An operator ideal generated by Orlicz spaces

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Abstract

Absolutely φ -summing operators between Banach spaces generated by Orlicz spaces are investigated. A variant of Pietsch's domination theorem is proved for these operators and applied to prove vector-valued inequalities. These results are used to prove asymptotic estimates of π_φ -summing norms of finite-dimensional operators and also diagonal operators between Banach sequence lattices for a wide class of Orlicz spaces based on exponential convex functions φ . The key here is the description of a space of coefficients of the Rademacher series in this class of Orlicz spaces, proved via interpolation methods. As by-product, some absolutely φ -summing operators on the Hilbert space ℓ_2 are characterized in terms of its approximation numbers.

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1 Introduction

The motivation for this paper comes from the theory of operator ideals. The notion of absolutely p -summing operator has a long history and concerns an important ideal of operators between Banach spaces. The theory of this class is rich and widely recognized as one of the most important developments in modern Banach space theory and has found applications in many areas of analysis (see, e.g., [4, 14, 16]). Motivated by various applications and generalizations, different classes were defined in a natural way in recent years. In this article we study a class of absolutely φ -summing operators between

To the memory of Władysław Orlicz.

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Banach spaces. This class forms an operator ideal $(\Pi_\varphi, \pi_\varphi)$ which based on the Orlicz space L_φ , generalizing the concept of absolutely p -summing related to L_p -spaces. The study of this class of operators was initiated in [2] and continued in [5, 6]. In [5] this variant of summability of operators was motivated by questions raised the study of the interesting Burkholder–Davis–Gundy inequalities for vector-valued martingales. In his remarkable paper, Geiss [5] used φ -summing operators generated by an exponential functions φ_q , given by $\varphi_q(t) = \exp(t^q) - 1$ for all $t \geq 0$ with $q \in [1, \infty)$, in proving the requisite inequalities. We point out that the notion of absolutely φ -summing was motivated by the consideration of majorizing measures for Gaussian processes. It is also connected to the notion of ℓ -norm which plays an important role in the local theory of Banach spaces (see [15]). Here, as a model example, an exponential function φ_2 plays a key role. We recall that if T is an operator on ℓ_2^n into a Banach space X , then $\ell(T) := (\mathbb{E} \|\sum_{i=1}^n g_i T(e_i)\|_X^2)^{1/2}$, where $(g_i)_{i=1}^n$ is a sequence of standard independent Gaussian variables, and $(e_i)_{i=1}^n$ is the unit standard basis vector basis for ℓ_2^n . In fact, it was observed in [5] that Talagrand's majorizing measure theorem for Gaussian variables (see [11, Theorem 12.10]), combined with properties of absolutely φ_2 -summing operators and the central limit theorem yields that $\pi_{\varphi_2}(T) \asymp \ell(T)$, where the equivalence constants are universal. It follows that having an estimate for $\pi_{\varphi_2}(T)$, we can estimate the Gelfand numbers of T^* . In fact from improvement of Sudakov's minoration Theorem (see [17, Theorem 5.5]), it follows that there exists a constant $C > 0$ such that for all Banach spaces X and, for each $n \geq 1$ and all operators $T: \ell_2^n \rightarrow X$, we have $\sup_{k \geq 1} \sqrt{k} c_k(T^*) \leq C \ell(T)$.

As was mentioned, the theory of absolutely p -summing operators is rich, however in the general case of absolutely φ -summing operators the theory is limited, due to numerous essential difficulties related to verification of involved requirements in the original vector-valued definition. This fact provided the initial impetus for our work. The main goal is to prove some new general results on absolutely φ -summing operators.

The paper is arranged in the following manner. Section 2 introduces the relevant definitions and background material. Section 3 is devoted to general properties of absolutely φ -summing operators. We strengthen certain results due to Geiss in [5]. In particular we give a more transparent variant of Pietsch's domination theorem. We also give general examples of this type of operators. We investigate the relationship between vector-valued inequalities associated to absolutely φ -summing operators. Section 4 contains asymptotic estimates of π_φ -summing norms of finite-dimensional operators in the case of functions of the form $\varphi(t) = \exp(\phi(t)) - 1$ for all $t \geq 0$, where ϕ belongs to a wide class of Orlicz functions ϕ . The asymptotic estimates for finite-dimensional operators are applied to estimates from below and above of π_φ -norms of the diagonal operators in Banach sequence lattices. As we will see, some interesting additional phenomena arise connected to the theory of Rademacher series and abstract interpolation theory. The proofs for diagonal operators involves a description of the space of coefficients of the Rademacher series from Orlicz spaces L_φ on $[0, 1]$ generated by functions φ from the above class. To get this description, we use interpolation methods. Combining it with characterization of spaces of multipliers between reasonable symmetric spaces, we prove that an operator $T: \ell_2 \rightarrow \ell_2$ is absolutely φ -summing if and only if T belongs to Schatten class $\mathcal{S}_{d(w, 2)}$, where $d(w, 2)$

is a Lorentz symmetric space generated by a decreasing sequence $w = (w_k)$ that depends on ϕ .

2 Notation and background

We use standard notation from Banach space theory. Let X, Y be Banach spaces. We denote by $L(X, Y)$ the space of all bounded linear operators $T: X \rightarrow Y$ with the usual operator norm. If we write $X \hookrightarrow Y$, then we assume that $X \subset Y$ and the inclusion map $\text{id}: X \rightarrow Y$ is bounded. If $X = Y$ with equality of norms, then we write $X \cong Y$. The identity operator on X is denoted by I_X . We denote by B_X the closed unit ball of X , and by X^* its Banach dual space. As usual $C(K)$ stands for the Banach space of real-valued continuous functions on a compact Hausdorff space K and is endowed with the supremum norm. Throughout the paper we use the following notation: Given two sequences (a_n) and (b_n) of nonnegative real numbers we write $a_n \prec b_n$, if there is a constant $c > 0$ such that $a_n \leq c b_n$ for all $n \in \mathbb{N}$, while $a_n \asymp b_n$ means that $a_n \prec b_n$ and $b_n \prec a_n$ holds. Analogously we use the symbols $f \prec g$ and $f \asymp g$ for nonnegative real functions. For the notion of Banach lattices we refer to [12]. Throughout the paper, for simplicity of representation by a measure space, we always understand by measure space, a complete σ -finite measure space.

Let $(\Omega, \mu) := (\Omega, \mathcal{A}, \mu)$ and let X be a Banach space. We denote by $L^0(\mu, X)$ the space of equivalence classes of strongly measurable X -valued functions on Ω , equipped with the topology of convergence in measure (on sets of finite μ -measure). In the case $X = \mathbb{R}$, we write $L^0(\mu)$ instead of $L^0(\mu, \mathbb{R})$. By a *Banach lattice* in $L^0(\mu)$ (or on (Ω, μ)), we shall mean a Banach space X which is a subspace of $L^0(\mu)$ such that there exists $u \in X$ with $u > 0$ a.e. and if $|x| \leq |y|$ a.e., where $y \in X$ and $x \in L^0(\mu)$, then $x \in X$ and $\|x\|_X \leq \|y\|_X$.

Let X be a Banach lattice in $L^0(\Omega, \mu)$. Then X is said to be *maximal* or equivalently have the *Fatou property* (resp., is said to be an *order-continuous*) if its unit ball B_X is a closed subset in $L^0(\mu)$ (resp., if for every sequence $x_n \downarrow 0$ we have $\|x_n\|_X \rightarrow 0$). The Köthe dual X' of X is defined by

$$X' = \left\{ x \in L^0(\mu); \|x\|_{X'} := \sup_{\|y\|_X \leq 1} \left| \int_{\Omega} xy \, d\mu \right| < \infty \right\}.$$

Note that X' is a Banach lattice under the norm $\|\cdot\|_{X'}$ which is lattice isometric to a closed subspace of the topological dual X^* . If X is an order-continuous, then X^* can be identified with the Köthe dual space X' .

By a Banach sequence lattice we mean a Banach lattice in $\omega(\mathbb{N}) := L^0(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where μ is the counting measure. A Banach sequence space E is said to be symmetric provided that $\|(x_n)\|_E = \|(x_n^*)\|_E$, where (x_n^*) denotes the decreasing rearrangement of the sequence $(|x_n|)$. The fundamental function of the symmetric sequence space E is defined by

$$\lambda_E(n) = \left\| \sum_{k=1}^n e_k \right\|_E, \quad n \in \mathbb{N};$$

throughout the paper (e_n) will denote the standard basis in c_0 .

In what follows, the notions of p -convexity and q -concavity of a Banach lattice are crucial. For $1 \leq p, q < \infty$ a Banach lattice X is called p -convex and q -concave, respectively if there exist constants $C_p > 0$ and $C_q > 0$ such that, for all $x_1, \dots, x_n \in X$, we have

$$\left\| \left(\sum_{k=1}^n |x_k|^p \right)^{1/p} \right\|_X \leq C^{(p)} \left(\sum_{k=1}^n \|x_k\|_X^p \right)^{1/p}$$

and

$$\left(\sum_{k=1}^n \|x_k\|_X^q \right)^{1/q} \leq C_{(q)} \left\| \left(\sum_{k=1}^n |x_k|^q \right)^{1/q} \right\|_X,$$

respectively. We denote by $M^{(p)}(X)$ and $M_{(q)}(X)$ the least constants $C^{(p)}$ and $C_{(q)}$, which satisfy the above inequalities, respectively. It is well known that a p -convex Banach lattice can always be renormed with a lattice norm in such a way that $M^{(p)}(X) = 1$ and $M_{(q)}(X) = 1$. We refer to [12, Ch. 1.d] for more information about the classical geometric concepts of p -convexity and q -concavity.

A Banach space X is said to be of (Rademacher) cotype q ($2 \leq q < \infty$) if for some $C > 0$ and any $x_1, \dots, x_n \in X$ we have

$$\left(\sum_{k=1}^n \|x_k\|_X^q \right)^{1/q} \leq C \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\|_X dt,$$

where (r_k) is the sequence of Rademacher functions on $[0, 1]$ given by $r_k(t) := \text{sign}(\sin(2^k t))$ for all $t \in [0, 1]$ and for each $k \in \mathbb{N}$. The least constant C that satisfies the above condition is denoted by $C_q(X)$.

For two Banach sequence spaces E and F the space $M(E, F)$ of multipliers from E into F consists of all scalar sequences $x = (x_n)$ such that the associated multiplication operator $(y_n) \mapsto (x_n y_n)$ is defined and bounded from E into F . $M(E, F)$ is a (maximal and symmetric provided that E and F are) Banach sequence space equipped with the norm

$$\|x\|_{M(E, F)} := \sup\{\|xy\|_F; y \in B_E\}.$$

It can be easily verified that $M(E, F) \cong M(F', E')$ whenever both E and F have the Fatou property. Note that if E is a Banach sequence space then $M(E, \ell_1) \cong E'$.

