Plemelj–Sokhotski isomorphism for quasicircles in Riemann surfaces and the Schiffer operators

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Received: 7 December 2018 / Revised: 24 June 2019 / Published online: 31 October 2019
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Abstract

Let R be a compact Riemann surface and Γ be a Jordan curve separating R into connected components Σ_1 and Σ_2 . We consider Calderón–Zygmund type operators $T(\Sigma_1, \Sigma_k)$ taking the space of L^2 anti-holomorphic one-forms on Σ_1 to the space of L^2 holomorphic one-forms on Σ_k for $k = 1, 2$, which we call the Schiffer operators. We extend results of Max Schiffer and others, which were confined to analytic Jordan curves Γ , to general quasicircles, and prove new identities for adjoints of the Schiffer operators. Furthermore, let V be the space of anti-holomorphic one-forms which are orthogonal to L^2 anti-holomorphic one-forms on R with respect to the inner product on Σ_1 . We show that the restriction of the Schiffer operator $T(\Sigma_1, \Sigma_2)$ to V is an isomorphism onto the set of exact holomorphic one-forms on Σ_2 . Using the relation between this Schiffer operator and a Cauchy-type integral involving Green’s function, we also derive a jump decomposition (on arbitrary Riemann surfaces) for quasicircles and initial data which are boundary values of Dirichlet-bounded harmonic functions and satisfy the classical algebraic constraints. In particular we show that the jump operator is an isomorphism on the subspace determined by these constraints.

Mathematics Subject Classification Primary 58C99 · 58A10 · 30F10 · 30F15 · 30F30 · 32W05; Secondary 58J40

To the memory of our friend Peter C. Greiner.

Communicated by Ngaiming Mok.

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1 Introduction

1.1 Results and literature

Let Γ be a sufficiently regular curve separating a compact surface into two components Σ_1 and Σ_2 . Given a sufficiently regular function h on that curve, it is well known that there are holomorphic functions h_k on Σ_k , for $k = 1, 2$, such that

$$h = h_2 - h_1$$

if and only if $\int_{\Gamma} h \alpha = 0$ for all holomorphic one forms on R . In the plane, this is a consequence of the Plemelj–Sokhotski jump formula (which is a more precise formula in terms of a principal value integral). The functions h_k are obtained by integrating h against the Cauchy kernel.

Different regularities of the curve and the function are possible. In this paper, we show that the jump formula holds for quasircles on compact Riemann surfaces, where the function h is taken to be the boundary values of a harmonic function of bounded Dirichlet energy on either Σ_1 or Σ_2 . In the case that Γ is analytic, this space agrees with the Sobolev $H^{1/2}$ space on Γ . We showed in an earlier paper [20] that the space of boundary values, for quasircles, is the same for both Σ_1 and Σ_2 , and the resulting map (which we call the transmission map) is bounded.

Since quasircles are non-rectifiable, we replace the Cauchy integral by a limit of integrals along level curves of Green’s function in Σ_k ; for quasircles, we show that this integral is the same whether one takes the limiting curves from within Σ_1 or Σ_2 . This relies on our transmission result mentioned above. We show that the map from the harmonic Dirichlet space $\mathcal{D}_{\text{harm}}(\Sigma_k)$ to the direct sum of holomorphic Dirichlet spaces $\mathcal{D}(\Sigma_1) \oplus \mathcal{D}(\Sigma_2)$ obtained from the jump integral is an isomorphism. We also consider a Calderón–Zygmund type integral operator on the space of one-forms which is one type of what we call a Schiffer operator. This was studied extensively by Schiffer and others in the plane and on Riemann surfaces (see Sect. 3.3 for a discussion of the literature). Schiffer discovered deep relations between these operators and inequalities in function theory, potential theory and Fredholm eigenvalues. We extend many known results from analytic boundary to quasircles, and derive some new identities for the adjoints of the Schiffer operators (Theorems 3.11, 3.12, and 3.13), as well as a complete set of identities relating the Schiffer operator to the Cauchy-type integral in higher genus (Theorem 4.2). The derivative of the Cauchy-type integral, when restricted to a finite co-dimension space of one-forms, equals a Schiffer operator which we denote below by $T(\Sigma_1, \Sigma_2)$. We prove that the restriction of this Schiffer operator to this finite co-dimensional space is an isomorphism (Theorem 4.22).

In the case of simply-connected domains in the plane (where the finite co-dimensional space is the full space of one-forms), the fact that aforementioned Schiffer operator $T(\Sigma_1, \Sigma_2)$ is an isomorphism is due to Napalkov and Yulmukhametov [8]. In fact, they showed that it is an isomorphism precisely for domains bounded by quasircles. This is closely related to a result of Shen [22], who showed that the Faber operator of approximation theory is an isomorphism precisely for domains bounded by quasircles. Indeed, using Shen’s result, the authors (at the time unaware of Napalkov and

Yulmukhametov's result) derived a proof that a jump operator and the Schiffer operator are isomorphisms precisely for quasicircles [18]. The isomorphism for the jump operator is what we call the Plemelj–Sokhotski isomorphism. As mentioned above, here we generalize the isomorphism theorem for $T(\Sigma_1, \Sigma_2)$ and the Plemelj–Sokhotski isomorphism (Theorem 4.26) to Riemann surfaces separated by quasicircles. We conjecture that the converse holds, as in the planar case; namely, if either of these is an isomorphism, then the separating curve is a quasicircle.

Let us conclude with a few remarks on technical issues and related literature. The main hindrance to the solution of the Riemann boundary problem and the establishment of the jump decomposition is that quasicircles are highly irregular, and are not in general rectifiable. Riemann–Hilbert problems on non-rectifiable curves have been studied extensively by Kats, see e.g. [6] for the case of Hölder continuous boundary values, and the survey article [5] and references therein. However the boundary values of Dirichlet bounded harmonic functions need not be Hölder continuous. For Dirichlet spaces boundary values exist for quasicircles and the jump formula can be expressed in terms of certain limiting integrals. A key tool here is our proof of the existence and boundedness of a transmission operator for harmonic functions in quasicircles [20] (which, in the plane, also characterizes quasicircles [17]). Indeed our approach to proving surjectivity of $T(\Sigma_1, \Sigma_2)$ relies on the equality of the limiting integral from both sides. We have also found that the transmission operator has a clarifying effect on the theory as a whole.

In this paper, approximation by functions which are analytic or harmonic on a neighbourhood of the closure plays an important role. We rely on an approximation result for Dirichlet space functions on nested doubly-connected regions in a Riemann surface. This is similar to a result of Askaripour and Barron [2] for L^2 k -differentials for nested surfaces satisfying certain conditions. Their result uses the density of polynomials in the Bergman space of a Carathéodory domain in the plane. The proof of our result is similar.

The results of this paper can be applied to families of operators over Teichmüller space, as we will pursue in future publications. Applications to a certain determinant line bundle occurring in conformal field theory appear in [11].

1.2 Outline of the paper

In Sect. 2 we establish notation and state preliminary results. We also outline previous results of the authors which are necessary here. In Sect. 3 we define the Schiffer operators, generalize known results to quasicircles, and establish some new identities for adjoints. In Sect. 4, we give identities relating one type of Schiffer operator to a Cauchy-type integral (in general genus), we relate it to the jump decomposition, and establish the isomorphism theorems for the Schiffer operator and the Cauchy-type integral. We call the latter isomorphism the Plemelj–Sokhotski isomorphism.

2 Notations and preliminaries

2.1 Forms and functions

We begin by establishing notation and terminology.

Let R be a Riemann surface, which we will always assume to be connected. For smooth real one-forms, define the dual of the almost complex structure $*$ by

$$*(a\,dx + b\,dy) = a\,dy - b\,dx$$

in a local holomorphic coordinate $z = x + iy$. This is independent of the choice of coordinates. Harmonic functions f on R are those C^2 functions which satisfy $d * df = 0$, while harmonic one-forms α are those C^1 one-forms which satisfy both $d\alpha = 0$ and $d * \alpha = 0$. Equivalently, harmonic one-forms are those which can be expressed locally as df for some harmonic function f . We consider complex-valued functions and forms. Denote complex conjugation of functions and forms with an overline, e.g. $\overline{\alpha}$.

Harmonic one-forms α can always be decomposed as a sum of a holomorphic and anti-holomorphic one-form. The decomposition is unique. On the other hand, harmonic functions do not possess such a decomposition.

The space of complex one-forms on R has the natural inner-product

$$(\omega_1, \omega_2) = \frac{1}{2} \iint_R \omega_1 \wedge * \overline{\omega_2}; \quad (2.1)$$

Denote by $L^2(R)$ the set of one-forms which are L^2 with respect to this inner product. The Bergman space of holomorphic one forms is

$$A(R) = \{\alpha \in L^2(R) : \alpha \text{ holomorphic}\}$$

and the set of antiholomorphic L^2 one-forms will be denoted by $\overline{A(R)}$. This notation is of course consistent, because $\beta \in \overline{A(R)}$ if and only if $\beta = \overline{\alpha}$ for some $\alpha \in A(R)$. We will also denote

$$A_{\text{harm}}(R) = \{\alpha \in L^2(R) : \alpha \text{ harmonic}\}.$$

If $\alpha, \beta \in A(R)$ then $*\overline{\beta} = i\beta$, from which we see that

$$(\alpha, \beta) = \frac{i}{2} \iint_R \alpha \wedge \overline{\beta}.$$

Observe that $A(R)$ and $\overline{A(R)}$ are orthogonal with respect to the aforementioned inner product.

If $F : R_1 \rightarrow R_2$ is a conformal map, then we denote the pull-back of $\alpha \in A_{\text{harm}}(R_2)$ under F by $F^*\alpha$.

We also define the Dirichlet spaces by

$$\begin{aligned}\mathcal{D}_{\text{harm}}(R) &= \{f : R \rightarrow \mathbb{C}, f \in C^2(R), : df \in L^2(R) \text{ and } d * df = 0\}, \\ \mathcal{D}(R) &= \{f : R \rightarrow \mathbb{C} : df \in A(R)\}, \text{ and} \\ \overline{\mathcal{D}(R)} &= \{f : R \rightarrow \mathbb{C} : df \in \overline{A(R)}\}.\end{aligned}$$

We can define a degenerate inner product on $\mathcal{D}_{\text{harm}}(R)$ by

$$(f, g)_{\mathcal{D}_{\text{harm}}(R)} = (df, dg)_{A_{\text{harm}}(R)}$$

where the right hand side is the inner product (2.1) restricted to elements of $A_{\text{harm}}(R)$.

If we denote

$$\mathcal{D}_{\text{harm}}(R)_q = \{f \in \mathcal{D}_{\text{harm}}(R) : f(q) = 0\}$$

for some $q \in R$, then the scalar product defined above is a genuine inner product on $\mathcal{D}_{\text{harm}}(R)_q$ and also makes it a Hilbert space. In what follows, a subscript q on a space of functions indicates the subspace of functions such that $f(q) = 0$.

If we now define the Wirtinger operators via their local coordinate expressions

$$\partial f = \frac{\partial f}{\partial z} dz, \quad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}} d\bar{z},$$

then the aforementioned inner product can be written as

$$(f, g)_{\mathcal{D}_{\text{harm}}(R)} = \frac{i}{2} \iint_R [\partial f \wedge \overline{\partial g} - \bar{\partial} f \wedge \partial \bar{g}]. \quad (2.2)$$

One can easily see from (2.2) that $\mathcal{D}(R)$ and $\overline{\mathcal{D}(R)}$ are orthogonal with respect to the inner product. We also note that if R is a planar domain and $f \in \mathcal{D}(R)$, then $(f, f)_{\mathcal{D}(R)} = \iint_R |f'(z)|^2 dA$ where dA denotes Lebesgue measure in the plane.

Finally, we will repeatedly use the following elementary fact.

Lemma 2.1 *Let $U \subset \mathbb{C}$ be an open set. For any compact subset K of U , there is a constant M_K such that*

$$\sup_{z \in K} |\alpha(z)| \leq M_K \|\alpha(z) dz\|_{A_{\text{harm}}(U)}$$

for all $\alpha(z) dz \in A_{\text{harm}}(U)$.

For any Riemann surface R , compact subset K of R , and fixed $q \in R$, there is a constant M_K such that

$$\sup_{z \in K} |h(z)| \leq M_K \|h\|_{\mathcal{D}_{\text{harm}}(R)_q}$$

for all $h \in \mathcal{D}_{\text{harm}}(R)_q$.

The first claim is classical and the second claim is an elementary consequence of the first.

2.2 Transmission of harmonic functions through quasicircles

In this section we summarize some necessary results of the authors. The proofs can be found in [20].

Let R be a compact Riemann surface. Let Γ be a Jordan curve in R , that is a homeomorphic image of $\mathbb{S}^1$. We say that U is a doubly-connected neighbourhood of Γ if U is an open set containing Γ , which is bounded by two non-intersecting Jordan curves each of which is homotopic to Γ within the closure of U . We say that a Jordan curve Γ is *strip-cutting* if there is a doubly-connected neighbourhood U of Γ and a conformal map $\phi : U \rightarrow \mathbb{A} \subseteq \mathbb{C}$ so that $\phi(\Gamma)$ is a Jordan curve in $\mathbb{C}$. We say that Γ is a *quasicircle* if $\phi(\Gamma)$ is a quasicircle in $\mathbb{C}$. By a quasicircle in $\mathbb{C}$ we mean the image of the circle $\mathbb{S}^1$ under a quasiconformal mapping of the plane. In particular a quasicircle is a strip-cutting Jordan curve. A closed analytic curve is strip-cutting by definition.

If R is a Riemann surface and $\Sigma \subset R$ is a proper open connected subset of R which is itself a Riemann surface, in such a way that the inclusion map is holomorphic, then we say that $g(w, z)$ is the Green's function for Σ if $g(w, \cdot)$ is harmonic on $R \setminus \{w\}$, $g(w, z) + \log |\phi(z) - \phi(w)|$ is harmonic in z for a local parameter $\phi : U \rightarrow \mathbb{C}$ in an open neighbourhood U of w , and $\lim_{z \rightarrow z_0} g(w, z) = 0$ for all $z_0 \in \partial \Sigma$ and $w \in \Sigma$. Green's function is unique and symmetric, provided that it exists. In this paper, we will consider only the case where R is compact and no boundary component of Σ reduces to a point, so Green's function of Σ exists; see for example Ahlfors and Sario [1, II.3 11H, III.1 4D].

Now let Σ be one of the connected components in R of the complement of Γ . Fix a point $q \in \Sigma$ and let g_q be Green's function of Σ with singularity at q . We associate to g_q a biholomorphism from a doubly-connected region in Σ , one of whose borders is Γ , onto an annulus as follows. Let γ be a smooth curve in Σ which is homotopic to Γ , and let $m = \int_{\gamma} *dg_q$. If $\tilde{g}$ denotes the multi-valued harmonic conjugate of g_q , then the function

$$\phi = \exp[-2\pi(g_q + i\tilde{g})/m]$$

is holomorphic and single-valued on some region A_r bounded by Γ and a level curve $\Gamma_r^q = \{z : g_q(z) = r\}$ of g_q for some $r > 0$. A standard use of the argument principle shows that ϕ is one-to-one and onto the annulus $\{z : e^{-2\pi r/m} < |z| < 1\}$. It can be shown that ϕ has a continuous extension to Γ which is a homeomorphism of Γ onto $\mathbb{S}^1$. By decreasing r , one can also arrange that ϕ extends analytically to a neighbourhood of Γ_r^q .

We call this the *canonical collar chart* with respect to (Σ, q) . It is uniquely determined up to a rotation and the choice of r in the definition of domain.

We say that a closed set $I \subseteq \Gamma$ is null with respect to (Σ, q) if $\phi(I)$ has logarithmic capacity zero in $\mathbb{S}^1$. The notion of a null set does not depend on the position of the singularity q . For quasicircles, it is also independent of the side of the curve.

