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Moduli spaces of twisted K3 surfaces and cubic fourfolds

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Abstract

Motivated by the relation between (twisted) K3 surfaces and special cubic fourfolds, we construct moduli spaces of polarized twisted K3 surfaces of any fixed degree and order. We do this by mimicking the construction of the moduli space of untwisted polarized K3 surfaces as a quotient of a bounded symmetric domain.

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Introduction

A twisted K3 surface is a pair (S, α) consisting of a K3 surface S and a Brauer class α on S. Using the isomorphism $\mathrm{Br}(S) \cong \mathrm{H}^2(S, \mathscr{O}_S^*)_{\mathrm{tors}}$, twisted K3 surfaces can be seen as a degree two version of polarized K3 surfaces. We may also view them from the perspective of Hitchin's generalized K3 surfaces [11], using α to change the volume form on S. This gives us a generalized Calabi–Yau structure, to which we associate a Hodge structure $\widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z})$ of K3 type on the full cohomology of S [17]. In this way, we can view (S, α) as a geometric realization of a point in the extended period domain for K3 surfaces.

This paper is concerned with polarized twisted K3 surfaces, that is, K3 surfaces together with a Brauer class and a primitive ample class in $\mathrm{H}^2(S, \mathbb{Z})$. Our first goal is to construct a moduli space of these objects, fixing the degree of the polarization and

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the order of the Brauer class. This can be done up to the following concession: when $\rho(S) > 1$, one parametrizes lifts of Brauer classes to $\mathrm{H}^2(S, \mathbb{Q})$, which gives a strictly bigger group than $\mathrm{Br}(S)$.

Theorem 1 (Def. 2.1, Prop. 2.4) *There exists a scheme* $\mathrm{M}_d[r]$ *which is a coarse moduli space for triples* (S, L, α) *where* S *is a K3 surface,* $L \in \mathrm{H}^2(S, \mathbb{Z})$ *is a polarization of degree* $(L)^2 = d$ *and* α *is an element of* $\mathrm{Hom}(\mathrm{H}^2(S, \mathbb{Z})_{\mathrm{pr}}, \mathbb{Z}/r\mathbb{Z})$*. This group has a surjection to* $\mathrm{Br}(S)[r]$*, which is an isomorphism if and only if* $\rho(S) = 1$.

We prove this by mimicking the construction of the moduli space of (untwisted) polarized K3 surfaces via the period domain. In particular, $\mathrm{M}_d[r]$ is a quasi-projective variety with at most finite quotient singularities, whose number of connected components can be bounded in terms of d and r (Proposition 2.5).

In the second part of the paper, we will concentrate on a Hodge-theoretic relation between twisted K3 surfaces and special cubic fourfolds. For untwisted K3 surfaces, this relation was first studied by Hassett [10]. He also constructed, for d satisfying a numerical condition $(**)$, rational maps

$$\mathrm{M}_d \dashrightarrow \mathcal{C}_d$$

from the moduli space of polarized K3 surfaces of degree d to the moduli space of special cubic fourfolds of discriminant d, sending a K3 surface to the cubic it is associated to.

Associated twisted K3 surfaces were studied by Huybrechts in [16], extending results of [1]. The numerical condition on the discriminant given by Huybrechts can be formulated as follows:

$$(**') \quad d' = dr^2 \text{ for some integers } d \text{ and } r, \text{ where } d \text{ satisfies } (**).$$

We give a full generalization of Hassett's results to the setting of twisted K3 surfaces.

Theorem 2 (Cor. 4.2) *A cubic fourfold* X *is in* $\mathcal{C}_{d'}$ *for some* d' *satisfying* $(**')$ *if and only if for every decomposition* $d' = dr^2$ *with* d *satisfying* $(**)$*,* X *has an associated polarized twisted K3 surface of degree* d *and order* r*.*

We also give the analogous construction of Hassett's rational maps to $\mathcal{C}_d$. Just like for untwisted K3 surfaces, these maps are either birational or of degree two. We end with a discussion of the covering involution in the degree two case, relating this paper to [6].

0.1 Notation

For basics on lattices, see e.g. [15, Chapter 14].

- U is the rank two lattice with intersection matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.
- E_8 is the unique positive definite even unimodular lattice of rank eight.
- $\Lambda := E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$ is the lattice isomorphic to the second cohomology $\mathrm{H}^2(S, \mathbb{Z})$ of a K3 surface S.

- $\widetilde{\mathrm{H}}(S,\mathbb{Z})$ is the full cohomology of S with the Mukai pairing, viewed as an ungraded ring.
- $\widetilde{\Lambda}:=\Lambda\oplus U$ is the lattice isomorphic to $\widetilde{\mathrm{H}}(S,\mathbb{Z})$.
- $\Lambda_d\subset\Lambda$ is the orthogonal complement of a primitive element $\ell_d\in\Lambda$ of square d, which is unique up to $\mathrm{O}(\Lambda)$.
- $\Lambda^\vee_{d,r}:=(\frac{1}{r}\Lambda^\vee_d)/\Lambda^\vee_d\cong\Lambda^\vee_d\otimes\mathbb{Z}/r\mathbb{Z}\cong\mathrm{Hom}(\Lambda_d,\mathbb{Z}/r\mathbb{Z})$.
- $\widetilde{\mathrm{O}}(\Lambda_d):=\mathrm{Ker}(\mathrm{O}(\Lambda_d)\to\mathrm{O}(\mathrm{Disc}\,\Lambda_d))$. This group acts naturally on $\Lambda^\vee_{d,r}$.
- For an isomorphism $\varphi\colon L\to L'$ of lattices, φ_r is the induced map $L^\vee\otimes\mathbb{Z}/r\mathbb{Z}\to(L')^\vee\otimes\mathbb{Z}/r\mathbb{Z}$.
- For a lattice L of signature (n_+,n_-) with $n_+\geq 2$, $\mathscr{D}(L)$ is the period domain $\{x\in\mathbb{P}(L\otimes\mathbb{C})\mid(x)^2=0,\ (x,\bar{x})>0\}$.
- $G[r]$ is the r-torsion subgroup of an abelian group G.
- Cohomology with coefficients in $\mathbb{G}_m$ means étale cohomology. Otherwise we always use the analytic topology.
- $\mathscr{M}_d$ is the moduli functor for polarized K3 surfaces of degree d, and $\Phi:\mathscr{M}_d\to\mathrm{M}_d$ is the associated coarse moduli space.

Remark 0.1 By a moduli functor $\mathscr{M}$, we will mean a functor on the category of schemes of finite type over $\mathrm{Spec}\,\mathbb{C}$. A coarse moduli space for $\mathscr{M}$ is a scheme M with a morphism $\xi:\mathscr{M}\to\mathrm{M}$ such that $\xi(\mathbb{C})$ is a bijection, and we have factorization over M of morphisms $\mathscr{M}\to T$ for T any $\mathbb{C}$-scheme of finite type.

1 Twisted K3 surfaces

1.1 Definitions

For references, see [14,13]. Recall that the *Brauer group* $\mathrm{Br}(X)$ of a scheme X is the group of sheaves of Azumaya algebras modulo Morita equivalence, with multiplication given by the tensor product. If X is quasi-compact and separated and has an ample line bundle, then $\mathrm{Br}(X)$ is isomorphic to the *cohomological Brauer group*

$$\mathrm{Br}(X)':=\mathrm{H}^2(X,\mathbb{G}_m)_{\mathrm{tors}},$$

which equals $\mathrm{H}^2(X,\mathbb{G}_m)$ when X is regular and integral. If X is a complex K3 surface, one can moreover show that

$$\mathrm{Br}(X)\cong\mathrm{H}^2(X,\mathscr{O}_X^*)_{\mathrm{tors}}\cong(\mathbb{Q}/\mathbb{Z})^{22-\rho(X)}.$$

A *twisted K3 surface* is a pair (S,α) where S is a K3 surface and $\alpha\in\mathrm{Br}(S)$. Two twisted K3 surfaces (S,α) and (S',α') are *isomorphic* if there exists an isomorphism $f\colon S\to S'$ such that $f^*\alpha'=\alpha$.

The exponential sequence on S induces the following exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{NS}(S) \arrow[r] & \mathrm{H}^2(S, \mathbb{Z}) \arrow[r] \arrow[d, phantom, "\cap"] & \mathrm{H}^2(S, \mathscr{O}_S) \arrow[r, "\exp"] \arrow[d, phantom, "\|\wr"] & \mathrm{H}^2(S, \mathscr{O}_S^*) \arrow[r] & 0 \\
 & & \mathrm{H}^2(S, \mathbb{C}) \arrow[r, "="'] & \mathrm{H}^{0,2}(S) \oplus \mathrm{H}^{1,1}(S) \oplus \mathrm{H}^{2,0}(S) & &
\end{tikzcd}$$

It follows that any Brauer class $\alpha \in \mathrm{H}^2(S, \mathscr{O}_S^*)_{\mathrm{tors}}$ is of the form $\exp(B^{0,2})$ for some $B \in \mathrm{H}^2(S, \mathbb{Q})$, unique up to $\mathrm{H}^2(S, \mathbb{Z})$ and $\mathrm{NS}(S) \otimes \mathbb{Q}$. Thus, denoting by $T(S)$ the transcendental lattice of S, intersecting with B gives a linear map $f_\alpha = (B, -) \colon T(S) \to \mathbb{Q}/\mathbb{Z}$ which only depends on α. One can show that $\alpha \mapsto f_\alpha$ yields an isomorphism $\mathrm{Br}(S) \cong \mathrm{Hom}(T(S), \mathbb{Q}/\mathbb{Z})$.

Given a lift $B \in \mathrm{H}^2(S, \mathbb{Q})$ of α, we define a weight two Hodge structure of K3 type $\widetilde{\mathrm{H}}(S, B, \mathbb{Z})$ on the full cohomology of S by

$$\widetilde{\mathrm{H}}^{2,0}(S, B) := \mathbb{C}[\exp(B)\sigma] \subset \widetilde{\mathrm{H}}(S, \mathbb{C}),$$

where σ is a nowhere degenerate holomorphic 2-form on S and $\exp(B)\sigma := \sigma + B \wedge \sigma$. This Hodge structure does not depend on our choice of B (up to non-canonical isomorphism [17, Section 2]), so we can define

$$\widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z}) := \widetilde{\mathrm{H}}(S, B, \mathbb{Z})$$

for any $B \in \mathrm{H}^2(X, \mathbb{Q})$ with $\exp(B^{0,2}) = \alpha$. When α is trivial, this gives back the usual Hodge structure on $\mathrm{H}^*(S, \mathbb{Z})$.

The *Picard group* of (S, α) is defined as $\widetilde{\mathrm{H}}^{1,1}(S, \alpha) \cap \widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z})$, so

$$\mathrm{Pic}(S, \alpha) = \{\delta \mid (\delta, \exp(B)\sigma) = 0\} \subset \widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z})$$

for $B \in \mathrm{H}^2(S, \mathbb{Q})$ lifting α. If α is trivial, then $\mathrm{Pic}(S, \alpha) = \mathrm{H}^0(S, \mathbb{Z}) \oplus \mathrm{Pic}(S) \oplus \mathrm{H}^4(S, \mathbb{Z})$. The *transcendental lattice* $T(S, \alpha)$ is defined as the orthogonal complement of $\mathrm{Pic}(S, \alpha)$ in $\widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z})$. If α is trivial, then $T(S, \alpha)$ is the transcendental lattice $T(S)$ of S. One can show that $T(S, \alpha)$ is isometric, as an abstract lattice, to

$$\mathrm{Ker}(f_\alpha \colon T(S) \to \mathbb{Q}/\mathbb{Z}) = \{x \in T(S) \mid (B, x) \in \mathbb{Z}\}.$$

Definition 1.1 A *polarized twisted K3 surface* is a triple (S, L, α), where S is a K3 surface, $L \in \mathrm{H}^2(S, \mathbb{Z})$ is a primitive ample class and $\alpha \in \mathrm{Br}(S)$. Two twisted polarized K3 surfaces (S, L, α) and (S', L', α') are *isomorphic* if there exists an isomorphism $f \colon S \to S'$ such that $f^*L' = L$ and $f^*\alpha' = \alpha$.

We define two invariants of (S, L, α): its *degree* $d = (L)^2$ and its *order* $r = \mathrm{ord}(\alpha)$ (also known as its *period*).

1.2 A non-existence result for moduli spaces

Ideally, one would like to find a (coarse) moduli space $\mathrm{N}_d[r]$ for the following functor:

$$\mathscr{N}_d[r]\colon (Sch/\mathbb{C})^o \to (Sets), T \mapsto \{(f\colon S \to T, L, \alpha)\}/\cong .$$

Here, $(f : S \to T, L \in \mathrm{H}^0(T, R^1 f_*\mathbb{G}_m)) \in \mathscr{M}_d(T)$ is a smooth proper family of polarized K3 surfaces of degree d and $\alpha \in \mathrm{H}^0(T, R^2 f_*\mathbb{G}_m)$ such that for any closed point $x \in T$, base change gives a Brauer class $\alpha_x \in \mathrm{H}^2(S_x, \mathbb{G}_m)[r]$.

