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Integrality of volumes of representations

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Abstract
Let M be an oriented complete hyperbolic n-manifold of finite volume. Using the definition of volume of a representation previously given by the authors in [3] we show that the volume of a representation $\rho\colon \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^n)$, properly normalized, takes integer values if n is even and ≥ 4. If M is not compact and 3-dimensional, it is known that the volume is not locally constant. In this case we give explicit examples of representations with volume as arbitrary as the volume of hyperbolic manifolds obtained from M via Dehn fillings.

1 Introduction

Let M be a connected oriented complete hyperbolic manifold of finite volume, which we represent as the quotient $M = \Gamma\backslash\mathbb{H}^n$ of real hyperbolic n-space $\mathbb{H}^n$ by a torsion-free lattice $\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$ in the group of orientation preserving isometries of $\mathbb{H}^n$.

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Given a representation $\rho\colon \Gamma \to \mathrm{Isom}^+(\mathbb{H}^n)$, our central object of study is the volume $\mathrm{Vol}(\rho)$ of ρ as defined in [6] for $n = 2$ and in general in [3]. This notion extends the classical one introduced in [15] for M compact and, as it was shown in [20], if M is of finite volume it coincides with definitions introduced by other authors [10,12,19].

We refer to Sect. 4.2 for the definition of $\mathrm{Vol}(\rho)$ and content ourselves with listing some of its main properties.

(1) The volume function is uniformly bounded on the representation variety $\mathrm{Hom}(\Gamma, \mathrm{Isom}^+(\mathbb{H}^n))$, that is

$$|\mathrm{Vol}(\rho)| \le \mathrm{Vol}(M)$$

and

$$\mathrm{Vol}(\mathrm{Id}_\Gamma) = \mathrm{Vol}(M)\,,$$

where $\mathrm{Id}_\Gamma\colon \Gamma \hookrightarrow \mathrm{Isom}^+(\mathbb{H}^n)$ is the canonical injection.

(2) (Rigidity) There is equality

$$\mathrm{Vol}(\rho) = \mathrm{Vol}(M)$$

if and only if either:

(a) $n = 2$ and ρ is the holonomy representation of a (possibly infinite volume) complete hyperbolization of the smooth surface underlying M, [5,14,21], or

(b) $n = 3$ and ρ is conjugate to Id_Γ, [3,4,10,11].

(3) The volume function is continuous on $\mathrm{Hom}(\Gamma, \mathrm{Isom}^+(\mathbb{H}^n))$ (see Proposition A.1 in Appendix A).

(4) If either M is compact and $n \ge 2$ [29] (see also [9]) or M has finite volume and $n \ge 4$ [20], the volume is constant on connected components of the representation variety. As a consequence it takes only finitely many values.

(5) If M is a non-compact surface, the range of Vol coincides with the interval $[-\chi(M), \chi(M)]$, where $\chi(M)$ is the Euler characteristic of M.

(6) If M is compact and n is even, then

$$\frac{2\mathrm{Vol}(\rho)}{\mathrm{Vol}(S^n)} \in \mathbb{Z}\,,$$

where here and in the sequel, $\mathrm{Vol}(S^n)$ is the volume of the n-sphere S^n of constant curvature 1.

Our main result is a generalization of the integrality property (6) to the case in which M is not compact, and n is even and ≥ 4. We remark that this is in sharp contrast with (5).

Theorem 1.1 *Let $\Gamma < \mathrm{Isom}^+(\mathbb{H}^{2m})$ be a torsion-free lattice and let $\rho\colon \Gamma \to \mathrm{Isom}^+(\mathbb{H}^{2m})$ be a representation. Assume that $2m \ge 4$.*

(1) If the manifold $M = \Gamma\backslash\mathbb{H}^{2m}$ has only toric cusps, then

$$\frac{2\mathrm{Vol}(\rho)}{\mathrm{Vol}(S^{2m})} \in \mathbb{Z}\,.$$

(2) In general

$$\frac{2\mathrm{Vol}(\rho)}{\mathrm{Vol}(S^{2m})} \in \frac{1}{B_{2m-1}} \cdot \mathbb{Z}\,,$$

where B_{2m-1} is the Bieberbach number in dimension $2m-1$.

We recall that the Bieberbach number is the smallest integer B_d such that any compact flat d-manifold has a covering of degree B_d that is a torus. Such d-manifolds occur as connected components of the boundary of a compact core. Recall in fact that, in the context of hyperbolic geometry a compact core N of M is a compact submanifold that is obtained as the quotient by Γ of the complement in $\mathbb{H}^{d+1}$ of a Γ-invariant family of pairwise disjoint open horoballs centered at the cusps.

The strategy of the proof of Theorem 1.1 builds on results in [6], where the authors studied the case in which $\dim M = 2$ and established congruence relations for $\mathrm{Vol}(\rho)$. In order to implement this strategy, we show that the continuous bounded class

$$\omega^{\mathrm{b}}_{2m} \in \mathrm{H}^{2m}_{\mathrm{cb}}(\mathrm{SO}(2m,1)^\circ, \mathbb{R})\,,$$

defined by the volume form on $\mathbb{H}^{2m}$, has a canonical representative

$$\varepsilon^{\mathrm{B,b}}_{2m} \in \mathrm{H}^{2m}_{\mathrm{b}}(\mathrm{SO}(2m,1)^\circ, \mathbb{Z})$$

in the bounded Borel cohomology of $\mathrm{SO}(2m,1)^\circ$ that, under the change of coefficients $\mathbb{Z} \to \mathbb{R}$, corresponds to $(-1)^m \frac{2}{\mathrm{Vol}(S^{2m})}\omega^{\mathrm{b}}_{2m}$. For $2m = 2$, $\varepsilon^{\mathrm{b}}_2$ coincides with the classical bounded Euler class as defined in [13]. Then we establish a congruence relation modulo $\mathbb{Z}$ for $(-1)^m \frac{2}{\mathrm{Vol}(S^{2m})}\mathrm{Vol}(\rho)$ in terms of invariants attached to the boundary components of N that are now assumed to be $(2m-1)$-tori. If T_i is a component of ∂N and $\rho_i : \mathbb{Z}^{2m-1} \to \mathrm{SO}(2m,1)^\circ$ is the restriction of ρ to $\pi_1(T_i) \simeq \mathbb{Z}^{2m-1}$, then the invariant attached to T_i is

$$\rho_i^*(\varepsilon^{\mathrm{b}}_{2m}) \in \mathrm{H}^{2m}_{\mathrm{b}}(\mathbb{Z}^{2m-1}, \mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z}\,.$$

In the case in which $m = 1$, $\rho_i^*(\varepsilon^{\mathrm{b}}_2)$ coincides with the negative of the rotation number of $\rho_i^*(1) \in \mathrm{SO}(2,1)^\circ$ and we show in § 5 that, if $m \geq 2$, $\rho_i^*(\varepsilon^{\mathrm{b}}_{2m})$ always vanishes.

Remark In general Γ has always a subgroup of finite index all whose cusps are toric. However little is known about which collections of compact flat $(2m-1)$-manifolds N are the components of the boundary of a compact core as above. If $\dim M = 4$, it is known that there are compact flat 3-manifolds that are not diffeomorphic to ∂N, [23, Corollary 1.4], while on the positive side there are hyperbolic 4-manifolds with ∂N that is a 3-torus, [22]. Which leads to the following:

Question If Λ is the fundamental group of a compact flat $(2m-1)$-manifold and $\rho\colon \Lambda \to \mathrm{SO}(2m,1)^\circ$ is any representation, does $\rho^*(\varepsilon^{\mathrm{b}}_{2m})$ vanish for $2m \geq 4$?

Thus it is really in all odd dimensions that the nature of the values of Vol remain mysterious, though, according to the results in [20], for $n \geq 4$ there are only finitely many possibilities.

We end this introduction by giving a result in dimension 3. In this case, the character variety of $\Gamma < \mathrm{Isom}^+(\mathbb{H}^3)$ is smooth near Id_Γ, and its complex dimension near Id_Γ equals the number h of cusps of M [31]. As a consequence of the volume rigidity theorem and the continuity of Vol, the image of Vol contains at least an interval $[\mathrm{Vol}(M)-\epsilon, \mathrm{Vol}(M)]$ for some $\epsilon > 0$. Special points in the character variety of Γ come from Dehn fillings of M. Let M_τ denote the compact manifold obtained from M by Dehn surgery along a choice of h simple closed loops $\tau = \{\tau_1, \ldots, \tau_h\}$. If the length of each geodesic loop τ_j is larger than 2π, M_τ admits a hyperbolic structure [31] and an analytic formula for $\mathrm{Vol}(M_\tau)$ depending on the length of the τ_j has been given in [27].

Proposition 1.2 *Let M_τ be the compact 3-manifold obtained by Dehn filling from the hyperbolic 3-manifold M. If $\rho_\tau\colon \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ is the representation obtained from the composition of the quotient homomorphism $\pi_1(M) \to \pi_1(M_\tau)$ with the holonomy representation of the hyperbolic structure on M_τ, then*

$$\mathrm{Vol}(\rho_\tau) = \mathrm{Vol}(M_\tau)\,.$$

Thus with our cohomological definition, $\mathrm{Vol}(\rho)$ gives a continuous interpolation between the special values $\mathrm{Vol}(M_\tau)$. A natural question here is whether Vol is real analytic.

The structure of the paper is as follows. In § 2 we summarize the main facts about various group cohomology theories used in this paper. In § 3 we define a Borel cohomology class $\varepsilon_{2m} \in \mathrm{H}^{2m}_{\mathrm{B}}(\mathrm{SO}(2m,1), \mathbb{Z}_\epsilon)$ with coefficients (see § 2) and relate it to an explicit multiple of the class $\omega_{2m} \in \mathrm{H}^{2m}_{\mathrm{B}}(\mathrm{SO}(2m,1), \mathbb{R}_\epsilon)$ defined by the volume form on $\mathbb{H}^{2m}$ (see (3.1)). In § 4 we first show that ε_{2m} has a unique bounded representative $\varepsilon^{\mathrm{b}}_{2m} \in \mathrm{H}^{2m}_{\mathrm{B,b}}(\mathrm{SO}(2m,1), \mathbb{Z}_\epsilon)$ (Proposition 4.2); in § 4.2 we proceed to define the volume $\mathrm{Vol}(\rho)$ of a representation $\rho\colon \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^{2m}) = \mathrm{SO}(2m,1)^\circ$ and use the bounded integral class $\varepsilon^{\mathrm{b}}_{2m}$ to establish a congruence relation for $\mathrm{Vol}(\rho)$ (Theorem 4.4). In § 5, which is the core of the paper, we show that the contributions from the various boundary components of a compact core of M to the congruence relation all vanish for $2m \geq 4$. In § 6 we relate the volume of the representations of $\pi_1(M)$ obtained by Dehn surgery to the volumes of the corresponding manifolds. In the Appendix we prove the continuity of the map $\rho \mapsto \mathrm{Vol}(\rho)$.

2 Various cohomology theories

We collect in this section cohomological results that will be used throughout the paper.

Given a locally compact second countable group G, we consider $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$ as trivial modules. If $\epsilon\colon G \to \{-1,+1\}$ is a continuous homomorphism, we denote by

$\mathbb{Z}_\epsilon$ and $\mathbb{R}_\epsilon$ the corresponding coefficient G-modules, where $g_*t = \epsilon(g)t$ for $t \in \mathbb{Z}, \mathbb{R}$ and by $\mathbb{R}_\epsilon/\mathbb{Z}_\epsilon$ the corresponding quotient module.

If A is any of the above G-modules, $\mathrm{H}^\bullet_{\mathrm{B}}(G, A)$ denotes the cohomology of the complex of Borel measurable A-valued cochains on G. If $A = \mathbb{R}, \mathbb{R}_\epsilon$, we will also need the continuous cohomology $\mathrm{H}^\bullet_{\mathrm{c}}(G, A)$ with coefficients in A and we will use that for $A = \mathbb{R}, \mathbb{R}_\epsilon$ the comparison map

$$\mathrm{H}^\bullet_{\mathrm{c}}(G, A) \xrightarrow{\simeq} \mathrm{H}^\bullet_{\mathrm{B}}(G, A) \tag{2.1}$$

is an isomorphism [1, Theorem A].

To compute the continuous cohomology of G we use repeatedly that if $G \times V \to V$ is a proper smooth action of a Lie group G on a smooth manifold V, there is an isomorphism

$$\mathrm{H}^\bullet_{\mathrm{c}}(G, \mathbb{R}) \simeq \mathrm{H}^\bullet(\Omega^\bullet(V)^G) \tag{2.2}$$

with the cohomology of the complex $\Omega^\bullet(V)^G$ of G-invariant differential forms on V, [17, Theorem 6.1]. If V is a symmetric space G/K, then

$$\mathrm{H}^\bullet_{\mathrm{c}}(G, \mathbb{R}) \simeq \Omega^\bullet(V)^G\,, \tag{2.3}$$

since every G-invariant differential form on G/K is closed.

Another result we will use is Wigner's isomorphism [32], or rather a special case thereof [1, Theorem E]: namely if $A = \mathbb{Z}$ or $A = \mathbb{Z}_\epsilon$, there is a natural isomorphism

$$\mathrm{H}^\bullet_{\mathrm{B}}(G, A) \simeq \mathrm{H}^\bullet_{\mathrm{sing}}(BG, A)\,, \tag{2.4}$$

where BG is the classifying space of G and $\mathrm{H}^\bullet_{\mathrm{sing}}$ refers to singular cohomology.

A vanishing theorem that is often used states that if L is compact, then

$$\mathrm{H}^k_{\mathrm{c}}(L, \mathbb{R}) = 0 \quad \text{and} \quad \mathrm{H}^k_{\mathrm{c}}(L, \mathbb{R}_\epsilon) = 0 \tag{2.5}$$

for all $k \geq 1$.

