigma \cap B_{2R} \cap \mathrm{supp}(\omega)$. By density arguments we can take ψ as test function in (2.2) and (2.3), so that, subtracting we obtain

$$\begin{aligned}
&\int_{\mathcal{C}(2R)} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla w)\varphi_R^2\, dx \\
&\quad \leq -2\int_{\mathcal{C}(2R)} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla \varphi_R) w\varphi_R\, dx \\
&\qquad + \int_{\mathcal{C}(2R)} [f(u) - f(v)] w\varphi_R^2\, dx\,.
\end{aligned} \tag{3.12}$$

From (3.12), using (3.1) and that $f'(u) \leq -L$ in $[-1, -1+\delta]$, we obtain

$$\begin{aligned}
&C_1 \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \varphi_R^2\, dx \\
&\quad \leq \int_{\mathcal{C}(2R)} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla w)\varphi_R^2\, dx \\
&\quad \leq -2\int_{\mathcal{C}(2R)} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla \varphi_R) w\varphi_R\, dx \\
&\qquad + \int_{\mathcal{C}(2R)} f'(\xi)(u-v)^+ w\varphi_R^2\, dx \\
&\quad \leq 2C_2 \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w|\, |\nabla \varphi_R| w\varphi_R\, dx \\
&\qquad - L\int_{\mathcal{C}(2R)} (u-v)^+ w\varphi_R^2\, dx,
\end{aligned} \tag{3.13}$$

where ξ is some point that belongs to (v, u). Using in (3.13) the weighted Young inequality (and the fact that $|\nabla u|, |\nabla v| \in L^\infty(\Sigma)$), we obtain

$$
\begin{aligned}
&C_1 \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \varphi_R^2 \, dx \\
&\leq 2C_2 \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{\frac{p-2}{2}} |\nabla w| \, (|\nabla u| + |\nabla v|)^{\frac{p-2}{2}} |\nabla \varphi_R| w \varphi_R \, dx \\
&\quad - L \int_{\mathcal{C}(2R)} w^2 \varphi_R^2 \, dx \\
&\leq C_2 \sigma \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \, dx \\
&\quad + \frac{C_2}{\sigma} \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla \varphi_R|^2 w^2 \varphi_R^2 \, dx \\
&\quad - L \int_{\mathcal{C}(2R)} w^2 \varphi_R^2 \, dx. \\
&\leq C_2 \sigma \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \, dx \\
&\quad + \left(\frac{C}{\sigma R^2} - L \right) \int_{\mathcal{C}(2R)} w^2 \varphi_R^2 \, dx,
\end{aligned}
\tag{3.14}
$$

where $C = C(p, \|\nabla u\|_{L^\infty(\Sigma)}, \|\nabla v\|_{L^\infty(\Sigma)})$ is a positive constant. Hence, up to redefine the constants, we have

$$
\begin{aligned}
\int_{\mathcal{C}(R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \, dx \leq C\sigma \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \, dx \\
+ \frac{1}{C_1} \left(\frac{C}{\sigma R^2} - L \right) \int_{\mathcal{C}(2R)} w^2 \varphi_R^2 \, dx.
\end{aligned}
\tag{3.15}
$$

Now we set

$$
\mathcal{L}(R) := \int_{\mathcal{C}(R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \, dx.
$$

By our assumption,$|\nabla u|, |\nabla v| \in L^\infty(\Sigma)$, it follows that $\mathcal{L}(R) \leq \dot{C} R^N$ for every $R > 0$ and for some $\dot{C} = \dot{C}(p, \|\nabla u\|_{L^\infty(\Sigma)}, \|\nabla v\|_{L^\infty(\Sigma)})$. Moreover, in equation (3.15), we take $\sigma = \sigma(p, N, \|\nabla u\|_{L^\infty(\Sigma)}, \|\nabla v\|_{L^\infty(\Sigma)}) > 0$ sufficiently small so that $C\sigma < 1/2^N$. Finally we fix $R_0 > 0$ such that

$$
\frac{C}{\sigma R^2} - L < 0
$$

for every $R \geq R_0$. Therefore by (3.15) we deduce that

$$
\begin{cases}
\mathcal{L}(R) \leq \vartheta \mathcal{L}(2R) & \forall R \geq R_0 \\
\mathcal{L}(R) \leq \dot{C} R^N & \forall R \geq R_0,
\end{cases}
\tag{3.16}
$$

where $\vartheta := C\sigma < 1/2^N$. By applying Lemma 2.1 in [14] it follows that $\mathcal{L}(R) = 0$ for all $R \geq R_0$. Hence $u \leq v$ in Σ. □

Let us recall a weak comparison principle in narrow domains that will be an essential tool in the proof of Theorem 1.1.

Theorem 3.2 [16] *Let* $1 < p < 2$ *and* $N > 1$. *Fix* $\lambda_0 > 0$ *and* $L_0 > 0$. *Consider* $a, b \in \mathbb{R}$, *with* $a < b$, $\tau, \epsilon > 0$ *and set*

$$\Sigma_{(a,b)} := \mathbb{R}^{N-1} \times (a, b).$$

Let $u, v \in C^{1,\alpha}_{loc}(\overline{\Sigma}_{(a,b)})$ *such that* $\|u\|_\infty + \|\nabla u\|_\infty \leq L_0$, $\|v\|_\infty + \|\nabla v\|_\infty \leq L_0$, f *fulfills* ($\boldsymbol{h_f}$) *and*

$$\begin{cases} -\Delta_p u \leq f(u) & \text{in } \Sigma_{(a,b)} \\ -\Delta_p v \geq f(v) & \text{in } \Sigma_{(a,b)} \\ u \leq v & \text{on } \partial\mathcal{S}_{(\tau,\epsilon)}, \end{cases} \tag{3.17}$$

where the open set $\mathcal{S}_{(\tau,\epsilon)} \subseteq \Sigma_{(a,b)}$ *is such that*

$$\mathcal{S}_{(\tau,\epsilon)} = \bigcup_{x' \in \mathbb{R}^{N-1}} I^{\tau,\epsilon}_{x'},$$

and the open set $I^{\tau,\epsilon}_{x'} \subseteq \{x'\} \times (a, b)$ *has the form*

$$I^{\tau,\epsilon}_{x'} = A^\tau_{x'} \cup B^\epsilon_{x'}, \text{ with } |A^\tau_{x'} \cap B^\epsilon_{x'}| = \emptyset$$

and, for x' *fixed,* $A^\tau_{x'}, B^\epsilon_{x'} \subset (a, b)$ *are measurable sets such that*

$$A^\tau_{x'}| \leq \tau \quad \text{and} \quad B^\epsilon_{x'} \subseteq \{x_N \in \mathbb{R} \mid |\nabla u(x', x_N)| < \epsilon, \ |\nabla v(x', x_N)| < \epsilon\}.$$

Then there exist

$$\tau_0 = \tau_0(N, p, a, b, L_0) > 0$$

and

$$\epsilon_0 = \epsilon_0(N, p, a, b, L_0) > 0$$

such that, if $0 < \tau < \tau_0$ *and* $0 < \epsilon < \epsilon_0$, *it follows that*

$$u \leq v \quad \text{in } \mathcal{S}_{(\tau,\epsilon)}.$$

The proof of this result is contained in [16, Theorem 1.6], where the authors proved the same result for a more general class of operators and nonlinearities and also in the presence of a first order term.

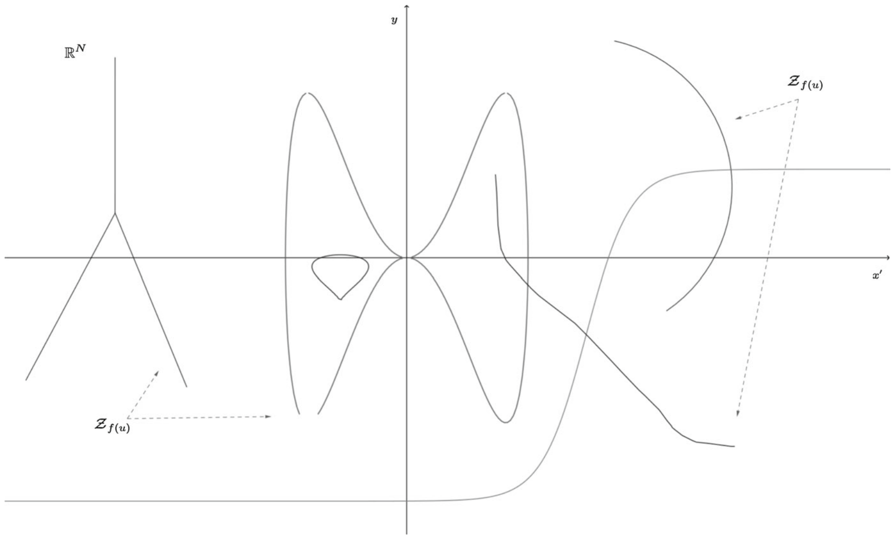

Fig. 1 The set $\mathcal{Z}_{f(u)}$

4 Monotonicity with respect to x_N

The purpose of this section consists in showing that all the non-trivial solutions u to ($\mathcal{P}$) that satisfies (1.2) are increasing in the x_N direction. Since in our problem the right hand side depends only on u, it is possible to define the following set

$$\mathcal{Z}_{f(u)} := \{x \in \mathbb{R}^N \mid u(x) \in \mathcal{N}_f\}.$$

Without any a priori assumption on the behaviour of ∇u, the set $\mathcal{Z}_{f(u)}$ may be very wild, see Fig. 1.