Theorem 2.2 *Let R be a compact Riemann surface and Γ be a strip-cutting Jordan curve separating R into two connected components Σ_1 and Σ_2 . Let I be a closed set in Γ .*

- (1) *I is null with respect to (Σ_1, q) for some $q \in \Sigma_1$ if and only if it is null with respect to (Σ_1, q) for all $q \in \Sigma_1$.*
- (2) *If Γ is a quasicircle, then I is null with respect to (Σ_1, q) for some $q \in \Sigma_1$ if and only if I is null with respect to (Σ_2, p) for all $p \in \Sigma_2$.*

Thus for quasicircles we can say “ I is null in Γ ” without ambiguity. For strip-cutting Jordan curves, we may say that “ I is null in Γ with respect to Σ ” without ambiguity.

Definition 2.3 Given a function f on an open neighbourhood of Γ in the closure of Σ , we say that the limit of f exists conformally non-tangentially at $p \in \Gamma$ with respect to (Σ, q) if $f \circ \phi^{-1}$ has non-tangential limits at $\phi(p)$ where ϕ is the canonical collar chart induced by Green’s function g_q of Σ . The conformal non-tangential limit of f at p is defined to be the non-tangential limit of $f \circ \phi^{-1}$.

We will abbreviate “conformally non-tangential” as CNT throughout the paper.

Theorem 2.4 *Let R be a compact Riemann surface and let Γ be a strip-cutting Jordan curve separating R into two connected components. Let Σ be one of these components. For any $H \in \mathcal{D}_{\text{harm}}(\Sigma)$, the CNT limit of H exists at every point in Γ except possibly on a null set with respect to Σ . For any q and q' in Σ , the boundary values so obtained agree except on a null set I in Γ . If $H_1, H_2 \in \mathcal{D}(\Sigma)$ have the same CNT boundary values except on a null set then $H_1 = H_2$.*

From now on, the terms “CNT boundary values” and “boundary values” of a Dirichlet-bounded harmonic function refer to the CNT limits thus defined except possibly on a null set. Also, if Γ is a quasicircle, we say that two functions h_1 and h_2 agree on Γ ($h_1 = h_2$) if they agree except on a null set. Outside of this section we will drop the phrase “except on a null set”, although it is implicit wherever boundary values are considered.

The set of boundary values of Dirichlet-bounded harmonic functions in a certain sense determined only by a neighbourhood of the boundary. For quasicircles, it is side-independent: that is, the set of boundary values of the Dirichlet spaces of Σ_1 and Σ_2 agree.

To make the first statement precise we define a kind of one-sided neighbourhood of Γ which we call a collar neighbourhood. Let Γ be a strip-cutting Jordan curve in a Riemann surface R . By a collar neighbourhood of Γ we mean an open set A , bounded by two Jordan curves one of which is Γ , and such that (1) the other Jordan curve Γ' is homotopic to Γ in the closure of A and (2) $\Gamma' \cap \Gamma$ is empty. For example, if U is a doubly-connected neighbourhood of Γ , and Γ separates a compact Riemann surface R into two connected components, the intersection of U with one of the components is a collar neighbourhood. Also, the domain of the canonical collar chart is a collar neighbourhood if the annulus $r < |z| < 1$ is chosen with r sufficiently close to one.

Theorem 2.5 *Let R be a compact Riemann surface and let Γ be a strip-cutting Jordan curve separating R into connected components Σ_1 and Σ_2 . Let h be a function defined on Γ , except possibly on a null set in Γ . The following are equivalent.*

- (1) *There is some $H \in \mathcal{D}_{\text{harm}}(\Sigma_1)$ whose CNT boundary values agree with h except possibly on a null set.*
- (2) *There is a collar neighbourhood A of Γ in Σ_1 , one of whose boundary components is Γ , and some $H \in \mathcal{D}(A)$ whose CNT boundary values agree with h except possibly on a null set with respect to Σ_1 .*

If Γ is a quasicircle, then the following may be added to the list of equivalences above.

- (3) *There is some $H \in \mathcal{D}_{\text{harm}}(\Sigma_2)$ whose CNT boundary values agree with h except possibly on a null set.*
- (4) *There is a collar neighbourhood A of Γ in Σ_2 , one of whose boundary components is Γ , and some $H \in \mathcal{D}(A)$ whose CNT boundary values agree with h except possibly on a null set.*

Thus, for a quasicircle Γ we may define $\mathcal{H}(\Gamma)$ to be the set of equivalence classes of functions $h : \Gamma \rightarrow \mathbb{C}$ which are boundary values of elements of $\mathcal{D}_{\text{harm}}(\Sigma_1)$ except possibly on a null set, where we define two such functions to be equivalent if they agree except possibly on a null set.

This theorem also induces a map from $\mathcal{D}_{\text{harm}}(\Sigma_1)$ to $\mathcal{D}_{\text{harm}}(\Sigma_2)$ as follows:

Definition 2.6 Let Γ be a quasicircle in a compact Riemann surface R , separating it into two connected components Σ_1 and Σ_2 . Given $H \in \mathcal{D}(\Sigma_1)$, let h be the CNT boundary values of H on Γ . Define $\mathfrak{D}(\Sigma_1, \Sigma_2)H$ ¹ to be the unique element of $\mathcal{D}_{\text{harm}}(\Sigma_2)$ with boundary values equal to h .

This operator enables one to transmit harmonic functions from one side of the Riemann surface to the other side through the quasicircle Γ .

Theorem 2.7 *Let R be a compact Riemann surface and Γ be a quasicircle separating R into components Σ_1 and Σ_2 . The map*

$$\mathfrak{D}(\Sigma_1, \Sigma_2) : \mathcal{D}_{\text{harm}}(\Sigma_1) \rightarrow \mathcal{D}_{\text{harm}}(\Sigma_2)$$

induced by Theorem 2.5 is bounded with respect to the Dirichlet semi-norm.

3 Schiffer's comparison operators

3.1 Assumptions

The following notation and assumptions will be in place throughout the rest of the paper (see the relevant sections for further explanations):

¹ The notation $\mathfrak{D}$ for this transmission operator stems from the first letter in the Old English word “oferferian” which means “to transmit” (or “to overfare”).

- R is a compact Riemann surface;
- Γ is a strip-cutting Jordan-like curve separating R ;
- Σ_1 and Σ_2 are the connected components of $R \setminus \Gamma$;
- Σ stands for an unspecified component Σ_1 or Σ_2 ;
- Γ is positively oriented with respect to Σ_1 ;
- $\Gamma_\epsilon^{p_k}$ the level curves of Green's function $g_{\Sigma_k}(\cdot, p_k)$ with respect to some fixed points $p_k \in \Sigma_k$;
- when an integrand depends on two variables, we will use the notation $\iint_{\Sigma, w}$ to specify that the integration takes place over the variable w .

We will sometimes alter the assumptions or repeat them for emphasis. When no assumptions are indicated at all, the above assumptions are in place.

3.2 Schiffer's comparison operators: definitions

Following for example Royden [12], we define Green's function of R to be the unique function $g(w, w_0; z, q)$ such that

- (1) g is harmonic in w on $R \setminus \{z, q\}$;
- (2) for a local coordinate ϕ on an open set U containing z , $g(w, w_0; z, q) + \log |\phi(w) - \phi(z)|$ is harmonic for $w \in U$;
- (3) for a local coordinate ϕ on an open set U containing q , $g(w, w_0; z, q) - \log |\phi(w) - \phi(q)|$ is harmonic for $w \in U$;
- (4) $g(w_0, w_0; z, q) = 0$ for all z, q, w_0 .

It can be shown that g exists, is uniquely determined by these properties, and furthermore satisfies the symmetry properties

$$g(w, w_1; z, q) = g(w, w_0; z, q) - g(w_1, w_0; z, q) \quad (3.1)$$

$$g(w_0, w; z, q) = -g(w, w_0; z, q) \quad (3.2)$$

$$g(z, q; w, w_0) = g(w, w_0; z, q). \quad (3.3)$$

In particular, g is also harmonic in z away from the poles.

We will treat w_0 as fixed throughout the paper, and notationally drop the dependence on w_0 as much as possible. In fact, it follows immediately from (3.1) that $\partial_w g$ is independent of w_0 . All formulas of consequence in this paper are independent of w_0 for this reason.

The following is an immediate consequence of the residue theorem and the fact that g is harmonic in w .

Theorem 3.1 *Let Γ be a closed analytic curve separating R , enclosing Σ , which is positively oriented with respect to Σ . If h is holomorphic on Σ , and $z, q \notin \Gamma$, then for any fixed $p \in \Sigma$*

$$-\lim_{\epsilon \searrow 0} \frac{1}{\pi i} \int_{\Gamma_\epsilon^p} h(w) \partial_w g(w, w_0; z, q) = \chi_\Sigma(z)h(z) - \chi_\Sigma(q)h(q)$$

where χ_Σ is the characteristic function of Σ .

We will also need the following well-known reproducing formula for Green's function of Σ .

Theorem 3.2 *Let R be a compact Riemann surface and Γ be a strip-cutting Jordan curve separating R . Let Σ be one of the components of the complement of Γ . For any $h \in \mathcal{D}_{\text{harm}}(\Sigma)$, we have*

$$h(z) = \lim_{\epsilon \searrow 0} -\frac{1}{\pi i} \int_{\Gamma_\epsilon^p} \partial_w g_\Sigma(w, z) h(w).$$

Next we turn to the definitions of the relevant kernel forms. Let R be a compact Riemann surface, and let $g(w, w_0; z, q)$ be the Green's function. Following [16], we define the Schiffer kernel to be the bi-differential

$$L_R(z, w) = \frac{1}{\pi i} \partial_z \partial_w g(w, w_0; z, q).$$

and the Bergman kernel to be the bi-differential

$$K_R(z, w) = -\frac{1}{\pi i} \partial_z \bar{\partial}_w g(w, w_0; z, q).$$

For non-compact surfaces Σ with border, with Green's function g , we define

$$L_\Sigma(z, w) = \frac{1}{\pi i} \partial_z \partial_w g(w, z).$$

and

$$K_\Sigma(z, w) = -\frac{1}{\pi i} \partial_z \bar{\partial}_w g(w, z).$$

Then the following identity holds. For any vector v tangent to Γ_ϵ^w at a point z , we have

$$\overline{K_\Sigma(z, w)}(\cdot, v) = -L_\Sigma(z, w)(\cdot, v) \quad (3.4)$$

This follows directly from the fact that the one form $\partial_z g(z, w) + \bar{\partial}_z g(z, w)$ vanishes on tangent vectors to the level curve Γ_ϵ^w .

It is well known that for all $h \in A(\Sigma)$

$$\iint_\Sigma K_\Sigma(z, w) \wedge h(w) = h(z). \quad (3.5)$$

For compact surfaces, the reproducing property of the Bergman kernel is established in [12].

Proposition 3.3 *Let R be a compact Riemann surface with Green's function $g(w, w_0; z, q)$. Then*

- (1) L_R and K_R are independent of q and w_0 .
- (2) K_R is holomorphic in z for fixed w , and anti-holomorphic in w for fixed z .
- (3) L_R is holomorphic in w and z , except for a pole of order two when $w = z$.
- (4) $L_R(z, w) = L_R(w, z)$.
- (5) $K_R(w, z) = -\overline{K_R(z, w)}$.

For non-compact Riemann surfaces Σ with Green's function, (2) – (5) hold with L_R and K_R replaced by L_Σ and K_Σ .

Remark 3.4 The symmetry statements (4) and (5) are formally expressed as follows. If $D : R \times R \rightarrow R \times R$ is the map $D(z, w) = (w, z)$ then $D^*L = L \circ D$ and $D^*K = \overline{K} \circ D$.

Proof It follows immediately from (3.1) that

$$\partial_w g(w, w_1; z, q) = \partial_w g(w, w_0; z, q) \quad \text{and} \quad \partial_{\bar{w}} g(w, w_1; z, q) = \partial_{\bar{w}} g(w, w_0; z, q),$$

so L_R and K_R are independent of w_0 . Applying (3.3) shows that similarly $\partial_w g$ and $\partial_{\bar{w}}$ are independent of q , and hence the same holds for L_R and K_R . This demonstrates that property (1) holds.

Since g is harmonic in w , $\partial_w \bar{\partial}_w g(w, w_0; z, q) = 0$ so K_R is anti-holomorphic in w . As observed above, (3.2) shows that g is also harmonic in z , so we similarly have that K_R is holomorphic in z . This demonstrates (2).

Similarly harmonicity of g in z and w implies that L_R is holomorphic in z and w . The fact that L_R has a pole of order two at z follows from the fact that g has a logarithmic singularity at $w = z$. This proves (3).

Properties (4) and (5) follow from Eq. (3.3) applied directly to the definitions of L_R and K_R .

The non-compact case follows similarly from the harmonicity with logarithmic singularity of g_Σ , and the symmetry $g_\Sigma(z, w) = g_\Sigma(w, z)$ $\square$

Remark 3.5 The reader might find the negative sign in (5) surprising, since the Bergman kernel should be skew-symmetric. However this is in agreement with the usual convention when one takes into account that one usually integrates against a measure, whereas the kernel K_R is a bi-differential to be integrated against one-forms. For example, if R is a region in the plane and $\alpha = h(w)dw$ is a one-form, then we have

$$\begin{aligned} \iint_R K_R(z, w) \wedge_w \alpha(w) &= -\frac{1}{\pi i} \iint_R \frac{\partial^2 g}{\partial \bar{w} \partial z}(w, z) d\bar{w} dz \wedge_w h(w) dw \\ &= -\frac{2}{\pi} \iint_R \frac{\partial^2 g}{\partial \bar{w} \partial z}(w, z) h(w) dA_w dz, \end{aligned}$$

where dA is Euclidean Lebesgue measure. Observe that the kernel of the final integral is in fact skew-symmetric.

One can find the constant at the pole of L from the definition. Expressed in a local holomorphic coordinates $\eta = \phi(w)$ near a fixed point $\zeta = \phi(z)$,

$$(\phi^{-1} \times \phi^{-1})^* L(z, w) = \left(-\frac{1}{2\pi i} \frac{1}{(\zeta - \eta)^2} + H(\eta) \right) d\zeta d\eta \quad (3.6)$$

where $H(\eta)$ is holomorphic in a neighbourhood of ζ . In most sources [3,4,8,13] the integral kernel is expressed as a function (rather than a form) to be integrated against the Euclidean area form $dA_\eta = d\bar{\eta} \wedge d\eta/2i$. For example, if $\overline{\alpha(w)}$ is a holomorphic one-form given in local coordinates by $(\phi^{-1})^* \alpha(\eta) = \overline{f(\eta)} d\bar{\eta}$ then we obtain the local expression

$$(\phi^{-1} \times \phi^{-1})^* L(z, w) \wedge_\eta \phi^{-1*} \alpha(w) = \left(\frac{1}{\pi} \frac{1}{(\zeta - \eta)^2} + H(\eta) \right) \overline{f(\eta)} d\zeta dA_\eta$$

which agrees with the classical normalization [3].

Now let R be a compact Riemann surface and let Γ be a strip-cutting Jordan curve. Assume that Γ separates R into two surfaces Σ_1 and Σ_2 . We will mostly be concerned with the case that Γ is a quasicircle.

Let $A(\Sigma_1 \cup \Sigma_2)$ denote the set of one-forms α on $\Sigma_1 \cup \Sigma_2$ which are holomorphic and square integrable, in the sense that their restrictions to Σ_k is in $A(\Sigma_k)$ for $k = 1, 2$; that is, $\|\alpha|_{\Sigma_1}\|_{\Sigma_1}^2 + \|\alpha|_{\Sigma_2}\|_{\Sigma_2}^2 < \infty$. Note that we do not require the existence of a holomorphic or continuous extension to the closure of $\Sigma_1 \cup \Sigma_2$. For $k = 1, 2$ define the restriction operators

$$\begin{aligned} \text{Res}(\Sigma_k) : A(R) &\rightarrow A(\Sigma_k) \\ \alpha &\mapsto \alpha|_{\Sigma_k} \end{aligned}$$

and

$$\begin{aligned} \text{Res}_0(\Sigma_k) : A(\Sigma_1 \cup \Sigma_2) &\rightarrow A(\Sigma_k) \\ \alpha &\mapsto \alpha|_{\Sigma_k}. \end{aligned}$$

It is obvious that these are bounded operators.