It is, however, not difficult to show that $\mathrm{N}_d[r]$ does not exist as a locally Noetherian scheme. Namely, suppose $\mathscr{N}_d[r] \to \mathrm{N}_d[r]$ exists. Consider the natural transformation $\xi\colon \mathscr{N}_d[r] \to \mathscr{M}_d$ which at a scheme T is defined by $(S \to T, L, \alpha) \mapsto (S \to T, L)$. By the properties of a coarse moduli space, there exists a unique morphism $\pi\colon\ \mathrm{N}_d[r] \to \mathrm{M}_d$ which makes the following diagram commute:

$$\begin{tikzcd}
\mathscr{N}_d[r] \arrow[r] \arrow[d, "\xi"'] & \mathrm{N}_d[r] \arrow[d, dashed, "\exists\pi"] \\
\mathscr{M}_d \arrow[r, "\Phi"'] & \mathrm{M}_d .
\end{tikzcd}$$

For a closed point $y \in \mathrm{N}_d[r]$ corresponding to a tuple (S, L, α), the image $\pi(y)$ should be the point x of M_d corresponding to (S, L). So the fibre of π over x is

$$(\mathrm{N}_d[r])_x = \{(S, L, \alpha) \mid \alpha \in \mathrm{Br}(S)[r]\}/\operatorname{Aut}(S, L).$$

For $d > 2$, let $U \subset \mathrm{M}_d$ be the open subset where $\operatorname{Aut}(S, L)$ is trivial. Over U, we have $(\mathrm{N}_d[r])_x \cong \mathrm{Br}(S)[r] \cong (\mathbb{Z}/r\mathbb{Z})^{22-\rho(S)}$. In particular, $\pi|_{\mathrm{N}_d[r]\times_{\mathrm{M}_d} U}$ is ramified exactly over the locus where $\rho(S) > 1$. Now this set is dense in U, thus not Zariski closed, giving a contradiction.

For $d = 2$, let $U \subset \mathrm{M}_2$ be the open subset where $\operatorname{Aut}(S, L) \cong \mathbb{Z}/2\mathbb{Z}$. Then over U, we have $2^{21-\rho(S)} \leq |(\mathrm{N}_2[r])_x| \leq 2^{22-\rho(S)}$. So $\pi|_{\mathrm{N}_2^r \times_{\mathrm{M}_2} U}$ is ramified (at least) over the locus where $\rho(S) > 2$, again a dense set in U, which leads to a contradiction.

When requiring that α has order r (on each connected component of T), non-existence is proven similarly. One obtains a morphism π to M_d such that over an open subset $U \subset \mathrm{M}_d$, the cardinality of the fibre of π over $(S, L) \in U$ is the number of elements of order r in $(\mathbb{Z}/r\mathbb{Z})^{22-\rho(S)}$ (or half this number when $d = 2$). Again, $\pi|_{\pi^{-1}(U)}$ is ramified exactly over the locus where $\rho(S) > 1$ (at least over the locus where $\rho(S) > 2$ when $d = 2$), a contradiction.

2 Moduli spaces of polarized twisted K3 surfaces

We will construct a slightly different moduli space $\mathrm{M}_d[r]$ mapping to M_d, whose fibre over $(S, L) \in \mathrm{M}_d$ parametrizes triples (S, L, α) with $\alpha \in \operatorname{Hom}(\mathrm{H}^2(S, \mathbb{Z})_{\mathrm{pr}}, \mathbb{Z}/r\mathbb{Z})$.

There is a surjective homomorphism from this group to $\mathrm{Br}(S)[r]$, which is an isomorphism if and only if $\rho(S) = 1$.

2.1 Definition of the moduli functor

The Kummer sequence $0 \to \mu_r \to \mathbb{G}_m \xrightarrow{(\cdot)^r} \mathbb{G}_m \to 0$ induces a short exact sequence

$$0 \to \mathrm{Pic}(S) \otimes \mathbb{Z}/r\mathbb{Z} \to \mathrm{H}^2(S, \mu_r) \to \mathrm{Br}(S)[r] \to 0.$$

If $L \in \mathrm{H}^2(S, \mathbb{Z})$ is a polarization, we have injections

$$\mathbb{Z}/r\mathbb{Z} \cdot L \hookrightarrow \mathrm{Pic}\, S \otimes \mathbb{Z}/r\mathbb{Z} \hookrightarrow \mathrm{H}^2(S, \mathbb{Z}/r\mathbb{Z}).$$

Hence, we get a surjective map

$$\begin{tikzcd}
\mathrm{H}^2(S, \mathbb{Z}/r\mathbb{Z})/(\mathbb{Z}/r\mathbb{Z} \cdot L) \arrow[r, two heads] \arrow[d, phantom, "\parallel"] & \mathrm{H}^2(S, \mathbb{Z}/r\mathbb{Z})/(\mathrm{Pic}\, S \otimes \mathbb{Z}/r\mathbb{Z}) \cong \mathrm{Br}(S)[r] \arrow[d, phantom, "\parallel"] \\
\mathrm{H}^2(S, \mathbb{Z})^\vee_{\mathrm{pr}} \otimes \mathbb{Z}/r\mathbb{Z} & T(S)^\vee \otimes \mathbb{Z}/r\mathbb{Z}
\end{tikzcd}$$

which is an isomorphism if and only if $\rho(S) = 1$.

We define a relative version of $\mathrm{H}^2(S, \mathbb{Z})^\vee_{\mathrm{pr}} \otimes \mathbb{Z}/r\mathbb{Z} \cong \mathrm{Hom}(\mathrm{H}^2(S, \mathbb{Z})_{\mathrm{pr}}, \mathbb{Z}/r\mathbb{Z})$ as follows. For a smooth proper family $(f\colon S \to T, L)$ of polarized K3 surfaces, set

$$R^2_{\mathrm{pr}} f_*\mathbb{Z} := \mathrm{Ker}\left(R^2 f_*\mathbb{Z} \xrightarrow{\cdot c_1(L)} R^4 f_*\mathbb{Z}\right)$$

where $c_1(L)$ is the image of L under $\mathrm{H}^0(T, R^1 f_*\mathbb{G}_m) \to \mathrm{H}^0(T, R^2 f_*\mathbb{Z})$. Let $\mathscr{F}[r]$ be the following local system:

$$\mathscr{F}[r] := \mathscr{H}om_{\mathrm{Ab}}(R^2_{\mathrm{pr}} f_*\mathbb{Z}, \underline{\mathbb{Z}/r\mathbb{Z}})$$

where $\mathscr{H}om_{\mathrm{Ab}}$ means morphisms of sheaves of abelian groups.

Definition 2.1 The moduli functor $\mathscr{M}_d[r]$ is defined as

$$\mathscr{M}_d[r]\colon (Sch/\mathbb{C})^o \to (Sets),\ T \mapsto \{(f\colon S \to T, L, \alpha)\}/_{\cong}$$

where $\left(f : S \to T, L \in \mathrm{H}^0(T, R^1 f_*\mathbb{G}_m)\right)$ is a smooth proper family of polarized K3 surfaces of degree d and $\alpha \in \mathrm{H}^0(T, \mathscr{F}[r])$. We define

$$\mathscr{M}^r_d\colon (Sch/\mathbb{C})^o \to (Sets)$$

to be the subfunctor sending a scheme T to the set of those tuples (f, L, α) for which α has order r on each connected component of T.

We will construct coarse moduli spaces for $\mathscr{M}_d[r]$ and $\mathscr{M}^r_d$.

2.2 Construction of the moduli space

We recall the construction of M_d as a subvariety of a quotient of a bounded symmetric domain (see e.g. [15]). The primitive cohomology $\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{pr}}$ of a degree-d polarized K3 surface only depends, as a lattice, on d, and is isomorphic to Λ_d. The moduli functor $\mathscr{M}_d^{\mathrm{mar}}$ of marked polarized K3 surfaces of degree d is given by

$$\mathscr{M}_d^{\mathrm{mar}}(T) = \{(f\colon S \to T, L \in \mathrm{H}^0(T, R^1 f_*\mathbb{G}_m), \varphi\colon R^2_{\mathrm{pr}} f_*\mathbb{Z} \cong \underline{\Lambda_d})\}/\cong,$$

where $(f\colon S \to T, L) \in \mathscr{M}_d(T)$. It has an analytic fine moduli space $\mathrm{M}_d^{\mathrm{mar}}$, which can be constructed as an open submanifold of the period domain $\mathscr{D}(\Lambda_d)$ of Λ_d. In particular, there exists a universal family

$$\left(f\colon S^{\mathrm{mar}} \to \mathrm{M}_d^{\mathrm{mar}}, L^{\mathrm{mar}}, \varphi^{\mathrm{mar}}\right).$$

We denote the morphism $\mathscr{M}_d^{\mathrm{mar}} \to \mathrm{M}_d^{\mathrm{mar}}$ by Φ^{mar}. The moduli space M_d is obtained from $\mathrm{M}_d^{\mathrm{mar}}$ by taking the quotient under the action of $\widetilde{\mathrm{O}}(\Lambda_d)$.

Note that φ^{mar} induces an isomorphism $\varphi_r^{\mathrm{mar}}\colon \mathscr{F}[r] \cong \underline{\mathrm{Hom}(\Lambda_d, \mathbb{Z}/r\mathbb{Z})} = \underline{\Lambda_{d,r}^{\vee}}$. Thus, $\mathscr{F}[r]$ is the sheaf of sections of the trivial finite cover

$$\begin{aligned}\mathrm{M}_d^{\mathrm{mar}}[r] &:= \underline{\mathrm{Spec}}\ \mathscr{H}om_{\mathrm{Sets}}(\underline{\Lambda_{d,r}^{\vee}}, \mathscr{O}_{\mathrm{M}_d^{\mathrm{mar}}}) \\ &= \mathrm{M}_d^{\mathrm{mar}} \times \Lambda_{d,r}^{\vee},\end{aligned}$$

where $\mathscr{H}om_{\mathrm{Sets}}$ means morphisms of sheaves of sets. The space $\mathrm{M}_d^{\mathrm{mar}}[r]$ is a coarse moduli space for the functor

$$\mathscr{M}_d^{\mathrm{mar}}[r]\colon (Sch/\mathbb{C})^o \to (Sets),\ T \mapsto \{(f\colon S \to T, L, \varphi, \alpha)\}/\cong$$

where $(f\colon S \to T, L, \varphi) \in \mathscr{M}_d^{\mathrm{mar}}(T)$ and $\alpha \in \mathrm{H}^0(T, \mathscr{F}[r])$. Namely, let

$$\Phi^{\mathrm{mar}}[r]\colon \mathscr{M}_d^{\mathrm{mar}}[r] \to \mathrm{M}_d^{\mathrm{mar}}[r]$$

be the morphism defined over a connected scheme T by

$$(S \to T, L, \varphi, \alpha) \mapsto (\Phi^{\mathrm{mar}}(S \to T, L, \varphi), \varphi_r(\alpha)),$$

so we have a commutative diagram

$$\begin{tikzcd}
\mathscr{M}_d^{\mathrm{mar}} \arrow[r, "\Phi^{\mathrm{mar}}"] & \mathrm{M}_d^{\mathrm{mar}} \\
\mathscr{M}_d^{\mathrm{mar}}[r] \arrow[u] \arrow[r, "\Phi^{\mathrm{mar}}{[r]}"'] & \mathrm{M}_d^{\mathrm{mar}}[r] \arrow[u]
\end{tikzcd}$$

Then $\Phi_d^{\mathrm{mar}}[r]$ is a bijection over $\operatorname{Spec}\mathbb{C}$. Moreover, suppose we have a map G from $\mathscr{M}_d^{\mathrm{mar}}[r]$ to a $\mathbb{C}$-scheme X. For any $\alpha \in \Lambda_{d,r}^{\vee}$, there is a map $G_\alpha \colon \mathscr{M}_d^{\mathrm{mar}} \to X$ defined over a connected scheme T by $(S \to T, L, \varphi) \mapsto G(S \to T, L, \varphi, \varphi_r^{-1}(\alpha))$. The G_α induce maps $g_\alpha \colon \mathrm{M}_d^{\mathrm{mar}} \to X$, which combine to a morphism $g \colon \mathrm{M}_d^{\mathrm{mar}}[r] \to X$ satisfying $g \circ \Phi^{\mathrm{mar}}[r] = G$.