We state now the main result of this paragraph in the following

Proposition 4.1 *Under the assumptions of Theorem 1.1, we have that*

$$\partial_{x_N} u \geq 0 \quad in\ \mathbb{R}^N \qquad and \qquad \partial_{x_N} u > 0 \quad in\ \mathbb{R}^N \setminus \mathcal{Z}_{f(u)}. \tag{4.1}$$

Proposition 4.1 will be proved at the end of Sect. 4 using different preliminary results. To this end we start by proving a lemma that we will use repeatedly in the sequel of the work.

Let us define the upper hemisphere

$$\mathbb{S}_+^{N-1} := \{\nu \in \mathbb{S}^{N-1} \mid (\nu, e_N) > 0\}. \tag{4.2}$$

Lemma 4.2 *Let $\mathcal{U}$ a connected component of $\mathbb{R}^N \setminus \mathcal{Z}_{f(u)}$, $\eta \in \mathbb{S}_+^{N-1}$ and let us assume that $\partial_\eta u \geq 0$ in $\mathcal{U}$. Then*

$$\partial_\eta u > 0 \quad in\ \mathcal{U}.$$

Proof Using Theorem 2.6 we deduce that either $\partial_\eta u > 0$ in $\mathcal{U}$ or $\partial_\eta u \equiv 0$ in $\mathcal{U}$. For contradiction, assume that $\partial_\eta u \equiv 0$ in $\mathcal{U}$. Pick $P_0 \in \mathcal{U}$ and let us define

$$r(t) = P_0 + t\eta, \quad t \in \mathbb{R}$$

and

$$t_0 = \inf \{t \in \mathbb{R} \,:\, r(\vartheta) \in \mathcal{U}, \ \forall \vartheta \in (t, 0]\}\,. \tag{4.3}$$

We note that the infimum in (4.3) is well defined, since by definition the connected component $\mathcal{U}$ is an open set, and that $t_0 \in [-\infty, 0)$.

In the case $t_0 = -\infty$, we deduce that $u(P_0) = -1$. Indeed u is constant on $r(t)$ for $t \in (-\infty, 0]$ (recall that $\partial_\eta u \equiv 0$ in $\mathcal{U}$) and (1.2) holds. But this is a contradiction, see Remark 2.7.

In the case $t_0 > -\infty$, we deduce that $r(t_0) \in \mathcal{Z}_{f(u)}$ and therefore $f(u(r(t_0))) = f(u(P_0 + t_0\eta)) = 0$. But u is constant on $r(t)$ for $t_0 \le t \le 0$, which implies that $f(u(P_0)) = f(u(P_0 + t_0\eta)) = 0$, namely $P_0 \in \mathcal{Z}_{f(u)}$. The latter clearly contradicts the assumption $P_0 \in \mathcal{U}$. Therefore $\partial_\eta u > 0$ in $\mathcal{U}$ as desired. □

Some arguments used in the proofs of the results of this section, are based on a nontrivial modification of the moving plane method. Let us recall some notations. We define the half-space Σ_λ and the hyperplane T_λ by

$$\Sigma_\lambda := \{x \in \mathbb{R}^N \mid x_N < \lambda\}, \qquad T_\lambda := \partial\Sigma_\lambda = \{x \in \mathbb{R}^N \mid x_N = \lambda\} \tag{4.4}$$

and the reflected function $u_\lambda(x)$ by

$$u_\lambda(x) = u_\lambda(x', x_N) := u(x', 2\lambda - x_N) \quad \text{in } \mathbb{R}^N.$$

We also define the critical set $\mathcal{Z}_{\nabla u}$ by

$$\mathcal{Z}_{\nabla u} := \{x \in \mathbb{R}^N \mid \nabla u(x) = 0\}. \tag{4.5}$$

The first step in the proof of the monotonicity is to get a property concerning the local symmetry regions of the solution, namely any $C \subseteq \Sigma_\lambda$ such that $u \equiv u_\lambda$ in C.

Having in mind these notations we are able to prove the following:

Proposition 4.3 *Under the assumption of Theorem 1.1, let us assume that u is a solution to $(\mathcal{P})$ satisfying (1.2), such that*

(i) *u is monotone non-decreasing in Σ_λ and*
(ii) *$u \le u_\lambda$ in Σ_λ.*

Then $u < u_\lambda$ in $\Sigma_\lambda \setminus \mathcal{Z}_{f(u)}$.

Proof By (1.2), given $0 < \delta_0 < 1$ there exists $M_0 = M_0(\delta_0) > 0$, with $\lambda > -M_0$, such that $u(x) = u(x', x_N) < -1 + \delta_0$ in $\{x_N < -M_0\}$ and $u_\lambda(x) = u(x', 2\lambda - x_N) > 1 - \delta_0$ in $\{x_N < -M_0\}$. We fix δ_0 sufficiently small such that $f'(u) < -L$ in $\{x_N < -M_0\}$, for some $L > 0$. Arguing by contradiction, let us assume that there

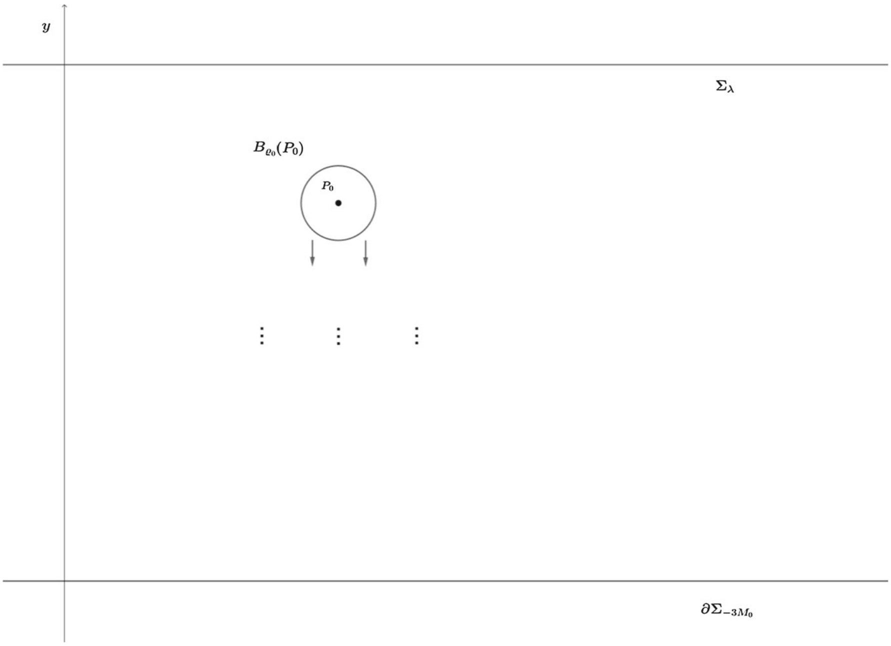

Fig. 2 The slided ball $B_{\rho_0}(P_0)$

exists $P_0 = (x'_0, x_{N,0}) \in \Sigma_\lambda \setminus \mathcal{Z}_{f(u)}$ such that $u(P_0) = u_\lambda(P_0)$. Let $\mathcal{U}_0$ the connected component of $\Sigma_\lambda \setminus \mathcal{Z}_{f(u)}$ containing P_0. By Theorem 2.4, since $u(P_0) = u_\lambda(P_0)$, we deduce that $\mathcal{U}_0$ is a local symmetry region, i.e. $u \equiv u_\lambda$ in $\mathcal{U}_0$.

We notice that, by construction, $u < u_\lambda$ in Σ_{-M_0}, since $u(x) < -1 + \delta_0$ and $u_\lambda(x) = u(x', 2\lambda - x_N) > 1 - \delta_0$ in Σ_{-M_0}. Since $\mathcal{U}_0$ is an open set of $\Sigma_\lambda \setminus \mathcal{Z}_{f(u)}$ (and also of $\mathbb{R}^N$) there exists $\rho_0 = \rho_0(P_0) > 0$ such that

$$B_{\rho_0}(P_0) \subset \mathcal{U}_0. \tag{4.6}$$

We can slide B_{ρ_0} in $\mathcal{U}_0$, towards to $-\infty$ in the y-direction and keeping its centre on the line $\{x' = x'_0\}$ (see Fig. 2), until it touches for the first time $\partial\mathcal{U}_0$ at some point $z_0 \in \mathcal{Z}_{f(u)}$. In Fig. 3, we show some possible examples of *first contact point* with the set $\mathcal{Z}_{f(u)}$.

