Mathematische Annalen (2019) 375:307–313
https://doi.org/10.1007/s00208-019-01859-8

Mathematische Annalen

PUBLISHER CORRECTION

Correction to: Some results on the $p(u)$-Laplacian problem

M. Chipot[1] · **H. B. de Oliveira**[2,3]

Published online: 4 July 2019

Correction to: Mathematische Annalen https://doi.org/10.1007/s00208-019-01803-w

In the Original Publication of the article, few errors have been identified in section 5 and acknowledgements section. The corrected section 5 and acknowledgements are given below:

5 Nonlocal problems

In this section we consider a real function p such that

$$p \text{ is continuous}, \quad 1 < \alpha < p \leq \beta, \tag{5.1}$$

for some constants α, β. We denote by b a mapping from $W_0^{1,\alpha}(\Omega)$ into $\mathbb{R}$ such that

$$b \text{ is continuous}, \quad b \text{ is bounded}, \tag{5.2}$$

i.e. b sends bounded sets of $W_0^{1,\alpha}(\Omega)$ into bounded sets of $\mathbb{R}$.

Definition 2 A function u is a weak solution to the problem (1.3) if

$$\begin{cases} u \in W_0^{1,p(b(u))}(\Omega), \\ \displaystyle\int_\Omega |\nabla u|^{p(b(u))-2}\nabla u \cdot \nabla v \, dx = \langle f, v\rangle \quad \forall\, v \in W_0^{1,p(b(u))}(\Omega), \end{cases} \tag{5.3}$$

The original article can be found online at https://doi.org/10.1007/s00208-019-01803-w.

✉ H. B. de Oliveira
holivei@ualg.pt

M. Chipot
m.m.chipot@math.uzh.ch

1 IMath, Universität Zürich, Winterthurerstrasse 190, 8057 Zurich, Switzerland

2 FCT, Universidade do Algarve, Campus de Gambelas, 8005-139 Faro, Portugal

3 CMAFCIO, Universidade de Lisboa, Campo Grande, 1749-016 Lisbon, Portugal

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $(W_0^{1,p(b(u))}(\Omega))'$ and $W_0^{1,p(b(u))}(\Omega)$.

One should notice that $p(b(u))$ is here a real number and not a function so that the Sobolev spaces involved are the classical ones. We refer to [5, 7–9] for more on nonlocal problems.

Then one has:

Theorem 5.1 *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain and assume that (5.1) and (5.2) hold together with*

$$f \in W^{-1,\alpha'}(\Omega).$$

Then there exists at least one weak solution to the problem (1.3) in the sense of Definition 2.

The proof of Theorem 5.1 is based on the following result.

Lemma 5.1 *For $n \in \mathbb{N}$, let u_n be the solution to the problem*

$$\begin{cases} u_n \in W_0^{1,p_n}(\Omega), \\ \displaystyle\int_\Omega |\nabla u_n|^{p_n-2} \nabla u_n \cdot \nabla v \, dx = \langle f, v \rangle \quad \forall \quad v \in W_0^{1,p_n}(\Omega), \end{cases} \tag{5.4}$$

where $\langle \cdot, \cdot \rangle$ denotes here the duality pairing between $(W_0^{1,p_n}(\Omega))'$ and $W_0^{1,p_n}(\Omega)$. Suppose that

$$p_n \to p, \quad as \quad n \to \infty, \quad where \quad p \in (1, \infty), \tag{5.5}$$

$$f \in W^{-1,q'}(\Omega) \quad for\ some \quad q < p. \tag{5.6}$$

Then

$$u_n \to u \quad in \quad W_0^{1,q}(\Omega), \quad as \quad n \to \infty, \tag{5.7}$$

where u is the solution to the problem

$$\begin{cases} u \in W_0^{1,p}(\Omega), \\ \displaystyle\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = \langle f, v \rangle \quad \forall \, v \in W_0^{1,p}(\Omega). \end{cases} \tag{5.8}$$

Proof of Lemma 5.1 We shall split this proof into two steps.

