Mathematische Annalen (2022) 382:1–36
https://doi.org/10.1007/s00208-020-02050-0

Mathematische Annalen

Renormalization and energy conservation for axisymmetric fluid flows

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Received: 4 October 2019 / Revised: 9 July 2020 / Published online: 25 July 2020

Abstract

We study vanishing viscosity solutions to the axisymmetric Euler equations without swirl with (relative) vorticity in L^p with $p > 1$. We show that these solutions satisfy the corresponding vorticity equations in the sense of renormalized solutions. Moreover, we show that the kinetic energy is preserved provided that $p > 3/2$ and the vorticity is nonnegative and has finite second moments.

Contents

1 Introduction

For axisymmetric incompressible flows without swirl, the (originally three-dimensional) Navier–Stokes and Euler equations can be reduced to two-dimensional

Communicated by Y. Giga.

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mathematical models which are obtained by assuming a cylindrical symmetry for both the physical space variables and the velocity components. Despite this simplification, such flows are still able to describe interesting physical phenomena like the motion and interaction of toroidal vortex rings. On the mathematical level, even though two-dimensional, the (vaguely defined) degree of difficulty of analyzing solution properties lies somewhere between that of the two-dimensional planar equations and the full three-dimensional model. Indeed, as we shall see later on, axisymmetric flows[1] do still feature vortex stretching and some of the standard global estimates have an unambiguous three-dimensional character. On the other hand, many of the features of the Biot–Savart kernel are typically two-dimensional even though some helpful symmetry properties are lost.

In the present work, our aim is to study renormalization and energy conservation of solutions to the Euler equations that are obtained as vanishing viscosity solutions from the axisymmetric Navier–Stokes equations. Here, renormalization is to be understood in the sense of DiPerna and Lions [24], that is, a solution is called renormalized if the chain rule of differentiation applies in a suitable way. We are particularly interested into solutions whose vorticity is merely L^p integrable in a sense that will be made precise later.

The analogous (though in some parts technically much simpler) studies for the two-dimensional planar equations have been conducted quite recently: As long as the vorticity is L^p integrable with exponent $p \geq 2$, DiPerna and Lions's theory for transport equations (combined with Calderón–Zygmund theory) ensures that the vorticity is a renormalized solution of the corresponding vorticity equation [39]. This fact is true regardless of the construction of the solution. If $p \in (1, 2)$, renormalization properties are proved in [18] for vanishing viscosity solutions. The argument in this work relies on a duality argument and exploits the DiPerna–Lions theory. This theory, however, does not apply to the $p = 1$ case, in which the associated velocity gradient is a singular integral of an L^1 function. Instead, a stability-based theory for continuity equations proposed in [42,43] can be suitably generalized in order to handle this situation and to extend the results from [18,39] to the limiting case $p = 1$; see [17].

Conservation of kinetic energy for vanishing viscosity solutions with L^p vorticity, $p > 1$, is established in [14] for the planar two-dimensional setting (on the torus). The corresponding three-dimensional problem gained much attention in recent years, particularly in connection with Onsager's conjecture [40], which states that the threshold Hölder regularity for the validity of energy conservation is the exponent $1/3$. Energy conservation for larger Hölder exponents was proved in [16], see also [26] for partial results and [13] for improvements. In particular, in the last paper the authors show conservation of energy for velocities in the Besov space $B^{1/3}_{3,s}$, for $1 \leq s < \infty$, which contains $W^{\frac{1}{3},3}$ for all $s \geq 3$. Note that by fractional Sobolev inequalities in $\mathbb{R}^3$ the $W^{1/3+,3}$ regularity holds for any vorticity in L^p with $p > 9/5$. More recently, the sharpness of the Hölder exponent was proved in [32], building up on the theory developed in [7–9,20,21].

Before discussing our precise findings and the relevant earlier results for the axisymmetric equations, we shall introduce the mathematical model. The Euler equations for

[1] From here on we shall omit the specification *without swirl* for convenience.

an ideal fluid in $\mathbb{R}^3$ are given by the system

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = 0, \tag{1}$$

$$\nabla_x \cdot u = 0, \tag{2}$$

where $u = u(t, x) \in \mathbb{R}^3$ is the fluid velocity and $p = p(t, x) \in \mathbb{R}$ is the pressure. In this formulation, the (constant) fluid density is set to 1. Whenever the fluid has locally finite kinetic energy, which will be the case in the regularity framework considered in this paper, the Euler equations can be interpreted in the sense of distributions.

Definition 1 Let $T > 0$ and $u_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^3)^3$ be given. A vector field $u \in L^2_{\mathrm{loc}}((0, T) \times \mathbb{R}^3)^3$ is called a distributional solution to the Euler equations (1), (2) if

$$\int_0^T \int_{\mathbb{R}^3} (\partial_t F \cdot u + \nabla_x F : u \otimes u)\ dxdt + \int_{\mathbb{R}^3} F(t = 0) \cdot u_0\, dx = 0$$

for any divergence-free vector field $F \in C_c^\infty([0, T) \times \mathbb{R}^3)^3$ and

$$\int_0^T \int_{\mathbb{R}^3} \nabla_x f \cdot u\, dxdt = 0$$

for any $f \in C_c^\infty([0, T) \times \mathbb{R}^3)$.

We restrict ourself to the case of axisymmetric solutions without swirl. That is, if (r, θ, z) are the cylindrical coordinates of a point $x \in \mathbb{R}^3$, i.e., $x = (r\cos\theta, r\sin\theta, z)^T$, we shall assume that

$$u = u(t, r, z), \quad \text{and} \quad u = u^r e_r + u^z e_z,$$

where e_r and e_z are the unit vectors in radial and vertical directions, which form together with the angular unit vector e_θ a basis of $\mathbb{R}^3$,

$$e_r = (\cos\theta, \sin\theta, 0)^T, \quad e_\theta = (-\sin\theta, \cos\theta, 0)^T, \quad e_z = (0, 0, 1)^T.$$

We remark that $u^\theta = u \cdot e_\theta$ is the swirl direction of the flow, that we assume to vanish identically. Under these hypotheses on the velocity field, the vorticity vector is unidirectional, $\nabla_x \times u = (\partial_z u^r - \partial_r u^z) e_\theta$, and we write $\omega = \partial_z u^r - \partial_r u^z$. A direct computation reveals that this quantity, that we will call vorticity from here on, satisfies the continuity equation

$$\partial_t \omega + \partial_r (u^r \omega) + \partial_z (u^z \omega) = 0 \tag{3}$$

on the half-space $\mathbb{H} = \left\{(r, z) \in \mathbb{R}^2 : r > 0\right\}$. We remark that ω is thus a conserved quantity, because the no-penetration boundary condition $u^r = 0$ on $\partial\mathbb{H}$ comes along with the symmetry assumptions. However, opposed to the situation for the two-dimensional planar Euler equations, the vorticity is not transported by the flow, as the divergence-free condition (2) becomes

$$r^{-1}\partial_r(ru^r) + \partial_z u^z = 0 \tag{4}$$

in cylindrical coordinates. Indeed, the continuity equation can be rewritten as a damped transport equation,

$$\partial_t\omega + u^r\partial_r\omega + u^z\partial_z\omega = \frac{1}{r}u^r\omega,$$

where the damping term on the right-hand side describes the phenomenon of vortex stretching, $\frac{1}{r}u^r\omega e_\theta = (\nabla_x \times u)\cdot\nabla_x u$. What is transported instead is the relative vorticity $\xi = \omega/r$,

$$\partial_t\xi + u^r\partial_r\xi + u^z\partial_z\xi = 0. \tag{5}$$

We remark that the flow is entirely determined by the (relative) vorticity, as the associated velocity field can be reconstructed with the help of the Biot–Savart law in $\mathbb{R}^3$,

$$u(t,x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{x-y}{|x-y|^3}\times e_\theta(y)\omega(t,y)\,dy. \tag{6}$$

A transformation into cylindrical coordinates and an analysis of the axisymmetric Biot–Savart law can be found, for instance, in [29].

Thanks to this relation, we may thus study (5), (6) instead of (1), (2). Working with the vorticity formulation has certain advantages: At least on a formal level, it is readily seen that the vorticity equation (5) preserves any L^p norm,

$$\|\xi(t)\|_{L^p(\mathbb{R}^3)} = \|\xi_0\|_{L^p(\mathbb{R}^3)} \quad \forall t \geq 0, \tag{7}$$

if ξ_0 is the initial relative vorticity.[2] This observation is crucial, for instance, in order to prove uniqueness in the case of bounded vorticity fields [19]. The drawback of working with (5) is that there is no direct way of giving a meaning to the transport term in low integrability settings (opposed to the momentum equation (1)). For instance, it is not obvious to us, how to extend common symmetrization techniques that allow for an alternative formulation of the transport nonlinearity in the planar two-dimensional setting, see, e.g., [6,22,48].

Whenever the product $u\xi$ is locally integrable, we can interpret the transport equation (5) in the sense of distributions.

[2] We caution the reader that throughout the manuscript, we carefully distinguish between the Lebesgue spaces on the full three-dimensional space, $L^p(\mathbb{R}^3)$, and those on the two-dimensional half-space $L^p(\mathbb{H})$. Notice also that the three-dimensional Lebesgue measure reduces to the weighted measure $2\pi r d(r,z)$ on $\mathbb{H}$ when restricted to axisymmetric configurations as in (7). In particular, $\|\xi\|_{L^1(\mathbb{R}^3)} = 2\pi\|\omega\|_{L^1(\mathbb{H})}$.

Definition 2 Let $T > 0$ and $p, q \in (1, \infty)$ be given with $\frac{1}{p} + \frac{1}{q} = 1$. Let $\xi_0 \in L^p_{\mathrm{loc}}(\mathbb{R}^3)$ and $u \in L^1((0,T); L^q_{\mathrm{loc}}(\mathbb{R}^3)^3)$ be such that $\nabla_x \cdot u = 0$. Then $\xi \in L^\infty((0,T); L^p_{\mathrm{loc}}(\mathbb{R}^3))$ is called a *distributional solution* to the transport equation (5) with initial datum ξ_0 if ξ is axisymmetric and

$$\int_0^T \int_{\mathbb{R}^3} \xi\, (\partial_t \varphi + u \cdot \nabla_x \varphi)\, dx dt + \int_{\mathbb{R}^3} \xi_0 \varphi(t=0)\, dx = 0$$

for any $\varphi \in C_c^\infty([0,T) \times \mathbb{H})$.

Notice that our formulation relies on the identity $u^r \partial_r \xi = u^1 \partial_1 \xi + u^2 \partial_2 \xi$ that allow us to switch between Cartesian and cylindrical coordinates. Moroever the definition provides a distributional formulation of the continuity equation (3) in which ω is replaced by $r\xi$.

Simple scaling arguments show that the local integrability of the product $u\xi$ can be expected to hold true only if $p \geq 4/3$. For this insight, it is crucial to observe that the Sobolev inequality

$$\|u\|_{L^{\frac{2p}{2-p}}(\mathbb{H})} \lesssim \|\omega\|_{L^p(\mathbb{H})} \tag{8}$$

is valid as in the planar two-dimensional setting, cf. [29, Proposition 2.3]. For vorticity fields with smaller integrability exponents, we propose the notion of renormalized solutions.

Definition 3 Let $T > 0$ be given. Let $\xi_0 \in L^1(\mathbb{R}^3)$ and $u \in L^1((0,T); L^1_{\mathrm{loc}}(\mathbb{R}^3)^3)$ be such that $\nabla_x \cdot u = 0$. Then $\xi \in L^\infty((0,T); L^1(\mathbb{R}^3))$ is called a *renormalized solution* to the transport equation (5) with initial datum ξ_0 if ξ is axisymmetric, $\xi(t, r, z)$ and

$$\int_0^T \int_{\mathbb{R}^3} \beta(\xi)\, (\partial_t \varphi + u \cdot \nabla_x \varphi)\, dx\, dt + \int_{\mathbb{R}^3} \beta(\xi_0) \varphi(t=0) dx = 0$$

for any $\varphi \in C_c^\infty([0,T) \times \mathbb{R}^3)$ and any bounded $\beta \in C^1(\mathbb{R})$ vanishing near zero.

We remark that the notion of renormalized solutions implies the conservation of the L^p integral of vorticity in the sense of (7) via a standard approximation argument. Moreover, it is shown in [3,24] that renormalized solutions are transported by the Lagrangian flow of the vector field u as in the smooth situation. We will further comment on this in Sect. 2 below. The relation between Lagrangian transport and the partial differential equations (3) and (5) was thoroughly reviewed in [4].

In the present paper, we study solutions to the vorticity equation (5) in the case where the initial (relative) vorticity can be unbounded, more precisely,

$$\xi_0 \in L^1 \cap L^p(\mathbb{R}^3) \tag{9}$$

for some $p \in (1, \infty)$. We are thus outside of the class of functions in which uniqueness is known to hold [2,19]. On the positive side, existence of distributional solutions to the

Euler equations (1), (2) was proved in [33] for initial vorticities satisfying (9) and under the additional assumption that the initial kinetic energy is finite, $u_0 \in L^2(\mathbb{R}^3)^3$. (Notice that local L^2 bounds on the initial velocity can be deduced from the integrability assumptions on the vorticity via Sobolev embeddings, cf. (8).) For larger integrability exponents and (near) vortex sheet initial data, (crucial insights on) existence results were previously obtained in [11,12,23,34,35,41,44,47]. To the best of our knowledge, renormalized solutions (Definition 3) have not been considered in the context of the axisymmetric Euler equations.

