Correction to: Poles and residues of standard L -functions attached to Siegel modular forms

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Abstract Correction to my paper on the poles of standard L -functions attached to Siegel modular forms.

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In the second corollary of Theorem 3 of the original paper [5, p. 601], I made a statement (without proof) on the orders of possible poles of the completed standard L -function $\Lambda(s, f, \text{St})$ attached to a Siegel modular form f of weight k and degree n in case $k < n$. It asserts in particular that the orders of poles are at most two, but this is not the case.

This error was pointed out by a mail from Professor Chenevier on October 27, 2018 (for details, see the remark below). After that I checked the paper again and noticed that I had made some mistakes in deducing the claim.

To state the corrected version the following notation will be used. For $n, k \in \mathbf{Z}_{>0}$ let S_k^n be the space of holomorphic cusp forms of weight k for

$$\Gamma^{(n)} := Sp(n, \mathbf{Z}) = Sp_{2n}(\mathbf{Z}).$$

Let $f \in S_k^n$ be a Hecke eigenform and $L(s, f, \text{St})$ be the standard L -function attached to f . Take $\varepsilon, \nu \in \{0, 1\}$ such that $n \equiv \varepsilon \pmod{2}$ and $k \equiv \nu \pmod{2}$. Let

$$\Gamma_{\mathbf{R}}(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right), \quad \Gamma_{\mathbf{C}}(s) := 2(2\pi)^{-s} \Gamma(s)$$

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and

$$\Lambda(s, f, \underline{\mathbf{St}}) := \Gamma_{\mathbf{R}}(s + \varepsilon) \prod_{j=1}^n \Gamma_{\mathbf{C}}(s + k - j) L(s, f, \underline{\mathbf{St}}).$$

By Böcherer [1], $\Lambda(s, f, \underline{\mathbf{St}})$ has a meromorphic continuation to the whole s -plane and is invariant under the substitution $s \mapsto 1 - s$. The symbol $\text{ord}_{s=c}$ stands for the order of zero of a meromorphic function at $s = c$. The largest integer $\leq x$ is denoted by $[x]$.

Now we state the correction. The second corollary of Theorem 3 in [5] should be replaced by the following

Proposition 1 *Suppose $k < n$. Then the possible poles of $\Lambda(s, f, \underline{\mathbf{St}})$ are contained in $[-n + k - \nu, n - k + \nu + 1] \cap \mathbf{Z}$ and for every $m \in [1, n - k + \nu + 1] \cap \mathbf{Z}$ we have*

$$\text{ord}_{s=m} \Lambda(s, f, \underline{\mathbf{St}}) \geq \left[\frac{m - n}{2} \right] + \frac{k - \nu}{2} - 1.$$

Proof Put

$$\Gamma_n(s) := \prod_{j=1}^n \Gamma\left(s - \frac{j-1}{2}\right), \quad C_n(s) := \prod_{j=1}^n \left(s + \frac{j-1}{2}\right) = \frac{\Gamma_n\left(s + \frac{n+1}{2}\right)}{\Gamma_n\left(s + \frac{n-1}{2}\right)},$$

and

$$\mu(n, k, s) := (-1)^{\frac{nk}{2}} 2^{\frac{n^2+3n}{2}-2ns-nk+1} \pi^{\frac{n(n+1)}{2}} \frac{\Gamma_n\left(s + k - \frac{n+1}{2}\right)}{\Gamma_n(s + k)}.$$

For $m \in \mathbf{Z}_{>0}$ let $\mathfrak{H}_m$ be the Siegel upper half space of degree m . For $w \in 2\mathbf{Z}_{\geq 0}$, $z \in \mathfrak{H}_m$ and $s \in \mathbf{C}$ with $\text{Re}(w + 2s) > m + 1$ let

$$E_w^{(m)}(z, s) := \det(\text{Im}(z))^s \sum_{\{c, d\}} \det(cz + d)^{-w} |\det(cz + d)|^{-2s}$$

be the nonholomorphic Eisenstein series of weight w for $\Gamma^{(m)}$. Here $\begin{pmatrix} * & * \\ c & d \end{pmatrix}$ runs over a complete set of representatives of $\left\{ \begin{pmatrix} * & * \\ 0^{(m)} & * \end{pmatrix} \right\} \setminus \Gamma^{(m)}$. The Eisenstein series $E_w^{(m)}(z, s)$ has a meromorphic continuation to the whole s -plane. By Böcherer [1, p. 157],