1. Weak convergence: We first observe that, in view of $p_n \to p$, as $n \to \infty$, and $q < p$, we may assume that

$$p + 1 > p_n > q \quad \forall \quad n \in \mathbb{N}. \tag{5.9}$$

Taking $v = u_n$ in the equation of (5.4) we get

$$\int_\Omega |\nabla u_n|^{p_n} dx \leq \|f\|_{-1,q'} \|\nabla u_n\|_q. \tag{5.10}$$

Recall that $\|f\|_{-1,q'}$ denotes the strong dual norm of f associated to the norm $\|\nabla\cdot\|_q$. On the other hand, by using Hölder's inequality and (5.9), we have

$$\|\nabla u_n\|_q \le C\|\nabla u_n\|_{p_n}, \tag{5.11}$$

for some positive constant $C = C(p, q, \Omega)$. Plugging (5.11) into (5.10) it comes

$$\|\nabla u_n\|_{p_n} \le C, \tag{5.12}$$

for some other positive constant $C = C(p, q, \Omega, f)$. Combining (5.11) with (5.12), it follows that

$$\|\nabla u_n\|_q \le C, \tag{5.13}$$

for some positive constant C independent of n. From (5.13) we deduce then that for some subsequence still labelled by n and for some $u \in W_0^{1,q}(\Omega)$

$$\nabla u_n \rightharpoonup \nabla u \quad \text{in} \quad L^q(\Omega), \quad \text{as} \quad n \to \infty. \tag{5.14}$$

Due to (5.5), (5.9), (5.12) and (5.14), we can also apply Lemma 3.1 so that

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_n|^{p_n} dx \ge \int_\Omega |\nabla u|^p dx.$$

As a consequence we have

$$u \in W_0^{1,p}(\Omega). \tag{5.15}$$

Clearly the equation in (5.4) is equivalent to

$$\int_\Omega |\nabla u_n|^{p_n-2}\nabla u_n \cdot \nabla(v - u_n)\, dx \ge \langle f, v - u_n\rangle \quad \forall\, v \in W_0^{1,p_n}(\Omega).$$

and by the Minty lemma to

$$\int_\Omega |\nabla v|^{p_n-2}\nabla v \cdot \nabla(v - u_n)\, dx \ge \langle f, v - u_n\rangle \quad \forall\, v \in W_0^{1,p_n}(\Omega). \tag{5.16}$$

Taking $v \in C_0^\infty(\Omega)$, one can use (5.5) and (5.14) to pass to the limit in (5.16), as $n \to \infty$, so that

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla(v - u)\, dx \ge \langle f, v - u\rangle \quad \forall\, v \in C_0^\infty(\Omega). \tag{5.17}$$

Using the density of $C_0^\infty(\Omega)$ in $W_0^{1,p}(\Omega)$, we see that (5.17) also holds for all $v \in W_0^{1,p}(\Omega)$. In this case, taking $v = u \pm \delta z$, with $z \in W_0^{1,p}(\Omega)$ and $\delta > 0$, and letting $\delta \to 0$ after simplifying the resulting inequality, one obtains

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla z\, dx = \langle f, z\rangle \quad \forall\, z \in W_0^{1,p}(\Omega).$$

Thus u is the solution to the problem (5.8).

2. Strong convergence: We want to show that the convergence (5.14) is in fact strong. To prove this, we first note that, taking $v = u_n$ in the equation of (5.4) and using (5.14) to pass to the limit, we obtain

$$\int_\Omega |\nabla u_n|^{p_n} dx = \langle f, u_n \rangle \to \langle f, u \rangle = \int_\Omega |\nabla u|^p dx, \quad \text{as} \quad n \to \infty. \tag{5.18}$$

Consider the case of the p_n's such that

$$p_n \geq p \quad \forall \quad n \in \mathbb{N}.$$

One has by Hölder's inequality

$$\int_\Omega |\nabla u_n|^p dx \leq \left(\int_\Omega |\nabla u_n|^{p_n} dx \right)^{\frac{p}{p_n}} |\Omega|^{1-\frac{p}{p_n}},$$

where $|\Omega|$ denotes the d-Lebesgue measure of Ω. Thus by (5.18) for such a sequence

$$\limsup_{n\to\infty} \int_\Omega |\nabla u_n|^p dx \leq \int_\Omega |\nabla u|^p dx \leq \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^p dx,$$

which shows (since $\|\nabla u_n\|_p \to \|\nabla u\|_p$, as $n \to \infty$)

$$u_n \to u \quad \text{strongly in } W_0^{1,p}(\Omega), \quad \text{as} \quad n \to \infty. \tag{5.19}$$

Since $W_0^{1,p}(\Omega) \subset W_0^{1,q}(\Omega)$, (5.19) implies (5.7).