Definition 3.6 For $k = 1, 2$, we define the Schiffer comparison operators

$$\begin{aligned} T(\Sigma_k) : \overline{A(\Sigma_k)} &\rightarrow A(\Sigma_1 \cup \Sigma_2) \\ \bar{\alpha} &\mapsto \iint_{\Sigma_k} L_R(\cdot, w) \wedge \overline{\alpha(w)}. \end{aligned}$$

and

$$S(\Sigma_k) : A(\Sigma_k) \rightarrow A(R)$$

$$\alpha \mapsto \iint_{\Sigma_k} K_R(\cdot, w) \wedge \alpha(w).$$

Also, we define for $j, k \in \{1, 2\}$

$$T(\Sigma_j, \Sigma_k) = \text{Res}_0(\Sigma_k)T(\Sigma_j) : \overline{A(\Sigma_j)} \rightarrow A(\Sigma_k).$$

We will also call these Schiffer comparison operators.

Note that the operator S is bounded and the image is clearly in $A(R)$. This can be seen from the fact that the kernel form is holomorphic in w and R is compact. On the other hand, for $j \neq k$, the integral kernel of the operator $T(\Sigma_j, \Sigma_k)$ is nonsingular, but if $j = k$, then the kernel has a pole of order 2 when $z = w$; thus the output of the operator $T(\Sigma_j)$ need not have a holomorphic continuation across Γ . In fact, the jump formula will show that it does not. We will show below that the image of $T(\Sigma_j, \Sigma_k)$ is in fact in $A(\Sigma_k)$, as the notation indicates.

Example 3.7 Let R be the Riemann sphere $\bar{\mathbb{C}}$, and let Γ be a Jordan curve in $\mathbb{C}$ dividing $\bar{\mathbb{C}}$ into two Jordan domains Σ_1 and Σ_2 ; assume that Σ_1 is the bounded domain. With the normalization $w_0 = \infty$, we have

$$g(w, \infty; z, q) = -\log \frac{|w - z|}{|w - q|}.$$

From this, it can be calculated that

$$K_{\mathbb{C}}(z, w) = 0.$$

Thus, $S(\Sigma_1) = 0$, as is expected as a consequence of the non-existence of non-trivial holomorphic one-forms on $\bar{\mathbb{C}}$. We can also calculate that

$$L_{\bar{\mathbb{C}}}(z, w) = -\frac{1}{2\pi i} \frac{dw dz}{(w - z)^2}.$$

Thus for $\overline{\alpha(w)} = \overline{h(w)}d\bar{w} \in \overline{A(\Sigma_1)}$, we have

$$[T(\Sigma_1, \Sigma_1 \cup \Sigma_2)\overline{\alpha}](z) = \frac{1}{\pi} \iint_{\Sigma_1} \frac{\overline{h(w)}}{(w - z)^2} \frac{d\bar{w} \wedge dw}{2i} \cdot dz.$$

If we choose for example $\Sigma_1 = \mathbb{D}$, we see that

$$g(z, w) = -\log \frac{|z - w|}{|1 - \bar{w}z|}.$$

So

$$L_{\mathbb{D}} = \frac{-1}{2\pi i} \frac{dw dz}{(w - z)^2}$$

and

$$K_{\mathbb{D}} = \frac{1}{2\pi i} \frac{d\bar{w} dz}{(1 - \bar{w}z)^2}.$$

First we require an identity of Schiffer. Although this identity was only stated for analytically bounded domains, it is easily seen to hold in greater generality.

Theorem 3.8 *For all $\bar{\alpha} \in \overline{A(\Sigma)}$*

$$\iint_{\Sigma, w} L_{\Sigma}(z, w) \wedge \overline{\alpha(w)} = 0.$$

Proof We assume momentarily that α has a holomorphic extension to the closure of Σ and that Γ is an analytic curve. Let $z \in \Sigma$ be fixed but arbitrary, and choose a chart ζ near z such that $\zeta(z) = 0$. Write $\bar{\alpha}$ locally as $\overline{f(\zeta)d\zeta}$ for some holomorphic function f . Let C_r be the curve $|\zeta| = r$, and denote its image in Σ by γ_r . Fixing $p \in \Sigma$ and using Stokes' theorem yield

$$\iint_{\Sigma, w} L_{\Sigma}(z, w) \wedge \overline{\alpha(w)} = \lim_{\epsilon \searrow 0} \frac{1}{\pi i} \int_{\Gamma_{\epsilon}^p} \partial_z g(w, z) \overline{\alpha(w)} - \lim_{r \searrow 0} \frac{1}{\pi i} \int_{C_r} \partial_z g(w, z) \overline{\alpha(w)}.$$

The first term goes to zero uniformly as $\epsilon \rightarrow 0$. Writing the second term in coordinates $\eta = \phi(w)$ in a neighbourhood of ζ for fixed ζ (see Eq. (3.6)) we obtain

$$\iint_{\Sigma, w} L_{\Sigma}(z, w) \wedge \overline{\alpha(w)} = \lim_{r \searrow 0} -\frac{1}{2\pi i} \int_{C_r} \left(\frac{1}{\eta} + h(\zeta) \right) \overline{f(\eta)d\eta}$$

where h is some harmonic function in a neighbourhood of 0. Now since both terms on the right hand side go to zero, we obtain the desired result.

Note that this shows that the principal value integral can be taken with respect to any local coordinate with the same result. Furthermore, the integral is conformally invariant. Thus, we may assume that Σ is a subset of its double and Γ is analytic. By [2, Proposition 2.2], the set of holomorphic one-forms on an open neighbourhood of the closure of Σ is dense in $A(\Sigma)$. The L^2 boundedness of the L_{Σ} operator yields the desired result. $\square$

This implies that for R , Γ , and Σ as in Theorem 3.8, we can write

$$[T(\Sigma, \Sigma)\alpha](z) = \iint_{\Sigma, w} (L_R(z, w) - L_{\Sigma}(z, w)) \wedge \overline{\alpha(w)}, \quad (3.7)$$

which has the advantage that the integral kernel is non-singular.

Remark 3.9 The above expression shows that the operator $T(\Sigma, \Sigma)$ is well-defined. The subtlety is that the principal value integral might depend on the choice of coordinates, which determines the ball which one removes in the neighbourhood of the

singularity. Since the integrand is not in L^2 , different exhaustions of Σ might in principle lead to different values of the integral.

However the proof of Theorem 3.8 shows that the integral of L_Σ is independent of the choice of coordinate near the singularity. Since the integrand of (3.7) is L^2 bounded, it is independent of the choice of exhaustion; combining this with Theorem 3.8 shows that the integral in the definition of $T(\Sigma, \Sigma)$ is independent of the choice of exhaustion. One may also obtain this fact from the general theory of Calderón–Zygmund operators on manifolds, see [21].

Theorem 3.10 *Let R be a compact Riemann surface, and Γ be a strip-cutting Jordan curve in R . Assume that Γ separates R into two surfaces Σ_1 and Σ_2 . Then $T(\Sigma_j)\bar{\alpha} \in A(\Sigma_1 \cup \Sigma_2)$ for all $\alpha \in A(\Sigma_j)$ for $j = 1, 2$. Furthermore for all $j, k \in \{1, 2\}$, $T(\Sigma_j)$ and $T(\Sigma_j, \Sigma_k)$ are bounded operators.*

Proof Fix j and let $k \in \{1, 2\}$ be such that $k \neq j$. By (3.7) we observe that

$$T(\Sigma_j)\bar{\alpha}(z) = \begin{cases} \iint_{\Sigma_j, w} L_R(z, w) \wedge \overline{\alpha(w)} & z \in \Sigma_k \\ \iint_{\Sigma_j, w} (L_R(z, w) - L_{\Sigma_j}(z, w)) \wedge \overline{\alpha(w)} & z \in \Sigma_j \end{cases} \quad (3.8)$$

The integrand in both terms (3.8) is non-singular and holomorphic in z for each $w \in \Sigma_j$, and furthermore both integrals are locally bounded in z . Therefore the holomorphicity of $T(\Sigma_j)\bar{\alpha}$ follows by moving the $\bar{\partial}$ inside (3.7), and using the holomorphicity of the integrand. This also implies the holomorphicity of $T(\Sigma_j, \Sigma_k)$.

Regarding the boundedness, the operator $T(\Sigma_j)$ is defined by integration against the L -Kernel which in local coordinates is given by $\frac{1}{\pi(\xi - \eta)^2}$, modulo a holomorphic function. Since the singular part of the kernel is a Calderón–Zygmund kernel we can use the theory of singular integral operators on general compact manifolds, developed by Seeley in [21] to conclude that, the operators with kernels such as $L_R(z, w)$ are bounded on L^p for $1 < p < \infty$. The boundedness of $T(\Sigma_j, \Sigma_k)$ follows from this and the fact that $R_0(\Sigma_j)$ is also bounded. $\square$

3.3 Attributions

The comparison operators $T(\Sigma_j, \Sigma_k)$ were studied extensively by Schiffer [13–15], and also together with other authors, e.g. Bergman and Schiffer [3]. In the setting of planar domains, a comprehensive outline of the theory was developed in a chapter in [4]. The comparison theory for Riemann surfaces can be found in Schiffer and Spencer [16]. See also our review paper [19].

In this section, we demonstrate some necessary identities for the Schiffer operators. Most of the identities were stated by for example Bergman and Schiffer [3], Schiffer [4], and Schiffer and Spencer [16] for the case of analytic boundaries. Versions can be found in different settings, for example multiply-connected domains in the sphere, nested multiply-connected domains, and Riemann surfaces.

On the other hand, we introduce here several identities involving the adjoints of the operators, which Schiffer seems not to have been aware of. These are Theorems 3.11,

3.12, and 3.13. The introduction of the adjoint operators has significant clarifying power. Proofs of the remaining identities are included because it is necessary to show that they hold for regions bordered by quasicircles.

Here are a few words on terminology. The Beurling transform in the plane is defined by

$$B_{\mathbb{C}}f(z) = \frac{-1}{\pi} \text{PV} \iint_{\mathbb{C}} \frac{f(\zeta)}{(z - \zeta)^2} dA(\zeta).$$

Schiffer refers to this operator as the Hilbert transform, due to the fact that the operator in question behaves like the actual Hilbert transform

$$\mathcal{H}f(x) := \text{PV} \int_{\mathbb{R}} \frac{f(y)}{x - y} dy.$$

The term “Hilbert transform” is also the one used in Lehto’s classical book on Teichmüller theory [7]. Indeed the integrands of both operators exhibit a similar type of singularity in their respective domains of integration and both fall into the general class of Calderón–Zygmund singular integral operators. For such operators, one has quite a complete and satisfactory theory, both in the plane and on differentiable manifolds.

We shall refer to the restriction of the Beurling transform to anti-holomorphic functions on fixed domain as a *Schiffer operator*. Here, of course, we express this equivalently as an operator on anti-holomorphic one-forms.

3.4 Identities for comparison operators

Theorem 3.11 *Let R be a compact surface and let Γ be a strip-cutting Jordan curve separating R into two components, one of which is Σ . Then $S(\Sigma) = \text{Res}(\Sigma)^*$, where $*$ denotes the adjoint operator.*

Proof Let $\alpha \in A(\Sigma)$ and $\beta \in A(R)$. Then, using the reproducing property of K_R and Proposition 3.3 we have

$$\begin{aligned} (S(\Sigma)\alpha, \beta)_R &= \frac{i}{2} \iint_{R,z} \iint_{\Sigma,\zeta} K_R(z, \zeta) \wedge_{\zeta} \alpha(\zeta) \wedge_z \overline{\beta(z)} \\ &= \frac{-i}{2} \iint_{\Sigma,\zeta} \iint_{R,z} \overline{K_R(\zeta, z)} \wedge_z \overline{\beta(z)} \wedge_{\zeta} \alpha(\zeta) \\ &= \frac{-i}{2} \iint_{\Sigma,\zeta} \overline{\beta(\zeta)} \wedge \alpha(\zeta) = (\alpha, \text{Res } \beta)_{\Sigma}. \end{aligned}$$

Note that interchange of order of integration is legitimate by Fubini’s theorem, due to the analyticity and boundedness of the Bergman kernel. $\square$

Define

$$\begin{aligned}\overline{T}(\Sigma_j, \Sigma_k) : A(\Sigma_j) &\rightarrow \overline{A(\Sigma_k)} \\ h &\mapsto \overline{T(\Sigma_j, \Sigma_k)h}.\end{aligned}$$

and similarly for $\overline{S}(\Sigma_k)$.

Theorem 3.12 *Let R be a compact surface. Let Γ be a strip-cutting Jordan curve with measure zero, and assume that the complement of Γ consists of two connected components Σ_1 and Σ_2 . Then*

$$T(\Sigma_j, \Sigma_k)^* = \overline{T}(\Sigma_k, \Sigma_j).$$

Proof If $j = k$, the claim follows from the non-singular integral representation (3.7) and interchanging the order of integration.

The claim essentially follows from the corresponding fact for planar domains, and we need only reduce the problem to this case using coordinates. Denote

$$\mathcal{L}_{\mathbb{C}}(z, w) = \frac{1}{\pi} \frac{1}{(z - w)^2}.$$

We first show that for $G, H \in L^2(\mathbb{C})$ one has

$$\begin{aligned}&\iint_{\mathbb{C}} \left(\iint_{\mathbb{C}} \mathcal{L}_{\mathbb{C}}(z, w) \overline{H(z)} dA(z) \right) \overline{G(w)} dA(w) \\ &= \iint_{\mathbb{C}} \left(\iint_{\mathbb{C}} \mathcal{L}_{\mathbb{C}}(z, w) \overline{G(w)} dA(w) \right) \overline{H(z)} dA(z)\end{aligned}\quad (3.9)$$

where the inside integral is understood as a principle value integral in both cases.

Now, for $f \in L^2(\mathbb{C})$, the Beurling transform is given by

$$B_{\mathbb{C}}f(z) = \text{PV} \iint_{\mathbb{C}} \mathcal{L}_{\mathbb{C}}(z, \zeta) f(\zeta) dA(\zeta) = \frac{-1}{\pi} \text{PV} \iint_{\mathbb{C}} \frac{f(\zeta)}{(z - \zeta)^2} dA(\zeta), \quad (3.10)$$

With this notation, and denoting $\overline{H}(w) = \overline{H(w)}$, (3.9) amounts to

$$\iint_{\mathbb{C}} B_{\mathbb{C}}\overline{H}(w) \overline{G}(w) dA(w) = \iint_{\mathbb{C}} B_{\mathbb{C}}\overline{G}(z) \overline{H}(z) dA(z). \quad (3.11)$$

If one defines the Fourier transform through

$$\widehat{f}(\xi, \eta) = \iint_{\mathbb{R}^2} e^{-2\pi i(x\xi + y\eta)} f(x + iy) dx dy,$$

then one has that $\widehat{B_{\mathbb{C}}f}(\xi, \eta) = \frac{\xi - i\eta}{\xi + i\eta} \widehat{f}(\xi, \eta)$.

Using Parseval's formula and the above Fourier multiplier representation of the Beurling transform, one has that

$$\begin{aligned} \iint_{\mathbb{C}} B_{\mathbb{C}} \overline{H}(w) \overline{G}(w) dA(w) &= \iint_{\mathbb{C}} \widehat{B_{\mathbb{C}} \overline{H}}(\xi, \eta) \widehat{\overline{G}}(\xi, \eta) d\xi d\eta \\ &= \iint_{\mathbb{C}} \frac{\xi - i\eta}{\xi + i\eta} \widehat{H}(\xi, \eta) \widehat{\overline{G}}(\xi, \eta) d\xi d\eta, \end{aligned}$$

and

$$\begin{aligned} \iint_{\mathbb{C}} B_{\mathbb{C}} \overline{G}(z) \overline{H}(z) dA(z) &= \iint_{\mathbb{C}} \widehat{B_{\mathbb{C}} \overline{G}}(\xi, \eta) \widehat{\overline{H}}(\xi, \eta) d\xi d\eta \\ &= \iint_{\mathbb{C}} \frac{\xi - i\eta}{\xi + i\eta} \widehat{\overline{G}}(\xi, \eta) \widehat{\overline{H}}(\xi, \eta) d\xi d\eta. \end{aligned}$$

This proves (3.11) and hence (3.9).