The action of $\widetilde{\mathrm{O}}(\Lambda_d)$ on $\mathrm{M}_d^{\mathrm{mar}}$ lifts to $\mathrm{M}_d^{\mathrm{mar}}[r]$ via

$$g(S, L, \varphi, \alpha) = \left(S, L, g \circ \varphi, \varphi_r^{-1} g \varphi_r(\alpha)\right).$$

Under $\mathrm{M}_d^{\mathrm{mar}}[r] \hookrightarrow \mathscr{D}(\Lambda_d) \times \Lambda_{d,r}^{\vee}$, this is the restriction of the natural action of $\widetilde{\mathrm{O}}(\Lambda_d)$ on $\mathscr{D}(\Lambda_d) \times \Lambda_{d,r}^{\vee}$. This action is properly discontinuous: it is on $\mathscr{D}(\Lambda_d)$ (see [15, Remark 6.1.10]), so also on the product with the finite group $\Lambda_{d,r}^{\vee}$. It follows that the quotient

$$\mathrm{M}_d[r] := \mathrm{M}_d^{\mathrm{mar}}[r] / \widetilde{\mathrm{O}}(\Lambda_d)$$

exists as a complex space. Similarly, let $\mathrm{M}_d^{\mathrm{mar},r} \subset \mathrm{M}_d^{\mathrm{mar}}[r]$ be the union of those components $\mathrm{M}_d^{\mathrm{mar}}[r] \times \{v\}$ for elements $v \in \Lambda_{d,r}^{\vee}$ of order r. Then the quotient

$$\mathrm{M}_d^r := \mathrm{M}_d^{\mathrm{mar},r} / \widetilde{\mathrm{O}}(\Lambda_d)$$

exists as a complex space.

We claim that $\mathrm{M}_d[r]$ and M_d^r are quasi-projective varieties. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{M}_d^{\mathrm{mar}}[r] = \mathrm{M}_d^{\mathrm{mar}} \times \Lambda_{d,r}^{\vee} \arrow[r, "\pi"] \arrow[d] & \Lambda_{d,r}^{\vee} \arrow[d] \\
\mathrm{M}_d[r] \arrow[r, "\bar{\pi}"] & \Lambda_{d,r}^{\vee} / \widetilde{\mathrm{O}}(\Lambda_d).
\end{tikzcd}$$

Giving the sets on the right side the discrete topology, all these maps are continuous. So under $\bar{\pi}$, each connected component of $\mathrm{M}_d[r]$ is mapped to a point. Vice versa, given $[w] \in \Lambda_{d,r}^{\vee}$, the inverse image of $\widetilde{\mathrm{O}}(\Lambda_d) \cdot [w] \in \Lambda_{d,r}^{\vee} / \widetilde{\mathrm{O}}(\Lambda_d)$ under $\bar{\pi}$ is

$$\mathrm{M}_w := \left(\mathrm{M}_d^{\mathrm{mar}} \times \widetilde{\mathrm{O}}(\Lambda_d) \cdot [w]\right) / \widetilde{\mathrm{O}}(\Lambda_d) \cong \left(\mathrm{M}_d^{\mathrm{mar}} \times \{[w]\}\right) / \operatorname{Stab}[w]$$

where $\operatorname{Stab}[w] \subset \widetilde{\mathrm{O}}(\Lambda_d)$ is the stabilizer of $[w]$ under the acion of $\widetilde{\mathrm{O}}(\Lambda_d)$ on $\Lambda_{d,r}^{\vee}$. Now $\operatorname{Stab}[w]$ contains the reflection s_δ for an element $\delta \in \Lambda_d$ of square -2 orthogonal to w and ℓ_d', which interchanges the two connected components of $\mathrm{M}_d^{\mathrm{mar}}$. Hence, M_w is connected; even irreducible. This shows that the connected components of $\mathrm{M}_d[r]$ are in one-to-one correspondence with $\Lambda_{d,r}^{\vee} / \widetilde{\mathrm{O}}(\Lambda_d)$. Each component M_w parametrizes triples (S, L, α) that admit a marking φ with $\varphi_r(\alpha) = [w]$. The components belonging to M_d^r are those M_w for which $[w]$ has order r.

Remark 2.2 Recall (see e.g. [15, Section 6.4.2]) that for ℓ large enough, there exists a fine moduli space $\mathrm{M}_d^{\mathrm{lev}}$ of polarized K3 surfaces (S, L) of degree d with a $\Lambda/\ell\Lambda$-*level structure*, i.e. an isometry $\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{pr}} \otimes \mathbb{Z}/\ell\mathbb{Z} \cong \Lambda_d \otimes \mathbb{Z}/\ell\mathbb{Z}$. The space $\mathrm{M}_d^{\mathrm{lev}}$ is a smooth quasi-projective variety which is a finite cover of M_d. We could have constructed $\mathrm{M}_d[r]$ as a quotient of $\mathrm{M}_d^{\mathrm{lev}} \times \Lambda_{d,r}^{\vee}$ instead, choosing ℓ to be a large enough multiple of r.

Corollary 2.3 *Every connected component of* $\mathrm{M}_d[r]$ *(and therefore of* M_d^r*) is an irreducible, quasi-projective variety with at most finite quotient singularities.*

Proof The finite index subgroup $\mathrm{Stab}\,[w] \subset \widetilde{\mathrm{O}}(\Lambda_d)$ being arithmetic, the quotient $\mathscr{D}(\Lambda_d)/\,\mathrm{Stab}\,[w]$ is a quasi-projective variety with finite quotient singularities, by [2] and [24, Lemma IV.7.2]. We will show that $\mathrm{M}_w = \big(\mathrm{M}_d^{\mathrm{mar}} \times \{[w]\}\big)/\,\mathrm{Stab}\,[w]$ is a Zariski open subset of it, using the same argument as for the algebraicity of the moduli space of untwisted polarized K3 surfaces (see e.g. [15, Section 6.4.1]).

Let ℓ be a large enough multiple of r such that there exists a fine moduli space $\mathrm{M}_d^{\mathrm{lev}}$ of polarized K3 surfaces with a $\Lambda/\ell\Lambda$-level structure, see Remark 2.2. For the universal family $\pi\colon S^{\mathrm{lev}} \to \mathrm{M}_d^{\mathrm{lev}}$, there exists a marking $R^2_{\mathrm{pr}}\pi_*\mathbb{Z} \otimes \underline{\mathbb{Z}/\ell\mathbb{Z}} \cong \underline{\Lambda_d \otimes \mathbb{Z}/\ell\mathbb{Z}}$. This induces a holomorphic map $\mathrm{M}_d^{\mathrm{lev}} \to \mathscr{D}(\Lambda_d)/\Gamma_\ell$, where

$$\Gamma_\ell = \{g \in \widetilde{\mathrm{O}}(\Lambda_d) \mid g \equiv \mathrm{id}\ \mathit{mod}\ \ell\} \subset \mathrm{Stab}\,[w].$$

The image of this map is $\mathrm{M}_d^{\mathrm{mar}}/\Gamma_\ell$. Dividing out further by $\mathrm{Stab}\,[w]$ yields a holomorphic map

$$\mathrm{M}_d^{\mathrm{lev}} \twoheadrightarrow \mathrm{M}_d^{\mathrm{mar}}/\,\mathrm{Stab}\,[w] \subset \mathscr{D}(\Lambda_d)/\,\mathrm{Stab}\,[w].$$

By a theorem of Borel [4] (and also [24, Lemma IV.7.2]), this map is algebraic, and therefore the image $\mathrm{M}_d^{\mathrm{mar}}/\,\mathrm{Stab}\,[w]$ is constructible. As it is also analytically open in $\mathscr{D}(\Lambda_d)/\,\mathrm{Stab}\,[w]$, it is Zariski open [9, Corollary XII.2.3]. □

One constructs a morphism $\Psi\colon \mathscr{M}_d[r] \to \mathrm{M}_d[r]$ in the following way. Consider a point $(f\colon S \to T, L, \alpha)$ in $\mathscr{M}_d[r](T)$. Proceeding as for untwisted polarized K3 surfaces, we pass to the (infinite) étale covering

$$\widetilde{T} := \mathrm{Isom}(R^2_{\mathrm{pr}} f_*\mathbb{Z}, \underline{\Lambda_d}) \xrightarrow{\eta} T,$$

which has a natural $\widetilde{\mathrm{O}}(\Lambda_d)$-action, satisfying $\widetilde{T}/\widetilde{\mathrm{O}}(\Lambda_d) \cong T$. Write $\widetilde{f}\colon \widetilde{S} \to \widetilde{T}$ for the pullback family. The local system $R^2_{\mathrm{pr}}\widetilde{f}_*\mathbb{Z}$ is trivial: there exists a canonical isomorphism $\varphi\colon R^2_{\mathrm{pr}}\widetilde{f}_*\mathbb{Z} \cong \underline{\Lambda_d}$. Now $\Phi^{\mathrm{mar}}[r]\big(\widetilde{S}, \eta^*L, \varphi, \eta^*\alpha\big)$ is an element of $\mathrm{M}_d^{\mathrm{mar}}[r](\widetilde{T})$, i.e. a holomorphic map $\widetilde{T} \to \mathrm{M}_d^{\mathrm{mar}}[r]$. This map is $\widetilde{\mathrm{O}}(\Lambda_d)$-equivariant, hence descends to a map $T \to \mathrm{M}_d[r]$. This map is algebraic by [4], thus defines a point in $\mathrm{M}_d[r](T)$. We let $\Psi(S \to T, L, \alpha)$ be this point.

Proposition 2.4 *The space* $\mathrm{M}_d[r]$ *is a coarse moduli space for the functor* $\mathscr{M}_d[r]$.

Proof By definition, there is a commutative diagram

$$\begin{tikzcd}
\mathscr{M}_d^{\mathrm{mar}}[r] \arrow[r, "\Phi^{\mathrm{mar}}[r]"] \arrow[d, "F"'] & \mathrm{M}_d^{\mathrm{mar}}[r] \arrow[d, "q"] \\
\mathscr{M}_d[r] \arrow[r, "\Psi"'] & \mathrm{M}_d[r]
\end{tikzcd}$$

where the map F forgets the marking and q is the quotient map. We need to show that $\Psi(\mathbb{C})\colon \mathscr{M}_d[r](\mathbb{C}) \to \mathrm{M}_d[r](\mathbb{C})$ is a bijection. For $x \in \mathrm{M}_d[r](\mathbb{C})$, let $y \in \mathrm{M}_d^{\mathrm{mar}}[r](\mathbb{C})$ such that $q(y) = x$. Set $\Psi(\mathbb{C})^{-1}(x){:=}F(\Phi^{\mathrm{mar}}[r](\mathbb{C})^{-1}(y))$; note that this does not depend on the choice of y. One checks that $\Psi(\mathbb{C})^{-1}$ defines a set-theoretic inverse to $\Psi(\mathbb{C})$.

For the universal property of Ψ, let $s\colon \mathscr{M}_d[r] \to T$ be a morphism to a finite type $\mathbb{C}$-scheme T. Then $s \circ F$ is a map from $\mathscr{M}_d^{\mathrm{mar}}[r]$ to T; since $\mathscr{M}_d^{\mathrm{mar}}[r] \to \mathrm{M}_d^{\mathrm{mar}}[r]$ is a coarse moduli space, this induces a unique holomorphic map $t\colon \mathrm{M}_d^{\mathrm{mar}}[r] \to T$ such that $t \circ \Phi^{\mathrm{mar}}[r] = s \circ F$. It follows from the uniqueness that t is equivariant, thus factors over a holomorphic map $\mathrm{M}_d[r] \to T$. We will show that this map is algebraic.

Like before, let ℓ be a large enough multiple of r such that there exists a fine moduli space $\mathrm{M}_d^{\mathrm{lev}}$ of K3 surfaces with a $\Lambda/\ell\Lambda$-level structure. The map $\mathrm{M}_d^{\mathrm{mar}}[r] \to T$ factors as

$$\mathrm{M}_d^{\mathrm{mar}}[r] \to \mathrm{M}_d^{\mathrm{lev}} \times \Lambda_{d,r}^{\vee} \to \mathrm{M}_d[r] \to T$$

(see Remark 2.2). The map $\mathrm{M}_d^{\mathrm{lev}} \times \Lambda_{d,r}^{\vee} \to T$ is algebraic and equivariant under the algebraic action of $\widetilde{\mathrm{O}}(\Lambda_d)$. The induced algebraic morphism $(\mathrm{M}_d^{\mathrm{lev}} \times \Lambda_{d,r}^{\vee})/\widetilde{\mathrm{O}}(\Lambda_d) \to T$ is the given map $\mathrm{M}_d[r] \to T$. □

The proof that M_d^r is a coarse moduli space for $\mathscr{M}_d^r$ is analogous.