We will use Orlicz spaces. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be an Orlicz function (that is, a convex increasing and continuous positive function with $\varphi(0) = 0$). The Orlicz space $L_\varphi(\mu)$ (L_φ for short) on a measure space $(\Omega, \mathcal{A}, \mu)$

is defined to be the space of $f \in L^0(\mu)$ such that $\int_{\Omega} \varphi(\lambda|f|) d\mu < \infty$ for some $\lambda > 0$ and is equipped with the norm

$$\|x\|_{L_{\varphi}} = \inf \left\{ \varepsilon > 0; \int_{\Omega} \varphi\left(\frac{|f|}{\varepsilon}\right) d\mu \leq 1 \right\}.$$

If Ω is a finite or countable set and $\mathcal{A} = 2^{\Omega}$, we will often write $\ell_{\varphi}(\mu)$ instead of $L_{\varphi}(\mu)$. Let L_{φ} be an Orlicz space on $(\Omega, \mathcal{A}, \mu)$. In what follows we will need two simple observations:

- For every $0 \neq f \in L_{\varphi}$, $\int_{\Omega} \varphi\left(\frac{|f|}{\|f\|_{L_{\varphi}}}\right) d\mu \leq 1$;
- If $g \in L^{\varphi}(\mu)$ satisfies $\int_{\Omega} \varphi\left(\frac{|g|}{\lambda}\right) d\mu \geq 1$, then $\lambda \leq \|g\|_{L_{\varphi}}$.

3 Absolutely φ -summing and φ semi-integral operators

Before we give a precise definition we note that in this section we study general properties of the absolutely φ -summing operators. We are motivated by interesting papers by Geiss [5,6] and references therein. Here in this section, among other things, we strengthen some results from [5]. We start from the discrete more natural definition, which has roots in the discrete definition of absolutely p -summing operators. At first we prove an extended variant of Pietsch's domination theorem. We next turn to the problem of vector-valued inequalities for these operators, which we apply to estimates of absolutely φ -summing norms of diagonal operators.

Now let φ be an Orlicz function. For a fixed $m \in \mathbb{N}$ and a positive sequence $\alpha = \{\alpha_j\} \in S_{\ell_1^m}$, we denote by $\ell_{\varphi}^m(\alpha)$ the m -dimensional Orlicz space over the probability measure $([m], 2^{[m]}, \mu)$, where $[m] := \{1, \dots, m\}$ and $\mu(\{j\}) = \alpha_j$ for each $j \in [m]$, endowed with the Luxemburg norm given by

$$\|\xi\|_{\ell_{\varphi}^m(\alpha)} = \inf \left\{ \lambda > 0; \sum_{j=1}^m \varphi(|\xi_j|/\lambda) \alpha_j \leq 1 \right\}, \quad \xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m.$$

Let X and Y be Banach spaces. We introduce the following definition: An operator $T: X \rightarrow Y$ is said to be *absolutely φ -summing* (*φ -summing* for short) if there exists a constant $C > 0$ such that regardless of the natural number m and regardless of the choice of $x_1, \dots, x_m \in X$ and every m -dimensional Orlicz space $\ell_{\varphi}^m(\alpha)$, we have

$$\|(\|Tx_1\|_Y, \dots, \|Tx_m\|_Y)\|_{\ell_{\varphi}^m(\alpha)} \leq C \sup_{x^* \in B_{X^*}} \|(x^*(x_1), \dots, x^*(x_m))\|_{\ell_{\varphi}^m(\alpha)}.$$

We denote by $\Pi_{\varphi}(X, Y)$ the space of all absolutely φ -summing linear operators from X into Y . It is a Banach space endowed with the norm $\pi_{\varphi}(T)$, that is defined to be the least constant C satisfying the above requirements.

As usual, for a Banach space X , we denote by κ_X the canonical embedding $\kappa_X: X \rightarrow C(B_{X^*})$, where B_{X^*} is equipped with the weak* topology.

Now we can state and prove the following result.

Theorem 3.1 *Let φ be a normalized Orlicz function (i.e., $\varphi(1) = 1$) and let $T: X \rightarrow Y$ be a φ -summing operator with $\pi_\varphi(T) \leq \gamma$. Suppose that $J: X \rightarrow C(K)$ is an isometric embedding of a Banach space X into $C(K)$. Then there exists a regular Borel probability measure μ on K such that, for all $x \in X$,*

$$\|Tx\|_Y \leq \gamma \|Jx\|_{L_\varphi(\mu)}.$$

Proof Consider the set

$$F_1 := \left\{ f \in C(K); \sup_{t \in K} f(t) < 1 \right\}.$$

For every $x \in X$, we define the function $f_x \in C(K)$ by $f_x := \varphi(\gamma|Jx|)$ and let

$$F_2 := \text{conv}\{f_x; x \in X, \|Tx\|_Y = 1\}.$$

Clearly, F_1 and F_2 are convex sets in $C(K)$ and F_1 is an open set. We prove that $F_1 \cap F_2 = \emptyset$. To see this, let $f \in F_2$. We prove that $f \notin F_1$. By the definition of F_2 , it follows that there are $\alpha_1, \dots, \alpha_m > 0$ and $x_1, \dots, x_m \in X$, such that $\sum_{j=1}^m \alpha_j = 1$, $\|Tx_j\|_Y = 1$ for each $1 \leq j \leq m$, and

$$f = \sum_{j=1}^m \alpha_j f_{x_j}.$$

Since $\pi_\varphi(T) \leq \gamma$, it follows that for $\alpha := (\alpha_1, \dots, \alpha_m)$, we have

$$\begin{aligned} \|(1, \dots, 1)\|_{\ell_\varphi^m(\alpha)} &= \|(\|Tx_1\|_Y, \dots, \|Tx_m\|_Y)\|_{\ell_\varphi^m(\alpha)} \\ &\leq \gamma \sup_{x^* \in B_{X^*}} \|(\langle x_1, x^* \rangle, \dots, \langle x_m, x^* \rangle)\|_{\ell_\varphi^m(\alpha)} \\ &= \gamma \sup_{\mu \in B_{C(K)^*}} \|(\langle Jx_1, \mu \rangle, \dots, \langle Jx_m, \mu \rangle)\|_{\ell_\varphi^m(\alpha)} \\ &= \gamma \sup_{t \in K} \|(Jx_1(t), \dots, Jx_m(t))\|_{\ell_\varphi^m(\alpha)}. \end{aligned} \quad (*)$$

We remark that the last equality follows by the fact that

$$\mu \longmapsto \|(\langle Jx_1, \mu \rangle, \dots, \langle Jx_m, \mu \rangle)\|_{\ell_\varphi^m(\alpha)}$$

is a $\sigma(C(K)^*, C(K))$ -continuous, convex function in variable μ on $B_{C(K)^*}$ and so attains its supremum on the set of extreme points $\{\delta_t; t \in K\}$.

We claim that $f \notin F_1$. Otherwise, we would have

$$\sup_{t \in K} \sum_{j=1}^m \varphi(\gamma|Jx_j(t)|) \alpha_j < 1.$$

Hence, by the definition of the norm in $\ell_\varphi^m(\alpha)$, we would get that

$$\sup_{t \in K} \|(Jx_1(t), \dots, Jx_m(t))\|_{\ell_\varphi^m(\alpha)} < 1/\gamma.$$

Clearly, $\varphi(1) = 1$ yields $\|(1, \dots, 1)\|_{\ell_\varphi^m(\alpha)} = 1$. Combining this with the inequality (*), we get a contradiction. This proves the claim.

Now we can apply both the Hahn–Banach and Riesz representation theorems to get the existence of a constant λ and a regular Borel measure μ on K such that

$$\int_K f \, d\mu \leq \lambda \quad \text{for all } f \in F_1$$

and

$$\int_K f \, d\mu \geq \lambda \quad \text{for all } f \in F_2.$$

Since F_1 contains all negative functions, μ has to be a positive measure. Thus, taking a normalization of μ , we get that $\lambda \geq 1$. Hence, if $x \in X$ with $\|Tx\|_Y = 1$, then

$$\int_K \varphi(\gamma|Jx|) \, d\mu \geq \lambda \geq 1$$

and so, for all $x \in X$ with $Tx \neq 0$, we have

$$\int_K \varphi\left(\frac{\gamma|Jx|}{\|Tx\|_Y}\right) \, d\mu \geq 1.$$

Thus we conclude that

$$\|Tx\|_Y \leq \gamma \|Jx\|_{L_\varphi(\mu)}, \quad x \in X.$$

This completes the proof. $\square$

An immediate consequence of Theorem 3.1 is a variant of the Pietsch domination theorem.

Theorem 3.2 *Let φ be a normalized Orlicz function. Suppose that $T: X \rightarrow Y$ is a φ -summing between Banach spaces with $\pi_\varphi(T) \leq \gamma$. Then there exists a regular Borel probability measure μ on B_{X^*} equipped with the weak* topology so that for every $x \in X$,*

$$\|Tx\|_Y \leq \gamma \|\kappa_X(x)\|_{L_\varphi(B_{X^*}, \mu)}.$$

Proof Since $J := \kappa_X$ is an isometric embedding of X into $C(K)$, where $K := B_{X^*}$ is equipped with the weak* topology, Theorem 3.1 applies. $\square$

The following definition is motivated by Theorem 3.2: Let X be Banach spaces and let $\mathcal{M}(B_{X^*})$ be the space of all Borel probability measures on B_{X^*} , endowed with the weak* topology. An operator $T: X \rightarrow Y$ is said to be $F(\mu)$ semi-integral whenever there exist a constant $C > 0$, a measure $\mu \in \mathcal{M}(B_{X^*})$ and a Banach lattice F in $L^0(B_{X^*}, \mu)$ such that, for all $x \in X$,

$$\|Tx\|_Y \leq C \|\kappa_X(x)\|_{F(\mu)}.$$

The infimum of the constant C for which the inequality holds is denoted by $\pi_{si, F}(T)$. In what follows μ is called a Pietsch's measure (or a representing measure) for T .

In what follows if T is $F(\mu)$ semi-integral, where $F(\mu) = L_\varphi(\mu)$ is an Orlicz space, then T is called φ semi-integral for short, and we write $\pi_{si, \varphi}$ instead of $\pi_{si, L_\varphi}(T)$. In this case we denote by $\Pi_{si, \varphi}(X, Y)$ the space of all φ semi-integral operators from X into Y .

An immediate consequence of Theorem 3.2 is the following.

Corollary 3.3 *Let φ be a normalized Orlicz function. If $T: X \rightarrow Y$ is a φ -summing operator, then T is φ semi-integral with $\pi_{si, \varphi}(T) \leq \pi_\varphi(T)$.*

We establish some properties of φ -summing operators. We recall that Dunford–Pettis operators between Banach spaces are those that carry weakly convergent sequences to norm convergent sequences. Clearly, every operator $T: X \rightarrow Y$ is a Dunford–Pettis operator if X has the Schur property (i.e., such that weakly compact sets in X are compact).

We will need the following characterization of the Dunford–Pettis operators in terms of weak Cauchy sequences (see [1, Theorem 19.1]).