Now let B be a doubly-connected neighbourhood of Γ and $\phi : B \rightarrow U \subseteq \mathbb{C}$ be a doubly-connected chart. Let $E = B \cap \Sigma_1$ and $E' = B \cap \Sigma_2$. Then $\Sigma_1 = D \cup E$ and $\Sigma_2 = D' \cup E'$ for some compact sets $D \subset \Sigma_1$ and $D' \subset \Sigma_2$ whose shared boundaries with E and E' are strip-cutting Jordan curves. We may choose these as regular as desired (say, analytic Jordan curves, which in particular have measure zero). Observe that we then have, for any forms $\alpha \in A(\Sigma_2)$ and $\beta \in A(\Sigma_1)$

$$\begin{aligned} &\iint_{\Sigma_1} \iint_{\Sigma_2} L(\zeta, \eta) \wedge_{\zeta} \overline{\alpha(\zeta)} \wedge_{\eta} \overline{\beta(\eta)} \\ &= \left(\iint_D \iint_{D'} + \iint_D \iint_{E'} + \iint_E \iint_{D'} + \iint_E \iint_{E'} \right) L(\zeta, \eta) \wedge_{\zeta} \overline{\alpha(\zeta)} \wedge_{\eta} \overline{\beta(\eta)}. \end{aligned} \quad (3.12)$$

and

$$\begin{aligned} &\iint_{\Sigma_2} \iint_{\Sigma_1} L(\zeta, \eta) \wedge_{\eta} \overline{\beta(\eta)} \wedge_{\zeta} \overline{\alpha(\zeta)} \\ &= \left(\iint_{D'} \iint_D + \iint_{D'} \iint_E + \iint_{E'} \iint_D + \iint_{E'} \iint_E \right) L(\zeta, \eta) \wedge_{\eta} \overline{\beta(\eta)} \wedge_{\zeta} \overline{\alpha(\zeta)}. \end{aligned} \quad (3.13)$$

We only need to show that one can interchange integrals in each term. The first three integrals in the right hand side of (3.12) are equal to their interchanged counterparts in the first three terms of (3.13). This follows from Fubini's theorem, using the fact that $L(\zeta, \zeta)$ is non-singular and in fact bounded on all of the six domains of integration involved in those integrals. Therefore it is enough to show that

$$\iint_E \iint_{E'} L(\zeta, \eta) \wedge_{\zeta} \overline{\alpha(\zeta)} \wedge_{\eta} \overline{\beta(\eta)} = \iint_{E'} \iint_E L(\zeta, \eta) \wedge_{\eta} \overline{\beta(\eta)} \wedge_{\zeta} \overline{\alpha(\zeta)}.$$

To show this, let ϕ be a local coordinate with $\eta = \phi(w)$ and $\zeta = \phi(z)$. We pull back the integral to the plane under $\psi = \phi^{-1}$ so that we reduce the problem to showing that

$$\begin{aligned} & \iint_{\phi(E)} \iint_{\phi(E')} (\psi \times \psi)^* L(\zeta, \eta) \wedge_{\zeta} \psi^* \overline{\alpha(\zeta)} \wedge_{\eta} \psi^* \overline{\beta(\eta)} \\ &= \iint_{\phi(E')} \iint_{\phi(E)} (\psi \times \psi)^* L(\zeta, \eta) \wedge_{\eta} \psi^* \overline{\beta(\eta)} \wedge_{\zeta} \psi^* \overline{\alpha(\zeta)}. \end{aligned} \quad (3.14)$$

Recall that in local coordinates by Eq. (3.6)

$$(\psi \times \psi)^* L(\zeta, \eta) = \left(-\frac{1}{2\pi i} \frac{1}{(\zeta - \eta)^2} + H(\eta) \right) d\zeta d\eta,$$

where $H(\eta)$ is holomorphic near ζ . For the holomorphic error term, we can just apply Fubini's theorem, so matters reduce to the demonstration of (3.14) for the principal term of $\mathcal{L}_{\mathbb{C}}(\zeta, \eta)$ which contains the singularity. We may write $\psi^* \alpha(z) = h(z)dz$ and $\psi^* \beta(w) = g(w)dw$ for some L^2 holomorphic functions g on E and h on E' . So the problem is reduced to showing that

$$\begin{aligned} & \iint_{\phi(E')} \iint_{\phi(E)} \mathcal{L}_{\mathbb{C}}(z, w) \wedge_{\zeta} \overline{h(z)} \wedge_{\eta} \overline{g(w)} dA(z) dA(w) \\ &:= \iint_{\phi(E')} \iint_{\phi(E)} \mathcal{L}_{\mathbb{C}}(z, w) \overline{h(z)} \overline{g(w)} dA(w) dA(z). \end{aligned}$$

Letting

$$G(z) = \begin{cases} g(z), & z \in E \\ 0, & z \in \mathbb{C} \setminus E \end{cases}$$

and

$$H(z) = \begin{cases} h(z), & z \in E' \\ 0, & z \in \mathbb{C} \setminus E' \end{cases}$$

then G and H are L^2 on $\mathbb{C}$ and the claim now follows directly from (3.9). $\square$

We also have the following identity.

Theorem 3.13 *If Γ is a quasicircle then*

$$T(\Sigma_1, \Sigma_1)^* T(\Sigma_1, \Sigma_1) + T(\Sigma_1, \Sigma_2)^* T(\Sigma_1, \Sigma_2) + \overline{S}(\Sigma_1)^* \overline{S}(\Sigma_1) = I.$$

Proof By Theorem 3.12, and interchange of order of integration (which can be justified as in the proof of Theorem 3.12) we have that

$$\begin{aligned} [T(\Sigma_1, \Sigma_2)^* T(\Sigma_1, \Sigma_2) \alpha](z) &= \iint_{\Sigma_2, \zeta} \overline{L_R(z, \zeta)} \wedge_{\zeta} \iint_{\Sigma_1, w} L_R(\zeta, w) \wedge_w \alpha(w) \\ &= \iint_{\Sigma_1, w} \left(\iint_{\Sigma_2, \zeta} \overline{L_R(z, \zeta)} \wedge_{\zeta} L_R(\zeta, w) \right) \wedge_w \alpha(w) \end{aligned}$$

so the integral kernel of $T(\Sigma_1, \Sigma_2)^* T(\Sigma_1, \Sigma_2)$ is

$$\iint_{\Sigma_2, \zeta} \overline{L_R(z, \zeta)} \wedge_{\zeta} L_R(\zeta, w).$$

Similarly, by Eq. (3.7) and Theorem 3.12, the integral kernel of $T(\Sigma_1, \Sigma_1)^* T(\Sigma_1, \Sigma_1)$ is

$$\iint_{\Sigma_1, \zeta} \left(\overline{L_R(z, \zeta)} - \overline{L_{\Sigma_1}(z, \zeta)} \right) \wedge (L_R(\zeta, w) - L_{\Sigma_1}(\zeta, w)).$$

Finally, by Theorem 3.11, the integral kernel of $\bar{S}(\Sigma_1)^* \bar{S}(\Sigma_1)$ is $\overline{K_R(z, w)}$.

Using this and the reproducing property of K_{Σ} we need only demonstrate the following identity:

$$\begin{aligned} & \iint_{\Sigma_1, \zeta} \left(\overline{L_R(z, \zeta)} - \overline{L_{\Sigma_1}(z, \zeta)} \right) \wedge (L_R(\zeta, w) - L_{\Sigma_1}(\zeta, w)) \\ & + \iint_{\Sigma_2, \zeta} \overline{L_R(z, \zeta)} \wedge L_R(\zeta, w) = \overline{K_{\Sigma_1}(z, w)} - \overline{K_R(z, w)}. \end{aligned} \quad (3.15)$$

Fix $w \in \Sigma_1$ and orient Γ_{ϵ}^w positively with respect to Σ_1 . For fixed w , $\partial_w g_{\Sigma_1}(\zeta, w)$ goes to zero uniformly as $\epsilon \rightarrow 0$. We then have that, applying Theorem 3.8,

$$\begin{aligned} & \iint_{\Sigma_1, \zeta} \left(\overline{L_R(z, \zeta)} - \overline{L_{\Sigma_1}(z, \zeta)} \right) \wedge (L_R(\zeta, w) - L_{\Sigma_1}(\zeta, w)) \\ & = \iint_{\Sigma_1, \zeta} \left(\overline{L_R(z, \zeta)} - \overline{L_{\Sigma_1}(z, \zeta)} \right) \wedge L_R(\zeta, w) \\ & = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi i} \int_{\Gamma_{\epsilon}^w} \left(\overline{L_R(z, \zeta)} - \overline{L_{\Sigma_1}(z, \zeta)} \right) \partial_w g(\zeta, w) \\ & = \frac{1}{\pi i} \lim_{\epsilon \rightarrow 0} \int_{\Gamma_{\epsilon}^w} \overline{L_R(z, \zeta)} \partial_w g(\zeta, w) + \frac{1}{\pi i} \lim_{\epsilon \rightarrow 0} \int_{\Gamma_{\epsilon}^w} K_{\Sigma_1}(z, \zeta) \partial_w g(\zeta, w) \end{aligned}$$

where we have applied Eq. (3.4) in the last step.

Applying Stokes' theorem to the first term, we see that

$$\frac{1}{\pi i} \lim_{\epsilon \rightarrow 0} \int_{\Gamma_{\epsilon}^w} \overline{L_R(z, \zeta)} \partial_w g(\zeta, w) = - \iint_{\Sigma_2, \zeta} \overline{L_R(z, \zeta)} \wedge L_R(\zeta, w).$$

Here we used the fact that quasicircles have measure zero. Note that Γ_{ϵ}^w is negatively oriented with respect to Σ_2 . For the second term, we have

$$\begin{aligned} \frac{1}{\pi i} \lim_{\epsilon \rightarrow 0} \int_{\Gamma_{\epsilon}^w} K_{\Sigma_1}(z, \zeta) \partial_w g(\zeta, w) & = \frac{1}{\pi i} \lim_{\epsilon \rightarrow 0} \int_{\Gamma_{\epsilon}^w} K_{\Sigma_1}(z, \zeta) (\partial_w g(\zeta, w) - \partial_w g_{\Sigma_1}(\zeta, w)) \\ & = - \iint_{\Sigma_1, \zeta} K_{\Sigma_1}(z, \zeta) \wedge (\overline{K_R(\zeta, w)} - \overline{K_{\Sigma_1}(\zeta, w)}) \\ & = -\overline{K_R(z, w)} + \overline{K_{\Sigma_1}(z, w)} \end{aligned}$$

where in the last term we have used part (5) of Proposition 3.3 and the reproducing property of Bergman kernel on Σ_1 . $\square$

Remark 3.14 Theorem 3.13 (in various settings) appears only as a norm equality in the literature.

4 Jump formula on quasicircles and related isomorphisms

4.1 The limiting integral in the jump formula

In this section, we show that the jump formula holds when Γ is a quasicircle. We also prove that in this case the Schiffer operator $T(\Sigma_1, \Sigma_2)$ is an isomorphism, when restricted to a certain subclass of $\overline{A(\Sigma_1)}$.

To establish a jump formula, we would like to define a Cauchy-type integral for elements $h \in \mathcal{H}(\Gamma)$. Since Γ is not necessarily rectifiable, instead we replace the integral over Γ with an integral over approximating curves $\Gamma_\epsilon^{p_1}$ (defined at the beginning of Sect. 3), and use the harmonic extensions $\tilde{h} \in \mathcal{D}_{\text{harm}}(\Sigma_1)$ of elements of $\mathcal{H}(\Gamma)$.

It is an arbitrary choice whether to approximate the curve from within Σ_1 or from within Σ_2 . Later, we will show that the result is the same in the case that Γ is a quasicircle. For now, we have chosen to approximate the curves from within Σ_1 .

Let $h \in \mathcal{D}_{\text{harm}}(\Sigma_1)$. Fix $q \in R \setminus \Gamma$ and define

$$J_q(\Gamma)h(z) = -\lim_{\epsilon \searrow 0} \frac{1}{\pi i} \int_{\Gamma_\epsilon^{p_1}} \partial_w g(w; z, q) h(w) \quad (4.1)$$

for $z \in R \setminus \Gamma$. Observe that, by definition, the curve $\Gamma_\epsilon^{p_1}$ depends on a fixed point $p_1 \in \Sigma_1$. However, we shall show that $J_q(\Gamma)$ is independent of p_1 in a moment.

First we show that the limit exists. There are several cases depending on the locations of z and q . Assume that $q \in \Sigma_2$, then for $z \in \Sigma_2$, we have by Stokes' theorem that

$$J_q(\Gamma)h(z) = -\frac{1}{\pi i} \iint_{\Sigma_1} \partial_w g(w; z, q) \wedge \bar{\partial} h(w) \quad (4.2)$$

so the limit exists and is independent of p_1 . For $z \in \Sigma_1$ we proceed as follows; let γ_r denote the circle of radius r centered at z , positively oriented with respect to z , in some fixed chart near z . By applying Stokes' theorem and the mean value property of harmonic functions we obtain

$$\begin{aligned} J_q(\Gamma)h(z) &= -\frac{1}{\pi i} \iint_{\Sigma_1} \partial_w g(w; z, q) \wedge \bar{\partial} h(w) - \lim_{r \searrow 0} \frac{1}{\pi i} \int_{\gamma_r} \partial_w g(w; z, q) h(w) \\ &= -\frac{1}{\pi i} \iint_{\Sigma_1} \partial_w g(w; z, q) \wedge \bar{\partial} h(w) + h(z). \end{aligned} \quad (4.3)$$

This shows that the limit exists for $z \in R \setminus \Gamma$ and $q \in \Sigma_2$ and is independent of p . In the case that $q \in \Sigma_1$, we obtain similar expressions, but with the term $h(q)$ added to both integrals.

This also shows that

Lemma 4.1 *For strip-cutting Jordan curves Γ , the limit (4.1) exists and is independent of the choice of p_1 .*

Therefore, in the following we will usually omit mention of the point p_1 in defining the level curves, and write simply Γ_ϵ .

Theorem 4.2 *Let Γ be a strip-cutting Jordan curve in R . For all $h \in \mathcal{D}_{\text{harm}}(\Sigma_1)$ and any $q \in R \setminus \Gamma$,*

$$\begin{aligned}\partial J_q(\Gamma)h(z) &= -T(\Sigma_1, \Sigma_2)\bar{\partial}h(z), & z \in \Sigma_2 \\ \partial J_q(\Gamma)h(z) &= \partial h(z) - T(\Sigma_1, \Sigma_1)\bar{\partial}h(z), & z \in \Sigma_1 \\ \bar{\partial} J_q(\Gamma)h(z) &= \bar{S}(\Sigma_1)\bar{\partial}h(z), & z \in \Sigma_1 \cup \Sigma_2.\end{aligned}$$

Proof Assume first that $q \in \Sigma_2$. The first claim follows from (4.2) and the fact that the integrand is non-singular. Similarly for $z \in \Sigma_2$, the third claim follows from (4.2).

The second claim follows from Stokes theorem:

$$\begin{aligned}\partial J_q(\Gamma)h(z) &= \partial_z \left(-\frac{1}{\pi i} \lim_{\epsilon \searrow 0} \int_{\Gamma_\epsilon} (\partial_w g(w; z, q) - \partial_w g_\Sigma(w, z)) h(w) \right) \\ &\quad - \partial_z \lim_{\epsilon \searrow 0} \frac{1}{\pi i} \int_{\Gamma_\epsilon} \partial_w g_\Sigma(w, z) h(w) \\ &= \partial_z \left(-\frac{1}{\pi i} \iint_{\Sigma_1} (\partial_w g(w; z, q) - \partial_w g_\Sigma(w, z)) \wedge_w \bar{\partial} h(w) \right) \\ &\quad - \partial_z \lim_{\epsilon \searrow 0} \frac{1}{\pi i} \int_{\Gamma_\epsilon} \partial_w g_\Sigma(w, z) h(w) \\ &= -\frac{1}{\pi i} \iint_{\Sigma_1} (\partial_z \partial_w g(w; z, q) - \partial_z \partial_w g_\Sigma(w, z)) \wedge_w \bar{\partial} h(w) + \partial h(z)\end{aligned}\tag{4.4}$$

where we have used Theorem 3.2. Also observe that the fact that the integrand of the first term is non-singular and holomorphic in z for each $w \in \Sigma_1$, and that

$$\iint_{\Sigma_1, w} |(\partial_w g(w; z, q) - \partial_w g_\Sigma(w, z)) \wedge_w \bar{\partial} h(w)|$$

is locally bounded in z , yield that derivation under the integral sign in the first term is legitimate.