Turning to bounded cohomology, $\mathrm{H}^\bullet_{\mathrm{B,b}}(G, A)$ denotes the cohomology of bounded A-valued Borel cochains on G. An important point is that if $A = \mathbb{R}$ or $A = \mathbb{R}_\epsilon$, the comparison map from continuous bounded to Borel bounded cohomology induces an isomorphism

$$\mathrm{H}^\bullet_{\mathrm{cb}}(G, A) \xrightarrow{\simeq} \mathrm{H}^\bullet_{\mathrm{B,b}}(G, A) \tag{2.6}$$

as can be readily verified using the regularization operators defined in [7, § 4].

Analogously to the vanishing of the continuous cohomology for compact groups, if P is amenable, then

$$\mathrm{H}^k_{\mathrm{cb}}(P,\mathbb{R}) \cong \mathrm{H}^k_{\mathrm{B,b}}(P,\mathbb{R}) = 0 \tag{2.7}$$

for all $k \geq 1$.

Consider now the two short exact sequences of coefficients

$$0 \longrightarrow \mathbb{Z} \hookrightarrow \mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0\,, \tag{2.8}$$

and

$$0 \longrightarrow \mathbb{Z}_\epsilon \hookrightarrow \mathbb{R}_\epsilon \twoheadrightarrow \mathbb{R}_\epsilon/\mathbb{Z}_\epsilon \longrightarrow 0\,.$$

Using that $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ admits a bounded Borel section, one obtains readily, both for the trivial and the nontrivial modules, long exact sequences in Borel and in bounded Borel cohomology with commutative squares coming from the comparison maps $c_\mathbb{Z}$ and $\mathrm{c}_\mathbb{R}$ between these two cohomology theories:

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}^{2m-1}_{\mathrm{B}}(G,\mathbb{R}/\mathbb{Z}) \arrow[r, "\delta^{\mathrm{b}}"] \arrow[d, equal] & \mathrm{H}^{2m}_{\mathrm{B,b}}(G,\mathbb{Z}) \arrow[r] \arrow[d, "c_\mathbb{Z}"] & \mathrm{H}^{2m}_{\mathrm{B,b}}(G,\mathbb{R}) \arrow[r] \arrow[d, "c_\mathbb{R}"] & \mathrm{H}^{2m}_{\mathrm{B}}(G,\mathbb{R}/\mathbb{Z}) \arrow[r] \arrow[d, equal] & \dots \\
\dots \arrow[r] & \mathrm{H}^{2m-1}_{\mathrm{B}}(G,\mathbb{R}/\mathbb{Z}) \arrow[r, "\delta"'] & \mathrm{H}^{2m}_{\mathrm{B}}(G,\mathbb{Z}) \arrow[r] & \mathrm{H}^{2m}_{\mathrm{B}}(G,\mathbb{R}) \arrow[r] & \mathrm{H}^{2m}_{\mathrm{B}}(G,\mathbb{R}/\mathbb{Z}) \arrow[r] & \dots,
\end{tikzcd} \tag{2.9}$$

and

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}^{2m-1}_{\mathrm{B}}(G,\mathbb{R}_\epsilon/\mathbb{Z}_\epsilon) \arrow[r, "\delta^{\mathrm{b}}"] \arrow[d, equal] & \mathrm{H}^{2m}_{\mathrm{B,b}}(G,\mathbb{Z}_\epsilon) \arrow[r] \arrow[d, "c_\mathbb{Z}"] & \mathrm{H}^{2m}_{\mathrm{B,b}}(G,\mathbb{R}_\epsilon) \arrow[r] \arrow[d, "c_\mathbb{R}"] & \mathrm{H}^{2m}_{\mathrm{B}}(G,\mathbb{R}_\epsilon/\mathbb{Z}_\epsilon) \arrow[r] \arrow[d, equal] & \dots \\
\dots \arrow[r] & \mathrm{H}^{2m-1}_{\mathrm{B}}(G,\mathbb{R}_\epsilon/\mathbb{Z}_\epsilon) \arrow[r, "\delta"'] & \mathrm{H}^{2m}_{\mathrm{B}}(G,\mathbb{Z}_\epsilon) \arrow[r] & \mathrm{H}^{2m}_{\mathrm{B}}(G,\mathbb{R}_\epsilon) \arrow[r] & \mathrm{H}^{2m}_{\mathrm{B}}(G,\mathbb{R}_\epsilon/\mathbb{Z}_\epsilon) \arrow[r] & \dots,
\end{tikzcd} \tag{2.10}$$

where δ and δ^{b} are the connecting homomorphisms.

3 Proportionality between volume and Euler class

Let $\mathrm{Isom}(\mathbb{H}^n)$ be the full group of isometries of real hyperbolic spaces $\mathbb{H}^n$, let $\epsilon\colon \mathrm{Isom}(\mathbb{H}^n) \to \{-1,1\}$ denote the homomorphism with kernel $\mathrm{Isom}^+(\mathbb{H}^n)$ and let $\mathbb{Z}_\epsilon \subset \mathbb{R}_\epsilon$ the corresponding modules. Using (2.1), [3, Proposition 2.1] and (2.3), we have isomorphisms

$$\mathrm{H}^n_{\mathrm{B}}(\mathrm{Isom}(\mathbb{H}^n),\mathbb{R}_\epsilon) \simeq \mathrm{H}^n_{\mathrm{c}}(\mathrm{Isom}(\mathbb{H}^n),\mathbb{R}_\epsilon) \simeq \mathrm{H}^n_{\mathrm{c}}(\mathrm{Isom}(\mathbb{H}^n)^\circ,\mathbb{R}) \simeq \Omega(\mathbb{H}^n)^{\mathrm{Isom}(\mathbb{H}^n)^\circ} \tag{3.1}$$

and denote by $\omega_n \in \mathrm{H}^n_{\mathrm{B}}(\mathrm{Isom}(\mathbb{H}^n), \mathbb{R}_\epsilon)$ the generator corresponding to the volume form on $\mathbb{H}^n$.

If $n = 2m$ is even, we can identify $\mathrm{Isom}(\mathbb{H}^{2m})$ with $\mathrm{SO}(2m, 1)$. The diagram of injections

$$\begin{tikzcd} & \mathrm{O}(2m) \arrow[dl, hook'] \arrow[dr, hook] & \\ \mathrm{GL}(2m, \mathbb{R}) & & \mathrm{SO}(2m, 1) \end{tikzcd}$$

realizes $\mathrm{O}(2m)$ as a maximal compact subgroup of both $\mathrm{SO}(2m, 1)$ and $\mathrm{GL}(2m, \mathbb{R})$ and induces homotopy equivalences

$$B\,\mathrm{GL}(2m, \mathbb{R}) \simeq B\mathrm{O}(2m) \simeq B\,\mathrm{SO}(2m, 1)\,.$$

The homomorphism

$$\mathrm{GL}(2m, \mathbb{R}) \to \{-1, 1\}$$

that to a matrix associates the sign of its determinant, coincides on $\mathrm{O}(2m)$ with the restriction of ϵ and will be denoted by ϵ. Thus we obtain isomorphisms

$$\mathrm{H}^{2m}_{\mathrm{sing}}(B\,\mathrm{GL}(2m, \mathbb{R}), \mathbb{Z}_\epsilon) \cong \mathrm{H}^{2m}_{\mathrm{sing}}(B\mathrm{O}(2m), \mathbb{Z}_\epsilon) \cong \mathrm{H}^{2m}_{\mathrm{sing}}(B\,\mathrm{SO}(2m, 1), \mathbb{Z}_\epsilon)\,. \tag{3.2}$$

The (universal) Euler class $\varepsilon^{\mathrm{univ}}_{2m} \in \mathrm{H}^{2m}_{\mathrm{sing}}(B\mathrm{GL}^+(2m, \mathbb{R}), \mathbb{Z})$ is a singular class in the integral cohomology of the classifying space $B\mathrm{GL}^+(2m, \mathbb{R})$ of oriented $\mathbb{R}^{2m}$-vector bundles (see [25, § 9]). It is the obstruction to the existence of a nowhere vanishing section. As it changes sign when the orientation is reversed, it extends to a class $\varepsilon^{\mathrm{univ}}_{2m} \in \mathrm{H}^{2m}_{\mathrm{sing}}(B\mathrm{GL}(2m, \mathbb{R}), \mathbb{Z}_\epsilon)$. Furthermore, if M is a closed oriented $2m$-dimensional manifold, its tangent bundle is classified by a (unique up to homotopy) classifying map

$$f\colon M \to B\mathrm{GL}^+(2m, \mathbb{R}) \hookrightarrow B\mathrm{GL}(2m, \mathbb{R}) \simeq B\mathrm{O}(2m)$$

inducing a map

$$f^*\colon \mathrm{H}^{2m}_{\mathrm{sing}}(B\mathrm{GL}(2m, \mathbb{R}), \mathbb{Z}_\epsilon) \cong \mathrm{H}^{2m}_{\mathrm{sing}}(B\mathrm{O}(2m)\mathbb{R}, \mathbb{Z}_\epsilon) \to \mathrm{H}^{2m}_{\mathrm{sing}}(M, \mathbb{Z})$$

and thus

$$\chi(M) = \langle f^*(\varepsilon^{\mathrm{univ}}_{2m}), [M]\rangle\,. \tag{3.3}$$

Now we use Wigner's isomorphism (2.4)

$$\mathrm{H}^{2m}_{\mathrm{sing}}(B\,\mathrm{SO}(2m, 1), \mathbb{Z}_\epsilon) \cong \mathrm{H}^{2m}_{\mathrm{B}}(\mathrm{SO}(2m, 1), \mathbb{Z}_\epsilon) \tag{3.4}$$

and call Euler class the group cohomology class

$$\varepsilon_{2m} \in \mathrm{H}^{2m}_{\mathrm{B}}(\mathrm{SO}(2m,1), \mathbb{Z}_\epsilon)$$

corresponding to $\varepsilon^{\mathrm{univ}}_{2m}$ under the composition of the isomorphisms in (3.2) and (3.4).

Since $\mathrm{H}^{2m}_{\mathrm{B}}(\mathrm{SO}(2m,1), \mathbb{R}_\epsilon)$ is one dimensional by (3.2), the image of ε_{2m} under the change of coefficients $\mathbb{Z}_\epsilon \hookrightarrow \mathbb{R}_\epsilon$ is a multiple of the volume class ω_{2m}. We show now that this multiple is given by

$$\begin{aligned} \mathrm{H}^{2m}_{\mathrm{B}}(\mathrm{SO}(2m,1), \mathbb{Z}_\epsilon) &\longrightarrow \mathrm{H}^{2m}_{\mathrm{B}}(\mathrm{SO}(2m,1), \mathbb{R}_\epsilon) \\ \varepsilon_{2m} \qquad &\longmapsto (-1)^m \frac{2}{\mathrm{Vol}(S^{2m})}\omega_{2m}. \end{aligned} \tag{3.5}$$

Indeed, let M be a closed oriented hyperbolic manifold. The lattice embedding $i\colon \pi_1(M) \hookrightarrow \mathrm{SO}(2m,1)$ induces classifying maps

$$\begin{tikzcd} M = K(\pi_1(M),1) \arrow[r, "Bi"] \arrow[rr, bend right, "\overline{Bi}"'] & B\mathrm{SO}(2m,1) \simeq & B\mathrm{O}(2m) \simeq B\mathrm{GL}(2m,\mathbb{R}). \end{tikzcd} \tag{3.6}$$

Set $\Gamma = i(\pi_1(M))$. The orthonormal frame bundle over M is naturally identified with

$$\Gamma\backslash\mathrm{SO}(2m,1)\,.$$

Extending the principal group structure from $\mathrm{O}(2m)$ to $\mathrm{SO}(2m,1)$ gives the principal $\mathrm{SO}(2m,1)$-bundle

$$(\Gamma\backslash\mathrm{SO}(2m,1)) \times_{\mathrm{O}(2m)} \mathrm{SO}(2m,1)\,.$$

The latter is isomorphic to

$$(\mathrm{SO}(2m,1)/\mathrm{O}(2m)) \times_\Gamma \mathrm{SO}(2m,1)\,,$$

which is the flat principal $\mathrm{SO}(2m,1)$-bundle associated to the lattice embedding $i\colon \pi_1(M) \hookrightarrow \mathrm{SO}(2m,1)$. It follows that the composition from (3.6)

$$\overline{Bi} : M \longrightarrow B\mathrm{O}(2m) \simeq B\mathrm{GL}(2m,\mathbb{R})$$

classifies the tangent bundle TM. As a consequence,

$$\chi(M) = \langle \overline{Bi}^*(\varepsilon^{\mathrm{univ}}_{2m}), [M]\rangle\,.$$

Thus, by the naturality of Wigner's isomorphism, the following diagram commutes:

$$\begin{tikzcd}
 & \mathrm{H}^{2m}_{\mathrm{sing}}(B\mathrm{GL}(2m,\mathbb{R}),\mathbb{Z}_\epsilon) \arrow[dl, "\overline{Bi}"'] \arrow[d, equal] \\
\mathrm{H}^{2m}_{\mathrm{sing}}(M,\mathbb{Z}) & \mathrm{H}^{2m}_{\mathrm{sing}}(B\mathrm{SO}(2m,1),\mathbb{Z}_\epsilon) \arrow[l, "Bi"] \arrow[d, equal] \\
\mathrm{H}^{2m}(\pi_1(M),\mathbb{Z}) \arrow[u, "\cong"] & \mathrm{H}^{2m}_{\mathrm{B}}(\mathrm{SO}(2m,1),\mathbb{Z}_\epsilon). \arrow[l, "i^*"]
\end{tikzcd}$$

We deduce that

$$\chi(M) = \langle i^*\varepsilon_{2m}, [M]\rangle \,.$$

Moreover, the hyperbolic volume of M is obviously given by

$$\mathrm{vol}(M) = \langle i^*\omega_{2m}, [M]\rangle \,.$$

The Chern–Gauss–Bonnet Theorem [30, Chapter 13, Theorem 26] implies that if M is a hyperbolic $(2m)$-dimensional manifolds then

$$\chi(M) = (-1)^m \frac{2}{\mathrm{Vol}(S^{2m})} \mathrm{vol}(M)\,,$$

where the proportionality constant is up to the sign $(-1)^m$ the quotient between the Euler characteristic and the volume of the compact dual of hyperbolic space, namely the $(2m)$-sphere of constant curvature $+1$. Finally, since the lattice embedding induces an injection

$$i^*\colon \mathrm{H}^{2m}_{\mathrm{B}}(\mathrm{SO}(2m,1),\mathbb{R}_\epsilon) \longrightarrow \mathrm{H}^{2m}(\pi_1(M),\mathbb{R})\,,$$

we obtain the value of the proportionality constant in (3.5).