Theorem 3.4 *An operator $T: X \rightarrow Y$ between two Banach spaces is a Dunford–Pettis operator if and only if it T carries weakly Cauchy sequences of X onto norm convergent sequences of Y .*

We will use the following observation that any Dunford–Pettis operator $T: X \rightarrow Y$ is compact whenever X does not contain an isomorphic copy of ℓ_1 . This fact is an immediate consequence of Theorem 3.4 and Rosenthal's result that states that Banach space X does not contain an isomorphic copy of ℓ_1 if and only if any norm bounded sequence in X contains a weak Cauchy subsequence.

We conclude our discussion about Dunford–Pettis operators with following application.

Theorem 3.5 *The following statements are true for any absolutely φ -summing operator $T: X \rightarrow Y$ between Banach spaces:*

- (i) T is a Dunford–Pettis operator;
- (ii) If the Banach space X does not contain isomorphic copy of ℓ_1 , then T is compact.

Proof (i). It is easily to check that for Orlicz space $L_\varphi(\mu)$ in $L^0(\Omega, \Sigma, \mu)$, we have $f_n \rightarrow f$ in L_φ if and only if for every $\lambda > 0$,

$$\lim_{n \rightarrow \infty} \int_{\Omega} \varphi(\lambda |f_n - f|) d\mu = 0.$$

From Theorem 3.2, it follows that T is φ semi-integral, i.e., there exists a probability Borel measure $\mu \in \mathcal{M}(B_{X^*})$, such that

$$\|Tx\|_Y \leq C \|\langle x, \cdot \rangle\|_{L_\varphi(\mu)}.$$

Let (x_n) be a weakly null sequence in X . Then for the sequence $(f_n)_{n=1}^\infty$ of continuous functions given by $f_n(x^*) := \langle x_n, x^* \rangle$ for all $x^* \in B_{X^*}$ and each $n \geq 1$, we have

$$\|Tx_n\|_Y \leq \|f_n\|_{L_\varphi(\mu)}.$$

Since $f_n \rightarrow 0$ holds pointwise on B_{X^*} and (f_n) is uniformly bounded sequence, it follows by Lebesgue's Dominated Convergence Theorem that $f_n \rightarrow 0$ in $L_\varphi(\mu)$. Thus, $\|Tx_n\|_Y \rightarrow 0$ as required.

Combining (i) with Theorem 3.4 yields the statement (ii). This completes the proof. $\square$

Now let us show give a fundamental example of a φ semi-integral operator.

Lemma 3.6 *Let K be a compact Hausdorff spaces, and let μ a positive regular Borel measure on K , and let φ be an Orlicz function. Then the canonical inclusion map*

$$j_\varphi: C(K) \longrightarrow L_\varphi(\mu)$$

is φ semi-integral with $\pi_{si, \varphi}(j_\varphi) \leq 1$.

Proof Let $f \in C(K)$ be such that $\lambda := \|j_\varphi f\|_{L_\varphi(\mu)} > 0$. Since f is continuous, it has an order continuous norm in $L_\varphi(\mu)$,

$$\int_K \varphi(|f|/\lambda) d\mu = 1.$$

Let $B_{C(K)^*}$ be equipped with the weak* topology. We consider the mapping $\delta: K \rightarrow B_{C(K)^*}$ given by $\delta(t) := \delta_t$, where as usual $\langle f, \delta_t \rangle = f(t)$ for all $f \in C(K)$ and all $t \in K$. Since δ is continuous and one-to-one (and so it is a homeomorphism onto $\delta(K)$), δ is a Borel mapping. Let $\nu := (\delta)\mu$ be the image measure of μ (via δ) on Borel sets of $B_{C(K)^*}$. Then

$$\begin{aligned} 1 &= \int_K \varphi(|f(t)|/\lambda) d\mu(t) = \int_K \varphi(|\kappa_{C(K)} f(\delta(t))|/\lambda) d\mu(t) \\ &\leq \int_{B_{C(K)^*}} \varphi(|\kappa_{C(K)} f(x^*)|/\lambda) d\nu(x^*) \end{aligned}$$

and so $\|j_\varphi f\|_{L_\varphi(\mu)} \leq \|\langle f, \cdot \rangle\|_{L_\varphi(\nu)}$, implying j_φ is φ semi-integral with $\pi_{si, \varphi}(j_\varphi) \leq 1$. $\square$

Below we will work with mixed Orlicz spaces. For the sake of completeness, we recall the definition of mixed Banach lattices. Let $(\Omega_1, \Sigma_1, \nu)$ and $(\Omega_2, \Sigma_2, \mu)$ be measure spaces and let E and F be Banach lattices in $L^0(\nu)$ and $L^0(\mu)$, respectively. Assume (for measurability reasons) that either the measure ν is discrete or the norm $\|\cdot\|_F$ is semi-continuous, i.e., if $0 \leq f_n \uparrow f$ ν -a.e., with $f \in F$, then $\|f_n\|_F \rightarrow \|f\|_F$. In what follows, for any every $f \in L^0(\Omega_1 \times \Omega_2, \nu \times \mu)$ and $(s, t) \in \Omega_1 \times \Omega_2$, we define $f_s \in L^0(\mu)$ and $f^t \in L^0(\nu)$ by $f_s(\cdot) = f(s, \cdot)$ and $f^t(\cdot) = f(\cdot, t)$. The mixed Banach lattice $E[F]$ in $L^0(\nu \times \mu)$ is defined to be the space of all $f \in L^0(\nu \times \mu)$ such that $f_s \in F$ with $s \mapsto \|f_s\|_F \in E$, and equipped with the norm

$$\|f\|_{E[F]} := \|\|f_s\|_F\|_E.$$

Similarly, we define $[E]F$ to be the Banach lattice in $L^0(\nu \times \mu)$ of all $f \in L^0(\nu \times \mu)$ such that $f^t \in E$ with $t \mapsto \|f^t\|_E \in F$, and equipped with the norm

$$\|f\|_{[E]F} := \|\|f^t\|_E\|_F.$$

In what follows we will prove vector-valued inequalities for φ semi-integral operators generated by Orlicz functions satisfying some minor conditions. As a by-product, we deduce that these operators are φ -summing. In the proof we will use the following characterization of the embeddings between mixed Orlicz spaces and Orlicz spaces defined on a product of measure spaces: *Let $(\mathcal{X}_1, \Sigma_1, \nu)$ and $(\mathcal{X}_2, \Sigma_2, \mu)$ be measure spaces. Then the inclusion map*

$$j: L_\varphi(\nu \times \mu) \hookrightarrow L_\varphi(\nu)[L_\varphi(\mu)]$$

is bounded with norm $\|j\| \leq C$ if and only if φ is C -supermultiplicative (i.e., $\varphi(Cuv) \geq \varphi(u)\varphi(v)$ for all $u, v > 0$ and some $C > 0$). If additionally μ and ν are finite measures, then the above statement holds if and only if φ is C -supermultiplicative at ∞ (i.e., there exists $C > 0$ such that $\varphi(Cuv) \geq \varphi(u)\varphi(v)$ for all $u, v \geq 1$).

Let us note that the above result was proven for the case of non-atomic finite measures in [20]. A minor modification of the proof gives the result for the general case.

We refer to [9] for examples of supermultiplicative Orlicz functions at ∞ . In what follows we will consider Orlicz functions φ given by the formula $\varphi(t) := e^{\phi(t)} - 1$ for all $t \geq 0$, where ϕ is any Orlicz function. Suppose that there exists $c > 0$ such that $\phi(cst) \geq \phi(s) + \phi(t)$ for all $s, t \geq 1$, then $\varphi(cst) \geq \varphi(s)\varphi(t)$ for all $s, t \geq 1$, i.e., φ is c -supermultiplicative at ∞ . In particular, if $\phi(cst) \geq \phi(s)\phi(t)$ for all $s, t \geq 1$ and $\phi(1) \geq 2$, then $\varphi(cst) \geq \varphi(s) + \varphi(t)$ for all $s, t \geq 1$.

We need the following lemma.

Lemma 3.7 *Let $(\mathcal{X}_1, \Sigma_1, \nu)$ and $(\mathcal{X}_2, \Sigma_2, \mu)$ be measure spaces with μ -finite. Suppose that φ is a C -supermultiplicative Orlicz function. Then the inclusion map*

$$j: [L_\varphi(\nu)]L_\infty(\mu) \hookrightarrow L_\varphi(\nu)[L_\varphi(\mu)]$$

is bounded with $\|j\| \leq C \max\{1, \mu(\mathcal{X}_2)\}$. Moreover, if μ and ν are finite measures, then the above statement holds whenever φ is C -supermultiplicative at ∞ .

Proof For any $f \in [L_\varphi(\nu)]L_\infty(\mu)$ with norm 1, we have $\|f^t\|_{L_\varphi(\nu)} \leq 1$ for μ -almost all $t \in \mathcal{X}_2$. Hence, for μ -almost all $t \in \mathcal{X}_2$, we have

$$\int_{\mathcal{X}_1} \varphi(|f(s, t)|) dv(s) = \int_{\mathcal{X}_1} \varphi(|f^t(s)|) dv(s) \leq \int_{\mathcal{X}_1} \varphi\left(\frac{|f^t(s)|}{\|f^t\|_{L_\varphi(\nu)}}\right) dv(s) \leq 1.$$

Combining with Fubini's Theorem, it follows that

$$\int_{\mathcal{X}_1 \times \mathcal{X}_2} \varphi(|f|) d(\nu \times \mu) = \int_{\mathcal{X}_2} \left(\int_{\mathcal{X}_1} \varphi(|f(s, t)|) dv(s) \right) d\mu(t) \leq \max\{1, \mu(\mathcal{X}_2)\}.$$

This shows that $[L_\varphi(\nu)]L_\infty(\mu) \subset L_\varphi(\nu \times \mu)$ with norm of the inclusion map less or equal than $\max\{1, \mu(\mathcal{X}_2)\}$. To finish, it is enough to apply the result mentioned above Lemma. $\square$

We are ready to prove a vector-valued estimate for φ semi-integral operators. At first we recall that for a given Banach lattice E on a measure space (Ω, Σ, ν) and a Banach space X , we denote by $E(X)$ the set of all $f \in L^0(\mu, X)$ such that $\|f(\cdot)\|_X \in E$. We note that $E(X)$ is a Banach space under pointwise operations and the natural norm $\|f\|_{E(X)} := \|\|f(\cdot)\|_X\|_E$.