Similarly removing the singularity using $\partial_w g_\Sigma$, and then applying Theorem 3.2 and Stokes' theorem yield that

$$\begin{aligned}\bar{\partial} J(\Gamma)h(z) &= -\bar{\partial}_z \frac{1}{\pi i} \lim_{\epsilon \searrow 0} \int_{\Gamma_\epsilon} (\partial_w g(w; z, q) - \partial_w g_\Sigma(w, z)) h(w) + \bar{\partial} h(z) \\ &= -\frac{1}{\pi i} \iint_{\Sigma_1} (\bar{\partial}_z \partial_w g(w; z, q) - \bar{\partial}_z \partial_w g_\Sigma(w, z)) \wedge_w \bar{\partial}_w h(w) + \bar{\partial} h(z).\end{aligned}$$

The third claim now follows by observing that the second term in the integral is just $-\bar{\partial} h$ because the integrand is just the complex conjugate of the Bergman kernel.

Now assume that $q \in \Sigma_1$. We show the second claim in the theorem. We argue as in Eq. (4.4), except that we must also add a term $\partial_w g_{\Sigma_1}(w; q)h(w)$. We obtain instead

$$\begin{aligned}\partial J(\Gamma)h &= \partial_z J(\Gamma)h = -\frac{1}{\pi i} \iint_{\Sigma_1} (\partial_z \partial_w g(w; z, q) - \partial_z \partial_w g_\Sigma(w; z)) \wedge_w \bar{\partial}_w h(w) \\ &\quad + \partial_z (h(z) + h(q))\end{aligned}$$

and the claim follows from $\partial_z h(q) = 0$. The remaining claims follow similarly. $\square$

Below, let $\overline{A(R)}^\perp$ denote the orthogonal complement in $A_{\text{harm}}(\Sigma_1)$ of the restrictions of $A(R)$ to Σ_1 .

Corollary 4.3 *Let Γ be a strip-cutting Jordan curve and assume that $q \in R \setminus \Gamma$.*

- (1) $J_q(\Gamma)$ is a bounded operator from $\mathcal{D}_{\text{harm}}(\Sigma_1)$ to $\mathcal{D}_{\text{harm}}(\Sigma_1 \cup \Sigma_2)$.
- (2) If $\bar{\partial} h \in \overline{A(R)}^\perp$ then $J_q(\Gamma)h \in \mathcal{D}(\Sigma_1 \cup \Sigma_2)$.

Proof The first claim follows immediately from Theorems 3.10 and 4.2. The second claim follows from Theorem 4.2 together with the fact that for fixed z $\bar{\partial}_z \partial_w g \in A(R)$. $\square$

4.2 Density theorems

In this section we show that certain subsets of the Dirichlet space are dense.

Our first density result parallels a general theorem of Askaripour and Barron [2], which asserts that L^2 holomorphic one-forms (in fact, more generally differentials) on a region in a Riemann surface can be approximated by holomorphic one-forms on a larger domain. We need a result of this type for the Dirichlet space, for doubly-connected regions.

Theorem 4.4 *Let R be a compact Riemann surface and Γ be a strip-cutting Jordan curve. Let U be a doubly-connected neighbourhood of Γ . Let $A_i = U \cap \Sigma_i$ for $i = 1, 2$, and let $\text{Res}_i : \mathcal{D}(U) \rightarrow \mathcal{D}(A_i)$ denote restriction for $i = 1, 2$. Then $\text{Res}_i \mathcal{D}(U)$ is dense in $\mathcal{D}(A_i)$ for $i = 1, 2$.*

Proof The proof proceeds in two steps. First, let A' be any doubly-connected domain in $\mathbb{C}$, bounded by two Jordan curves Γ_1 and Γ_2 . Let B_1 and B_2 be connected components

of the complements of Γ_1 and Γ_2 respectively, chosen so that B_1 and B_2 both contain A' ; thus $A' = B_1 \cap B_2$. We claim that every $h \in \mathcal{D}(A')$ can be written $h = h_1 + h_2$ where $h_1 \in \mathcal{D}(B_1)$ and $h_2 \in \mathcal{D}(B_2)$. To see this, one may take level curves Γ_{ϵ, p_k}^k of Green's function of B_k for $k = 1, 2$, and define

$$h_k(z) = \frac{1}{2\pi i} \lim_{\epsilon \searrow 0} \int_{\Gamma_{\epsilon, p_k}^k} \frac{h(\zeta)}{\zeta - z} d\zeta$$

(where we assume that Γ_k are positively oriented with respect to B_k for $k = 1, 2$, and therefore also with respect to A'). Then h_1 and h_2 are clearly holomorphic and $h = h_1 + h_2$.

We now show that they are in $\mathcal{D}(B_k)$ for $k = 1, 2$. Let $C \subseteq B_1$ be a collar neighbourhood of Γ_1 , and let $D \subset B_1$ be an open set whose closure is in B_1 , which furthermore contains the closure of $B_1 \setminus C$. Since $C \subset A'$, we have that $h \in \mathcal{D}(C)$. Since the closure of C is contained in B_2 , we see that $h_2 \in \mathcal{D}(C)$. Thus using $h_1 = h - h_2$ we see that $h_1 \in \mathcal{D}(C)$. Now since the closure of D is contained in B_1 , $h_1 \in \mathcal{D}(D)$. This proves that $h_1 \in \mathcal{D}(B_1)$. The proof that $h_2 \in \mathcal{D}(B_2)$ is obtained by interchanging the indices 1 and 2 above.

Next we claim that the linear space $\mathbb{C}[z, z^{-1}]$ of polynomials in z and z^{-1} is dense in $\mathcal{D}(A')$. To see this, assume for definiteness that B_1 is the bounded domain and B_2 is the unbounded domain. Since polynomials in z are dense in $\mathcal{D}(B_1)$ and polynomials in z^{-1} are dense in $\mathcal{D}(B_2)$, this proves the claim.

Returning to the statement of the theorem, observe that we can assume that U is an annulus $\mathbb{A} = \{z : r < |z| < 1/r\}$. This is because we can map U conformally onto $\mathbb{A}$, and every space in the statement of the theorem is conformally invariant. But since $\mathbb{C}[z, z^{-1}]$ is dense in both $\mathcal{D}(A_1)$ and $\mathcal{D}(A_2)$, and $\mathbb{C}[z, z^{-1}] \subset \mathcal{D}(U)$, this completes the proof. $\square$

We will also need a density result of another kind. Let Γ be a strip-cutting Jordan curve in a compact Riemann surface R , which separates R into two components Σ_1 and Σ_2 . Let A be a collar neighbourhood of Γ in Σ_1 . By Theorem 2.5 the boundary values of $\mathcal{D}_{\text{harm}}(A)$ exist conformally non-tangentially in Σ_1 and are themselves CNT boundary values of an element of $\mathcal{D}_{\text{harm}}(\Sigma_1)$. We then define

$$\begin{aligned} \mathfrak{G} : \mathcal{D}_{\text{harm}}(A) &\rightarrow \mathcal{D}_{\text{harm}}(\Sigma_1) \\ h &\mapsto \tilde{h} \end{aligned} \quad (4.5)$$

where $\tilde{h}$ is the unique element of $\mathcal{D}_{\text{harm}}(\Sigma_1)$ with CNT boundary values equal to those of h . We have the following result:

Theorem 4.5 [20] *Let Γ be a strip-cutting Jordan curve in a compact Riemann surface R . Assume that Γ separates R into two components, one of which is Σ . Let A be a collar neighbourhood of Γ in Σ . Then the associated map $\mathfrak{G} : \mathcal{D}_{\text{harm}}(A) \rightarrow \mathcal{D}_{\text{harm}}(\Sigma)$ is bounded.*

Theorem 4.6 *Let Γ , R , A and Σ be as above. The image of $\mathcal{D}(A)$ under $\mathfrak{G}$ is dense in $\mathcal{D}_{\text{harm}}(\Sigma_1)$.*

Proof First, we prove this in the case that $A = \mathbb{A}$ is an annulus with outer boundary $\mathbb{S}^1$ and $\Sigma_1 = \mathbb{D}$, and $\mathfrak{G}$ is

$$\mathfrak{G}(\mathbb{A}, \mathbb{D}) : \mathcal{D}_{\text{harm}}(\mathbb{A}) \rightarrow \mathcal{D}_{\text{harm}}(\mathbb{D}).$$

Now the set of Laurent polynomials $\mathbb{C}[z, z^{-1}]$ are contained in $\mathcal{D}_{\text{harm}}(\mathbb{A})$, and

$$\mathfrak{G}(\mathbb{A}, \mathbb{D})z^n = z^n \quad \text{and} \quad \mathfrak{G}(\mathbb{A}, \mathbb{D})z^{-n} = \bar{z}^n.$$

Since the set $\mathbb{C}[z, \bar{z}]$ of polynomials in $z, \bar{z}$ is dense in $\mathcal{D}_{\text{harm}}(\mathbb{D})$, this proves the claim.

Next, let $F : \mathbb{A} \rightarrow A$ be a conformal map. Define the composition map

$$\begin{aligned} \mathcal{C}_F : \mathcal{D}_{\text{harm}}(A) &\rightarrow \mathcal{D}_{\text{harm}}(\mathbb{A}) \\ h &\mapsto h \circ F, \end{aligned}$$

which is bounded by conformal invariance of the Dirichlet norm, and furthermore is a bijection with bounded inverse $\mathcal{C}_{F^{-1}}$. Similarly the restriction of $\mathcal{C}_F$ to $\mathcal{D}(A)$ is a bounded bijection onto $\mathcal{D}(\mathbb{A})$. Thus, the image of $\mathcal{D}(A)$ under $\mathfrak{G}(\mathbb{A}, \mathbb{D})\mathcal{C}_F$ is dense in $\mathcal{D}_{\text{harm}}(\mathbb{D})$.

Now denote the restriction map from $\mathcal{D}_{\text{harm}}(\mathbb{D})$ to $\mathcal{D}_{\text{harm}}(\mathbb{A})$ by $\text{Res}(\mathbb{D}, \mathbb{A})$ and similarly for $\text{Res}(\Sigma_1, A)$. Define the linear map

$$\rho = \mathfrak{G}(A, \Sigma_1)\mathcal{C}_{F^{-1}}\text{Res}(\mathbb{D}, \mathbb{A}) : \mathcal{D}_{\text{harm}}(\mathbb{D}) \rightarrow \mathcal{D}_{\text{harm}}(\Sigma_1).$$

This is obviously bounded, with bounded inverse

$$\rho^{-1} = \mathfrak{G}(\mathbb{A}, \mathbb{D})\mathcal{C}_F\text{Res}(\Sigma_1, A)$$

by uniqueness of Dirichlet bounded harmonic extensions of elements of $\mathcal{H}(\mathbb{S}^1)$ and $\mathcal{H}(\Gamma)$ to $\mathcal{D}_{\text{harm}}(\mathbb{D})$ and $\mathcal{D}_{\text{harm}}(\Sigma_1)$ respectively.

Now by definition of $\mathfrak{G}(\mathbb{A}, \mathbb{D})$, for any $h \in \mathcal{D}_{\text{harm}}(A)$, the CNT boundary values of

$$\mathcal{C}_{F^{-1}}\text{Res}(\mathbb{D}, \mathbb{A})\mathfrak{G}(\mathbb{A}, \mathbb{D})\mathcal{C}_F h$$

equal those of h . Thus we obtain the following factorization of $\mathfrak{G}(A, \Sigma_1)$:

$$\rho \mathfrak{G}(\mathbb{A}, \mathbb{D})\mathcal{C}_F = \mathfrak{G}(A, \Sigma_1)\mathcal{C}_{F^{-1}}\text{Res}(\mathbb{D}, \mathbb{A})\mathfrak{G}(\mathbb{A}, \mathbb{D})\mathcal{C}_F = \mathfrak{G}(A, \Sigma_1).$$

Since the image of $\mathcal{D}(A)$ under $\mathfrak{G}(\mathbb{A}, \mathbb{D})\mathcal{C}_F$ is dense in $\mathcal{D}_{\text{harm}}(\mathbb{D})$, and ρ is a bounded bijection with bounded inverse, this completes the proof. $\square$

4.3 Limiting integrals from two sides

In this section, we show that for quasicircles, the limiting integral defining $J_q(\Gamma)$ can be taken from either side of Γ , with the same result.

We will need to write the limiting integral in terms of holomorphic extensions to collar neighbourhoods. The integral in the definition of $J_q(\Gamma)$ is easier to work with when restricting to $\mathcal{D}(A)$. To make use of this simplification, we must first show that the limiting integrals of $\mathfrak{G}h$ and h are equal.

For $h \in \mathcal{D}(A)$, letting Γ_ϵ be level curves of Green's function of Σ_1 with respect to some fixed point $p \in \Sigma_1$, denote

$$J_q(\Gamma)'h(z) = -\frac{1}{\pi i} \lim_{\epsilon \searrow 0} \int_{\Gamma_\epsilon} \partial_w g(w; z, q) h(w) \quad (4.6)$$

for q fixed in Σ_2 . We use the notation $J_q(\Gamma)'$ to distinguish it from the operator $J_q(\Gamma)$, which applies only to elements of $\mathcal{D}_{\text{harm}}(\Sigma_1)$. For ϵ in some interval $(0, R)$ the curve Γ_ϵ lies entirely in A , so this makes sense. Because the integrand is holomorphic, the integral is independent of ϵ for $\epsilon \in (0, R)$.

We first require a more general theorem, which shows that the limiting integral is the same for any functions with the same CNT boundary values.

Theorem 4.7 *Let Γ be a quasicircle and let $\beta \in A(B)$ where B is a collar neighbourhood of Γ in Σ_1 . Let Γ_ϵ be level curves of Green's function in Σ_1 . If $h \in \mathcal{D}_{\text{harm}}(B)$ has CNT boundary values zero, then*

$$\lim_{\epsilon \searrow 0} \int_{\Gamma_\epsilon} \beta(w) h(w) = 0.$$

Proof Since B contains a canonical collar neighbourhood, it is enough to prove this for the case that B is a canonical collar neighbourhood. Let $\phi : B \rightarrow \mathbb{A}$ be a canonical collar chart onto an annulus $\mathbb{A} = \{z : R < |z| < 1\}$ for some $R \in (0, 1)$. The level curves Γ_ϵ map onto circles $|z| = r = e^{-\epsilon}$ for all ϵ sufficiently close to zero. A change of variables reduces the problem to showing that

$$\lim_{r \nearrow 1} \int_{|z|=r} \alpha H = 0$$

for $\alpha = (\phi^{-1})^* \beta \in A(\mathbb{A})$ and $H = h \circ \phi^{-1} \in \mathcal{D}_{\text{harm}}(\mathbb{A})$.

We demonstrate this first for α of the form $a(z)dz$ for Laurent polynomials $a(z) \in \mathbb{C}[z, z^{-1}]$. By Corollary 2.14 in [20] there is a $c \in \mathbb{C}$, a $H_1 \in \mathcal{D}_{\text{harm}}(\mathbb{D})$ and a $H_2 \in \mathcal{D}(\Omega)$ where $\Omega = \{z : R < |z|\} \cup \{\infty\}$, such that

$$H = cg_0 + H_1 + H_2$$

where g_0 is Green's function with singularity at 0 (that is, $g_0(z) = -\log |z|$). Now since H_2 and g_0 are continuous up to $\mathbb{S}^1$, and α is continuous on $\mathbb{S}^1$, we have that

$$\lim_{r \rightarrow 1} \int_{|z|=r} \alpha H_2 = \lim_{r \rightarrow 1} \int_{|z|=r} \alpha \mathfrak{G}(\mathbb{A}, \mathbb{D}) H_2$$

and

$$\lim_{r \rightarrow 1} \int_{|z|=r} \alpha c g_0 = \lim_{r \rightarrow 1} \int_{|z|=r} \alpha \mathfrak{G}(\mathbb{A}, \mathbb{D}) c g_0.$$

But we also have that $\mathfrak{G}(\mathbb{A}, \mathbb{D}) H_1 = H_1$. Therefore

$$\begin{aligned} \lim_{r \rightarrow 1} \int_{|z|=r} \alpha \mathfrak{G}(\mathbb{A}, \mathbb{D}) H &= \lim_{r \rightarrow 1} \int_{|z|=r} \alpha \mathfrak{G}(\mathbb{A}, \mathbb{D}) (H_1 + c g_0 + H_2) \\ &= \lim_{r \rightarrow 1} \int_{|z|=r} \alpha (H_1 + c \mathfrak{G}(\mathbb{A}, \mathbb{D}) g_0 + \mathfrak{G}(\mathbb{A}, \mathbb{D}) H_2) \\ &= \lim_{r \rightarrow 1} \int_{|z|=r} \alpha (H_1 + c g_0 + H_2) = \lim_{r \rightarrow 1} \int_{|z|=r} \alpha H. \end{aligned}$$

Since if $H = 0$ then $\mathfrak{G}(\mathbb{A}, \mathbb{D}) H = 0$, this proves the claim for the special case of α of the above form.