Remark 3.1 Recall that the orientation cocycle

$$\mathrm{Or}\colon (S^1)^3 \longrightarrow \{+1, 0, -1\} \tag{3.7}$$

assigns the value ± 1 to distinct triple of points according to their orientation, and 0 otherwise. Identifying S^1 with $\partial\mathbb{H}^2$ we obtain by evaluation a Borel cocycle and a cohomology class

$$[\mathrm{Or}] \in \mathrm{H}^2_{\mathrm{B}}(\mathrm{Isom}(\mathbb{H}^2),\mathbb{R}_\epsilon)\,.$$

Since the area of ideal hyperbolic triangles in $\mathbb{H}^2$ is $\pm\pi$ depending on the orientation, this cocycle represents $(1/\pi)\omega_2 = -2\varepsilon_2$, where the last equality comes from (3.5).

Thus the Euler class ε_2 is represented by $-\frac{1}{2}\mathrm{Or}$, [18, Lemma 2.1] and [8, Proposition 8.4].

4 The bounded Euler class

In this section we recall that the volume class of hyperbolic n-space admits a unique bounded representative and establish for n even the analogous result for the Euler class, which leads to the definition of the bounded Euler class in Proposition 4.2. In § 4.2 we define the volume of a representation and use the existence of the bounded Euler class in even dimensions to prove in Theorem 4.4 a congruence relation for the volume of a representation.

4.1 Bounded volume and Euler classes

Lemma 4.1 *Let* $G := \mathrm{Isom}(\mathbb{H}^n)$. *In the commuting diagram*

$$\begin{tikzcd} \mathrm{H}^n_{\mathrm{B,b}}(G,\mathbb{Z}_\epsilon) \arrow[r] \arrow[d, "c_\mathbb{Z}"'] & \mathrm{H}^n_{\mathrm{B,b}}(G,\mathbb{R}_\epsilon) \arrow[d, "c_\mathbb{R}"] \\ \mathrm{H}^n_{\mathrm{B}}(G,\mathbb{Z}_\epsilon) \arrow[r] & \mathrm{H}^n_{\mathrm{B}}(G,\mathbb{R}_\epsilon) \end{tikzcd}$$

the vertical maps are isomorphisms.

Proof The fact that $c_\mathbb{R}$ is an isomorphism follows from [3, Proposition 2.1] and the identifications between the (bounded) Borel and the (bounded) continuous cohomology.

To show that $c_\mathbb{Z}$ is an isomorphism, we will do diagram chases in (2.10).

Surjectivity of $c_\mathbb{Z}$: Let $\alpha \in \mathrm{H}^n_\mathrm{B}(G,\mathbb{Z}_\epsilon)$. Denote by $\alpha_\mathbb{R} \in \mathrm{H}^n_\mathrm{B}(G,\mathbb{R}_\epsilon)$ its image under the change of coefficients and $\alpha^\mathrm{b}_\mathbb{R} = (c_\mathbb{R})^{-1}(\alpha_\mathbb{R}) \in \mathrm{H}^n_{\mathrm{B,b}}(G,\mathbb{R}_\epsilon)$. By exactness of the lines in (2.10), the image of $\alpha_\mathbb{R}$ in $\mathrm{H}^n_\mathrm{B}(G,\mathbb{R}_\epsilon/\mathbb{Z}_\epsilon)$ vanishes. And thus the same holds for the image of $\alpha^\mathrm{b}_\mathbb{R}$. Thus there is $\beta \in \mathrm{H}^n_\mathrm{B}(G,\mathbb{Z}_\epsilon)$ with image $\alpha^\mathrm{b}_\mathbb{R}$. But $\mathrm{c}_\mathbb{Z}(\beta)-\alpha$ goes to zero in $\mathrm{H}^n_\mathrm{B}(G,\mathbb{R}_\epsilon)$, hence $\mathrm{c}_\mathbb{Z}(\beta)-\alpha = \delta(\eta)$ for some $\eta \in \mathrm{H}^{n-1}_\mathrm{B}(G,\mathbb{R}_\epsilon/\mathbb{Z}_\epsilon)$. Thus $c_\mathbb{Z}(\beta+\delta^\mathrm{b}(\eta)) = \alpha$.

Injectivity of $c_\mathbb{Z}$: Observe that $\mathrm{H}^{n-1}_\mathrm{B}(G,\mathbb{R}_\epsilon) = \mathrm{H}^{n-1}_\mathrm{c}(G,\mathbb{R}_\epsilon) = 0$. Indeed this group injects into $\mathrm{H}^{n-1}_\mathrm{c}(G^\circ,\mathbb{R})$ by restriction, and the latter vanishes, taking into account (2.2), since there are no $\mathrm{SO}(n)$-invariant $(n-1)$-forms on $\mathbb{R}^n$ and hence no G°-invariant differential $(n-1)$-forms on $\mathbb{H}^n$.

Let now $\alpha \in \mathrm{H}^n_{\mathrm{B,b}}(G,\mathbb{Z}_\epsilon)$ with $c_\mathbb{Z}(\alpha) = 0$. Since $c_\mathbb{R}$ is an isomorphism, we have that the image $\alpha_\mathbb{R} \in \mathrm{H}^n_{\mathrm{B,b}}(G,\mathbb{R}_\epsilon)$ vanishes. Hence there is $\beta \in \mathrm{H}^{n-1}_\mathrm{B}(G,\mathbb{R}_\epsilon/\mathbb{Z}_\epsilon)$ with $\delta^\mathrm{b}(\beta) = \alpha$. But then $\delta(\beta) = c_\mathbb{Z}(\alpha) = 0$. By exactness, this implies that β is in the image of $\mathrm{H}^{n-1}_\mathrm{B}(G,\mathbb{R}_\epsilon)$. Since $\mathrm{H}^{n-1}_\mathrm{B}(G,\mathbb{R}_\epsilon) = 0$, then $\beta = 0$, which finally implies that $\alpha = 0$. $\square$

As a consequence of the fact that $c_{\mathbb{R}}$ is an isomorphism and that the volume of geodesic simplices in hyperbolic n-space is bounded, the volume cocycle defines also a bounded Borel class

$$\omega_n^{\mathrm{b}} \in \mathrm{H}^n_{\mathrm{B,b}}(G, \mathbb{R}_\epsilon)$$

corresponding to the volume class ω_n.

As an immediate consequence of Lemma 4.1 and the correspondence (3.5) we obtain:

Proposition 4.2 *Let $G = \mathrm{Isom}(\mathbb{H}^{2m})$. The Euler class $\varepsilon_{2m} \in \mathrm{H}^{2m}_{\mathrm{B}}(G, \mathbb{Z}_\epsilon)$ has a bounded representative $\varepsilon^{\mathrm{b}}_{2m} \in \mathrm{H}^{2m}_{\mathrm{B,b}}(G, \mathbb{Z}_\epsilon)$ that has the following properties:*

(1) it is unique, and
(2) under the change of coefficients $\mathbb{Z}_\epsilon \to \mathbb{R}_\epsilon$ it corresponds to

$$(-1)^m \frac{2}{\mathrm{Vol}(S^{2m})} \omega^{\mathrm{b}}_{2m} \in \mathrm{H}^{2m}_{\mathrm{B,b}}(G, \mathbb{R}_\epsilon)\,.$$

Remark 4.3 (1) With a slight abuse of notation we denote equally by $\varepsilon^{\mathrm{b}}_{2m} \in \mathrm{H}^{2m}_{\mathrm{B,b}}(\mathrm{O}(2m), \mathbb{Z}_\epsilon)$ and by $\varepsilon^{\mathrm{b}}_{2m} \in \mathrm{H}^{2m}_{\mathrm{B,b}}(\mathrm{SO}(2m), \mathbb{Z})$ the restriction of $\varepsilon^{\mathrm{b}}_{2m}$ respectively to $\mathrm{O}(2m)$ and to $\mathrm{SO}(2m)$.
(2) If $m = 1$ and with the usual slight abuse of notation, the restriction $\varepsilon^{\mathrm{b}}_2 \in \mathrm{H}^2_{\mathrm{B,b}}(\mathrm{Isom}(\mathbb{H}^2)^\circ, \mathbb{Z})$ is the usual bounded Euler class, where $\mathrm{Isom}(\mathbb{H}^2)^\circ$ is considered as a group of orientation preserving homeomorphisms of the circle.

4.2 Definition of volume and congruence relations

Let M be a complete finite volume hyperbolic n-dimensional manifold and let $\rho\colon \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^n)$ be a homomorphism. Given a compact core N of M we consider ρ as a representation of $\pi_1(N)$ and use the pullback via ρ in bounded cohomology together with the isomorphism

$$\mathrm{H}^n_{\mathrm{b}}(\pi_1(N), \mathbb{R}) \cong \mathrm{H}^n_{\mathrm{b}}(N, \mathbb{R})$$

to obtain a bounded singular class in $\mathrm{H}^n_{\mathrm{b}}(N, \mathbb{R})$, denoted $\rho^*(\omega^{\mathrm{b}}_n)$ by abuse of notation. Using the isometric isomorphism

$$j\colon \mathrm{H}^{2m}_{\mathrm{b}}(N, \partial N, \mathbb{R}) \longrightarrow \mathrm{H}^{2m}_{\mathrm{b}}(N, \mathbb{R}) \tag{4.1}$$

[2, Theorem 1.2], the volume of ρ is defined by

$$\mathrm{Vol}(\rho) := \langle j^{-1}(\rho^*(\omega^{\mathrm{b}}_{2m})), [N, \partial N]\rangle\,.$$

If $n = 2m$, by the same abuse of notation, and by considering again the pullback in bounded cohomology via ρ, this time together with the isomorphism

$$\mathrm{H}_{\mathrm{b}}^{2m}(\pi_1(N), \mathbb{Z}) \cong \mathrm{H}_{\mathrm{b}}^{2m}(N, \mathbb{Z}) ,$$

we obtain a class $\rho^*(\varepsilon_{2m}^{\mathrm{b}}) \in \mathrm{H}_{\mathrm{b}}^{2m}(N, \mathbb{Z})$, which is a bounded singular integral class. Thus, denoting by δ^{b} the connecting homomorphism in the long exact sequence in bounded singular cohomology

$$\delta^{\mathrm{b}} \colon \mathrm{H}^{2m-1}(\partial N, \mathbb{R}/\mathbb{Z}) \longrightarrow \mathrm{H}_{\mathrm{b}}^{2m}(\partial N, \mathbb{Z}) \tag{4.2}$$

(which is in fact an isomorphism), we have:

Theorem 4.4 *Let M be a complete hyperbolic manifold of finite volume and even dimension $n = 2m$ and N a compact core of M. If $\rho \colon \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^{2m})$, then*

$$(-1)^m \frac{2}{\mathrm{Vol}(S^{2m})} \cdot \mathrm{Vol}(\rho) \equiv -\langle (\delta^{\mathrm{b}})^{-1} \rho^*(\varepsilon_{2m}^{\mathrm{b}})|_{\partial N}, [\partial N] \rangle \bmod \mathbb{Z} .$$

Proof In fact, (4.1) and (4.2) are part of the following diagram

$$\begin{tikzcd}
\mathrm{H}^{2m-1}(\partial N, \mathbb{R}/\mathbb{Z}) \arrow[r, "\delta^{\mathrm{b}}"] & \mathrm{H}_{\mathrm{b}}^{2m}(\partial N, \mathbb{Z}) \arrow[r] & \mathrm{H}_{\mathrm{b}}^{2m}(\partial N, \mathbb{R}) = 0 \\
& \mathrm{H}_{\mathrm{b}}^{2m}(N, \mathbb{Z}) \arrow[u] \arrow[r] & \mathrm{H}_{\mathrm{b}}^{2m}(N, \mathbb{R}) \arrow[u] \\
& & \mathrm{H}_{\mathrm{b}}^{2m}(N, \partial N, \mathbb{R}) , \arrow[u, "j"']
\end{tikzcd}$$

where the rows are obtained from the long exact sequence

$$\dots \longrightarrow \mathrm{H}_{\mathrm{b}}^{n-1}(X, \mathbb{R}/\mathbb{Z}) \longrightarrow \mathrm{H}_{\mathrm{b}}^{n}(X, \mathbb{Z}) \longrightarrow \mathrm{H}_{\mathrm{b}}^{n}(X, \mathbb{R}) \longrightarrow \mathrm{H}_{\mathrm{b}}^{n}(X, \mathbb{R}/\mathbb{Z}) \longrightarrow \dots ,$$

with $X = \partial N$ and $X = N$, induced by the change of coefficients in (2.8) and from the fact that $\mathrm{H}_{\mathrm{b}}^{\bullet}(\partial N, \mathbb{R}) = 0$ since $\pi_1(\partial N)$ is amenable; the columns on the other hand follow from the long exact sequence in relative bounded cohomology associated to the inclusion of pairs $(N, \varnothing) \hookrightarrow (N, \partial N)$ (see [6, § 2.2]).