In what follows, if $T: X \rightarrow Y$ is an operator between Banach spaces, then we denote by $\vec{T}$ a linear mapping $\vec{T}: L^0(X, \nu) \rightarrow L^0(Y, \nu)$ given, for all $f \in L^0(X, \nu)$ by the formula:

$$\vec{T}f(\omega) := T(f(\omega)), \quad \omega \in \Omega.$$

Theorem 3.8 *Let $T: X \rightarrow Y$ be a φ semi-integral operator with $\pi_{si, \varphi}(T) \leq \gamma$, where φ is a C -supermultiplicative Orlicz function, and let (Ω, Σ, ν) be a measure space. Suppose that $L_\varphi(\nu)$ is an Orlicz space in $L^0(\Omega, \Sigma, \nu)$. Then the vector-valued estimate holds:*

$$\|\vec{T}f\|_{L_\varphi(\nu, Y)} \leq \gamma C \sup_{x^* \in B_{X^*}} \|\langle f(\cdot), x^* \rangle\|_{L_\varphi(\nu)}$$

for all $f \in L^0(\nu, X)$ such that the value of the expression on the right hand side of the inequality is finite. If the measure ν is finite, then the above statement is true if φ is supermultiplicative at ∞ .

Proof We assume that φ is C -supermultiplicative; that is, there is $C > 0$ such that $\varphi(Cuv) \geq \varphi(u)\varphi(v)$ for all $u, v > 0$. Our hypothesis implies that there exists a probability Borel measure μ on B_{X^*} equipped with the weak* topology such that for all $x \in X$,

$$\|Tx\|_Y \leq \gamma \|\langle x, \cdot \rangle\|_{L_\varphi(\mu)}.$$

An application of Lemma 3.7 gives the required estimate. Indeed,

$$\begin{aligned} \| \|T(f(\cdot))\|_Y \|_{L_\varphi(v)} &\leq \gamma \| \langle f(\cdot), \cdot \rangle \|_{L_\varphi(v)[L_\varphi(\mu)]} \\ &\leq \gamma C \| \langle f(\cdot), \cdot \rangle \|_{[L_\varphi(v)]L_\infty(\mu)} \\ &= \gamma C \sup_{x^* \in B_{X^*}} \| \langle f(\cdot), x^* \rangle \|_{L_\varphi(v)}. \end{aligned}$$

In the case when ν is a finite measure the proof is similar and so we omit it. $\square$

The following result is an immediate consequence of Theorems 3.1 and 3.8.

Corollary 3.9 *Let φ be an Orlicz function that is C -supermultiplicative at ∞ . Then an operator $T: X \rightarrow Y$ is absolutely φ -summing if and only if T is φ semi-integral.*

Combining Theorems 3.1, 3.2 and 3.8, we obtain the following variant of Pietsch's characterization of φ -summing operators.

Theorem 3.10 *Let φ be an Orlicz function that is normalized and 1-supermultiplicative at ∞ . The following statements are equivalent for an operator $T: X \rightarrow Y$:*

- (i) T is φ -summing with $\pi_\varphi(T) \leq \gamma$;
- (ii) T is φ semi-integral with $\pi_{si,\varphi}(T) \leq \gamma$;
- (iii) *For every (equivalently, for some) isometric embedding $J: X \rightarrow C(K)$ there exists a Borel probability measure μ on a compact Hausdorff space K and a constant $\gamma > 0$ such that, for all $x \in X$,*

$$\|Tx\|_Y \leq \gamma \|Jx\|_{L_\varphi(\mu)}.$$

4 Norm estimates of certain φ -summing operators

In this section we apply the previous results to derive asymptotic estimates of φ -summing norms of the identity maps on finite dimension spaces for some class of Orlicz spaces. We apply these results in the study of φ -summing norms of diagonal operators on Banach sequence lattices.

We first prove some lemmas which will be used for the proof of our main estimates. We start with some useful facts that we use in our considerations. For a given positive integer m let E^m be $\mathbb{R}^m$ equipped with a lattice norm $\|\cdot\|_{E^m}$ and let $(E^m)'$ be the dual equipped with the lattice norm

$$\|(a_i)\|_{(E^m)'} = \sup_{(b_i) \in B_{E^m}} \left| \sum_{i=1}^m a_i b_i \right|.$$

A simple argument gives that for any $x_1, \dots, x_m$ in the Banach space X , we have

$$\sup_{x^* \in B_{X^*}} \| (x^*(x_1), \dots, x^*(x_m)) \|_{E^m} = \sup_{a \in B_{(E^m)'}} \left\| \sum_{i=1}^m a_i x_i \right\|_X,$$

Using the above formula for $X = L_\infty$ on a measure space (Ω, Σ, μ) , we conclude on the basis of similar arguments as in the case $E = \ell_p$ (see [4, pp. 41–42] that, for all $f_1, \dots, f_m \in L_\infty(\mu)$,

$$\sup_{x^* \in B_{L_\infty}^*} \|(x^*(f_1), \dots, x^*(f_m))\|_{E^m} = \operatorname{ess\,sup}_{\omega \in \Omega} \|(f_1(\omega), \dots, f_m(\omega))\|_{E^m}.$$

In what follows the following natural example of an absolutely φ -summing operator will be useful.

Lemma 4.1 *Let (Ω, Σ, μ) be a finite measure space. Suppose that φ is an Orlicz function that is C -supermultiplicative at ∞ . Then the formal inclusion map*

$$j: L_\infty(\mu) \longrightarrow L_\varphi(\mu)$$

is φ -summing with $\pi_\varphi(j) \leq C \max\{1, \mu(\Omega)\}$.

Proof Let $0 < \alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ with $\alpha_1 + \dots + \alpha_m = 1$. From Lemma 3.7, applied for $L_\varphi(v) := \ell_\varphi^m(\alpha)$ and $E^m := \ell_\varphi^m(\alpha)$, it follows that

$$j: [\ell_\varphi^m(\alpha)]_{L_\infty(\mu)} \hookrightarrow \ell_\varphi^m(\alpha)[L_\varphi(\mu)]$$

with $\|id\| \leq \tilde{C} := C \max\{1, \mu(\Omega)\}$. To finish, we combine this continuous inclusion with discussed above facts to get that for all $f_1, \dots, f_m \in L_\infty(\mu)$,

$$\begin{aligned} \|(\|f_1\|_{L_\varphi(\mu)}, \dots, \|f_m\|_{L_\varphi(\mu)})\|_{\ell_\varphi^m(\alpha)} &\leq \tilde{C} \operatorname{ess\,sup}_{\omega \in \Omega} \|(f_1(\omega), \dots, f_m(\omega))\|_{\ell_\varphi^m(\alpha)} \\ &= \tilde{C} \sup_{x^* \in B_{L_\infty}^*} \|(x^*(f_1), \dots, x^*(f_m))\|_{\ell_\varphi^m(\alpha)}. \end{aligned}$$

This completes the proof. $\square$

We will need also the following technical lemma.

Lemma 4.2 *Let ϕ be an Orlicz function and let $L_\varphi(\mu)$ be an Orlicz sequence space, where $\varphi(t) = e^{\phi(t)} - 1$ for all $t \geq 0$ and μ is the probability measure on $\mathcal{P}(\mathbb{N})$ given by $\mu(\{k\}) := 1/k(k+1)$ for each $k \in \mathbb{N}$. Then $L_\phi(\mu) = \ell_\infty(v)$ with $v = (1/\phi^{-1}(\log(1+k)))_{k=1}^\infty$ and moreover*

$$\frac{1}{2} \|\cdot\|_{\ell_\infty(v)} \leq \|\cdot\|_{L_\varphi(\mu)} \leq 5 \|\cdot\|_{\ell_\infty(v)}.$$

Proof Let $0 \neq \xi = (\xi_k) \in L_\varphi(\mu)$. Then for $\lambda := \|\xi\|_{L_\varphi(\mu)}$, we have

$$\int_{\mathbb{N}} \varphi(|\xi|/\lambda) d\mu = \sum_{k=1}^{\infty} (e^{\phi(|\xi_k|/\lambda)} - 1) \frac{1}{k(k+1)} \leq 1.$$

This implies that, for each $k \in \mathbb{N}$,

$$e^{\phi(|\xi_k|/\lambda)} \leq 1 + k(k+1) < (k+1)^2.$$

Hence

$$\sup_{k \geq 1} \frac{|\xi_k|}{\phi^{-1}(\log(1+k))} \leq 2\lambda.$$

and so $L_\varphi(\mu) \hookrightarrow \ell_\infty(v)$ with $\|\text{id}\| \leq 2$.

To show the converse inclusion, take $\xi = (\xi_k) \in B_{\ell_\infty(v)}$. Then $|\xi_k| \leq \phi^{-1}(\log(1+k))$ yields by convexity of ϕ that for each $k \in \mathbb{N}$,

$$\phi\left(\frac{|\xi_k|}{2}\right) \leq \log \sqrt{1+k}$$

and hence

$$e^{\phi(|\xi_k|/2)} - 1 \leq \sqrt{1+k} - 1 \leq \sqrt{k}.$$

Consequently,

$$\begin{aligned} \int_{\mathbb{N}} \varphi(|\xi|/2) d\mu &= \sum_{k=1}^{\infty} (e^{\phi(|\xi_k|/2)} - 1) \frac{1}{k(k+1)} \leq \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}(k+1)} \\ &\leq \frac{1}{2} + \int_1^{\infty} \frac{1}{\sqrt{t}(1+t)} dt = \frac{1}{2}(1+\pi). \end{aligned}$$

By convexity of ψ , it follows that for $c = 1 + \pi$,

$$\int_{\mathbb{N}} \varphi(|\xi|/c) d\mu \leq 1.$$

This yields the required continuous inclusion $\ell_\infty(v) \hookrightarrow L_\varphi(\mu)$ with $\|\text{id}\| \leq c < 5$. $\square$

We now turn to the problem of asymptotic estimates of φ -summing norms of the identity map on finite dimensional spaces. We start with the following lemma.

Lemma 4.3 *For every c -supermultiplicative normalized Orlicz function φ and for each $n \in \mathbb{N}$ one has*

$$\varphi^{-1}(n) \leq c \pi_\varphi(\text{id}: \ell_\infty^n \rightarrow \ell_\infty^n).$$

Proof Let $L_\varphi(v)$ be an Orlicz space on $[0, 1]$ with the Lebesgue measure v . We consider $f \in L^0(v, \ell_\infty^n)$ given by $f = (f_i)_{i=1}^n$ with $f_i = \chi_{[\frac{i-1}{n}, \frac{i}{n}]}$ for each $1 \leq i \leq n$. Here χ_A denotes the characteristic function of A . From Theorem 3.8, it follows that

$$\|\vec{\text{id}}(f)\|_{L_\varphi(v, \ell_\infty^n)} \leq c \pi_\varphi(\text{id}: \ell_\infty^n \rightarrow \ell_\infty^n) \sup_{a \in B_{(\ell_\infty^n)'}} \|\langle f, a \rangle\|_{L_\varphi(v)}.$$

This estimate, combined with $\|\chi_A\|_{L_\varphi(v)} = 1/\phi^{-1}(1/\nu(A))$ for any measurable set A , yields

$$\begin{aligned} 1 &= \|\chi_{[0,1]}\|_{L_\varphi(v)} = \left\| \sum_{i=1}^n \chi_{[\frac{i-1}{n}, \frac{i}{n})} \right\|_{L_\varphi(v)} = \left\| \sup_{1 \leq i \leq n} |f_i| \right\|_{L_\varphi(v)} \\ &\leq c \pi_\varphi(\text{id}: \ell_\infty^n \rightarrow \ell_\infty^n) \sup_{(a_i) \in B_{\ell_1^n}} \left\| \sum_{i=1}^n a_i \chi_{[\frac{i-1}{n}, \frac{i}{n})} \right\|_{L_\varphi(v)} \\ &\leq c \pi_\varphi(\text{id}: \ell_\infty^n \rightarrow \ell_\infty^n) \sup_{(a_i) \in B_{\ell_1^n}} \sum_{i=1}^n a_i \|\chi_{[\frac{i-1}{n}, \frac{i}{n})}\|_{L_\varphi(v)} \\ &= c \pi_\varphi(\text{id}: \ell_\infty^n \rightarrow \ell_\infty^n) \frac{1}{\phi^{-1}(n)}, \end{aligned}$$

which concludes the proof. $\square$

Lemma 4.4 *Let φ be an Orlicz function such that $\varphi(t) = e^{\phi(t)} - 1$ for all $t \geq 0$ is c -supermultiplicative at ∞ . Then for each $n \in \mathbb{N}$ one has*

$$\pi_\varphi(\text{id}: \ell_\infty^n \rightarrow \ell_\infty^n) \asymp \phi^{-1}(\log(1+n)),$$

where the constants of equivalence depend on c and ϕ only.