Proposition 2.5 *The space* M_d^r *has at most* $r \cdot \gcd(r, d)$ *many connected components.*

This follows directly from the following lemma. Denote $\Lambda = E_8(-1)^{\oplus 2} \oplus U_1 \oplus U_2 \oplus U_3$. Let $\{e_i, f_i\}$ be the standard basis for the i-th copy of U. Fix $\ell_d{:=}e_3 + \frac{d}{2} f_3$ and $\ell_d'{:=}e_3 - \frac{d}{2} f_3$, so $\Lambda_d^{\vee} \cong E_8(-1)^{\oplus 2} \oplus U_1 \oplus U_2 \oplus \langle \frac{1}{d}\ell_d' \rangle$. For integers n, k, we let

$$w_{n,k}{:=}\tfrac{1}{r}(e_1 + n f_1 + \tfrac{k}{d}\ell_d') \in \tfrac{1}{r}\Lambda_d^{\vee}.$$

Lemma 2.6 *Every element of order* r *in* $\Lambda_{d,r}^{\vee}$ *is equivalent under the action of* $\widetilde{\mathrm{O}}(\Lambda_d)$ *to* $[w_{n,k}]$ *for some* $n, k \in \mathbb{Z}$. *Moreover, if* $n \equiv n' \bmod r$ *and* $k \equiv k' \bmod \gcd(r, d)$, *then* $[w_{n,k}]$ *and* $[w_{n',k'}]$ *are equivalent.*

Proof Elements in $\Lambda_{d,r}^{\vee}$ of order r are of the form $m[\frac{1}{r}x]$ where $\gcd(m, r) = 1$ and $x \in \Lambda_d^{\vee}$ is primitive, so $x = sy + \frac{t}{d}\ell_d'$ for some primitive $y \in E_8(-1)^{\oplus 2} \oplus U_1 \oplus U_2$

and integers s, t with $\gcd(s,t) = 1$. Write $d = d_0 \cdot \gcd(d,t)$ and $t = t_0 \cdot \gcd(d,t)$. Then $d_0 x = d_0 sy + t_0 \ell'_d \in \Lambda_d$ is primitive and

$$\begin{aligned}(d_0 x, \Lambda_d) &= \gcd\Big((d_0 sy, E_8(-1)^{\oplus 2} \oplus U_1 \oplus U_2), (t_0 \ell'_d, \mathbb{Z}\ell'_d)\Big)\\ &= \gcd(d_0 s, d t_0)\\ &= d_0.\end{aligned}$$

By Eichler's criterion [8, Proposition 3.3], $d_0 x$ is equivalent under $\widetilde{\mathrm{O}}(\Lambda_d)$ to $d_0(e_1 + nf_1) + t_0 \ell'_d$ for some n. So $\frac{1}{r}x$ is equivalent to $\frac{1}{r}(e_1 + nf_1 + \frac{t}{d}\ell'_d) = w_{n,t}$.

Now $\frac{m}{r}x \equiv m w_{n,t}$ is equivalent modulo $\Lambda_d^\vee$ to $\frac{1}{r}(me_1 + (mn+r)f_1 + \frac{mt}{d}\ell'_d)$. As $\gcd(r,m) = 1$, the element $y = me_1 + (mn+r)f_1 + \frac{mt}{d}\ell'_d \in \Lambda_d^\vee$ is primitive, so by the above, $\frac{1}{r}y$ is equivalent under $\widetilde{\mathrm{O}}(\Lambda_d)$ to some $w_{n',t'}$. It follows that $m[\frac{1}{r}x] \in \Lambda_{d,r}^\vee$ is equivalent to $[w_{n',t'}]$.

Next, note that if $t' \equiv t \bmod d$, then $w_{n,t}$ is equivalent to $w_{n',t'}$ for some n' (by Eichler's criterion). In particular, writing $\gcd(r,d) = pr + qd$, the class

$$[w_{n,\gcd(r,d)+t}] = [\tfrac{1}{r}(e_1 + nf_1 + (pr + qd + t)\tfrac{1}{d}\ell'_d)] = [\tfrac{1}{r}(e_1 + nf_1 + (qd+t)\tfrac{1}{d}\ell'_d)]$$

in $\Lambda_{d,r}^\vee$ is equivalent to $[\frac{1}{r}(e_1 + n'f_1 + \frac{t}{d}\ell'_d)] = [w_{n',t}]$ for some n'. This shows that every $[w_{n,k}]$ is equivalent to some $[w_{n',k'}]$ with $0 \le n' < r$ and $0 \le k' < \gcd(r,d)$. □

3 Period maps

We show how to construct period maps on the connected components of M_d^r, which will be an important ingredient in relating twisted K3 surfaces to cubic fourfolds in Sect. 4.

3.1 Construction

We have seen that the connected components of M_d^r are of the form

$$\mathrm{M}_w = (\mathrm{M}_d^{\mathrm{mar}} \times \{[w]\}) / \operatorname{Stab}[w]$$

for $[w] \in \Lambda_{d,r}^\vee$ of order r. We will construct a period map from $\mathrm{M}_d^{\mathrm{mar}} \times \{[w]\}$ to the period domain $\mathscr{D}(T_w)$ of the lattice

$$T_w := \operatorname{Ker}((w, -) \colon \Lambda_d \to \mathbb{Q}/\mathbb{Z}).$$

Let $(S, L, \varphi, [w]) \in \mathrm{M}_d^{\mathrm{mar}} \times \{[w]\}$. The corresponding twisted Hodge structure $\widetilde{\mathrm{H}}(S, [w], \mathbb{Z})$ on S is given as follows. Let

$$w' = \varphi_{\mathbb{Q}}^{-1}(w) \in \tfrac{1}{r}\,\mathrm{H}^2(S, \mathbb{Z})_{\mathrm{pr}}^\vee \subset \mathrm{H}^2(S, \mathbb{Q}).$$

Then $\widetilde{\mathrm{H}}^{2,0}(S,[w])$ is $\mathbb{C}[\sigma + w' \wedge \sigma]$, where σ is a non-degenerate holomorphic 2-form on S. Let $\widetilde{\Lambda} = \Lambda \oplus U_4$ be the extended K3 lattice. We can extend φ to an isomorphism $\widetilde{\varphi}\colon \widetilde{\mathrm{H}}(S,\mathbb{Z}) \to \widetilde{\Lambda}$ by sending $1 \in \mathrm{H}^0(S,\mathbb{Z})$ to $e_4 \in U_4$ and $1 \in \mathrm{H}^4(S,\mathbb{Z})$ to $f_4 \in U_4$. Then

$$\widetilde{\varphi}(\sigma + w' \wedge \sigma) = \varphi(\sigma) + (w, \varphi(\sigma)) f_4.$$

Recall that for an even lattice N and $B \in N$, the *B-field shift* $\exp(B) \in \mathrm{O}(N \oplus U)$ is defined by

$$z \mapsto z - (B.z)f, \quad e \mapsto e + B - \tfrac{(B)^2}{2} f, \quad f \mapsto f$$

for $z \in N$, where $\{e, f\}$ is the standard basis of the hyperbolic plane U. For $B \in N_{\mathbb{Q}}$, we define $\exp(B) \in \mathrm{O}((N \oplus U)_{\mathbb{Q}})$ by linear extension. The discussion above shows that $\widetilde{\varphi}(\sigma + w' \wedge \sigma) = \exp(w)\varphi(\sigma)$ (note: $U \cong U(-1) = \langle e, -f\rangle$). We thus obtain a map

$$\mathscr{Q}_w\colon\ \mathrm{M}_d^{\mathrm{mar}} \times \{[w]\} \to \mathscr{D}\big((\exp(w)\Lambda_d) \cap \widetilde{\Lambda}\big)$$

sending $(S, L, \varphi, [w])$ to $[\widetilde{\varphi}(\widetilde{\mathrm{H}}^{2,0}(S,[w]))]$.

The above depends on the choice of a representative $w \in \frac{1}{r}\Lambda_d^\vee$ of $[w] \in \Lambda_{d,r}^\vee$. We can get rid of this choice in the following way. First, the lattice T_w is a finite index sublattice of Λ_d, so we have $\mathscr{D}(T_w) = \mathscr{D}(\Lambda_d)$. Second, note that the map $\exp(w)$ gives an isomorphism $T_w \cong (\exp(w)\Lambda_d) \cap \widetilde{\Lambda}$. We see that $\mathscr{Q}_w$ factors over the usual period map $\mathscr{P}$ for $\mathrm{M}_d^{\mathrm{mar}}$: the diagram

$$\begin{tikzcd}
\mathrm{M}_d^{\mathrm{mar}} \times \{[w]\} \cong \mathrm{M}_d^{\mathrm{mar}} \arrow[r, "\mathscr{P}"] \arrow[d, "\mathscr{Q}_w"'] & \mathscr{D}(\Lambda_d) \arrow[d, "="] \\
\mathscr{D}\big((\exp(w)\Lambda_d) \cap \widetilde{\Lambda}\big) \arrow[r, "\cong"'] & \mathscr{D}(T_w)
\end{tikzcd}$$

commutes. Denote by $\mathscr{P}_w$ the composition from $\mathrm{M}_d^{\mathrm{mar}} \times \{[w]\}$ to $\mathscr{D}(T_w)$. It follows from the above diagram that $\mathscr{P}_w$ is holomorphic and injective.

Just like M_w, the quotient $\mathscr{D}(T_w)/\operatorname{Stab}[w]$ is a quasi-projective variety by [2]. There is a commutative diagram

$$\begin{tikzcd}
\mathrm{M}_d^{\mathrm{mar}} \times \{[w]\} \arrow[r, hook, "\mathscr{P}_w"] \arrow[d] & \mathscr{D}(T_w) \arrow[d] \\
\mathrm{M}_w \arrow[r, hook, "\overline{\mathscr{P}}_w"] & \mathscr{D}(T_w)/\operatorname{Stab}[w]
\end{tikzcd}$$

where $\overline{\mathscr{P}}_w$ is algebraic by the same argument as in Corollary 2.3 (note that when ℓ is a multiple of $r^2 d$, the group $\Gamma_\ell = \{g \in \widetilde{\mathrm{O}}(\Lambda_d) \mid g \equiv \mathrm{id} \ \mathit{mod}\ \ell\}$ is contained in $\operatorname{Stab}[w]$).

Recall (see e.g. [15, Remark 6.4.5]) that $\mathscr{D}(T_w)\setminus \operatorname{im}\mathscr{P}_w = \mathscr{D}(\Lambda_d)\setminus \operatorname{im}\mathscr{P}$ is a union of hyperplanes $\bigcup_{\delta\in\Delta(\Lambda_d)}\delta^{\perp}$, where $\Delta(\Lambda_d)$ is the set of (-2)-classes in Λ_d. It follows that $\mathscr{D}(T_w)$ parametrizes periods of twisted *quasi-polarized* K3 surfaces, i.e. twisted K3 surfaces with a line bundle that is nef and big (however, the corresponding moduli stack is not separated). Hence, the quotient $\mathscr{D}(T_w)/\operatorname{Stab}[w]$ can be viewed as a moduli space of quasi-polarized twisted K3 surfaces.

3.2 The discriminant group of T_w

We collect some results about the lattice T_w, in preparation of Sect. 4. Let $w \in \frac{1}{r}\Lambda_d^{\vee}$ such that $[w] \in \Lambda_{d,r}^{\vee}$ has order r. We will describe the group $\operatorname{Disc} T_w$ and the quadratic form on it. Note that if $g \in \widetilde{\mathrm{O}}(\Lambda_d)$, then g induces an isomorphism $T_w \cong T_{g(w)}$. So by Lemma 2.6, we can assume that

$$w = w_{n,k} = \tfrac{1}{r}(e_1 + nf_1 + \tfrac{k}{d}\ell_d')$$

for some n, k. Then $T_w = E_8(-1)^{\oplus 2} \oplus U_2 \oplus T_0$, where

$$T_0 = \{y \in U_1 \oplus \mathbb{Z}\ell_d' \mid (y, w) \in \mathbb{Z}\} = \langle e_1 - nf_1, rf_1, kf_1 + \ell_d'\rangle.$$

Since $E_8(-1)^{\oplus 2}\oplus U_2$ is unimodular, $\operatorname{Disc} T_w$ is isomorphic to $\operatorname{Disc} T_0$. The intersection matrix of T_0 is (compare [23, Lemma 2.12])

$$M = \begin{pmatrix} -2n & r & k \\ r & 0 & 0 \\ k & 0 & -d \end{pmatrix}$$

As the map $T_0 \to T_0^{\vee}$ is given by the matrix $M^t = M$, we have

$$\operatorname{Disc} T_0 = \mathbb{Z}/g_1\mathbb{Z} \times \mathbb{Z}/g_2\mathbb{Z} \times \mathbb{Z}/g_3\mathbb{Z}$$

where the invariant factors g_i can be computed using the $i \times i$-minors of M [5, Satz 2.9.6]:

$$g_1 = \gcd(2n, r, k, d),\ \ g_2 = \gcd(r^2, kr, rd, 2nd - k^2)/g_1,\ \ g_3 = dr^2/g_1g_2.$$

We will be interested in the following two cases:

Proposition 3.1 *Let* $w = w_{n,k} \in \frac{1}{r}\Lambda_d^{\vee}$.

(i) The group $\operatorname{Disc} T_w$ *is cyclic if and only if* $\gcd(r, 2nd - k^2) = 1$.
(ii) We have

$$\operatorname{Disc} T_w \cong \mathbb{Z}/(r^2d/3)\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$$

if and only if $\gcd(r, 2nd - k^2) = 3$, *and if* $3|d$ *then* $9 \nmid nd$.

