



The fibration method over real function fields

Ambrus Pál¹ · Endre Szabó²

Received: 12 June 2019 / Revised: 17 July 2020 / Published online: 15 August 2020
© The Author(s) 2020

Abstract

Let $\mathbb{R}(C)$ be the function field of a smooth, irreducible projective curve over \mathbb{R} . Let X be a smooth, projective, geometrically irreducible variety equipped with a dominant morphism f onto a smooth projective rational variety with a smooth generic fibre over $\mathbb{R}(C)$. Assume that the cohomological obstruction introduced by Colliot-Thélène is the only one to the local-global principle for rational points for the smooth fibres of f over $\mathbb{R}(C)$ -valued points. Then we show that the same holds for X , too, by adopting the fibration method similarly to Harpaz–Wittenberg.

Mathematics Subject Classification 14G05 · 14P05

1 Introduction

Let C be a smooth, geometrically irreducible projective curve over \mathbb{R} . Let $\mathbb{R}(C)$ denote the function field of C , and for every $x \in C(\mathbb{R})$ let $\mathbb{R}(C)_x$ be the completion of $\mathbb{R}(C)$ with respect to the valuation furnished by x . Now let \mathcal{V} be a class of geometrically irreducible projective varieties over $\mathbb{R}(C)$. We say that \mathcal{V} satisfies the *local-global principle for rational points* if for every X in \mathcal{V} the following holds:

$$\prod_{x \in C(\mathbb{R})} X(\mathbb{R}(C)_x) \neq \emptyset \text{ implies that } X(\mathbb{R}(C)) \neq \emptyset.$$

There are classes of varieties when this local-global principle holds:

Communicated by Wei Zhang.

✉ Ambrus Pál
a.pal@imperial.ac.uk
Endre Szabó
szabo.endre@renyi.mta.hu

¹ Department of Mathematics, Imperial College, 180 Queen's Gate, London SW7 2AZ, UK

² Alfréd Rényi Institute of Mathematics, Reáltanoda u. 13-15, Budapest 1053, Hungary

Theorem 1.1 (Witt) *The local-global principle for rational points holds for smooth quadric hypersurfaces of dimension at least one over $\mathbb{R}(C)$.*

Proof See [15,16]. □

There is an even more general result due to Scheiderer:

Theorem 1.2 (Scheiderer) *The local-global principle for rational points holds for smooth compactifications of homogeneous spaces over connected linear groups over $\mathbb{R}(C)$.*

Proof See [14]. □

However similarly to varieties over number fields, there are some reasonably simple counter-examples to this local-global principle. The following counterexample is due to Racinet: let $C = \mathbb{P}_{\mathbb{R}}^1$ and let U be the affine surface over $\mathbb{R}(C) = \mathbb{R}(t)$ given by the following equation:

$$a^2 + b^2 = (c^2 + t)(tc^2 + c - 1)$$

in the variables a, b and c , and let X be a smooth projective model of U . Then $X(\mathbb{R}(X)_x)$ is non-empty for every $x \in C(\mathbb{R})$, but $X(\mathbb{R}(C))$ is empty. The failure of the local-global principle for this X was explained using a simple obstruction, analogous to the Brauer–Manin obstruction in the number field case, by Colliot-Thélène in [3] around 20 years ago, using unramified cohomology groups. He constructed a subset

$$\left(\prod_{x \in C(\mathbb{R})} X(\mathbb{R}(C)_x) \right)^{CT} \subseteq \prod_{x \in C(\mathbb{R})} X(\mathbb{R}(C)_x)$$

which contains $X(\mathbb{R}(C))$. We will review this construction in the next section. Later Ducros has found a simple topological obstruction equivalent to this obstruction which we will describe next. By resolution of singularities there is an integral, smooth, projective variety \mathcal{X} equipped with a projective dominant morphism $p : \mathcal{X} \rightarrow C$ over \mathbb{R} whose generic fibre is $X \rightarrow \text{Spec}(\mathbb{R}(C))$. Then we have:

Theorem 1.3 (Ducros) *Let X be as above and assume that $\prod_{x \in C(\mathbb{R})} X(\mathbb{R}(C)_x)$ is non-empty. Then $(\prod_{x \in C(\mathbb{R})} X(\mathbb{R}(C)_x))^{CT}$ is non-empty if and only if there is a continuous semi-algebraic section of p on $C(\mathbb{R})$ whose image lies in the smooth locus of p .*

Proof This is Théorème 4.3 of [7] on page 86. □

Let again \mathcal{V} be a class of geometrically irreducible projective varieties over $\mathbb{R}(C)$. We say that for \mathcal{V} the *CT obstruction is the only one to the local-global principle for rational points* if for every X in \mathcal{V} the following holds:

$$\left(\prod_{x \in C(\mathbb{R})} X(\mathbb{R}(C)_x) \right)^{CT} \neq \emptyset \text{ implies that } X(\mathbb{R}(C)) \neq \emptyset.$$

Ducros produced a class of geometrically rational smooth projective surfaces over the rational function field $\mathbb{R}(t)$ for which the local-global principle might fail, in fact Racinet's example belongs to this class, but the CT obstruction is the only one to the local-global principle for rational points:

Theorem 1.4 (Ducros) *The CT obstruction is the only one to the local-global principle for rational points for smooth projective varieties over $\mathbb{R}(C)$ which are fibrations in Brauer–Severi varieties over $\mathbb{P}_{\mathbb{R}(C)}^1$.*

Proof The special case of conic fibrations is proved in [7]. The general case is proved in [8]. \square

The main result of this article, Theorem 5.3, implies the following generalisation of the theorem above:

Theorem 1.5 *Let $f : X \rightarrow Y$ be a dominant morphism between smooth, proper, geometrically irreducible varieties over $\mathbb{R}(C)$ with a geometrically irreducible and smooth generic fibre, and assume that Y is rational. If the CT obstruction is the only one to the local-global principle for rational points for the smooth fibres of f over all but finitely many $\mathbb{R}(C)$ -valued points, then the same holds for X , too.*

Clearly Theorem 1.4 is a consequence of this result by Theorem 1.1. The result also implies that the CT obstruction is the only one to the local-global principle for rational points for X when the generic fibre of f is the smooth compactification of a homogeneous space by a connected linear algebraic group by Theorem 1.2. We will actually prove a stronger claim incorporating weak approximation (see Theorem 5.3).

Remark 1.6 The reader might wonder why we only consider real points in the formulation of the local-global principle. Note that our main interest is the case of rationally connected varieties (see Conjecture 2.6 below), when by the Graber–Harris–Starr theorem (see [10]) we have a rational point over the completion of $\mathbb{R}(X)$ with respect to every complex point of C . So we do not get additional restrictions by considering all closed points of C . However this is not true in general, genus one curves give easy counterexamples.

Remark 1.7 It is natural to ask if this result can be applied inductively. The way we formulate our main result does not lend itself naturally to iteration, since having a rational point does not guarantee X is rational itself. However see [1], which became available after our preprint appeared, where a similar idea is pursued.

Contents 1.8 In the next section we will recall the notion of unramified cohomology and the definition of Colliot-Thélène's obstruction, then we will show a useful lemma on the birational invariance of this obstruction. Then we review the work of Ducros on the topological reinterpretation of the cohomological obstruction in the third section. In the fourth section we prove a refined version of the Stone–Weierstrass approximation theorem, which also incorporates interpolation. In the fifth section we formulate a stronger form of our main result, Theorem 5.3, and reduce the general case to fibrations over the projective line. We use arguments inspired by the work of Harpaz–Wittenberg in [11] to prove Theorem 5.3 in the sixth section, using the approximation theorem from the fourth section as a crucial ingredient.

2 The local-global principle and Colliot-Thélène’s obstruction

Definition 2.1 Let X be a smooth, irreducible projective variety defined over a field F . Let $F(X)$ and $X(d)$ denote the function field of X and the set of points of X of codimension d , respectively. For every $x \in X(1)$ let $\mathcal{O}_{X,x}$ and $F(x)$ denote the discrete valuation ring in $F(X)$ corresponding to x and the residue field of $\mathcal{O}_{X,x}$, respectively. Assume that $\text{char}(F) \neq 2$. For the sake of simplicity let $\mathbb{Z}/2$ denote the group of order two. We will also denote by the same symbol the constant étale sheaf of order 2 over any scheme. The unramified cohomology group $H_{nr}^i(F(X)/F, \mathbb{Z}/2)$ of X over F is by definition (see section 1.1 of [5] on page 6):

$$H_{nr}^i(F(X)/F, \mathbb{Z}/2) = \text{Ker}(H_{\text{ét}}^i(F(X), \mathbb{Z}/2) \xrightarrow{\oplus_{x \in X(1)} \partial_x} \bigoplus_{x \in X(1)} H_{\text{ét}}^{i-1}(F(x), \mathbb{Z}/2))$$

where

$$\partial_x : H_{\text{ét}}^i(F(X), \mathbb{Z}/2) \longrightarrow H_{\text{ét}}^{i-1}(F(x), \mathbb{Z}/2)$$

is the residue map associated to the discrete valuation ring $\mathcal{O}_{X,x}$.

Next we need some basic facts about the Galois cohomology of function fields of real algebraic curves, and some form of a residue theorem for them.

Definition 2.2 Let C be a smooth, geometrically irreducible projective curve over \mathbb{R} . Let $\mathbb{R}(C)$ denote the function field of C , as above, and for every $x \in C(\mathbb{R})$ let $\mathbb{R}(C)_x$ be the completion of $\mathbb{R}(C)$ with respect to the valuation furnished by x . Then we have a residue map

$$\partial_x : H_{\text{ét}}^i(\mathbb{R}(C)_x, \mathbb{Z}/2) \longrightarrow H_{\text{ét}}^{i-1}(\mathbb{R}, \mathbb{Z}/2)$$

as the residue field $\mathbb{R}(x)$ of $\mathbb{R}(C)_x$ is \mathbb{R} . Note that as a graded algebra:

$$H_{\text{ét}}^*(\mathbb{R}, \mathbb{Z}/2) \cong \mathbb{Z}/2[t],$$

where t is the generator of the group $H_{\text{ét}}^1(\mathbb{R}, \mathbb{Z}/2)$ of order two. In particular we have a canonical isomorphism $H_{\text{ét}}^i(\mathbb{R}, \mathbb{Z}/2) \cong \mathbb{Z}/2$. So the residue map is a homomorphism:

$$\partial_x : H_{\text{ét}}^i(\mathbb{R}(C)_x, \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2.$$

By slight abuse of notation let ∂_x denote also the composition of the pull-back

$$H_{\text{ét}}^i(\mathbb{R}(C), \mathbb{Z}/2) \longrightarrow H_{\text{ét}}^i(\mathbb{R}(C)_x, \mathbb{Z}/2)$$

and this residue map.

The residue theorem, also called the reciprocity law, is the following

Proposition 2.3 *Let $V \subseteq C(\mathbb{R})$ be a connected component, let i be at least 2, and let $h \in H_{\text{ét}}^i(\mathbb{R}(C), \mathbb{Z}/2)$. Then*

$$\sum_{x \in V} \partial_x(h) = 0. \square$$

Remark 2.4 It is easy to see that all but finitely many terms of the sum above are zero, so the left hand side is well-defined. For a proof of this fact, and the proposition, see for example Proposition 3.7 of [3] on pages 157–158.

For the sake of simple notation let \mathbb{A}_C denote the direct product $\prod_{x \in C(\mathbb{R})} \mathbb{R}(C)_x$. It is an algebra over $\mathbb{R}(C)$. Now let X be a smooth, irreducible projective variety defined over $\mathbb{R}(C)$. Clearly

$$X(\mathbb{A}_C) = \prod_{x \in C(\mathbb{R})} X(\mathbb{R}(C)_x).$$

Now we are ready to define Colliot-Thélène’s obstruction.

Definition 2.5 Consider the set

$$X(\mathbb{A}_C)^{CT} \subseteq X(\mathbb{A}_C),$$

whose elements

$$\prod_{x \in C(\mathbb{R})} M_x \in \left(\prod_{x \in C(\mathbb{R})} X(\mathbb{R}(C)_x) \right)^{CT}$$

are subject to the following condition:

$$\sum_{x \in V} \partial_x(M_x^*(h)) = 0$$

for every $h \in H_{nr}^{\dim(X)+2}(\mathbb{R}(C)(X)/\mathbb{R}(C), \mathbb{Z}/2)$ and for every $V \subseteq C(\mathbb{R})$ connected component, where

$$M_x^* : H_{nr}^i(\mathbb{R}(C)(X)/\mathbb{R}(C), \mathbb{Z}/2) \rightarrow H_{\text{ét}}^i(\mathbb{R}(C)_x, \mathbb{Z}/2)$$

is the pull-back with respect to the map M_x . Note that all but finitely many terms of the sum above are zero, so the left hand side is well-defined, and the image of $X(\mathbb{R}(C))$ in $X(\mathbb{A}_C)$ under the diagonal embedding is in $X(\mathbb{A}_C)^{CT}$ by Proposition 2.3 above. One may justify the usage of the more mysterious group $H_{nr}^i(\mathbb{R}(C)(X)/\mathbb{R}(C), \mathbb{Z}/2)$ instead of $H_{\text{ét}}^i(X, \mathbb{Z}/2)$ as follows: for proper varieties having a smooth rational point is a birationally invariant property, so the obstruction should also have birational invariance. This holds for the former group, but not the latter.