Proof Without loss of generality we may assume that ϕ is a normalized Orlicz function (i.e., $\phi(1) = 1$). Since $\varphi^{-1}(t) = \phi^{-1}(\log(1+t))$ for all $t \geq 0$, Lemma 4.3 yields the following estimate

$$c^{-1} \phi^{-1}(\log(1+n)) \leq \pi_\varphi(\text{id}: \ell_\infty^n \rightarrow \ell_\infty^n).$$

Using the notation from Lemma 4.2, we see that the identity map $I: L_\infty(\mu) \rightarrow L_\varphi(\mu)$ is an isomorphism. This implies that $\text{id}: \ell_\infty^n \rightarrow \ell_\infty^n$ admits the following factorization:

$$\text{id}: \ell_\infty^n \xrightarrow{J_n} L_\infty(\mu) \xrightarrow{I} L_\varphi(\mu) \xrightarrow{P_n} \ell_\infty^n,$$

where $J_n(\xi_1, \dots, \xi_n) := (\xi_1, \dots, \xi_n, 0, 0, \dots)$ and $P_n \xi := (\xi_1, \dots, \xi_n)$ for all $\xi = (\xi_k) \in L_\varphi(\mu)$. Then we get from Lemma 4.2

$$\begin{aligned} \|P_n \xi\|_{\ell_\infty^n} &= \sup_{1 \leq k \leq n} |\xi_k| \leq \phi^{-1}(\log(1+n)) \sup_{k \geq 1} \frac{|\xi_k|}{\phi^{-1}(\log(1+k))} \\ &\leq 5 \phi^{-1}(\log(1+n)) \|\xi\|_{L_\varphi(\mu)}. \end{aligned}$$

Finally the ideal property combined with Lemma 3.7 yields

$$\begin{aligned}\pi_\varphi(\text{id}: \ell_\infty^n \rightarrow \ell_\infty^n) &\leq \|J_n\| \pi_\varphi(I: L_\infty(\mu) \rightarrow L_\varphi(\mu)) \|P_n\| \\ &\leq 5c \phi^{-1}(\log(1+n)).\end{aligned}$$

This completes the proof. $\square$

Our final application of the above lemma is the following result.

Corollary 4.5 *Let φ be an Orlicz function such that $\varphi(t) = e^{\phi(t)} - 1$ for all $t \geq 0$ is supermultiplicative at ∞ . Then*

$$\pi_\varphi(I_X) \leq C \phi^{-1}(n)$$

for every Banach space X with $\dim(X) = n$, where $C > 0$ depends on ϕ only.

Proof By [8, Lemma 13.8] it follows that there exist $N \leq 4^n$ (resp., $N \leq 4^{2n}$ in the complex case) and an operator $A: X \rightarrow \ell_\infty^N$ such that

$$\frac{1}{3} \|x\|_X \leq \|Ax\|_{\ell_\infty^N} \leq \|x\|_X \quad \text{for all } x \in X.$$

If we let $Y := A(X)$, then I_X has a factorization

$$I_X: X \xrightarrow{A} Y \xrightarrow{A^{-1}} X.$$

Combining with Lemma 4.5, we obtain the following estimate:

$$\begin{aligned}\pi_\varphi(I_X) &\leq \pi_\varphi(A: X \rightarrow Y) \|A^{-1}: Y \rightarrow X\| \leq 3 \pi_\varphi(A: X \rightarrow Y) \\ &= 3 \pi_\varphi(A: X \rightarrow \ell_\infty^N) \leq 3 \pi_\varphi(I: \ell_\infty^N \rightarrow \ell_\infty^N) \\ &\leq 3c \phi^{-1}((1 + \log 16)n) \leq C \phi^{-1}(n),\end{aligned}$$

where $C = 3c(1 + \log 16)$. This completes the proof. $\square$

In what follows, for a given Banach sequence lattice E we let E^n to be $\mathbb{R}^n$ equipped with the norm

$$\|\xi\|_{E^n} := \left\| \sum_{i=1}^n \xi_i e_i \right\|_E, \quad \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n.$$

Now we can state the following result.

Theorem 4.6 *Let E be an r -convex maximal Banach sequence lattice and let φ be an Orlicz function that is supermultiplicative at ∞ and such that $t^r < \phi(t)$ for large enough t . Suppose that $0 = \gamma_0 < \gamma_1 \leq \gamma_2 \leq \dots$ with $\pi_\varphi(I_{E^n}) \leq \gamma_n$ for each $n \in \mathbb{N}$.*

Then, there exists a constant $C > 0$ such that for every diagonal operator D_λ on E with $\lambda_1 \geq \lambda_2 \geq \dots > 0$, we have

$$\pi_\varphi(D_\lambda: E \rightarrow E) \leq C \left(\sum_{k=1}^{\infty} \lambda_k^r (\gamma_k^r - \gamma_{k-1}^r) \right)^{1/r}.$$

Proof We will use the well known result (see [15, Lemma 2.4], [21, Proposition 1.2]), which states that if X is an r -convex Banach function lattice in $L^0(\Omega, \Sigma, \mu)$ with the Fatou property, then there exists a collection $W \subset L^0(\mu)$ of nonnegative measurable functions such that

$$\|x\|_X \asymp \sup_{w \in W} \left(\int_{\Omega} |x|^r w d\mu \right)^{1/r}, \quad x \in X,$$

where the constants of equivalence depend on the r -convexity constant of X .

It follows from Theorem 3.2 that for each $k \in \mathbb{N}$ there exists a probability Borel measure μ_k on $(B_{E^k})^*$ such that

$$\|x\|_{E^k} \leq C\pi_\varphi(I_{E^k})\|\langle x, \cdot \rangle\|_{L_\varphi(B_{(E^k)^*}, \mu_k)} \leq C\gamma_k\|\langle x, \cdot \rangle\|_{L_\varphi(B_{(E^n)^*}, \mu_k)},$$

for all $x \in E^k$, where C depends on φ only. Now if we denote by ν_k be the image measure of μ_k with respect to the natural canonical isometrical embedding $J_k: (E^k)^* \rightarrow E^*$, we have

$$\|x\|_{E^k} \leq C\gamma_k\|\langle x, \cdot \rangle\|_{L_\varphi(B_{E^*}, \nu_k)}, \quad x \in E^k.$$

Fix $N \in \mathbb{N}$. Combining the above facts, we conclude that for some subset W of nonnegative nonnegative sequences and for all $\xi = (\xi_i) \in E$, we have

$$\begin{aligned} \left\| \sum_{k=1}^N \lambda_k \xi_k e_k \right\|_E &= \sup_{(w_i) \in W} \left(\sum_{k=1}^N \lambda_k^r |\xi_k|^r w_k \right)^{1/r} \\ &= \sup_{(w_i) \in W} \left(\sum_{k=1}^N (\lambda_k^r - \lambda_{k+1}^r) (|\xi_1|^r w_1 + \dots + |\xi_k|^r w_k) \right)^{1/r} \\ &\leq \left(\sum_{k=1}^N (\lambda_k^r - \lambda_{k+1}^r) \sup_{(v_i) \in W} (|\xi_1|^r v_1 + \dots + |\xi_k|^r v_k) \right)^{1/r} \\ &= \left(\sum_{k=1}^N (\lambda_k^r - \lambda_{k+1}^r) \|(\xi_1, \dots, \xi_k)\|_{E^k}^r \right)^{1/r} \end{aligned}$$

$$\begin{aligned}
&\leq C \left(\sum_{k=1}^N (\lambda_k^r - \lambda_{k+1}^r) \gamma_k^r \|\langle \xi, \cdot \rangle\|_{L_\varphi(B_{E^*}, v_k)}^r \right)^{1/r} \\
&= C \left(\sum_{k=1}^N (\gamma_k^r (\lambda_k^r - \lambda_{k-1}^r)) \right) \left(\sum_{k=1}^N \theta_k \|\langle \xi, \cdot \rangle\|_{L_\varphi(B_{E^*}, v_k)}^r \right)^{1/r},
\end{aligned}$$

where

$$\theta_k := \frac{(\lambda_k - \lambda_{k+1}) \gamma_k^r}{\sum_{i=1}^N (\lambda_i - \lambda_{i+1}) \gamma_i^r}, \quad 1 \leq i \leq N.$$

Applying Lemma 4.11 from [5], we conclude that for each positive integer N ,

$$\begin{aligned}
\left\| \sum_{k=1}^N \lambda_k \xi_k e_k \right\|_E &\leq C \left(\sum_{k=1}^N (\lambda_k^r - \lambda_{k+1}^r) \gamma_k^r \right)^{1/r} \|\langle \xi, \cdot \rangle\|_{L_\varphi(B_{E^*}, \mu)} \\
&= C \left(\sum_{k=1}^N (\lambda_k^r (\gamma_k^r - \gamma_{k-1}^r)) \right) \left(\sum_{k=1}^N \theta_k \|\langle \xi, \cdot \rangle\|_{L_\varphi(B_{E^*}, v_k)}^r \right)^{1/r},
\end{aligned}$$

where $\mu := \sum_{i=1}^N \theta_i v_i$ is a probability Borel measure on B_{E^*} equipped with weak* topology.