In order to determine the quadratic form on Disc T_w, we write down explicit generators. Consider the following elements of Disc T_0:

$$\begin{aligned}
[f_1] &= [\tfrac{1}{r}(rf_1)] \\
[\ell'_d/d] &= [\tfrac{1}{d}(kf_1 + \ell'_d) - \tfrac{k}{rd}(rf_1)] \\
[w] &= [\tfrac{1}{r}(e_1 - nf_1) + \tfrac{2nd-k^2}{r^2 d}(rf_1) + \tfrac{k}{rd}(kf_1 + \ell'_d)].
\end{aligned}$$

The order of $[x]$ is the smallest natural number a such that $ax \in T_w$, that is, $(ax, w) \in \mathbb{Z}$. For the elements above, this gives

$$\operatorname{ord}[f_1] = r,\ \operatorname{ord}[\ell'_d/d] = \tfrac{rd}{\gcd(k,rd)},\ \operatorname{ord}[w] = \tfrac{r^2 d}{\gcd(r^2 d, 2nd-k^2)}.$$

The class of $x \in T_w^\vee$ in Disc T_w is

$$[x] = (x, rf_1)[w] - (x, kf_1 + \ell'_d)[\ell'_d/d] + (x, e_1 - nf_1)[f_1].$$

This shows that Disc T_w is generated by $[f_1]$, $[\ell'_d/d]$ and $[w]$.

Lemma 3.2 *If* $\gcd(d, k, r) = s$*, then there is an integer* p *such that* $\gcd(d, k+pr) = s$.

Proof Let $d = d_0 \gcd(r, d)$, so $\gcd(r, sd_0) = s$. Write $xsd_0 + yr = s$. Then

$$\begin{aligned}
\gcd\left(d, k + (1 - \tfrac{k}{s})yr\right) &= \gcd\left(sd_0, k + (1 - \tfrac{k}{s})(s - xsd_0)\right) \\
&= s \gcd\left(d_0, 1 - xd_0 + xd_0\tfrac{k}{s}\right) \\
&= s.
\end{aligned}$$

□

First assume Disc T_w is cyclic, so $\gcd(r, 2nd - k^2) = 1$. In particular, we have $\gcd(r, d, k) = 1$. By Lemma 3.2 there exists a p such that $\gcd(d, k + pr) = 1$. Since $T_{w_{n,k}} \cong T_{w_{n,k+pr}}$, we can replace k by $k + pr$. Then we have $\gcd(r^2 d, 2nd - k^2) = 1$; hence, $[w]$ generates Disc T_w. So the quadratic form q_{T_w} on Disc T_w is determined by

$$q_{T_w}([w]) = [(w)^2] = \tfrac{1}{r^2 d}(2nd - k^2)\, mod\, 2\mathbb{Z}.$$

Next, assume Disc $T_w \cong \mathbb{Z}/(r^2 d/3)\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. If $3|d$, then $\gcd(r, 2nd - k^2) = 3$, and $9 \nmid nd$ implies $9 \nmid 2nd - k^2$. It follows that $\gcd(r^2, 2nd - k^2) = 3$, so $[w]$ generates $\mathbb{Z}/(dr^2/3)\mathbb{Z}$. As a generator of the factor $\mathbb{Z}/3\mathbb{Z}$, we take the element

$$u{:=}\tfrac{k}{3}[f_1] - \tfrac{d}{3}[\ell'_d/d] = \tfrac{1}{3}[kf_1 + \ell'_d],$$

which satisfies $q_{T_w}(u) = -\frac{d}{9}\, mod\, 2\mathbb{Z}$. If u were a multiple $m[w]$ of $[w]$, we would have $q_{T_w}(u) \equiv m^2 \frac{2nd-k^2}{3}\, mod\, 2$; multiplying by $\frac{r^2 d}{3}$ gives $-\left(\frac{d}{3}\right)^2 \frac{r^2}{3} \equiv$

$m^2\frac{2nd-k^2}{3}\,mod\,\frac{2r^2d}{3}$. This implies that $m = rm_0$ for some m_0; hence $-\left(\frac{d}{3}\right)^2 \equiv 3m_0^2\frac{2nd-k^2}{3}\,mod\,2d$. This is not possible as 3 does not divide the left hand side.

If $3 \nmid d$, we may have $9|2nd-k^2$, but this implies $9 \nmid r$. Using that $T_{w_{n,k}} \cong T_{w_{n+r,k}}$, we may replace n by $n+r$ and obtain $9 \nmid 2nd-k^2$. This gives $\gcd(r^2, 2nd-k^2) = 3$, so $[w]$ generates $\mathbb{Z}/(dr^2/3)\mathbb{Z}$. As a generator of the factor $\mathbb{Z}/3\mathbb{Z}$, we take

$$u' := \tfrac{rd}{3}[w] - \tfrac{2nd-k^2}{3}[f_1] = \tfrac{1}{3}([d(e_1 - nf_1) + k(kf_1 + \ell'_d)]).$$

We have $q_{T_w}(u') = -\frac{d}{9}(2nd-k^2)\,mod\,2\mathbb{Z}$. Like before, if u' were a multiple $m[w]$ of $[w]$, we would find $-d^2\left(\frac{r}{3}\right)^2\frac{2nd-k^2}{3} \equiv m^2\frac{2nd-k^2}{3}\,mod\,\frac{2r^2d}{3}$. It follows that $m = m_0r/3$ for some m_0, and hence $\frac{2nd-k^2}{3}(m_0^2+d^2) \equiv 0\,mod\,6d$. But this is not possible, since the left hand side is not divisible by 3.

Corollary 3.3 *Let* $w = w_{n,k} \in \frac{1}{r}\Lambda_d^\vee$.

(i) If Disc T_w *is cyclic, there exists a generator* t *such that*

$$q_{T_w}(t) = \tfrac{1}{r^2d}(2nd-k^2)\,mod\,2\mathbb{Z}.$$

(ii) If Disc $T_w \cong \mathbb{Z}/(r^2d/3)\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ *and* $3 \nmid d$, *there exist generators* $(1, 0)$ *and* $(0, 1)$ *such that*

$$q_{T_w}(1, 0) = \tfrac{1}{r^2d}(2nd-k^2)\,mod\,2\mathbb{Z}$$

and

$$q_{T_w}(0, 1) = -\tfrac{d}{9}(2nd-k^2)\,mod\,2\mathbb{Z}.$$

(iii) If Disc $T_w \cong \mathbb{Z}/(r^2d/3)\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ *and* $3|d$, *there exist generators* $(1, 0)$ *and* $(0, 1)$ *such that*

$$q_{T_w}(1, 0) = \tfrac{1}{r^2d}(2nd-k^2)\,mod\,2\mathbb{Z}$$

and

$$q_{T_w}(0, 1) = -\tfrac{d}{9}\,mod\,2\mathbb{Z}.$$

4 Twisted K3 surfaces and cubic fourfolds

A smooth cubic fourfold X is *special* if the lattice $\mathrm{H}^{2,2}(X) \cap \mathrm{H}^4(X, \mathbb{Z})$ has rank at least two. Hassett [10] showed that special cubic fourfolds form a countably infinite union of irreducible divisors $\mathscr{C}_d$ in the moduli space of cubic fourfolds. Here $\mathscr{C}_d \neq \emptyset$ if and only if $d > 6$ and $d \equiv 0, 2\,mod\,6$. Hassett proved that X is in $\mathscr{C}_d$ with d satisfying

$(**)$ d is even and not divisible by 4, 9, or any odd prime $p \equiv 2$ mod 3 if and only if there exists a polarized K3 surface (S, L) of degree d whose primitive cohomology $\mathrm{H}^2(S, \mathbb{Z})_{\mathrm{pr}}$ can be embedded Hodge-isometrically into $\mathrm{H}^4(X, \mathbb{Z})_{\mathrm{pr}}$, up to a sign and a Tate twist.

In this section, we will generalize this result to twisted K3 surfaces.

4.1 Associated twisted K3 surfaces

We denote by $\mathrm{H}^4(X, \mathbb{Z})(1)$ the middle cohomology of a cubic fourfold X with the intersection product changed by a sign and the weight of the Hodge structure shifted by two. There is a lattice isometry

$$\mathrm{H}^4(X, \mathbb{Z})(1) \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus \mathbb{Z}(-1)^{\oplus 3}. \tag{1}$$

The isometry can be chosen such that the square of the hyperplane class on X is mapped to $h{:=}(1, 1, 1) \in \mathbb{Z}(-1)^{\oplus 3}$. We denote the orthogonal complement to h by Γ, so Γ is isomorphic to $\mathrm{H}^4(X, \mathbb{Z})_{\mathrm{pr}}(1)$.

The cubic X lies in the divisor $\mathscr{C}_d$ if and only if there exists a primitive sublattice

$$K \subset \mathrm{H}^{2,2}(X) \cap \mathrm{H}^4(X, \mathbb{Z})(1)$$

of rank two and discriminant d containing the square of the hyperplane class. The orthogonal complement $K^\perp \subset \mathrm{H}^4(X, \mathbb{Z})(1)$ has an induced Hodge structure which determines X when $X \in \mathscr{C}_d$ is very general. As abstract lattices, K and $K^\perp$ only depend on d.

Under the isometry (1), the lattice K corresponds to a primitive sublattice of $E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus \mathbb{Z}(-1)^{\oplus 3}$ of rank two and discriminant d containing h. Such a sublattice is unique up to the action of the stable orthogonal group $\widetilde{\mathrm{O}}(\Gamma) = \mathrm{Ker}(\mathrm{O}(\Gamma) \to \mathrm{O}(\mathrm{Disc}\, \Gamma))$. We fix one such sublattice for each discriminant d and denote it by K_d. Its orthogonal complement $K_d^\perp$ is contained in Γ.

Hassett proved that d satisfies $(**)$ if and only if there is an isometry $K_d^\perp \cong \Lambda_d$. In generalizing this to the situation of twisted K3 surfaces, we replace Λ_d by T_w and $(**)$ by the condition $(**')$ introduced in [16]:

$$(**') \quad d' = dr^2 \text{ for some integers } d \text{ and } r, \text{ where } d \text{ satisfies } (**).$$

Theorem 4.1 *The number d' satisfies $(**')$ if and only if for every decomposition $d' = dr^2$ with d satisfying $(**)$, there exists an element $[w] \in \Lambda_{d,r}^\vee$ of order r such that $K_{d'}^\perp$ is isomorphic to* $\mathrm{Ker}((w, -)\colon \Lambda_d \to \mathbb{Q}/\mathbb{Z})$.

For a cubic fourfold $X \in \mathscr{C}_{d'}$, the inclusion $K_{d'}^\perp \subset \mathrm{H}^4(X, \mathbb{Z})(1)$ gives an induced Hodge structure of K3 type on $K_{d'}^\perp$ and thus on $T_w = \mathrm{Ker}\,(w, -)$, yielding a point x in the period domain $\mathscr{D}(T_w)$. In [25], it was shown that for a smooth cubic fourfold X, there are no classes in $\mathrm{H}^4(X, \mathbb{Z})_{\mathrm{pr}} \cap \mathrm{H}^{2,2}(X)$ of square 2. It follows that the class of x in $\mathscr{D}(T_w)/\,\mathrm{Stab}\,[w]$ lies in the image of the period map $\mathscr{P}_w$. As a consequence, we obtain

Corollary 4.2 *A cubic fourfold X is in $\mathscr{C}_{d'}$ for some d' satisfying $(**')$ if and only if for every decomposition $d' = dr^2$ with d satisfying $(**)$, there exists a polarized K3 surface (S, L) of degree d and an element $\alpha \in \operatorname{Hom}(\mathrm{H}^2(S, \mathbb{Z})_{\mathrm{pr}}, \mathbb{Q}/\mathbb{Z})$ of order r such that $K_{d'}^{\perp}$ is Hodge isometric to* $\operatorname{Ker}\alpha$.

We say that the twisted K3 surface in Corollary 4.2 is *associated* to X.

Remark 4.3 This notion of associated twisted K3 surfaces almost coincides with the one given by Huybrechts [16]. He relates the full cohomology $\widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z})$ to the Hodge structure $\widetilde{\mathrm{H}}(\mathscr{A}_X, \mathbb{Z})$ of K3 type associated to the K3 category $\mathscr{A}_X \subset \mathrm{D}^{\mathrm{b}}(X)$, which was introduced in [1].

To be precise, Huybrechts shows that a cubic X is in $\mathscr{C}_{d'}$ for some d' satisfying $(**')$ if and only if there is a twisted K3 surface (S, α) such that $\widetilde{\mathrm{H}}(\mathscr{A}_X, \mathbb{Z})$ is Hodge isometric to $\widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z})$. One can show that a Hodge isometry $K_{d'}^{\perp} \cong \operatorname{Ker}(\alpha\colon\ \mathrm{H}^2(S, \mathbb{Z})_{\mathrm{pr}} \to \mathbb{Q}/\mathbb{Z})$ always extends to $\widetilde{\mathrm{H}}(\mathscr{A}_X, \mathbb{Z}) \cong \widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z})$, see Proposition 5.6.