Now we consider the function

$$w_0(x) := u(x) - u(z_0)$$

and we observe that $w_0(x) \neq 0$ for every $x \in B_{\rho_0}(\hat{P}_0)$, where $\hat{P}_0$ is the new centre of the slided ball. In fact, if this is not the case there would exist a point $\bar{z} \in B_{\rho_0}(\hat{P}_0)$ such that $w_0(\bar{z}) = 0$, but this is in contradiction with the fact that $\mathcal{U}_0 \cap \mathcal{Z}_{f(u)} = \emptyset$. We have to distinguish two cases. Since $p < 2$ and f is locally Lipschitz, we have that

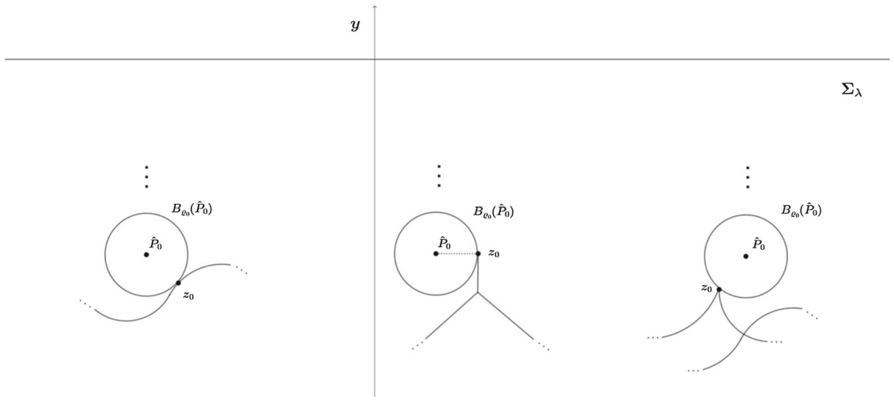

Fig. 3 The first contact point z_0

Case 1 If $w_0(x) > 0$ in $B_{\rho_0}(\hat{P}_0)$, then

$$\begin{cases} \Delta_p w_0 \le C w_0^{p-1} & \text{in } B_{\rho_0}(\hat{P}_0) \\ w_0 > 0 & \text{in } B_{\rho_0}(\hat{P}_0) \\ w(z_0) = 0 & z_0 \in \partial B_{\rho_0}(\hat{P}_0), \end{cases}$$

where C is a positive constant.

Case 2 If $w_0(x) < 0$ in $B_{\rho_0}(\hat{P}_0)$, setting $v_0 = -w_0$ we have

$$\begin{cases} \Delta_p v_0 \le C v_0^{p-1} & \text{in } B_{\rho_0}(\hat{P}_0) \\ v_0 > 0 & \text{in } B_{\rho_0}(\hat{P}_0) \\ v_0(z_0) = 0 & z_0 \in \partial B_{\rho_0}(\hat{P}_0), \end{cases}$$

where C is a positive constant.

In both cases, by the Höpf boundary lemma (see e.g. [22,28]), it follows that $|\nabla w(z_0)| = |\nabla u(z_0)| \neq 0$.

Using the Implicit Function Theorem we deduce that the set $\{u = u(z_0)\}$ is a smooth manifold near z_0. Now we want to prove that

$$u_{x_N}(z_0) > 0$$

and actually that the set $\{u = u(z_0)\}$ is a graph in the y-direction near the point z_0. By our assumption we know that $u_{x_N}(z_0) := u_y(z_0) \geq 0$. According to [7,8] (see also Sect. 2), the linearized operator of $(\mathcal{P})$ is well defined

$$L_u(u_y, \varphi) \equiv \int_{\Sigma_\lambda} [|\nabla u|^{p-2}(\nabla u_y, \nabla \varphi) + (p-2)|\nabla u|^{p-4}(\nabla u, \nabla u_y)(\nabla u, \nabla \varphi)]\, dx \\ - \int_{\Sigma_\lambda} f'(u) u_y \varphi \, dx \tag{4.7}$$

for every $\varphi \in C^1_c(\Sigma_\lambda)$. Moreover u_y satisfies the linearized equation (2.15), i.e.

$$L_u(u_y, \varphi) = 0 \quad \forall \varphi \in C^1_c(\Sigma_\lambda). \tag{4.8}$$

Let us set $z_0 = (z'_0, y_0)$. We have two possibilities: $u_y(z_0) = 0$ or $u_y(z_0) > 0$.

Claim: We show that the case $u_y(z_0) = 0$ is not possible.

If $u_y(z_0) = 0$, then $u_y(x) \equiv 0$ in all $B_{\hat{\rho}}(z_0)$ for some positive $\hat{\rho}$; to prove this we use the fact that $|\nabla u(z_0)| \neq 0$, $u \in C^{1,\alpha}$ and that Theorem 2.5 holds.

By construction of z_0 there exists $\varepsilon_1 > 0$ such that every point $z \in \mathcal{S}_1 := \{(z'_0, t) \in \mathcal{U}_0 : y_0 < t < y_0 + \varepsilon_1\}$ has the following properties:

(1) $z \in \overline{\mathcal{U}_0}$, since the ball is sliding along the segment $\mathcal{S}_1$;
(2) $z \notin \partial\mathcal{U}_0$, since z_0 is the first contact point with $\partial\mathcal{U}_0$.

In particular, for every $z \in \mathcal{S}_1$ we have

$$z \in \overline{\mathcal{U}_0} \setminus \partial\mathcal{U}_0 = \mathcal{U}_0. \tag{4.9}$$

Since $|\nabla u(z_0)| \neq 0$ and $u \in C^{1,\alpha}$, by Theorem 2.5 it follows that there exists $0 < \varepsilon_2 < \varepsilon_1$ such that

$$u_y(x) = 0 \quad \forall x \in B_{\varepsilon_2}(z_0).$$

Let us consider $\mathcal{S}_2 := \{(z'_0, t) \in \mathcal{U}_0 : y_0 < t < y_0 + \varepsilon_2\}$; by definition $\mathcal{S}_2 \subset \mathcal{S}_1$ and every point of $\mathcal{S}_2$ belongs also to $\mathcal{Z}_{f(u)}$, since $u(z) = u(z_0)$ for every $z \in \mathcal{S}_2$ and $z_0 \in \mathcal{Z}_{f(u)}$ by our assumptions. But this gives a contradiction with (4.9).

From what we have seen above, we have $|\nabla u(z_0)| \neq 0$ and hence there exists a ball $B_r(z_0)$ where $|\nabla u(x)| \neq 0$ for every $x \in B_r(z_0)$. By Theorem 2.3 it follows that $u \equiv u_\lambda$ in $B_r(z_0)$ namely $u \equiv u_\lambda$ in a neighborhood of the point $z_0 \in \partial\mathcal{U}_0$. Since $u_y(z_0) > 0$ and $\mathcal{N}_f$ is finite

$$B_r(z_0) \cap \big((\Sigma_\lambda \setminus \mathcal{Z}_{f(u)}) \setminus \mathcal{U}_0\big) \neq \emptyset$$

and $u_y(x) > 0$ in $B_r(z_0)$, as consequence, the set $\{u = u(z_0)\}$ is a graph in the y-direction in a neighborhood of the point z_0. Now we have to distinguish two cases:

Case 1 $u(z_0) = \min[\mathcal{N}_f \setminus \{-1\}]$.
Define the sets

$$\mathcal{C}_1 := \{x \in \mathbb{R}^N : x' \in (B_r(z_0) \cap \{y := y_0\}) \quad \text{and} \quad u(x) < u(z_0)\}$$

$$\mathcal{C}_2 := B_r(z_0) \cup ((B_r(z_0) \cap \{y := y_0\}) \times (-\infty, y_0))$$

and

$$\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2.$$

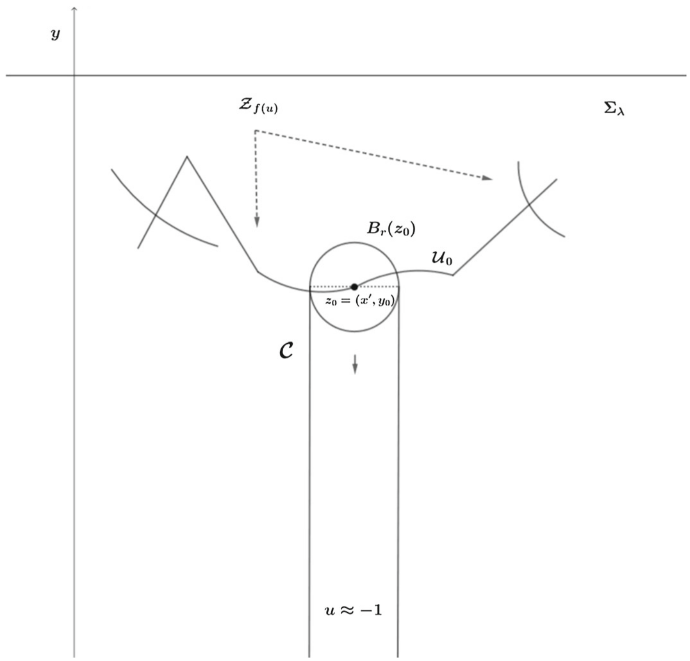

Fig. 4 Case 1: $u(z_0) = \min[\mathcal{N}_f \setminus \{-1\}]$

We observe that $\mathcal{C}$ is an open unbounded path-connected set (actually a deformed cylinder), see Fig. 4. Since $f(u(z_0))$ has the right sign, by Theorem 2.4 it follows that $u \equiv u_\lambda$ in $\mathcal{C}$ and this in contradiction with the uniform limit conditions (1.2).