Next we show that α 's of this form are dense in $A(\mathbb{A})$. To see this observe that $\mathbb{C}[z, z^{-1}]$ is dense in $\mathcal{D}(\mathbb{A})$. Thus, Laurent polynomials of the form

$$\frac{a_{-n}}{z^n} + \cdots + \frac{a_{-2}}{z^2} + a_0 + a_1 z + \cdots + a_m z^m \quad (4.7)$$

(that is, the set of derivatives of Laurent polynomials) are dense in the set of exact one-forms on $A(\mathbb{A})$.

Now let $\alpha \in A(\mathbb{A})$ be arbitrary and let $c = \int_{|z|=r} \alpha$ where $r \in (R, 1)$. Then

$$\alpha_0 = \alpha - \frac{c}{2\pi i z}$$

is exact, and hence for any $\epsilon > 0$ it can be approximated within ϵ in $A(\mathbb{A})$ by a Laurent polynomial p of the form (4.7). Then the Laurent polynomial $p(z) + c/(2\pi i z)$ approximates α within ϵ in $A(\mathbb{A})$.

The proof will be complete if it can be shown that for fixed $H \in \mathcal{D}_{\text{harm}}(\mathbb{A})$ the functional

$$\alpha \mapsto \lim_{r \nearrow 1} \int_{|z|=r} H \alpha$$

on $A(\mathbb{A})$ is bounded (where we fix the orientation in the integral to be positive with respect to zero). Denote $\alpha(z) = a(z)dz$ as above (but now with no extra assumptions

on $a(z)$. Fix an $s \in (R, 1)$ and let $M = \sup_{|z|=s} |H(z)|$. Then denoting $B_s = \{z : s < |z| < 1\}$, we have by Stokes' theorem that

$$\begin{aligned} \left| \lim_{r \nearrow 1} \int_{|z|=r} H\alpha \right| &= \left| \int_{|z|=s} \alpha H + \iint_{B_s} \alpha \wedge \bar{\partial} H \right| \\ &\leq 2\pi s \cdot M \cdot \sup_{|z|=s} |a(z)| + \|\alpha\|_{A(B_s)} \|\bar{\partial} H\|_{A(B_s)}. \end{aligned}$$

Now since $|z| = s$ is compact, by Lemma 2.1 there is a constant C which is independent of α such that

$$\sup_{|z|=s} |a(z)| \leq C \|\alpha\|_{A(\mathbb{A})}.$$

Inserting this estimate in the line above we obtain

$$\begin{aligned} \left| \lim_{r \nearrow 1} \int_{|z|=r} H\alpha \right| &\leq 2\pi s \cdot M \cdot C \cdot \|\alpha(z)\|_{A(\mathbb{A})} + \|\alpha\|_{A(B_s)} \|H\|_{\mathcal{D}(B_s)} \\ &\leq (2\pi s \cdot M \cdot C + \|H\|_{\mathcal{D}(\mathbb{A})}) \|\alpha\|_{A(\mathbb{A})}. \end{aligned}$$

Thus for fixed H the integral is a bounded functional on $A(\mathbb{A})$, which completes the proof. $\square$

We then have the following immediate consequence.

Theorem 4.8 *Let Γ be a quasicircle, α be a holomorphic one-form on a collar neighbourhood B of Γ in Σ_1 . If Γ_ϵ are the level curves of Green's function in Σ_1 and if $h_1, h_2 \in \mathcal{D}_{\text{harm}}(B)$ have the same CNT boundary values on Γ , then*

$$\lim_{\epsilon \searrow 0} \int_{\Gamma_\epsilon} \alpha(w) h_1(w) = \lim_{\epsilon \searrow 0} \int_{\Gamma_\epsilon} \alpha(w) h_2.$$

In particular, if $\mathfrak{G}$ is given by (4.5) then

$$\lim_{\epsilon \searrow 0} \int_{\Gamma_\epsilon} \alpha(w) h(w) = \lim_{\epsilon \searrow 0} \int_{\Gamma_\epsilon} \alpha(w) \mathfrak{G}h(w).$$

The following special case will allow us to make convenient use of the density of $\mathfrak{G}\mathcal{D}(B)$ in $\mathcal{D}_{\text{harm}}(B)$, as was mentioned above.

Theorem 4.9 *Let Γ be a quasicircle and A be a collar neighbourhood of Γ in Σ_1 . Then for fixed $q \in R \setminus \Gamma$ and all $h \in \mathcal{D}_{\text{harm}}(B)$ and $z \in R \setminus \Gamma$*

$$J_q(\Gamma)'h(z) = J_q(\Gamma)\mathfrak{G}h(z) \quad (4.8)$$

where $\mathfrak{G}$ is as in (4.5) and $J_q(\Gamma)'$ is as in (4.6).

Proof By restricting to a smaller canonical collar neighbourhood, we can assume that B does not contain z or q in its closure. For fixed z and q set

$$\alpha(w) = -\frac{1}{\pi i} \partial_w g(w; z, q) \Big|_B.$$

Since the right hand side is holomorphic on an open neighbourhood of the closure of B , $\alpha \in A(B)$. Applying Theorem 4.8 proves the theorem. $\square$

We now show that for quasicircles, one can define the jump operator $J(\Gamma)$ using either limiting integrals from within Σ_1 or from within Σ_2 with the same result. We use the following temporary notation. For $q \in R \setminus \Gamma$ let $J_q(\Gamma, \Sigma_i) : \mathcal{D}_{\text{harm}}(\Sigma_1) \rightarrow \mathcal{D}_{\text{harm}}(\Sigma_1 \cup \Sigma_2)$ be defined by

$$J_q(\Gamma, \Sigma_i)h(z) = -\lim_{\epsilon \searrow 0} \frac{1}{\pi i} \int_{\Gamma_\epsilon^{p_i}} \partial_w g(w; z, q)h(w).$$

For definiteness, we assume that all curves $\Gamma_\epsilon^{p_i}$ are oriented positively with respect to Σ_1 . Aside from this change of sign, all previous theorems apply equally to $J_q(\Gamma, \Sigma_1)$ and $J_q(\Gamma, \Sigma_2)$.

Theorem 4.10 *Let Γ be a quasicircle. Then for all $q \in R \setminus \Gamma$*

$$J_q(\Gamma, \Sigma_1) = J_q(\Gamma, \Sigma_2)\mathfrak{D}(\Sigma_1, \Sigma_2).$$

Proof Let U be a doubly-connected neighbourhood of Γ , bounded by $\Gamma_i \subset \Sigma_i$. Let $A_i = U \cap \Sigma_i$. Then A_i are collar neighbourhoods of Γ in Σ_i . Let $\mathfrak{G}_i : \mathcal{D}(A_i) \rightarrow \mathcal{D}_{\text{harm}}(\Sigma_i)$ be induced by these collar neighbourhoods for $i = 1, 2$.

For any $h \in \mathcal{D}(U)$, let $\text{Res}_i h = h|_{A_i}$. It follows immediately from the definition of $\mathfrak{G}_i$ that

$$\mathfrak{G}_2 \text{Res}_2 h = \mathfrak{D}(\Sigma_1, \Sigma_2)\mathfrak{G}_1 \text{Res}_1 h. \quad (4.9)$$

Therefore

$$\begin{aligned} \lim_{\epsilon \searrow 0} \int_{\Gamma_\epsilon^{p_1}} \partial_w g(w; z, q)\mathfrak{G}_1 h(w) &= \lim_{\epsilon \searrow 0} \int_{\Gamma_\epsilon^{p_1}} \partial_w g(w; z, q)h(w) \\ &= \lim_{\epsilon \searrow 0} \int_{\Gamma_\epsilon^{p_2}} \partial_w g(w; z, q)h(w) \\ &= \lim_{\epsilon \searrow 0} \int_{\Gamma_\epsilon^{p_2}} \partial_w g(w; z, q)\mathfrak{G}_2 h(w), \end{aligned}$$

where we have used holomorphicity of the integrand in the second equality, and Proposition 4.9 to obtain the first and the third equalities. Thus for all $h \in \mathfrak{G}_i \text{Res}_i \mathcal{D}(U)$,

$$J(\Gamma, \Sigma_1)h = J(\Gamma, \Sigma_2)\mathfrak{D}(\Sigma_1, \Sigma_2)h.$$

Now by Theorem 4.4 $\text{Res}_i \mathcal{D}(U)$ is dense in $\mathcal{D}(A_i)$ for $i = 1, 2$, and therefore by Theorem 4.6 part (2) $\mathfrak{G}_i \text{Res}_i \mathcal{D}(U)$ is dense in $\mathcal{D}_{\text{harm}}(\Sigma_i)$. Since Res_i , $\mathfrak{G}_i$ and $J(\Gamma, \Sigma_i)$ are all bounded, this completes the proof. $\square$

Thus one may think of $J(\Gamma)$ as an operator on $\mathcal{H}(\Gamma)$.

In the rest of the paper, we return to the convention that $J_q(\Gamma)$ is an operator on $\mathcal{D}_{\text{harm}}(\Sigma_1)$. However, Theorem 4.10 plays an important role in the proof that $T(\Sigma_1, \Sigma_2)$ is surjective.

Also, by using Theorem 4.8 and proceeding exactly as in the proof of Theorem 4.10 we obtain

Theorem 4.11 *Let Γ be a quasicircle. Let α be a holomorphic one-form in an open neighbourhood of Γ . For any $h \in \mathcal{D}_{\text{harm}}(\Sigma_1)$*

$$\lim_{\epsilon \searrow 0} \int_{\Gamma_\epsilon^{p1}} h(w) \alpha(w) = \lim_{\epsilon \searrow 0} \int_{\Gamma_\epsilon^{p2}} [\mathfrak{D}(\Sigma_1, \Sigma_2)h](w) \alpha(w)$$

4.4 A transmission formula

In this section we prove an explicit formula for the transmission operator $\mathfrak{D}$ on the image of the jump operator.

Definition 4.12 We denote by W_k the linear subspace of $\mathcal{D}_{\text{harm}}(\Sigma_i)$ given by

$$W_k = \left\{ h \in \mathcal{D}_{\text{harm}}(\Sigma_k) : \lim_{\epsilon \searrow 0} \int_{\Gamma_\epsilon^{pk}} h(w) \alpha(w) = 0 \right\}$$

for all $\alpha \in A(R)$ and for $k = 1, 2$. The elements of W_k are the admissible functions for the jump problem.

Let

$$J(\Gamma)_{\Sigma_k} h = J(\Gamma)h|_{\Sigma_k}$$

for $k = 1, 2$. We have the following result:

Theorem 4.13 *Let R be a compact surface and Γ be a quasicircle separating R into components Σ_1 and Σ_2 . Let $q \in R \setminus \Gamma$. If $h \in W_1$ then*

$$-\mathfrak{D}(\Sigma_2, \Sigma_1)J_q(\Gamma)_{\Sigma_2}h = h - J_q(\Gamma)_{\Sigma_1}h.$$

To prove this theorem we need a lemma.

Lemma 4.14 *Let Γ be a quasicircle and let A be a collar neighbourhood of Γ in Σ_1 . Fix a smooth curve Γ' in A homotopic to Γ , and assume that $h \in \mathcal{D}(A)$ satisfies*

$$\int_{\Gamma'} h \alpha = 0 \tag{4.10}$$

for all $\alpha \in A(R)$. Then $\mathfrak{G}h \in W_1$ and

$$-\mathfrak{D}(\Sigma_2, \Sigma_1)J_q(\Gamma)_{\Sigma_2}\mathfrak{G}h = \mathfrak{G}h - J_q(\Gamma)_{\Sigma_1}\mathfrak{G}h.$$

Proof The fact that $\mathfrak{G}h \in W_1$ follows immediately from Theorem 4.8. By Royden [12, Theorem 4] and the explicit formula on the following page, there are holomorphic functions H_1 on Σ_1 and H_2 on $\text{cl } \Sigma_2 \cup A$ such that $H_1 - H_2 = h$ on A . Furthermore, these functions are given by the restrictions of $J_q(\Gamma)h$ to Σ_1 and Σ_2 . Thus, by Proposition 4.9, we have that

$$H_k = J_q(\Gamma)|_{\Sigma_k} \mathfrak{G}h \quad (4.11)$$

for $k = 1, 2$ (where H_2 has a holomorphic extension to $\text{cl } \Sigma_2 \cup A$).

Since H_1, H_2 and h are all in $\mathcal{D}(A)$, they have conformally non-tangential boundary values in $\mathcal{H}(\Gamma)$ with respect to Σ_1 . Since $H_1 - H_2 = h$ on A , then the boundary values also satisfy this equation. Thus

$$H_1 - \mathfrak{D}(\Sigma_2, \Sigma_1)H_2 = \mathfrak{G}h$$

by definition of $\mathfrak{G}$ and $\mathfrak{D}(\Sigma_2, \Sigma_1)$. Finally Eq. (4.11) completes the proof. $\square$

We continue with the proof of Theorem 4.13.

Proof Let E be the linear subspace of $\mathcal{D}(A)$ consisting of those elements of $\mathcal{D}(A)$ for which (4.10) is satisfied. It is enough to show that $\mathfrak{G}E$ is dense in W_1 .

Fix a basis $\alpha_1, \dots, \alpha_g$ for $A(R)$. Let $\mathcal{P} : \mathcal{D}(A) \rightarrow E$ denote the orthogonal projection in $\mathcal{D}(A)$.

For $u \in \mathcal{D}(A)$ define

$$\mathcal{Q}(u) = \left(\int_{\Gamma'} u \alpha_1, \dots, \int_{\Gamma'} u \alpha_g \right).$$

By Lemma 2.1 and the fact that $\mathcal{Q}(u + c) = \mathcal{Q}(u)$ for any constant c , it follows that each component of $\mathcal{Q}$ is a bounded linear functional on $\mathcal{D}(A)$. Once again, a simple argument based on Riesz representation theorem and the Gram-Schmidt process yields that there is a C such that

$$\|\mathcal{P}u - u\|_{\mathcal{D}(A)} \leq C \|\mathcal{Q}(u)\|_{\mathbb{C}^g}. \quad (4.12)$$

For $H \in \mathcal{D}_{\text{harm}}(\Sigma_1)$ define now

$$\mathcal{Q}_1(H) = \lim_{\epsilon \searrow 0} \left(\int_{\Gamma_\epsilon^{p_1}} H \alpha_1, \dots, \int_{\Gamma_\epsilon^{p_1}} H \alpha_g \right).$$

We have that there is a C' such that

$$\|\mathcal{Q}_1(H)\|_{\mathbb{C}^g} \leq C' \|H\|_{\mathcal{D}_{\text{harm}}(\Sigma_1)}.$$

This follows by applying Stokes' theorem to each component:

$$\lim_{\epsilon \searrow 0} \int_{\Gamma_\epsilon^{p_1}} H \alpha_k = \iint_{\Sigma_1} \bar{\partial} H \wedge \alpha_k$$

which is proportional to $(\bar{\partial} H, \bar{\alpha}_k)_{A(\Sigma_1)}$. Observe also that $Q_1(\mathfrak{G}u) = Q(u)$ for all $u \in \mathcal{D}(A)$ by Proposition 4.9.