We are particularly interested into solutions that are obtained as the vanishing viscosity limit from the Navier–Stokes equations, which are, in fact, physically meaningful approximations to the Euler equation. Hence, for any viscosity constant $\nu > 0$, we consider solutions (u_ν, p_ν) to the Navier–Stokes equations

$$\partial_t u_\nu + u_\nu \cdot \nabla_x u_\nu + \nabla_x p_\nu = \nu \Delta_x u_\nu, \tag{10}$$

$$\nabla_x \cdot u_\nu = 0. \tag{11}$$

We furthermore impose fixed initial conditions, $u_\nu(0) = u_0$ and shall assume that u_ν is axisymmetric, that is, $u_\nu = u_\nu(t, r, z)$ and $u_\nu = (u_\nu)^r e_r + (u_\nu)^z e_z$.

Instead of working with the momentum equation (10), will mostly study its vorticity formulation, which is a viscous version of (3) (or (5)), see (22) (or (23)) below. It was shown in [29] that under the assumption (9) on the initial data, which implies that $\omega_0 \in L^1(\mathbb{H})$, there exists a unique global (mild) solution $\omega^\nu \in C^0([0,\infty) \times L^1(\mathbb{H})) \cap C^0((0,\infty) \times L^\infty(\mathbb{H}))$ to the viscous vorticity equation.

Starting from this solution to the Navier–Stokes equations, our first result addresses compactness and convergence to the Euler equations.

Theorem 1 (Compactness and convergence to Euler) *Let u_ν be the unique solution to the Navier–Stokes equations (10), (11) with initial datum $u_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ such that the associated relative vorticity ξ_0 belongs to $L^1 \cap L^p(\mathbb{R}^3)$ for some $p > 1$. Then there exist $u \in C([0,T]; L^2_{\mathrm{loc}}(\mathbb{R}^3)^3)$ with $\nabla_x u \in L^\infty((0,T); L^1_{\mathrm{loc}}(\mathbb{R}^3)^{3\times 3})$ and $\xi \in L^\infty((0,T); L^1 \cap L^p(\mathbb{R}^3))$ and a subsequence $\{\nu_k\}_{k=0}^\infty$ such that*

$$u_{\nu_k} \to u \quad \textit{strongly in } C([0,T]; L^2_{\mathrm{loc}}(\mathbb{R}^3)^3)$$

and

$$\xi_{\nu_k} \to \xi \quad \textit{weakly} - \star \textit{ in } L^\infty((0,T); L^p(\mathbb{R}^3)).$$

Moreover, u is a distributional solution to the Euler equations (1), (2) and $\omega = r\xi$ is the corresponding vorticity that is (in a distributional sense) related to u by the Biot–Savart law (6).

The vanishing viscosity limit was studied for finite energy solutions with mollified initial datum satisfying the bound (9) in [33]. The novelty in the above result is the kinetic energy may be unbounded. For earlier and related convergence results for non-classical solutions, we refer to [1,31,35,47,49] and references therein.

Our next statement concerns the renormalization property of the relative vorticity.

Theorem 2 (Renormalization) *Let u and ξ be the velocity field and relative vorticity, respectively, from Theorem 1. Then ξ is a renormalized solution to the transport equation (5) with velocity u. In particular, it holds that*

$$\|\xi(t)\|_{L^p(\mathbb{R}^3)} = \|\xi_0\|_{L^p(\mathbb{R}^3)}$$

and ξ is transported by the regular Lagrangian flow of u in $\mathbb{R}^3$.

To the best of our knowledge, in this result, renormalized solutions to the axisymmetric Euler equations are considered for the first time. We recall from the above discussion that for $p \in (1, 4/3)$, the interpretation of the transport equation (5) as a distributional solution does not apply anymore as the transport nonlinearity is no longer integrable. In particular, while for $p \geq 4/3$ our result implies that distributional and renormalized solutions coincide, in the low integrability range, we show the existence of renormalized solutions. We also recall that for $p \geq 2$, the result in Theorem 2 is already covered in DiPerna and Lions's original paper [24]. In Sect. 2, we recall the theory from [24] and explain what we mean by ξ being transported by a flow. For a precise definition of regular Lagrangian flows, we refer to [3,4].

Our final result addresses the conservation of the kinetic energy.

Theorem 3 *Let $p \geq \frac{3}{2}$. Suppose that the fluid has finite kinetic energy, $u_0 \in L^2(\mathbb{R}^3)^3$, and that ω_0 is nonnegative and has finite impulse,*

$$\int_{\mathbb{H}} \omega_0 r^2 \, d(r, z) < \infty.$$

Then the kinetic energy is preserved,

$$\|u(t)\|_{L^2(\mathbb{R}^3)} = \|u_0\|_{L^2(\mathbb{R}^3)}.$$

In order to show conservation of energy, the growth of vorticity at infinity has to be suitably controlled. Here, we choose a growth condition that is natural as it can be interpreted as the control of the fluid impulse. Notice that the latter is conserved by the evolution, cf. Lemma 8. This is in principle not required by our method of proving Theorem 3, and any estimate of the form $\|r^2\omega(t)\|_{L^1(\mathbb{H})} \lesssim \|r^2\omega_0\|_{L^1(\mathbb{H})}$ would be sufficient. It is, however, not clear to us whether such an estimate holds true under our integrability assumptions apart from the special case considered in Lemma 8, that is, for nonnegative (or nonpositive) vorticity fields. Also, if higher order moments could be controlled, our method shows that the value of p could be lowered (at least up to $p > \frac{6}{5}$). See, for instance, [12] for similar results in the setting with $p > 3$ (and general solutions). We leave this issue for future research and consider the simplest case here.

From the result in Theorem 3, it follows that we are outside of the range in which Kolmogorov's celebrated K41 theory of three-dimensional turbulence applies, since, similar to the case of planar two-dimensional turbulence, there cannot be anomalous diffusion.

From here on, we will simplify the notation by writing $\nabla = \binom{\partial_r}{\partial_z}$, with the interpretation that $\nabla \cdot f = \partial_r f^r + \partial_z f^z$ while $\nabla_x \cdot f = \partial_1 f^1 + \partial_2 f^2 + \partial_3 f^3$ is the divergence with respect to a Cartesian basis. The advective derivatives $f \cdot \nabla$ and $f \cdot \nabla_x$ are to be interpreted correspondingly.

The remainder of the article is organized as follows: In Sect. 2 we recall the parts of the DiPerna–Lions theory for transport equations and explain how the results apply to the setting under consideration. In Sect. 3 we provide estimates for the velocity field that are essentially based on the Biot–Savart law. Section 4 contains global estimates for the axisymmetric Navier–Stokes equations, while the proof of Theorems 1, 2 and 3 are given in Sects. 5, 6 and 7, respectively. This work, finally, contains an appendix in which a helpful interpolation estimate is provided.

2 Renormalized solutions for linear transport equations

In this section, we shall briefly recall DiPerna and Lions's theory for linear transport equations [24] in the general setting of transport equations in $\mathbb{R}^3$, thus neglecting the assumption of axisymmetry for a moment. We are particularly interested into well-posedness and renormalization properties of the vorticity equation (5), which we shall now treat as a (linear) passive scalar equation

$$\partial_t \theta + u \cdot \nabla \theta = 0 \tag{12}$$

for some scalar quantity θ and a velocity field u that does not depend on θ. Notice that working in cylindrical coordinates would at this point become problematic as the cylindrical divergence of the velocity field u in our fluid dynamics problem might in general be unbounded opposed to the Cartesian divergence, which vanishes identically. In order to apply the DiPerna–Lions theory, in which that boundedness is a crucial assumption, it is therefore advantageous to go back to the Cartesian formulation and rewrite (12) as

$$\partial_t \theta + u \cdot \nabla_x \theta = 0. \tag{13}$$

Both formulations are indeed equivalent if u and θ are axisymmetric, because then $u^r \partial_r \theta = u^1 \partial_1 \theta + u^2 \partial_2 \theta$. If, in addition, u is Sobolev regular, as is the case for the axisymmetric Euler equations under the integrability assumption (7) on the vorticity, the theory in [24] applies. We summarize some of the main results, not aiming for the most general assumptions.

Theorem 4 ([24]) *Let $T > 0$ and $p \in (1, \infty)$ be given and $\theta_0 \in L^p(\mathbb{R}^3)$ and $u \in L^1((0,T); W^{1,1}_{\mathrm{loc}}(\mathbb{R}^3)^3)$ be such that $\nabla_x \cdot u = 0$ and*

$$\frac{|u|}{1+|x|} \in L^1((0,T)\times\mathbb{R}^3) + L^\infty((0,T)\times\mathbb{R}^3). \tag{14}$$

(i) *There exists a unique renormalized solution* $\theta \in L^\infty((0,T); L^p(\mathbb{R}^3))$ *of the transport equation* (12) *with initial datum* θ_0.

(ii) *This solution is stable under approximation in the following sense: Let* $\{\theta_0^k\}_{k\in\mathbb{N}}$ *be a sequence that approximates* θ_0 *in* $L^p(\mathbb{R}^3)$ *and* $\{u^k\}_{k\in\mathbb{N}}$ *a sequence that approximates* u *in* $L^1((0,T); W^{1,1}_{\mathrm{loc}}(\mathbb{R}^3)^3)$ *and such that* $\nabla_x \cdot u = 0$. *Let* θ^k *denote the corresponding renormalized solution. Then* $\theta^k \to \theta$ *strongly in* $C([0,T]; L^p(\mathbb{R}^3))$.

(iii) *If* $q \in (1,\infty)$ *is such that* $\frac{1}{p} + \frac{1}{q} \le 1$ *and* $u \in L^1((0,T); W^{1,q}_{\mathrm{loc}}(\mathbb{R}^3)^3)$, *then distributional solutions are renormalized solutions and vice versa.*

It has been proved in [3,24] that renormalized solutions are in fact transported by the (regular) Lagrangian flow of the vector field u, and this feature carries over to the cylindrical setting. Hence, it holds that $\theta(t,\phi(t,x)) = \theta_0(x)$, where ϕ satisfies a suitably generalized formulation of the ordinary differential equation

$$\partial_t \phi(t,x) = u(t,\phi(t,x)), \quad \phi(0,x) = x.$$

In terms of the vorticity, the transport identity can be rewritten as $\omega(t,\phi(t,x)) = \omega_0(x)\phi^r(t,x)/r$, and thus, $r/\phi^r(t,x)$ is the Jacobian. See also [4] for a review of the connection between the Lagrangian and Eulerian descriptions of transport by non-smooth velocity fields.

Following [17,18], our strategy for proving that vanishing viscosity solutions to the axisymmetric Euler equations are renormalized solutions relies on duality arguments both in the viscous and in the inviscid setting. In the latter, we quote a suitable duality theorem from DiPerna and Lions's original work.

Lemma 1 ([24]) *Let* $p, q \in (1,\infty)$ *be given such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Let* u *satisfy the general assumptions of Theorem 4 and let* $\theta \in L^\infty((0,T); L^p(\mathbb{R}^3))$ *be the renormalized solution to the transport equation* (12) *with initial datum* $\theta_0 \in L^p(\mathbb{R}^3)$. *Let* $\chi \in L^1((0,T); L^q(\mathbb{R}^3))$ *be given and let* $f \in L^\infty((0,T); L^q(\mathbb{R}^3))$ *be a renormalized solution of the backwards transport equation*

$$-\partial_t f - u\cdot\nabla_x f = \chi. \tag{15}$$

Then it holds

$$\int_0^T \int_{\mathbb{R}^3} \theta\,\chi\,dxdt = \int_{\mathbb{R}^3} \theta(0,x) f(0,x)\,dx - \int_{\mathbb{R}^3} \theta(T,x) f(T,x)\,dx.$$

3 Estimates on the velocity field

In this section, we provide some estimates on the velocity field that turn out to be helpful in the subsequent analysis. We continue denoting by ω and ξ the vorticity and relative vorticity, respectively, of a given (steady) axisymmetric velocity field u, that is, $\omega = \partial_z u^r - \partial_r u^z$ and $\xi = \omega/r$ independently from the Euler or Navier–Stokes background. In particular, any of the following estimates are consequences of the explicit definitions or follow from suitable properties of the Biot–Savart kernel.

We start by verifying that the velocity field generated by (relative) vorticities in the class (9) satisfies the growth condition needed to apply DiPerna and Lions's theory recalled in Theorem 4.

Lemma 2 *Let $\omega \in L^1(\mathbb{H})$. Then it holds that*

$$\frac{|u|}{1+|x|} \in L^1(\mathbb{R}^3) + L^\infty(\mathbb{R}^3). \tag{16}$$

Proof It is proved in [29] that the axisymmetric Biot–Savart kernel satisfies similar decay estimates as the planar two-dimensional one, namely, if G is obtained from restricting the three-dimensional Biot–Savart kernel to the axisymmetric setting, so that

$$u(r,z) = \int_{\mathbb{H}} G(r,z,\bar r,\bar z)\omega(\bar r,\bar z)\, d(\bar r,\bar z),$$

it holds that

$$|G(r,z,\bar r,\bar z)| \lesssim \frac{1}{|r-\bar r| + |z-\bar z|},$$

cf. [29, Eq. (2.11)]. We now denote by G_1 the restriction of G to the unit half ball $B_1(0)$ and set $G_2 = G - G_1$, and decompose $u = u_1 + u_2$ accordingly. Then, on the one hand, by Young's convolution inequality, it holds

$$\|u_1\|_{L^1(\mathbb{H})} \lesssim \|(\chi_{B_1(0)}\frac{1}{|\cdot|}) * |\omega|\|_{L^1(\mathbb{H})} \le \|\chi_{B_1(0)}\frac{1}{|\cdot|}\|_{L^1(\mathbb{H})}\|\omega\|_{L^1(\mathbb{H})} \lesssim \|\omega\|_{L^1(\mathbb{H})}.$$

Thus, in view of the trivial bound $\|u_1/|x|\|_{L^1(\mathbb{R}^3)} \lesssim \|u_1\|_{L^1(\mathbb{H})}$, we verify the growth condition for u_1. On the other hand, since $\chi_{B_1(0)^c}\frac{1}{|\cdot|}$ is uniformly bounded,

$$\|u_2/(1+|x|)\|_{L^\infty(\mathbb{R}^3)} \le \|u_2\|_{L^\infty(\mathbb{H})} \lesssim \|(\chi_{B_1(0)^c}\frac{1}{|\cdot|}) * |\omega|\|_{L^\infty(\mathbb{H})} \le \|\omega\|_{L^1(\mathbb{H})},$$

we deduce a uniform control on u_2. This establishes (16). □

Our second result is a fairly standard identity for the enstrophy, that is, the (square of the) L^2 norm of the velocity gradient.