Now that we made explicit what we mean by the CT obstruction, we can make the following bold conjecture, which is motivated by theorems of Witt, Scheiderer and Ducros mentioned in the introduction, and in analogy with Colliot-Thélène’s celebrated conjecture (see [4]) saying that the Brauer–Manin obstruction is the only one to the local-global principle for rational points for smooth projective rationally connected varieties over number fields:

Conjecture 2.6 *The CT obstruction is the only one to the local-global principle for rational points for smooth projective rationally connected varieties over $\mathbb{R}(C)$.*

All known classes of varieties for which CT obstruction is the only one to the local-global principle for rational points are rationally connected. Our main result Theorem 1.5 is a contribution to this conjecture.

We finish this section with a lemma which will be used in the proof of Theorem 5.3. Let $f : X \rightarrow Y$ be a morphism between smooth, projective, irreducible varieties over $\mathbb{R}(C)$. The morphism f induces a map $X(\mathbb{R}(C)_x) \rightarrow Y(\mathbb{R}(C)_x)$ for every $x \in C(\mathbb{R})$, which in turn induces a map $X(\mathbb{A}_C) \rightarrow Y(\mathbb{A}_C)$, which we will denote also by f by slight abuse of notation.

Lemma 2.7 *Assume that $f : X \rightarrow Y$ is birational and let $\underline{M} \in X(\mathbb{A}_C)$. Then*

$$\underline{M} \in X(\mathbb{A}_C)^{CT} \text{ if and only if } f(\underline{M}) \in Y(\mathbb{A}_C)^{CT}.$$

Proof Let d be the common dimension of X and Y . By assumption the pull-back with respect to f induces an isomorphism $\mathbb{R}(C)(Y) \rightarrow \mathbb{R}(C)(X)$ and hence an isomorphism $f^* : H_{\text{ét}}^{d+2}(\mathbb{R}(C)(Y), \mathbb{Z}/2) \rightarrow H_{\text{ét}}^{d+2}(\mathbb{R}(C)(X), \mathbb{Z}/2)$. Under this isomorphism the subgroup $H_{nr}^{d+2}(\mathbb{R}(C)(Y)/\mathbb{R}(C), \mathbb{Z}/2)$ maps isomorphically onto $H_{nr}^{d+2}(\mathbb{R}(C)(X)/\mathbb{R}(C), \mathbb{Z}/2)$. Write \underline{M} as

$$\underline{M} = \prod_{x \in C(\mathbb{R})} M_x$$

and let $h \in H_{nr}^{d+2}(\mathbb{R}(C)(Y)/\mathbb{R}(C), \mathbb{Z}/2)$ be arbitrary. Then

$$M_x^*(f^*(h)) = f(M_x)^*(h)$$

for every $x \in C(\mathbb{R})$ by naturality, so

$$\sum_{x \in V} \partial_x(M_x^*(f^*(h))) = 0 \text{ if and only if } \sum_{x \in V} \partial_x(f(M_x)^*(h)) = 0$$

for every connected component $V \subseteq C(\mathbb{R})$. The claim is now clear. □

3 The topological reinterpretation of the obstruction due to Ducros

Notation 3.1 By resolution of singularities there is an integral, smooth, projective variety \mathcal{X} equipped with a projective dominant morphism $p : \mathcal{X} \rightarrow C$ over \mathbb{R} whose

generic fibre is $X \rightarrow \text{Spec}(\mathbb{R}(C))$. As usual we will call \mathcal{X} a model of X over C . For every closed point x of C let \mathcal{O}_x be the valuation ring of $\mathbb{R}(C)_x$, let \mathcal{X}_x denote the fibre of p over x , let $\mathcal{X}_{sm} \subseteq \mathcal{X}$ be the smooth locus of p , and let $\mathcal{X}_{x,sm} = \mathcal{X}_{sm} \cap \mathcal{X}_x$ be the smooth locus of \mathcal{X}_x .

Definition 3.2 Equip each connected component of $C(\mathbb{R})$ with an orientation. For every pair of points $x, y \in C(\mathbb{R})$ lying on the same connected component $V \subseteq C(\mathbb{R})$ the interval $]xy[$ is the set of points of V which lies to the right of x and lies to the left of y with respect to the chosen orientation. We also set $[xy]$ to be the union of $]xy[$ and the points x, y . Now let $P \in \mathcal{X}_{x,sm}(\mathbb{R})$, $Q \in \mathcal{X}_{y,sm}(\mathbb{R})$ and let $\mathcal{X}'_{x,y} \subseteq \mathcal{X}_{sm}$ denote the open subscheme which is the complement of the union of $\mathcal{X}_{x,sm}$ and $\mathcal{X}_{y,sm}$. We say that a connected component V of $\mathcal{X}'_{x,y}(\mathbb{R})$ touches P on the right and Q on the left if the image of V under p is $]xy[$, and both P and Q lie in the closure of V .

Definition 3.3 For every map $f : A \rightarrow B$ of schemes over \mathbb{R} let $f(\mathbb{R}) : A(\mathbb{R}) \rightarrow B(\mathbb{R})$ denote the map induced by f on \mathbb{R} -valued points. Let σ be a set-theoretical section of the map $p(\mathbb{R}) : \mathcal{X}(\mathbb{R}) \rightarrow C(\mathbb{R})$. We say that the map σ is weakly continuous if for every $x \in C(\mathbb{R})$ the point $\sigma(x)$ lies in $\mathcal{X}_{x,sm}(\mathbb{R})$ and if the following holds: let x and y be two different \mathbb{R} -valued points of C lying in the same connected component of $C(\mathbb{R})$. Then there is a semialgebraic connected component V of $\mathcal{X}'_{x,y}(\mathbb{R})$ touching $\sigma(x)$ on the left and $\sigma(y)$ on the right such that for every $z \in]xy[$ the point $\sigma(z)$ lies in V .

This terminology is justified by the fact that every continuous section $C(\mathbb{R}) \rightarrow \mathcal{X}_{sm}(\mathbb{R})$ is indeed weakly continuous. The converse is not true, because we have the following

Proposition 3.4 *Let σ be a weakly continuous section of $p(\mathbb{R})$. Let σ' be another set-theoretical section of $p(\mathbb{R})$ such that for every $x \in C(\mathbb{R})$ the points $\sigma(x)$ and $\sigma'(x)$ are in the same connected component of $\mathcal{X}_{x,sm}(\mathbb{R})$. Then σ' is also weakly continuous.*

Proof This is Proposition 3.3 of [7] on page 82. □

Notation 3.5 By the valuative criterion of properness there is a bijection between the set of sections of the fibre product $\mathcal{X} \times_C \text{Spec}(\mathcal{O}_x) \rightarrow \text{Spec}(\mathcal{O}_x)$ and the set of $\mathbb{R}(C)_x$ -valued points of X . Given an

$$\underline{M} = \prod_{x \in C(\mathbb{R})} M_x \in \prod_{x \in C(\mathbb{R})} X(\mathbb{R}(C)_x)$$

let m_x be the special fibre of the section $\text{Spec}(\mathcal{O}_x) \rightarrow \mathcal{X} \times_C \text{Spec}(\mathcal{O}_x)$ associated to M_x for every $x \in C(\mathbb{R})$. Since \mathcal{X} is regular, the point m_x lies in $\mathcal{X}_{x,sm}(\mathbb{R})$. Whenever convenient, we will denote the map $x \mapsto m_x$ on $C(\mathbb{R})$ by $\sigma(\underline{M})$.

The topological reformulation of the CT obstruction due to Ducros is the following

Theorem 3.6 (Ducros) *The following are equivalent:*

- (i) we have $\underline{M} \in X(\mathbb{A}_C)^{CT}$,

(ii) the map $x \mapsto m_x$ on $C(\mathbb{R})$ is weakly continuous.

Proof This is Théorème 3.5 of [7] on page 83. \square

Proposition 3.7 Let σ be a weakly continuous section of $p(\mathbb{R})$. Then there is a continuous semi-algebraic section σ' of $p(\mathbb{R})$ such that for every $x \in C(\mathbb{R})$ the points $\sigma(x)$ and $\sigma'(x)$ are in the same connected component of $\mathcal{X}_{x,sm}(\mathbb{R})$.

Remark 3.8 Note that Theorem 1.3 is an immediate corollary of this proposition.

Proof of Proposition 3.7 This is essentially Proposition 4.1 of [7] on page 85, but there it is stated in a weaker form. However the proof actually shows the stronger form above. We will give an even stronger version incorporating interpolation, see Proposition 3.19 below, using essentially the same methods. \square

Definition 3.9 Let B be a smooth variety over \mathbb{R} . For every closed point x of B let $\mathcal{I}(x) \subset \mathcal{O}_B$ be the corresponding ideal sheaf. For every effective zero cycle $S = \sum n_x x$ on B the zero-dimensional closed subscheme of B defined by S is the zero scheme of the ideal sheaf $\mathcal{I}_S = \prod \mathcal{I}(x)^{n_x} \subset \mathcal{O}_B$. By slight abuse of notation we will let the symbol S denote this closed subscheme, too. When B is a curve this construction furnishes a bijective correspondence between the set of zero-dimensional closed subschemes of B whose closed points are all real and the set of effective zero cycles on B which are supported on $B(\mathbb{R})$. In this case we will identify these two sets in all that follow.

Definition 3.10 Let $f : A \rightarrow B$ be a map of smooth varieties over \mathbb{R} . An *interpolation condition* (for the map f) is a section $\phi : S \rightarrow A \times_B S$ (of the base change of f), defined over \mathbb{R} , where S is a zero-dimensional closed subscheme of B defined by an effective zero cycle supported on $B(\mathbb{R})$. We say that ϕ is of order $\leq k$, where k is a positive integer, if the coefficient of every point in the effective zero cycle defining S is at most $k + 1$. We say that an interpolation condition $\tilde{\phi} : \tilde{S} \rightarrow A \times_B \tilde{S}$ *subsumes* the interpolation condition $\phi : S \rightarrow A \times_B S$ if S is a closed subscheme of \tilde{S} and the pull-back of $\tilde{\phi}$ with respect to the closed immersion $S \hookrightarrow \tilde{S}$ is ϕ . We say that a set-theoretical section σ of the map $f(\mathbb{R}) : A(\mathbb{R}) \rightarrow B(\mathbb{R})$ and an interpolation condition $\phi : S \rightarrow A \times_B S$ are *weakly compatible* with each other if for every $x \in S(\mathbb{R})$ we have $\sigma(x) = \phi(x)$. We say that a section $\sigma : B \rightarrow A$ of f and an interpolation condition $\phi : S \rightarrow A \times_B S$ are *compatible* with each other if the composition of the closed immersion $S \hookrightarrow B$ and σ is ϕ .

Definition 3.11 As usual, we say that a function $f : M \rightarrow N$ between two smooth manifolds is a C^k -map if it is k -times continuously differentiable. For every M, N as above and for every point p of M let $C_p^k(M, N)$ denote the set of C^k -maps $f : U \rightarrow N$ where U is some open neighbourhood of p in M . For every $p \in \mathbb{R}$ and for every $f \in C_p^k(\mathbb{R}, \mathbb{R})$ the k -jet of f at the point x_0 is defined to be the polynomial

$$(J_p^k f)(z) = f(p) + f'(p)z + \cdots + \frac{f^{(k)}(p)}{k!} z^k.$$

We say that two maps $f, g \in \mathcal{C}_p^k(M, N)$ are k -equivalent at p if $f(p) = g(p)$ and for every pair of maps $\gamma \in \mathcal{C}_0^k(\mathbb{R}, M)$ and $\phi \in \mathcal{C}_{f(p)}^k(M, \mathbb{R})$ such that $\gamma(0) = p$, we have $J_0^k(\phi \circ f \circ \gamma) = J_0^k(\phi \circ g \circ \gamma)$.

Definition 3.12 Now let $f : A \rightarrow B$ be as in Definition 3.10. Then $f(\mathbb{R}) : A(\mathbb{R}) \rightarrow B(\mathbb{R})$ is a morphism of Nash manifolds. Now let $S = \sum n_x x$ be an effective zero cycle on B supported on $B(\mathbb{R})$, and let $\phi : S \rightarrow A \times_B S$ be an interpolation condition of order $\leq k$. Let $U \subset B$ be an open, affine subvariety such that there is a section $\tilde{\phi} : U \rightarrow A \times_B U$ compatible with ϕ . We say that a \mathcal{C}^k -section $\sigma : B(\mathbb{R}) \rightarrow A(\mathbb{R})$ of $f(\mathbb{R})$ and ϕ are compatible with each other if σ and $\tilde{\phi}$ are $(n_x - 1)$ -equivalent at x for every $x \in B(\mathbb{R})$ in the support of S . Since the notion introduced in Definition 3.11 is indeed an equivalence relation, this notion of compatibility is independent of the choice of $\tilde{\phi}$, and so it is well-defined.