Vice versa, assume $\widetilde{\mathrm{H}}(\mathscr{A}_X, \mathbb{Z}) \cong \widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z})$. When S has Picard number one, it follows that $K_d^{\perp} \subset \mathrm{H}^4(X, \mathbb{Z})(1)$ is Hodge isometric to $\operatorname{Ker}(\alpha\colon\ \mathrm{H}^2(S, \mathbb{Z})_{\mathrm{pr}} \to \mathbb{Q}/\mathbb{Z})$ (these are the transcendental parts of $\widetilde{\mathrm{H}}(\mathscr{A}_X, \mathbb{Z})$ and $\widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z})$). When $\rho(S) > 1$, there exists a Hodge isometry $K_{d'}^{\perp} \cong \operatorname{Ker}(\alpha')$ for a possibly different K3 surface S' and $\alpha' \in \operatorname{Hom}(\mathrm{H}^2(S', \mathbb{Z})_{\mathrm{pr}}, \mathbb{Q}/\mathbb{Z})$ that satisfies $\widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z}) \cong \widetilde{\mathrm{H}}(S', \alpha', \mathbb{Z})$.

The above is completely analogous to the untwisted situation. Note that Corollary 4.2 implies a strengthening of Huybrechts' result, replacing "there is a twisted K3 surface" by "for any decomposition $d' = dr^2$ with d satisfying $(**)$, there is a twisted K3 surface of degree d and order r".

Finally, we should mention that these Hodge-theoretical notions (both twisted and untwisted) have a categorical counterpart due to [1,16,3]: There exists a Hodge isometry $\widetilde{\mathrm{H}}(\mathscr{A}_X, \mathbb{Z}) \cong \widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z})$ if and only if the category $\mathscr{A}_X$ is equivalent to the bounded derived category $\mathrm{D}^{\mathrm{b}}(S, \alpha)$ of α-twisted sheaves on S.

4.2 Proof of Theorem 4.1

We have seen that the discriminant group of T_w can always be generated by three elements. As T_w has signature (2, 19), it follows that T_w is determined by its discriminant group and the quadratic form on it [22, Corollary 1.13.3]. To prove Theorem 4.1, it thus suffices to determine when

$$\big(\operatorname{Disc} T_w, q_{T_w}\big) \cong \big(\operatorname{Disc} K_{dr^2}^{\perp}, q_{K_{dr^2}^{\perp}}\big).$$

Write $d' = dr^2$. We will use the following result by Hassett (using our sign convention):

Proposition 4.4 [10, Proposition 3.2.5] *When $d' \equiv 0 \bmod 6$, then* $\operatorname{Disc}(K_{d'}^{\perp})$ *is isomorphic to $\mathbb{Z}/\frac{d'}{3}\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, which is cyclic unless 9 divides d'. One can choose generators* $(1, 0)$ *and* $(0, 1)$ *such that the quadratic form $q_{K_{d'}^{\perp}}$ satisfies $q_{K_{d'}^{\perp}}(1, 0) = 3/d' \bmod 2\mathbb{Z}$*

and $q_{K_{d'}^{\perp}}(0,1) = -2/3 \mod 2\mathbb{Z}$*. When* $d' \equiv 2 \mod 6$*, then* $\mathrm{Disc}(K_{d'}^{\perp})$ *is* $\mathbb{Z}/d'\mathbb{Z}$*. One can choose a generator* u *such that* $q_{K_{d'}^{\perp}}(u) = (1-2d')/3d' \mod 2\mathbb{Z}$*.*

We prove Theorem 4.1 by comparing the quadratic forms on Disc $K_{d'}^{\perp}$ and Disc T_w. We distinguish the cases when the groups are cyclic and non-cyclic. We will use the following statements, which follow from quadratic reciprocity [10, proof of Proposition 5.1.4].

Lemma 4.5 *When* $d \equiv 2 \mod 6$*, then* d *satisfies* $(**)$ *if and only if* -3 *is a square modulo* $2d$*. When* $d \equiv 0 \mod 6$*, write* $d = 6t$*. Then* d *satisfies* $(**)$ *if and only if* -3 *is a square modulo* $4t$ *and* $4t$ *is a square modulo 3.*

4.2.1 Cyclic case

Assuming that Disc $K_{d'}^{\perp}$ is cyclic, we will show that $d' = dr^2$ with d satisfying $(**')$ if and only if there exists a $[w] \in \Lambda_{d,r}^{\vee}$ of order r such that $K_{d'}^{\perp} \cong T_w$. The proof consists of Propositions 4.6 and 4.7.

Proposition 4.6 *Assume that* Disc $K_{d'}^{\perp}$ *is cyclic. If there is a* $[w] \in \Lambda_{d,r}^{\vee}$ *of order* r *such that* $K_{d'}^{\perp} \cong T_w$ *(so in particular,* $d' = dr^2$*), then* d *satisfies* $(**)$*.*

Proof First assume that 3 does not divide d. By Proposition 4.4 and Corollary 3.3, we have $K_{d'}^{\perp} \cong T_{w_{n,k}}$ if and only if there is an x such that

$$\tfrac{x^2}{r^2 d}(k^2 - 2nd) \equiv \tfrac{2dr^2-1}{3dr^2} \mod 2.$$

Multiplying by $3dr^2$ gives

$$3x^2(k^2 - 2nd) \equiv 2dr^2 - 1 \mod 6dr^2$$

which is equivalent to

$$3x^2(k^2 - 2nd) \equiv -1 \mod 2dr^2. \tag{2}$$

It follows that -3 is a square modulo $2d$, so by Lemma 4.5, d satisfies $(**)$.

Next we assume $3|d$. By Proposition 4.4 and Corollary 3.3, we have $K_{d'}^{\perp} \cong T_{w_{n,k}}$ if and only if there is an x such that

$$\tfrac{x^2}{r^2 d}(k^2 - 2nd) \equiv \tfrac{2}{3} - \tfrac{3}{dr^2} \mod 2.$$

Writing $d = 6t$ and multiplying by dr^2 gives

$$x^2(k^2 - 12nt) \equiv 4tr^2 - 3 \mod 12tr^2. \tag{3}$$

In particular, -3 is a square modulo $4t$, and we have $4tr^2 \equiv x^2k^2 \mod 3$. Since 3 does not divide r, this implies that $4t$ is a square modulo 3. It follows from Lemma 4.5 that d satisfies $(**)$. □

Write $r = 2^s q r_0$ where q consists of all prime factors of r which are 1 modulo 3, and r_0 consists of all odd prime factors of r which are 2 modulo 3. In particular, dq^2 still satisfies $(**)$, and we have $\gcd(r_0, dq^2) = 1$.

Proposition 4.7 *There exists an n such that for $w = w_{nq^2,r_0} \in \frac{1}{r}\Lambda_d^\vee$, we have $K_{d'}^\perp \cong T_w$.*

Proof We first assume $3 \nmid d$. By (2) we have to show that for some x and some n,

$$f_n(x) := 3x^2(r_0^2 - 2ndq^2) + 1 \equiv 0 \, mod \, m \tag{4}$$

where $m = 2dr^2$.

Since dq^2 satisfies $(**)$, the number -3 is a square modulo $2dq^2$. As $3r_0$ is invertible in $\mathbb{Z}/2dq^2\mathbb{Z}$, we get $-3 \equiv (3r_0x)^2 \, mod \, 2dq^2$ for some $x \in \mathbb{Z}$. This gives $3x^2r_0^2 + 1 \equiv 0 \, mod \, 2dq^2$, which shows that (4) has a solution modulo $m = 2dq^2$, for any n. In particular, it has solutions modulo $dq^2/2$ and modulo 4.

It follows that $(f_n/2)(x) \equiv 0$ has a solution modulo 2. Also, $(f_n/2)'(x) = 3x(r_0^2 - 2ndq^2)$ is always odd. By Hensel's lemma, $(f_n/2)(x) = 0$ has a solution modulo 2^l for any $l \geq 1$. It follows that (4) has a solution modulo 2^l for any $l \geq 2$.

By the Chinese remainder theorem, there exists a solution x for (4) modulo $m = 2d(2^s q)^2$. We can assume $\gcd(x, r_0) = 1$: otherwise, write $ar_0 + b \cdot 2d(2^s q)^2 = 1$ and replace x by $x + b \cdot 2d(2^s q)^2(1 - x) = 1 + ar_0(x - 1)$.

Now we have $\gcd(r_0^2, 6x^2dq^2) = 1$, so there exist a and b such that $ar_0^2 + b \cdot 6x^2dq^2 = 1$. In particular, r_0^2 divides $3x^2 \cdot -2bdq^2 + 1$. We see that for $n = b$, there is a solution to (4) modulo $m = r_0^2$. By the Chinese remainder theorem, there exists a solution modulo $2dr^2$.

Next, assume $3|d$. Write $d = 6t$. By (3) we have to show that for some x and n,

$$g_n(x) := x^2(r_0^2 - 12ntq^2) - 4tr^2 + 3 \equiv 0 \, mod \, m \tag{5}$$

where $m = 12tr^2$.

Since dq^2 satisfies $(**)$, first, $4tq^2$ is a square modulo 3, so also $4tr^2 = 4t(2^s q r_0)^2$ is a square modulo 3. Second, -3 is a square modulo $4tq^2$. Since 3 does not divide $4tq^2$, it follows that $4tr^2 - 3$ is a square modulo $12tq^2$.

Now r_0 is invertible in $\mathbb{Z}/12tq^2\mathbb{Z}$, which implies that $4tr^2 - 3 \equiv (xr_0)^2 \, mod \, 12tq^2$ for some x. So $x^2r_0^2 - 4tr^2 + 3$ is divisible by $12tq^2$, which shows that (5) has a solution modulo $m = 12tq^2$, for any n. In particular, there exist solutions modulo $3tq^2$ and modulo 4.

Like before, it follows from Hensel's lemma that (5) has a solution modulo 2^l for any $l \geq 2$.

By the Chinese remainder theorem, there exists a solution x for (5) modulo $12t(2^s q)^2$. Like before, if $\gcd(x, r_0) \neq 1$, take a and b such that $ar_0 + b \cdot 12t(2^s q)^2 = 1$ and replace x by $x + b \cdot 12t(2^s q)^2 \cdot (1 - x) = 1 + ar_0(x - 1)$.

Now we have $\gcd(r_0^2, 4tx^2q^2) = 1$, so we can write $3ar_0^2 + bx^2 \cdot 12tx^2q^2 = 3$ for some a and b. So for $n = b$, we find that r_0^2 divides $x^2 \cdot -12ntq^2 + 3$, hence (5) has a solution modulo $m = r_0^2$. By the Chinese remainder theorem, it has a solution modulo $2dr^2$. □

4.2.2 Non-cyclic case

We now assume $\mathrm{Disc}(K_{d'}^{\perp}) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/\frac{d'}{3}\mathbb{Z}$, and we again show that $d' = dr^2$ with d satisfying $(**')$ if and only if there exists a $[w] \in \Lambda_{d,r}^{\vee}$ of order r such that $K_{d'}^{\perp} \cong T_w$. The proof consists of Propositions 4.8, 4.9 and 4.10.

Proposition 4.8 *Assume that* $\mathrm{Disc}\, K_{d'}^{\perp} \cong \mathbb{Z}/(d'/3)\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. *If there is a* $[w] \in \Lambda_{d,r}^{\vee}$ *of order* r *such that* $K_{d'}^{\perp} \cong T_w$ *(so in particular,* $d' = dr^2$*), then* d *satisfies* $(**)$.

Proof Consider the factor $\mathbb{Z}/(d'/3)\mathbb{Z} = \mathbb{Z}/(dr^2/3)\mathbb{Z}$. By Proposition 4.4 and Corollary 3.3, there exists an x such that $x^2 \frac{2nd-k^2}{r^2d}$ is congruent to $\frac{3}{dr^2}$ modulo 2. Multiplying both expressions with $-dr^2$ gives

$$x^2(k^2 - 2nd) \equiv -3 \, mod \, 2dr^2. \tag{6}$$

We see that -3 is a square modulo $2d$, which implies that d satisfies $(**)$. □

Write $r = 2^s q r_0$, where q consists of all prime factors of r which are congruent to 1 modulo 3, and r_0 consists of all other odd prime factors of r. In particular, dq^2 still satisfies $(**)$ and $\gcd(r_0, dq^2)$ is 1 or 3. Note that 3 divides r_0.

Proposition 4.9 *Suppose that* 3 *does not divide* d. *There exists an integer* n *such that for* $w = w_{3nq^2, r_0} \in \frac{1}{r}\Lambda_{d}^{\vee}$, *we have* $K_{d'}^{\perp} \cong T_w$.

Proof By (6), we need n and x such that

$$x^2(r_0^2 - 6ndq^2) + 3 \equiv 0 \, mod \, m \tag{7}$$

where $m = 2dr^2$.