Next, consider the p_n's such that

$$q < p_n < p \quad \forall\, n \in \mathbb{N} \tag{5.20}$$

and set

$$A_n := \int_\Omega \left(|\nabla u_n|^{p_n-2} \nabla u_n - |\nabla u|^{p_n-2} \nabla u \right) \cdot (\nabla u_n - \nabla u)\; dx. \tag{5.21}$$

Due to the monotonicity, $A_n \geq 0$ and, because of (5.4), one has

$$A_n = \langle f, u_n - u \rangle - \int_\Omega |\nabla u|^{p_n-2} \nabla u \cdot \nabla(u_n - u)\, dx.$$

From (5.6) and (5.14), we have

$$\langle f, u_n - u \rangle \to 0, \quad \text{as } n \to \infty. \tag{5.22}$$

Moreover, from (5.15) one easily gets

$$\left| |\nabla u|^{p_n-2} \nabla u \right| \leq \max\{1, |\nabla u|\}^{p-1} \in L^{p'}(\Omega). \tag{5.23}$$

Hence, (5.20), (5.22) and (5.23) ensure that

$$A_n \to 0, \quad \text{as} \quad n \to \infty. \tag{5.24}$$

Assume first that

$$p_n \geq 2.$$

This allows us to use property (3.10) of Lemma 3.2 in (5.21) so that

$$A_n \geq \frac{1}{2^{p_n-1}} \int_\Omega |\nabla(u_n - u)|^{p_n}\, dx. \tag{5.25}$$

Since, by (5.20), $p_n > q$, we have by Hölder's inequality, (5.20), (5.24) and (5.25)

$$\int_\Omega |\nabla(u_n - u)|^q\, dx \leq \left(\int_\Omega |\nabla(u_n - u)|^{p_n}\, dx\right)^{\frac{q}{p_n}} |\Omega|^{1-\frac{q}{p_n}} \to 0,$$

when $n \to \infty$. This proves (5.7) in this case.

Consider now the case when

$$p_n < 2.$$

Here, we use Hölder's inequality as follows

$$\begin{aligned} &\int_\Omega |\nabla(u_n - u)|^{p_n}\, dx \\ &= \int_\Omega |\nabla(u_n - u)|^{p_n} (|\nabla u_n| + |\nabla u|)^{\frac{(p_n-2)p_n}{2}} (|\nabla u_n| + |\nabla u|)^{\frac{(2-p_n)p_n}{2}}\, dx \\ &\leq \left[\int_\Omega |\nabla(u_n - u)|^2 (|\nabla u_n|+|\nabla u|)^{p_n-2}\, dx\right]^{\frac{p_n}{2}} \left[\int_\Omega (|\nabla u_n|+|\nabla u|)^{p_n}\, dx\right]^{1-\frac{p_n}{2}}. \end{aligned} \tag{5.26}$$

Using property (3.11) of Lemma 3.2 we have

$$A_n \geq C \int_\Omega |\nabla(u_n - u)|^2 (|\nabla u_n| + |\nabla u|)^{p_n-2}\, dx, \tag{5.27}$$

for some positive constant $C = C(p_n)$. Now, by using (5.26), (5.27) together with (5.12) we deduce that

$$\int_\Omega |\nabla(u_n - u)|^{p_n}\, dx \to 0, \quad \text{as } n \to \infty.$$

Thus, as above, (5.7) holds true also in this case. □

Let us now show how Lemma 5.1 can be applied to prove the existence of weak solutions to the nonlocal problem (1.3).