Remarks 3.13 A section $\sigma : B \rightarrow A$ and an interpolation condition $\phi : S \rightarrow A \times_B S$ are compatible in the sense of Definition 3.10 if and only if the underlying Nash section $\sigma(\mathbb{R}) : B(\mathbb{R}) \rightarrow A(\mathbb{R})$ and ϕ are compatible in the sense of Definition 3.12. Therefore the latter is a generalisation of the former. If a section $\sigma : B \rightarrow A$ and an interpolation condition $\phi : S \rightarrow A \times_B S$ are compatible then $\sigma(\mathbb{R}) : B(\mathbb{R}) \rightarrow A(\mathbb{R})$ and ϕ are weakly compatible. In particular this terminology is justified. Finally note that when $A = B \times D$ for some smooth variety D over \mathbb{R} and $f : A \rightarrow B$ is the projection onto the first factor, sections of f are in a natural bijection correspondence with morphisms $A \rightarrow D$ of schemes over \mathbb{R} . Similarly an interpolation condition $\phi : S \rightarrow A \times_B S$ is the same data as a morphism $S \rightarrow D$ of schemes over \mathbb{R} , while \mathcal{C}^k -sections of $f(\mathbb{R})$ can be identified with \mathcal{C}^k -maps $A(\mathbb{R}) \rightarrow D(\mathbb{R})$. Therefore we will freely apply the concepts of Definitions 3.10 and 3.12 to such functions in all that follows.

Proposition 3.14 Let σ be a weakly continuous section of $p(\mathbb{R})$ and let $\phi : S \rightarrow \mathcal{X} \times_C S$ be an interpolation condition of order $\leq k$ weakly compatible with σ . Then there is a \mathcal{C}^k -section σ' of $p(\mathbb{R})$ such that for every $x \in C(\mathbb{R})$ the points $\sigma(x)$ and $\sigma'(x)$ are in the same connected component of $\mathcal{X}_{x,sm}(\mathbb{R})$ and σ' is compatible with ϕ .

Proof Note that the fibre of $p(\mathbb{R})$ over x is a Nash manifold for all but finitely many $x \in C(\mathbb{R})$. So by the Nash version of the stratification theorem (see Theorem A of [6] on page 349) there is a finite subset \mathcal{P} of $C(\mathbb{R})$, and for every semi-algebraic connected component U of $C(\mathbb{R}) - \mathcal{P}$ a Nash manifold F_U such that $p(\mathbb{R})^{-1}(U)$ is Nash-isomorphic to $F_U \times U$ and the restriction of $p(\mathbb{R})$ onto $p(\mathbb{R})^{-1}(U)$ is, modulo the given isomorphism, is the projection onto the second coordinate. By the nature of our construction for every point P in $C(\mathbb{R}) - \mathcal{P}$ the fibre \mathcal{X}_P is smooth. By adding finitely many points to the set \mathcal{P} , if it is necessary, we may assume that \mathcal{P} has at least two points in each semi-algebraic connected component of $C(\mathbb{R})$. Similarly we may assume that \mathcal{P} contains every closed point of S without loss of generality.

Set $\tilde{S} = (k + 1) \sum_{P \in \mathcal{P}} P$ and let the same symbol denote the unique closed subscheme defined by this zero cycle. Since there is an interpolation condition $\tilde{\phi} : \tilde{S} \rightarrow \mathcal{X} \times_C \tilde{S}$ which subsumes ϕ , we may assume without loss of generality that the

zero cycle defined by S is indeed $(k + 1) \sum_{P \in \mathcal{P}} P$. Write S as a coproduct:

$$S = \coprod_{P \in \mathcal{P}} S_P,$$

where S_P is a closed subscheme of S supported on P for each $P \in \mathcal{P}$ (possibly empty). For every such P let $\phi_P : S_P \rightarrow \mathcal{X} \times_C S_P$ be the interpolation condition which is the pull-back of ϕ with respect to the closed imbedding $S_P \hookrightarrow S$. Let P, Q be a pair of consecutive points of \mathcal{P} . Since $\sigma(P)$ and $\sigma(Q)$ lie in the smooth locus of p , by the implicit function theorem there are two points P', Q' in the open interval $]PQ[$ such that P' lies before Q' , and $p(\mathbb{R})$ has a C^∞ -section σ_P (resp. σ_Q) defined over some open neighbourhood of $[PP']$ (resp. of $[Q'Q]$) such that $\sigma_P(P) = \sigma(P)$ (resp. $\sigma_Q(Q) = \sigma(Q)$), and σ_P is compatible with ϕ_P (resp. σ_Q is compatible with ϕ_Q). Since σ is weakly continuous, there is a semi-algebraic connected component $V_{P,Q}$ of $p(\mathbb{R})^{-1}(]PQ[)$ which touches P on the right and Q on the left. Clearly $\sigma_P(P')$ and $\sigma_Q(Q')$ lie in $V_{P,Q}$.

On the other hand, because of the way the set \mathcal{P} was constructed, the restriction of $p(\mathbb{R})$ onto $p(\mathbb{R})^{-1}(]PQ[)$ is a projection, up to a Nash isomorphism. Therefore there is a C^∞ -section $\sigma_{P,Q}$ of $p(\mathbb{R})$ defined over some open neighbourhood of $[P'Q']$ such that $\sigma_{P,Q}$ and σ_P are k -equivalent at P' , and similarly $\sigma_{P,Q}$ and σ_Q are k -equivalent at Q' . Therefore the concatenation of $\sigma_P, \sigma_{P,Q}$ and σ_Q (restricted to $[PP']$, $[P'Q']$ and $[Q'Q]$, respectively) is a C^k -section $\widehat{\sigma}_{P,Q}$ of $p(\mathbb{R})$ defined over $[PQ]$ such that $\widehat{\sigma}_{P,Q}(P) = \sigma(P)$ and $\widehat{\sigma}_{P,Q}(Q) = \sigma(Q)$.

Now let P, Q, R be three consecutive points of \mathcal{P} (where $P = R$ is allowed). Since both $\widehat{\sigma}_{P,Q}$ and $\widehat{\sigma}_{Q,R}$ have extensions to an open neighbourhood of their definitions which are compatible with ϕ_Q , we get that their concatenation is C^k at Q . We get that the concatenation of the different sections $\widehat{\sigma}_{P,Q}$ for all couples P, Q of consecutive points of \mathcal{P} is a C^k -section $\widehat{\sigma}$ of $p(\mathbb{R})$ defined over all of $C(\mathbb{R})$ such that for each point $x \in C(\mathbb{R})$ the point $\widehat{\sigma}(x)$ lies in the same connected component of $\mathcal{X}_{x,sm}(\mathbb{R})$ as $\sigma(x)$, and $\widehat{\sigma}$ is compatible with ϕ . □

We will need a variant of the claim above with Nash sections, since for technical reasons it will be more convenient to work with the latter in the next section. In order to do so we will show two interpolation lemmas first.

Lemma 3.15 *Let $V \subseteq \mathbb{A}_{\mathbb{R}}^n$ be a non-singular closed affine subvariety such that $V(\mathbb{R})$ is compact. Let $\phi : Y \rightarrow \mathbb{A}_{\mathbb{R}}^m$ be an interpolation condition for some subscheme $Y \subseteq V$. Then every C^∞ -function $g : V(\mathbb{R}) \rightarrow \mathbb{R}^m$ compatible with ϕ can be approximated in the C^∞ -topology by regular functions $V \rightarrow \mathbb{A}_{\mathbb{R}}^m$ compatible with ϕ .*

Proof Let $\pi_j : \mathbb{A}_{\mathbb{R}}^m \rightarrow \mathbb{A}_{\mathbb{R}}^1$ be the projection onto the j -th coordinate, where $j = 1, 2, \dots, m$. It will be sufficient to show the claim for $\phi_j = \pi_j \circ \phi$ and $g_j = \pi_j \circ g$ for each j . In other words we may assume that $m = 1$ without loss of generality. Since V is affine there is a regular map $\psi : V \rightarrow \mathbb{A}_{\mathbb{R}}^1$ compatible with ϕ . By replacing g with $g - \psi$ we may assume without loss of generality that ϕ is the zero map. In this form the claim is a mild variant of Lemma 12.5.5 of [2] on page 321. We include the proof for the reader's convenience. Let h_1, h_2, \dots, h_n be the generators of the defining ideal

of Y . Since V is non-singular, we can represent the germ of g at a point $x \in V$ in the form $g_x = \lambda_{1,x}h_1 + \dots + \lambda_{n,x}h_n$, where the $\lambda_{i,x}$ are the germs of C^∞ -functions at x . Using a partition of unity and the compactness of $V(\mathbb{R})$, this allows us to represent g globally as $g = \lambda_1h_1 + \dots + \lambda_nh_n$, where $\lambda_i \in C^\infty(V(\mathbb{R}))$. Then it suffices to apply Nachbin’s version of the Stone–Weierstrass theorem to the functions λ_i (see [13]). \square

Definition 3.16 Let V be a nonsingular variety over \mathbb{R} , and let $W \subset \mathbb{A}^m(\mathbb{R}) = \mathbb{R}^m$ be a Nash manifold. An *interpolation condition* $\phi : S \rightarrow W$ for some subscheme $S \subset V$ of the type considered above is an interpolation condition $\phi : S \rightarrow \mathbb{A}^m$ which is compatible with some smooth function $U \rightarrow W$, where U is a neighborhood of the support of S in $V(\mathbb{R})$.

Lemma 3.17 Let $V \subseteq \mathbb{A}^n_{\mathbb{R}}$ be a non-singular closed affine subvariety such that $V(\mathbb{R})$ is compact, and let $W \subset \mathbb{R}^m$ be a compact Nash submanifold. Let $\phi : Y \rightarrow W$ be an interpolation condition of order $\leq k$ for some subscheme $Y \subseteq V$. Then every C^∞ -function $g : V(\mathbb{R}) \rightarrow W$ compatible with ϕ can be approximated in the C^∞ -topology by Nash functions $V(\mathbb{R}) \rightarrow W$ compatible with ϕ .

Proof (Compare with Corollary 8.9.7 of [2] on page 200.) By the Nash version of the tubular neighbourhood theorem (see Corollary 8.9.5 of [2] on page 199) there is an open semi-algebraic neighbourhood U of W in \mathbb{R}^m and a Nash retraction $\rho : U \rightarrow W$. Since for C^∞ -functions $h : V(\mathbb{R}) \rightarrow \mathbb{R}^m$ the condition of having image in U is open for the C^1 -topology, by Lemma 3.15 above any given function $g : V(\mathbb{R}) \rightarrow W$ compatible with ϕ can be approximated in the C^∞ -topology by a sequence g_n of Nash functions $V(\mathbb{R}) \rightarrow U$ compatible with $\iota \circ \phi$, where $\iota : W \rightarrow U$ is the inclusion map. The compositions $\rho \circ g_n$ are Nash functions $V(\mathbb{R}) \rightarrow W$ compatible with ϕ . Since W is compact, each derivative of ρ is bounded in some fixed ϵ -neighbourhood of W , and hence sequence $\rho \circ g_n$ approximates g in the C^∞ -topology. \square

Remark 3.18 Note that for every pair of conjugate point $P, \bar{P} \in C(\mathbb{C}) - C(\mathbb{R})$ the complement $C' = C - P - \bar{P}$ is an affine curve, and $C'(\mathbb{R}) = C(\mathbb{R})$, so this set is compact. Therefore we may apply Lemmas 3.15 and 3.17 to C' . In particular the conclusion of Lemma 3.17 holds for C , too.

Proposition 3.19 Let σ be a weakly continuous section of $p(\mathbb{R})$ and let $\phi : S \rightarrow \mathcal{X} \times_C S$ be an interpolation condition weakly compatible with σ . Then there is a Nash section σ' of $p(\mathbb{R})$ such that for every $x \in C(\mathbb{R})$ the points $\sigma(x)$ and $\sigma'(x)$ are in the same connected component of $\mathcal{X}_{x,sm}(\mathbb{R})$ and σ' is compatible with ϕ .

Proof We may assume that ϕ is an interpolation condition of order $\leq k$, where k is a positive integer. By Lemma 3.14 there is a C^k -section $\tilde{\sigma}$ of $p(\mathbb{R})$ such that for every $x \in C(\mathbb{R})$ the points $\sigma(x)$ and $\tilde{\sigma}(x)$ are in the same connected component of $\mathcal{X}_{x,sm}(\mathbb{R})$ and $\tilde{\sigma}$ is compatible with ϕ . By the usual approximation theorems in theory of smooth manifolds the section $\tilde{\sigma}$ can be arbitrarily well approximated by C^∞ -maps $s : C(\mathbb{R}) \rightarrow \mathcal{X}_{sm}(\mathbb{R})$ which are compatible with ϕ .

Note that if such an s is sufficiently close to $\tilde{\sigma}$ in the C^1 -topology then $p(\mathbb{R}) \circ s$ is a diffeomorphism and its inverse is very close to the identity map of $C(\mathbb{R})$. Therefore $s \circ (p(\mathbb{R}) \circ s)^{-1}$ is a C^∞ -section of $p(\mathbb{R})$ which can be arbitrarily close to $\tilde{\sigma}$. Also note

that $(p(\mathbb{R}) \circ s)^{-1}$, if it exists, is Nash if s is. For any smooth section $\sigma' : C(\mathbb{R}) \rightarrow \mathcal{X}_{sm}(\mathbb{R})$ sufficiently close to $\tilde{\sigma}$ in the C^0 -topology and for every $x \in C(\mathbb{R})$ the points $\tilde{\sigma}(x)$ and $\sigma'(x)$ are in the same connected component of $\mathcal{X}_{x,sm}(\mathbb{R})$, and hence the same holds for $\sigma(x)$ and $\sigma'(x)$. The claim now follows at once from Lemma 3.17, as we explained in Remark 3.18. \square

We finish this section with a convenient condition for weak continuity.