Since dq^2 satisfies $(**)$, -3 is a square modulo $2dq^2$, and as r_0 is divisible in $\mathbb{Z}/2dq^2\mathbb{Z}$, we have $-3 \equiv (r_0x)^2 \, mod \, 2dq^2$ for some x. This shows that (7) has a solution modulo $2dq^2$ for any n. In particular, there exist solutions modulo $dq^2/2$ and modulo 4.

Using Hensel's lemma again, one shows that there exist solutions modulo 2^{ℓ} for any $\ell \geq 2$, and by the Chinese remainder theorem, there exists a solution x modulo $m = 2d(2^s q)^2$. We may assume that $\gcd(x, r_0) = 1$ by writing $ar_0 + b \cdot 2d(2^s q)^2 = 1$ and replacing x by $x + b \cdot 2d(2^s q)^2(1 - x) = 1 + ar_0(x - 1)$.

Now $\gcd(r_0^2/3, 2x^2dq^2) = 1$; take a and b such that $ar_0^2/3 + b \cdot 2x^2dq^2 = 1$. Then r_0^2 divides $-6bx^2dq^2 + 3$, so for $n = b$, there exists a solution to (7) modulo r_0^2. By the Chinese remainder theorem, there is a solution modulo $m = 2dr^2$.

We still need to check that for the generator u of $\mathbb{Z}/3\mathbb{Z} \subset \mathrm{Disc}\, T_w$, there exists $y \in \mathbb{Z}$ such that $y^2(u)^2 = y^2 \cdot \frac{d}{9}(r_0^2 - 6nq^2d)$ is congruent to $-2/3$ modulo 2. Multiplying both expressions by $3/2$ gives

$$y^2 \frac{d}{2} \frac{r_0^2 - 6nq^2d}{3} \equiv -1 \, mod \, 3.$$

Now note that $d/2 \equiv 1 \, mod \, 3$, so taking y such that 3 does not divide y, we have

$$y^2 \cdot \tfrac{d}{2} \cdot \tfrac{r_0^2 - 2nq^2 d}{3} \equiv \tfrac{r_0^2 - 6nq^2 d}{3} \, mod \, 3.$$

The element on the right hand side is

$$3(r_0/3)^2 - 2bq^2 d \equiv -b \, mod \, 3$$

where b was defined by the equation $ar_0^2/3 + b \cdot 2x^2 dq^2 = 1$. Reducing this modulo 3, we indeed find $b \equiv 1 \, mod \, 3$. □

We are left with the case $3|d$.

Proposition 4.10 *Suppose 3 divides d. There is an n such that for* $w = w_{nq^2, 3r_0} \in \frac{1}{r}\Lambda_d^{\vee}$, *we have* $K_{d'}^{\perp} \cong T_w$.

Proof By (6), we need n and x such that

$$x^2((3r_0)^2 - 2ndq^2) + 3 \equiv 0 \, mod \, 2dr^2.$$

Write $d = 6t$, then this is equivalent to

$$x^2(3r_0^2 - 4ntq^2) + 1 \equiv 0 \, mod \, m \tag{8}$$

where $m = 4tr^2$.

As dq^2 satisfies $(**)$, Lemma 4.5 tells us that -3 is a square modulo $4tq^2$. Since $\gcd(3r_0, 4tq^2) = 1$, it follows that $-3 \equiv (3r_0 x)^2 \, mod \, 4tq^2$ for some x. So we have $3x^2 r_0^2 + 1 \equiv 0 \, mod \, 4tq^2$, which shows that (8) has a solution modulo $m = 4tq^2$.

By Hensel's lemma once more, (8) also has a solution modulo 2^ℓ for all $\ell \geq 2$, and by the Chinese remainder theorem it then has a solution x modulo $4t(2^s q)^2$. Like before, we may assume $\gcd(x, r_0) = 1$ by writing $ar_0 + b \cdot 2d(2^s q)^2 = 1$ and replacing x by $x + b \cdot 2d(2^s q)^2(1 - x) = 1 + ar_0(x - 1)$.

Now note that $\gcd(r_0^2, 4tx^2q^2) = 1$ and take a, b such that $ar_0^2 + b \cdot 4tx^2q^2 = 1$. Then r_0^2 divides $-b \cdot 4tx^2q^2 + 1$, showing that for $n = b$, (8) has a solution modulo $m = r_0^2$. By the Chinese remainder theorem, there exists a solution modulo $4tr^2$.

Finally, we need to check that for the generator u' of $\mathbb{Z}/3\mathbb{Z} \subset \mathrm{Disc}\, T_w$, there exists a y such that $y^2(u')^2 = -y^2 d/9$ is congruent to $-2/3$ modulo 2. Multiplying by $-3/2$, we get

$$y^2 d/6 \equiv 1 \, mod \, 3$$

which is true whenever 3 does not divide y. □

5 Rational maps to $\mathscr{C}_{d'}$

For untwisted K3 surfaces, an isomorphism $\Lambda_d \cong K_d^\perp$ can be used to construct a rational map $\mathrm{M}_d \dashrightarrow \mathscr{C}_d$. We will generalize these maps to the situation of twisted K3 surfaces.

Throughout this section, we will assume d' satisfies $(**')$ and fix a decomposition $d' = dr^2$ with d satisfying $(**)$. Moreover, we fix $[w] \in \Lambda_{d,r}^\vee$ as in Theorem 4.1 and choose an isomorphism $K_{d'}^\perp \cong T_w = \mathrm{Ker}\,(w, -)$.

5.1 Construction

Note that the group $\widetilde{\mathrm{O}}(K_{d'}^\perp)$ can be viewed as a subgroup of $\widetilde{\mathrm{O}}(\Gamma)$: any element $f \in \widetilde{\mathrm{O}}(K_{d'}^\perp)$ can be extended to an orthogonal transformation $\tilde{f}$ of the unimodular lattice $E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus \mathbb{Z}(-1)^{\oplus 3}$ such that $\tilde{f}|_{K_{d'}}$ is the identity. Then restrict to Γ to get an element of $\widetilde{\mathrm{O}}(\Gamma)$.

On the level of the period domain, we have a commutative diagram

$$\begin{tikzcd}
\mathscr{D}(T_w) \arrow[r, "\cong"] \arrow[d] & \mathscr{D}(K_{d'}^\perp) \arrow[rr, hook] \arrow[d] & & \mathscr{D}(\Gamma) \arrow[d] \\
\mathscr{D}(T_w)/\widetilde{\mathrm{O}}(T_w) \arrow[r, "\cong"] & \mathscr{D}(K_{d'}^\perp)/\widetilde{\mathrm{O}}(K_{d'}^\perp) \arrow[r] & \overline{\mathscr{C}}_{d'} \arrow[r, hook] & \mathscr{D}(\Gamma)/\widetilde{\mathrm{O}}(\Gamma)
\end{tikzcd}$$

where $\overline{\mathscr{C}}_{d'}$ is the image of $\mathscr{D}(K_{d'}^\perp)$ under $\mathscr{D}(\Gamma) \to \mathscr{D}(\Gamma)/\widetilde{\mathrm{O}}(\Gamma)$. Embedding the moduli space $\mathscr{C}$ of smooth cubic fourfolds into $\mathscr{D}(\Gamma)/\widetilde{\mathrm{O}}(\Gamma)$ via the period map, one shows that $\overline{\mathscr{C}}_{d'}$ is the closure of $\mathscr{C}_{d'} \subset \mathscr{C}$ in $\mathscr{D}(\Gamma)/\widetilde{\mathrm{O}}(\Gamma)$.

Lemma 5.1 *The group* $\widetilde{\mathrm{O}}(T_w)$ *is a subgroup of* $\mathrm{Stab}\,[w] \subset \widetilde{\mathrm{O}}(\Lambda_d)$.

Proof Let $g \in \widetilde{\mathrm{O}}(T_w)$. By assumption, $g^\vee$ sends any $x \in T_w^\vee$ to $x + y$ for some $y \in T_w \subset \Lambda_d$. In particular, this holds for $x \in \Lambda_d \subset T_w^\vee$, which shows that $g^\vee$ preserves Λ_d. Moreover, $g^\vee$ induces the identity on $\mathrm{Disc}\,\Lambda_d$, so $g^\vee|_{\Lambda_d}$ is an element of $\widetilde{\mathrm{O}}(\Lambda_d)$.

Now $w \in \frac{1}{r}\Lambda_d^\vee$ lies in $T_w^\vee$, so we also have $g^\vee(w) = w + y$ for some $y \in T_w \subset \Lambda_d^\vee$. This implies that when acting on $\Lambda_{d,r}^\vee$, the map $g^\vee|_{\Lambda_d}$ stabilizes $[w]$. □

The period map $\mathscr{P}_w\colon\ \mathrm{M}_d^{\mathrm{mar}} \times \{[w]\} \to \mathscr{D}(T_w)$ induces an embedding of

$$\widetilde{\mathrm{M}}_w := (\mathrm{M}_d^{\mathrm{mar}} \times \{[w]\})/\widetilde{\mathrm{O}}(T_w)$$

into $\mathscr{D}(T_w)/\widetilde{\mathrm{O}}(T_w)$. This map is algebraic, which is shown similarly as for the embedding $\mathrm{M}_w \hookrightarrow \mathscr{D}(T_w)/\,\mathrm{Stab}\,[w]$. The space $\widetilde{\mathrm{M}}_w$ parametrizes tuples (S, L, α, f) where (S, L, α) is in M_w and f is an isomorphism from $\mathrm{Disc}(\mathrm{Ker}\,\alpha)$ to $\mathrm{Disc}\,T_w$. The composition

$$\widetilde{\mathrm{M}}_w \to \mathscr{D}(T_w)/\widetilde{\mathrm{O}}(T_w) \to \overline{\mathscr{C}}_{d'}$$

induces a rational map $\widetilde{\mathrm{M}}_w \dashrightarrow \mathscr{C}_{d'}$, which is regular on an open subset that maps surjectively (by Corollary 4.2) to $\mathscr{C}_{d'}$. Hassett showed that $\mathscr{D}(K_{d'}^{\perp})/\widetilde{\mathrm{O}}(K_{d'}^{\perp}) \to \overline{\mathscr{C}}_{d'}$ generically has degree one when $d' \equiv 2 \, mod \, 6$, and degree two when $d' \equiv 0 \, mod \, 6$. Hence, $\widetilde{\mathrm{M}}_w \dashrightarrow \mathscr{C}_{d'}$ is birational in the first case and has degree two in the second case; see also Sect. 5.3.

The map $\gamma \colon \widetilde{\mathrm{M}}_w \dashrightarrow \mathscr{C}_{d'}$ is in general not unique: it depends on the choice of an isomorphism $T_w \cong K_{d'}^{\perp}$. To be precise, let $\iota \colon \mathrm{O}(T_w) \to \mathrm{Aut}(\mathscr{D}(T_w))$ send an isometry of T_w to the induced action on the period domain. Then γ is unique up to $\iota(\mathrm{O}(T_w))/\iota(\widetilde{\mathrm{O}}(T_w))$. We can compute this group as in [12, Lemma 3.1]: there is a short exact sequence

$$0 \to \widetilde{\mathrm{O}}(T_w) \to \mathrm{O}(T_w) \to \mathrm{O}(\mathrm{Disc}\, T_w) \to 0.$$

Using $\iota(\widetilde{\mathrm{O}}(T_w)) \cong \widetilde{\mathrm{O}}(T_w)$ and $\iota(\mathrm{O}(T_w)) \cong \mathrm{O}(T_w)/\pm \mathrm{id}$, we find that

$$\iota(\mathrm{O}(T_w))/\widetilde{\mathrm{O}}(T_w) \cong \mathrm{O}(\mathrm{Disc}\, T_w)/\pm \mathrm{id}\,.$$

Corollary 5.2 *The map* $\widetilde{\mathrm{M}}_w \dashrightarrow \mathscr{C}_{d'}$ *is unique up to elements of* $\mathrm{O}(\mathrm{Disc}\, T_w)/\pm \mathrm{id}$.

When $\mathrm{Disc}\, T_w \cong \mathbb{Z}/d'\mathbb{Z}$, this group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus \tau(d'/2)-1}$, where $\tau(d'/2)$ is the number of prime factors of $d'/2$.

We have seen that there is a difference to the untwisted situation: the rational map to $\mathscr{C}_{d'}$ can only be defined after taking a finite covering $\pi : \widetilde{\mathrm{M}}_w \to \mathrm{M}_w$. We give an upper bound for the degree of this covering.

Corollary 5.3 *The degree of the quotient map* $\pi : \widetilde{\mathrm{M}}_w \to \mathrm{M}_w$ *is at most*

$$I = |\, \mathrm{O}(\mathrm{Disc}\, T_w)/\pm \mathrm{id}\,|.$$

If $\mathrm{Disc}\, T_w$ *is cyclic, then* $I = 2^{\tau(d'/2)-1}$.