Case 2 $u(z_0) > \min[\mathcal{N}_f \setminus \{-1\}]$.
In this case the open ball $B_r(z_0)$ must intersect another connected component (i.e. $\not\equiv \mathcal{U}_0$) of $\Sigma_\lambda \setminus \mathcal{Z}_{f(u)}$, such that $u \equiv u_\lambda$ in a such component, see Fig. 5. Here we used the fact that near the (new) first contact point, the corresponding level set is a graph in the y-direction. Now, it is clear that repeating a finite number of times the argument leading to the existence of the touching point z_0, we can find a touching point z_m such that

$$u(z_m) = \min[\mathcal{N}_f \setminus \{-1\}].$$

The contradiction then follows exactly as in Case 1.

Hence $u < u_\lambda$ in $\Sigma_\lambda \setminus \mathcal{Z}_{f(u)}$. □

To prove Proposition 4.1 we need the following result:

Lemma 4.4 *Under the assumption of Theorem 1.1, let u be a solution to ($\mathcal{P}$). Then there exist $M_0 = M_0(p, f, N, \|\nabla u\|_{L^\infty(\mathbb{R}^N)}) > 0$ sufficiently large such that for every*

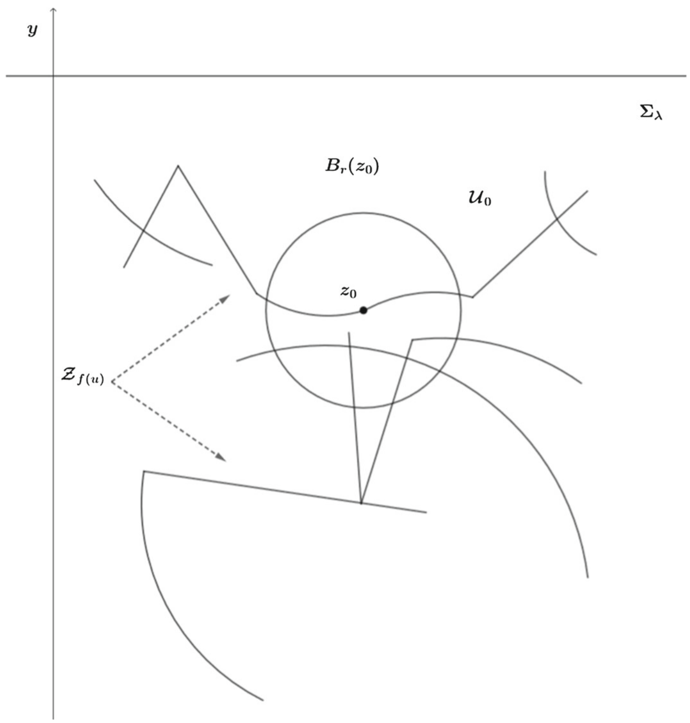

Fig. 5 Case 2: $u(z_0) > \min[\mathcal{N}_f \setminus \{-1\}]$

$M \geq M_0$ there exits a constant $C^* = C^*(M) > 0$ such that

$$|\nabla u| \geq \partial_{x_N} u \geq C^* > 0 \quad in \ \{-M - 1 < x_N < -M + 1\}. \tag{4.10}$$

Proof Performing the moving plane procedure, using (1.2) and (h_f), by the Proposition 3.1 with $v = u_\lambda$ and $\Sigma = \Sigma_\lambda$, we infer that there exists a constant $M_0 = M_0(p, f, N, \|\nabla u\|_{L^\infty(\mathbb{R}^N)}) > 0$ such that $\partial_{x_N} u \geq 0$ in $\{x_N < -M_0 + 1\}$. Now we can assume

$$\mathcal{Z}_{f(u)} \cap \{x_N < -M_0 + 1\} = \emptyset,$$

then by Theorem 2.6 it follows that $\partial_{x_N} u > 0$ in $\{x_N < -M_0 + 1\}$, since the case $\partial_{x_N} u = 0$ would imply a contradiction, i.e. $u(x) = -1$ in $\{x_N < -M_0 + 1\}$. We observe that in particular it holds $|\nabla u| \geq \partial_{x_N} u > 0$ in $\{-M_0 - 1 < x_N < -M_0 + 1\}$. We want to prove that for all $M \geq M_0$, there exists $C^* = C^*(M) > 0$ such that $\partial_{x_N} u \geq C^* > 0$ in $\{-M - 1 < x_N < -M + 1\}$.

Arguing by contradiction let us assume that there exists a sequence of point $P_n = (x_n', x_{N,n})$, with $-M - 1 < x_{N,n} < -M + 1$ for every $n \in \mathbb{N}$, such that $\partial_{x_N} u(P_n) \to 0$ as $n \to +\infty$ in $\{-M - 1 < x_N < -M + 1\}$. Up to subsequences, let us assume that

$$x_{N,n} \to \bar{x}_N \text{ with } -M - 1 \leq \bar{x}_N \leq -M + 1.$$

Let us now define

$$\tilde{u}_n(x', x_N) := u(x' + x'_n, x_N)$$

so that $\|\tilde{u}_n\|_\infty = \|u\|_\infty \leq 1$. By the second claim of Remark 2.7, we have that

$$\|\tilde{u}_n\|_{C^{1,\alpha}(\mathbb{R}^N)} \leq C_0$$

for some $0 < \alpha < 1$. By Ascoli's Theorem we have

$$\tilde{u}_n \overset{C^{1,\alpha'}_{loc}(\mathbb{R}^N)}{\longrightarrow} \tilde{u}$$

up to subsequences, for $\alpha' < \alpha$. By construction $\partial_{x_N}\tilde{u} \geq 0$ and $\partial_{x_N}\tilde{u}(0, \bar{x}_N) = 0$, hence by Theorem 2.5 it follows that $\partial_{x_N}\tilde{u} = 0$ in $\{-M - 1 < x_N < -M + 1\}$ and therefore $\partial_{x_N}\tilde{u} = 0$ in all $\{(x', x_n) : x_N < -M + 1\}$ by Theorem 2.6, since $\mathcal{Z}_{f(u)} \cap \{x_N < -M_0 + 1\} = \emptyset$. This gives a contradiction (by Theorem 2.5) with the fact that $\lim\limits_{x_N \to -\infty} u(x', x_N) = -1$ (this implies that $\lim\limits_{x_N \to -\infty} \tilde{u}(x', x_N) = -1$), see Remark 2.7. □

With the notation introduced above, we set

$$\Lambda := \{\lambda \in \mathbb{R} \mid u \leq u_t \text{ in } \Sigma_t \ \forall t < \lambda\}. \tag{4.11}$$

Note that, by Proposition 3.1 (with $v = u_t$), it follows that $\Lambda \neq \emptyset$, hence we can define

$$\bar{\lambda} := \sup \Lambda. \tag{4.12}$$

Moreover it is important to say that by the continuity of u and u_λ, it follows that

$$u \leq u_{\bar{\lambda}} \quad \text{in } \Sigma_{\bar{\lambda}}.$$

The proof of the fact that $u(x', x_N)$ is monotone increasing in the x_N-direction in the entire space $\mathbb{R}^N$ is done once we show that $\bar{\lambda} = +\infty$. To do this we assume by contradiction that $\bar{\lambda} < +\infty$, and we prove a crucial result, which allows us to localize the support of $(u - u_{\bar{\lambda}})^+$. This localization, that we are going to obtain, will be useful to apply the weak comparison principle given by Proposition 3.1 and Theorem 3.2.

Proposition 4.5 *Under the assumption of Theorem 1.1, let u be a solution to ($\mathcal{P}$). Assume that $\bar{\lambda} < +\infty$ (see (4.12)) and set*

$$W_\varepsilon := (u - u_{\bar{\lambda}+\varepsilon})\chi_{\{x_N \leq \bar{\lambda}+\varepsilon\}}.$$

Let $M, \kappa > 0$ be such that $M > 2|\bar{\lambda}|$. Then for all $\mu \in (0, (\bar{\lambda} + M)/2)$ there exists $\bar{\varepsilon} > 0$ such that for every $0 < \varepsilon < \bar{\varepsilon}$

$$\operatorname{supp} W_\varepsilon^+ \subset \{x_N \leq -M\} \cup \{\bar{\lambda} - \mu \leq x_N \leq \bar{\lambda} + \varepsilon\} \cup \{|\nabla u| \leq \kappa\}. \tag{4.13}$$

Proof Assume by contradiction that (4.13) is false, so that there exists $\mu > 0$ in such a way that, given any $\bar{\varepsilon} > 0$, we find $0 < \varepsilon \leq \bar{\varepsilon}$ so that there exists a corresponding $x_\varepsilon = (x'_\varepsilon, x_{N,\varepsilon})$ such that

$$u(x'_\varepsilon, x_{N,\varepsilon}) \geq u_{\bar{\lambda}+\varepsilon}(x'_\varepsilon, x_{N,\varepsilon}),$$

with $x_\varepsilon = (x'_\varepsilon, x_{N,\varepsilon})$ belonging to the set

$$\{(x', x_N) \in \mathbb{R}^N \ : \ M < x_{N,\varepsilon} < \bar{\lambda} - \mu\}$$

and such that $|\nabla u(x_\varepsilon)| \geq \kappa$.