Definition 3.20 We say that a set-theoretical section σ of the map $p(\mathbb{R})$ is *mildly continuous at $x \in C(\mathbb{R})$* if $\sigma(x)$ lies in $\mathcal{X}_{x,sm}(\mathbb{R})$ and the intersection of the closure of the image of σ with $\mathcal{X}_x(\mathbb{R})$ lies in the connected component of $\mathcal{X}_{x,sm}(\mathbb{R})$ containing $\sigma(x)$. Note that σ is mildly continuous at $x \in C(\mathbb{R})$ if it is continuous there, since in the latter case the image of σ is closed, and its intersection with $\mathcal{X}_x(\mathbb{R})$ is $\sigma(x)$. Therefore the terminology is justified.

Proposition 3.21 *Let σ be a semi-algebraic section of $p(\mathbb{R})$ which is mildly continuous at every $x \in C(\mathbb{R})$. Then σ is weakly continuous.*

Proof Let x and y be two arbitrary different \mathbb{R} -valued points of C lying in the same connected component of $C(\mathbb{R})$. We will need the following \square

Lemma 3.22 *Let V be a semi-algebraic connected component of $\mathcal{X}'_{x,y}(\mathbb{R})$ and let \bar{V} be its closure in $\mathcal{X}(\mathbb{R})$. Then the intersection $\bar{V} \cap \mathcal{X}_{x,sm}(\mathbb{R})$ (respectively $\bar{V} \cap \mathcal{X}_{y,sm}(\mathbb{R})$) is the union of connected components of $\mathcal{X}_{x,sm}(\mathbb{R})$ (respectively of $\mathcal{X}_{y,sm}(\mathbb{R})$).*

Proof We may assume without loss of generality that V lies in $p(\mathbb{R})^{-1}(]xy[)$; otherwise we only need to reverse the roles of x and y . It is also enough to prove the claim for the fibre above x ; the proof for the fibre above y is similar. Since \bar{V} is closed, the intersection $\bar{V} \cap \mathcal{X}_{x,sm}(\mathbb{R})$ is closed in $\mathcal{X}_{x,sm}(\mathbb{R})$. Therefore it will be enough to show that it is also open in $\mathcal{X}_{x,sm}(\mathbb{R})$. Let n be the relative dimension of \mathcal{X} over C and let $z \in \bar{V} \cap \mathcal{X}_{x,sm}(\mathbb{R})$ be arbitrary. By the implicit function theorem there is a small connected open neighbourhood $I \subset C(\mathbb{R})$ of x , a connected open set $O \subset \mathbb{R}^n$, an open neighbourhood $U \subset \mathcal{X}_{sm}(\mathbb{R})$ of z , and a diffeomorphism $\phi : I \times O \rightarrow U$ such that the composition $p(\mathbb{R}) \circ \phi$ is the projection $I \times O \rightarrow I$ onto the first factor.

Let $J \subset I$ be the set of points in I lying to the right of x . By shrinking I , if it is necessary, we may assume that $J = I \cap]xy[$. By assumption

$$\phi(J \times O) \cap V = \phi(I \times O) \cap p(\mathbb{R})^{-1}(]xy[) \cap V = \phi(I \times O) \cap V = U \cap V$$

is non-empty. Since both J and O are connected, the product $J \times O$ is also connected, and hence $\phi(J \times O)$ is connected, too. Moreover $\phi(J \times O)$ lies in $p(\mathbb{R})^{-1}(]xy[)$, so the connected component V must contain it. Therefore \bar{V} contains the closure of $\phi(J \times O)$, which contains $\phi(\{x\} \times O)$. The latter is an open neighbourhood of z in $\mathcal{X}_{x,sm}(\mathbb{R})$. \square

Since σ is semi-algebraic, it is continuous at all but finitely many points of $C(\mathbb{R})$. Therefore there is a finite sequence of points $z_1, z_2, \dots, z_n \in]xy[$ such that σ is continuous at every $z \in]xy[$ not on this list. We may even assume that $z_1 = x, z_n = y$,

and z_i lies to the left of z_j for every pair of indices $i < j$. For every $i = 1, 2, \dots, n$ let E_i denote the connected component of $\mathcal{X}'_{z_i,sm}(\mathbb{R})$ containing $\sigma(z_i)$ and for every $i = 1, 2, \dots, n - 1$ let J_i denote the closure of the image of $]z_i z_{i+1}[$ with respect to σ .

Since the restriction of σ onto $]z_i z_{i+1}[$ is continuous for every index $i < n$, the image of $]z_i z_{i+1}[$ with respect to σ is connected, and hence its closure J_i is connected, too. Since J_i is also compact, its image under $p(\mathbb{R})$ is the closure $[z_i z_{i+1}]$ of $]z_i z_{i+1}[$. Therefore $J_i \cap X_{z_i}(\mathbb{R})$ is non-empty. By our assumptions the latter intersection lies in E_i , hence $J_i \cap E_i$ is non-empty, too. A similar argument shows that $J_i \cap E_{i+1}$ is also non-empty. Therefore the set

$$L = p(\mathbb{R})^{-1}(\text{Jxy}) \cap \left(\bigcup_{i=1}^{n-1} J_i \cup \bigcup_{i=2}^{n-1} E_i \right)$$

is connected. Let V be the unique semi-algebraic connected component of $\mathcal{X}'_{x,y}(\mathbb{R})$ containing L and let \bar{V} be its closure in $\mathcal{X}(\mathbb{R})$. Since

$$\bar{V} \cap E_{z_1} \supseteq J_1 \cap E_{z_1} \neq \emptyset,$$

by Lemma 3.22 the intersection $\bar{V} \cap \mathcal{X}_{x,sm}(\mathbb{R})$ contains E_{z_1} , and hence $\sigma(x)$. We may argue similarly to deduce that $\bar{V} \cap \mathcal{X}_{y,sm}(\mathbb{R})$ contains $\sigma(y)$. In other words V touches $\sigma(x)$ on the right and $\sigma(y)$ on the left. Since x and y are arbitrary, we get that σ is weakly continuous. □

4 The Stone–Weierstrass approximation theorem with interpolation

Definition 4.1 Let $\pi : C \times \mathbb{P}^1(\mathbb{R}) = C(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R}) \rightarrow C(\mathbb{R})$ denote the projection onto the first factor. Let $R \subseteq C(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ be a semi-algebraic subset. We say that R is *admissible* if it is the union of an open semi-algebraic set and finitely many points. The *kissing points* of an admissible semi-algebraic set R as above are all points of R which are not in the interior of R . We say that R *does not have topological obstruction* if there is a Nash section $s : C(\mathbb{R}) \rightarrow C \times \mathbb{P}^1(\mathbb{R})$ whose image lies in R .

The key result we need is an analogue of Conjecture 9.1 in [11] which we will formulate next. It is essentially a refined version of the classical the Stone–Weierstrass approximation theorem with interpolation conditions.

Theorem 4.2 *Let $R \subseteq C(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ be an admissible semi-algebraic subset, and for some closed subscheme $S \subset C$ let $\phi : S \rightarrow (\mathbb{P}^1 \times C) \times_C S = \mathbb{P}^1 \times S$ be an interpolation condition such that there is a Nash section $s : C(\mathbb{R}) \rightarrow C \times \mathbb{P}^1(\mathbb{R})$ compatible with ϕ and whose image lies in R . Then there is a regular section $f : C \rightarrow C \times \mathbb{P}^1_{\mathbb{R}}$ of the first projection compatible with ϕ such that $f(C(\mathbb{R}))$ lies in R .*

In particular we get that if $R \subseteq C(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ is an admissible semi-algebraic subset which does not have topological obstruction then there is a morphism $f : C \rightarrow \mathbb{P}^1_{\mathbb{R}}$

of schemes over \mathbb{R} such that $f(C(\mathbb{R}))$ lies in R . We are going to prove the theorem above via a sequence of lemmas.

Lemma 4.3 *For some closed subscheme $Z \subset C$ let $h : Z \rightarrow \mathbb{P}_{\mathbb{R}}^1$ be an interpolation condition. Then there is a morphism $f : C \rightarrow \mathbb{P}_{\mathbb{R}}^1$ of schemes over \mathbb{R} such that f is compatible with h and f has no poles on $C(\mathbb{R})$ outside of Z .*

Proof First assume that h is actually a map $h : Z \rightarrow \mathbb{A}_{\mathbb{R}}^1$. By our usual abuse of notation let Z also denote the effective divisor defining this closed subscheme and let d denote its degree. Choose an effective real divisor D on C which is supported outside of $C(\mathbb{R})$ and whose degree is bigger than $2g + d$ where g is the genus of C . Then by the Riemann–Roch theorem we have $\dim H^1(C, \mathcal{O}_C(D)) = 0$, so the pull-back induces a surjection:

$$H^0(C, \mathcal{O}_C(D + Z)) \longrightarrow H^0(Z, \mathcal{O}_C(D + Z)|_Z).$$

Therefore there is a real rational function f on C compatible with h whose polar divisor is a sub-divisor of D .

Now consider the general case. We may assume without loss of generality that Z is non-empty. Since Z is a finite scheme over \mathbb{R} there is an interpolation condition $\tilde{h} : Z \rightarrow \mathbb{A}_{\mathbb{R}}^2 - \{0\}$ such that the composition of \tilde{h} and the projection $\mathbb{A}_{\mathbb{R}}^2 - \{0\} \rightarrow \mathbb{P}_{\mathbb{R}}^1$ given by the rule $(a, b) \mapsto (a : b)$ is h . Let $\tilde{h}_1 : Z \rightarrow \mathbb{A}_{\mathbb{R}}^1$ and $\tilde{h}_2 : Z \rightarrow \mathbb{A}_{\mathbb{R}}^1$ denote the first, respectively the second, coordinate of \tilde{h} . By the above there is a rational function f_2 on C compatible with \tilde{h}_2 . Let O be the zero divisor of f_2 . We can write it as $O = O_1 + O_2 + O_3$, where O_1 is supported on the support of Z , the divisor O_2 is supported on the complement of the support of Z in $C(\mathbb{R})$, and O_3 is supported outside of $C(\mathbb{R})$. Now let

$$h_1 : Z \coprod 2O_2 \rightarrow \mathbb{A}_{\mathbb{R}}^1$$

be the unique map such that $h_1|_Z$ is \tilde{h}_1 , and $h_1|_{2O_2}$ is $f_2|_{2O_2}$. By the above there is a rational function f_1 on C compatible with h_1 whose polar divisor is supported outside of $C(\mathbb{R})$. Since Z is non-empty, the functions f_1, f_2 are not both identically zero, and hence there is a non-empty Zariski-open $U \subseteq C$ such that the rule $z \mapsto (f_1, f_2)$ furnishes a map $\tilde{f} : U \rightarrow \mathbb{A}_{\mathbb{R}}^2 - \{0\}$. The composition of \tilde{f} and the projection $\mathbb{A}_{\mathbb{R}}^2 - \{0\} \rightarrow \mathbb{P}_{\mathbb{R}}^1$ extends uniquely to a morphism $f : C \rightarrow \mathbb{P}_{\mathbb{R}}^1$ by the valuative criterion of properness. By our choice of f_1 and f_2 it is clear that f is compatible with h . It is also clear that on $C(\mathbb{R}) - Z$ the rational function f can only have poles on the support of O_2 . However the zero divisors of f_1 and f_2 are equal on the support of O_2 , in fact it is O_2 in both cases, so f does not have a pole on the support of O_2 either. \square

Lemma 4.4 *Let A and B be smooth irreducible algebraic varieties over \mathbb{R} , and let $f : A(\mathbb{R}) \rightarrow B(\mathbb{R})$ be a Nash map, and let $Z \subset B$ be a closed subscheme of positive codimension. Then the intersection of $f^{-1}(Z)(\mathbb{R})$ with each connected component of $A(\mathbb{R})$ is either a semi-algebraic subset of positive codimension, or the connected component itself.*

Proof Let $D \subset A(\mathbb{R})$ be a semi-algebraic connected component, and let $D_1 \subset D$ be the collection of all points $x \in D$ such that x has a semi-algebraic open neighbourhood $U \subset D$ such that the semi-algebraic set $U \cap f^{-1}(Z)$ has dimension strictly less than D . Clearly D_1 is open (in the usual semi-algebraic topology). Set $D_2 = f^{-1}(Z) - D_1$; since $f^{-1}(Z)$ is closed, the set D_2 is closed, too. It will be sufficient to show that D_2 is also open. Clearly we only need to verify the latter Zariski-locally, that is, we may assume without loss of generality that A is affine.