Proof of Theorem 5.1 Note that $f \in (W_0^{1,\alpha}(\Omega))' \subset (W_0^{1,\delta}(\Omega))'$ for any $\delta > \alpha$. Thus for each $\lambda \in \mathbb{R}$, there exists a unique solution $u = u_\lambda$ to the $p(\lambda)$-Laplacian problem

$$\begin{cases} u \in W_0^{1,p(\lambda)}(\Omega), \\ \int_\Omega |\nabla u|^{p(\lambda)-2} \nabla u \cdot \nabla v \, dx = \langle f, v \rangle \quad \forall\, v \in W_0^{1,p(\lambda)}(\Omega). \end{cases} \tag{5.28}$$

Taking $v = u = u_\lambda$ in (5.28) one derives

$$\int_\Omega |\nabla u_\lambda|^{p(\lambda)} \, dx \leq \|f\|_{-1,\alpha'} \|\nabla u_\lambda\|_\alpha. \tag{5.29}$$

By Hölder's inequality one has

$$\|\nabla u_\lambda\|_\alpha \leq \|\nabla u_\lambda\|_{p(\lambda)} |\Omega|^{\frac{1}{\alpha} - \frac{1}{p(\lambda)}}. \tag{5.30}$$

Thus by (5.29) it comes

$$\|\nabla u_\lambda\|_{p(\lambda)}^{p(\lambda)-1} \leq \|f\|_{-1,\alpha'} |\Omega|^{\frac{1}{\alpha} - \frac{1}{p(\lambda)}}. \tag{5.31}$$

Gathering (5.30) and (5.31), and using (5.1) we obtain

$$\|\nabla u_\lambda\|_\alpha \leq \|f\|_{-1,\alpha'}^{\frac{1}{p(\lambda)-1}} |\Omega|^{\left(\frac{1}{\alpha} - \frac{1}{p(\lambda)}\right)\frac{p(\lambda)}{p(\lambda)-1}} \leq \max_{p \in [\alpha,\beta]} \|f\|_{-1,\alpha'}^{\frac{1}{p-1}} |\Omega|^{\left(\frac{1}{\alpha} - \frac{1}{p}\right)\frac{p}{p-1}} = C, \tag{5.32}$$

for some positive constant $C = C(\alpha, \beta, \Omega, f)$. Due to the boundedness of b, see (5.2), and to (5.32), there exists $L \in \mathbb{R}$ such that

$$b(u_\lambda) \in [-L, L] \quad \forall\, \lambda \in \mathbb{R}.$$

Let us now consider the map

$$\lambda \mapsto b(u_\lambda), \tag{5.33}$$

from $[-L, L]$ into itself. This map is continuous. Indeed, if $\lambda_n \to \lambda$ as $n \to \infty$, due to (5.1), we have $p(\lambda_n) \to p(\lambda)$. Applying now Lemma 5.1 with $p_n = p(\lambda_n)$, it follows that

$$u_{\lambda_n} \to u_\lambda \quad \text{in} \quad W_0^{1,\alpha}(\Omega), \quad \text{as} \quad n \to \infty.$$

Now, b being continuous (see (5.2)), it follows that $b(u_{\lambda_n}) \longrightarrow b(u_\lambda)$, as $n \to \infty$, and thus the map (5.33) is also continuous. It has then a fixed point λ_0 and u_{λ_0} is then solution to (5.3). □

The original article has been corrected.

Acknowledgements We are very grateful to the referees for their constructive remarks. This work was performed when the first author was visiting the USTC in Hefei and during a part time employment at the S. M. Nikolskii Mathematical Institute of RUDN University, 6 Miklukho-Maklay St, Moscow, 117198, supported by the Ministry of Education and Science of the Russian Federation. He is grateful to these institutions for their support. Main part of this work was carried out also during the visit of the second author to the University of Zurich during the first quarter of 2018. Besides the Grant SFRH/BSAB/135242/2017 of the Portuguese Foundation for Science and Technology (FCT), Portugal, which made this visit possible, the second author also wishes to thank to Prof. Michel Chipot who kindly welcomed him at the University of Zurich.