Taking $\bar{\varepsilon} = 1/n$, then there exists $\varepsilon_n \leq 1/n$ going to zero, and a corresponding sequence

$$x_n = (x'_n, x_{N,n}) = (x'_{\varepsilon_n}, x_{N,\varepsilon_n})$$

such that

$$u(x'_n, x_{N,n}) \geq u_{\bar{\lambda}+\varepsilon_n}(x'_n, x_{N,n})$$

with $-M < x_{N,n} < \bar{\lambda} - \mu$. Up to subsequences, let us assume that

$$x_{N,n} \to \bar{x}_N \text{ with } -M \leq \bar{x}_N \leq \bar{\lambda} - \mu.$$

Let us define

$$\tilde{u}_n(x', x_N) := u(x' + x'_n, x_N)$$

so that $\|\tilde{u}_n\|_\infty = \|u\|_\infty \leq 1$. By the second claim of Remark 2.7, we have that

$$\|\tilde{u}_n\|_{C^{1,\alpha}(\mathbb{R}^N)} \leq C_0$$

for some $0 < \alpha < 1$. By Ascoli's Theorem we have

$$\tilde{u}_n \overset{C^{1,\alpha'}_{loc}(\mathbb{R}^N)}{\longrightarrow} \tilde{u}$$

up to subsequences, for $\alpha' < \alpha$. By construction it follows that

- $\tilde{u} \leq \tilde{u}_{\bar{\lambda}}$ in $\Sigma_{\bar{\lambda}}$;
- $\tilde{u}(0, \bar{x}_N) = \tilde{u}_{\bar{\lambda}}(0, \bar{x}_N)$;
- $|\nabla \tilde{u}(0, \bar{x}_N)| \geq \kappa$.

Since $|\nabla \tilde{u}(0, \bar{x}_N)| \geq \kappa$ there exists $\rho > 0$ and a ball $B_\rho(0, \bar{x}_N) \subset \Sigma_{\bar{\lambda}}$ such that $|\nabla u(x)| \neq 0$ for every $x \in B_\rho(0, \bar{x}_N)$. Now, if $\tilde{u}(0, \bar{x}_N) \in \mathcal{Z}_{f(u)}$, since $\tilde{u}$ is non constant in $B_\rho(0, \bar{x}_N)$, there exists $P_0 \in B_\rho(0, \bar{x}_N)$ such that $u(P_0) \notin \mathcal{Z}_{f(u)}$. By Theorem 2.3 it follows that

$$\tilde{u} \equiv \tilde{u}_{\bar{\lambda}} \quad \text{in } B_\rho(0, \bar{x}_N). \tag{4.14}$$

On the other hand, by Proposition 4.3 it follows that

$$\tilde{u} < \tilde{u}_{\bar{\lambda}} \qquad \text{in } \Sigma_{\bar{\lambda}} \setminus \mathcal{Z}_{f(u)}.$$

This gives a contradiction with (4.14). Hence we have (4.13). □

Proof of Proposition 4.1 Let us assume by contradiction that $\bar{\lambda} < +\infty$, see (4.12). Let $\hat{M} > 0$ be such that Proposition 3.1 and Lemma 4.4 apply. Let $C^* = C^*(\hat{M})$ be the constant given in Lemma 4.4. By Proposition 4.5 (choose $M = 4\hat{M} + 1$ there, redefining $\hat{M}$ if necessary) we have that

$$\mathit{supp}\, W_\varepsilon^+ \subset \{x_N \le -4\hat{M} - 1\} \cup \{-4\hat{M} + 1 \le x_N \le \bar{\lambda} + \varepsilon\}, \tag{4.15}$$

where $W_\varepsilon := (u - u_{\bar{\lambda}+\varepsilon})\chi_{\{x_N \le \bar{\lambda}+\varepsilon\}}$. In particular, to get (4.15), we choose κ in Proposition 4.5 such that $2\kappa = C^*$. Then we deduce that

$$u \le u_{\bar{\lambda}+\varepsilon} \text{ in } \{(x, x_N) \in \mathbb{R}^N \,:\, -4\hat{M} - 1 < x_N < -4\hat{M} + 1\}. \tag{4.16}$$

Using (4.16), we can apply Proposition 3.1 in $\{x_N < -4\hat{M} - 1\}$ and therefore, together Lemma 4.4 and Proposition 4.5, we actually deduce

$$\mathit{supp}\, W_\varepsilon^+ \subset \{-4\hat{M} + 1 \le x_N \le \bar{\lambda} + \varepsilon\}.$$

In particular, if we look to (4.13), we deduce that $\mathit{supp}\, W_\varepsilon^+$ must belong to the set

$$A := \Big\{\{\bar{\lambda} - \mu \le x_N \le \bar{\lambda} + \varepsilon\} \cup \{|\nabla u| \le \kappa\}\Big\} \cap \Big\{x_N \ge -4\hat{M} + 1\Big\}.$$

We now apply Theorem 3.2 in the set A. Let us choose (in Theorem 3.2)

$$L_0 = 1 + \|\nabla u\|_{L^\infty(\mathbb{R}^N)}$$

and take $\tau_0 = \tau_0(p, \bar{\lambda}, \hat{M}, N, L_0) > 0$ and $\epsilon_0 = \epsilon_0(p, \bar{\lambda}, \hat{M}, N, L_0) > 0$ as in Theorem 3.2. Let μ, ε in Proposition 4.5 such that $2(\mu + \varepsilon) < \tau_0$ and let us redefine κ eventually such that $\kappa := \min\{C^*/2, \epsilon_0\}$. We finally apply Theorem 3.2 concluding that actually $W_\varepsilon^+ = 0$ in the set A. This gives a contradiction, in view of the definition (4.12) of $\bar{\lambda}$. Consequently we deduce that $\bar{\lambda} = +\infty$. This implies the monotonicity of u, that is $\partial_{x_N} u \ge 0$ in $\mathbb{R}^N$. By Theorem 2.6, it follows that

$$\partial_{x_N} u > 0 \quad \text{in } \mathbb{R}^N \setminus \mathcal{Z}_{f(u)},$$

since by Lemma 4.2, the case $\partial_{x_N} u \equiv 0$ in some connected component, say $\mathcal{U}$, of $\mathbb{R}^N \setminus \mathcal{Z}_{f(u)}$ can not hold. □

5 1-Dimensional symmetry

In this section we pass from the monotonicity in x_N to the monotonicity in all the directions of the upper hemisphere $\mathbb{S}_+^{N-1}$ defined in (4.2). We refer to [10] for the case of the Laplacian operator, where in the proof the linearity of the operator was crucial. Here we have to take into account the singular nature and the nonlinearity of the operator p-Laplacian.

Lemma 5.1 *Under the same assumption of Theorem 1.1, given $\rho > 0$ and $k > 0$, we define*

$$\Sigma_k^\rho := \{x \in \mathbb{R}^N \mid -k < x_N < k\} \cap \{|\nabla u| > \rho\}.$$

Assume $\eta \in \mathbb{S}_+^{N-1}$ and suppose that

$$\partial_\eta u \geq 0 \quad in\ \mathbb{R}^N \quad and \quad \partial_\eta u > 0 \quad in\ \mathbb{R}^N \setminus \mathcal{Z}_{f(u)}. \tag{5.1}$$

Then, there exists an open neighbourhood $\mathcal{O}_\eta$ of η in $\mathbb{S}_+^{N-1}$, such that

$$\partial_\nu u = (\nabla u, \nu) > 0 \quad in\ \Sigma_k^\rho, \tag{5.2}$$

for every $\nu \in \mathcal{O}_\eta$.

Proof Arguing by contradiction let us assume that there exist two sequences $\{P_m\} \in \mathbb{R}^N$ and $\{\nu_m\} \in \mathbb{S}_+^{N-1}$ such that, for every $m \in \mathbb{N}$ we have that $P_m = (x'_m, x_{N,m}) \in \Sigma_k^\rho$, $|(\nu_m, \eta) - 1| < 1/m$ and $\partial_{\nu_m} u(P_m) \leq 0$. Since $-k < x_{N,m} < k$ for every $m \in \mathbb{N}$, then up to subsequences $x_{N,m} \to \bar{x}_N$. Now, let us define

$$\tilde{u}_m(x', x_N) := u(x' + x'_m, x_N)$$

so that $\|\tilde{u}_m\|_\infty = \|u\|_\infty \leq 1$. By the second claim of Remark 2.7, we have that

$$\|\tilde{u}_m\|_{C^{1,\alpha}(\mathbb{R}^N)} \leq C_0.$$

By Ascoli's Theorem, via a standard diagonal process, we have, up to subsequences

$$\tilde{u}_m \overset{C^{1,\alpha'}_{loc}(\mathbb{R}^N)}{\longrightarrow} \tilde{u},$$

for some $0 < \alpha' < \alpha$.