Now let $x \in D_2$ be arbitrary; then there are a Zariski-open affine neighbourhood $V \subset B$ of $f(x)$ and a finite set of regular maps g_1, g_2, \dots, g_n from V to $\mathbb{A}^1_{\mathbb{R}}$ such that $Z \cap V$ is the common zero locus of g_1, g_2, \dots, g_n . Let $U \subset f^{-1}(V(\mathbb{R}))$ be a connected semi-algebraic open neighbourhood of x in D . Then $g_i(\mathbb{R}) \circ f : U \rightarrow \mathbb{R}$ is a Nash function for each index i , so by Proposition 8.1.10 of [2] on page 166 its zero set $Z_i \subset U$ has either dimension strictly less than U , or this set Z_i is equal to U . Since Z_i contains $f^{-1}(Z)(\mathbb{R}) \cap U$ the former is not possible, so $Z_i = U$ for every index i . This implies that $f^{-1}(Z)(\mathbb{R}) \cap U$ also equal to U . \square

Lemma 4.5 *In the proof of Theorem 4.2 we may assume that $R \subseteq C(\mathbb{R}) \times \mathbb{A}^1(\mathbb{R})$ without loss of generality.*

Proof By Lemma 4.4 for every connected component $D \subset C(\mathbb{R})$ either s is constant on D , or s takes every value only finitely many times on D . Therefore for all but finitely many $x \in \mathbb{P}^1(\mathbb{R})$ the function s takes x only finitely many times on $C(\mathbb{R})$, and hence after applying an automorphism of \mathbb{P}^1 over \mathbb{R} , if this is necessary, we may assume that the set T of points $t \in C(\mathbb{R})$ where $s(t) = \infty$ is finite without loss of generality. We may even assume that ϕ does not take ∞ as a value. Since Nash maps are analytic, there are an effective zero divisor Z with support on $C(\mathbb{R})$ and an interpolation condition $h : Z \rightarrow \mathbb{P}^1_{\mathbb{R}}$ compatible with s such that for every Nash map $r : C(\mathbb{R}) \rightarrow \mathbb{P}^1(\mathbb{R})$ compatible with h the limit

$$\lim_{x \rightarrow t} (s(x) - r(x)) \tag{4.5.1}$$

exists and finite for every $t \in T$. We may even assume that h subsumes ϕ without loss of generality by choosing a Z such that $Z - S$ is effective. By Lemma 4.3 there is a morphism $r : C \rightarrow \mathbb{P}^1_{\mathbb{R}}$ of schemes over \mathbb{R} such that r is compatible with h and r has no poles on $C(\mathbb{R})$ outside of Z .

By our assumptions both s and r take values in \mathbb{R} on $C(\mathbb{R}) - T$, so their difference $\tilde{s} = s - r$ is a Nash function $C(\mathbb{R}) - T \rightarrow \mathbb{R}$. Since the limit (4.5.1) exists and finite for every $t \in T$, this map \tilde{s} extends uniquely to a continuous map $C(\mathbb{R}) \rightarrow \mathbb{R}$ which we will also denote by \tilde{s} by slight abuse of notation. The latter is also Nash and it is compatible with a unique interpolation condition $\tilde{\phi} : Z \rightarrow \mathbb{A}^1_{\mathbb{R}}$. Let $\tilde{R} \subseteq C(\mathbb{R}) \times \mathbb{A}^1(\mathbb{R})$ be the union of the image of \tilde{s} and the set

$$\{(x, y) \in C(\mathbb{R}) \times \mathbb{A}^1(\mathbb{R}) \mid x \notin T \text{ and } (x, y + r(x)) \in R\}.$$

The set \tilde{R} is admissible, and it contains the image of \tilde{s} . So if we assume that the claim of the theorem holds for \tilde{R} then there is a map $\tilde{f} : C \rightarrow \mathbb{P}^1_{\mathbb{R}}$ of schemes over \mathbb{R}

compatible with $\tilde{\phi}$ such that $\tilde{f}(C(\mathbb{R}))$ lies in \tilde{R} . The rational function $\tilde{f}+r : C \dashrightarrow \mathbb{P}^1_{\mathbb{R}}$ extends to a map $f : C \rightarrow \mathbb{P}^1_{\mathbb{R}}$ which is compatible with ϕ and $f(C(\mathbb{R}))$ lies in R . \square

Proposition 4.6 *Let $R \subseteq C \times \mathbb{A}^1(\mathbb{R})$ be an admissible semi-algebraic subset, and let $s : C(\mathbb{R}) \rightarrow C \times \mathbb{A}^1(\mathbb{R})$ be a Nash section whose image lies in R . Then there are an open neighbourhood U of s in the C^∞ -topology, a finite subscheme $Y \subset C$ and an interpolation condition $\psi : Y \rightarrow C \times \mathbb{A}^1_{\mathbb{R}}$ compatible with s with the following property: for every C^∞ -section $r : C(\mathbb{R}) \rightarrow C \times \mathbb{A}^1(\mathbb{R})$ which lies in U and is compatible with ψ , the image of r lies in R .*

Proof Let $s_2, r_2 : C(\mathbb{R}) \rightarrow \mathbb{R}$ denote the second coordinate functions of s and r . The set of kissing points is $Z = R \setminus R^\circ$. Let $T = \pi(Z) \subset C(\mathbb{R})$, and let $V \subset C(\mathbb{R})$ be an open neighborhood of T which is the union of disjoint open intervals. We fix a real analytic imbedding $V \hookrightarrow (0, 1) \subset \mathbb{R}$, this induces a real analytic imbedding of $G = V \times \mathbb{A}^1(\mathbb{R})$ into the Euclidean plane. In particular, U and G inherit the Euclidean metric. Let $A = G \setminus R^\circ$, it is a closed semi-algebraic set in G . Let $K \subset V$ be a compact neighborhood of T in V , and let S denote the maximum of $|r_2(t)|$ for $t \in K$. Let $E \subset A$ denote the compact subset $E = (K \times [-S - 1, S + 1]) \setminus R^\circ$. Consider the real analytic function

$$f : G = V \times \mathbb{A}^1(\mathbb{R}) \rightarrow \mathbb{R}, \quad f(t, a) = |a - s_2(t)|^2.$$

The zero set of f is $s(V)$, hence the zero set of the restriction $f|_A$ is just Z . By the Łojasiewicz inequality (in the form of Corollaire to Théorème 1 of section 18 in [12]) applied to our f, G, A, E , and Z , there are $d, N > 0$ such that

$$|f(x)| \geq d \cdot \text{dist}(x, Z)^N \quad \text{for all } x \in E. \tag{4.2}$$

Choose an integer $M \geq N/2$, and set $\tilde{d} = \min(\sqrt{d}, 1)$. Then

$$|a - s_2(t)| \geq \tilde{d} \cdot \text{dist}(t, T)^M \quad \text{whenever } t \in K, (t, a) \in A. \tag{4.3}$$

Indeed, the projection π is a contraction, and $\text{dist}(t, T) \leq 1$ since U was imbedded into $(0, 1)$. So (4.2) implies (4.3) for $(t, a) \in E$. On the other hand if $(t, a) \notin E$ then $|a| > S + 1$, hence $|a - s_2(t)| > 1$, and (4.3) follows again.

With the divisor $Y = MT$ let $\psi : Y \rightarrow C \times \mathbb{A}^1_{\mathbb{R}}$ be the (unique) interpolation condition (of order M) compatible with s . Suppose now that a section r is compatible with ψ , and its C^{M+1} -distance from s is less than, say, 1. If D denotes the maximum of $|s_2^{(M+1)}(t)|$ for $t \in K$, then $|r_2^{(M+1)}(t)| < D + 1$ for all $t \in K$. By Taylor’s theorem (with the Lagrange form of remainder)

$$|r_2(t) - s_2(t)| < 2 \frac{D + 1}{(M + 1)!} \text{dist}(t, T)^{M+1} \quad \text{for all } t \in K. \tag{4.4}$$

Combining (4.3) and (4.4) we obtain that, after possibly shrinking K , $r(t) = (t, r_2(t)) \notin A$ for $t \in K \setminus T$, hence $r(K) \subset R$.

On the other hand $Q = s(C(\mathbb{R}) \setminus K^\circ)$ is a compact set in R° . If the C^0 -distance of r from s is smaller than $\text{dist}(Q, A)$, then $r(C(\mathbb{R}) \setminus K) \subset R$ as well. \square

Proof of Theorem 4.2 By Lemma 4.5 we may assume that $R \subseteq C(\mathbb{R}) \times \mathbb{A}^1(\mathbb{R})$ without loss of generality. By Proposition 4.6 there is an open neighbourhood U of s in the C^∞ -topology and an interpolation condition $\psi : Y \rightarrow \mathbb{A}^1_{\mathbb{R}}$ compatible with s with the following property: for every C^∞ -section $r : C(\mathbb{R}) \rightarrow C \times \mathbb{A}^1(\mathbb{R})$ which lies in U and compatible with ψ , the image of r lies in R . Let T be a zero-dimensional closed subscheme of C whose closed points are all real and which contains both S and Y as a closed subscheme. Let $\tilde{\phi} : T \rightarrow \mathbb{A}^1_{\mathbb{R}}$ be the unique interpolation condition compatible with s . Since s is compatible both with ϕ and ψ , the interpolation condition $\tilde{\phi}$ subsumes both ϕ and ψ . By Lemma 3.15 there is a regular section $r : C \rightarrow C \times \mathbb{A}^1$ compatible with $\tilde{\phi}$ such that $r(\mathbb{R})$ lies in U . Since $\tilde{\phi}$ subsumes ϕ , the section r is compatible with ϕ . Since $\tilde{\phi}$ subsumes ψ , the section r is compatible with ψ , too. Therefore the image of $r(\mathbb{R})$ lies in R . \square

5 The main theorem and some easy reductions

Definition 5.1 Note that for every $x \in C(\mathbb{R})$ the discrete valuation of $\mathbb{R}(C)_x$ induces a topology on the projective space $\mathbb{P}^n(\mathbb{R}(C)_x)$, and hence on the $\mathbb{R}(C)_x$ -valued points of any quasi-projective variety defined over $\mathbb{R}(C)_x$. Moreover this topology is canonical in the sense that it does not depend on the choice of the embedding into a projective space. We will call this the x -adic topology. Now let X be again a smooth, irreducible projective variety defined over $\mathbb{R}(C)$. We will equip the direct product

$$X(\mathbb{A}_C) = \prod_{x \in C(\mathbb{R})} X(\mathbb{R}(C)_x)$$

with the direct product of the x -adic topologies.

Remark 5.2 Let X be as above, and let \mathcal{X} be a model of X over C . It is possible to give a simple description of a basis for the topology on $X(\mathbb{A}_C)$ defined above in terms of \mathcal{X} as follows. By slight abuse of notation let $\mathcal{X}(\mathcal{O}_x)$ denote the set of sections $\text{Spec}(\mathcal{O}_x) \rightarrow \mathcal{X} \times_C \text{Spec}(\mathcal{O}_x)$ for every $x \in C(\mathbb{R})$. As we already noted we have a bijection $X(\mathbb{R}(C)_x) \cong \mathcal{X}(\mathcal{O}_x)$ for every $x \in C(\mathbb{R})$ by the valuative criterion of properness, so we have a bijection

$$X(\mathbb{A}_C) \cong \prod_{x \in C(\mathbb{R})} \mathcal{X}(\mathcal{O}_x), \tag{5.2.1}$$

too. For every interpolation condition $\phi : S \rightarrow \mathcal{X} \times_C S$ let

$$U(S, \phi) \subseteq \prod_{x \in C(\mathbb{R})} \mathcal{X}(\mathcal{O}_x)$$

be the subset of all those sections whose pull-back under the closed immersion

$$S \rightarrow \coprod_{x \in C(\mathbb{R})} \text{Spec}(\mathcal{O}_x)$$

is ϕ . These sets form a basis for the topology of $X(\mathbb{A}_C)$ under the map in (5.2.1).

For every morphism $f : X \rightarrow Y$ of varieties over $\mathbb{R}(C)$ and for every $c \in Y(\mathbb{R}(C))$ let X_c denote the fibre of f above c . Our main result Theorem 1.5 follows from the following

Theorem 5.3 *Assume that $C(\mathbb{R})$ is non-empty. Let Y be a smooth, projective, rational variety over $\mathbb{R}(C)$, and let X be a smooth, projective, irreducible variety over $\mathbb{R}(C)$ endowed with a dominant morphism $f : X \rightarrow Y$ with a geometrically irreducible and smooth generic fibre. Let*

$$\underline{M} = \prod_{x \in C(\mathbb{R})} M_x \in X(\mathbb{A}_C)^{CT},$$

and let $U \subset X(\mathbb{A}_C)$ be an open neighbourhood of \underline{M} . Then there exist

- (i) a point $c \in Y(\mathbb{R}(C))$ such that X_c is smooth, and
- (ii) an $\underline{N} \in X_c(\mathbb{A}_C)^{CT}$ such that $\underline{N} \in U$.

Remark 5.4 Note that Theorem 1.5 is trivially true when $C(\mathbb{R})$ is empty. Indeed in this case there is no CT obstruction both for the smooth fibres of f over rational points and for X itself. By assumption all such fibres will have rational points, so X has rational points, too.