Let z be a $\mathbb{Z}$-valued singular bounded cocycle representing $\rho^*(\varepsilon_{2m}^{\mathrm{b}}) \in \mathrm{H}_{\mathrm{b}}^{2m}(N, \mathbb{Z})$. Restricting z to the boundary ∂N we obtain a $\mathbb{Z}$-valued singular bounded cocycle $z|_{\partial N}$, which we know is a coboundary when considered as a $\mathbb{R}$-valued cocycle since $\mathrm{H}_{\mathrm{b}}^{2m}(\partial N, \mathbb{R}) = 0$. Thus there must exist a bounded $\mathbb{R}$-valued singular $(2m-1)$-cochain b on ∂N such that

$$(z|_{\partial N})_{\mathbb{R}} = db ,$$

where d is the differential operator on (bounded $\mathbb{R}$-valued) singular cochains.

On the one hand, we note that since db is $\mathbb{Z}$-valued, the cochain $b \bmod \mathbb{Z}$ is a $\mathbb{R}/\mathbb{Z}$-valued $(2m-1)$-cocycle on ∂N whose cohomology class is mapped to

$$[\rho^*(\varepsilon^{\mathrm{b}}_{2m})|_{\partial N}] = [z|_{\partial N}] = \delta^{\mathrm{b}}([b \bmod \mathbb{Z}])$$

by the connecting homomorphism δ^{b} in (4.2). On the other hand, define a bounded $\mathbb{R}$-valued singular $(2m-1)$-cochain $\overline{b}$ on N by extending b to N,

$$\overline{b}(\sigma) := \begin{cases} b(\sigma) & \text{if } \sigma \subset \partial N, \\ 0 & \text{otherwise.} \end{cases}$$

Then $[z_{\mathbb{R}} - d\overline{b}] = [z_{\mathbb{R}}] \in \mathrm{H}^{2m}_{\mathrm{b}}(N, \mathbb{R})$ and, since $z_{\mathbb{R}} - d\overline{b}$ vanishes on ∂N, we have actually constructed a cocycle representing the relative bounded class $j^{-1}((\rho^*(\varepsilon^{\mathrm{b}}_{2m}))_{\mathbb{R}}) \in \mathrm{H}^{2m}_{\mathrm{b}}(N, \partial N, \mathbb{R})$.

It remains to evaluate $j^{-1}((\rho^*(\varepsilon^{\mathrm{b}}_{2m}))_{\mathbb{R}})$ on $[N, \partial N]$ by using this specific cocycle. Let t be a singular chain representing the relative fundamental class $[N, \partial N]$ over $\mathbb{Z}$. In particular, ∂t is a cycle representing the fundamental class $[\partial N]$. Then we obtain

$$\begin{aligned} \langle j^{-1}((\rho^*(\varepsilon^{\mathrm{b}}_{2m}))_{\mathbb{R}}), [N, \partial N]\rangle \bmod \mathbb{Z} &= \langle z_{\mathbb{R}} - d\overline{b}, t\rangle \bmod \mathbb{Z} \\ &= \underbrace{\langle z_{\mathbb{R}}, t\rangle}_{\in \mathbb{Z}} - \langle \overline{b}, \partial t\rangle \bmod \mathbb{Z} \\ &= -\langle b \bmod \mathbb{Z}, [\partial N]\rangle \\ &= -\langle (\delta^{\mathrm{b}})^{-1}(\rho^*(\varepsilon_{2m})|_{\partial N}), [\partial N]\rangle. \end{aligned}$$

□

5 Vanishing of the bounded Euler class on tori and the proof of Theorem 1.1

The goal of this section is to prove Theorem 1.1. From Theorem 4.4, we know that $(-1)^m 2/\mathrm{Vol}(S^{2m})$ times the volume of a representation is determined, mod $\mathbb{Z}$, by the restriction to the cusps of the pullback of the bounded Euler class. The main result of this section will be to prove, in Theorem 5.1, that the pullback of the bounded Euler class by any representation $\rho\colon \mathbb{Z}^{2m-1} \to \mathrm{SO}(2m, 1)^\circ$ is identically zero for $m \geq 2$. We conclude the section by showing how Theorems 4.4 and 5.1 imply Theorem 1.1.

The bounded class $\varepsilon^{\mathrm{b}}_{2m} \in \mathrm{H}^{2m}_{\mathrm{B,b}}(\mathrm{SO}(2m, 1), \mathbb{Z}_\epsilon)$ defined in the previous section restricts to a class on $\mathrm{SO}(2m, 1)^\circ$ with trivial $\mathbb{Z}$-coefficients. If $m = 1$ and

$$\rho\colon \mathbb{Z} \to \mathrm{SO}(2, 1)^\circ$$

is a homomorphism, then $\rho^*(\varepsilon^{\mathrm{b}}_2) \in \mathrm{H}^2_{\mathrm{b}}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$ is the negative of the rotation number of $\rho(1)$, [13]. In contrast to this, in higher dimension we have the following:

Theorem 5.1 *Let $m \geq 2$ and let $\rho \colon \mathbb{Z}^{2m-1} \to \mathrm{SO}(2m,1)^\circ$ be a homomorphism. Then $\rho^*(\varepsilon^{\mathrm{b}}_{2m})$ vanishes in $\mathrm{H}^{2m}_{\mathrm{b}}(\mathbb{Z}^{2m-1}, \mathbb{Z})$.*

Before proving the theorem we need some information about Abelian subgroups of $\mathrm{SO}(2m,1)^\circ$. To fix the notation, recall that

$$\mathrm{SO}(2m,1) := \Big\{ A \in \mathrm{GL}(2m+1,\mathbb{R}) : \det A = 1, \\ A \text{ preserves } q(x) := \sum_{i=1}^{2m} x_i^2 - x_{2m+1}^2 \Big\}.$$

Then the maximal compact subgroup $K < \mathrm{SO}(2m,1)$ is the image of $\mathrm{O}(2m)$ under the homomorphism

$$\begin{aligned} \mathrm{O}(2m) &\longrightarrow \mathrm{SO}(2m,1) \\ A &\longmapsto \begin{pmatrix} A & 0 \\ 0 & \det A \end{pmatrix}, \end{aligned}$$

and the image K° of $\mathrm{SO}(2m)$ is the maximal compact subgroup of $\mathrm{SO}(2m,1)^\circ$. If T is the image of

$$\begin{aligned} \mathrm{O}(2)^m &\longrightarrow \mathrm{SO}(2m,1) \\ (A_1,\dots,A_m) &\longmapsto \begin{pmatrix} A_1 & & & \\ & \ddots & & \\ & & A_m & \\ & & & \prod_{i=1}^m \det A_i \end{pmatrix}, \end{aligned}$$

we define

$$T_0 := T \cap K^\circ .$$

Since $\mathrm{SO}(2m,1)^\circ$ preserves each connected component of the two-sheeted hyperboloid

$$x_1^2 + \cdots + x_{2m}^2 - x_{2m+1}^2 = -1\,,$$

the parabolic subgroup $P = \mathrm{Stab}_{\mathrm{SO}(2m,1)^\circ}(\mathbb{R}(e_1 - e_{2m+1}))$ admits the decomposition $P = MAN$, where

$$M := \left\{ m(U) := \begin{pmatrix} 1 & & \\ & U & \\ & & 1 \end{pmatrix} : U \in \mathrm{SO}(2m-1) \right\}$$

$$A := \left\{ a(t) := \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & \mathrm{Id} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} : t \in \mathbb{R} \right\}$$

and

$$N := \left\{ n(x) := \begin{pmatrix} 1 - \frac{\|x\|^2}{2} & -x & -\frac{\|x\|^2}{2} \\ {}^t x & I & {}^t x \\ \frac{\|x\|^2}{2} & x & 1 + \frac{\|x\|^2}{2} \end{pmatrix} : x \in \mathbb{R}^{2m-1} \right\} .$$

We can now outline the proof of Theorem 5.1. According to Lemma 5.2 below, there are up to conjugation two cases to consider for $\rho : \mathbb{Z}^{2m-1} \to \mathrm{SO}(2m, 1)^\circ$:

(1) ρ takes values in P: then, building on Lemma 5.4, Lemma 5.5 shows that $\varepsilon^{\mathrm{b}}_{2m}|_P \in \mathrm{H}^{2m}_{\mathrm{B,b}}(P, \mathbb{Z})$ vanishes and this implies Theorem 5.1 in this case.
(2) ρ takes values in T_0: this case splits into two subcases, namely $\rho(\mathbb{Z}^{2m-1}) \not\subset T^\circ$, which is dealt with in Lemma 5.6, and $\rho(\mathbb{Z}^{2m-1}) \subset T^\circ$. In the latter case we represent $\varepsilon^{\mathrm{b}}_{2m}|_{\mathrm{SO}(2)^{2m}}$ as the image under the connecting homomorphism of an explicit class in $\mathrm{H}^{2m-1}_{\mathrm{B}}(\mathrm{SO}(2)^{2m}, \mathbb{R}/\mathbb{Z})$ whose pullback via ρ we evaluate in Lemma 5.8 on an explicit cycle representing the fundamental class $[\mathbb{Z}^{2m-1}]$ of $\mathrm{H}_{2m-1}(\mathbb{Z}^{2m-1}, \mathbb{Z})$ constructed in Lemma 5.7.

Lemma 5.2 *Let $B < \mathrm{SO}(2m, 1)^\circ$ be an Abelian group. Then up to conjugation one of the following holds:*

(1) $B < P$;
(2) $B < T_0$.

Proof Since B is Abelian, it is elementary [28, Lemma 1, § 5.5], that is, it is either of elliptic type or of parabolic type or of hyperbolic type.

(i) If B is of parabolic type, it fixes a point in $\partial\mathbb{H}^{2m}$ [28, Theorem 5.5.1] and thus it can be conjugated into P.
(ii)) If B is of elliptic type it fixes a point in $\mathbb{H}^{2m}$ [28, Theorem 5.5.3] and hence it can be conjugated into $K^\circ \cong \mathrm{SO}(2m)$. Since it is Abelian, it can be simultaneously reduced to a diagonal 2×2 bloc form and hence can be conjugated into T_0.
(iii) If B is of hyperbolic type, every union of finite orbits in $\overline{\mathbb{H}^{2m}}$ consists of two points, [28, Theorem 5.5.6]. Thus there is a geodesic $g \in \mathbb{H}^{2m}$ that is left setwise invariant by B. If $\{g_-, g_+\}$ are its endpoints, then either $Bg_+ = g_+$ and we are in case (i) above, or there is $b \in B$ with $bg_+ = g_-$. But then B fixes the unique b-fixed point $g_0 \in g$, hence we are in case (ii). □

Remark 5.3 Since Theorems 5.5.1, 5.5.3 and 5.5.6 in [28] are characterizations respectively of elliptic, parabolic and hyperbolic groups of isometries, it follows from Lemma 5.2 that an Abelian group $B < \mathrm{SO}(2m, 1)^\circ$ cannot be hyperbolic.

Lemma 5.4 *Let $P < \mathrm{SO}(2m, 1)^\circ$ be the minimal parabolic subgroup as above. Then:*

$$\mathrm{H}^k_{\mathrm{B}}(P, \mathbb{R}) = 0$$

for $k = 2m - 1$ and $k = 2m$.

Proof Combining (2.1) and (2.2), we show the vanishing assertion by showing the vanishing of the cohomology of P-invariant differential forms on $M\backslash P$ in degrees $2m$ and $2m-1$. This follows if we establish the following:

Claim $\Omega^{2m-1}(M\backslash P)^P = \mathbb{R}\alpha$, where α is an explicit $(2m-1)$ form with $d\alpha \neq 0$.

Let us first see how the claim implies the vanishing. It follows from the claim that

$$d\colon \Omega^{2m-1}(M\backslash P)^P \to \Omega^{2m}(M\backslash P)^P$$

is injective, which implies the vanishing in degree $2m-1$. Since P acts transitively on $M\backslash P$ and $\dim(M\backslash P) = 2m$, then $\dim \Omega^{2m}(M\backslash P)^P \leq 1$. But then $d\alpha \neq 0$ implies that $\dim \Omega^{2m}(M\backslash P)^P = 1$ and the image of d is $\Omega^{2m}(M\backslash P)^P$, which implies the vanishing in degree $2m$.

We now prove the claim. Observe that

$$\Omega^{2m-1}(M\backslash P)^P \simeq \Lambda^{*(2m-1)} T_{x_0}(M\backslash P)^M \,,$$

where $\Lambda^{*(2m-1)} T_{x_0}(M\backslash P)^M$ denotes the space of M-invariant $(2m-1)$-multilinear forms on the tangent space to $M\backslash P$ at the basepoint $x_0 = [M] \in M\backslash P$.