By uniform convergence and (5.1) it follows that

$$\partial_\eta \tilde{u}(0, \bar{x}_N) = 0 \quad \text{and} \quad |\nabla \tilde{u}(0, \bar{x}_N)| \geq \rho.$$

- If $P_0 := (0, \bar{x}_N) \in \mathcal{Z}_{f(\tilde{u})}$, since $|\nabla \tilde{u}(0, \bar{x}_N)| \geq \rho$, then there exists a ball $B_r(P_0)$ such that $|\nabla \tilde{u}(x)| \neq 0$ for every $x \in B_r(P_0)$. By Theorem 2.5, applied having in mind that $|\nabla \tilde{u}(x)| \neq 0$ in $B_r(P_0)$, it follows that $\partial_\eta \tilde{u}(x) = 0$ for every $x \in B_r(P_0)$. In particular $\partial_\eta \tilde{u}(x) = 0$ for every $x \in B_r(P_0) \cap \left(\Sigma_k^\rho \setminus \mathcal{Z}_{f(\tilde{u})}\right)$, hence by Theorem 2.6 we deduce that $\partial_\eta \tilde{u} \equiv 0$ in the connected component $\mathcal{U}$ of $\Sigma_k^\rho \setminus \mathcal{Z}_{f(\tilde{u})}$ containing $B_r(P_0)$ (possibly redefining r), but this is in contradiction with Lemma 4.2.
- If $P_0 \in \Sigma_k^\rho \setminus \mathcal{Z}_{f(\tilde{u})}$ by Theorem 2.6 it follows that $\partial_\eta \tilde{u} > 0$ in the connected component of $\mathbb{R}^N \setminus \mathcal{Z}_{f(\tilde{u})}$ containing the point P_0. Indeed the case $\partial_\eta \tilde{u} \equiv 0$ in the connected component of $\mathbb{R}^N \setminus \mathcal{Z}_{f(\tilde{u})}$ containing P_0 can not hold since Lemma 4.2.

Hence we deduce (5.2). □

Having in mind the previous lemma, now we are able to prove the monotonicity in a small cone of direction around η in the entire space.

Proposition 5.2 *Under the assumption of Theorem 1.1, assume $\eta \in \mathbb{S}_+^{N-1}$ such that $\partial_\eta u > 0$ in $\mathbb{R}^N \setminus \mathcal{Z}_{f(u)}$. Then, there exists an open neighbourhood $\mathcal{O}_\eta$ of η in $\mathbb{S}_+^{N-1}$, such that*

$$\partial_\nu u = u_\nu \geq 0 \quad in\ \mathbb{R}^N \quad and \quad \partial_\nu u = u_\nu > 0 \quad in\ \mathbb{R}^N \setminus \mathcal{Z}_{f(u)}, \tag{5.3}$$

for every $\nu \in \mathcal{O}_\eta$.

Proof We fix $\tilde{\delta} > 0$ and let $k = k(\tilde{\delta}) > 0$ be such that $u < -1 + \tilde{\delta}$ in $\{x_N < -k\}$, $u > 1 - \tilde{\delta}$ in $\{x_N > k\}$ and (3.3) holds in $\{|x_N| > k\}$. By Lemma 5.1 it follows that for all $\rho > 0$ one has

$$\text{supp}\,(u_\nu^-) \subseteq (\{|x_N| \geq k\} \cup (\{-k < x_N < k\} \cap \{|\nabla u| \leq \rho\}))\,.$$

For simplicity of exposition we set

$$A := \{|x_N| \geq k\} \quad \text{and} \quad D := (\{-k < x_N < k\} \cap \{|\nabla u| \leq \rho\})\,.$$

Our claim is to show that $u_\nu^- = 0$ in $A \cup D$. In order to do this we split the proof in two part.

Step 1. We show that $u_\nu^- = 0$ in A.

We set

$$\varphi := (u_\nu^-)^\alpha \varphi_R^2 \chi_{\mathcal{A}(2R)} \tag{5.4}$$

where $\alpha > 1$, $R > 0$ large, $\mathcal{A}(2R) := A \cap B_{2R}$ and φ_R is a standard cutoff function such that $0 \leq \varphi_R \leq 1$ on $\mathbb{R}^N$, $\varphi_R = 1$ in B_R, $\varphi_R = 0$ outside B_{2R}, with $|\nabla \varphi_R| \leq 2/R$ in $B_{2R} \setminus B_R$. First of all we notice that φ belongs to $W_0^{1,2}(\mathcal{A}(2R))$ (we remark that by the regularity results [8,23,24] it follows that $u \in W_{loc}^{2,2}(\mathbb{R}^N)$, if $p < 3$). To see this, use the definition of φ_R and note that by Lemma 4.4 and Lemma 5.1, it follows that $u_\nu^- = 0$ on the hyperplanes $|x_N| = k$, namely on ∂A.

According to [7,8] (see also Sect. 2) the linearized operator is well defined

$$L_u(u_\nu,\varphi)\equiv\int_{\mathbb{R}^N}[|\nabla u|^{p-2}(\nabla u_\nu,\nabla\varphi)+(p-2)\ \ |\nabla u|^{p-4}(\nabla u,\nabla u_\nu)(\nabla u,\nabla\varphi)]\,dx \\ -\int_{\mathbb{R}^N} f'(u)u_\nu\varphi\,dx \tag{5.5}$$

for every $\varphi\in W_c^{1,2}(\mathbb{R}^N)$. Moreover it satisfies the following equation

$$L_u(u_\nu,\varphi)=0\quad\forall\varphi\in W_c^{1,2}(\mathbb{R}^N). \tag{5.6}$$

Taking $\varphi\in W_0^{1,2}(\mathcal{A}(2R))\cap W_c^{1,2}(\mathbb{R}^N)$ defined in (5.4) in the previous equation, we obtain

$$\begin{aligned}&\alpha\int_{\mathcal{A}(2R)}|\nabla u|^{p-2}(\nabla u_\nu,\nabla u_\nu^-)(u_\nu^-)^{\alpha-1}\varphi_R^2\\&\quad+2\int_{\mathcal{A}(2R)}|\nabla u|^{p-2}(\nabla u_\nu,\nabla\varphi_R)(u_\nu^-)^{\alpha}\varphi_R\\&\quad+\alpha(p-2)\int_{\mathcal{A}(2R)}|\nabla u|^{p-4}(\nabla u,\nabla u_\nu)(\nabla u,\nabla u_\nu^-)(u_\nu^-)^{\alpha-1}\varphi_R^2\,dx\\&\quad+2(p-2)\int_{\mathcal{A}(2R)}|\nabla u|^{p-4}(\nabla u,\nabla u_\nu)(\nabla u,\nabla\varphi_R)(u_\nu^-)^{\alpha}\varphi_R\,dx\\&=\int_{\mathcal{A}(2R)}f'(u)(u_\nu^-)^{\alpha+1}\varphi_R^2\,dx\end{aligned} \tag{5.7}$$

Making some computations we obtain

$$\begin{aligned}&\alpha\int_{\mathcal{A}(2R)}|\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\varphi_R^2\,dx\\&\quad=-2\int_{\mathcal{A}(2R)}|\nabla u|^{p-2}(\nabla u_\nu^-,\nabla\varphi_R)(u_\nu^-)^{\alpha}\varphi_R\,dx\\&\qquad+\alpha(2-p)\int_{\mathcal{A}(2R)}|\nabla u|^{p-4}(\nabla u,\nabla u_\nu^-)^2(u_\nu^-)^{\alpha-1}\varphi_R^2\,dx\\&\qquad+2(2-p)\int_{\mathcal{A}(2R)}|\nabla u|^{p-4}(\nabla u,\nabla u_\nu^-)(\nabla u,\nabla\varphi_R)(u_\nu^-)^{\alpha}\varphi_R\,dx\\&\qquad+\int_{\mathcal{A}(2R)}f'(u)(u_\nu^-)^{\alpha+1}\varphi_R^2\,dx\\&\quad\le\alpha(2-p)\int_{\mathcal{A}(2R)}|\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\varphi_R^2\,dx\\&\qquad+2(3-p)\int_{\mathcal{A}(2R)}|\nabla u|^{p-2}|\nabla u_\nu^-|\,|\nabla\varphi_R|(u_\nu^-)^{\alpha}\varphi_R\,dx\\&\qquad+\int_{\mathcal{A}(2R)}f'(u)(u_\nu^-)^{\alpha+1}\varphi_R^2\,dx.\end{aligned} \tag{5.8}$$

Now it is possible to rewrite (5.8) as follows

$$
\begin{aligned}
&\alpha(p-1)\int_{\mathcal{A}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\varphi_R^2\,dx \\
&\quad\le 2(3-p)\int_{\mathcal{A}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|\,|\nabla\varphi_R|(u_\nu^-)^{\alpha}\varphi_R\,dx \\
&\qquad+\int_{\mathcal{A}(2R)} f'(u)(u_\nu^-)^{\alpha+1}\varphi_R^2\,dx.
\end{aligned}
\tag{5.9}
$$

Exploiting the weighted Young inequality we obtain

$$
\begin{aligned}
&\alpha(p-1)\int_{\mathcal{A}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\varphi_R^2\,dx \\
&\quad\le 2(3-p)\int_{\mathcal{A}(2R)} |\nabla u|^{\frac{p-2}{2}}|\nabla u_\nu^-|\,(u_\nu^-)^{\frac{\alpha-1}{2}}|\nabla u|^{\frac{p-2}{2}}|\nabla\varphi_R|(u_\nu^-)^{\frac{\alpha+1}{2}}\varphi_R\,dx \\
&\qquad+\int_{\mathcal{A}(2R)} f'(u)(u_\nu^-)^{\alpha+1}\varphi_R^2\,dx \\
&\quad\le \sigma(3-p)\int_{\mathcal{A}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\,dx \\
&\qquad+\frac{3-p}{\sigma}\int_{\mathcal{A}(2R)} |\nabla u|^{p-2}|\nabla\varphi_R|^2(u_\nu^-)^{\alpha+1}\varphi_R^2\,dx \\
&\qquad+\int_{\mathcal{A}(2R)} f'(u)(u_\nu^-)^{\alpha+1}\varphi_R^2\,dx.
\end{aligned}
\tag{5.10}
$$