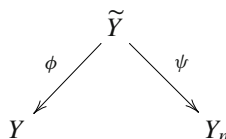
Proof We start the proof with two easy reduction steps. For the sake of simple notation set

$$Y_n = \underbrace{\mathbb{P}_{\mathbb{R}(C)}^1 \times \mathbb{P}_{\mathbb{R}(C)}^1 \times \cdots \times \mathbb{P}_{\mathbb{R}(C)}^1}_{n\text{-times}}.$$

□

Proposition 5.5 *We may assume that $Y = Y_n$ for some n without loss of generality.*

Proof Using resolutions of singularities it follows from the assumption that there is a diagram



of birational morphisms between smooth projective varieties over $\mathbb{R}(C)$ (for some n). Let $\tilde{X} \rightarrow X \times_Y \tilde{Y}$ be a desingularisation of the pull-back $X \times_Y \tilde{Y}$ of X via ϕ which is isomorphic over the nonsingular part, and let $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ be the composition of this desingularisation and the base change of f with respect to ϕ . Let $V \subseteq \tilde{Y}$ be a non-empty Zariski-open subset such that the restrictions of both ϕ and ψ onto V are isomorphisms onto their images, and f is smooth over $\phi(V)$. Then we have a commutative diagram:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\rho} & X \\ \downarrow \tilde{f} & & \downarrow f \\ \tilde{Y} & \xrightarrow{\phi} & Y \end{array}$$

such that ρ maps $\tilde{f}^{-1}(V)$ isomorphically onto $f^{-1}(\phi(V))$. In particular ρ is birational. Let $Z \subset X$ be the complement of $f^{-1}(\phi(V))$, and let $\tilde{Z} \subset \tilde{X}$ be the complement of $\tilde{f}^{-1}(V)$. Now let \underline{M} be an element of $X(\mathbb{A}_C)^{CT}$, as in the claim above. We may assume without loss of generality that its given open neighbourhood U is of the form

$$U = \prod_{x \in C(\mathbb{R})} U_x$$

where U_x is an x -adic open neighbourhood of M_x in $X(\mathbb{R}(C)_x)$ for every $x \in C(\mathbb{R})$. Let $I \subseteq C(\mathbb{R})$ be the set of those $x \in C(\mathbb{R})$ where $U_x \neq X(\mathbb{R}(C)_x)$. The set I is finite.

Now let \mathcal{X} be a model of X over C and for every $x \in C(\mathbb{R})$ let $T_x \subseteq X(\mathbb{R}(C)_x)$ be the x -adic open neighbourhood of M_x which under the bijection $X(\mathbb{R}(C)_x) \cong \mathcal{X}(\mathcal{O}_x)$ corresponds to those sections $\text{Spec}(\mathcal{O}_x) \rightarrow \mathcal{X} \times_C \text{Spec}(\mathcal{O}_x)$ whose special fibre is the same as the special fibre of the section corresponding to M_x . Since Z is a proper Zariski-closed subscheme of X , the set $Z(\mathbb{R}(C)_x)$ is nowhere dense in $X(\mathbb{R}(C)_x)$ with respect to the x -adic topology, so there is a non-empty $W_x \subseteq U_x \cap T_x$, open with respect to the x -adic topology, such that W_x and $Z(\mathbb{R}(C)_x)$ have empty intersection, for every $x \in C(\mathbb{R})$. By the above for every

$$\underline{M}' = \prod_{x \in C(\mathbb{R})} M'_x \in \prod_{x \in C(\mathbb{R})} W_x$$

we have $\sigma(\underline{M}') = \sigma(\underline{M})$ (see Notation 3.5). Pick such an \underline{M}' . Then by Theorem 3.6 we have $\underline{M}' \in X(\mathbb{A}_C)^{CT}$. Clearly it will be enough to show the claim for \underline{M}' and for any of its open neighbourhoods contained in

$$W = \prod_{x \in I} W_x \times \prod_{x \in C(\mathbb{R})-I} X(\mathbb{R}(C)_x).$$

Therefore we may assume, without loss of generality, that $M_x \notin Z(\mathbb{R}(C)_x)$ for every $x \in C(\mathbb{R})$, the set I is non-empty, and $U_x \cap Z(\mathbb{R}(C)_x) = \emptyset$ for every $x \in I$. Since ρ

maps $\tilde{f}^{-1}(V)$ isomorphically onto $f^{-1}(\phi(V))$ we get that there is a $\tilde{M}_x \in \tilde{X}(\mathbb{R}(C)_x)$ for every $x \in C(\mathbb{R})$ such that $\tilde{M}_x \notin \tilde{Z}(\mathbb{R}(C)_x)$ and $\rho(\tilde{M}_x) = M_x$. By Lemma 2.7 we know that

$$\underline{\tilde{M}} = \prod_{x \in C(\mathbb{R})} \tilde{M}_x \in \tilde{X}(\mathbb{A}_C)^{CT}.$$

Moreover for every $x \in I$ there is an x -adic open neighbourhood $\tilde{U}_x \subseteq \tilde{X}(\mathbb{R}(C)_x)$ of \tilde{M}_x such that $\tilde{U}_x \cap \tilde{Z}(\mathbb{R}(C)_x) = \emptyset$ and \tilde{U}_x maps to U_x homeomorphically with respect to ρ . Set

$$\tilde{U} = \prod_{x \in I} \tilde{U}_x \times \prod_{x \in C(\mathbb{R})-I} \tilde{X}(\mathbb{R}(C)_x);$$

it is an open neighbourhood of $\underline{\tilde{M}}$. Since the map $\psi \circ \tilde{f} : \tilde{X} \rightarrow Y_n$ satisfies the conditions of theorem, we get that there is a point $c \in Y_n(\mathbb{R}(C))$ such that \tilde{X}_c is smooth, and an $\underline{\tilde{N}} \in \tilde{X}_c(\mathbb{A}_C)^{CT}$ such that $\underline{\tilde{N}} \in \tilde{U}$. Let x be now an element of I . Then c lies in the image of \tilde{U}_x with respect to $\psi \circ \tilde{f}$, so it must lie in $\psi(V(\mathbb{R}(C)_x))$. Therefore there is a unique $\tilde{c} \in V(\mathbb{R}(C))$ such that $\tilde{c} = \psi(\tilde{c})$. Set $c = \phi(\tilde{c})$ and $\underline{N} = \rho(\underline{\tilde{N}})$. Clearly $c \in \phi(V(\mathbb{R}(C)))$ and hence X_c is smooth. Moreover ρ maps \tilde{X}_c isomorphically onto X_c , so $\underline{N} \in X_c(\mathbb{A}_C)^{CT}$. Finally $\underline{N} \in U$ since ρ maps \tilde{U} into U . □

Lemma 5.6 *We may assume that $Y = \mathbb{P}_{\mathbb{R}(C)}^1$ without loss of generality.*

Proof We may immediately reduce the case when $Y = Y_n$ to the case when $Y = \mathbb{P}_{\mathbb{R}(C)}^1$ via an easy induction on n , so the claim follows from the proposition above. □

6 The fibration method

Let us begin the main part of the proof of Theorem 5.3. By the above we may assume without loss of generality that $Y = \mathbb{P}_{\mathbb{R}(C)}^1$. By resolution of singularities there is an integral, smooth, projective variety \mathcal{X} equipped with a projective dominant morphism $\mathbf{f} : \mathcal{X} \rightarrow C \times \mathbb{P}_{\mathbb{R}}^1$ over \mathbb{R} whose generic fibre is $f : X \rightarrow \mathbb{P}_{\mathbb{R}(C)}^1$. In particular \mathcal{X} is a model of X over C with respect to the composition p of \mathbf{f} with the projection $\pi : C \times \mathbb{P}_{\mathbb{R}}^1 \rightarrow C$ onto the first factor.

Since the generic fibre of f is smooth, the same holds for \mathbf{f} , too. Therefore there is a closed subscheme $\mathcal{Z} \subset C \times \mathbb{P}_{\mathbb{R}}^1$ of positive codimension such that $\mathbf{f}|_{\mathbf{f}^{-1}(\mathcal{U})} : \mathbf{f}^{-1}(\mathcal{U}) \rightarrow \mathcal{U}$ is smooth, where $\mathcal{U} = C \times \mathbb{P}_{\mathbb{R}}^1 - \mathcal{Z}$ is the complement of \mathcal{Z} . Now let \underline{M} be an element of $X(\mathbb{A}_C)^{CT}$, and let $U \subset X(\mathbb{A}_C)$ be an open neighbourhood of \underline{M} , as in the claim above. We may assume without loss of generality that there is an interpolation condition $\phi : S \rightarrow \mathcal{X} \times_C S$ such that the open neighbourhood U given is of the form $U(S, \phi)$ by Remark 5.2.

Proposition 6.1 *There is an $\underline{M}' \in X(\mathbb{A}_C)^{CT} \cap U$ such that*

- (i) the section $\sigma(\underline{M}') : C(\mathbb{R}) \rightarrow \mathcal{X}_{sm}(\mathbb{R})$ is Nash,
- (ii) the section $\sigma(\underline{M}')$ is compatible with ϕ ,
- (iii) the section $\mathbf{f}(\mathbb{R}) \circ \sigma(\underline{M}')$ only intersects $\mathcal{Z}(\mathbb{R})$ in finitely many points.

Proof By our assumption U is of the form

$$U = \prod_{x \in C(\mathbb{R})} U_x$$

where U_x is an x -adic open neighbourhood of M_x in $X(\mathbb{R}(C)_x)$ for every $x \in C(\mathbb{R})$. Let $I \subseteq C(\mathbb{R})$ be the set of those $x \in C(\mathbb{R})$ where $U_x \neq X(\mathbb{R}(C)_x)$. The set I is finite. For every connected component $D \subset C(\mathbb{R})$ for all but finitely many $x \in D$ the intersection $\mathcal{Z} \cap \pi^{-1}(x)$ is a zero dimensional scheme, since \mathcal{Z} has positive codimension in $C \times \mathbb{P}_{\mathbb{R}}^1$. Since p is generically smooth and has geometrically irreducible fibres, for all but finitely many $x \in D$ the fibre \mathcal{X}_x is smooth and geometrically irreducible. Therefore for all D as above we may choose a point $x(D) \in D$ which does not lie in I , the intersection $\mathcal{Z}(\mathbb{R}) \cap \pi^{-1}(x(D))(\mathbb{R})$ is finite, and $\mathcal{X}_{x(D)}$ is smooth and geometrically irreducible.

For every D as above let $E_D \subset \mathcal{X}_{x(D)}(\mathbb{R})$ denote the connected component containing $m_{x(D)} = \sigma(\underline{M})(x(D)) \in \mathcal{X}_{x(D)}(\mathbb{R})$. We claim that there is an $n_D \in E_D$ such that $n_D \notin \mathbf{f}^{-1}(\mathcal{Z})(\mathbb{R})$. Assume that this is not the case for some $D \in \pi_0(C(\mathbb{R}))$. Then the image of E_D under $\mathbf{f}(\mathbb{R})$ lies in the finite set $\mathcal{Z}(\mathbb{R}) \cap \pi^{-1}(x(D))(\mathbb{R})$. But E_D is connected, so is its image under the continuous map $\mathbf{f}(\mathbb{R})$, which therefore must be a point $p_D \in \mathcal{Z}(\mathbb{R}) \cap \pi^{-1}(x(D))(\mathbb{R})$. Note that E_D is Zariski-dense in $\mathcal{X}_{x(D)}$. Indeed suppose that this is not the case; as the latter is geometrically irreducible, the Zariski-closure of E_D has dimension strictly less than the dimension d of $\mathcal{X}_{x(D)}$. Therefore the dimension of the semi-algebraic set E_D is also less than d . But $\mathcal{X}_{x(D)}$ is smooth, so the dimension of E_D is d by the inverse function theorem, which is a contradiction. We get that $\mathcal{X}_{x(D)}$ lies in $\mathbf{f}^{-1}(p_D)$, and hence the fibre of \mathbf{f} over any other point in $\pi^{-1}(x(D))$ is empty. But this is a contradiction, so our original assumption on the image of E_D with respect to $\mathbf{f}(\mathbb{R})$ is false.

For every D as above choose an $N_{x(D)} \in X(\mathbb{R}(C)_{x(D)})$ such that the special fibre of the section $\text{Spec}(\mathcal{O}_{x(D)}) \rightarrow \mathcal{X} \times_C \text{Spec}(\mathcal{O}_{x(D)})$ associated to $N_{x(D)}$ is the point $n_D \in E_D - \mathbf{f}^{-1}(\mathcal{Z})(\mathbb{R})$ found above. (This is possible by Hensel’s lemma.) Let $\underline{N} \in X(\mathbb{A}_C)^{CT}$ be the point we get from \underline{M} by replacing $M_{x(D)}$ with $N_{x(D)}$ for every D as above. Clearly $\underline{N} \in U$, and by Proposition 3.4 we have $\underline{N} \in X(\mathbb{A}_C)^{CT}$. Let T be the effective zero cycle $S + \sum_{D \in \pi_0(C(\mathbb{R}))} x_D$ and let $\psi : T \rightarrow \mathcal{X} \times_C T$ be the unique interpolation condition which subsumes ϕ and $\psi(y)$ is the special fibre of the section $\text{Spec}(\mathcal{O}_y) \rightarrow \mathcal{X} \times_C \text{Spec}(\mathcal{O}_y)$ associated to N_y for every $y \in T$. This interpolation condition ψ is weakly compatible with $\sigma(\underline{N})$, so there is a Nash section $\rho : C(\mathbb{R}) \rightarrow \mathcal{X}_{sm}(\mathbb{R})$ compatible with ψ by Proposition 3.19. Since $\rho(x_D) \notin \mathcal{Z}(\mathbb{R})$ for every $D \in \pi_0(C(\mathbb{R}))$ by construction, by Lemma 4.4 the section ρ only intersects $\mathcal{Z}(\mathbb{R})$ in finitely many points. Again by Hensel’s lemma there is an $\underline{M}' \in X(\mathbb{A}_C)^{CT}$ such that $\sigma(\underline{M}') = \rho$ and $M'_x = N_x = M_x$ for every $x \in I$. Clearly $\underline{M}' \in U$ and by Theorem 3.6 we have $\underline{M}' \in X(\mathbb{A}_C)^{CT}$. \square

Replacing \underline{M} by \underline{M}' we may assume without loss of generality that $\sigma(\underline{M})$ is Nash, compatible with ϕ , and $\mathbf{f}(\mathbb{R}) \circ \sigma(\underline{M})$ only intersects $\mathcal{Z}(\mathbb{R})$ in finitely many points. Recall that a Nash map $a : A \rightarrow B$ of Nash manifolds over \mathbb{R} is *Nash trivial* if there is a Nash manifold L and a Nash diffeomorphism $b : L \times B \rightarrow A$ (over \mathbb{R}) such that $a \circ b : L \times B \rightarrow B$ is the projection onto the second coordinate.