Lemma 3 *It holds that*

$$\|\nabla_x u\|_{L^2(\mathbb{R}^3)} = \|\omega\|_{L^2(\mathbb{R}^3)}.$$

We provide the argument for this standard identity for the convenience of the reader.

Proof From the definition of the vorticity, we infer that

$$\begin{aligned}\frac{1}{2\pi}\|\omega\|^2_{L^2(\mathbb{R}^3)} &= \int_{\mathbb{H}} (\partial_z u^r - \partial_r u^z)^2 r\, d(r,z)\\ &= \int_{\mathbb{H}} (\partial_z u^r)^2\, r\, d(r,z) + \int_{\mathbb{H}} (\partial_r u^z)^2\, r\, d(r,z) - 2\int_{\mathbb{H}} \partial_z u^r\, \partial_r u^z\, r\, d(r,z).\end{aligned}$$

We have to identify the third term on the right-hand side: It holds that

$$-2\int_{\mathbb{H}} \partial_z u^r\, \partial_r u^z\, r\, d(r,z) = \int_{\mathbb{H}} (\partial_r u^r)^2\, r\, d(r,z) + \int_{\mathbb{H}} \frac{(u^r)^2}{r}\, d(r,z) + \int_{\mathbb{H}} (\partial_z u^z)^2\, r\, d(r,z).$$

Indeed, using the no-penetration boundary condition $u^r = 0$ on $\partial\mathbb{H}$ together with the incompressibility condition (4), a multiple integration by parts reveals on the one hand that

$$\begin{aligned}\int_{\mathbb{H}} \partial_z u^r\, \partial_r u^z\, r\, d(r,z) &= -\int_{\mathbb{H}} u^r\, \partial_z\partial_r u^z\, r\, d(r,z)\\ &= -\int_{\mathbb{H}} u^r\, \partial_r(-\partial_r u^r - \frac{1}{r}u^r)\, r\, d(r,z)\\ &= -\int_{\mathbb{H}} (\partial_r u^r)^2\, r\, d(r,z) - \int \frac{(u^r)^2}{r}\, d(r,z)\,.\end{aligned}$$

On the other hand, it holds that

$$\begin{aligned}\int_{\mathbb{H}} \partial_z u^r\, \partial_r u^z\, r\, d(r,z) &= -\int_{\mathbb{H}} \partial_r(\partial_z u^r\, r) u^z\, d(r,z)\\ &= -\int_{\mathbb{H}} \partial_r\partial_z u^r u^z\, r\, d(r,z) - \int_{\mathbb{H}} \partial_z u^r u^z\, d(r,z)\\ &= -\int_{\mathbb{H}} \partial_z(-\partial_z u^z - \frac{1}{r}u^r) u^z\, r\, d(r,z) - \int_{\mathbb{H}} \partial_z u^r u^z\, d(r,z)\\ &= -\int_{\mathbb{H}} (\partial_z u^z)^2\, r\, d(r,z)\,.\end{aligned}$$

It remains to notice that

$$|\nabla_x u|^2 = (\partial_r u^r)^2 + \frac{1}{r^2}(u^r)^2 + (\partial_z u^r)^2 + (\partial_r u^z)^2 + (\partial_z u^z)^2 \tag{17}$$

to conclude the statement of the lemma. □

In the following lemma, we provide a maximal regularity estimate for the velocity gradient in terms of the relative vorticity. Our proof relies on the classical theories by Calderón, Zygmund and Muckenhoupt.

Lemma 4 *For $p \in (1,2)$ it holds that*

$$\|\frac{1}{r}\nabla_x u\|_{L^p(\mathbb{R}^3)} \lesssim \|\xi\|_{L^p(\mathbb{R}^3)}. \tag{18}$$

Proof We note that in view of the Biot–Savart law (6), the velocity gradient can be represented as a singular integral of convolution type, $\nabla_x u = K * (\omega e_\theta)$, where $|K(x)| \sim \frac{1}{|x|^3}$. It is well-known that Calderón–Zygmund theory guarantees that

$$\|\nabla_x u\|_{L^p(\mathbb{R}^3)} \lesssim \|\omega\|_{L^p(\mathbb{R}^3)}$$

for any $p \in (1,\infty)$. Our goal is to produce a weighted version of this estimate, namely

$$\int_{\mathbb{R}^3} |\nabla_x u|^p \, m \, dx \lesssim \int_{\mathbb{R}^3} |\omega|^p \, m \, dx$$

with $m = m(r) = \frac{1}{r^p}$ and $r = \sqrt{x_1^2 + x_2^2}$, which is nothing but (18). We are thus led to the theory of Muckenhoupt weights: If $p \in (1,\infty)$ and m is in the class of Muckenhoupt weights A_p then the weighted-maximal regularity estimate (18) holds. We recall that a nonnegative locally integrable functions m is in the class A_p if there exists a constant $C > 0$ such that for all balls B in $\mathbb{R}^3$ the condition

$$\left(\fint_B m(x)\,dx\right)\left(\fint_B m(x)^{-\frac{q}{p}}\,dx\right)^{\frac{p}{q}} \le C \tag{19}$$

holds with $q \in (1,\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. This well known result that can be found, for instance, in [25] (Theorem 7.11, Chapter 7).

We thus have to show that $m = m(r) = r^{-p}$ satisfies (19) for $p \in (1,2)$. For this, consider a ball in $\mathbb{R}^3$ with radius R and centered in a generic point $X = (X_1, X_2, X_3) \in \mathbb{R}^3$, i.e., $B = B_R(X)$. We denote by d the distance of X to the z-axis, that is, $d = \sqrt{X_1^2 + X_2^2}$. We split our argumentation into the two cases when $d \ge 2R$ (far field) and $d < 2R$ (near field).

Let us first consider the case where $d \ge 2R$. Notice that we have $d - R \le \sqrt{x_2^1 + x_2^2} \le d + R$ for any $x \in B$ by the triangle inequality, and thus

$$\frac{1}{(x_1^2 + x_2^2)^{\frac{p}{2}}} \le \frac{1}{(d-R)^p} \quad \text{and} \quad (x_1^2 + x_2^2)^{\frac{q}{2}} \le (d+R)^q \,.$$

For $m(x) = (x_1^2 + x_2^2)^{-\frac{p}{2}}$, we now compute

$$\fint_B m(x)\,dx \le \frac{1}{(d-R)^p},$$

and

$$\left(\fint_B m(x)^{-\frac{q}{p}}\,dx\right)^{\frac{p}{q}} \le (d+R)^p.$$

Making use of the fact that $\frac{d+R}{d-R} \le 3$ for all $d \ge 2R$, we deduce that

$$\left(\fint_B m(x)\,dx\right)\left(\fint_B m(x)^{-\frac{q}{p}}\,dx\right)^{\frac{p}{q}} \le \left(\frac{d+R}{d-R}\right)^p \le 3^p.$$

We now turn to the case where $d < 2R$. We first observe that $\sqrt{x_1^2+x_2^2} < d+R$ and $|x_3 - X_3| < R$ for all $x \in B$, and we may thus bound the integral over the ball by an integral over the cylinder. Making relative transformations in cylindrical coordinates, we then have the estimates

$$\fint_B m(x)\,dx \lesssim \frac{1}{R^2}\int_0^{d+R} \frac{1}{r^{p-1}}\,dr \lesssim \frac{(d+R)^{2-p}}{R^2},$$

provided that $p < 2$, and

$$\left(\fint_B m(x)^{-\frac{q}{p}}\,dx\right)^{\frac{p}{q}} \lesssim \left(\frac{1}{R^2}\int_0^{d+R} r^{q+1}\,dr\right)^{\frac{p}{q}} \lesssim \left(\frac{(d+R)^{q+2}}{R^2}\right)^{\frac{p}{q}}.$$

Taking the product and using that $\frac{d+R}{R} \le 3$ for all $d < 2R$, we conclude that

$$\left(\fint_B m(x)\,dx\right)\left(\fint_B m(x)^{-\frac{q}{p}}\,dx\right)^{\frac{p}{q}} \lesssim \left(\frac{d+R}{R}\right)^{2p} \le 3^{2p}.$$

Hence, in either cases, we proved (19) and, thus, the proof is over. □

We conclude this section with an estimate on the velocity field in large annuli.

Lemma 5 *Let $\alpha > \beta > 0$ be given. For $p \in \left(\frac{11}{9}, 2\right)$ and $r \in \left[3, \frac{6p-2}{3-p}\right)$, it holds that*

$$\int_{B_{\alpha R}(0)\setminus B_{\beta R}(0)\cap\mathbb{H}} |u|^r\,d(r,z) \lesssim R^{-\frac{3r-4}{2}}\|u\|_{L^2(\mathbb{R}^3)}^r + R^{-\frac{p(r+6)-3r-2}{2(p-1)}}\|\xi\|_{L^p(\mathbb{R}^3)}^{\frac{p(r-2)}{2(p-1)}}\|\omega r^2\|_{L^1(\mathbb{H})}^{\frac{rp-2r+2p}{2(p-1)}},$$

for any $R > 0$.

Notice that the interval $\left[3, \frac{6p-2}{3-p}\right)$ is well-defined and nonempty precisely if $p \in \left(\frac{11}{9}, 3\right)$.

Proof In the following, we will use the notation B_R for both the ball in $\mathbb{R}^3$, that is $B_R(0) \subset \mathbb{R}^3$, and the half ball in $\mathbb{H}$, that is $B_R(0) \cap \mathbb{H} \subset \mathbb{H}$. It should be clear from the situation, which one is considered.

We start by using the Sobolev embedding in two dimensions and find

$$\begin{aligned}&\int_{B_{\alpha R}\setminus B_{\beta R}} |u|^r \, d(r,z)\\&\lesssim \left(R^{-\frac{2r}{r+2}} \int_{B_{\alpha R}\setminus B_{\beta R}} |u|^{\frac{2r}{r+2}} \, d(r,z) + \int_{B_{\alpha R}\setminus B_{\beta R}} |\nabla u|^{\frac{2r}{r+2}} \, d(r,z)\right)^{\frac{r+2}{2}}. \qquad (20)\end{aligned}$$

To estimate the gradient term in (20), we write the velocity field with the help of a vector stream function, $u = \nabla_x \times \psi$, where $-\Delta_x \psi = \omega e_\theta$. Then recalling (17), we find that

$$\int_{B_{\alpha R}\setminus B_{\beta R}} |\nabla u|^{\frac{2r}{r+2}} \, d(r,z) \lesssim \frac{1}{R}\int_{B_{\alpha R}\setminus B_{\beta R}} |\nabla_x u|^{\frac{2r}{r+2}} \, dx \lesssim \frac{1}{R}\int_{B_{\alpha R}\setminus B_{\beta R}} |\nabla_x^2 \psi|^{\frac{2r}{r+2}} \, dx.$$

Thanks to local Calderon–Zygmund estimates (see, e.g., Theorem 9.11 in [30]), we have a control on the term on right-hand side in terms of the vorticity,

$$\frac{1}{R}\int_{B_{\alpha R}\setminus B_{\beta R}} |\nabla_x^2 \psi|^{\frac{2r}{r+2}} \, dx \lesssim \frac{1}{R}\int_{B_{2\alpha R}\setminus B_{\frac{\beta R}{2}}} |\omega|^{\frac{2r}{r+2}} \, dx + \frac{1}{R^{1+\frac{4r}{r+2}}}\int_{B_{2\alpha R}\setminus B_{\frac{\beta R}{2}}} |\psi|^{\frac{2r}{r+2}} \, dx.$$

Notice that $\psi = \psi^\theta e_\theta$ by construction. Moreover, as the stream function is unique up to additive constants, we may, without loss of generality, assume that ψ^θ has zero average in the annulus $B_{2\alpha R}\setminus B_{\frac{\beta R}{2}} \subset \mathbb{R}^3$, so that an application of the Poincaré inequality implies

$$\begin{aligned}\int_{B_{2\alpha R}\setminus B_{\frac{\beta R}{2}}} |\psi|^{\frac{2r}{r+2}} \, dx &\lesssim R^{1-\frac{2r}{r+2}} \int_{B_{2\alpha R}\setminus B_{\frac{\beta R}{2}}} |r\psi^\theta|^{\frac{2r}{r+2}} \, d(r,z)\\&\lesssim R \int_{B_{2\alpha R}\setminus B_{\frac{\beta R}{2}}} |\nabla(r\psi^\theta)|^{\frac{2r}{r+2}} \, d(r,z)\\&\lesssim R^{1+\frac{2r}{r+2}} \int_{B_{2\alpha R}\setminus B_{\frac{\beta R}{2}}} |\frac{1}{r}\psi^\theta + \partial_r\psi^\theta|^{\frac{2r}{r+2}} + |\partial_z\psi^\theta|^{\frac{2r}{r+2}} \, d(r,z)\end{aligned}$$