Since $u_\nu = (\nabla u, \nu)$, where $\|\nu\| = 1$, we have

$$
\begin{aligned}
&\alpha(p-1)\int_{\mathcal{A}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\varphi_R^2\,dx \\
&\quad\le \sigma(3-p)\int_{\mathcal{A}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\,dx \\
&\qquad+\frac{3-p}{\sigma}\int_{\mathcal{A}(2R)} |\nabla u|^{p-1}|\nabla\varphi_R|^2(u_\nu^-)^{\alpha}\varphi_R^2\,dx \\
&\qquad+\int_{\mathcal{A}(2R)} f'(u)(u_\nu^-)^{\alpha+1}\varphi_R^2\,dx \\
&\quad\le \sigma(3-p)\int_{\mathcal{A}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\,dx \\
&\qquad+\hat{C}\int_{\mathcal{A}(2R)} |\nabla\varphi_R|(u_\nu^-)^{\alpha}\varphi_R^2|\nabla\varphi_R|\,dx \\
&\qquad-L\int_{\mathcal{A}(2R)} (u_\nu^-)^{\alpha+1}\varphi_R^2\,dx,
\end{aligned}
\tag{5.11}
$$

where we used (3.3) and where $\hat{C} := 3 - p/\sigma \|\nabla u\|_\infty^{p-1}$. Exploiting the Young inequality with exponents $(\alpha+1)/\alpha$ and $\alpha+1$ we obtain

$$\begin{aligned}
&\alpha(p-1)\int_{\mathcal{A}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\varphi_R^2\,dx\\
&\le \sigma(3-p)\int_{\mathcal{A}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\,dx\\
&\quad+\frac{\hat{C}}{\alpha+1}\int_{\mathcal{A}(2R)} |\nabla \varphi_R|^{\alpha+1}\,dx\\
&\quad+\frac{\hat{C}(\alpha+1)}{\alpha}\int_{\mathcal{A}(2R)} |\nabla \varphi_R|^{\frac{\alpha+1}{\alpha}}(u_\nu^-)^{\alpha+1}\varphi_R^{2\frac{\alpha+1}{\alpha}}\,dx\\
&\quad-L\int_{\mathcal{A}(2R)} (u_\nu^-)^{\alpha+1}\varphi_R^2\,dx,
\end{aligned} \tag{5.12}$$

Since $|\nabla\varphi_R| \le 2/R$ in $B_{2R}\setminus B_R$, $0\le\varphi_R\le 1$ in $\mathbb{R}^N$ and $\varphi_R = 1$ in B_R, we obtain

$$\begin{aligned}
\int_{\mathcal{A}(R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\,dx \le \vartheta\int_{\mathcal{A}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\,dx&\\
+\frac{1}{\alpha(p-1)}\left(\frac{\hat{C}(\alpha+1)}{\alpha R^{\frac{\alpha+1}{\alpha}}}-L\right)&\\
\times\int_{\mathcal{A}(2R)} (u_\nu^-)^{\alpha+1}\varphi_R^2\,dx,&\\
+\frac{\bar{C}}{R^{\alpha-(N-1)}},&
\end{aligned} \tag{5.13}$$

where $\vartheta := \sigma(3-p)/\alpha(p-1)$ and $\bar{C} := 2\hat{C}/\alpha(\alpha+1)(p-1)$. Now we fix $\alpha > 0$ such that $\alpha > N-1$, $\sigma > 0$ sufficiently small such that $\vartheta < 2^{-N}$ and finally $R_0 > 0$ such that $\hat{C}(\alpha+1)/\alpha R^{\frac{\alpha+1}{\alpha}} - L < 0$. Having in mind all these fixed parameters let us define

$$\mathcal{L}(R) := \int_{\mathcal{A}(R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\,dx.$$

It is easy to see that $\mathcal{L}(R) \le CR^N$. By (5.13) we deduce that holds

$$\mathcal{L}(R) \le \vartheta\mathcal{L}(2R) + \frac{\bar{C}}{R^{\alpha-(N-1)}}$$

for every $R \ge R_0$. By applying Lemma 2.1 in [14] it follows that $\mathcal{L}(R) = 0$ for all $R \ge R_0$. Hence passing to the limit we obtain that $u_\nu^- = 0$ in A.

Step 2. $u_\nu^- = 0$ in D.

Let us denote by B' the $(N-1)$ dimensional ball in $\mathbb{R}^{N-1}$ and $\psi_R(x', x_N) = \psi_R(x') \in C_c^\infty(\mathbb{R}^{N-1})$ is a standard cutoff function such that

$$\begin{cases} \psi_R \equiv 1, & \text{in } B^{'}(0,R) \subset \mathbb{R}^{N-1}, \\ \psi_R \equiv 0, & \text{in } \mathbb{R}^{N-1} \setminus B^{'}(0,2R), \\ |\nabla \psi_R| \le \frac{2}{R}, & \text{in } B^{'}(0,2R) \setminus B^{'}(0,R) \subset \mathbb{R}^{N-1}. \end{cases} \tag{5.14}$$

Let us define the cylinder

$$\mathcal{C}(R) := \left\{ (x', x_N) \in \mathbb{R}^N \ : \ \{x \in \mathbb{R}^N \mid -k < x_N < k\} \cap \overline{\{B'(0,R) \times \mathbb{R}\}} \right\}.$$

We set

$$\psi := (u_\nu^-)^\beta \psi_R^2 \chi_{\mathcal{C}(2R)} \tag{5.15}$$

where $\beta > 1$. First of all we notice that ψ belongs to $W_0^{1,2}(\mathcal{C}(2R))$ by (5.14) and since $u_\nu^- = 0$ on ∂A (as above, see Lemmas 4.4 and 5.1). Recalling (5.5) we have also in this case that

$$L_u(u_\nu, \psi) = 0 \quad \forall \psi \in W_c^{1,2}(\mathbb{R}^N). \tag{5.16}$$

Taking $\psi \in W_0^{1,2}(\mathcal{C}(2R)) \cap W_c^{1,2}(\mathbb{R}^N)$ defined in (5.15) in the previous equation, we obtain

$$\begin{aligned} \beta & \int_{\mathcal{C}(2R)} |\nabla u|^{p-2} (\nabla u_\nu, \nabla u_\nu^-)(u_\nu^-)^{\beta-1} \psi_R^2 \, dx \\ & + 2 \int_{\mathcal{C}(2R)} |\nabla u|^{p-2} (\nabla u_\nu, \nabla \psi_R)(u_\nu^-)^\beta \psi_R \, dx \\ & + \beta(p-2) \int_{\mathcal{C}(2R)} |\nabla u|^{p-4} (\nabla u, \nabla u_\nu)(\nabla u, \nabla u_\nu^-)(u_\nu^-)^{\beta-1} \psi_R^2 \, dx \\ & + 2(p-2) \int_{\mathcal{C}(2R)} |\nabla u|^{p-4} (\nabla u, \nabla u_\nu)(\nabla u, \nabla \varphi_R)(u_\nu^-)^\beta \psi_R \, dx \\ = & \int_{\mathcal{C}(2R)} f'(u)(u_\nu^-)^{\beta+1} \psi_R^2 \, dx. \end{aligned} \tag{5.17}$$

Repeating verbatim the same argument of (5.8), (5.9) and (5.10), starting by (5.17) we obtain

$$\begin{aligned} \beta(p-1) & \int_{\mathcal{C}(2R)} |\nabla u|^{p-2} |\nabla u_\nu^-|^2 (u_\nu^-)^{\beta-1} \psi_R^2 \, dx \\ & \le \sigma(3-p) \int_{\mathcal{C}(2R)} |\nabla u|^{p-2} |\nabla u_\nu^-|^2 \, (u_\nu^-)^{\beta-1} \, dx \\ & \quad + \frac{(3-p)}{\sigma} \int_{\mathcal{C}(2R)} |\nabla u|^{p-2} |\nabla \psi_R|^2 (u_\nu^-)^{\beta+1} \psi_R^2 \, dx \\ & \quad + \int_{\mathcal{C}(2R)} f'(u)(u_\nu^-)^{\beta+1} \psi_R^2 \, dx. \end{aligned} \tag{5.18}$$