Proposition 6.2 *There is an admissible semi-algebraic subset $R \subseteq C(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ such that*

- (i) *the image \mathbf{I} of $\mathbf{f}(\mathbb{R}) \circ \sigma(\underline{M})$ lies in R ,*
- (ii) *the set R^o lies in $\mathcal{U}(\mathbb{R})$,*
- (iii) *there is a Nash retraction $\rho : R^o \rightarrow R^o \cap \mathbf{I}$,*
- (iv) *for every semi-algebraic connected component $O \subset R^o$ the map*

$$\mathbf{f}(\mathbb{R})|_{\mathbf{f}(\mathbb{R})^{-1}(O)} : \mathbf{f}(\mathbb{R})^{-1}(O) \rightarrow O$$

is Nash-trivial.

Proof Since $\mathcal{U}(\mathbb{R})$ is a Nash manifold over \mathbb{R} , there is a finite covering $\{V_j\}_{j \in J}$ of $\mathcal{U}(\mathbb{R})$ by open semi-algebraic subsets which are Nash diffeomorphic to an affine space over \mathbb{R} , necessarily of dimension 2, by Lemma 3.2 of [9] on page 1217. For every $j \in J$ the restriction

$$\mathbf{f}(\mathbb{R})|_{\mathbf{f}(\mathbb{R})^{-1}(V_j)} : \mathbf{f}(\mathbb{R})^{-1}(V_j) \rightarrow V_j$$

is a proper surjective submersion of Nash manifolds, so it is Nash trivial by Proposition 5.2 of [6] on page 368 and the main theorem of [9] on page 1209. For every $j \in J$ the intersection $V_j \cap \mathbf{I}$ is semi-algebraic and open in \mathbf{I} , so its image Q_j under the projection $\pi(\mathbb{R})$ is semi-algebraic and open in $C(\mathbb{R})$. Therefore Q_j is the union of finitely many pairwise disjoint open intervals or circles, i.e. open and connected sets. By possibly refining the covering $\{V_j\}_{j \in J}$ we may achieve that we have only intervals, no circles.

Let B be the union of the end-points of these open intervals for all $j \in J$, where by end-points we mean accumulation points not in the interval. Since the set of these intervals is finite, the set B is also finite. The complement of B in $C(\mathbb{R})$ is the union of finitely many pair-wise disjoint open intervals; let K denote the set of these open intervals. Since the sets $\{V_j\}_{j \in J}$ cover all but finitely many points of \mathbf{I} , for every $H \in K$ the pre-image $\pi(\mathbb{R})^{-1}(H) \cap \mathbf{I}$ lies in V_j for some $j \in J$. For every such H fix such a V_j and let $W_H \subset V_j$ be a semi-algebraic tubular neighbourhood of $\pi(\mathbb{R})^{-1}(H) \cap \mathbf{I}$ in $\pi(\mathbb{R})^{-1}(H) \cap V_j$. (The latter exists by Corollary 8.9.5 of [2] on page 199.)

Let R be the union of the $\{W_H\}_{H \in K}$ and \mathbf{I} . Since B is finite, the set R is admissible, and (i) holds for R by construction. Since the elements of K are pair-wise disjoint, the elements of $\{W_H\}_{H \in K}$ are also pair-wise disjoint. For every $H \in K$ the semi-algebraic set $\pi(\mathbb{R})^{-1}(H) \cap \mathbf{I}$ is connected, so the same holds for its tubular neighbourhood W_H . As $R^o = \bigcup_{H \in K} W_H$, we get that the sets $\{W_H\}_{H \in K}$ are the connected components of R^o . Since $W_H \subset V_j$ for some j , condition (iv) holds. As the sets $\{V_j\}_{j \in J}$ lie in

$\mathcal{U}(\mathbb{R})$, condition (ii) is true. Moreover W_H is a semi-algebraic tubular neighbourhood of $W_H \cap \mathbf{I} = \pi(\mathbb{R})^{-1}(H) \cap \mathbf{I}$, so there is a Nash retraction $\rho_H : W_H \rightarrow W_H \cap \mathbf{I}$ for every $H \in K$. Therefore (iii) holds, too. \square

Definition 6.3 For every $x \in C \times \mathbb{P}^1(\mathbb{R})$ and every 1-dimensional subspace L of the tangent space of $C \times \mathbb{P}^1$ at x let $\mathcal{X}_{L,sm} \subset \mathcal{X}_x$ be the largest open sub-scheme such that for every closed point P of $\mathcal{X}_{L,sm}$ the image of the differential of \mathbf{f} at P contains L . Let $\sigma : C(\mathbb{R}) \rightarrow \mathcal{X}(\mathbb{R})$ be a Nash section of $p(\mathbb{R})$ and let $R_+ \subseteq C(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ be an admissible semi-algebraic subset which contains the image $\text{Im}(\mathbf{f}(\mathbb{R}) \circ \sigma)$ of $\mathbf{f}(\mathbb{R}) \circ \sigma$. For every point x on $\text{Im}(\mathbf{f}(\mathbb{R}) \circ \sigma)$ let $L(x)$ denote the tangent line of $\mathbf{f}(\mathbb{R}) \circ \sigma$ at x ; it is a 1-dimensional subspace of the tangent space of $C \times \mathbb{P}^1$ at x . A butterfly extension of σ on R_+ is a semi-algebraic section $\beta : R_+ \rightarrow \mathcal{X}(\mathbb{R})$ of $\mathbf{f}(\mathbb{R})$ such that σ restricted to R_+^o is continuous, the composition $\beta \circ \mathbf{f}(\mathbb{R}) \circ \sigma$ is σ , and for every kissing point x of R_+ on $\text{Im}(\mathbf{f}(\mathbb{R}) \circ \sigma)$ the intersection of the closure of the image of β with $\mathcal{X}_x(\mathbb{R})$ lies in the connected component of $\beta(x)$ in $\mathcal{X}_{L(x),sm}(\mathbb{R})$. (Note that $\beta(x) = \sigma(\pi(\mathbb{R})(x))$, so $\beta(x)$ does lie in $\mathcal{X}_{L(x),sm}(\mathbb{R})$, since σ is a Nash section.)

Proposition 6.4 *There is an admissible semi-algebraic subset $R_+ \subseteq R$ containing $\text{Im}(\mathbf{f}(\mathbb{R}) \circ \sigma(\underline{M})) = \mathbf{I}$ such that there is a butterfly extension $s : R_+ \rightarrow \mathcal{X}(\mathbb{R})$ of $\sigma(\underline{M})$.*

Proof We will need the following easy semi-algebraic separation lemma. \square

Lemma 6.5 *Let F and G be two disjoint closed semi-algebraic subsets of $\mathcal{X}(\mathbb{R})$. Then there are two disjoint open semi-algebraic subsets $A, B \subset \mathcal{X}(\mathbb{R})$ such that $F \subset A$ and $G \subset B$.*

Proof Since \mathcal{X} is projective, by Theorem 3.4.4 of [2] on page 72 there is a continuous semi-algebraic embedding $\iota : \mathcal{X}(\mathbb{R}) \rightarrow \mathbb{R}^m$ for some positive integer m . Because \mathcal{X} is projective, the semi-algebraic set $\mathcal{X}(\mathbb{R})$ is compact, so the same holds for its closed subsets F and G . Therefore $\iota(F)$ and $\iota(G)$ are closed in \mathbb{R}^m , and by elimination of quantifiers these sets are also semi-algebraic. Then $d = \text{dist}(F, G) > 0$, where dist stands for the Euclidean distance. The sets

$$A = \left\{ \underline{x} \in \mathbb{R}^m \mid \text{dist}(\underline{x}, F) < \frac{d}{3} \right\}, \quad B = \left\{ \underline{x} \in \mathbb{R}^m \mid \text{dist}(\underline{x}, G) < \frac{d}{3} \right\}$$

are open, semi-algebraic, contain F and G , respectively, and $A \cap B = \emptyset$. \square

By removing every kissing point of R which does not lie on \mathbf{I} we may assume that every kissing point of R lies on \mathbf{I} without loss of generality. Let $\rho : R^o \rightarrow R^o \cap \mathbf{I}$ be a Nash retraction. For every semi-algebraic connected component $O \subset R^o$ let M_O be a compact Nash manifold such that there is a Nash diffeomorphism $b_O : M_O \times O \rightarrow \mathbf{f}(\mathbb{R})^{-1}(O)$ such that $\mathbf{f}(\mathbb{R}) \circ b_O : M_O \times O \rightarrow O$ is the projection onto the first coordinate. For every such O let $a_O : M_O \times O \rightarrow M_O$ be the projection onto the second coordinate, let $\rho_O : O \rightarrow O \cap \mathbf{I}$ be the restriction of ρ onto O , let $r_O : O \rightarrow M_O$ denote the composition:

$$a_O \circ b_O^{-1} \circ \sigma(\underline{M}) \circ \pi(\mathbb{R}) \circ \rho_O,$$

and let $s_O : O \rightarrow \mathbf{f}(\mathbb{R})^{-1}(O)$ be the composition of the map:

$$O \rightarrow O \times M_O \text{ given by } t \mapsto (t, r_O)$$

and b_O . Let s^o be the coproduct of the maps s_O for all such O . The map s^o is a Nash section of

$$\mathbf{f}(\mathbb{R})|_{\mathbf{f}(\mathbb{R})^{-1}(R^o)} : \mathbf{f}(\mathbb{R})^{-1}(R^o) \rightarrow R^o$$

which extends $\sigma(\underline{M}) \circ \pi(\mathbb{R})|_{R^o \cap \mathbf{I}}$. It has a unique extension $s : R \rightarrow \mathcal{X}(\mathbb{R})$ such that $s \circ f(\mathbb{R}) \circ \sigma(\underline{M})$ is $\sigma(\underline{M})$.

For every point x on \mathbf{I} let $L(x)$ denote the tangent line of $\mathbf{f}(\mathbb{R}) \circ \sigma(\underline{M})$ at x . Note that for every $x \in \mathbf{I}$ we have $s(x) = \sigma(\underline{M})(\pi(\mathbb{R})(x))$, and $\sigma(\underline{M})$ is a Nash section, so $s(x)$ lies in $\mathcal{X}_{L(x),sm}(\mathbb{R})$. Therefore it will be enough to show that there is an admissible semi-algebraic subset $R_+ \subseteq R$ containing \mathbf{I} such that for every kissing point x of R_+ on \mathbf{I} the intersection of the closure of the image of s with $\mathcal{X}_x(\mathbb{R})$ lies in the connected component of $s(x)$ in $\mathcal{X}_{L(x),sm}$. Let K denote the set of kissing points of R .

For every x on \mathbf{I} let $\mathcal{X}_{L(x),bad}$ be the complement of $\mathcal{X}_{L(x),sm}$ in \mathcal{X}_x . For every $x \in K$ the subscheme $\mathcal{X}_{L(x),bad}$ of \mathcal{X} is Zariski-closed, so the semi-algebraic set $\mathcal{X}_{L(x),bad}(\mathbb{R})$ is closed in $\mathcal{X}(\mathbb{R})$, and does not contain $s(x)$, hence by Lemma 6.5 we may pick two disjoint open semi-algebraic subsets $A_x, B_x \subset \mathcal{X}(\mathbb{R})$ such that $\mathcal{X}_{L(x),bad}(\mathbb{R}) \subset A_x$ and $s(x) \in B_x$. Now let E be any semi-algebraic connected component of $\mathcal{X}_{L(x),sm}(\mathbb{R})$ which does not contain $s(x)$, and let \bar{E} denote its closure in $\mathcal{X}_x(\mathbb{R})$. Since E is closed in $\mathcal{X}_{L(x),sm}(\mathbb{R})$, and $s(x)$ lies in $\mathcal{X}_{L(x),sm}(\mathbb{R})$, the set \bar{E} does not contain $s(x)$, so by Lemma 6.5 we may pick two disjoint open semi-algebraic subsets $A_E, B_E \subset \mathcal{X}(\mathbb{R})$ such that $E \subset A_E$ and $s(x) \in B_E$.