Hence, because we can write $u = \partial_z\psi^\theta e_r + \left(\frac{1}{r}\psi^\theta + \partial_r\psi^\theta\right) e_z$, we have proved that

$$\int_{B_{2\alpha R}\setminus B_{\frac{\beta R}{2}}} |\psi|^{\frac{2r}{r+2}} \, dx \lesssim R^{1+\frac{2r}{r+2}} \int_{B_{2\alpha R}\setminus B_{\frac{\beta R}{2}}} |u|^{\frac{2r}{r+2}} \, d(r,z).$$

A combination of the previous estimates yields

$$\int_{B_{\alpha R}\setminus B_{\beta R}} |\nabla u|^{\frac{2r}{r+2}}\, d(r,z) \lesssim R^{-\frac{2r}{r+2}} \int_{B_{2\alpha R}\setminus B_{\frac{\beta R}{2}}} |u|^{\frac{2r}{r+2}}\, d(r,z) + \int_{B_{2\alpha R}\setminus B_{\frac{\beta R}{2}}} |\omega|^{\frac{2r}{r+2}}\, d(r,z).$$

Plugging this estimate into (20), we arrive at

$$\begin{aligned}\int_{B_{\alpha R}\setminus B_{\beta R}} &|u|^r\, d(r,z)\\ &\lesssim \left(R^{-\frac{2r}{r+2}} \int_{B_{2\alpha R}\setminus B_{\frac{\beta R}{2}}} |u|^{\frac{2r}{r+2}}\, d(r,z) + \int_{B_{2\alpha R}\setminus B_{\frac{\beta R}{2}}} |\omega|^{\frac{2r}{r+2}}\, d(r,z)\right)^{\frac{r+2}{2}}. \end{aligned} \tag{21}$$

With regard to the first term in (21), we notice that by Jensen's inequality, it holds that

$$\begin{aligned} R^{-\frac{2r}{r+2}} \int_{B_{2\alpha R}\setminus B_{\frac{\beta R}{2}}} |u|^{\frac{2r}{r+2}}\, d(r,z) &\lesssim R^{2-\frac{4r}{r+2}} \left(\int_{B_{2\alpha R}\setminus B_{\frac{\beta R}{2}}} |u|^2\, d(r,z)\right)^{\frac{r}{r+2}}\\ &\lesssim R^{2-\frac{5r}{r+2}} \|u\|_{L^2(\mathbb{R}^3)}^{\frac{2r}{r+2}}, \end{aligned}$$

and, thus, in view of our assumptions on r, the first term in (21) vanishes as $R \to \infty$.

For the second term, we appeal to Hölder's inequality,

$$\begin{aligned}\int_{B_{2\alpha R}\setminus B_{\frac{\beta R}{2}}} &|\omega|^{\frac{2r}{r+2}}\, d(r,z)\\ &\le \left(\int_{B_{2\alpha R}\setminus B_{\frac{\beta R}{2}}} |\omega|^p\, d(r,z)\right)^{\frac{r-2}{(r+2)(p-1)}} \left(\int_{B_{2\alpha R}\setminus B_{\frac{\beta R}{2}}} |\omega|\, d(r,z)\right)^{\frac{rp-2r+2p}{(r+2)(p-1)}}, \end{aligned}$$

where we use the fact that $r < \frac{2p}{2-p}$, which holds true because $\frac{2p}{2-p} > \frac{6p-2}{3-p}$ for any $p < 2$. We can easily smuggle in some weights to the effect that

$$\int_{B_{2\alpha R}\setminus B_{\frac{\beta R}{2}}} |\omega|^{\frac{2r}{r+2}}\, d(r,z) \lesssim R^{-\frac{-p(r+6)+3r+2}{(r+2)(p-1)}} \|\xi\|_{L^p(\mathbb{R}^3)}^{\frac{p(r-2)}{(r+2)(p-1)}} \|\omega r^2\|_{L^1(\mathbb{H})}^{\frac{rp-2r+2p}{(r+2)(p-1)}}.$$

It remains to observe that the exponent on R is negative by our assumption on r. This shows the convergence of the second term in (21). The proof is over. □

4 Global estimates for the axisymmetric Navier–Stokes equations

In this section, we provide some global estimates for solutions to the Navier–Stokes equations that will turn out to be helpful later on. We start by rewriting the momentum equation (10) in terms of the vorticity $\omega_\nu = \partial_z u_\nu^r - \partial_r u_\nu^z$ and the relative vorticity $\xi_\nu = \omega_\nu / r$. The evolution equation for the vorticity is given by

$$\partial_t \omega_\nu + \nabla \cdot (u_\nu \omega_\nu) = \nu \left(\Delta \omega_\nu + \frac{1}{r} \partial_r \omega_\nu - \frac{1}{r^2} \omega_\nu \right), \tag{22}$$

and is equipped with homogeneous Dirichlet conditions on the boundary of the half-space, i.e. $\omega_\nu = 0$ on $\partial \mathbb{H}$. It follows that the vorticity equation is conservative, as expected, because $r^{-1}\partial_r \omega_\nu - r^{-2}\omega_\nu = \partial_r (r^{-1}\omega_\nu)$. The relative vorticity satisfies the nonconservative equation

$$\partial_t \xi_\nu + u_\nu \cdot \nabla \xi_\nu = \nu \left(\Delta \xi_\nu + \frac{3}{r} \partial_r \xi_\nu \right) \tag{23}$$

which is supplemented with homogeneous Neumann boundary conditions, $\partial_r \xi_\nu = 0$ on $\partial \mathbb{H}$. We will mostly work with the latter equation. For initial data $\xi_\nu(0) = \xi_0$ in $L^1(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, cf. (9), well-posedness for either formulation can be inferred from the theory developed by Gallay and Šverák [29]. In the following, ω_ν will always be the unique mild solution to the vorticity equation (22) in the class $C([0,T); L^1(\mathbb{H})) \cap C((0,T); L^\infty(\mathbb{H}))$ and $\xi_\nu = \omega_\nu / r$. We start by recalling some useful properties which can be found in various references. Yet, we provide their short proofs for the convenience of the reader. Our first concern is an L^p estimate.

Lemma 6 *It holds that*

$$\|\xi_\nu\|_{L^\infty((0,T);L^p(\mathbb{R}^3))} \le \|\xi_0\|_{L^p(\mathbb{R}^3)}. \tag{24}$$

Proof We can perform a quite formal computation as solutions can be assumed to be smooth by standard approximation procedures. A direct calculation yields

$$\frac{d}{dt}\frac{1}{p}\int_{\mathbb{H}} |\xi_\nu|^p \, r \, d(r,z) = \nu \int_{\mathbb{H}} |\xi_\nu|^{p-2} \xi_\nu \Delta \xi_\nu \, r d(r,z) + 3\nu \int_{\mathbb{H}} |\xi_\nu|^{p-2} \xi_\nu \partial_r \xi_\nu \, d(r,z),$$

where we made use of the no-penetration boundary conditions on the velocity field u, i.e., $u^r = 0$ at $\partial \mathbb{H}$, to eliminate the advection term. The Cartesian Laplacian $\Delta_x = \Delta + \frac{1}{r}\partial_r$ is coercive, because

$$\int_{\mathbb{H}} |\xi_\nu|^{p-2} \xi_\nu \left(\Delta \xi_\nu + \frac{1}{r} \partial_r \xi_\nu \right) r d(r,z) = -(p-1) \int_{\mathbb{H}} |\xi_\nu|^{p-2} |\nabla_x \xi_\nu|^2 \, r d(r,z) \le 0$$

as can be seen by an integration by parts. Another integration by parts reveals that the first order term is nonpositive and can thus be dropped,

$$\int_{\mathbb{H}} |\xi_\nu|^{p-2}\xi_\nu \partial_r \xi_\nu \, d(r,z) = \frac{1}{p}\int_{\mathbb{H}} \partial_r |\xi_\nu|^p \, d(r,z) = -\frac{1}{p}\int_{\partial\mathbb{H}} |\xi_\nu|^p \, d(r,z) \le 0.$$

A combination of the previous estimates yields

$$\frac{d}{dt}\frac{1}{p}\int_{\mathbb{H}} |\xi_\nu|^p \, r \, d(r,z) + \nu(p-1)\int_{\mathbb{H}} |\xi_\nu|^{p-2}|\nabla\xi_\nu|^2 \, r d(r,z) \le 0, \tag{25}$$

and an integration in time yields the desired estimate (24). □

Our next estimate quantifies integrability improving features of the advection-diffusion equation (23) by suitably extending the estimates on the L^p norm established in the previous lemma to any $q \in [p, \infty)$.

Lemma 7 *For any $q \in [p, \infty]$, it holds that*

$$\|\xi_\nu(t)\|_{L^q(\mathbb{R}^3)} \lesssim \left(\frac{1}{\nu t}\right)^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|\xi_0\|_{L^p(\mathbb{R}^3)} \quad \forall t > 0. \tag{26}$$

Proof Our proof is a small modification of the argument of Feng and Šverák in [27, Lemma 3.8], where the case $p = 1$ is considered. We define $E_q(t) = \|\xi_\nu(t)\|^q_{L^q(\mathbb{R}^3)}$ for some $q \in [p, \infty)$ and claim that

$$\frac{d}{dt}\left[E_q(t)^{-\frac{2}{3}}\right] \gtrsim \nu \left(\int_{\mathbb{R}^3} |\xi_\nu|^{\frac{q}{2}} \, dx\right)^{-\frac{4}{3}}. \tag{27}$$

Let us postpone the proof of this estimate a bit and explain first how it implies (26). Notice that, by interpolation of Lebesgue spaces, it is enough to show (26) for exponents $q = 2^k p$ with $k \in \mathbb{N}_0$ and $q = \infty$. We first treat the case for finite exponents, which will be achieved by induction. We start by observing that the base case $k = 0$ is settled in Lemma 6 above. The induction step from k to $k+1$ is based on estimate (27). We set $\tilde q = 2^k$ and $q = 2^{k+1} = 2\tilde q$. Plugging (26) with $\tilde q = \frac{q}{2}$ into (27), we find

$$\frac{d}{dt}\left[E_q(t)^{-\frac{2}{3}}\right] \gtrsim \nu(\nu t)^{2\left(\frac{\tilde q}{p}-1\right)} \|\xi_0\|^{-\frac{4}{3}\tilde q}_{L^p(\mathbb{R}^3)} = \nu^{\frac{q}{p}-1} t^{\frac{q}{p}-2} \|\xi_0\|^{-\frac{2}{3}q}_{L^p(\mathbb{R}^3)}.$$

Integrating in time yields

$$\begin{aligned}(E_q(t))^{-\frac{2}{3}} \ge (E_q(t))^{-\frac{2}{3}} - (E_q(0))^{-\frac{2}{3}} &\gtrsim \nu^{\frac{q}{p}-1}\|\xi_0\|^{-\frac{2}{3}q}_{L^p(\mathbb{R}^3)} \int_0^t s^{\frac{q}{p}-2} \, ds \\ &\sim \nu^{\frac{q}{p}-1}\|\xi_0\|^{-\frac{2}{3}q}_{L^p(\mathbb{R}^3)} t^{\frac{q}{p}-1},\end{aligned}$$

where we have used that $q > p$. Notice that all constants can be chosen uniformly in q. We have thus proved (26) for $q = 2\tilde{q}$, which settles the case where $q = 2^k p$.

If $q = \infty$, we may now simply take the limit in (26) and use the convegence of the Lebesgue norms, $\|\cdot\|_{L^\infty} = \lim_{q\to\infty} \|\cdot\|_{L^q}$.

It remains to provide the argument for (27). We start by recalling that

$$-\frac{d}{dt}E_q(t) \overset{(25)}{\geq} q(q-1)\nu \int_{\mathbb{R}^3} |\xi_\nu|^{q-2}|\nabla\xi_\nu|^2\,dx \sim \frac{q-1}{q}\nu \int_{\mathbb{R}^3} \left|\nabla|\xi_\nu|^{\frac{q}{2}}\right|^2 dx.$$

Notice that the constants in the estimate can be chosen independently of q as $q > 1$, and can thus be dropped. We estimate the right-hand-side with the help the 3D Nash inequality $\|f\|_{L^2(\mathbb{R}^3)} \lesssim \|f\|_{L^1(\mathbb{R}^3)}^{2/5}\|\nabla f\|_{L^2(\mathbb{R}^3)}^{3/5}$, and obtain

$$-\frac{d}{dt}E_q(t) \gtrsim \nu \left(\int_{\mathbb{R}^3} |\xi_\nu|^{\frac{q}{2}}\,dx\right)^{-\frac{4}{3}} \left(\int_{\mathbb{R}^3} |\xi_\nu|^q\,dx\right)^{\frac{5}{3}},$$

which can be rewritten as (27). □

We also note that the fluid impulse is conserved along the viscous flow.

Lemma 8 *Suppose that $r^2\omega_0 \in L^1(\mathbb{H})$. Then*

$$\int_{\mathbb{H}} \omega_\nu(t) r^2\,d(r,z) = \int_{\mathbb{H}} \omega_0 r^2\,d(r,z).$$

This identity can be seen in several ways, see, for instance [29, Lemma 6.4] for a proof that is based on the symmetry properties of the Biot–Savart kernel and applies to our regularity setting. We omit the proof and remark only that

$$\int_{\mathbb{R}^3} u^z\,dx = \pi \int_{\mathbb{H}} \omega r^2\,d(r,z),$$

whenever u is an axisymmetric vector field and ω the associated scalar vorticity. The conservation of momentum follows immediately from the Euler equations (1), (2).