Let $h \in W_1 \subseteq \mathcal{D}_{\text{harm}}(\Sigma_1)$ be arbitrary. By density of $\mathfrak{G}\mathcal{D}(A)$, there is a $u \in \mathcal{D}(A)$ such that

$$\|\mathfrak{G}u - h\|_{\mathcal{D}_{\text{harm}}(\Sigma_1)} < \varepsilon.$$

We then have

$$\begin{aligned} \|\mathfrak{G}\mathcal{P}u - h\|_{\mathcal{D}_{\text{harm}}(\Sigma_1)} &\leq \|\mathfrak{G}\mathcal{P}u - \mathfrak{G}u\|_{\mathcal{D}_{\text{harm}}(\Sigma_1)} + \|\mathfrak{G}u - h\|_{\mathcal{D}_{\text{harm}}(\Sigma_1)} \\ &\leq \|\mathfrak{G}\| \|\mathcal{P}u - u\|_{\mathcal{D}(A)} + \|\mathfrak{G}u - h\|_{\mathcal{D}_{\text{harm}}(\Sigma_1)}. \end{aligned}$$

Now

$$\|Q(u)\| = \|Q_1(\mathfrak{G}u)\| = \|Q_1(\mathfrak{G}u - h)\| \leq C' \|\mathfrak{G}u - h\| < C' \varepsilon$$

so (4.12) yields that

$$\|\mathcal{P}u - u\|_{\mathcal{D}(A)} \leq CC' \varepsilon.$$

Thus

$$\|\mathfrak{G}\mathcal{P}u - h\|_{\mathcal{D}_{\text{harm}}(\Sigma_1)} \leq (CC' \|\mathfrak{G}\| + 1) \varepsilon.$$

□

We also define a transmission operator for exact one-forms as follows. For spaces $A(\Sigma)$, $A_{\text{harm}}(\Sigma)$, etc., denote the subset of exact one-forms with a subscript e , i.e. $A_e(\Sigma)$, $A_{\text{harm}}(\Sigma)_e$, etc.

Definition 4.15 For an exact one-form $\alpha \in A_{\text{harm}}(\Sigma_2)_e$ let h_2 be a harmonic function on Σ_2 such that $dh_2 = \alpha$. Let h_1 be the unique element of $\mathcal{D}_{\text{harm}}(\Sigma_1)$ with boundary values agreeing with h_2 . Then we define

$$\begin{aligned} \mathfrak{D}_e(\Sigma_2, \Sigma_1) : A_{\text{harm}}(\Sigma_2)_e &\rightarrow A_{\text{harm}}(\Sigma_1)_e \\ \alpha &\mapsto dh_1. \end{aligned}$$

The transmission from $A_{\text{harm}}(\Sigma_1)_e$ to $A_{\text{harm}}(\Sigma_2)_e$ is defined similarly.

To prove the transmission formula for $\mathfrak{D}_e$, we require the following elementary lemma.

Lemma 4.16 *Let Σ be a Riemann surface of finite genus g bordered by a curve homeomorphic to a circle. Let $\bar{\alpha} \in \overline{A(\Sigma)}$. There is an $h \in \mathcal{D}_{\text{harm}}(\Sigma)$ such that $\bar{\partial}h = \bar{\alpha}$. If $\tilde{h} \in \mathcal{D}_{\text{harm}}(\Sigma)$ is any other such function, then $\tilde{h} - h \in \mathcal{D}(\Sigma)$.*

Proof Let R be the double of Σ ; so $A(R)$ has dimension $2g$ where g is the genus of Σ . Let $a_1, \dots, a_{2g}$ be a collection of smooth curves which generate the fundamental group of Σ . Let

$$c_k = \int_{a_k} \bar{\alpha}$$

for $k = 1, \dots, 2g$. Since $A(R)$ has dimension $2g$, there is a $\beta \in A(R)$ such that

$$\int_{a_k} \beta = -c_k$$

for $k = 1, \dots, 2g$. Thus $\bar{\alpha} + \beta$ is exact in Σ and hence is equal to dh for some $h \in \mathcal{D}_{\text{harm}}(\Sigma)$. But clearly $\bar{\partial}h = \bar{\alpha}$.

If $\tilde{h}$ is any other such function then $\bar{\partial}(\tilde{h} - h) = 0$, which completes the proof. $\square$

Recall that $\overline{A(R)}^\perp$ denotes the set of elements in $A_{\text{harm}}(\Sigma)$ which are orthogonal, with respect to $(\cdot, \cdot)_{A_{\text{harm}}(\Sigma)}$, to the restrictions to Σ of elements of $\overline{A(R)}$.

Definition 4.17 Given R and Σ_i as above, let

$$V_k = \overline{A(\Sigma_k)} \cap \overline{A(R)}^\perp,$$

and

$$V'_k = \{\bar{\alpha} + \beta \in A_{\text{harm}}(\Sigma_k)_e : \bar{\alpha} \in V_k\},$$

for $k = 1, 2$.

Theorem 4.18 *Let R be a compact Riemann surface and let Γ be a quasicircle separating R into components Σ_1 and Σ_2 . If $\bar{\alpha} \in V_1$ then*

$$\mathfrak{D}_e(\Sigma_2, \Sigma_1)T(\Sigma_1, \Sigma_2)\bar{\alpha} = \bar{\alpha} + T(\Sigma_1, \Sigma_1)\bar{\alpha}.$$

Proof Let $\bar{\alpha} \in V_1$, then by Lemma 4.16 there is an $h \in \mathcal{D}_{\text{harm}}(\Sigma_1)$ such that $\bar{\partial}h = \bar{\alpha}$.

Since $\bar{\partial}h = \bar{\alpha} \in \overline{A(R)}^\perp$, $\bar{S}(\Sigma_1)\bar{\partial}h = 0$, so by Theorem 4.2 $\bar{\partial}J(\Gamma)h = 0$.

Applying d to both sides of Theorem 4.13 and using this fact yields

$$-\mathfrak{D}_e(\Sigma_2, \Sigma_1)\partial J(\Gamma)_{\Sigma_2}h = dh - \partial J(\Gamma)_{\Sigma_1}h.$$

The Theorem now follows from the remaining relations in Theorem 4.2. $\square$

For $k = 1, 2$ denote by

$$\begin{aligned} P(\Sigma_k) : A_{\text{harm}}(\Sigma_k) &\rightarrow A(\Sigma_k) \\ \overline{P}(\Sigma_k) : A_{\text{harm}}(\Sigma_k) &\rightarrow \overline{A(\Sigma_k)} \end{aligned}$$

the orthogonal projections onto the holomorphic and anti-holomorphic parts of a given harmonic one-form.

Corollary 4.19 *Let R be a compact Riemann surface and Γ be a quasicircle separating R into components Σ_1 and Σ_2 . Then $\overline{P}(\Sigma_1)\mathfrak{D}(\Sigma_2, \Sigma_1)$ is a left inverse of $T(\Sigma_1, \Sigma_2)|_{V_1}$. In particular, the restriction of $T(\Sigma_1, \Sigma_2)$ to V_1 is injective.*

Proof This follows immediately from Theorem 4.18 and the fact that for $\overline{\alpha} \in V_1$, $T(\Sigma_1, \Sigma_1)\overline{\alpha}$ and $T(\Sigma_1, \Sigma_2)\overline{\alpha}$ are holomorphic. $\square$

As another consequence of Theorem 4.18 we are able to prove an inequality analogous to the strengthened Grunsky inequality for quasicircles [10].

Theorem 4.20 *Let R be a compact Riemann surface and Γ be a quasicircle separating R into disjoint components Σ_1 and Σ_2 . Then $\|T(\Sigma_1, \Sigma_1)|_{V_1}\| < 1$.*

Proof Since $d : \mathcal{D}_{\text{harm}}(\Sigma_k) \rightarrow A_{\text{harm}}(\Sigma_k)$ is norm-preserving (with respect to the Dirichlet semi-norm), it follows from Theorem 2.7 that there is a $c \in (0, 1)$ which is independent of $\overline{\alpha}$ such that

$$\|\mathfrak{D}_e(\Sigma_2, \Sigma_1)T(\Sigma_1, \Sigma_2)\overline{\alpha}\|^2 \leq \frac{1+c}{1-c} \|T(\Sigma_1, \Sigma_2)\overline{\alpha}\|^2. \quad (4.13)$$

We will insert the identity

$$\mathfrak{D}_e(\Sigma_2, \Sigma_1)T(\Sigma_1, \Sigma_2)\overline{\alpha} = \overline{\alpha} + T(\Sigma_1, \Sigma_1)\overline{\alpha} \quad (4.14)$$

of Theorem 4.18 into (4.13).

In the following computation, we need two observations. First, if a function H is holomorphic on a domain Ω , then $\|H\|_{\mathcal{D}_{\text{harm}}(\Omega)}^2 = 2\|\text{Re}(H)\|_{\mathcal{D}(\Omega)}^2$. Second, if H_2 is a primitive of $T(\Sigma_1, \Sigma_2)\overline{\alpha}$ and if we let $H_1 = \mathfrak{D}(\Sigma_2, \Sigma_1)H_2$ (so that H_1 is a primitive of $\overline{\alpha} + T(\Sigma_1, \Sigma_1)\overline{\alpha}$ by definition), then we observe that $\mathfrak{D}(\Sigma_2, \Sigma_1)\text{Re}(H_2) = \text{Re}(H_1)$, and therefore the boundedness of transmission estimate applies to $\text{Re}(H_i)$.

Since $\alpha + T(\Sigma_1, \Sigma_1)\overline{\alpha}$ has the same real part as the right hand side of (4.14), combining with (4.13) (applied to the real part of the primitives) we obtain

$$\begin{aligned} \frac{1+c}{1-c} \|T(\Sigma_1, \Sigma_2)\overline{\alpha}\|^2 &= \frac{1+c}{1-c} 2\|\text{Re}(H_2)\|_{\mathcal{D}_{\text{harm}}(\Sigma_2)}^2 \\ &\geq 2\|\text{Re}(H_1)\|_{\mathcal{D}_{\text{harm}}(\Sigma_1)}^2 \\ &= 2\|d\text{Re}(H_1)\|_{A_{\text{harm}}(\Sigma_1)}^2 = 2\|\text{Re}(dH_1)\|_{A_{\text{harm}}(\Sigma_1)}^2 \end{aligned}$$

$$\begin{aligned} &= 2\|\operatorname{Re}(\bar{\alpha} + T(\Sigma_1, \Sigma_1)\bar{\alpha})\|^2 = 2\|\operatorname{Re}(\alpha + T(\Sigma_1, \Sigma_1)\bar{\alpha})\|^2 \\ &= \|\bar{\alpha}\|^2 + 2\operatorname{Re}(T(\Sigma_1, \Sigma_1)\bar{\alpha}, \alpha) + \|T(\Sigma_1, \Sigma_1)\bar{\alpha}\|^2. \end{aligned} \quad (4.15)$$

where we have used the fact that $\alpha + T(\Sigma_1, \Sigma_1)\bar{\alpha}$ is holomorphic. By Theorem 3.13 we have that

$$\begin{aligned} \|\bar{\alpha}\|^2 &= (\bar{\alpha}, \bar{\alpha}) = (\bar{\alpha}, T(\Sigma_1, \Sigma_1)^*T(1, 1)\bar{\alpha} + T(\Sigma_1, \Sigma_2)^*T(\Sigma_1, \Sigma_2)\bar{\alpha}) \\ &= (\bar{\alpha}, T(\Sigma_1, \Sigma_1)^*T(\Sigma_1, \Sigma_1)\bar{\alpha}) + (T(\Sigma_1, \Sigma_2)^*T(\Sigma_1, \Sigma_2)\bar{\alpha}) \\ &= \|T(\Sigma_1, \Sigma_1)\bar{\alpha}\|^2 + \|T(\Sigma_1, \Sigma_2)\bar{\alpha}\|^2. \end{aligned}$$

Combining this with (4.15) yields

$$\frac{1-c}{1+c}\operatorname{Re}(\alpha, T(\Sigma_1, \Sigma_1)\bar{\alpha}) \leq \frac{c}{1+c}\|\bar{\alpha}\|^2 - \frac{1}{1+c}\|T(\Sigma_1, \Sigma_1)\bar{\alpha}\|^2.$$

Hence

$$\begin{aligned} \operatorname{Re}(\alpha, T(\Sigma_1, \Sigma_1)\bar{\alpha}) &\leq \frac{c}{1+c}\|\bar{\alpha}\|^2 + \frac{2c}{1+c}\operatorname{Re}(\alpha, T(\Sigma_1, \Sigma_1)\bar{\alpha}) - \frac{1}{1+c}\|T(\Sigma_1, \Sigma_1)\bar{\alpha}\|^2 \\ &= c\|\bar{\alpha}\|^2 - \frac{1}{1+c}\|T(\Sigma_1, \Sigma_1)\bar{\alpha} - c\alpha\|^2. \end{aligned}$$

Applying this to $e^{i\theta}\alpha$, we see that the same inequality holds with the left hand side replaced by $e^{-2i\theta}\operatorname{Re}(\alpha, T(\Sigma_1, \Sigma_1)\bar{\alpha})$ for any θ . So

$$|\operatorname{Re}(\alpha, T(\Sigma_1, \Sigma_1)\bar{\alpha})| \leq c\|\bar{\alpha}\|^2.$$

Together with the fact that $T(\Sigma_1, \Sigma_1)^* = \overline{T(\Sigma_1, \Sigma_1)}$ this proves the theorem. $\square$

Remark 4.21 This gives another proof that $T(\Sigma_1, \Sigma_2)$ is injective. Let $\nu = \|T(\Sigma_1, \Sigma_1)\| < 1$. Observe that if $\bar{\alpha} \in \overline{A(\Sigma_1)}$ is in V_1 , then since the kernel of the operator $S(\Sigma_1)$ is holomorphic we have that $\bar{\alpha} \in \operatorname{Ker} \bar{S}(\Sigma_1)$. Thus by Theorem 3.13

$$\begin{aligned} \|T(\Sigma_1, \Sigma_2)\bar{\alpha}\|_{A(\Sigma_2)}^2 &= \|\bar{\alpha}\|_{\overline{A(\Sigma_1)}}^2 - \|T(\Sigma_1, \Sigma_1)\bar{\alpha}\|_{\overline{A(\Sigma_1)}}^2 \\ &\geq (1 - \nu^2)\|\bar{\alpha}\|_{\overline{A(\Sigma_1)}}^2. \end{aligned}$$

Since $1 - \nu^2 > 0$ this completes the proof.

4.5 Isomorphism theorem for the Schiffer operator

In this section, we prove the isomorphism theorem for the Schiffer operators. Theorem 4.22 shows that $T(\Sigma_1, \Sigma_2)$ is an isomorphism between $V_1 \subset \overline{A(\Sigma_1)}$ and the space $A(\Sigma_2)_e$ of exact one-forms on Σ_2 , thus generalizing Napalkov and Yulmukhametov's theorem to compact Riemann surfaces. In Proposition 4.24 we establish that for harmonic Dirichlet space functions h on Σ_1 such that $\bar{\partial}h \in V_1$, $\partial h + T(\Sigma_1, \Sigma_1)\bar{\partial}h$ is

exact. These two facts, combined with the identities of Theorem 4.2, allow us to give, in Theorem 4.25, an isomorphism between $V'_1 \subset A_{\text{harm}}(\Sigma_1)_e$ and $A(\Sigma_1)_e \oplus A(\Sigma_2)_e$. This last theorem is the “derivative” of the Plemelj–Sokhotski isomorphism, which will be given in the final section of the paper.

Theorem 4.22 *Let Γ be a quasicircle. Then the restriction of $T(\Sigma_1, \Sigma_2)$ to V_1 is an isomorphism onto $A(\Sigma_2)_e$.*

Proof Injectivity of $T(\Sigma_1, \Sigma_2)$ is Corollary 4.19.