Since $u_\nu = (\nabla u, \nu)$ and $|\nabla u| \leq \rho$ in $\mathcal{C}(2R)$ we have

$$\begin{aligned}\int_{\mathcal{C}(2R)} &|\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\beta-1}\psi_R^2\,dx\\ &\leq \vartheta \int_{\mathcal{C}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2\,(u_\nu^-)^{\beta-1}\,dx\\ &\quad+\hat{C}\rho^{p-1}\int_{\mathcal{C}(2R)}|\nabla\psi_R|^2(u_\nu^-)^{\beta}\psi_R^2\,dx\\ &\quad+C_k\int_{\mathcal{C}(2R)}(u_\nu^-)^{\beta+1}\psi_R^2\,dx.\end{aligned} \tag{5.19}$$

where $\vartheta := \sigma(3-p)/\beta(p-1)$, $\hat{C} := (3-p)/\sigma\beta(p-1)$ and $\tilde{C} := \|f'\|_{L^\infty((-1,1))}$ $\beta(p-1)$. Exploiting the Young inequality with exponents $(\beta+1)/\beta$ and $\beta+1$ we obtain

$$\begin{aligned}\int_{\mathcal{C}(2R)} &|\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\beta-1}\psi_R^2\,dx\\ &\leq \vartheta \int_{\mathcal{C}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2\,(u_\nu^-)^{\beta-1}\,dx\\ &\quad+\frac{\hat{C}\rho^{p-1}}{\beta+1}\int_{\mathcal{C}(2R)}|\nabla\psi_R|^{\beta+1}\,dx\\ &\quad+\frac{\hat{C}\rho^{p-1}(\beta+1)}{\beta}\int_{\mathcal{C}(2R)}|\nabla\psi_R|^{\frac{\beta+1}{\beta}}(u_\nu^-)^{\beta+1}\psi_R^{2\frac{\beta+1}{\beta}}\,dx\\ &\quad+\tilde{C}\int_{\mathcal{C}(2R)}(u_\nu^-)^{\beta+1}\psi_R^2\,dx.\end{aligned} \tag{5.20}$$

Since $|\nabla\psi_R| \leq 2/R$ in $B'_{2R}\setminus B'_R$, $0 \leq \psi_R \leq 1$ in $\mathbb{R}^N$ and $\psi_R = 1$ in B'_R, we obtain

$$\begin{aligned}\int_{\mathcal{C}(2R)} &|\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\beta-1}\psi_R^2\,dx\\ &\leq \vartheta \int_{\mathcal{C}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2\,(u_\nu^-)^{\beta-1}\,dx+\bar{C}_R\\ &\times\int_{\mathcal{C}(2R)}(u_\nu^-)^{\beta+1}\psi_R^2\,dx+\frac{2^{\beta+1}\hat{C}\rho^{p-1}}{(\beta+1)R^{\beta-(N-2)}}\\ &\leq \vartheta \int_{\mathcal{C}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2\,(u_\nu^-)^{\beta-1}\,dx+\bar{C}_R\\ &\times\int_{B'(0,2R)}\left(\int_{-k}^{k}\left[(u_\nu^-)^{\frac{\beta+1}{2}}\right]^2\,dx_N\right)\psi_R^2(x')\,dx'\\ &\quad+\frac{2^{\beta+1}\hat{C}\rho^{p-1}}{(\beta+1)R^{\beta-(N-2)}}\\ &\leq \vartheta \int_{\mathcal{C}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2\,(u_\nu^-)^{\beta-1}\,dx+\bar{C}_R C_p(k)^2\frac{(\beta+1)^2}{4}\end{aligned}$$

$$\times \int_{\mathcal{C}(2R)} |\partial_{x_N} u_\nu^-|^2 (u_\nu^-)^{\beta-1} \psi_R^2 \, dx + \frac{2^{\beta+1} \hat{C} \rho^{p-1}}{(\beta+1) R^{\beta-(N-2)}}$$

$$\leq \vartheta \int_{\mathcal{C}(2R)} |\nabla u|^{p-2} |\nabla u_\nu^-|^2 \, (u_\nu^-)^{\beta-1} \, dx$$

$$+ \bar{C}_R C_p(k)^2 \frac{(\beta+1)^2}{4} \rho^{2-p} \int_{\mathcal{C}(2R)} |\nabla u|^{p-2} |\nabla u_\nu^-|^2 (u_\nu^-)^{\beta-1} \psi_R^2 \, dx$$

$$+ \frac{2^{\beta+1} \hat{C} \rho^{p-1}}{(\beta+1) R^{\beta-(N-2)}},$$

with $\bar{C}_R := \hat{C}\rho^{p-1}(\beta+1)/\beta R^{\frac{\beta+1}{\beta}} + \tilde{C}$. We point out that in (5.21) we used a Poincaré inequality in the set $[-k, k]$ (denoting with C_p the associated constant) together with the fact that $\psi_R = \psi_R(x')$.

Finally we choose $\beta > 0$ such that $\beta > N - 2$, $\vartheta > 0$ sufficiently small such that $\vartheta < 2^{-N+1}$ and $\rho > 0$ sufficiently small such that

$$\bar{C}_R C_p(k)^2 \frac{(\beta+1)^2}{2} \rho^{2-p} < 1.$$

Having in mind all these fixed parameters let us define

$$\mathcal{L}(R) := \int_{\mathcal{C}(R)} |\nabla u|^{p-2} |\nabla u_\nu^-|^2 (u_\nu^-)^{\beta-1} \, dx.$$

It is easy to see that $\mathcal{L}(R) \leq C R^{N-1}$. By (5.21) (up to a redefining of the constant involved) we deduce that

$$\mathcal{L}(R) \leq \vartheta \mathcal{L}(2R) + \frac{C}{R^{\beta-(N-2)}} \tag{5.21}$$

holds for every $R > 0$. By applying Lemma 2.1 in [14] it follows that $\mathcal{L}(R) = 0$ for all $R > 0$. Since $p < 2$, passing to the limit in (5.21), we deduce that for a.e. $x \in D$

$$\text{either } u_\nu^-(x) = 0, \qquad \text{or } |\nabla u_\nu^-(x)| = 0. \tag{5.22}$$

This actually implies that $u_\nu^-(x) = 0$ in D. Indeed let us suppose that would exist a point $P \in D$ such that $u_\nu^-(P) \neq 0$. Let us consider the connected component $\mathcal{U}$ of $D \setminus \{x \in D \ : \ u_\nu^-(x) = 0\}$ containing P. By the continuity of u_ν^-, it follows that $u_\nu^- = 0$ on the boundary $\partial\mathcal{U}$. On the other hand u_ν^- must be constant in $\mathcal{U}$ (since by (5.22) $|\nabla u_\nu^-| = 0$ there) .This is a contradiction.

By this two step we deduce that $u_\nu \geq 0$ in $\mathbb{R}^N$. Finally by Lemma 4.2 we get (5.3). □

Proof of Theorem 1.1 Using Proposition 4.1 we get that the solution is monotone increasing in the y-direction and this implies that $\partial_y u \geq 0$ in $\mathbb{R}^N$. In particular we have $\partial_y u > 0$ in $\mathbb{R}^N \setminus \mathcal{Z}_{f(u)}$ by (4.1). By Proposition 5.2, actually we obtain that the solution is increasing in a cone of directions close to the y-direction. This allows us to show that in fact, for $i = 1, 2, \cdots, N-1$, $\partial_{x_i} u = 0$ in $\mathbb{R}^N$, just exploiting the arguments in [10] (see also [17]). We provide the details for the sake of completeness. Let Ω be the set of the directions $\eta \in \mathbb{S}^{N-1}_+$ for which there exists an open neighborhood $\mathcal{O}_\eta \subset \mathbb{S}^{N-1}_+$ such that

$$\partial_\nu u = u_\nu \geq 0 \quad \text{in } \mathbb{R}^N \quad \text{and} \quad \partial_\nu u = u_\nu > 0 \quad \text{in } \mathbb{R}^N \setminus \mathcal{Z}_{f(u)},$$

for every $\nu \in \mathcal{O}_\eta$. The set Ω is non-empty, since $e_N \in \Omega$, and it is also open by Proposition 5.2. Now we want to show that it is also closed. Let $\bar{\eta} \in \mathbb{S}^{N-1}_+$ and let us consider the sequence $\{\eta_n\}$ in Ω such that $\eta_n \to \bar{\eta}$ as $n \to +\infty$ in the topology of $\mathbb{S}^{N-1}_+$. Since by our assumptions $\partial_{\eta_n} u \geq 0$ in $\mathbb{R}^N$, passing to the limit we obtain that $\partial_{\bar{\eta}} u \geq 0$ in $\mathbb{R}^N$. By Lemma 4.2 it follows that $\partial_{\bar{\eta}} u > 0$ in $\mathbb{R}^N \setminus \mathcal{Z}_{f(u)}$. By Proposition 5.2 there exists an open neighborhood $\mathcal{O}_{\bar{\eta}}$ such that (5.3) is true for every $\nu \in \mathcal{O}_{\bar{\eta}}$; hence $\bar{\eta} \in \Omega$ and this implies that Ω is also closed. Now, since $\mathbb{S}^{N-1}_+$ is a path-connected set, we have that $\Omega = \mathbb{S}^{N-1}_+$. Then there exists $v \in C^{1,\alpha}_{loc}(\mathbb{R})$ such that $u(x', y) = v(y)$. Now we prove (1.3). We proceed by contradiction. Let us assume that there exists $b \in \mathcal{Z}_{f(u)} \setminus \{-1, 1\}$ such that $v'(b) = 0$. Then, by $u_y \geq 0$, the level set $\{v = v(b)\}$ must be a bounded closed interval (possibly reduced to a single point), i.e., there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha \leq \beta$ such that

$$\{v = v(b)\} = [\alpha, \beta].$$

Therefore, by Höpf's Lemma, we have that $v'(\beta) > 0$. The latter clearly implies that $\{v = v(b)\} = \{\beta\} = \{b\}$ and so $v'(b) > 0$, which is in contradiction with our initial assumption. Hence we deduce that $\partial_y u > 0$ in $\mathbb{R}^N$, concluding the proof. $\square$

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