The last global estimate concerns the energy balance law, for which we assume that the initial kinetic energy is bounded.

Lemma 9 *Suppose that $\|u_0\|_{L^2(\mathbb{R}^3)} < \infty$. Then*

$$\|u_\nu(t)\|_{L^2(\mathbb{R}^3)}^2 + \nu \int_0^t \|\nabla_x u_\nu\|_{L^2(\mathbb{R}^3)}^2\,dt = \|u_0\|_{L^2(\mathbb{R}^3)}^2 \quad \textit{for all } t > 0. \tag{28}$$

It is a classical result by Leray that for any divergence-free initial datum u_0 in $L^2(\mathbb{R}^3)$, there exists a weak solution to the Navier–Stokes equations (10), (11) satisfying the energy inequality

$$\|u_\nu(t)\|_{L^2(\mathbb{R}^3)}^2 + \nu \int_0^t \|\nabla_x u_\nu\|_{L^2(\mathbb{R}^3)}^2\,dt \leq \|u_0\|_{L^2(\mathbb{R}^3)}^2, \tag{29}$$

cf. [38]. Whether there is an energy *equality* (28) for such solutions is an important open problem. There are various conditions available in the literature under which an equality can be established, most notably, Serrin's condition $u \in L^q((0,T); L^p(\mathbb{R}^d))$ with $\frac{d}{p} + \frac{2}{q} \le 1$ or Shinbrot's criterion $\frac{2}{p} + \frac{2}{q} \le 1$ and $p \ge 4$, cf. [45,46]. We refer to [15] for an extension of the previous results to a larger class of function spaces and to [5] for a recent improvement based on assumptions on the gradient of the velocity.

It is not difficult to see that we can construct mild solutions in the setting of [29] that satisfy the inequality (29), and thus, thanks to the uniqueness in that setting, our solutions do as well. We remark that in [10] Buckmaster and Vicol construct weak solutions for the three-dimensional Navier for which the energy inequality is not automatically achieved. Unfortunately, it is not obvious how to check Serrin's or Shinbrot's integrability conditions to ensure an energy equality in the axisymmetric setting. The problem is the appearance of weights as, for instance, in (18) and in suitable Sobolev inequalities. For this reason, we provide a proof of (28) that is tailored to our needs but still mimics the original arguments in [45,46].

Proof By interpolation between Lebesgue spaces, we may without loss of generality assume that $p \in (1,2)$. Thanks to the well-posedness result in [29], we may suppose that (29) holds true in our setting. In particular, we deduce

$$u_\nu \in L^\infty((0,T); L^2(\mathbb{R}^3)^3) \quad \text{and} \quad \nabla_x u_\nu \in L^2((0,T); L^2(\mathbb{R}^3)^{3\times 3}) \,. \tag{30}$$

In addition, thanks to the L^p bound on the vorticity in Lemma 6 and the weighted maximal regularity estimate in Lemma 4, it holds that

$$\frac{1}{r}\nabla_x u_\nu \in L^\infty((0,T); L^p(\mathbb{R}^3)^{3\times 3}). \tag{31}$$

By standard density arguments, we may thus find a sequence $\{u_\nu^\delta\}_{\delta\downarrow 0}$ of axisymmetric divergence-free functions in $C_c^\infty((0,T)\times\mathbb{R}^3)^3$ which satisfy (30) and (31) and that converges towards u_ν in $L^2((0,T); H^1(\mathbb{R}^3)^3)$, staying bounded in all the spaces in which u_ν is contained. We furthermore denote by η^ε a standard mollifier on $\mathbb{R}$. Because

$$F(t,x) = \int_0^T \eta^\varepsilon(t-\tau) u_\nu^\delta(\tau,x)\, d\tau$$

is an admissible test function in the definition of distributional solution of the Navier–Stokes equations, we find that

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} \eta^\varepsilon(T-\tau) u_\nu^\delta(\tau,x)\cdot u_\nu(T,x)\, dx d\tau \\
&\quad = \int_0^T \int_{\mathbb{R}^3} \eta^\varepsilon(-\tau) u_\nu^\delta(\tau,x)\cdot u_0(x)\, dx d\tau \\
&\qquad + \int_0^T \int_0^T \int_{\mathbb{R}^3} \frac{d\eta^\varepsilon}{dt}(t-\tau) u_\nu^\delta(\tau,x)\cdot u_\nu(t,x)\, dx d\tau dt
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^T\int_0^T\int_{\mathbb{R}^3}\eta^\varepsilon(t-\tau)u_\nu^\delta(\tau,x)\cdot(u_\nu(t,x)\cdot\nabla_x)u_\nu(t,x)\,dxd\tau dt\\
&-\nu\int_0^T\int_0^T\int_{\mathbb{R}^3}\eta^\varepsilon(t-\tau)\nabla_x u_\nu^\delta(\tau,x):\nabla_x u_\nu(t,x)\,dxd\tau dt.
\end{aligned}
$$

In a first step, we send δ to zero with $\varepsilon>0$ fixed. The convergence is obvious for all but the nonlinear term. It is enough to show that the nonlinear term vanishes when u_ν^δ is replaced by $v^\delta=u_\nu^\delta-u_\nu$. Performing an integration by parts, we can throw the derivative on one of the $u_\nu(t,x)$. Hölder's inequality then yields

$$
\begin{aligned}
&\left|\int_0^T\int_0^T\int_{\mathbb{R}^3}\eta^\varepsilon(t-\tau)v_\nu^\delta(\tau,x)\cdot(u_\nu(t,x)\cdot\nabla_x)u_\nu(t,x)\,dxd\tau dt\right|\\
&\quad\le\int_0^T\|\eta^\varepsilon*v^\delta\|_{L^4(\mathbb{R}^3)}\|u_\nu\|_{L^4(\mathbb{R}^3)}\|\nabla_x u_\nu\|_{L^2(\mathbb{R}^3)}\,dt,
\end{aligned}
$$

where by $*$ we denote the convolution-type operation between η^ε and v^δ. We now have to make use of the interpolation inequality in Lemma 15 in the appendix and notice that $|\nabla u|\le|\nabla_x u|$ for any axisymmetric velocity field u. We find that

$$
\begin{aligned}
&\int_0^T\|\eta^\varepsilon*v^\delta\|_{L^4(\mathbb{R}^3)}\|u_\nu\|_{L^4(\mathbb{R}^3)}\|\nabla_x u_\nu\|_{L^2(\mathbb{R}^3)}\,dt\\
&\quad\lesssim\int_0^T\|\eta^\varepsilon*v^\delta\|_{L^2(\mathbb{R}^3)}^\lambda\|\eta^\varepsilon*\nabla_x v^\delta\|_{L^2(\mathbb{R}^3)}^{\frac12}\left\|\frac1r\eta^\varepsilon*\nabla_x v^\delta\right\|_{L^p(\mathbb{R}^3)}^{\frac12-\lambda}\\
&\qquad\times\|u_\nu\|_{L^2(\mathbb{R}^3)}^\lambda\|\nabla_x u_\nu\|_{L^2(\mathbb{R}^3)}^{\frac32}\left\|\frac1r\nabla_x u_\nu\right\|_{L^p(\mathbb{R}^3)}^{\frac12-\lambda}\,dt,
\end{aligned}
$$

where $\lambda=\frac{3p-3}{7p-6}$. Using Hölder's and Young's convolution inequality, we then infer that

$$
\begin{aligned}
&\int_0^T\|\eta^\varepsilon*v^\delta\|_{L^4(\mathbb{R}^3)}\|u_\nu\|_{L^4(\mathbb{R}^3)}\|\nabla_x u_\nu\|_{L^2(\mathbb{R}^3)}\,dt\\
&\quad\lesssim\|v^\delta\|_{L^\infty(L^2(\mathbb{R}^3))}^\lambda\left\|\frac1r\nabla_x v^\delta\right\|_{L^\infty(L^p(\mathbb{R}^3))}^{\frac12-\lambda}\|\nabla_x v^\delta\|_{L^2((0,T)\times\mathbb{R}^3)}^{\frac12}\\
&\qquad\times\|u_\nu\|_{L^\infty(L^2(\mathbb{R}^3))}^\lambda\|\nabla_x u_\nu\|_{L^2((0,T)\times\mathbb{R}^3)}^{\frac32}\left\|\frac1r\nabla_x u_\nu\right\|_{L^\infty(L^p(\mathbb{R}^3))}^{\frac12-\lambda}.
\end{aligned}
$$

Notice that from the definition of v^δ, the triangle inequality and estimate (31) applied to u_ν^δ, it follows that $\|\frac1r\nabla_x v^\delta\|_{L^\infty(L^p(\mathbb{R}^3))}^{\frac12-\lambda}$ is bounded for all δ. From (30) and (31) and the assumptions on v^δ, we deduce that the right-hand side in the above estimate is vanishing as $\delta\to0$. Passing to the limit in the weak formulation of the Navier–Stokes equations above thus yields

$$
\int_0^T\int_{\mathbb{R}^3}\eta^\varepsilon(T-\tau)u_\nu(\tau,x)\cdot u_\nu(T,x)\,dxd\tau
$$

$$
\begin{aligned}
&= \int_0^T \int_{\mathbb{R}^3} \eta^\varepsilon(-\tau) u_\nu(\tau,x) \cdot u_0(x)\, dx d\tau \\
&\quad - \int_0^T \int_0^T \int_{\mathbb{R}^3} \eta^\varepsilon(t-\tau) u_\nu(\tau,x) \cdot (u_\nu(t,x)\cdot\nabla_x) u_\nu(t,x)\, dx d\tau dt \\
&\quad - \nu \int_0^T \int_0^T \int_{\mathbb{R}^3} \eta^\varepsilon(t-\tau) \nabla_x u_\nu(\tau,x) : \nabla_x u_\nu(t,x)\, dx d\tau dt.
\end{aligned}
$$

Notice that the term that involved the time derivative on η^ε dropped out by imposing that η^ε is an even function.

We finally send ε to zero and may thus choose $\varepsilon < T$ from here on. Notice first that

$$
\nu \int_0^T \int_0^T \int_{\mathbb{R}^3} \eta^\varepsilon(t-\tau) \nabla_x u_\nu(\tau,x) : \nabla_x u_\nu(t,x)\, dx d\tau dt \to \nu \int_0^T \|\nabla_x u_\nu\|^2_{L^2(\mathbb{R}^3)}\, dt
$$

thanks to standard convergence properties of the mollifier. For the convergence of the end-point integrals, we make use of the fact that our solutions are continuous in time with respect to the weak topology in $L^2(\mathbb{R}^3)$, see, e.g., [46, Corollary 3.2]. Because η^ε is chosen even, Lebesgue's convergence theorem then yields

$$
\begin{aligned}
\int_0^T \int_{\mathbb{R}^3} \eta^\varepsilon(T-\tau) u_\nu(\tau,x) \cdot u_\nu(T,x)\, dx d\tau &\to \frac{1}{2}\|u_\nu(T)\|^2_{L^2(\mathbb{R}^3)}, \\
\int_0^T \int_{\mathbb{R}^3} \eta^\varepsilon(-\tau) u_\nu(\tau,x) \cdot u_0(x)\, dx d\tau &\to \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^3)}.
\end{aligned}
$$

It remains to argue that the nonlinear term is vanishing. Notice first that

$$
\int_0^T \int_{\mathbb{R}^3} u_\nu^\delta \cdot (u_\nu \cdot \nabla_x) u_\nu^\delta\, dx dt = \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} u_\nu \cdot \nabla_x |u_\nu^\delta|^2\, dx dt = 0
$$

for any δ if u_ν^δ is defined as above. This identity carries over to the limit $\delta \to 0$ as can be seen by using the same kind of estimates that we used above in order to control the nonlinear term. We may thus rewrite the nonlinear term above as

$$
\int_0^T \int_{\mathbb{R}^3} (u_\nu - \eta^\varepsilon * u_\nu) \cdot (u_\nu \cdot \nabla_x) u_\nu\, dx dt,
$$

and, by applying the same kind of estimates again, we observe that this term vanishes as $\varepsilon \to 0$ by the convergence properties of the mollifier. $\square$

5 Vanishing viscosity limit. Proof of Theorem 1

In this section, we turn to the proof of Theorem 1. The compactness argument is based on the a priori estimate (24) on the relative vorticity and local estimates on the velocity field. The latter are provided by the following two lemmas.

Lemma 10 *For any $R > 0$, $p_* \in (1, p] \cap (1, 2)$ and $q_* \in (2, \frac{2p}{2-p}) \cap (2, \infty)$, there exists a constant $C(R)$ such that*

$$\begin{aligned} &\|u_\nu\|_{L^\infty((0,T);L^{q_*}(B_R(0)\cap\mathbb{H}))} + \|\nabla_x u_\nu\|_{L^\infty((0,T);L^{p_*}(B_R(0)\cap\mathbb{H}))} \\ &\quad \le C(R)\left(\|\xi_0\|_{L^p(\mathbb{R}^3)} + \|\omega_0\|_{L^1(\mathbb{H})}\right). \end{aligned} \tag{32}$$

Proof By standard interpolation between Lebesgue spaces, we may without loss of generality assume that $p = p_* < 2$. The bound on the gradient is an immediate consequence of the maximal regularity estimate in Lemma 4,

$$\|\nabla_x u_\nu\|_{L^\infty((0,T);L^p(B_R))} \le R^{1-\frac{1}{p}} \|\frac{1}{r}\nabla_x u_\nu\|_{L^\infty((0,T);L^p(\mathbb{R}^3))} \lesssim R^{1-\frac{1}{p}} \|\xi_0\|_{L^p(\mathbb{R}^3)}$$

where $B_R = B_R(0) \cap \mathbb{H}$ denotes the open half ball of radius R centered at 0 in the half-space $\mathbb{H}$.