We show that $T(\Sigma_1, \Sigma_2)(V_1) \subseteq A(\Sigma_2)_e$. If we take $\bar{\alpha} \in V_1$, then since

$$\iint_{\Sigma_1, w} \partial_{\bar{z}} \partial_w g(w, w_0; z, q) \wedge \overline{\alpha(w)} = 0,$$

for any fixed $q \in \Sigma_2$ we have (without loss of generality, because $T(\Sigma_1, \Sigma_2)$ is independent of q)

$$\begin{aligned} T(\Sigma_1, \Sigma_2)\bar{\alpha}(z) &= \frac{1}{\pi i} \iint_{\Sigma_1, z} \partial_z \partial_w g(w, w_0; z, q) \wedge \overline{\alpha(w)} \\ &= d_z \frac{1}{\pi i} \iint_{\Sigma_1, z} \partial_w g(w, w_0; z, q) \wedge \overline{\alpha(w)} \in A(\Sigma_2)_e, \end{aligned}$$

and therefore $T(\Sigma_1, \Sigma_2)(V_1) \subseteq A(\Sigma_2)_e$.

To show that $T(\Sigma_1, \Sigma_2)(V_1)$ contains $A(\Sigma_2)_e$, let $\beta \in A(\Sigma_2)_e$, and let h be the unique element of $\mathcal{D}(\Sigma_2)_q$ such that $\partial h = \beta$. By Theorem 2.7 there is an $H \in \mathcal{D}_{\text{harm}}(\Sigma_1)$ such that h and H have the same boundary values on Γ . Now $dH = \delta_1 + \delta_2$ for $\delta_1, \delta_2 \in A(\Sigma_1)$ (specifically, $\delta_1 = \partial H$ and $\bar{\delta}_2 = \bar{\partial} H$). Now by Theorem 4.10 we have

$$\begin{aligned} \beta(z) &= -\partial_z \lim_{\epsilon \searrow 0} \frac{1}{\pi i} \int_{\Gamma_\epsilon^{p_2}} \partial_w g(z, q; w) h(w) \\ &= -\partial_z \lim_{\epsilon \searrow 0} \frac{1}{\pi i} \int_{\Gamma_\epsilon^{p_1}} \partial_w g(z, q; w) H(w) \\ &= -\partial_z \frac{1}{\pi i} \iint_{\Sigma_1} \partial_w g(z, q; w) \wedge \bar{\delta}_2(w) \\ &= -\frac{1}{\pi i} \iint_{\Sigma_1} \partial_z \partial_w (z, q; w) \wedge \bar{\delta}_2(w) \end{aligned}$$

which proves that $A(\Sigma_2)_e \subseteq \text{Im}(T_R(\Sigma_1, \Sigma_2))$. Now we need to show that $\bar{\partial} H \in V_1$.

Since h is holomorphic by assumption, we have that $\partial h = dh$, hence

$$\begin{aligned} -\frac{1}{\pi i} \iint_{\Sigma_1} \partial_z \partial_w g(w, w_0; z, q) \wedge \bar{\partial} H(w) &= \partial h(z) = dh(z) \\ &= -\frac{1}{\pi i} \iint_{\Sigma_1} \partial_z \partial_w g(w, w_0; z, q) \wedge \bar{\partial} H(w) - \frac{1}{\pi i} \iint_{\Sigma_1} \bar{\partial}_z \partial_w g(w, w_0; z, q) \wedge \bar{\partial} H(w). \end{aligned}$$

Thus

$$-\frac{1}{\pi i} \iint_{\Sigma_1} \bar{\partial}_z \partial_w g(w, w_0; z, q) \wedge \bar{\partial} H(w) = 0 \quad (4.16)$$

for all $z \in \Sigma_2$. If we now let $\bar{\alpha} \in \overline{A(R)}$, then we have

$$\begin{aligned} (\bar{\partial} H, \bar{\alpha})_{\Sigma_1} &= -\frac{i}{2} \iint_{\Sigma_1} \bar{\partial} H(w) \wedge \alpha(w) \\ &= -\frac{i}{2} \iint_{\Sigma_1, w} \bar{\partial} H(w) \wedge_w \iint_{R, z} K_R(w; z) \wedge_z \alpha(z) \\ &= -\frac{i}{2} \iint_{R, z} \alpha(z) \wedge_z \iint_{\Sigma_1, w} \overline{K_R(z; w)} \wedge_w \bar{\partial} H(w) \end{aligned}$$

which is zero by (4.16). Thus $\bar{\partial} H \in V_1$ as claimed. $\square$

Remark 4.23 Although we have only proven that $T(\Sigma_1, \Sigma_2)$ is injective for quasicircles, we conjecture that this is true in greater generality, as in Napalkov and Yulmukhametov [8] in the planar case. It would also be of interest to give a proof of surjectivity using their approach. One would use the adjoint identity of Theorem 3.12 in place of the symmetry of the L kernel, which is used implicitly in their proof. One would also need to take into account the topological obstacles as we did above.

Proposition 4.24 *Let R be a compact Riemann surface and let Γ be a quasicircle separating R into components Σ_1 and Σ_2 . For any $h \in \mathcal{D}_{\text{harm}}(\Sigma)$ such that $\bar{\partial} h \in V_1$*

$$-T(\Sigma_1, \Sigma_1) \bar{\partial} h + \partial h \in A(\Sigma_1)_e.$$

Proof By Corollary 4.3 we need only show that $-T(\Sigma_1, \Sigma_1) \bar{\partial} h + \partial h$ is exact. As usual let Γ_ϵ be level curves of g_{Σ_1} for fixed z . Since L_R and hence $T(\Sigma_1, \Sigma_1)$ is independent of q , we can assume that $q \in \Sigma_2$. By Stokes' theorem

$$\begin{aligned} &-\frac{1}{\pi i} \lim_{\epsilon \rightarrow 0} \int_{\Gamma_\epsilon} (\partial_w g(w; z, q) - \partial_w g_{\Sigma_1}(w, z)) h(w) \\ &= -\frac{1}{\pi i} \iint_{\Sigma_1} (\partial_w g(w; z, q) - \partial_w g_{\Sigma_1}(w, z)) \wedge \bar{\partial} h(w) =: \omega(z). \end{aligned}$$

The integral on the left hand side exists by (4.1) and Theorem 3.2. Thus the right hand side is a well-defined function $\omega(z)$ on Σ_1 .

Thus

$$\begin{aligned} -T(\Sigma_1, \Sigma_1) \bar{\partial} h(z) &= \partial \omega(z) \\ &= d\omega(z) - \bar{\partial} \omega(z) \\ &= d\omega(z) + \frac{1}{\pi i} \iint_{\Sigma_1} \partial_{\bar{z}} \partial_w g(w; z, q) \wedge \bar{\partial} h(w) \\ &\quad - \frac{1}{\pi i} \iint_{\Sigma_1} \partial_{\bar{z}} \partial_w g_{\Sigma_1}(w, z) \wedge \bar{\partial} h(w) \end{aligned}$$

$$= d\omega(z) + 0 + \bar{\partial}h$$

where the middle term vanishes because $\bar{\partial}h \in V_1$, and we have observed that the last term is just the conjugate of the Bergman kernel applied to $\bar{\partial}h$. Thus $-T(\Sigma_1, \Sigma_1)\bar{\partial}h - \bar{\partial}h$ is exact. Since $dh = \partial h + \bar{\partial}h$ is exact, the claim follows. $\square$

The following theorem is in some sense a derivative of the jump decomposition.

Theorem 4.25 *Let R be a compact Riemann surface and let Γ be a quasicircle separating R into components Σ_1 and Σ_2 and V'_1 be given as in Definition 4.17.*

$$\begin{aligned}\hat{\mathfrak{H}} : V'_1 &\rightarrow A(\Sigma_1)_e \oplus A(\Sigma_2)_e \\ dh &\mapsto (\partial h - T(\Sigma_1, \Sigma_1)\bar{\partial}h, -T(\Sigma_1, \Sigma_2)\bar{\partial}h)\end{aligned}$$

is an isomorphism.

Proof First we show surjectivity. Let $(\alpha, \beta) \in A(\Sigma_1)_e \oplus A(\Sigma_2)_e$. By Theorem 4.22, $T(\Sigma_1, \Sigma_2)$ is surjective so there is a $\bar{\delta} \in V_1$ such that $T(\Sigma_1, \Sigma_2)\bar{\delta} = \beta$. By Lemma 4.16 there is a $\tilde{h} \in \mathcal{D}_{\text{harm}}(\Sigma_1)$ such that $\bar{\partial}\tilde{h} = -\bar{\delta}$.

Now set $\mu = \alpha - \partial\tilde{h} + T(\Sigma_1, \Sigma_1)\bar{\partial}\tilde{h}$. By construction μ is holomorphic and it is exact by Proposition 4.24. Let $u \in \mathcal{D}(\Sigma_1)$ be such that $\partial u = \mu$. Setting $h = \tilde{h} + u$ we see that

$$\begin{aligned}\hat{\mathfrak{H}}(dh) &= (\partial h - T(\Sigma_1, \Sigma_1)\bar{\partial}h, -T(\Sigma_1, \Sigma_2)\bar{\partial}h) \\ &= (\partial\tilde{h} + \mu - T(\Sigma_1, \Sigma_1)\bar{\partial}\tilde{h}, -T(\Sigma_1, \Sigma_2)\bar{\partial}\tilde{h}) \\ &= (\alpha, \beta).\end{aligned}$$

Thus $\hat{\mathfrak{H}}$ is surjective.

Now assume that $\hat{\mathfrak{H}}(dh) = 0$. The vanishing of the second component yields that $-T(\Sigma_1, \Sigma_2)\bar{\partial}h = 0$, so by Theorem 4.22 we have that $\bar{\partial}h = 0$. Thus the vanishing of the first component of $\hat{\mathfrak{H}}(dh)$ yields that $\partial h = 0$, hence $dh = 0$. $\square$

4.6 The jump isomorphism

In this section we establish the existence of a jump decomposition for functions in $\mathcal{H}(\Gamma)$. The first theorem phrases the decomposition in terms of an isomorphism, which we call the Plemelj–Sokhotski isomorphism.

Theorem 4.26 *Let R be a compact Riemann surface, and let Γ be a quasicircle separating R into two connected components Σ_1 and Σ_2 . Fix $q \in \Sigma_2$ and let W_1 be given as in Definition 4.12. Then the map*

$$\begin{aligned}\mathfrak{H} : \mathcal{D}_{\text{harm}}(\Sigma_1) &\rightarrow \mathcal{D}(\Sigma_1) \oplus \mathcal{D}(\Sigma_2)_q \\ h &\mapsto \left(J_q(\Gamma)h|_{\Sigma_1}, J_q(\Gamma)h|_{\Sigma_2} \right)\end{aligned}$$

is a bounded isomorphism from W_1 to $\mathcal{D}(\Sigma_1) \oplus \mathcal{D}(\Sigma_2)_q$.

Proof By Corollary 4.3 we have that the image of the map is in $\mathcal{D}(\Sigma_1) \oplus \mathcal{D}(\Sigma_2)$. Now since $g(w_0, w_0; z, q) = 0$ by definition of g , (3.3) yields that $g(w, w_0; q, q) = 0$. Therefore $\partial_w g(w; q, q) = 0$ and so

$$J_q(\Gamma)h(q) = -\frac{1}{\pi i} \lim_{\epsilon \rightarrow 0} \int_{\Gamma_\epsilon} \partial_w g(w, w_0; q, q)h(w) = 0.$$

Thus the image of the map is in $\mathcal{D}(\Sigma_1) \oplus \mathcal{D}_q(\Sigma_2)$.

By Theorem 4.2 $\partial \mathfrak{H}h = \hat{\mathfrak{H}}dh$, so since $\hat{\mathfrak{H}}$ is an isomorphism by Theorem 4.25, we only need to deal with constants. If $J_q(\Gamma)h = 0$ then $dh = 0$ so h is constant on Σ_1 . Since the second component of $\mathfrak{H}h$ vanishes at q we see that $h = 0$, so $\mathfrak{H}$ is injective. Now observe that $\mathfrak{H}(h+c) = \mathfrak{H}h + (c, 0)$ for any constant c . Thus surjectivity follows from surjectivity of $\hat{\mathfrak{H}}$. $\square$

Proposition 4.27 *Let R be a compact Riemann surface, and let Γ be a quasicircle separating R into components Σ_1 and Σ_2 . Assume that Γ is positively oriented with respect to Σ_1 . For $q \in \Sigma_2$, let $J_q(\Gamma)$ be defined using limiting integrals from within Σ_1 . If $h \in \mathcal{D}(\Sigma_1)$ then $J_q(\Gamma)h = (h, 0)$, and if $h \in \mathcal{D}_q(\Sigma_2)$ then $J_q(\Gamma)\mathfrak{D}(\Sigma_2, \Sigma_1)h = (0, -h)$.*

Proof The first claim follows immediately from Theorem 3.2. The second claim follows from Theorems 3.2 and 4.10 (note that Γ is negatively oriented with respect to Σ_2). $\square$

We then have a version of the Plemelj–Sokhotski jump formula.

Corollary 4.28 *Let R , Γ , Σ_1 and Σ_2 be as above. Let $H \in \mathcal{H}(\Gamma)$ be such that its extension h to $\mathcal{D}_{\text{harm}}(\Sigma_1)$ is in W_1 . There are unique $h_k \in \mathcal{D}(\Sigma_k)$, $k = 1, 2$ so that if H_k are their CNT boundary values then $H = -H_2 + H_1$. These unique h_i 's are given by*

$$h_k = J_q(\Gamma)h|_{\Sigma_k}$$

for $k = 1, 2$.

Proof We claim that $h = -\mathfrak{D}(\Sigma_2, \Sigma_1)h_2 + h_1$, which would imply that $H = -H_2 + H_1$. Proposition 4.27 yields

$$\mathfrak{H}(-\mathfrak{D}(\Sigma_2, \Sigma_1)h_2 + h_1) = (h_1, h_2) = \mathfrak{H}h.$$

Thus by Theorem 4.25 the claim follows.

We need only show that the solution is unique. Given any other solution $(\tilde{h}_1, \tilde{h}_2)$ we have that $-\mathfrak{D}(\Sigma_2, \Sigma_1)(\tilde{h}_2 - h_2) + (\tilde{h}_1 - h_1) \in \mathcal{D}_{\text{harm}}(\Sigma_1)$ has boundary values zero, so by uniqueness of the extension it is zero. Thus

$$0 = \mathfrak{H}\left(-\mathfrak{D}(\Sigma_2, \Sigma_1)(\tilde{h}_2 - h_2) + (\tilde{h}_1 - h_1)\right) = (\tilde{h}_1 - h_1, \tilde{h}_2 - h_2)$$

which proves the claim. $\square$

Finally, we show that the condition for existence of a jump formula is independent of the choice of side of Γ .

Theorem 4.29 *Let Γ be a quasicircle and V_k, V'_k be as in Definition 4.17 and W_k as in Definition 4.12. Then*

$$\begin{aligned}\mathfrak{D}(\Sigma_1, \Sigma_2)W_1 &= W_2 \\ \mathfrak{D}_e(\Sigma_1, \Sigma_2)V'_1 &= V'_2.\end{aligned}$$

Proof The first claim follows immediately from Theorem 4.11. Assume that $\overline{\alpha_k} + \beta_k \in A(\Sigma_k)_e$ for $k = 1, 2$ are such that

$$\mathfrak{D}_e(\Sigma_1, \Sigma_2)(\overline{\alpha_1} + \beta_1) = \overline{\alpha_2} + \beta_2.$$

In other words, there are $h_k \in \mathcal{D}_{\text{harm}}(\Sigma_k)$ such that $dh_k = \overline{\alpha_k} + \beta_k$ and $\mathfrak{D}(\Sigma_1, \Sigma_2)h_1 = h_2$. By Stokes' theorem, we have that for any $\overline{\alpha} \in \overline{A(R)}$

$$\begin{aligned}(\overline{\alpha_k}, \overline{\alpha})_{A_{\text{harm}}(\Sigma_k)} &= \frac{1}{2i} \iint_{\Sigma_k} \alpha \wedge \overline{\alpha_k} = \frac{1}{2i} \iint_{\Sigma_k} \alpha \wedge dh_k \\ &= \lim_{\epsilon \searrow 0} \int_{\Gamma_\epsilon^{p_k}} h_k(w) \alpha(w).\end{aligned}$$

Theorem 4.11 yields that $\overline{\alpha_1} \in V_1$ if and only if $\overline{\alpha_2} \in V_2$. $\square$

Acknowledgements Open access funding provided by Uppsala University. The authors are grateful to the referee, whose insightful comments have improved the overall presentation of the paper.

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