$$\begin{aligned}
& L(s, f, \underline{\text{St}})f(Z) \\
&= \left(2^n C_n \left(1 - \frac{s+k+v+n}{2} \right) \right)^{-\delta(\frac{v-1}{2})} \mu \left(n, k, \frac{s-k+v+n}{2} \right)^{-1} \\
&\quad \cdot \zeta(s+n) \prod_{j=0}^{n-1} \zeta(2s+2j) \\
&\quad \cdot \left(f(W), \left(\left(D^\nu E_{k-\nu}^{(2n)} \right) \left(*, \frac{\bar{s}-k+v+n}{2} \right) \right) \begin{pmatrix} -\bar{Z} & 0 \\ 0 & W \end{pmatrix} \right) \quad (a)
\end{aligned}$$

for every $Z \in \mathfrak{H}_n$. Here $\delta(x) = 1$ or 0 according as $x \in \mathbf{Z}$ or not; for

$$\mathcal{Z} = \begin{pmatrix} Z & U \\ {}^t U & W \end{pmatrix} \in \mathfrak{H}_{2n} \quad \text{with } Z, W \in \mathfrak{H}_n \text{ and } U = (u_{ij}),$$

the differential operator D is defined by

$$D := \det \left(\frac{\partial}{\partial u_{ij}} \right)_{1 \leq i, j \leq n}.$$

By the functional equation of $\Lambda(s, f, \underline{\text{St}})$ we restrict our attention to the poles of the functions in (a) in the right half plane $\text{Re}(s) > 0$. In the following let $g_j(s)$ ($j = 1, 2, \dots$) be some functions which are holomorphic and nonzero in $\text{Re}(s) > 0$. Note that

$$C_n \left(1 - \frac{s+k+v+n}{2} \right) = (-1)^n C_n \left(\frac{s+k+v-1}{2} \right) = g_1(s)$$

and

$$\mu \left(n, k, \frac{s-k+v+n}{2} \right)^{-1} = g_2(s) \cdot \Gamma_n \left(\frac{s+k+v-1}{2} \right)^{-1}.$$

For $N \in 2\mathbf{Z}_{>0}$ and $z \in \mathfrak{H}_m$ let $E^*(z, s; w, \text{triv}_N, N)$ be the Eisenstein series of level N defined in Feit [3, p. 11], with triv_N being the trivial Dirichlet character modulo N . By Shimura [8, Proposition 2.1],

$$E_w^{(m)}(z, s) = \text{Tr}_1^N \left(E^*(z, s; w, \text{triv}_N, N)|_w \eta_m \right),$$

where $\eta_m := \begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix}$ and Tr_1^N is the trace map from $\Gamma_0^{(m)}(N)$ to $\Gamma^{(m)}$ defined as in [5, p. 592]. Rewriting this in the notation of Feit [3, p. 48], we have

$$E_{k-v}^{(2n)} \left(\mathcal{Z}^{(2n)}, \frac{s-k+v+n}{2} \right) \\ = g_3(s) \zeta(2s)^{-1} \mathrm{Tr}_1^N \left(D \left(\mathcal{Z}, \frac{s-k+v+n}{2}; k-v, \mathrm{triv}_N, N \right) \right) \Big|_{k-v} \eta_{2n} \Big). \quad (a)$$

Hence by (a), for every $c \in \mathbf{C}$ with $\mathrm{Re}(c) > 0$ we have

$$\mathrm{ord}_{s=c} L(s, f, \underline{\mathrm{St}}) \geq \inf_{\mathcal{Z} \in \mathfrak{H}_{2n}} \mathrm{ord}_{s=c} \frac{D \left(\mathcal{Z}, \frac{s-k+v+n}{2}; k-v, \mathrm{triv}_N, N \right)}{\Gamma_n \left(\frac{s+k+v-1}{2} \right)}, \quad (b)$$

where $N \in 2\mathbf{Z}_{>0}$ is arbitrary. By [3, p. 49], every possible pole of

$$D \left(\mathcal{Z}^{(2n)}, \frac{s-k+v+n}{2}; k-v, \mathrm{triv}_N, N \right)$$