Since E is maximal, for each $\xi = (\xi_i) \in E$, we get that

$$\|D_\lambda \xi\|_E = \lim_{N \rightarrow \infty} \left\| \sum_{k=1}^N \lambda_k \xi_k e_k \right\|_E.$$

Thus the desired estimate now follows from the above inequality and Theorem 3.2. $\square$

5 The diagonal absolutely φ -summing operators

In what follows we will apply properties of Rademacher series in symmetric spaces. Recall that the sequence $(r_k)_{k \in \mathbb{N}}$ of the Rademacher functions on $[0, 1]$ is given by $r_k(t) := \text{sign}(\sin(2^k \pi t))$ for all $t \in [0, 1]$ and for each $k \in \mathbb{N}$. It is well known (see [23, Theorem 8.2, p. 212]) that, for every $(a_k) \in \ell_2$, the series $\sum_{k=1}^\infty a_k r_k$ converges for almost all $t \in [0, 1]$. For every symmetric space X on $[0, 1]$, we assign the sequence space R_X of Rademacher coefficients $(a_k) \in \ell_2$ of all functions $f \in X$ such that, for a.e. $t \in [0, 1]$, we have $f(t) = \sum_{k=1}^\infty a_k r_k(t)$. The space R_X is a Banach symmetric sequence space equipped with norm (see [18])

$$\|(a_k)\|_{R_X} := \left\| \sum_{k=1}^\infty a_k r_k \right\|_X.$$

In the case $X := L_\varphi$ is an Orlicz space on $[0, 1]$, the space R_{L_φ} is denoted by R_φ for short.

Lemma 5.1 *Let φ be Orlicz function that is a supermultiplicative at ∞ and let $T : X \rightarrow Y$ be absolutely φ -summing operator between Banach spaces. Suppose that Y has cotype $r \in [2, \infty)$. Then, for any finite collection $x_1, \dots, x_n \in X$,*

$$\left(\sum_{k=1}^n \|Tx_k\|_Y^r \right)^{1/r} \leq c C_r(Y) \pi_\varphi(T) \sup_{\|x^*\|_{X^*} \leq 1} \|(\langle x_1, x^* \rangle, \dots, \langle x_n, x^* \rangle)\|_{R_\varphi^n},$$

where the constant $c > 0$ depends on φ only.

Proof Without loss of generality we may assume that φ is a normalized Orlicz function. We consider $[0, 1]$ with Lebesgue measure ν . Fix any finite collection $x_1, \dots, x_n \in X$. Let $f \in L^0([0, 1], Y)$ be given by $f(t) := \sum_{k=1}^n r_k(t)x_k$ for all $t \in [0, 1]$. Our hypothesis combined with Theorem 3.8 applied for f yields that there exists $c > 0$ that depends on φ such that for $C := c C_r(Y) \pi_\varphi(T)$, we have

$$\begin{aligned} \left(\sum_{k=1}^n \|Tx_k\|_Y^r \right)^{1/r} &\leq C_r(Y) \int_0^1 \left\| \sum_{k=1}^n r_k(t)Tx_k \right\|_Y dt \leq C_r(Y) \left\| T \left(\sum_{k=1}^n r_k x_k \right) \right\|_{L_\varphi(\nu, Y)} \\ &\leq c C_r(Y) \pi_\varphi(T) \sup_{\|x^*\|_{X^*} \leq 1} \left\| \left\langle \sum_{k=1}^n r_k x_k, x^* \right\rangle \right\|_{L_\varphi(\nu)} \\ &= C \sup_{\|x^*\|_{X^*} \leq 1} \left\| \sum_{k=1}^n r_k \langle x_k, x^* \rangle \right\|_{L_\varphi(\nu)} \\ &= C \sup_{\|x^*\|_{X^*} \leq 1} \|(\langle x_1, x^* \rangle, \dots, \langle x_n, x^* \rangle)\|_{R_\varphi^n}. \end{aligned}$$

This completes the proof. $\square$

Corollary 5.2 *Let φ be an Orlicz function that is supermultiplicative at ∞ . Suppose $D_\lambda : E \rightarrow F$ is a diagonal operator between Banach sequence lattices, where E is order continuous and F has cotype $r \in [2, \infty)$. If $b := \inf_{n \geq 1} \|e_n\|_F > 0$ and $D_\lambda : E \rightarrow F$ is absolutely φ -summing, then $\lambda \in X := M(M(E', R_\varphi), \ell_r)$ and*

$$\|\lambda\|_X \leq C \pi_\varphi(D_\lambda),$$

where $C > 0$ depends on φ and F only.

Proof Fix $\xi = (\xi_k) \in M(E', R_\varphi)$. Since E is order continuous, E^* is isometrically identified in a natural way with the Köthe dual space E' of E . Thus, Lemma 5.1 applied for $x_1 = \xi_1 e_1, \dots, x_n = \xi_n e_n$ yields

$$b \left(\sum_{k=1}^n |\lambda_k \xi_k|^r \right)^{1/r} \leq \left(\sum_{k=1}^n \|Dx_k\|_F^r \right)^{1/r} \leq c C_r(F) \pi_\varphi(T) \sup_{\|(\eta_k)\|_{E'} \leq 1} \|(\eta_k \xi_k)\|_{R_\varphi}.$$

Since $\xi = (\xi_k) \in M(E', R_\varphi)$ was arbitrary, the required statement follows. $\square$

Similarly we prove the following lemma.

Lemma 5.3 *Let φ be an Orlicz function that is supermultiplicative at ∞ . Suppose $D_\lambda: E \rightarrow F$ is a diagonal operator between Banach sequence lattices, where E is order continuous. If $D_\lambda: E \rightarrow F$ is absolutely φ -summing, then $\lambda \in X := M(M(E', R_\varphi), F)$ and*

$$\|\lambda\|_X \leq C\pi_\varphi(D_\lambda),$$

where $C > 0$ depends on φ and F only.

Proof Without loss of generality we assume that φ is a normalized Orlicz function. We consider $[0, 1]$ with Lebesgue measure ν . Fix $\xi = (\xi_k) \in M(E', R_\varphi)$. Let $f_\xi \in L^0([0, 1], R_\varphi)$ be given by $f_\xi(t) := (r_k(t)\xi_k)$ for all $t \in [0, 1]$. Our hypothesis, combined with Theorem 3.8, yields

$$\|\vec{D}_\lambda(f_\xi)\|_{L_\varphi(\nu, F)} \leq c\pi_\varphi(D_\lambda) \sup_{x^* \in B_{E^*}} \|\langle f_\xi, x^* \rangle\|_{L_\varphi(\nu)},$$

where c depends on φ . Since E is order continuous, we get that

$$\begin{aligned} \|(\lambda_k \xi_k)\|_F &\leq C\pi_\varphi(D_\lambda) \sup_{(\eta_k) \in B_{E'}} \left\| \sum_{k=1}^{\infty} \xi_k \eta_k r_k(t) \right\|_{L_\varphi(\nu)} \\ &\leq C\pi_\varphi(D_\lambda) \sup_{(\eta_k) \in B_{E'}} \|(\xi_k \eta_k)\|_{R_\varphi} \\ &= C\pi_\varphi(D_\lambda) \|\xi\|_{M(E', R_\varphi)}. \end{aligned}$$

Since $\xi = (\xi_k) \in M(E', R_\varphi)$ was arbitrary, the required statement follows. $\square$

As an application, we obtain the following corollary. In the proof we use the well known result due to Rodin and Semyonov [18], which states that if X is a symmetric space on $[0, 1]$, then sequence the Rademacher (r_k) is equivalent to the unit vector basis (e_k) in ℓ_2 if and only if $G \hookrightarrow X$, where G denotes the closure of the simple functions in the Orlicz space corresponding to the Orlicz function $\phi(t) = \exp(t^2) - 1$ for all $t \geq 0$.

Corollary 5.4 *Let φ be an Orlicz function that is supermultiplicative at ∞ and such that $\varphi(t) < \exp(t^2) - 1$ for large enough t and let E be an order continuous Banach sequence lattice with the Fatou property. Suppose that the diagonal operator $D_\lambda: E \rightarrow E$ is absolutely φ -summing. Then $\lambda \in X := M(M(\ell_2, E), E)$ and*

$$\|\lambda\|_X \leq C\pi_\varphi(D_\lambda),$$

where $C > 0$ depends on φ . In particular if E is 2-concave, then $\lambda \in \ell_2$ and $\|\lambda\|_{\ell_2} \leq C\pi_\varphi(D_\lambda)$, where $C > 0$ depends on φ and 2-concavity constant of E .

Proof We will use the above-mentioned result of Rodin and Semenov in an equivalent form: $R_X \cong \ell_2$. Since our hypothesis on φ implies $G \hookrightarrow L_\varphi(v)$, it follows that $R_\varphi = \ell_2$.

It is easily to verified that for any Banach sequence lattice with the Fatou property we have $M(E', \ell_2) \cong M(\ell_2, E)$. Combining this with Lemma 5.2 yields $\lambda \in X := M(M(\ell_2, E), E)$ and

$$\|\lambda\|_X \leq C\pi_\varphi(D_\lambda),$$

where $C > 0$ depends on φ .

Now suppose that E is 2-concave. Clearly, E cannot contain a copy of c_0 and so E is order continuous. To finish we apply the formula (see [21, Theorem 4.1 (ii)])

$$M(M(\ell_2, E), E) = \ell_2$$

(see [21, Theorem 4.1 (ii)]), where the constants of equivalence of norms depend on the 2-concavity constant of E . This completes the proof. $\square$

We will need some relationships between symmetric Lorentz and Marcinkiewicz sequence spaces. Let $w = (w_n)$ be a nonincreasing positive sequence. The Lorentz sequence symmetric space $d(w, 1)$ consists of all sequences $\xi \in \omega(\mathbb{N})$ equipped with the norm,

$$\|\xi\|_{d(w,1)} := \sum_{k=1}^{\infty} \xi_k^* w_k.$$

If $1 < p < \infty$, then the p -convexification of $d(w, 1)$ is denoted by $d(w, p)$.

It is well known that the dual $d(w, 1)$ can be naturally identified as the Köthe dual $d(w, 1)' \cong m_w$, where m_w is the Marcinkiewicz symmetric sequence space consisting of all sequences $(\eta_k) \in \omega(\mathbb{N})$ equipped with the norm

$$\|\eta\|_{m_w} := \sup_{n \in \mathbb{N}} \frac{\eta_1^* + \cdots + \eta_n^*}{w_1 + \cdots + w_n}.$$

In what follows we say that a positive nondecreasing sequence $v = (v_n)_{n=1}^{\infty}$ is a concave sequence if $(v_n - v_{n-1})_{n=1}^{\infty}$ with $v_0 := 0$ is a nonincreasing sequence. Clearly, if $\psi: [0, \infty) \rightarrow [0, \infty)$ is a concave function with $\psi(0) = 0$, then $v := (\psi(n) - \psi(n-1))_{n=1}^{\infty}$ is a concave sequence. We point out that in this case, we have

$$\sup_{n \geq 1} \frac{n}{\psi(n)} \xi_n^* \leq \|\xi\|_{m_v} \leq C \sup_{n \geq 1} \frac{n}{\psi(n)} \xi_n^*, \quad \xi \in m_v$$

for some constant $C > 1$ if and only if $\liminf_{n \rightarrow \infty} \frac{\psi(2n)}{\psi(n)} > 1$. We also note that in the case when $p \in (1, \infty)$ and $\psi(n) = n^{1-1/p}$ for each $n \in \mathbb{N}$, the space m_v

coincides with the classical Marcinkiewicz space $\ell_{p,\infty}$ and, in the above estimate, $c = p/(p-1)$.