In order to deduce an estimate on the velocity field itself, we first invoke the Poincaré estimate for mean-zero functions, formula (17) and the previous bound to observe that

$$\begin{aligned} \|u_\nu\|_{L^p(B_R)} &\lesssim R\|\nabla u_\nu\|_{L^p(B_R)} + R^{\frac{2}{p}-2}\|u_\nu\|_{L^1(B_R)} \\ &\lesssim R^{2-\frac{1}{p}}\|\xi_0\|_{L^p(\mathbb{R}^3)} + R^{\frac{2}{p}-2}\|u_\nu\|_{L^1(B_R)} \end{aligned} \tag{33}$$

uniformly in time. It remains to bound the L^1 norm of u. For this purpose, we make use of the decay behavior of the Biot–Savart kernel. In [29], the authors show that the decay of the axisymmetric Biot–Savart kernel is identical (in scaling) to that of the planar Biot–Savart kernel, that is, if we rewrite (6) as

$$u_\nu(r, z) = \int_{\mathbb{H}} K(r, z, \bar r, \bar z)\omega_\nu(\bar r, \bar z)\, d(\bar r, \bar z),$$

then the kernel K obeys the estimate

$$|K(r, z, \bar r, \bar z)| \lesssim \frac{1}{|r - \bar r| + |z - \bar z|}.$$

We thus write

$$
\begin{aligned}
\int_{B_R} |u_\nu(r,z)|\,d(r,z) &\lesssim \int_{B_R}\int_{\mathbb{H}} \frac{|\omega_\nu(\bar r,\bar z)|}{|r-\bar r|+|z-\bar z|}\,d(\bar r,\bar z)d(r,z)\\
&= \int_{B_R}\int_{B_R(r,z)\cap\mathbb{H}} \frac{|\omega_\nu(\bar r,\bar z)|}{|r-\bar r|+|z-\bar z|}\,d(\bar r,\bar z)d(r,z)\\
&\quad + \int_{B_R}\int_{B_R(r,z)^c\cap\mathbb{H}} \frac{|\omega_\nu(\bar r,\bar z)|}{|r-\bar r|+|z-\bar z|}\,d(\bar r,\bar z)d(r,z).
\end{aligned}
$$

For the near-field, we use Fubini's theorem, Young's convolution estimate and Lemma 6 to deduce

$$
\begin{aligned}
\int_{B_R}\int_{B_R(r,z)\cap\mathbb{H}} \frac{|\omega_\nu(\bar r,\bar z)|}{|r-\bar r|+|z-\bar z|}\,d(\bar r,\bar z)d(r,z) &\lesssim \|\omega_\nu * \left(\frac{1}{|\cdot|}\chi_{B_R(0)}\right)\|_{L^1(\mathbb{H})}\\
&\le \int_{B_R(0)} \frac{1}{\sqrt{r^2+|z|^2}}\,d(r,z)\|\omega_\nu\|_{L^1(\mathbb{H})}\\
&\lesssim R\|\xi_0\|_{L^1(\mathbb{R}^3)}.
\end{aligned}
$$

For the far-field, we simply observe that the kernel is bounded below, and thus

$$
\begin{aligned}
&\int_{B_R}\int_{B_R(r,z)^c\cap\mathbb{H}} \frac{|\omega_\nu(\bar r,\bar z)|}{|r-\bar r|+|z-\bar z|}\,d(\bar r,\bar z)d(r,z)\\
&\lesssim \frac{1}{R}\int_{B_R}\int_{B_R(r,z)^c\cap\mathbb{H}} |\omega_\nu(\bar r,\bar z)|\,d(\bar r,\bar z)\,d(r,z)\\
&\lesssim R\|\omega_\nu\|_{L^1(\mathbb{H})} \le R\|\xi_0\|_{L^1(\mathbb{R}^3)}.
\end{aligned}
$$

Plugging the previous bounds into (33) and recalling the already established gradient bounds yields a uniform in time control on $\|u_\nu\|_{W^{1,p}(B_R)}$. From this, we deduce the estimate in L^{q_*} in (32) via standard two-dimensional Sobolev embedding. $\square$

Lemma 11 *For any $R>0$, it holds that*

$$
\|\partial_t u_\nu\|_{L^\infty((0,T);W_\sigma^{-1,1}(B_R(0))} \le C(R)\left(\|\xi_0\|_{L^p(\mathbb{R}^3)} + \|\omega_0\|_{L^1(\mathbb{H})}\right),
$$

where $W_\sigma^{-1,1}(B_R(0))^3$ is the Banach space that is dual to the space of divergence-free vector fields in $W_0^{1,\infty}(B_R(0))^3$.

The proof of this estimate is fairly standard. We sketch the argument for the convenience of the reader.

Proof Let F be a divergence-free vector field in $W_0^{1,\infty}(B_R(0))^3$. Then

$$
\begin{aligned}
(u_\nu\cdot\nabla u_\nu, F)_{W_\sigma^{-1,1}(B_R(0))\times W_0^{1,\infty}(B_R(0))} &= -\int_{B_R(0)} u_\nu\otimes u_\nu : \nabla F\,dx\\
&\le \|u_\nu\|_{L^2(B_R(0))}^2\|F\|_{W^{1,\infty}(B_R(0))},
\end{aligned}
$$

and a similar bound holds for the dissipation term $-\nu\Delta u_\nu$. The statement thus follows directly from the momentum equation and Lemma 10. □

We are now in the position to prove the compactness result.

Proof of Theorem 1 Notice that the norms in the statement of Lemma 10 can be replaced by the corresponding norms on the three-dimensional balls $B_R(0)$, because $dx \leq 2\pi R d(r, z)$ in the domain of integration. Therefore, thanks to Lemmas 10 and 11, the sequence of velocity fields $\{u_\nu\}_{\nu\downarrow 0}$ satisfies the hypotheses of the Aubin–Lions Lemma because $W^{1,p_*}(B_R(0)) \cap L^{\frac{2p_*}{2-p_*}}(B_R(0))$ is compactly embedded in $L^2(B_R(0))$ and the latter in is continuously embedded in $W^{-1,1}_\sigma(B_R(0))$. Therefore, for any $R > 0$, there exists a subsequence that converges strongly in $C([0, T]; L^2(B_R(0)))$. By applying a diagonal sequence argument, this convergence carries over to the space $C([0, T]; L^2(K))$ for any compact K in $\mathbb{R}^3$. Hence, there exists a subsequence (not relabelled) and a vector field $u \in C([0, T]; L^2_{\text{loc}}(\mathbb{R}^3)^3)$ such that

$$u_\nu \to u \quad \text{strongly in } C([0, T]; L^2_{\text{loc}}(\mathbb{R}^3)).$$

It is readily checked that u is a distributional solution to the Euler equations (1), (2).

Moreover, from the a priori estimate on the relative vorticity in Lemma 6, we deduce that there exists a function $\xi \in L^\infty((0, T); L^p(\mathbb{R}^3))$ such that, upon taking a further subsequence,

$$\xi_\nu \to \xi \quad \text{weakly} - \star \text{ in } L^\infty((0, T); L^p(\mathbb{R}^3)).$$

We finally notice that the velocity field u and the vorticity $\omega = r\xi$ are related by the Biot–Savart law that holds true in the sense of distributions. □

6 Renormalization. Proof of Theorem 2

In this section, we provide the argument for the renormalization property of the relative vorticity obtained as the vanishing viscosity solution of the Navier–Stokes equations in Theorem 1. Our approach is based on the duality formula in Lemma 1 established in [24] and follows closely the argumentation from [17,18]. By interpolation of Lebesques spaces, we may without loss of generality assume that $p < 2$ in (9).

We now show a compactness result for a backwards advection-diffusion equation, that is, as we will see, dual to the vorticity formulation (23) of the Navier–Stokes equations.

Lemma 12 *Let $q \in (2, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} \leq 1$ and let $\chi \in L^1((0, T); L^q(\mathbb{R}^3))$ be a given axisymmetric function, $\chi = \chi(r, z)$. Let f_ν denote the unique solution in the class $L^\infty((0, T); L^q(\mathbb{R}^3))$ with $\nabla |f_\nu|^{\frac{q}{2}} \in L^2((0, T); L^2(\mathbb{R}^3))$ to the backwards advection-diffusion equation*

$$-\partial_t f_\nu - u_\nu \cdot \nabla f_\nu = \chi + \nu \left(\Delta f_\nu - \frac{1}{r} \partial_r f \right)$$

in $\mathbb{H}$ with final datum $f_\nu(T) = 0$ and homogeneous Dirichlet boundary conditions $f_\nu = 0$ on $\partial\mathbb{H}$. Then there exists a subsequence $\{\nu_k\}_{k\in\mathbb{N}}$ (the same as in Theorem 1) such that

$$f_{\nu_k} \to f \quad weakly - \star in L^\infty((0,T); L^q(\mathbb{R}^3)),$$

where f is the unique solution to the backwards transport equation (15).

We remark that renormalized solutions to advection-diffusion equations have been considered, for instance, in [24,28,37].

Proof We start with an a priori estimate. A direct computation reveals that

$$\begin{aligned} \frac{d}{dt}\frac{1}{q}\int_{\mathbb{R}^3} |f_\nu|^q \, dx &= -\int_{\mathbb{R}^3} |f_\nu|^{q-2} f_\nu \chi \, dx + \nu(q-1)\int_{\mathbb{R}^3} |f_\nu|^{q-2}|\nabla f_\nu|^2 \, dx \\ &\geq -\|f_\nu\|^{q-1}_{L^q(\mathbb{R}^3)}\|\chi\|_{L^q(\mathbb{R}^3)}, \end{aligned} \tag{34}$$

where we have used the Dirichlet boundary conditions on f_ν, and thus $\frac{d}{dt}\|f_\nu\|_{L^q} \geq -\|\chi\|_{L^q}$. Via integration and by our choice of the final datum, we deduce that

$$\|f_\nu\|_{L^\infty((0,T);L^q(\mathbb{R}^3))} \leq \|\chi\|_{L^1((0,T);L^q(\mathbb{R}^3))}. \tag{35}$$

Hence, there exists a subsequence $\{\nu_k\}_{k\in\mathbb{N}}$ that can be chosen as a subsequence of the one found in Theorem 1 and an $\tilde{f} \in L^\infty((0,T); L^q(\mathbb{R}^3))$ such that

$$f_{\nu_k} \to \tilde{f} \quad \text{weakly} - \star \text{ in } L^\infty((0,T); L^q(\mathbb{R}^3)).$$

Since at the same time

$$u_{\nu_k} \to u \quad \text{strongly in } L^2((0,T); L^2_{\text{loc}}(\mathbb{R}^3)),$$

by the virtue of Theorem 1 and $q \geq 2$, we find in the limit that $\tilde{f}$ solves the backward advection equation (15) in the sense of distributions. From Theorem 1 and Lemma 4 we have $u \in L^1((0,T); W^{1,p}_{\text{loc}}(\mathbb{R}^3))$ for $1 < p < 2$ and thus, by Theorem 4, the solution is renormalized and thus unique; hence $\tilde{f} = f$. In particular, the convergence result holds true for the subsequence from Theorem 1. □

We finally turn to the proof of the renormalization property.

Proof of Theorem 2 Let $\chi \in C^\infty_c((0,T)\times\mathbb{R}^3)$ be an arbitrarily fixed axisymmetric function and f_ν a solution to the backwards advection-diffusion equation considered in Lemma 12. From the statement of the lemma, it follows that $\{f_{\nu_k}\}_{k\in\mathbb{N}}$ converges to f weakly-$\star$ in $L^\infty((0,T); L^q(\mathbb{R}^3))$. Moreover, there exists a positive number s such that for any radius R, the sequence of time derivatives $\{\partial_t f_{\nu_k}\}_{k\in\mathbb{N}}$ is bounded in $L^1((0,T); H^{-s}(B_R(0)))$, and thus, invoking an Arzelà–Ascoli-type argument, we conclude that the convergence of $\{f_{\nu_k}\}_{k\in\mathbb{N}}$ can be upgraded to hold in $C([0,T],L^q_{\text{weak}}(\mathbb{R}^3))$, that is,

$$\sup_{[0,T]}\int_{\mathbb{R}^3}(f_{\nu_k}(t)-f(t))\zeta\,dx\to 0\quad\forall\zeta\in L^{\tilde q}(\mathbb{R}^3),\tag{36}$$

where $1/q+1/\tilde q=1$.