Under the identification

$$\begin{aligned} M\backslash P \quad &\longrightarrow \mathbb{R} \times \mathbb{R}^{2m-1} \\ Ma(t)n(x) &\longmapsto \ (t, x) \end{aligned} \tag{5.1}$$

the point x_0 corresponds to $(0, 0)$, so that

$$T_{x_0}(M\backslash P) \simeq T_{(0,0)}(\mathbb{R} \times \mathbb{R}^{2m-1}) \simeq \mathbb{R} \times \mathbb{R}^{2m-1}$$

and the M-action on $T_{x_0}(M\backslash P)$ corresponds to the action on $\mathbb{R} \times \mathbb{R}^{2m-1}$ via $\mathrm{Id} \times \mathrm{SO}(2m-1)$

$$(t, x) \xrightarrow{\mathrm{Id}\times m(U)} (t, xU) \,.$$

Let $\{e_0, e_1, \dots, e_{2m-1}\}$ be the canonical basis of $\mathbb{R} \times \mathbb{R}^{2m-1}$ (with e_0 spanning $\mathbb{R}$) and let $\{e_0^*, e_1^*, \dots, e_{2m-1}^*\}$ be the dual basis. Every $(2m-1)$-multilinear form α has a canonical decomposition

$$\alpha = e_0^* \wedge \beta + \omega \,,$$

where β is a $(2m-2)$-multilinear form on $\mathbb{R}^{2m-1}$ and ω is the pullback to $\mathbb{R} \times \mathbb{R}^{2m-1}$ of a $(2m-1)$-multilinear form on $\mathbb{R}^{2m-1}$. By uniqueness of the decomposition, α is $\mathrm{Id} \times \mathrm{SO}(2m-1)$-invariant if and only if β and ω are $\mathrm{SO}(2m-1)$-invariant. Thus ω is a multiple of the determinant

$$\omega = \lambda e_1^* \wedge \cdots \wedge \epsilon_{2m-1}^*$$

and if we show that there are no SO$(2m-1)$-invariant $(2m-2)$-multilinear forms on $\mathbb{R}^{2m-1}$ we will have shown that $\Omega^{2m-1}(M\backslash P)^P$ is one-dimensional.

The fact that $\Lambda^{*2m-1}(\mathbb{R}^{2m-1})$ is one-dimensional and the pairing

$$\begin{array}{ccc} \Lambda^{*2m-2}(\mathbb{R}^{2m-1}) \times (\mathbb{R}^{2m-1})^* & \to & \Lambda^{*2m-1}(\mathbb{R}^{2m-1}) \\ (\beta, \lambda) & \longmapsto & \beta \wedge \lambda \end{array}$$

show that there is an isomorphism

$$\begin{array}{ccc} \Lambda^{*2m-2}(\mathbb{R}^{2m-1}) & \longrightarrow & (\mathbb{R}^{2m-1})^* \\ e_1^* \wedge \cdots \wedge \widehat{e_j^*} \wedge \cdots \wedge e_{2m-1}^* & \mapsto & e_j^* \end{array}$$

that is SO$(2m-1)$-equivariant. Since the SO$(2m-1)$-action on $(\mathbb{R}^{2m-1})^*$ is irreducible, there are no SO$(2m-1)$-invariant $(2m-2)$-multilinear forms on $\mathbb{R}^{2m-1}$, thus showing that $\Omega^{2m-1}(M\backslash P)^P$ is one-dimensional.

We show now that the exterior derivative on $\Omega^{2m-1}(M\backslash P)^P$ does not vanish. With the identification (5.1), the right translation $R_{(t,x)}$ by an element $m(U)a(t)n(x)$ is given by

$$R_{(t,x)}(s, y) = (s + t, e^t yU + x)\,.$$

In fact

$$\begin{aligned} R_{(t,x)}(s, y) &= (Ma(s)n(y))m(U)a(t)n(x) \\ &= \underbrace{Mm(U)}_{=M}\,\underbrace{m(U)^{-1}a(s)}_{=a(s)m(U)^{-1}}\,n(y)m(U)a(t)n(x) \\ &= Ma(s)\,\underbrace{m(U)^{-1}n(y)m(U)}_{=n(yU)}\,a(t)n(x) \\ &= Ma(s)n(yU)a(t)n(x) \\ &= Ma(s)a(t)\,\underbrace{a(t)^{-1}n(yU)a(t)}_{=n(e^t yU)}\,n(x) \\ &= Ma(s+t)n(e^t yU + x)\,. \end{aligned}$$

In particular

$$R_{(t,x)}(0, 0) = Ma(t)n(x) \simeq (t, x),$$

so that $\omega \in \Omega^{2m-1}(\mathbb{R} \times \mathbb{R}^{2m-1})$ can be extended to a P-invariant differential $(2m-1)$-form

$$\omega_{(0,0)}((s_1, y_1), \ldots, (s_{2m-1}, y_{2m-1})) = ((R_{(t,x)})^*\omega)_{(0,0)}((s_1, y_1), \ldots, (s_{2m-1}, y_{2m-1}))$$

for $(s_j, t_j) \in \mathbb{R} \times \mathbb{R}^{2m-1} \simeq T_{(0,0)}(\mathbb{R} \times \mathbb{R}^{2m-1})$, $j = 1, \dots, 2m-1$.
Since

$$\begin{aligned}
&\omega_{(0,0)}((s_1, y_1), \dots, (s_{2m-1}, y_{2m-1})) \\
&\quad = \omega_{(t,x)}(D_{(0,0)}R_{(t,x)}((s_1, y_1)), \dots, D_{(0,0)}R_{(t,x)}((s_{2m-1}, y_{2m-1}))) \\
&\quad = \omega_{(t,x)}((s_1, e^t y_1), \dots, (s_{2m-1}, e^t y_{2m-1})) \\
&\quad = e^{(2m-1)t}\omega_{(t,x)}((s_1, y_1), \dots, (s_{2m-1}, y_{2m-1})) ,
\end{aligned}$$

then

$$\omega_{(t,x)} = e^{-(2m-1)t}\omega_{(0,0)} ,$$

so that

$$d\omega = -(2m-1)e^{-(2m-1)t} dt \wedge \omega$$

is not vanishing. This shows the claim and completes the proof of the lemma. □

Lemma 5.5 *If $m \geq 2$ then the restriction*

$$\varepsilon^{\mathrm{b}}_{2m}|_P \in \mathrm{H}^{2m}_{\mathrm{B,b}}(P, \mathbb{Z})$$

vanishes.

Proof Considering the long exact sequence in bounded and in ordinary cohomology associated to (2.8), and taking into account Lemma 5.4 and the vanishing of bounded real cohomology of P (2.7), we obtain the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^{2m-1}_{\mathrm{B}}(P, \mathbb{R}/\mathbb{Z}) \arrow[r, "\delta^{\mathrm{b}}"] \arrow[d, equal] & \mathrm{H}^{2m}_{\mathrm{B,b}}(P, \mathbb{Z}) \arrow[r] \arrow[d, "c_{\mathbb{Z}}"] & 0 \\
0 \arrow[r] & \mathrm{H}^{2m-1}_{\mathrm{B}}(P, \mathbb{R}/\mathbb{Z}) \arrow[r, "\delta"'] & \mathrm{H}^{2m}_{\mathrm{B}}(P, \mathbb{Z}) \arrow[r] & 0 ,
\end{tikzcd}$$

where we used Lemma 5.4 in the bottom row and the vanishing of bounded cohomology with real coefficients in the top row. Thus $\mathrm{H}^{2m}_{\mathrm{B,b}}(P, \mathbb{Z}) \cong \mathrm{H}^{2m}_{\mathrm{B}}(P, \mathbb{Z})$. If we show that $\varepsilon_{2m}|_P = 0$, this will imply that $\varepsilon^{\mathrm{b}}_{2m}|_P = 0$.

To see that $\varepsilon_{2m}|_P = 0$, by naturality of Wigner's isomorphism (2.4) and the fact that M is maximal compact in P, the restriction to M induces an isomorphism $\mathrm{H}^{2m}_{\mathrm{B}}(P, \mathbb{Z}) \cong \mathrm{H}^{2m}_{\mathrm{B}}(M, \mathbb{Z})$. Since however the Euler class ε_{2m} of $\mathrm{SO}(2m)$ restricted to $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} : U \in \mathrm{SO}(2m-1) \right\}$ vanishes, we conclude that $\varepsilon_{2m}|_P = 0$. □

We deduce then using Lemma 5.5 that Theorem 5.1 holds if the image of ρ is contained in P. We must therefore turn to the case in which $\rho(\mathbb{Z}^{2m-1})$ lies in T_0. Let then

$$\pi_i : T \longrightarrow \mathrm{O}(2)$$

be the projection on the i-th factor of T and let

$$\varepsilon_{(i)} := \pi_i^*(\varepsilon_2) \quad \text{and} \quad \varepsilon_{(i)}^{\mathrm{b}} := \pi_i^*(\varepsilon_2^{\mathrm{b}}),$$

where ε_2 and $\varepsilon_2^{\mathrm{b}}$ are respectively the Euler class and the bounded Euler class of O(2). Observe that if L is compact, the long exact sequence (2.10) gives

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}_{\mathrm{B}}^{\bullet}(L, \mathbb{R}_\epsilon/\mathbb{Z}_\epsilon) \arrow[r, "\cong"] \arrow[d, equal] & \mathrm{H}_{\mathrm{B,b}}^{\bullet}(L, \mathbb{Z}_\epsilon) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{H}_{\mathrm{B}}^{\bullet}(L, \mathbb{R}_\epsilon/\mathbb{Z}_\epsilon) \arrow[r, "\cong"] & \mathrm{H}_{\mathrm{B}}^{\bullet}(L, \mathbb{Z}_\epsilon) \arrow[r] & 0\,. \end{tikzcd} \tag{5.2}$$

Since the ordinary Euler class is a characteristic class and T is a product, we have

$$\varepsilon_{2m}|_T = \varepsilon_{(1)} \cup \cdots \cup \varepsilon_{(m)}\,, \tag{5.3}$$

hence it follows from (5.2) that

$$\varepsilon_{2m}^{\mathrm{b}}|_T = \varepsilon_{(1)}^{\mathrm{b}} \cup \cdots \cup \varepsilon_{(m)}^{\mathrm{b}}\,. \tag{5.4}$$

Observe that the image of $\mathrm{SO}(2)^m$ in T is its connected component T°.

Lemma 5.6 *If $\rho\colon \mathbb{Z}^{2m-1} \to T_0$ does not take values in T°, then $\rho^*(\varepsilon_{2m}^{\mathrm{b}}) = 0$.*

Proof An Abelian subgroup of O(2) not contained in SO(2) is of the form $\{1, \sigma\}$, with $\sigma^2 = e$. Thus if $\rho(\mathbb{Z}^{2m-1}) \not\subset T^\circ$, there is π_i such that $\pi_i\rho(\mathbb{Z}^{2m-1}) \subset \{1, \sigma\}$. Since $\mathrm{H}_{\mathrm{b}}^2(\{1, \sigma\}, \mathbb{Z}_\epsilon) = 0$, then $\varepsilon_2^{\mathrm{b}}$ vanishes on $\{1, \sigma\}$, and hence $\rho^*(\varepsilon_{2m}^{\mathrm{b}}) = 0$ by (5.4). □

Thus we are reduced to analyze homomorphisms

$$\rho\colon \mathbb{Z}^{2m-1} \longrightarrow T^\circ \cong \mathrm{SO}(2)^m\,.$$

As before, since $\mathrm{SO}(2)^m$ is compact and hence amenable, the connecting homomorphism

$$\delta^{\mathrm{b}}\colon \mathrm{H}_{\mathrm{B}}^{2m-1}(\mathrm{SO}(2)^m, \mathbb{R}/\mathbb{Z}) \longrightarrow \mathrm{H}_{\mathrm{B,b}}^{2m}(\mathrm{SO}(2)^m, \mathbb{Z})$$

is an isomorphism. We fix the orientation preserving identification

$$\mathrm{SO}(2) \longrightarrow \mathbb{R}/\mathbb{Z}\,,$$

and, for $m = 1$, we define the homogeneous 1-cocycle

$$\begin{aligned} \mathrm{Rot}\colon\ \mathrm{SO}(2) \times \mathrm{SO}(2) &\longrightarrow \mathrm{SO}(2) \cong \mathbb{R}/\mathbb{Z} \\ (g, h) &\longmapsto g^{-1}h \end{aligned} \tag{5.5}$$

and

$$[\mathrm{Rot}] \in \mathrm{H}^1_{\mathrm{B}}(\mathrm{SO}(2), \mathbb{R}/\mathbb{Z})$$

the corresponding class. Then $\delta^{\mathrm{b}}([\mathrm{Rot}]) = \varepsilon_2^{\mathrm{b}}$ and from this we deduce that if

$$\vartheta := \pi_1^*([\mathrm{Rot}]) \in \mathrm{H}^1_{\mathrm{B}}(\mathrm{SO}(2)^m, \mathbb{R}/\mathbb{Z})\,,$$

then δ^{b} maps the class $\vartheta \cup \varepsilon_{(2)} \cup \cdots \cup \varepsilon_{(m)} \in \mathrm{H}^{2m-1}_{\mathrm{B}}(\mathrm{SO}(2)^m, \mathbb{R}/\mathbb{Z})$ to the class $\varepsilon^{\mathrm{b}}_{(1)} \cup \cdots \cup \varepsilon^{\mathrm{b}}_{(m)} \in \mathrm{H}^{2m}_{\mathrm{B,b}}(\mathrm{SO}(2)^m, \mathbb{Z})$. As a result, it follows from the commutativity of the square

$$\begin{tikzcd} \mathrm{H}^{2m-1}_{\mathrm{B}}(\mathrm{SO}(2)^m, \mathbb{R}/\mathbb{Z}) \arrow[r, "\delta^{\mathrm{b}}"] \arrow[d, "\rho^*"'] & \mathrm{H}^{2m}_{\mathrm{B,b}}(\mathrm{SO}(2)^m, \mathbb{Z}) \arrow[d, "\rho^*"] \\ \mathrm{H}^{2m-1}(\mathbb{Z}^{2m-1}, \mathbb{R}/\mathbb{Z}) \arrow[r, "\delta^{\mathrm{b}}"] & \mathrm{H}^{2m}_{\mathrm{b}}(\mathbb{Z}^{2m-1}, \mathbb{Z}) \end{tikzcd}$$

that

$$\rho^*(\varepsilon^{\mathrm{b}}_{2m}) = \delta^{\mathrm{b}}(\rho^*(\vartheta \cup \varepsilon_{(2)} \cup \cdots \cup \varepsilon_{(m)}))\,. \tag{5.6}$$

In order to prove the vanishing of the left hand side, we are going to show that $\rho^*(\vartheta \cup \varepsilon_{(2)} \cup \cdots \cup \varepsilon_{(m)}) = 0$. To this end we use that the pairing

$$\begin{aligned} \mathrm{H}^{2m-1}(\mathbb{Z}^{2m-1}, \mathbb{R}/\mathbb{Z}) &\longrightarrow \mathbb{R}/\mathbb{Z} \\ \beta &\longmapsto \langle \beta, [\mathbb{Z}^{2m-1}] \rangle \end{aligned} \tag{5.7}$$

is an isomorphism, where $[\mathbb{Z}^{2m-1}] \in \mathrm{H}_{2m-1}(\mathbb{Z}^{2m-1}, \mathbb{Z}) \cong \mathbb{Z}$ denotes the fundamental class.