For every $x \in K$ let W_x be the intersection of B_x and the B_E for all E as above. The set of connected components of $\mathcal{X}_{L(x),sm}(\mathbb{R})$ is finite, therefore the set W_x is an open semi-algebraic neighbourhood of $s(x)$ such the intersection of its closure with $\mathcal{X}_x(\mathbb{R})$ lies in the connected component of $s(x)$ in $\mathcal{X}_{L(x),sm}$. Since $\sigma(\underline{M})$ is continuous, for every x as above $\pi(\mathbb{R})(x)$ has an open connected neighbourhood V_x in $C(\mathbb{R})$ such that $\sigma(\underline{M})$ maps V_x into W_x . We may assume that the sets V_x are pair-wise disjoint by shrinking them, if this is necessary. For every $x \in K$ let R_x be the intersection of $\pi(\mathbb{R})^{-1}(V_x)$ with the image of $W_x \cap \mathbf{f}(\mathbb{R})^{-1}(R^o)$ with respect to \mathbf{f} . Since $R^o \subset \mathcal{U}(\mathbb{R})$, the restriction of \mathbf{f} onto $W_x \cap \mathbf{f}(\mathbb{R})^{-1}(R^o)$ is a Nash submersion, and hence open by the implicit function theorem. Hence R_x is open and semi-algebraic. Since $\sigma(\underline{M})$ is a section, the set R_x also contains the image of $V_x - \pi(\mathbb{R})(x)$ with respect to $\mathbf{f}(\mathbb{R}) \circ \sigma(\underline{M})$.

Let T be the interior of the complement of the union of V_x for all $x \in K$ in $C(\mathbb{R})$. Then the set

$$R_{++} = \left(\pi(\mathbb{R})^{-1}(T) \cap R^o \right) \cup \bigcup_{x \in K} R_x \subset R^o$$

is open, semi-algebraic and contains all but finitely many points of \mathbf{I} . Therefore $R_+ = R_{++} \cup \mathbf{I}$ is an admissible subset of R . For every $x \in K$ the intersection of the closure of the image of $s|_{R_+}$ with $\mathcal{X}_x(\mathbb{R})$ lies in the connected component of $s(x)$ in $\mathcal{X}_{L(x),sm}$

by construction. If x is a kissing point of R_+ not in K , then $x \in R^o$, so s is continuous at x , and hence the intersection of the closure of the image of $s|_{R_+}$ with $\mathcal{X}_x(\mathbb{R})$ is just $s(x)$. \square

Let $X_{sm} \subseteq X$ be the smooth locus of f .

Lemma 6.6 *Let x be a point in $C(\mathbb{R})$, let $y \in X_{sm}(\mathbb{R}(C)_x)$ be a point, and let $\mathbf{U} \subset X(\mathbb{R}(C)_x)$ be an open neighbourhood of y in the x -adic topology. Then there is an open neighbourhood \mathbf{V} of $f(y)$ in the x -adic topology such that for every $z \in \mathbf{V}$ the set $f^{-1}(z)(\mathbb{R}(C)_x) \cap \mathbf{U}$ is non-empty.*

Proof By the implicit function theorem there is an open neighbourhood \mathbf{W} of $f(y)$ in the x -adic topology and a section $\mathbf{s} : \mathbf{W} \rightarrow X_{sm}(\mathbb{R}(C)_x)$ of the x -adic analytic map $X_{sm}(\mathbb{R}(C)_x) \rightarrow \mathbb{P}^1(\mathbb{R}(C)_x)$ induced by f such that $\mathbf{s}(f(y)) = y$. This is continuous with respect to the x -adic topology. Therefore there is an open neighbourhood $\mathbf{V} \subset \mathbf{W}$ of $f(y)$ in the x -adic topology such that $\mathbf{s}(\mathbf{V}) \subset \mathbf{U}$. For every $z \in \mathbf{V}$ we have $\mathbf{s}(z) \in f^{-1}(z)(\mathbb{R}(C)_x) \cap \mathbf{U}$. \square

For every $x \in I$ let P_x denote the formal completion of the Nash section $\sigma(\underline{M})$ around $\sigma(\underline{M})(x)$; it is a section $\text{Spec}(\mathcal{O}_x) \rightarrow \mathcal{X} \times_C \text{Spec}(\mathcal{O}_x)$. By slight abuse of notation let the same symbol P_x denote its generic fibre, too. Since $\sigma(\underline{M})$ only intersects $\mathcal{Z}(\mathbb{R})$ in finitely many points, we have $P_x \in X_{sm}(\mathbb{R}(C)_x)$. By property (ii) of Proposition 6.1 we have $P_x \in U_x$ for every $x \in I$, so by Lemma 6.6 for every $x \in I$ there is an open neighbourhood V_x of $f(P_x)$ in the x -adic topology such that for every $z \in V_x$ the set $f^{-1}(z)(\mathbb{R}(C)_x) \cap U_x$ is non-empty. We may assume that $V_x = \mathbb{P}^1(\mathbb{R}(C)_x)$ whenever $U_x = X(\mathbb{R}(C)_x)$. There is an interpolation condition $\tilde{\kappa} : \tilde{T} \rightarrow \mathbb{P}^1$ compatible with $\mathbf{f}(\mathbb{R}) \circ \sigma(\underline{M})$ such

$$U(\tilde{T}, \tilde{\kappa}) \subset \prod_{x \in I} V_x \times \prod_{x \in C(\mathbb{R}) - I} \mathbb{P}^1(\mathbb{R}(C)_x).$$

Now let $R_+ \subseteq R$ be an admissible semi-algebraic subset containing the image \mathbf{I} of $\mathbf{f}(\mathbb{R}) \circ \sigma(\underline{M})$ and let $s : R_+ \rightarrow \mathcal{X}(\mathbb{R})$ be a butterfly extension of $\sigma(\underline{M})$. By removing every kissing point of R_+ which does not lie on \mathbf{I} we may even assume that every kissing point of R_+ lies on \mathbf{I} . Let $\kappa : T \rightarrow \mathbb{P}^1$ be an interpolation condition subsuming $\tilde{\kappa} : T \rightarrow \mathbb{P}^1$, compatible with $\mathbf{f}(\mathbb{R}) \circ \sigma(\underline{M})$ such that for every kissing point z of R_+ the point $\pi(\mathbb{R})(z)$ has coefficient at least 2 in T .

By Theorem 4.2 there is a regular map $\mathbf{c} : C \rightarrow \mathbb{P}^1$ compatible with κ such that the graph $\Gamma_{\mathbf{c}}$ of \mathbf{c} in $C \times \mathbb{P}^1(\mathbb{R})$ lies in R_+ . Let $c \in \mathbb{P}^1(\mathbb{R}(C))$ be the generic point of \mathbf{c} . Let \mathcal{X}_c denote the closed subscheme $\mathbf{f}^{-1}(\Gamma_{\mathbf{c}}) \subset \mathcal{X}$ and let $\pi_c : \mathcal{X}_c \rightarrow C$ be the composition of \mathbf{f} and π . Since all but finitely many points of $\Gamma_{\mathbf{c}}$ lie in $R_+^o \subset \mathcal{U}(\mathbb{R})$, the generic fibre X_c of \mathcal{X}_c (with respect to π_c) is smooth. Let $\mathcal{X}_{c,sm} \subseteq \mathcal{X}_c$ be the smooth locus of π_c . By resolution of singularities there is a sequence of blow-ups $r : \tilde{\mathcal{X}}_c \rightarrow \mathcal{X}_c$ such that $\tilde{\mathcal{X}}_c$ is a model of X_c over C and contains $\mathcal{X}_{c,sm}$ as an open sub C -scheme, i.e. the restriction $r|_{r^{-1}(\mathcal{X}_{c,sm})} : r^{-1}(\mathcal{X}_{c,sm}) \rightarrow \mathcal{X}_{c,sm}$ is an isomorphism.

Lemma 6.7 *The image of $s \circ \mathbf{c}(\mathbb{R})$ lies in $\mathcal{X}_{c,sm}(\mathbb{R})$ and the composition*

$$s \circ \mathbf{c}(\mathbb{R}) : C(\mathbb{R}) \rightarrow \mathcal{X}_{c,sm}(\mathbb{R}) \subset \tilde{\mathcal{X}}_c(\mathbb{R})$$

is weakly continuous.

Proof Since restriction of \mathbf{f} onto $\mathbf{f}^{-1}(\mathcal{U})$ is smooth, and R_+^o lies in $\mathcal{U}(\mathbb{R})$, for every $z \in \Gamma_c(\mathbb{R}) \cap R_+^o$ the point $s(z)$ lies in $\mathcal{X}_{c,sm}(\mathbb{R})$. If $z \in \Gamma_c(\mathbb{R}) - R_+^o$ then z is a kissing point of R_+ lying on \mathbf{I} . Therefore the point $\pi(\mathbb{R})(z)$ has coefficient at least 2 in T , so the tangent lines of Γ_c and \mathbf{I} at z are the same; let L denote this tangent line. Note that the fibre of $\mathcal{X}_{c,sm}$ above z is $\mathcal{X}_{L,sm}$. Since $\sigma(\underline{M})$ is a Nash section, its tangent vector at $s(x)$ maps non-trivially into L , and we get that $s(x)$ lies in $\mathcal{X}_{c,sm}$ in this case, too. Also note that the closure of the image of $s \circ \mathbf{c}(\mathbb{R})$, considered as a map into $\tilde{\mathcal{X}}_c(\mathbb{R})$, lies in the pre-image of the closure of the image of $s \circ \mathbf{c}(\mathbb{R})$, considered as a map into $\mathcal{X}_c(\mathbb{R})$, with respect to $r(\mathbb{R})$. Therefore the second half of the claim follows from Proposition 3.21. \square

Now we can complete the proof of Theorem 5.3. Since $c \in U(T, \kappa)$, for every $x \in I$ the set $f^{-1}(c)(\mathbb{R}(C)_x) \cap U_x$ is non-empty, and we may choose N_x from this set. Using Hensel's lemma for every $x \in C(\mathbb{R}) - I$ we get that there is an $N_x \in X_c(\mathbb{R}(C)_x)$ such that the special fibre of the section $\text{Spec}(\mathcal{O}_x) \rightarrow \mathcal{X} \times_C \text{Spec}(\mathcal{O}_x)$ associated to N_x is the point $s \circ \mathbf{c}(\mathbb{R})(x)$. Set $\underline{N} = \prod_{x \in C(\mathbb{R})} N_x$. Clearly $\underline{N} \in U$. Moreover $\sigma(\underline{N}) = s \circ \mathbf{c}(\mathbb{R})$, which is weakly continuous by Lemma 6.7, so $\underline{N} \in X_c(\mathbb{A}_C)^{CT}$ by Theorem 3.6. \square

Acknowledgements A. Pál wishes to acknowledge the generous support of the Imperial College Mathematics Department's Platform Grant. E. Szabó was supported by the National Research, Development and Innovation Office (NKFIH) Grants K120697, K115799. The project leading to this application has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Grant agreement no. 741420).

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Benoist, O., Wittenberg, O.: The tight approximation property (2019). [arXiv:1907.10859](https://arxiv.org/abs/1907.10859)
2. Bochnak, J., Coste, M., Roy, M.-F.: Real Algebraic Geometry, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 36. Springer, Berlin (1998)
3. Colliot-Thélène, J.-L.: Groupes linéaires sur les corps de fonctions de courbes réelles. *J. Reine Angew. Math.* **474**, 139–167 (1996)
4. Colliot-Thélène, J.-L.: Points rationnels sur les fibrations. Higher Dimensional Varieties and Rational Points (Budapest, 2001). Bolyai Society Mathematical Studies, vol. 12, pp. 171–221. Springer, Berlin (2003)

5. Colliot-Thélène, J.-L., Kahn, B.: Cycles de codimension 2 et H^3 non-ramifié pour les variétés sur les corps finis. *J. K-Theory* **11**, 1–53 (2013)
6. Coste, M., Shiota, M.: Nash triviality in families of Nash manifolds. *Invent. Math.* **108**, 349–368 (1992)
7. Ducros, A.: L'obstruction de réciprocity à l'existence de points rationnels pour certaines variétés sur le corps des fonctions d'une courbe réelle. *J. Reine Angew. Math.* **504**, 73–114 (1998)
8. Ducros, A.: Fibrations en variétés de Severi-Brauer au-dessus de la droite projective sur le corps des fonctions d'une courbe réelle. *C. R. Acad. Sci. Paris Sér. I Math.* **327**, 71–75 (1998)
9. Escribano, J.: Nash triviality in families of Nash mappings. *Ann. Inst. Fourier* **51**, 1209–1228 (2001)
10. Graber, T., Harris, J., Starr, J.: Families of rationally connected varieties. *J. AMS* **16**, 57–67 (2003)
11. Harpaz, Y., Wittenberg, O.: On the fibration method for zero-cycles and rational points. *Ann. Math.* **183**, 229–295 (2016)
12. Łojasiewicz, S.: Ensembles semi-analytiques. Institut des Hautes Études Scientifiques, Bures-sur-Yvette (1965). <https://perso.univ-rennes1.fr/michel.coste/Lojasiewicz.pdf>
13. Nachbin, L.: Sur les algèbres denses de fonctions différentiables sur une variété. *C. R. Acad. Sci. Paris* **228**, 1549–1551 (1949)
14. Scheiderer, C.: Hasse principles and approximation theorems for homogeneous spaces over fields of virtual cohomological dimension one. *Invent. Math.* **125**, 307–365 (1996)
15. Witt, E.: Zerlegung reeller algebraischer Funktionen in Quadrate. Schiefkörper über reellem Funktionkörper. *J. Reine Angew. Math.* **171**, 4–11 (1934)
16. Witt, E.: Theorie der quadratischen Formen in beliebigen Körpern. *J. Reine Angew. Math.* **176**, 31–44 (1937)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.