Upon a standard approximation argument, f_ν can be considered as a test function in the distributional formulation of the vorticity formulation (23) of the Navier–Stokes equations. Thus

$$\begin{aligned}\int_{\mathbb{R}^3} f_\nu(0)\xi_0\,dx &= \int_0^T\int_{\mathbb{R}^3}\xi_\nu\left(\partial_t f_\nu+u_\nu\cdot\nabla f_\nu+\nu\left(\Delta f_\nu-\frac{1}{r}\partial_r f_\nu\right)\right)dxdt\\ &=\int_0^T\int_{\mathbb{R}^3}\xi_\nu\chi\,dxdt.\end{aligned}$$

As a consequence of Theorem 1, Lemma 12 and (36), we can pass to the limit in this identity and find

$$\int_{\mathbb{R}^3} f(0)\xi_0\,dx=\int_0^T\int_{\mathbb{R}^3}\xi\chi\,dxdt.$$

On the other hand, because u satisfies the general assumptions of Theorem 4, see Lemma 2, there exists a unique renormalized solution $\tilde\xi\in L^\infty((0,T);L^p(\mathbb{R}^3))$ to the transport equation (5) with u being the given solution to the Euler equations and with initial datum ξ_0. Of course, $\tilde\xi$ is axisymmetric. By Lemma 1, we then find that

$$\int_{\mathbb{R}^3} f(0)\xi_0\,dx=\int_0^T\int_{\mathbb{R}^3}\tilde\xi\chi\,dxdt,$$

and thus,

$$\int_0^T\int_{\mathbb{R}^3}(\xi-\tilde\xi)\chi\,dxdt=0.$$

Because χ was an arbitrarily fixed smooth axisymmetric function and ξ and $\tilde\xi$ are both axisymmetric, we infer that $\tilde\xi=\xi$ almost everywhere, and thus, ξ coincides almost everywhere with the renormalized solution $\tilde\xi$. □

7 Energy conservation. Proof of Theorem 3

We now prove Theorem 3. Throughout this section, we thus suppose that ω_ν is non-negative and has finite impulse.

One of the main ingredients of the proof is the convergence of the kinetic energy that is established in the following lemma.

Lemma 13 *Let $p > \frac{11}{9}$. Let $\{\nu_k\}_{k\in\mathbb{N}}$ be the subsequence found in Theorem 1. Then it holds that*

$$\lim_{k\to\infty} \|u_{\nu_k}(t)\|_{L^2(\mathbb{R}^3)} = \|u(t)\|_{L^2(\mathbb{R}^3)}$$

for any $t \in [0, T]$.

Proof We have already seen in Theorem 1 that u_{ν_k} converges to u strongly in $C(0, T; L^2_{\text{loc}}(\mathbb{R}^3))$. We have to turn this result into a global convergence result. In fact, it is enough to show that

$$\sup_k \|u_{\nu_k}(t)\|_{L^2(\mathbb{R}^3\setminus B_R(0))} \to 0 \quad \text{as } R \to \infty. \tag{37}$$

Indeed, if (37) holds true, given $\varepsilon > 0$, we can find a radius $R \geq 1$ such that

$$\sup_k \|u_{\nu_k}(t, \cdot + h)\|_{L^2(\mathbb{R}^3\setminus B_{2R}(0))} \leq \varepsilon \quad \text{for any } |h| \leq 1.$$

Moreover, thanks to the strong convergence in $B_{2R}(0)$, we have that

$$\sup_k \|u_{\nu_k}(t) - u_{\nu_k}(t, \cdot + h)\|_{L^2(B_R(0))} \leq \varepsilon \quad \text{for } |h| \text{ sufficiently small.}$$

Combining both estimates, we find that

$$\sup_k \|u_{\nu_k}(t) - u_{\nu_k}(t, \cdot + h)\|_{L^2(\mathbb{R}^3)} \leq 3\varepsilon \quad \text{for } |h| \text{ sufficiently small.}$$

By Riesz' compactness criterion, the latter result together with (37) and the standard energy estimate (29) imply strong convergence in $L^2(\mathbb{R}^3)$ for ant $t \in [0, T]$.

We now give the argument for (37). For notational convenience, we write u and ν instead of u_{ν_k} and ν_k. We consider a smooth cut-off function η_R that is 1 in $B_R = B_R(0)$ and 0 outside $B_{2R} = B_{2R}(0)$. Testing the Navier–Stokes equations with $(1 - \eta_R)^2 u$ and integrating by parts yields

$$\frac{d}{dt}\frac{1}{2}\int (1-\eta_R)^2|u|^2\,dx + \nu\int (1-\eta_R)^2|\nabla u|^2\,dx \tag{38}$$

$$= \int (\eta_R - 1)\nabla\eta_R \cdot u|u|^2\,dx + 2\int (\eta_R - 1)\nabla\eta_R \cdot up\,dx \tag{39}$$

$$+ 2\nu\int (1-\eta_R)(\nabla\eta_R\cdot\nabla)u\cdot u\,dx. \tag{40}$$

The error term in (40) is quite easily estimated. Indeed, using the Cauchy–Schwarz inequality together with the elementary inequality $2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon}b^2$, we can absorb the gradient term in (40) in the second term in (38) and we are left with an error term of the form $\frac{\nu}{R^2}\|u\|^2_{L^2(\mathbb{R}^3)}$. In view of the energy inequality for the Navier–Stokes equations, this term is obviously vanishing as $R \to \infty$ uniformly in t.

As a next step, we address the first error term in (39). Using the properties of the cut-off function, this term is bounded as follows:

$$\int (\eta_R - 1)\nabla\eta_R \cdot u|u|^2\,dx \lesssim \frac{1}{R}\int_{B_{2R}\setminus B_R} |u|^3\,dx \lesssim \int_{B_{2R}\setminus B_R} |u|^3\,d(r,z). \tag{41}$$

Here, we have used the same notation for both the ball in $\mathbb{R}^3$ and the half ball in $\mathbb{H}$. It should be clear from the situation, which one is considered. The expression on the right-hand side vanishes as $R \to \infty$ as a consequence of Lemmas 5,6, 8, and 9.

We finally turn to the term that involves the pressure, that is, the second term in (39). We choose $r \in \left(3, \frac{6p-p}{3-r}\right)$ and assume for convenience that $p \le \frac{5}{3}$, which can be achieved by interpolation between Lebesgue spaces. Using the properties of the cut-off function and Hölder's inequality, we observe that

$$\int (\eta_R - 1)\nabla\eta_R \cdot up\,dx \lesssim \left(\frac{1}{R}\int_{B_{2R}\setminus B_R} |u|^r\,dx\right)^{\frac{1}{r}} \left(\frac{1}{R}\int_{B_{2R}\setminus B_R} |p|^{\frac{r}{r-1}}\,dx\right)^{\frac{r-1}{r}}.$$

Again, the velocity term vanishes by the virtue of Lemmas 5,6, 8, and 9, however, this time we need to extract a small amount of more information on the rate of decay, namely

$$\frac{1}{R}\int_{B_{2R}\setminus B_R} |u|^r\,dx \lesssim R^{-\frac{p(r+6)-3r-2}{2(p-1)}}, \tag{42}$$

which is the leading order term for large R, because $p \le \frac{5}{3} \le \frac{3}{2}$. It is thus enough to consider the pressure term and to show that

$$\lim_{R\to\infty} R^{-1-\frac{p(r+6)-3r-2}{2(p-1)(r-1)}} \int_{B_{2R}\setminus B_R} |p|^{\frac{r}{r-1}}\,dx = 0. \tag{43}$$

For this, we recall that p solves the Poisson equation $-\Delta_x p = \nabla_x^2 : u \otimes u$, and thus, we have that $p = \sum_{ij} \partial_{x_i}\partial_{x_j} G * (u_i u_j)$, where G is the Newtonian potential in $\mathbb{R}^3$, i.e., $G(x) = \frac{1}{4\pi}\frac{1}{|x|}$. Let us write $f = G * (u_i u_j)$, so that $|p| \lesssim |\nabla_x^2 f|$. The localized Calderón–Zygmund estimates (see, e.g., Theorem 9.11 in [30]) yield

$$\|\nabla_x^2 f\|_{L^{\frac{r}{r-1}}(B_{2R}\setminus B_R)} \lesssim \|u_i u_j\|_{L^{\frac{r}{r-1}}(B_{3R}\setminus B_{\frac{R}{2}})} + R^{-2}\|f\|_{L^{\frac{r}{r-1}}(B_{3R}\setminus B_{\frac{R}{2}})},$$

and thus, (43) is a consequence of the two estimates

$$\lim_{R\to\infty} R^{-\frac{rp-8p+r+4}{4(p-1)r}} \|u\|_{L^{\frac{2r}{r-1}}(B_{2R}\setminus B_R)} = 0, \tag{44}$$

$$\lim_{R\to\infty} R^{-\frac{rp+10p-3r-6}{2(p-1)r}} \|f\|_{L^\infty(B_{2R}\setminus B_R)} = 0. \tag{45}$$

To prove (44), we use Jensen's inequality on the annulus and estimate (42),

$$\|u\|_{L^{\frac{2r}{r-1}}(B_{3R}\setminus B_{\frac{R}{2}})} \lesssim R^{\frac{3r-9}{2r}} \|u\|_{L^r(B_{3R}\setminus B_{\frac{R}{2}})} \lesssim R^{\frac{2pr-13p-9}{2(p-1)r}}.$$

We observe that

$$\frac{2pr-13p-9}{2(p-1)r} < \frac{rp-8p+r+4}{4(p-1)r} \iff r < \frac{18p-14}{3p-1},$$

and the latter condition can be achieved by our choice of r's because $\frac{18p-14}{3p-1} > 3$ for any $p > \frac{11}{9}$.

To prove (45), we use the kernel representation of f and estimate and decompose

$$|f(x)| \lesssim \int_{B_{\frac{R}{4}}(x)} \frac{1}{|x-y|}|u(y)|^2\,dy + \int_{B_{\frac{R}{4}}(x)^c} \frac{1}{|x-y|}|u(y)|^2\,dy.$$

To estimate the first term, we use Jensen's inequality and the fact that $B_{\frac{R}{4}(x)} \subset B_{4R}\setminus B_{\frac{R}{4}}$ for any $x \in B_{3R}\setminus B_{\frac{R}{2}}$,

$$\begin{aligned}\int_{B_{\frac{R}{4}}(x)} \frac{1}{|x-y|}|u(y)|^2\,dy &\lesssim \left(\int_{B_{\frac{R}{4}}(x)} \frac{1}{|x-y|^{\frac{r}{r-2}}}\,dy\right)^{\frac{r-2}{r}} \left(\int_{B_{\frac{R}{4}}(x)} |u(y)|^r\,dy\right)^{\frac{2}{r}}\\ &\lesssim R^{\frac{2r-6}{r}} \left(\int_{B_{4R}\setminus B_{\frac{R}{4}}} |u(y)|^r\,dy\right)^{\frac{2}{r}}.\end{aligned}$$

Note that in this step, the assumption $r > 3$ is crucial. Plugging in the bound in (42), we deduce

$$\int_{B_{\frac{R}{4}}(x)} \frac{1}{|x-y|}|u(y)|^2\,dy \lesssim R^{\frac{pr-10p+r+6}{r(p-1)}}.$$

We observe that

$$\frac{pr-10p+r+6}{r(p-1)} < \frac{rp+10p-3r-6}{2(p-1)r} \iff r < \frac{30p-18}{p+5},$$

and the latter condition can be achieved by our choice of r's because $\frac{30p-18}{p+5} > 3$ for any $p > \frac{11}{9}$. For the second term, we simply estimate

$$\int_{B_{\frac{R}{4}}(x)^c} \frac{1}{|x-y|}|u(y)|^2\,dy \lesssim \frac{1}{R}\|u\|_{L^2(\mathbb{R}^3)}^2 \lesssim R^{-1},$$

by Lemma 9, and we notice that

$$\frac{rp + 10p - 3r - 6}{2(p-1)r} + 1 > 0 \quad \Longleftrightarrow \quad r < \frac{10p - 6}{5 - 3p},$$

where we have used the assumption that $p < \frac{5}{3}$. The latter statement can again be fulfilled by our r's because $\frac{10p-6}{5-3p} > 3$ for any $p > \frac{21}{19}$, and thus, in particular, for any $p > \frac{11}{9}$. We have thus established (45). □

With these preparations, we are now in the position to prove Theorem 3. Our short proof is strongly inspired by [14].

Proof of Theorem 3 In order to prove conservation of energy, we choose a subsequence as in Theorem 1, which we will not relabel for notational convenience, and recall the energy identity in Lemma 9, which we rewrite as

$$0 \geq \|u_\nu(t)\|^2_{L^2(\mathbb{R}^3)} - \|u_0\|^2_{L^2(\mathbb{R}^3)} = -2\nu \int_0^t \|\nabla_x u_\nu(s)\|^2_{L^2(\mathbb{R}^3)}\, ds.$$

Thanks to Lemmas 3, 8 and 7, we observe that

$$\|\nabla_x u_\nu(s)\|^2_{L^2(\mathbb{R}^3)} = \|r\xi_\nu(s)\|^2_{L^2(\mathbb{R}^3)} \leq \|r^2\xi_\nu(s)\|_{L^1(\mathbb{R}^3)} \|\xi_\nu(s)\|_{L^\infty(\mathbb{R}^3)} \lesssim \left(\frac{1}{\nu s}\right)^{\frac{3}{2p}},$$

and thus, the energy identity implies that

$$0 \geq \|u_\nu(t)\|^2_{L^2(\mathbb{R}^3)} - \|u_0\|^2_{L^2(\mathbb{R}^3)} \geq -C(\nu t)^{1-\frac{3}{2p}},$$

because $p > \frac{3}{2}$. Sending ν to zero, we conclude that

$$\lim_{\nu\to\infty} \|u_\nu(t)\|_{L^2(\mathbb{R}^3)} = \|u_0\|_{L^2(\mathbb{R}^3)},$$

and the statement of the theorem follows upon applying Lemma 13, in which the convergence of the kinetic energy is established. □

Acknowledgements Open Access funding provided by Projekt DEAL. CN thanks Stefano Spirito for pointing out references. CS acknowledges discussions with Helena Nussenzveig Lopes. This work is partially funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2044 –390685587, Mathematics Münster: Dynamics–Geometry–Structure.