in $\mathrm{Re}(s) > 0$ is simple and contained in $[1, n-k+v+1] \cap \mathbf{Z}$. In particular, the possible poles of $\Lambda(s, f, \underline{\mathrm{St}})$ are contained in $\mathbf{Z}$. Now let $m \in \mathbf{Z}_{>0}$. Then

$$\mathrm{ord}_{s=m} \Gamma_n \left(\frac{s+k+v-1}{2} \right) \\ = -\delta(m \leq n-k-v) \left(\left[\frac{n-m}{2} \right] - \frac{k+v}{2} + 1 \right),$$

where $\delta(C) := 1$ or 0 according as the condition C is satisfied or not. Hence from (b) it follows that

$$\mathrm{ord}_{s=m} L(s, f, \underline{\mathrm{St}}) \geq \delta(m \leq n-k-v) \left(\left[\frac{n-m}{2} \right] - \frac{k+v}{2} + 1 \right) \\ - \delta(m \leq n-k+v+1). \quad (c)$$

On the gamma factor of $\Lambda(s, f, \underline{\mathrm{St}})$ we have

$$\mathrm{ord}_{s=m} \Gamma_{\mathbf{R}}(s + \varepsilon) \prod_{j=1}^n \Gamma_{\mathbf{C}}(s + k - j) = \delta(m \leq n-k)(m - n + k - 1). \quad (d)$$

From (c) and (d) the assertion of Proposition 1 follows. $\square$

Remark (1) Proposition 1 follows also from Shimura [9, Theorem 6.1], which is much more extensive.

(2) By the above-mentioned mail from Professor Chenevier I was informed of the following: Let $F \in S_{13}^{24}$ be the Hecke eigenform discovered by Freitag [4], Satz 5.2. Then by the theory of theta lift for the orthogonal group $O(24)$ of Chenevier–Lannes [2] combined with the Eichler commutation relations of Rallis [7], the equality

$$L(s, F, \underline{\text{St}}) = \prod_{i=0}^{11} L(s + 11 - i, \Delta) \prod_{j=0}^{24} \zeta(s + 12 - j)$$

holds, where $L(s, \Delta)$ is the Hecke L -function attached to Ramanujan's $\Delta \in S_{12}^1$ which satisfies a functional equation under $s \mapsto 12 - s$. This implies in particular that $\Lambda(s, F, \underline{\text{St}})$ has a pole of order $7 - [m/2]$ at every $s = m \in [1, 13] \cap \mathbf{Z}$. Note that Proposition 1 gives the best possible bound in this case.

Professor Chenevier pointed out also that the Langlands standard gamma factor for $L(s, f, \underline{\text{St}})$ in case $k < n$ is

$$\gamma_k^{(n)}(s) := \Gamma_{\mathbf{R}}(s + \nu) \prod_{i=1}^k \Gamma_{\mathbf{C}}(s + k - i) \prod_{j=1}^{n-k} \Gamma_{\mathbf{R}}(s + \nu + j) \Gamma_{\mathbf{R}}(s + \nu - j)$$

according to Mœglin–Renard [6], Proposition 9.1. Hence let

$$\Lambda^*(s, f, \underline{\text{St}}) := \gamma_k^{(n)}(s) L(s, f, \underline{\text{St}}),$$

which also satisfies

$$\Lambda^*(s, f, \underline{\text{St}}) = \Lambda^*(1 - s, f, \underline{\text{St}}). \quad (\text{e})$$

For every $m \in \mathbf{Z}_{>0}$ we have

$$\text{ord}_{s=m} \gamma_k^{(n)}(s) = -\delta(m \leq n - k - \nu) \left(\left\lfloor \frac{n - m}{2} \right\rfloor - \frac{k + \nu}{2} + 1 \right). \quad (\text{f})$$

Hence (c), (e), and (f) imply that in terms of $\Lambda^*(s, f, \underline{\text{St}})$ Proposition 1 takes the following form:

Proposition 2 *For any Hecke eigenform $f \in S_k^n$ with $k < n$, every possible pole of $\Lambda^*(s, f, \underline{\text{St}})$ is simple and contained in $[-n + k - \nu, n - k + \nu + 1] \cap \mathbf{Z}$.*

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