Proof The degree of π is the index of $\iota(\widetilde{\mathrm{O}}(T_w)) \cong \widetilde{\mathrm{O}}(T_w)$ in $\iota(\mathrm{Stab}\,[w])$. This is at most the index I of $\widetilde{\mathrm{O}}(T_w)$ in $\iota(\mathrm{O}(T_w))$. □

5.2 Example

We consider the case $d = r = 2$, so $d' = 8$. The cubic fourfolds in $\mathscr{C}_8$ are those containing a plane. For a generic such cubic X, it was shown already in [19] that X has an associated twisted K3 surface (S, α) in the categorical sense (see Remark 4.3). In this special case, there is a geometric construction for (S, α), which was used before by Voisin in her proof of the Torelli theorem for cubic fourfolds [25]. As explained in Remark 4.3, (S, α) is also Hodge-theoretically associated to X.

By Lemma 2.6, the moduli space M_2^2 has at most four connected components, corresponding to the vectors $w_{n,k} = \frac{1}{2}(e_1 + nf_1 + \frac{k}{2}\ell_2')$ with $n, k \in \{0, 1\}$. Now by

Eichler's criterion, e_1 is equivalent to $e_1 + f_1 + \ell'_2$ under $\widetilde{\mathrm{O}}(\Lambda_2)$, and this is equivalent to $e_1 + f_1$ modulo $2\Lambda_2^\vee$. Thus, the components $\mathrm{M}_{w_{0,0}}$ and $\mathrm{M}_{w_{1,0}}$ are the same.

The discriminant group of $K_8^\perp$ is cyclic, and one can choose a generator u such that $q_{K_8^\perp}(u) = -\frac{5}{8}\, mod\, 2\mathbb{Z}$. By Proposition 3.1, the discriminant group of $T_{w_{n,k}}$ is cyclic if and only if $k = 1$. By Corollary 3.3, $T_{w_{n,1}}$ is isomorphic to $K_8^\perp$ if and only if there exists an $x \in \mathbb{Z}$ such that $\frac{x^2(4n-1)}{2} \equiv -\frac{5}{8}\, mod\, 2$. For $n = 0$, we have

$$\frac{x^2(4n-1)}{2} = -\frac{x^2}{8}$$

which is never equivalent to $-\frac{5}{8}$ modulo 2. For $n = 1$, we have

$$\frac{x^2(4n-1)}{2} = \frac{3x^2}{8}$$

which is equivalent to $-\frac{5}{8}$ modulo 2 when $x = 3$.

We see that for $w = w_{1,1}$, there exists a rational map $\widetilde{\mathrm{M}}_w \dashrightarrow \mathscr{C}_{d'}$ as above. Since $d'/2 = 4$ has only one prime factor, Corollary 5.2 tells us that there is a unique choice for the rational map $\widetilde{\mathrm{M}}_{w_{1,1}} \dashrightarrow \mathscr{C}_8$. Moreover, it follows from Corollary 5.3 that $\pi : \widetilde{\mathrm{M}}_{w_{1,1}} \to \mathrm{M}_{w_{1,1}}$ is an isomorphism. Hence, we obtain a rational map

$$\mathrm{M}_{w_{1,1}} \dashrightarrow \mathscr{C}_8$$

which gives an inverse to the geometric construction of associated twisted K3 surfaces over the locus where $\rho(S) = 1$.

Remark 5.4 The three types of Brauer classes occurring in this example have been studied before by Van Geemen [7] (see also [20, Section 2]). He relates the twisted K3 surfaces in the components $\mathrm{M}_{w_{0,0}}$ and $\mathrm{M}_{w_{0,1}}$ to certain double covers of $\mathbb{P}^2 \times \mathbb{P}^2$ and to complete intersections of three quartics in $\mathbb{P}^4$, respectively.

Remark 5.5 In general, the component $\mathrm{M}_w \subset \mathrm{M}_d^r$ for which a rational map $\widetilde{\mathrm{M}}_w \dashrightarrow \mathscr{C}_{d'}$ exists is not unique, because the class $[w] \in \Lambda_{d,r}^\vee$ satisfying $T_w \cong K_{d'}^\perp$ is not unique modulo $\widetilde{\mathrm{O}}(\Lambda_d)$. We work out an example.

Let $d = 14$ and $r = 7$, so Disc $K_{d'}^\perp$ is cyclic. Since r divides d, [20, Theorem 9] tells us that for $[w] \in \Lambda_{d,r}^\vee$ of order r, there is only one isomorphism class of lattices T_w with cyclic discriminant group. By Theorem 4.1, these T_w are isomorphic to $K_{d'}^\perp$.

Consider $w_{0,1} = \frac{1}{7}(e_1 + \ell'_{14}/14)$ and $w_{1,3} = \frac{1}{7}(e_1 + f_1 + 3\ell'_{14}/14)$. By Proposition 3.1, Disc $T_{w_{0,1}}$ and Disc $T_{w_{1,3}}$ are both cyclic. By the above, we have $T_{w_{0,1}} \cong T_{w_{1,3}} \cong K_{14\cdot 7^2}^\perp$. We show that $[w_{0,1}] \not\equiv [w_{1,3}]$ in $\Lambda_{d,r}^\vee/\widetilde{\mathrm{O}}(\Lambda_d)$.

Namely, suppose $[w_{1,3}]$ lies in the orbit $\widetilde{\mathrm{O}}(\Lambda_{14}) \cdot [w_{0,1}] \subset \Lambda_{14,7}^\vee$. Then there exists $z \in \Lambda_{14}^\vee$ such that $f_7(w_{0,1}) = w_{1,3} + z$ for some $f \in \widetilde{\mathrm{O}}(\Lambda_{14})$, that is, $f(14 \cdot 7 w_{0,1}) = 14 \cdot 7(w_{1,3} + z)$. Write $z = z_0 + \frac{t}{14}\ell'_{14}$ for some $z_0 \in E_8(-1)^{\oplus 2} \oplus U_1 \oplus U_2$ and $t \in \mathbb{Z}$, so

$$14 \cdot 7(w_{1,3} + z) = 14(e_1 + f_1) + 14 \cdot 7 z_0 + (3 + 7t)\ell'_{14}.$$

The square of the right hand side should be equal to $(14 \cdot 7w_{0,1})^2 = -14$. This gives

$$-14 = 2 \cdot 14^2 + 14^3(e_1 + f_1, z_0) + (14 \cdot 7)^2(z_0)^2 - 14(9 + 6 \cdot 7t + (7t)^2)$$

which simplifies to

$$8 = 2 \cdot 14 + 14^2(e_1 + f_1, z_0) + 14 \cdot 7^2(z_0)^2 - (6 \cdot 7t + (7t)^2).$$

Reducing modulo 7, one sees that this is not possible.

5.3 Pairs of associated twisted K3 surfaces

In [6], we studied the covering involution of Hassett's rational map $\mathrm{M}_d \dashrightarrow \mathscr{C}_d$ in the case this has degree two. We showed that if $(S, L) \in \mathrm{M}_d$ is mapped to (S^τ, L^τ) under this involution, then S^τ is isomorphic to a moduli space of stable sheaves on S with Mukai vector $(3, L, d/6)$. In this section, we discuss the analogous twisted situation.

We denote the bounded derived category of α-twisted coherent sheaves on S by $\mathrm{D^b}(S, \alpha)$. When $\alpha \in \mathrm{Hom}(\mathrm{H}^2(S, \mathbb{Z})_{\mathrm{pr}}, \mathbb{Z}/r\mathbb{Z})$, then by α-twisted sheaves we mean $\overline{\alpha}$-twisted sheaves, where $\overline{\alpha}$ is the image of α in $\mathrm{Hom}(T(S), \mathbb{Z}/r\mathbb{Z}) = \mathrm{Br}(S)[r]$. Similarly, $\widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z})$ means $\widetilde{\mathrm{H}}(S, \overline{\alpha}, \mathbb{Z})$.

Assume that 3 divides $d' = dr^2$. Hassett showed (see also [6]) that the map $\mathscr{D}(K_{d'}^\perp)/\widetilde{\mathrm{O}}(K_{d'}^\perp) \to \overline{\mathscr{C}}_{d'}$ is a composition $\nu \circ f$, where ν is the normalization of $\overline{\mathscr{C}}_{d'}$ and f is generically of degree two, induced by an element in $\mathrm{O}(K_{d'}^\perp)$ of order two. The corresponding element $g \in \mathrm{O}(T_w)$ induces a covering involution

$$\tau \colon \mathscr{D}(K_{d'}^\perp)/\widetilde{\mathrm{O}}(K_{d'}^\perp) \to \mathscr{D}(K_{d'}^\perp)/\widetilde{\mathrm{O}}(K_{d'}^\perp)$$

that preserves $\widetilde{\mathrm{M}}_w$. We claim that g extends to an orthogonal transformation of $\widetilde{\Lambda}$. This follows from [22, Corollary 1.5.2] and the following statement. We embed $T_w \subset \Lambda_d$ primitively into $\widetilde{\Lambda}$ using the map $\exp(w)$, as in Sect. 3.1.

Proposition 5.6 *Let $S_w := T_w^\perp \subset \widetilde{\Lambda}$. The map $\mathrm{O}(S_w) \to \mathrm{O}(\mathrm{Disc}\, S_w)$ is surjective.*

Proof The lattice S_w has rank three. When $\mathrm{Disc}\, T_w \cong \mathrm{Disc}\, S_w$ is cyclic, the statement follows from [22, Theorem 1.14.2]. When $\mathrm{Disc}\, T_w$ is $\mathbb{Z}/(d'/3)\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, it follows from Corollary VIII.7.3 in [21]. □

This implies that when τ maps $(S, L, \alpha, f) \in \widetilde{\mathrm{M}}_w$ to (S', L', α', f'), then there is a Hodge isometry

$$\widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z}) \cong \widetilde{\mathrm{H}}(S', \alpha', \mathbb{Z}).$$

This map might not preserve the orientation of the four positive directions. However, by [16, Lemma 2.3], there exists an orientation reversing Hodge isometry in $\mathrm{O}(\widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z}))$. By composing with it, we see that there exists a Hodge isometry

$g\colon \widetilde{\mathrm{H}}(S,\alpha,\mathbb{Z}) \to \widetilde{\mathrm{H}}(S',\alpha',\mathbb{Z})$ which is orientation preserving. By [18], g is induced by a Fourier–Mukai equivalence

$$\Phi_{\mathscr{E}}\colon \mathrm{D^b}(S,\alpha) \to \mathrm{D^b}(S',\alpha')$$

for some $\mathscr{E} \in \mathrm{D^b}(S \times S', \alpha^{-1} \boxtimes \alpha')$, that is, the associated cohomological Fourier–Mukai transform $\Phi^H_{\mathscr{E}} : \widetilde{\mathrm{H}}(S,\alpha,\mathbb{Z}) \to \widetilde{\mathrm{H}}(S',\alpha',\mathbb{Z})$ equals g. Now S' is a moduli space of stable complexes of α-twisted sheaves on S with Mukai vector

$$v = (\Phi^H_{\mathscr{E}})^{-1}(v(k(x))) = (\Phi^H_{\mathscr{E}})^{-1}(0,0,1),$$

where x is any closed point in S'. It is a coarse moduli space: the universal family on $S \times S'$ exists as an $\alpha^{-1} \boxtimes \alpha'$-twisted sheaf, which is an untwisted sheaf if and only if α' is trivial.

In fact, one can show that S' is isomorphic to a moduli space of stable α-twisted *sheaves* on S. Namely, by [26] (see also [18]), there exists a (coarse) moduli space $M(v)$ of stable α-twisted sheaves on S with Mukai vector v. By precomposing $\Phi_{\mathscr{E}}$ with autoequivalences of $\mathrm{D^b}(S,\alpha)$, we may assume $M(v)$ is non-empty [18, Section 2]. Hence, as $(v)^2 = 0$, the space $M(v)$ is a K3 surface.

For some B-field $\beta \in \mathrm{H}^2(M(v),\mathbb{Q})$, there exists a universal family $\mathscr{E}_v$ on $S \times M(v)$ which is an $\alpha^{-1} \boxtimes \beta$-twisted sheaf. It induces an equivalence of categories $\Phi_{\mathscr{E}_v} : \mathrm{D^b}(S,\alpha) \to \mathrm{D^b}(M(v),\beta)$ whose associated cohomological Fourier–Mukai transform $\Phi^H_{\mathscr{E}_v}$ sends v to $(0,0,1) \in \widetilde{\mathrm{H}}(M(v),\beta,\mathbb{Z})$. The composition

$$\Phi^H_{\mathscr{E}_v} \circ (\Phi^H_{\mathscr{E}})^{-1} : \widetilde{\mathrm{H}}(S',\alpha',\mathbb{Z}) \to \widetilde{\mathrm{H}}(M(v),\beta,\mathbb{Z})$$

is a Hodge isometry that sends $(0,0,1)$ to $(0,0,1)$ and is orientation preserving, since both $\Phi^H_{\mathscr{E}_v}$ and $\Phi^H_{\mathscr{E}}$ are (for $\Phi^H_{\mathscr{E}_v}$, see [17]). It follows from [18, Section 2] that S' is isomorphic to $M(v)$.

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