To prove the main result of this section, we will need to describe the space $R_\varphi = R_{L_\varphi}$ of Rademacher coefficients of Rademacher series generated by Orlicz spaces L_φ on $[0, 1]$ with $\varphi(t) = e^{\phi(t)} - 1$ for all $t \geq 0$, where ϕ belongs to some class of Orlicz functions. Our approach is based upon interpolation theory combined with a remarkable result from Astashkin's paper [3]. We need to recall some key definitions. Let E be a Banach sequence lattice of two-sided sequences such that $(\min\{1, 2^k\})_{k \in \mathbb{Z}} \in E$. If (X_0, X_1) is a Banach couple, then the space $(X_0, X_1)_E$ of the real K -method of interpolation is a Banach space of all $x \in X_0 + X_1$ equipped with the norm

$$\|x\| := \|(K(2^k, x; X_0, X_1))_k\|_E,$$

where K is the Peetre functional given for any $x \in X_0 + X_1$ and all $t > 0$ by

$$K(t, x; X_0, X_1) := \inf \{ \|x_0\|_{X_0} + t\|x_1\|_{X_1}; x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1 \}.$$

In the special case when $\psi: (0, \infty) \rightarrow (0, \infty)$ is a quasi-concave function (i.e., $\psi(s) \leq \max\{1, s/t\}\psi(t)$ for all $s, t > 0$) and $E := \ell_\infty(1/\psi(2^n))$, then the space $(X_0, X_1)_E$ is denoted by $(X_0, X_1)_{\psi, \infty}$. We note that for the couple (L_1, L_∞) on $[0, 1]$ equipped with Lebesgue measure, we have

$$(L_1, L_\infty)_{\psi, \infty} = M(\psi)$$

with universal constants of equivalence of norms, where $M(\psi)$ is the Marcinkiewicz space of all $f \in L^1$ equipped with the norm

$$\|f\|_{M(\psi)} := \sup_{0 < t < 1} \frac{1}{\psi(t)} \int_0^t f^*(s) ds.$$

We will use a remarkable theorem due to Astashkin [3, Theorem 1.4] which states: If F is an interpolation space between ℓ_1 and ℓ_2 given by $F = (\ell_1, \ell_2)_E$, then for the symmetric space $X = (L_\infty, G)_E$ on $[0, 1]$, we have $R_X = F$.

Before we state the next result we recall that a positive sequence $(a_n)_{n=1}^\infty$ is said to be almost decreasing if there exists $C > 0$ such that $a_{n+1} \leq Ca_n$ for each $n \in \mathbb{N}$. Similarly a positive function defined on an interval $I \subset (0, \infty)$ is called almost decreasing if there is $C > 0$ such that $f(t) \leq Cf(s)$ for all $s, t \in I$ with $s < t$.

Theorem 5.5 *Let ϕ be an Orlicz function and let L_φ be an Orlicz space on $[0, 1]$ with $\varphi(t) = e^{\phi(t)} - 1$ for all $t \geq 0$. Suppose that $(\phi^{-1}(2^{2k})/2^k)_{k=1}^\infty$ is an almost decreasing sequence such that $(\sum_{j=k}^\infty (\frac{\phi^{-1}(2^{2j})}{2^j})^2)^{1/2} \prec \frac{\phi^{-1}(2^{2k})}{2^k}$. Then*

$$R_{L_\varphi} = (\ell_1, \ell_2)_F = m_w,$$

where $F = \ell_\infty(\omega_k)$ with $\omega_k = 1$ for each $k < 0$, $\omega_k = \phi(2^{2k})$ for each $k \geq 0$ and $w := (\phi^{-1}(k) - \phi^{-1}(k-1))_{k=1}^\infty$.

Proof We put $\rho(t) := \phi^{-1}(t^2)$ for all $t \geq 0$. We first observe that for every $\xi \in (\ell_1, \ell_2)_F$, we have

$$\|\xi\|_{(\ell_1, \ell_2)_F} = \sup_{k \geq 0} \frac{K(2^k, \xi; \ell_1, \ell_2)}{\rho(2^k)}.$$

We claim that $m_w \hookrightarrow (\ell_1, \ell_2)_F$, where $w = (\phi^{-1}(k) - \phi^{-1}(k-1))_{k=1}^\infty$. We will apply the Holmstedt formula [7] which states that, for all $\xi \in \ell_2$ and all $t > 0$,

$$C \left(\sum_{i=1}^{[t^2]} \xi_i^* + t \left(\sum_{i=[t^2]+1}^\infty (\xi_i^*)^2 \right)^{1/2} \right) \leq K(t, \xi; \ell_1, \ell_2) \leq \sum_{i=1}^{[t^2]} \xi_i^* + t \left(\sum_{i=[t^2]+1}^\infty (\xi_i^*)^2 \right)^{1/2},$$

where $C > 0$ is an absolute constant. Fix $\xi \in m_w$ with $\|\xi\|_{m_w} = 1$. Clearly, for each $n \in \mathbb{N}$, we have

$$\xi_n^* \leq \frac{1}{n} \sum_{i=1}^n \xi_i^* \leq \frac{\phi^{-1}(n)}{n}.$$

Hence, our hypothesis on ϕ implies that for all $k \geq 0$,

$$\begin{aligned} \sum_{i=2^{2k}+1}^\infty (\xi_i^*)^2 &= \sum_{j=k}^\infty \sum_{i=2^{2j}+1}^{2^{2(j+1)}} (\xi_i^*)^2 \leq 3 \sum_{j=k}^\infty 2^{2j} (\xi_{2^{2j}}^*)^2 \\ &\leq 3 \sum_{j=k}^\infty \left(\frac{\phi^{-1}(2^{2j})}{2^j} \right)^2 \leq 3c \left(\frac{\phi^{-1}(2^{2k})}{2^k} \right)^2, \end{aligned}$$

where $c > 1$ is a constant that depends on ϕ . Combining the above estimates with Holmstedt's formula yields

$$K(2^k, \xi; \ell_1, \ell_2) \leq (1 + \sqrt{3c}) \phi^{-1}(2^{2k}) = (1 + \sqrt{3c}) \rho(2^k), \quad k \geq 0.$$

This completes the proof of the claim.

We show the opposite inclusion, $(\ell_1, \ell_2)_F \hookrightarrow m_w$. Let $x \in (\ell_1, \ell_2)_{\rho, \infty}$. For a given $k \in \mathbb{N}$, we choose an integer $j \geq 0$ such that $2^{2j} + 1 \leq k < 2^{2(j+1)}$. Then by Holmstedt's formula, we obtain

$$\begin{aligned} \sum_{i=1}^k \xi_i^* &\leq \sum_{i=1}^{2^{2(j+1)}} \xi_i^* \leq \frac{1}{C} K(2^{j+1}, \xi; \ell_1, \ell_2) \leq \frac{1}{C} \rho(2^{j+1}) \|\xi\|_F \\ &\leq \frac{1}{2C} \rho(2^j) \|\xi\|_F = \frac{1}{2C} \phi^{-1}(2^{2j}) \|\xi\|_F \\ &\leq \frac{1}{2C} \phi^{-1}(k) \|\xi\|_F. \end{aligned}$$

This gives the required continuous inclusion.

Now we prove that $(L_\infty, G)_{\rho, \infty} = L_\varphi$. To accomplish this we first note that it is easy to see that if X is a symmetric space on $[0, 1]$ such that $1/\psi_X \in X$, then $X = M(t/\psi_X)$. Simple calculations shows that for the fundamental function of $X := L_\varphi$, we have $1/\psi_X(t) = \phi^{-1}(\log(1 + \frac{1}{t}))$ for all $t \in (0, 1]$. Hence

$$L_\varphi([0, 1]) = M(\psi),$$

where $\psi(t) := t\phi^{-1}(\log(1 + \frac{1}{t}))$ for all $t > 0$.

We use a general reiteration formula for the generalized Marcinkiewicz spaces. For any Banach couple (X_0, X_1) and any quasi-concave functions f_0, f_1 and f , we have (see, e.g., [13])

$$((X_0, X_1)_{f_0, \infty}, (X_0, X_1)_{f_1, \infty})_F = (X_0, X_1)_{f_0 f(f_1/f_0), \infty}.$$

Since $(L_1, L_\infty)_{f_0, \infty} = L_\infty$ and $(L_1, L_\infty)_{f_1, \infty} = L_N$ with $f_0(t) = t$, $f_1(t) = t \log^{1/2}(1 + \frac{1}{t})$ and $N(t) = e^{t^2} - 1$ for all $t \geq 0$, the reiteration formula applied for $(X_0, X_1) = (L_1, L_\infty)$ and $f(t) := \max\{\rho(1), \rho(t)\}$ yields (by $(L_\infty, G)_F = (L_\infty, L_N)_F$)

$$(L_\infty, G)_F = (L_1, L_\infty)_{f_0 \tilde{\rho}(f_1/f_0), \infty} = M(\psi) = L_\varphi.$$

Now we are in a position to apply the above-mentioned result by Astashkin to get that $R_{L_\varphi} = (\ell_1, \ell_2)_F$. This completes the proof. $\square$

As an application, we obtain the following corollary. It provides a condition in terms of Matuszewska–Orlicz indices of ϕ that assures that the assumptions of Theorem 5.5 are satisfied. We recall that if $\psi: (0, \infty) \rightarrow (0, \infty)$ is such that a function $\mathcal{M}_\psi$ given by $\mathcal{M}_\psi(t) := \sup_{s>0} \frac{\psi(st)}{\psi(s)}$ for all $t > 0$ is finite measurable or nondecreasing, then the following indices exist (see [10, Theorem 1.3, p. 53]),

$$\gamma_\psi := \lim_{t \rightarrow 0+} \frac{\log \mathcal{M}_\psi(t)}{\log t}, \quad \delta_\psi := \lim_{t \rightarrow \infty} \frac{\log \mathcal{M}_\psi(t)}{\log t}.$$

Corollary 5.6 *Let ϕ be an Orlicz function such that $t \mapsto \phi(\sqrt{t})$ is equivalent to an Orlicz function on $[0, \infty)$ and that $2 < \gamma_\phi$. Then, for an Orlicz space L_φ on $[0, 1]$ with $\varphi(t) = e^{\phi(t)} - 1$ for all $t \geq 0$, we have*

$$R_\varphi = m_w,$$

where $w := (\phi^{-1}(k) - \phi^{-1}(k-1))_{k=1}^\infty$.