Appendix: Two auxiliary inequalities

We conclude this paper with two auxiliary inequalities, that are weighted versions of standard Sobolev and interpolation inequalities.

Lemma 14 *Let* $1 \le s \le t < \infty$ *and* $\alpha, \beta \in \mathbb{R}$ *be such that*

$$\alpha > -1, \quad \beta \in [\alpha(1-\frac{s}{2}), s+\alpha], \quad \frac{2+\alpha}{t} = \frac{2-s+\beta}{s}.$$

Then

$$\left(\int_{\mathbb{H}} |f|^t r^\alpha \, d(r,z)\right)^{\frac{1}{t}} \lesssim \left(\int_{\mathbb{H}} |\nabla f|^s r^\beta \, d(r,z)\right)^{\frac{1}{s}},$$

for any $f \in C_c^\infty(\bar{\mathbb{H}})$*, provided that the right-hand side is bounded.*

This estimate is proved, for instance, in [36]. We recall the argument for completeness.

Proof *Step 1.* We first treat the special case $s = 1$, and thus

$$\frac{2+\alpha}{t} = 1+\beta.$$

We set $\gamma = \frac{\alpha}{t}$ and let $g \in C_c^\infty(\mathbb{H})$ be defined by $g(r,z) = f(2r,z)$ and $A = [R,2R] \times \mathbb{R}$ and $B = [R,4R] \times \mathbb{R}$ be two subsets of $\mathbb{H}$ for some $R > 0$ fixed. By Hölder's inequality, we then have that

$$\int_A (f-g)^2 r^{2\gamma} \, d(r,z) \le \int_R^{2R} \|r^\gamma (f-g)\|_{L^1(dz)} \|r^\gamma (f-g)\|_{L^\infty(dz)} \, dr.$$

We now use the embedding $\dot{W}^{1,1} \subset L^\infty$, that holds true in one space dimension, in each variable. On the one hand, using the embedding in r (in form of the fundamental theorem of calculus), we have

$$\sup_{r \in (R,2R)} \|r^\gamma (f-g)\|_{L^1(dz)} \le \sup_{r \in (R,2R)} r^\gamma \int_r^{2r} \|\partial_\rho f(\rho)\|_{L^1(dz)} d\rho \lesssim \int_B |\nabla f| r^\gamma \, d(r,z).$$

On the other hand, it holds that

$$\int_R^{2R} \|r^\gamma (f-g)\|_{L^\infty(dz)} \, dr \lesssim \int_R^{2R} r^\gamma \|\partial_z (f-g)\|_{L^1(dz)} \, dr \lesssim \int_B |\nabla f| r^\gamma \, d(r,z),$$

where we have used the triangle inequality and a rescaling argument in the last inequality. Combining the previous three estimates, we find that

$$\left(\int_A (f-g)^2 r^{2\gamma} \, d(r,z)\right)^{\frac{1}{2}} \lesssim \int_B |\nabla f| r^\gamma \, d(r,z). \tag{46}$$

Our next goal is the Hardy-type inequality

$$\int_A |f-g| r^\gamma \, d(r,z) \lesssim \int_B |\nabla f| r^{\gamma+1} \, d(r,z). \tag{47}$$

It can be established as follows: Using the fundamental theorem again, we observe that

$$\int_A |f-g| r^\gamma \, d(r,z) \le \int_R^{2R} r^\gamma \int_r^{2r} \|\partial_\rho f(\rho)\|_{L^1(dz)} \, d\rho dr \lesssim R^{\gamma+1} \int_B |\partial_r f| \, d(r,z),$$

which implies (47) because the prefactor $R^{\gamma+1}$ can be smuggled into the integrand.

Towards the weighted Sobolev inequality with $s=1$, we set $A_k = [2^k, 2^{k+1}] \times \mathbb{R}$ and $B_k = [2^k, 2^{k+2}] \times \mathbb{R}$ and estimate with the help of the triangle inequality

$$\left(\int_{\mathbb{H}} |f-g|^t r^\alpha \, d(r,z) \right)^{\frac{1}{t}} \le \sum_{k\in\mathbb{Z}} \left(\int_{A_k} |f-g|^t r^\alpha \, d(r,z) \right)^{\frac{1}{t}}.$$

Interpolation between Lebesgue spaces and an application of (46) and (47) yields

$$\begin{aligned} &\left(\int_{A_k} |f-g|^t r^\alpha \, d(r,z) \right)^{\frac{1}{t}} \\ &\le \left(\int_{A_k} |f-g| r^\gamma \, d(r,z) \right)^{\beta-\gamma} \left(\int_{A_k} (f-g)^2 r^{2\gamma} \, d(r,z) \right)^{\frac{1-\beta+\gamma}{2}} \\ &\lesssim \left(\int_{B_k} |\nabla f| r^{\gamma+1} \, d(r,z) \right)^{\beta-\gamma} \left(\int_{B_k} |\nabla f| r^{\gamma} \, d(r,z) \right)^{1-\beta+\gamma} \\ &\sim \int_{B_k} |\nabla f| r^\beta \, d(r,z), \end{aligned}$$

because $\beta - \gamma = \frac{2}{t} - 1 \in [0,1]$ since $s=1$ by our assumption on β. Summation over k yields

$$\left(\int_{\mathbb{H}} |f-g|^t r^\alpha \, d(r,z) \right)^{\frac{1}{t}} \le C \int_{\mathbb{H}} |\nabla f| r^\beta \, d(r,z)$$

for some universal constant C. It remains to apply the triangle inequality and a change of variables to the effect that

$$\begin{aligned} \left(\int_{\mathbb{H}} |f|^t r^\alpha \, d(r,z) \right)^{\frac{1}{t}} &\le \left(\int_{\mathbb{H}} |g|^t r^\alpha \, d(r,z) \right)^{\frac{1}{t}} + C \int_{\mathbb{H}} |\nabla f| r^\beta \, d(r,z) \\ &= \frac{1}{2^{\frac{\alpha+1}{t}}} \left(\int_{\mathbb{H}} |f|^t r^\alpha \, d(r,z) \right)^{\frac{1}{t}} + C \int_{\mathbb{H}} |\nabla f| r^\beta \, d(r,z). \end{aligned}$$

We can absorb the first term on the right-hand side by the left-hand side because $\alpha + 1 > 0$ and obtain

$$\left(\int_{\mathbb{H}} |f|^{\frac{2+\alpha}{1+\beta}} r^{\alpha}\, d(r,z)\right)^{\frac{1+\beta}{2+\alpha}} \lesssim \int_{\mathbb{H}} |\nabla f| r^{\beta}\, d(r,z). \tag{48}$$

Step 2. The general case $s > 1$ follows from the special case $s = 1$. Indeed, choosing $h = |f|^{\frac{1+\gamma}{2+\alpha} t}$ for some $\delta \in \left[\frac{\alpha}{2}, 1+\alpha\right]$ and substituting f by h and β by γ in (48), we find

$$\left(\int_{\mathbb{H}} |h|^{t} r^{\alpha}\, d(r,z)\right)^{\frac{1}{t}\frac{t(1+\delta)}{2+\alpha}} \lesssim \int_{\mathbb{H}} |h|^{\frac{(1+\delta)t}{2+\alpha}-1} |\nabla h| r^{\delta}\, d(r,z), .$$

Apply Hölder's inequality in the right-hand side to obtain

$$\left(\int_{\mathbb{H}} |h|^{t} r^{\alpha}\, d(r,z)\right)^{\frac{(1+\delta)}{2+\alpha}} \lesssim \left(\int_{\mathbb{H}} |h|^{(\frac{(1+\delta)t}{2+\alpha}-1)p} r^{\theta\delta p}\, d(r,z)\right)^{\frac{1}{p}} \left(\int_{\mathbb{H}} |\nabla h|^{q} r^{(1-\theta)\delta q} d(r,z)\right)^{\frac{1}{q}},$$

with (p,q) such that $\frac{1}{p} + \frac{1}{q} = 1$. Now set $q = s,\ (1-\theta)\delta q = \beta,\ p\theta\delta = \alpha$ and combining these conditions with $p = \frac{s}{s-1}$ we find $\delta = \frac{\beta}{s} + \frac{\alpha(s-1)}{s}$. Finally, it is easy to verify that the condition $\frac{(1+\delta)tp}{2+\alpha} - p = t$ holds exactly when $\frac{\beta+2-s}{s} = \frac{1+\alpha}{t}$. The statement follows by absorbing the first term of the right-hand side in the left-hand side.

□

We finally provide an interpolation inequality.

Lemma 15 *Let $p \in (1,2]$ and $\lambda = \frac{3p-3}{7p-6}$. Then*

$$\begin{aligned}&\left(\int_{\mathbb{H}} |f|^{4} r\, d(r,z)\right)^{\frac{1}{4}}\\ &\lesssim \left(\int_{\mathbb{H}} |f|^{2} r\, d(r,z)\right)^{\frac{\lambda}{2}} \left(\int_{\mathbb{H}} |\nabla f|^{2} r\, d(r,z)\right)^{\frac{1}{4}} \left(\int_{\mathbb{H}} |\nabla f|^{p} r^{1-p}\, d(r,z)\right)^{\frac{1-2\lambda}{2p}}\end{aligned} \tag{49}$$

for any $f \in C_c^{\infty}(\bar{\mathbb{H}})$.

Proof *Step 1*: It is enough to prove that

$$\left(\int_{\mathbb{H}} |f|^{4} r\, d(r,z)\right)^{\frac{1}{4}} \lesssim \left(\int_{\mathbb{H}} |f|^{2} r\, d(r,z)\right)^{\frac{\lambda}{2}} \left(\int_{\mathbb{H}} |\nabla f|^{q} r^{\gamma}\, d(r,z)\right)^{\frac{1-\lambda}{q}}, \tag{50}$$

where $\gamma = \frac{5p-4}{7p-4}$ and $q = \frac{16p-12}{7p-4}$.

Indeed, the statement in (49) immediately follows from (50) and Hölder's inequality. Let a and b be Hölder dual exponents given by $a = \frac{4(1-\lambda)}{q} = \frac{7p-4}{7p-6}$ and $b = \frac{4(1-\lambda)}{4(1-\lambda)-q} = \frac{7p-4}{2}$. We write and estimate

$$\begin{aligned}\int_{\mathbb{H}} |\nabla f|^q r^\gamma \, d(r,z) &= \int_{\mathbb{H}} \left(|\nabla f|^2 r\right)^{\frac{q}{4-4\lambda}} |\nabla f|^{\frac{1-2\lambda}{2-2\lambda}q} r^{\gamma - \frac{q}{4-4\lambda}} \, d(r,z) \\ &\le \left(\int_{\mathbb{H}} \left(|\nabla f|^2 r\right)^{\frac{qa}{4-4\lambda}} \, d(r,z)\right)^{\frac{1}{a}} \left(\int_{\mathbb{H}} |\nabla f|^{\frac{1-2\lambda}{2-2\lambda}qb} r^{\gamma b - \frac{qb}{4-4\lambda}} \, d(r,z)\right)^{\frac{1}{b}} \\ &= \left(\int_{\mathbb{H}} |\nabla f|^2 r \, d(r,z)\right)^{\frac{1}{a}} \left(\int_{\mathbb{H}} |\nabla f|^p r^{1-p} \, d(r,z)\right)^{\frac{1}{b}}.\end{aligned}$$

Now, plugging the resulting estimate into (50) yields (49).

Step 2. The interpolation inequality (50) follows from the weighted Sobolev inequality from Lemma 14 in the formulation

$$\left(\int_{\mathbb{H}} |f|^t r \, d(r,z)\right)^{\frac{1}{t}} \lesssim \left(\int_{\mathbb{H}} |\nabla f|^s r^\beta \, d(r,z)\right)^{\frac{1}{s}}, \tag{51}$$

where $t = \frac{16p-12}{7p-6}$, $s = \frac{16p-12}{13p-10}$ and $\beta = \frac{11p-10}{13p-10}$ via a Ladyshenskaya-type argument. Notice that t, s and β satisfy the conditions

$$\beta \in \left[1 - \frac{s}{2}, s+1\right], \quad t = \frac{3s}{2 - s + \beta},$$

because $p \ge 1$. Indeed, substituting $|f|^{\frac{4}{t}}$ for f in (51) implies that

$$\left(\int_{\mathbb{H}} |f|^4 r \, d(r,z)\right)^{\frac{1}{t}} \lesssim \left(\int_{\mathbb{H}} |f|^{(\frac{4}{t}-1)s} |\nabla f|^s r^\beta \, d(r,z)\right)^{\frac{1}{s}}.$$

We now use Hölder's inequality with dual exponents $a = \frac{13p-10}{6p-6}$ and $b = \frac{13p-10}{7p-4}$ and get, since $r^\beta = r^{\frac{1}{a}} r^{\beta - 1 + \frac{1}{b}}$, that

$$\begin{aligned}\int_{\mathbb{H}} |f|^{(\frac{4}{t}-1)s} |\nabla f|^s r^\beta \, d(r,z) &\le \left(\int_{\mathbb{H}} |f|^{(\frac{4}{t}-1)sa} r \, d(r,z)\right)^{\frac{1}{a}} \left(\int_{\mathbb{H}} |\nabla f|^{sb} r^{(\beta - 1 + \frac{1}{b})b} \, d(r,z)\right)^{\frac{1}{b}} \\ &= \left(\int_{\mathbb{H}} |f|^2 r \, d(r,z)\right)^{\frac{1}{a}} \left(\int_{\mathbb{H}} |\nabla f|^q r^\gamma \, d(r,z)\right)^{\frac{1}{b}}.\end{aligned}$$

Combining the previous two estimates, it is straightforward to deduce (50). This completes the proof. □

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