We will need the following

Lemma 5.7 *Let n be even and $n \geq 2$ and let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{Z}^n$. Then the group chain*

$$z = \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sign}(\sigma)[0, e_{\sigma(1)}, e_{\sigma(1)} + e_{\sigma(2)}, \ldots, e_{\sigma(1)} + \cdots + e_{\sigma(n)}]$$

is a representative of the fundamental class $[\mathbb{Z}^n] \in \mathrm{H}_n(\mathbb{Z}^n, \mathbb{Z}) \cong \mathbb{Z}$.

Proof We first check that $\partial z = 0$ so that z is indeed a cycle. Let

$$s_\sigma := [0, e_{\sigma(1)}, e_{\sigma(1)} + e_{\sigma(2)}, \dots, e_{\sigma(1)} + \dots + e_{\sigma(n)}] .$$

Observe that, for $1 \leq i \leq n-1$, the sum of each of the i-th faces of the simplices appearing in z over all permutations of $\sigma \in \mathrm{Sym}(n)$ vanishes since

$$\partial_i s_\sigma = [0, e_{\sigma(1)}, e_{\sigma(1)} + e_{\sigma(2)}, \dots, e_{\sigma(1)} \widehat{+ \dots +} e_{\sigma(i)}, \dots, e_{\sigma(1)} + \dots + e_{\sigma(n)}]$$

is invariant under the odd transposition $\tau_i := (\sigma(i), \sigma(i+1))$. Indeed if $1 \leq i \leq n-1$,

$$\begin{aligned}\sum_{\sigma\in\mathrm{Sym}(n)} \mathrm{sign}(\sigma)\partial_i s_\sigma &= \sum_{\sigma\in\mathrm{Sym}(n)} \mathrm{sign}(\sigma)\partial_i s_{\sigma\tau_i}\\ &= \sum_{\sigma\in\mathrm{Sym}(n)} \mathrm{sign}(\sigma\tau_i^{-1})\partial_i s_\sigma\\ &= -\sum_{\sigma\in\mathrm{Sym}(n)} \mathrm{sign}(\sigma)\partial_i s_\sigma ,\end{aligned}$$

which hence implies that $\sum_{\sigma\in\mathrm{Sym}(n)} \mathrm{sign}(\sigma)\partial_i s_\sigma = 0$. Thus

$$\partial z = \sum_{\sigma\in\mathrm{Sym}(n)} \mathrm{sign}(\sigma)\partial_0 s_\sigma + \sum_{\sigma\in\mathrm{Sym}(n)} \mathrm{sign}(\sigma)\partial_n s_\sigma \tag{5.8}$$

and it remains to see that the 0-th and n-th face cancel each other. By $\mathbb{Z}^n$-invariance,

$$\begin{aligned}\partial_0 s_\sigma &= [e_{\sigma(1)}, e_{\sigma(1)} + e_{\sigma(2)}, \dots, e_{\sigma(1)} + \dots + e_{\sigma(n)}]\\ &= [0, e_{\sigma(2)}, \dots, e_{\sigma(2)} + \dots + e_{\sigma(n)}]\\ &= [0, e_{\sigma\tau(1)}, \dots, e_{\sigma\tau(1)} + \dots + e_{\sigma\tau(n-1)}]\\ &= \partial_n s_{\sigma\tau} ,\end{aligned} \tag{5.9}$$

where $\tau = (1, 2, \dots, n)$ is the cyclic permutation of signature $\mathrm{sign}(\tau) = (-1)^{n-1}$. From (5.8) and (5.9) it follows that

$$\begin{aligned}\partial z &= \sum_{\sigma\in\mathrm{Sym}(n)} \mathrm{sign}(\sigma)\partial_n s_{\sigma\tau} + \sum_{\sigma\in\mathrm{Sym}(n)} \mathrm{sign}(\sigma)\partial_n s_\sigma\\ &= \sum_{\sigma\in\mathrm{Sym}(n)} (-1)^{n-1}\mathrm{sign}(\sigma)\partial_n s_\sigma + \sum_{\sigma\in\mathrm{Sym}(n)} \mathrm{sign}(\sigma)\partial_n s_\sigma = 0\end{aligned}$$

since n is even.

Let $\omega_{\mathbb{R}^n} \in \mathrm{H}^n(\mathbb{Z}^n, \mathbb{R})$ be the Euclidean volume class. Note that the volume class evaluates to 1 on the fundamental class, as the n-torus generated by the canonical basis has volume 1. A cocycle representing $\omega_{\mathbb{R}^n}$ is given by $V_n \colon (\mathbb{Z}^n)^{n+1} \to \mathbb{R}$ sending

$v_0, v_1, \ldots, v_n$ to the signed volume of the affine simplex with vertices in the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$, that is

$$V_n(v_0, v_1, \ldots, v_n) = \frac{1}{n!}\det(v_1 - v_0, \ldots, v_n - v_0)\,.$$

In order to show that

$$\langle \omega_{\mathbb{R}^n}, [\mathbb{Z}^n] \rangle = V_n(z) = 1$$

we need to evaluate V_n on each summand of z. By definition,

$$\begin{aligned}
&V_n(0, e_{\sigma(1)}, e_{\sigma(1)} + e_{\sigma(2)}, \ldots, e_{\sigma(1)} + \cdots + e_{\sigma(n)}) \\
&\quad = \frac{1}{n!}\det(e_{\sigma(1)}, e_{\sigma(1)} + e_{\sigma(2)}, \ldots, e_{\sigma(1)} + \cdots + e_{\sigma(n)}) \\
&\quad = \operatorname{sign}(\sigma)\frac{1}{n!}\det(e_1, e_1 + e_2 \ldots, e_1 + \cdots + e_n) \\
&\quad = \operatorname{sign}(\sigma)\frac{1}{n!}\,,
\end{aligned}$$

where we have used for the second equality the fact that the determinant is alternating with respect to line permutations. Summing up, we obtain

$$\langle \omega_{\mathbb{R}^n}, [\mathbb{Z}^n] \rangle = \sum_{\sigma \in \mathrm{Sym}(n)} \frac{1}{n!} = 1\,,$$

thus proving the lemma. □

Lemma 5.8 *Let $m \geq 2$. Then*

$$\langle \rho^*(\vartheta \cup \varepsilon_{(2)} \cup \cdots \cup \varepsilon_{(m)}), [\mathbb{Z}^{2m-1}] \rangle = 0\,.$$

Proof We use as a representative of $\varepsilon_2 \in \mathrm{H}^2(\mathrm{SO}(2), \mathbb{Z})$ the multiple $-\frac{1}{2}$ of the orientation cocycle (see Remark 3.1). This cocycle takes values in $\frac{1}{2}\mathbb{Z}$ but represents an integral class and in particular evaluates to an integer on a fundamental class. Hence a representative for $\kappa := \vartheta \cup \varepsilon_{(2)} \cup \cdots \cup \varepsilon_{(m)} \in \mathrm{H}^{2m-1}_{\mathrm{B}}(\mathrm{SO}(2)^m, \mathbb{R}/\mathbb{Z})$ is given by the cocycle mapping $g_0, \ldots, g_{2m-1} \in \mathrm{SO}(2)^m$ to the product

$$\frac{(-1)^{m-1}}{2^{m-1}} \cdot \mathrm{Rot}_1(g_0, g_1) \cdot \mathrm{Or}_2(g_1, g_2, g_3) \cdot \cdots \cdot \mathrm{Or}_m(g_{2m-3}, g_{2m-2}, g_{2m-1})\,,$$

where Rot_1 denotes the pullback to $\mathrm{SO}(2)^m$ via the first projection π_1 of the homogeneous 1-cocycle Rot defined in (5.5), while Or_j denotes the pullback to $\mathrm{SO}(2)^m$ via π_j of the orientation cocycle

$$\mathrm{Or}\colon\ \mathrm{SO}(2)^3 \longrightarrow \{-1, 0, 1\}\,.$$

We now evaluate the pullback $\rho^*(\kappa)$ on the cycle z of Lemma 5.7 and, writing $f_i = \rho(e_i) \in \mathrm{SO}(2)^m$, we obtain

$$\begin{aligned}
&\langle \rho^*(\kappa), [\mathbb{Z}^{2m-1}]\rangle \\
&= \frac{(-1)^{m-1}}{2^{m-1}} \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sign}(\sigma)\mathrm{Rot}_1(0, f_{\sigma(1)}) \cdot \mathrm{Or}_2(f_{\sigma(1)}, f_{\sigma(1)} f_{\sigma(2)}, f_{\sigma(1)} f_{\sigma(2)} f_{\sigma(3)}) \\
&\quad \cdot (\mathrm{Or}_3 \quad \cdots \quad \mathrm{Or}_m)(f_{\sigma(1)} f_{\sigma(2)} f_{\sigma(3)}, \dots, f_{\sigma(1)} \cdot \cdots \cdot f_{\sigma(n)}).
\end{aligned} \tag{5.10}$$

To prove that this expression vanishes first observe that

$$\mathrm{Or}(f_1, f_2, f_3) = -\mathrm{Or}(f_1^{-1}, f_2^{-1}, f_3^{-1})$$

holds for any f_1, f_2, f_3 in $\mathrm{SO}(2)$ (but not in $\mathrm{SL}(2, \mathbb{R})$). Specializing to $f_1 = g^{-1}$, $f_2 = \mathrm{Id}$, $f_3 = h$ and using that Or is alternating gives

$$\mathrm{Or}(g^{-1}, \mathrm{Id}, h) = \mathrm{Or}(h^{-1}, \mathrm{Id}, g)$$

for any g, h in $\mathrm{SO}(2)$. Finally, using the invariance of Or, we can multiply the variables on the left hand side of the latter equality by fg, and the variables on the right hand side by fh to obtain

$$\mathrm{Or}(f, fg, fgh) = \mathrm{Or}(f, fh, fhg) \tag{5.11}$$

for any f, g, h in $\mathrm{SO}(2)$.

Thus from (5.10) and (5.11) and with $\tau_2 = (23)$, we obtain the asserted vanishing as

$$\begin{aligned}
&\langle \rho^*(\kappa), [\mathbb{Z}^{2m-1}]\rangle \\
&= \frac{(-1)^{m-1}}{2^{m-1}} \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sign}(\sigma)\mathrm{Rot}_1(0, f_{\sigma(1)}) \cdot \mathrm{Or}_2(f_{\sigma(1)}, f_{\sigma(1)} f_{\sigma(3)}, f_{\sigma(1)} f_{\sigma(3)} f_{\sigma(2)}) \\
&\quad \cdot (\mathrm{Or}_3 \quad \cdots \quad \mathrm{Or}_m)(f_{\sigma(1)} f_{\sigma(2)} f_{\sigma(3)}, \dots, f_{\sigma(1)} \cdot \cdots \cdot f_{\sigma(n)}) \\
&= \mathrm{sign}(\tau_2) \frac{(-1)^{m-1}}{2^{m-1}} \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sign}(\sigma)\mathrm{Rot}_1(0, f_{\sigma(1)}) \\
&\quad \cdot \mathrm{Or}_2(f_{\sigma(1)}, f_{\sigma(1)} f_{\sigma(2)}, f_{\sigma(1)} f_{\sigma(2)} f_{\sigma(3)}) \\
&\quad \cdot (\mathrm{Or}_3 \quad \cdots \quad \mathrm{Or}_m)(f_{\sigma(1)} f_{\sigma(2)} f_{\sigma(3)}, \dots, f_{\sigma(1)} \cdot \cdots \cdot f_{\sigma(n)}) \\
&= -\langle \rho^*(\kappa), [\mathbb{Z}^{2m-1}]\rangle\,.
\end{aligned}$$

□

Proof of Theorem 5.1 Let $\rho\colon \mathbb{Z}^{2m-1} \to \mathrm{SO}(2m, 1)^\circ$ be a homomorphism. By Lemma 5.2 either $\rho(\mathbb{Z}^{2m-1}) < P$ up to conjugation and then $\rho^*(\varepsilon^{\mathrm{b}}_{2m}) = 0$ by Lemma 5.5 or $\rho^*(\mathbb{Z}^{2m-1}) < T_0$ up to conjugacy. Then either $\rho(\mathbb{Z}^{2m-1}) \not\subset T^\circ$ and the vanishing follows from Lemma 5.6 or $\rho(\mathbb{Z}^{2m-1}) < T^\circ$, in which case Lemma 5.8 and (5.6) imply that $\rho^*(\vartheta \cup \varepsilon_{(2)} \cup \cdots \cup \varepsilon_{(m)}) = 0$ and hence $\rho^*(\varepsilon^{\mathrm{b}}_{2m}) = 0$ by (5.7). □

Proof of Theorem 1.1 Let N be a compact core of M and let $C_1, \dots, C_h$ the connected components of the boundary of ∂N. It follows from Theorem 4.4 that

$$(-1)^m \frac{2}{\mathrm{Vol}(S^{2m})}\mathrm{Vol}(\rho) \equiv -\sum_{i=1}^{h}\langle(\delta^{\mathrm{b}})^{-1}(\rho^*(\varepsilon^{\mathrm{b}}_{2m})|_{C_i}, [C_i]\rangle \mod \mathbb{Z}\,,$$

with the usual abuse of notation that $\rho^*(\varepsilon^{\mathrm{b}}_{2m})|_{C_i}$ refers to the element in $\mathrm{H}^{2m}_{\mathrm{b}}(C_i, \mathbb{Z})$ corresponding to $(\rho|_{\pi_1(C_i)})^*(\varepsilon^{\mathrm{b}}_{2m}) \in \mathrm{H}^{2m}_{\mathrm{b}}(\pi_1(C_i), \mathbb{Z})$.