Proof Since $f(\cdot) := \phi(\sqrt{\cdot})$ is equivalent to an Orlicz function, $f^{-1} = (\phi^{-1})^2$ is equivalent to a concave function. Hence $(0, \infty) \ni t \mapsto f^{-1}(t)/t$ is an almost decreasing function. Let $\psi(t) := (\phi^{-1}(t^2)/t)^2$ for all $t > 0$. Then it is easily verified that

$$\gamma_\psi = 2(2\gamma_{\phi^{-1}} - 1) = \frac{4}{\delta_\phi} - 2, \quad \delta_\psi = \frac{4}{\gamma_\phi} - 2.$$

Since $1 \leq \delta_\phi \leq \infty$ (by convexity of ϕ), it follows that $-\infty < \gamma_\psi \leq \delta_\psi < 0$ is equivalent to $2 < \gamma_\phi$. Applying [10, Corollary, p. 57] yields that,

$$\int_t^\infty \frac{\psi(\tau)}{\tau} d\tau \asymp \psi(t), \quad t > 0.$$

Combining the above, we conclude that $(\phi^{-1}(2^{2k})/2^k)_{k=1}^\infty$ is almost decreasing sequence and

$$\begin{aligned} \sum_{j=k}^\infty \left(\frac{\phi^{-1}(2^{2j})}{2^j} \right)^2 &< \sum_{j=k}^\infty \int_{2^j}^{2^{j+1}} \frac{\psi(\tau)}{\tau} d\tau = \int_{2^k}^\infty \frac{\psi(\tau)}{\tau} d\tau \\ &< (\psi(2^k))^2 = \left(\frac{\phi^{-1}(2^{2k})}{2^k} \right)^2. \end{aligned}$$

This completes the proof by Theorem 5.5. $\square$

We remark that from Theorem 5.5 and Corollary 5.6 we recover known results. We mention here the case L_ϕ of an Orlicz function on $[0, 1]$ with $\varphi(t) = e^{\phi(t)} - 1$ for all $t \geq 0$, where ϕ is an Orlicz function given, respectively by $\phi(t) = t^q$ for $q \in (2, \infty)$, $\phi(t) = e^t - 1$ for $\alpha \in [1, \infty)$. Then $R_\phi = \ell_{p,\infty}$ with $1/p = 1 - 1/q$ and $R_\phi = m_w$ with $w = (\log(1+k) - \log k)_{k=1}^\infty$, respectively.

We prove auxiliary results regarding multipliers which we will need later in the proof of the main result of this section. The following lemma is surely well-known to specialists, but we include a proof.

Lemma 5.7 *Let E be a symmetric sequence space on $\mathbb{N}$ with fundamental function λ_E and let $w = (\lambda_E(k) - \lambda_E(k-1))_{k=1}^\infty$ with $\lambda_E(0) := 0$. Then the space of multipliers $M(d(w, 1), E) = F$, where F is a Banach symmetric sequence space of all $\xi = (\xi_k)_{k=1}^\infty$ equipped with the norm*

$$\|\xi\|_F := \sup_{n \in \mathbb{N}} \frac{\left\| \sum_{k=1}^n \xi_k^* e_k \right\|_E}{\lambda_E(n)}.$$

Proof We may assume that both $w \in c_0$ and $\sup_{k \geq 1} \lambda_E(k) = \infty$, otherwise the statement is obvious (by $d(w, 1) = \ell_1$ if $w \notin c_0$ and $d(w, 1) = \ell_\infty$). For simplicity of notation let $X := M(d(w, 1), E)$. Clearly, X is a symmetric sequence space. Observe that $\eta_n := \frac{1}{\lambda(n)} \sum_{k=1}^n e_k \in d(w, 1)$ with $\|\eta_n\|_{d(w, 1)} = 1$ for each $n \in \mathbb{N}$. Hence, for all $\xi \in X$, we have

$$\sup_{n \in \mathbb{N}} \frac{\left\| \sum_{k=1}^n \xi_k^* e_k \right\|_E}{\lambda_E(n)} = \sup_{n \in \mathbb{N}} \|\xi^* \eta_n\|_E \leq \|\xi\|_X$$

and so $X \hookrightarrow F$ with $\|\text{id}: X \rightarrow F\| \leq 1$.

Now we prove that $F \hookrightarrow X$ with $\|\text{id}: F \rightarrow X\| \leq 1$. To see this fix $\xi \in F$ and observe that, for every finite index set $I \subset \mathbb{N}$ and for each $k \in \mathbb{N}$, we have $(\xi^* \chi_I)_k^* \leq x_k^* \chi_{\{1, \dots, \text{card}(I)\}}(k)$. Hence

$$\|\xi^* \chi_I\|_E = \left\| \sum_{i \in I} \xi_i^* e_i \right\|_E \leq \left\| \sum_{k=1}^{\text{card}(I)} \xi_k^* e_k \right\|_E \leq \|\xi\|_{F \lambda_E(\text{card}(I))}.$$

By [10, Theorem 2.5.7], this implies that, for every finitely supported sequence $\eta = (\eta_k)$, we have

$$\|\xi^* \eta\|_E \leq 2 \|\xi\|_F \|\eta\|_{d(w, 1)}.$$

Since (e_n) form a basis in $d(w, 1)$, the space of finitely supported sequences is dense in $d(w, 1)$. In consequence, the above inequality holds for all $\eta \in d(w, 1)$ whence $\xi^* \in X$ with $\|\xi^*\|_X \leq 2 \|\xi\|_F$. This completes the proof by the obvious fact that F is a Banach symmetric sequence space. $\square$

Corollary 5.8 *Let $\psi: (0, \infty) \rightarrow (0, \infty)$ be a concave function such that $w_k := \psi(k) - \psi(k-1) \rightarrow 0$ as $k \rightarrow \infty$. Suppose ρ , given by $\rho(t) = \psi(t)^2$ for all $t \geq 0$, is equivalent to a quasi-concave function $\tilde{\rho}$. Then*

$$M(M(\ell_2, m_w), \ell_2) = d(v, 2),$$

where $w = (w_k)_{k=1}^\infty$ and $v = (\tilde{\rho}(k) - \tilde{\rho}(k-1))_{k=1}^\infty$.

Proof Since $M(\ell_2, m_w) \cong M((m_w)', \ell_2)$, it follows from the duality formula and Proposition 5.7 that

$$M(\ell_2, m_w) \cong M(d(w, 1), \ell_2) =: X,$$

where X is a symmetric space equipped with the norm

$$\|\xi\|_X := \sup_{n \geq 1} \frac{(\sum_{k=1}^n (\xi_k^*)^2)^{1/2}}{\psi(n)}.$$

This, and our hypothesis on ψ yields that $X = (m_v)^{(2)}$ coincides up to equivalence of norms with 2-convexification of the Marcinkiewicz space m_v , where $v = (\tilde{\rho}(k))_{k=1}^\infty$. Combining the above, we conclude by the duality formula and the obvious fact that $(m_v)^{(2)}$ has the Fatou property that

$$\begin{aligned} M(M(\ell_2, m_w), \ell_2) &= M((m_v)^{(2)}, \ell_2) \cong M(m_v, \ell_1)^{(2)} \\ &\cong ((m_v)')^{(2)} \cong d(v, 1)^{(2)} \cong d(v, 2). \end{aligned}$$

This completes the proof. $\square$

We recall that the n -th *approximation number* of an operator $T: X \rightarrow Y$ is defined by

$$a_n(T) := \inf \{ \|T - A\|_{X \rightarrow Y}; \operatorname{rank}(A) < n \}.$$

If E is a symmetric sequence space and H is a Hilbert space, then the Schatten-von-Neumann class $\mathcal{S}_E(H)$ consists of all operators $T: H \rightarrow H$ such that

$$(a_n(T))_{n=1}^\infty \in E.$$

As a final application of the results of this section we are ready to prove the following theorem, which gives an extension of a result in the case of the exponential Orlicz function $\varphi(t) = \exp(t^q) - 1$ for all $t \geq 0$ with $2 < q < \infty$ proved by Geiss (see [5, Theorem 4.2]).

Theorem 5.9 *Let ϕ an Orlicz function such that $t \mapsto \phi(\sqrt{t})$ is an Orlicz function and $\varphi(t) = e^{\phi(t)} - 1$ for all $t \geq 1$ is supermultiplicative at ∞ . Suppose that $2 < \gamma_\phi \leq \delta_\phi < \infty$. Then the following statements are equivalent with $w := (\tilde{\rho}(k) - \tilde{\rho}(k-1))_{k=1}^\infty$ generated by $\rho(t) := \phi^{-1}(t)^2$ for all $t \geq 0$, where $\tilde{\rho}$ is a concave majorant of ρ .*

- (i) *An operator $T: \ell_2 \rightarrow \ell_2$ is absolutely φ -summing;*
- (ii) *T belongs to the Schatten-von-Neumann class $\mathcal{S}_{d(w,2)}(\ell_2)$.*

Proof (i) $\Rightarrow$ (ii). From Theorem 3.4, it follows that T is a compact operator. Thus by the well known fact T has a Schmidt monotonic representation:

$$T\xi = \sum_{n=1}^{\infty} \lambda_n(\xi|u_n)v_n, \quad \xi \in \ell_2,$$

where (u_n) and (v_n) are orthonormal sequences in ℓ_2 and $\lambda = (\lambda_n) = (a_n(T))$. Then, the diagonalization procedure gives the following factorizations:

$$T: \ell_2 \xrightarrow{U} \ell_2 \xrightarrow{D_\lambda} \ell_2 \xrightarrow{V} \ell_2$$

and

$$D_\lambda: \ell_2 \xrightarrow{V^*} \ell_2 \xrightarrow{T} \ell_2 \xrightarrow{U^*} \ell_2,$$

where $U\xi := (\xi|u_n)_{k=1}^\infty$ and $V\xi := \sum_{k=1}^\infty \xi_k v_k$ for all $\xi = (\xi_k) \in \ell_2$. From the second factorization, we conclude that the diagonal operator D_λ is absolutely φ -summing. From Corollary 5.2, we get that

$$\lambda \in M(M(\ell_2, R_\varphi), \ell_2).$$

Observe that our hypothesis on indices of ϕ combined with obvious formulas $\gamma_\rho = 2/\delta_\phi$ and $\delta_\rho = 2/\gamma_\phi$ implies that $0 < \gamma_\rho \leq \delta_\rho < 1$. This implies that ρ is

equivalent to its concave majorant $\tilde{\rho}$ (see [10, Corollary 2, p. 55]). Now it follows from Corollaries 5.6 and 5.8 that $\lambda \in d(w, 2)$.

(ii) $\Rightarrow$ (i). It follows from Lemma 4.3 that

$$\gamma_n := \pi_\varphi(\text{id}: \ell_2^n \rightarrow \ell_2^n) < \phi^{-1}(n).$$

Assume that $\lambda := (a_n(T)) \in d(w, 2)$. Then Theorem 4.6 implies

$$\pi_\varphi(D_\lambda: \ell_2 \rightarrow \ell_2) \leq C \|\lambda\|_{d(w, 2)}$$

and whence D_λ is an absolutely φ -summing operator. Since D_λ is compact, $T = U D_\lambda V$ and so T is also absolutely φ -summing by the ideal property. This completes the proof. $\square$

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