If now all the C_i's are tori, the above congruence relation and Theorem 5.1 imply that $(-1)^m \frac{2}{\mathrm{Vol}(S^{2m})}\mathrm{Vol}(\rho) \in \mathbb{Z}$.

In the general case, let $p_i : C_i' \to C_i$ be a covering of degree B_{2m-1} that is a torus. Then

$$\begin{aligned}B_{2m-1}\langle(\delta^{\mathrm{b}})^{-1}(\rho^*(\varepsilon^{\mathrm{b}}_{2m})|_{C_i}), [C_i]\rangle &= \langle(\delta^{\mathrm{b}})^{-1}(\rho^*(\varepsilon^{\mathrm{b}}_{2m})|_{C_i}), p_{i*}([C_i'])\rangle \\ &= \langle(\delta^{\mathrm{b}})^{-1}p_i^*(\rho^*(\varepsilon^{\mathrm{b}}_{2m})|_{C_i}), [C_i']\rangle\,.\end{aligned}$$

Now observe that $p_i^*(\rho^*(\varepsilon^{\mathrm{b}}_{2m})|_{C_i}) \in \mathrm{H}^{2m}_{\mathrm{b}}(C_i', \mathbb{Z})$ corresponds to the class

$$(\rho \circ p_{i*})^*(\varepsilon^{\mathrm{b}}_{2m}) = (\rho|^*_{\pi_1(C_i')})(\varepsilon^{\mathrm{b}}_{2m}) \in \mathrm{H}^{2m}_{\mathrm{b}}(\pi_1(C_i'), \mathbb{Z})\,,$$

which vanishes by Theorem 5.1. □

6 Examples of nontrivial and non-maximal representations

In this section we give examples of volumes of representations. More precisely:

- In § 6.1 we set ourselves in dimension 3. Here we show in particular that the volume of a Dehn filling of a finite volume hyperbolic manifold coincides with the volume of the filling representation. In fact Proposition 6.1 deals with a more general case.
- In § 6.2, by glueing appropriately copies of a hyperbolic manifold of arbitrary dimensions with totally geodesic boundary, we construct manifolds M_k and representations of $\pi_1(M_k)$ whose volume is a rational multiple of $\mathrm{Vol}(M_k)$.

6.1 Dimension 3: representations given by Dehn filling

Let M be a complete finite volume hyperbolic 3-manifold, which, for simplicity, we assume has only one cusp. If N is a compact core of M, its boundary ∂N is Euclidean with the induced metric and hence there is an isometry $\varphi\colon \partial N \to \mathbb{T}^2$ to a two-dimensional torus for an appropriate flat metric on $\mathbb{T}^2$. We obtain then a decomposition of M as a connected sum

$$M = N\#(\mathbb{T}^2 \times \mathbb{R}_{\geq 0})\,,$$

where the identification is via φ. We are now going to fill in a solid two-torus to obtain a compact manifold. To this end, let $\tau \subset \partial N$ be a simple closed geodesic and let us choose a diffeomorphism $\varphi_\tau : \partial N \to S^1 \times S^1$, in such a way that $\varphi_\tau(\tau) = S^1 \times \{*\}$. Then M_τ is the connected sum

$$M_\tau : = N\#(\mathbb{D}^2 \times S^1)\,,$$

identified via φ_τ.

Denote by $j_\tau : N \hookrightarrow M_\tau$ the canonical inclusion and by $p\colon M \to N$ the canonical projection given by the cusp retraction $\mathbb{T}^2 \times \mathbb{R}_{>0} \to \mathbb{T}^2$. The composition

$$f_\tau = j_\tau \circ p\colon M \longrightarrow M_\tau$$

induces a map

$$(f_\tau)_*\colon \Gamma \longrightarrow \Gamma_\tau$$

between the fundamental groups $\Gamma = \pi_1(M)$ and $\Gamma_\tau = \pi_1(M_\tau)$.

Proposition 6.1 *Let M_τ be the compact 3-manifold obtained by Dehn filling from the hyperbolic 3-manifold M with one cusp. Let $\rho\colon \Gamma_\tau \to \mathrm{SO}(3,1)$ be any representation of Γ_τ and let $\rho_\tau : = \rho \circ f_\tau : \Gamma \to \mathrm{SO}(3,1)$. Then*

$$\mathrm{Vol}(\rho_\tau) = \mathrm{Vol}(\rho)\,.$$

By Gromov–Thurston's (2π)-Theorem [16], for all geodesic curves τ for which the induced length is greater than 2π in the induced Euclidean metric on ∂N, the compact manifold M_τ admits a hyperbolic structure. Proposition 1.2 is then an immediate consequence of Proposition 6.1.

To prove the proposition, recall that by definition, the volume of the representation ρ_τ is equal to

$$\mathrm{Vol}(\rho_\tau) = \langle c \circ \Psi^{-1} \circ f^* \circ \rho^*(\omega_3^{\mathrm{b}}), [N, \partial N]\rangle\,,$$

where all maps involved can be read in the diagram below. We will start by defining a map $F\colon \mathrm{H}^3(M_\tau) \to \mathrm{H}^3(N, \partial N)$ that will turn the diagram below into a commutative diagram (Lemma 6.3) and which will induce a canonical isomorphism (Lemma 6.2).

$$\begin{tikzcd}
\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}(3,1)) \arrow[r, "\rho^*"'] \arrow[d, "c"] \arrow[rr, bend left, "\rho_\tau^*"] & \mathrm{H}^3_{\mathrm{b}}(\Gamma_\tau) \arrow[r, "f_\tau^*"'] \arrow[d, "c"] & \mathrm{H}^3_{\mathrm{b}}(\Gamma) & \mathrm{H}^3_{\mathrm{b}}(N, \partial N) \arrow[l, "\cong", "\Psi"'] \arrow[d, "c"] \\
\mathrm{H}^3_{\mathrm{c}}(\mathrm{SO}(3,1)) \arrow[r, "\rho^*"'] & \mathrm{H}^3(\Gamma_\tau) \arrow[r, "\cong", "g"'] & \mathrm{H}^3(M_\tau) \arrow[r, dotted, "F"'] & \mathrm{H}^3(N, \partial N).
\end{tikzcd} \tag{6.1}$$

The inclusions

$$\begin{tikzcd}[column sep=small] M_\tau \arrow[dr, hook, "i"'] & & (N, \partial N) \arrow[dl, hook', "(j_\tau, \varphi_\tau)"] \\ & (M_\tau, \mathbb{D}^2 \times S^1) & \end{tikzcd}$$

induce the following homology and cohomology maps

$$\begin{tikzcd}[column sep=small] \mathrm{H}_\bullet(M_\tau, \mathbb{Z}) \arrow[dr, "i_*"'] & & \mathrm{H}_\bullet((N, \partial N), \mathbb{Z}) \arrow[dl, "(j_\tau, \varphi_\tau)_*"] \\ & \mathrm{H}_\bullet((M_\tau, \mathbb{D}^2 \times S^1), \mathbb{Z}) & \end{tikzcd} \tag{6.2}$$

and

$$\begin{tikzcd}[column sep=small] \mathrm{H}^\bullet(M_\tau, \mathbb{Z}) & & \mathrm{H}^\bullet((N, \partial N), \mathbb{Z}) \\ & \mathrm{H}^\bullet((M_\tau, \mathbb{D}^2 \times S^1), \mathbb{Z}) \arrow[ul, "i_*"] \arrow[ur, "(j_\tau, \varphi_\tau)_*"'] & \end{tikzcd} \tag{6.3}$$

Lemma 6.2 *In degree* 3 *the maps in* (6.2) *and* (6.3) *are canonical isomorphisms and the composition*

$$F = (j_\tau, \varphi_\tau)^* \circ (i^*)^{-1} \colon \mathrm{H}^3(M_\tau, \mathbb{Z}) \longrightarrow \mathrm{H}^3(N, \partial N, \mathbb{Z})$$

maps the dual β_{M_τ} of the fundamental class of M_τ to the dual $\beta_{[N,\partial N]}$ of the fundamental class of $(N, \partial N)$.

Proof It is enough to show the statement in homology where we show that fundamental classes are mapped to each other by showing the existence of a compatible triangulation of the three manifolds. Start with a triangulation of the boundary torus $S^1 \times S^1 = \partial N$, extend it on the one hand to the filled torus $\mathbb{D}^2 \times S^1$ and on the other hand to N, [26, Theorem 10.6]. This produces compatible triangulations representing $[M_\tau]$, $[M_\tau, \mathbb{D}^2 \times S^1]$ and $[N, \partial N]$. □

Lemma 6.3 *The diagram (6.1) commutes.*

Proof We only need to show that the right rectangle commutes. For this, we will decompose the diagram in subdiagrams as follows:

$$
\begin{tikzcd}
\mathrm{H}^3_{\mathrm{b}}(\Gamma) \arrow[r, "\cong"] & \mathrm{H}^3_{\mathrm{b}}(M) & \mathrm{H}^3_{\mathrm{b}}(N) \arrow[l, "\cong", "p^*"'] & \\
\mathrm{H}^3_{\mathrm{b}}(\Gamma_\tau) \arrow[u, "f^*"] \arrow[r, "\cong"] \arrow[dd, "c"] & \mathrm{H}^3_{\mathrm{b}}(M_\tau) \arrow[u, "f^*"] \arrow[ur, "j_\tau^*"'] \arrow[dd, "c"] & & \mathrm{H}^3_{\mathrm{b}}(N, \partial N) \arrow[ul, "i^*_{|N}"'] \arrow[dd, "c"] \arrow[uulll, bend right, "\Psi"'] \\
& & \mathrm{H}^3_{\mathrm{b}}(M_\tau, \mathbb{D}^2 \times S^1) \arrow[ul, "i^*"'] \arrow[ur, "(j_\tau,\varphi_\tau)^*"] \arrow[dd, "c"] & \\
\mathrm{H}^3(\Gamma_\tau) \arrow[r, "\cong", "g"'] & \mathrm{H}^3(M_\tau) \arrow[rr, "F"] & & \mathrm{H}^3(N, \partial N) \\
& & \mathrm{H}^3(M_\tau, \mathbb{D}^2 \times S^1)\,. \arrow[ul, "i^*"] \arrow[ur, "(j_\tau,\varphi_\tau)^*"'] &
\end{tikzcd}
$$

Since by naturality, all subdiagrams commute, the lemma follows. □

Proof of Proposition 6.1 Using the commutativity of the diagram (6.1) and the fact that $c(\omega_3^{\mathrm{b}}) = \omega_3$ and $g \circ \rho^*(\omega_3) = \mathrm{Vol}(\rho) \cdot \beta_{M_\tau}$ we compute

$$
\begin{aligned}
c \circ \Psi^{-1} \circ f_\tau^* \circ \rho^*(\omega_3^{\mathrm{b}}) &= F \circ g \circ \rho^* \circ c(\omega_3^{\mathrm{b}}) \\
&= F \circ g \circ \rho^*(\omega_3) \\
&= F(\mathrm{Vol}(\rho) \cdot \beta_{M_\tau}) \\
&= \mathrm{Vol}(\rho) \cdot \beta_{[N,\partial N]}.
\end{aligned}
$$

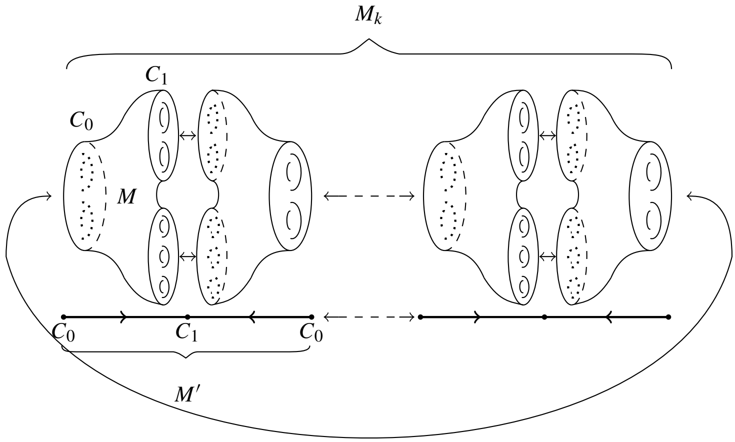
It is immediate that

$$
\begin{aligned}
\mathrm{Vol}(\rho_\tau) &= \langle c \circ \Psi^{-1} \circ f^* \circ \rho^*(\omega_3)^{\mathrm{b}}, [N, \partial N] \rangle = \langle \mathrm{Vol}(\rho) \cdot \beta_{[N,\partial N]}, [N, \partial N] \rangle \\
&= \mathrm{Vol}(\rho)\,,
\end{aligned}
$$

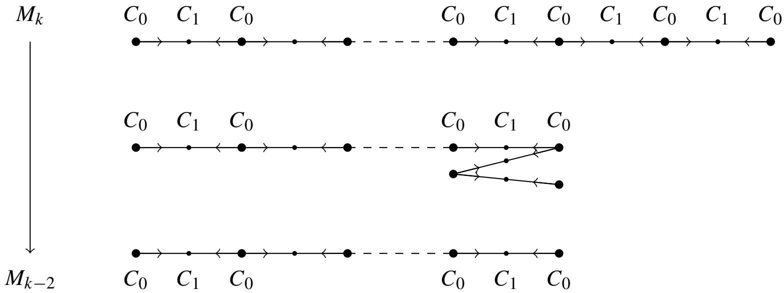
which finishes the proof of the proposition. □

6.2 Representations giving rational multiples of the maximal representation

Let M be a n-dimensional hyperbolic manifold with nonempy totally geodesic boundary (possibly with cusps) (see for example [24]). Suppose that the boundary of M has at least two connected components, which can be achieved by taking appropriate coverings of a manifold with a connected boundary. Decompose $\partial M = C_0 \sqcup C_1$. Let M' be the double of M along C_1. Observe that M' has as boundary two copies of C_0 with opposite orientation. Glueing these two copies, we obtain a complete hyperbolic manifold M_1. We can repeat the procedure as follows: Take k copies of M', glue the two copies of C_0 two by two so as to obtain a connected closed hyperbolic manifold M_k. Observe that $\mathrm{Vol}(M_k) = 2k\,\mathrm{Vol}(M)$.



For any even $0 < \ell < k$, there are degree one maps $f\colon M_k \to M_{k-\ell}$ obtained by folding ℓ copies of M' in M_k along its boundary. These maps send the last $\ell + 1$ copies of M' inside M_k to the last copy of M' in $M_{k-\ell}$ as illustrated in the following picture for $\ell = 2$:



The induced representation of $\pi_1(M_k)$ obtained by the induced map on fundamental groups composed with the lattice embedding of $\pi_1(M_{k-\ell})$ in $\mathrm{Isom}(\mathbb{H}^n)$ has volume equal to the volume of $M_{k-\ell}$, that is $(k-\ell)/k$ times the volume of the maximal representation.

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Appendix A. On the continuity of the volume of a representation

The goal of this section is to prove the following:

Proposition A.1 *Let* $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ *be any torsion-free lattice. The function*

$$\begin{array}{ccc} \mathrm{Hom}(\Gamma, \mathrm{Isom}(\mathbb{H}^n)) & \longrightarrow & \mathbb{R} \\ \rho & \longmapsto & \mathrm{Vol}(\rho) \end{array}$$

is continuous.

We begin with some preliminaries. Let G be a locally compact group, $\Gamma < G$ a lattice and L a locally compact group. We denote by $\mathrm{C_b}(X)$ the continuous bounded real valued functions on a topological space X. We define a map

$$\begin{array}{ccc} \mathrm{C_b}(L^{n+1}) \times \mathrm{Rep}(\Gamma, L) & \longrightarrow & \mathrm{C_b}(\Gamma^{n+1}) \\ (c, \pi) & \longmapsto & \pi^*(c), \end{array}$$

where

$$\pi^*(c)(\gamma_0, \dots, \gamma_n) = c(\pi(\gamma_0), \dots, \pi(\gamma_n))\,.$$

We endow $\mathrm{C_b}(L^{n+1})$ with the topology of uniform convergence and $\mathrm{C_b}(\Gamma^{n+1})$ with the topology of pointwise convergence with control of norms: in other words $\alpha_n \to \alpha$ in $\mathrm{C_b}(\Gamma^{n+1})$ if it converges pointwise and $\sup_n \|\alpha_n\|_\infty < \infty$. With these topologies and with the pointwise convergence topology on $\mathrm{Rep}(\Gamma, L)$ the above map is continuous.

We proceed to implement the transfer from Γ to G. For this, let $s\colon \Gamma\backslash G \to G$ be a Borel section and $r\colon G \to \Gamma$ be defined by

$$g = r(g) \cdot s(p(g))\,,$$

where $p\colon G \to \Gamma\backslash G$ denotes the canonical projection. Given a Γ-invariant cochain $\alpha \in \mathrm{C_b}(\Gamma^{n+1})^\Gamma$, define

$$T\alpha(g_0, \dots, g_n) = \int_{\Gamma\backslash G} \alpha(r(gg_0), \dots, r(gg_n))d\mu(g)\,,$$

where μ is the Haar measure on G normalized so that $\mu(\Gamma\backslash G) = 1$.

Proposition A.2 *Suppose that the Borel section has the property that images of compact subsets are precompact. Then*

(1) $T\alpha$ *is continuous, hence* $T\alpha \in \mathrm{C_b}(G^{n+1})^G$*,*

(2) $T\colon \mathrm{C_b}(\Gamma^{n+1})^\Gamma \to \mathrm{C_b}(G^{n+1})^G$ *is continuous for pointwise convergence on* Γ *with control of norms and uniform convergence on compact sets on* G^{n+1}.

Proof Let $D = s(\Gamma\backslash G)$. Choose $\epsilon > 0$ and $C \subset D$ compact with $\mu(D\backslash C) < \epsilon$. Then

$$| \, T\alpha(g_0, \dots, g_n) - \int_C \alpha(r(gg_0), \dots, r(gg_n)) d\mu(g) \, | < \epsilon \|\alpha\|_\infty \,.$$

Now we write

$$\begin{aligned}&\int_C \alpha(r(gg_0), \dots, r(gg_n)) d\mu(g)\\ &\quad = \sum_{\gamma_0,\dots,\gamma_n} \alpha(\gamma_0, \dots, \gamma_n)\mu(C \cap \gamma_0 D g_0^{-1} \cap \dots \cap \gamma_n D g_n^{-1})\,.\end{aligned}$$

Before we continue with the proof of the proposition, we need to show that for every compact subset $K \subset G$ the number F_K of translates of the fundamental domain D that K intersects is finite:

Lemma A.3 *For any compact subset* $K \subset G$*, the set*

$$F_K := \{\gamma \in \Gamma \mid K \cap \gamma D \neq \emptyset\}$$

is finite.

Note that the lemma is wrong for arbitrary fundamental domains, even for cocompact Γ. Indeed, start by writing the standard fundamental domain $(0, 1]$ of $\mathbb{Z}$ in $\mathbb{R}$ as

$$D_0 = \sqcup_{n=1}^{+\infty}(1/2^n, 1/2^{n-1}]\,,$$

and perturb it by translating each of the disjoint interval of D_0 by a different translation, for example obtaining the new fundamental domain

$$D = \sqcup_{n=1}^{+\infty} n + (1/2^n, 1/2^{n-1}]\,.$$

Take as compact set the closed interval $C = [0, 1]$. Then for every $-n \le 0$, the intersection $C \cap (-n + D)$ is nonempty.

Proof Set $F := \cup_{\eta\in\Gamma}\eta K$ and observe that $F \cap D = s(p(F))$ is relatively compact by our choice of Borel section. Since $\gamma K \cap D = \gamma K \cap (F \cap D)$ and K and $F \cap D$ are relatively compact, the lemma follows by the discreteness of Γ. □

Going back to the proof of the proposition, fix compact subsets $C_0, \dots, C_n$ of G such that $g_i \in C_i$. Observe that $F_{Cg_i} \subset F_{CC_i}$ and if $\gamma_i \in F_{CC_i}\backslash F_{Cg_i}$ then the measure of $C \cap \gamma_0 D g_0^{-1} \cap \cdots \cap \gamma_n D g_n^{-1}$ is zero. We can thus rewrite the above sum as

$$\begin{aligned}\int_C & \alpha(r(gg_0), \ldots, r(gg_n)) d\mu(g) \\ &= \sum_{\gamma_i \in F_{CC_i}} \alpha(\gamma_0, \ldots, \gamma_n) \mu(C \cap \gamma_0 D g_0^{-1} \cap \cdots \cap \gamma_n D g_n^{-1}),\end{aligned}$$

for any $(g_0, \ldots, g_n) \in C_0 \times \ldots C_n$.

The point (2) of Proposition follows since if $\alpha_n \to \alpha$ with pointwise convergence and $\sup_n \|\alpha_n\|_\infty < +\infty$ then $T\alpha_n \to T\alpha$ uniformly on compact sets.

Finally, we show (1) by showing that the function

$$(g_0, \ldots, g_n) \mapsto \mu(\{C \cap \gamma_0 D g_0^- 1 \cap \cdots \cap \gamma_n D g_n^{-1}\})$$

is continuous. To estimate the difference

$$\mu(C \cap \bigcap_{i=1}^n \gamma_i D g_i^{-1}) - \mu(C \cap \bigcap_{i=1}^n \gamma_i D h_i^{-1})$$

we introduce the notation

$$A(x_0, \ldots, x_n) := C \cap \bigcap_{i=0}^n \gamma_i D x_i^{-1},$$

for any $x_0, \ldots, x_n \in G$. The above difference thus becomes

$$\mu(A(g_0, \ldots, g_n)) - \mu(A(h_0, \ldots, h_n))$$

which we rewrite as a telescopic sum

$$\sum_{i=0}^n (\mu(A(h_0, \ldots, h_{i-1}, g_i, g_{i+1}, \ldots g_n)) - \mu(A(h_0, \ldots, h_{i-1}, h_i, g_{i+1}, \ldots g_n))) .$$

Setting

$$B_j := C \cap \bigcap_{\ell=0}^{i-1} \gamma_\ell D h_\ell^{-1} \cap \bigcap_{\ell=i+1}^{n} \gamma_\ell D g_\ell^{-1},$$

the telescopic sum becomes

$$\sum_{j=0}^n (\mu(B_j \cap \gamma_j D g_j^{-1}) - \mu(B_j \cap \gamma_j D h_j^{-1})) .$$

Using the simple set theoretical inequality valid for any sets B, E, E'

$$|\mu(B \cap E) - \mu(B \cap E')| \leq \mu((B \cap E)\Delta(B \cap E')) \leq \mu(E \Delta E'),$$

we obtain for each summand the estimate

$$|\mu(B_j \cap \gamma_j D g_j^{-1}) - \mu(B_j \cap \gamma_j D h_j^{-1})| \leq \mu(\gamma_j D g_j^{-1} \Delta \gamma_j D h_j^{-1}) = \|\chi_{D g_j^{-1} h_j} - \chi_D\|_1 .$$

Thus

$$\begin{aligned} &|\mu(C \cap \bigcap_{i=1}^{n} \gamma_i D g_i^{-1}) - \mu(C \cap \bigcap_{i=1}^{n} \gamma_i D h_i^{-1})| \\ &\leq \sum_{j=0}^{n} |\mu(B_j \cap \gamma_j D g_j^{-1}) - \mu(B_j \cap \gamma_j D h_j^{-1})| \\ &\leq \sum_{j=0}^{n} \|\chi_{D g_j^{-1} h_j} - \chi_D\|_1 . \end{aligned}$$

The continuity of the right regular action of G on $L^1(G)$ concludes the proof of the proposition. □

Proof of Proposition A.1 Consider Γ as a lattice in the full isometry group $\mathrm{Isom}(\mathbb{H}^n)$ and denote by $\varepsilon \colon \mathrm{Isom}(\mathbb{H}^n) \to \{-1, +1\}$ the homomorphism sending an isometry to $+1$ if it preserves orientation and -1 otherwise. By what precedes, the cohomology class $\mathrm{transf}(\rho^*(\omega_{\mathbb{H}^n}))$ can be represented by the continuous cocycle sending $(g_0, \ldots, g_n) \in \mathrm{Isom}(\mathbb{H}^n)^{n+1}$ to

$$\int_{\Gamma\backslash \mathrm{Isom}(\mathbb{H}^n)} \varepsilon(g)\omega_n(\rho(r(gg_0)), \ldots, \rho(r(gg_n)))d\mu(g) .$$

Note that the cocycle stays continues after transferring from $\mathrm{Isom}^+(\mathbb{H}^n)$ to $\mathrm{Isom}(\mathbb{H}^n)$. Integrating over a maximal compact subgroup K in $\mathrm{Isom}(\mathbb{H}^n)$ we obtain a continuous cocycle $(\mathbb{H}^n)^{n+1} \to \mathbb{R}$ that sends an $(n+1)$-tuple of points $g_0K, \ldots, g_nK \in \mathrm{Isom}(\mathbb{H}^n)/K \cong \mathbb{H}^n$ to

$$\int_{K^{n+1}} \prod_{i=0}^{n} dk_i \int_{\Gamma\backslash \mathrm{Isom}(\mathbb{H}^n)} \varepsilon(g)\omega_n(\rho(r(gg_0k_0)), \ldots, \rho(r(gg_nk_n)))d\mu(g) . \qquad (6.4)$$

We showed in [3, Proposition 3.3] that

$$\mathrm{transf}(\rho^*(\omega_n)) = \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)} \cdot \omega_{\mathbb{H}^n} \in \mathrm{H}^n_{\mathrm{cb}}(\mathrm{Isom}(\mathbb{H}^n), \mathbb{R}_\varepsilon) .$$

Since there are no coboundaries in degree n for $\mathrm{Isom}(\mathbb{H}^n)$-equivariant continuous bounded cochains on $\mathbb{H}^n$, this implies that we have a strict equality between (6.4) and

$$\frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)} \cdot \omega_n(g_0K, \ldots, g_nK) .$$

Since (6.4) varies continuously in ρ, so does $\mathrm{Vol}(\rho)$. ◻

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