On q -de Rham cohomology via Λ -rings

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Received: 14 August 2017 / Revised: 11 January 2019 / Published online: 5 February 2019
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Abstract

We show that Aomoto's q -deformation of de Rham cohomology arises as a natural cohomology theory for Λ -rings. Moreover, Scholze's $(q - 1)$ -adic completion of q -de Rham cohomology depends only on the Adams operations at each residue characteristic. This gives a fully functorial cohomology theory, including a lift of the Cartier isomorphism, for smooth formal schemes in mixed characteristic equipped with a suitable lift of Frobenius. If we attach p -power roots of q , the resulting theory is independent even of these lifts of Frobenius, refining a comparison by Bhatt, Morrow and Scholze.

Introduction

The q -de Rham cohomology of a polynomial ring is a $\mathbb{Z}[q]$ -linear complex given by replacing the usual derivative with the Jackson q -derivative $\nabla_q(x^n) = [n]_q x^{n-1} dx$, where $[n]_q$ is Gauss' q -analogue $\frac{q^n - 1}{q - 1}$ of the integer n . In [13], Scholze discussed the $(q - 1)$ -adic completion of this theory for smooth rings, explaining relations to p -adic Hodge theory and singular cohomology, and conjecturing that it is independent of co-ordinates, so functorial for smooth algebras over a fixed base [13, Conjectures 1.1, 3.1 and 7.1].

We show that q -de Rham cohomology with q -connections naturally arises as a functorial invariant of Λ -rings (Theorems 1.17, 1.23 and Proposition 1.25), and that its $(q - 1)$ -adic completion depends only on a Λ_p -ring structure (Theorem 2.8), for P the set of residue characteristics; a Λ_p -ring has a lift of Frobenius for each $p \in P$. This recovers the known equivalence between de Rham cohomology and complete q -de Rham cohomology over the rationals, while giving no really new functoriality statements for smooth schemes over $\mathbb{Z}$. However, in mixed characteristic, it means that

Communicated by Toby Gee.

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complete q -de Rham cohomology depends only on a lift Ψ^p of absolute Frobenius locally generated by co-ordinates with $\Psi^p(x_i) = x_i^p$. Given such data, we construct (Proposition 2.10) a quasi-isomorphism between Hodge cohomology and q -de Rham cohomology modulo $[p]_q$, extending the local lift of the Cartier isomorphism in [13, Proposition 3.4].

Taking the Frobenius stabilisation of the complete q -de Rham complex of A yields a complex resembling the de Rham–Witt complex. We show (Theorem 3.11) that up to $(q^{1/p^\infty} - 1)$ -torsion, the p -adic completion of this complex depends only on the p -adic completion of $A[\zeta_{p^\infty}]$ (where ζ_n denotes a primitive n th root of unity), with no requirement for a lift of Frobenius or a choice of co-ordinates. The main idea is to show that the stabilised q -de Rham complex is in a sense given by applying Fontaine’s period ring construction A_{inf} to the best possible perfectoid approximation to $A[\zeta_{p^\infty}]$. As a consequence, this shows (Corollary 3.13) that after attaching all p -power roots of q , q -de Rham cohomology in mixed characteristic is independent of choices, which was already known after base change to a period ring, via the comparisons of [4] between q -de Rham cohomology and their theory $A\Omega$.

The cohomology theories we construct thus depend either on Adams operations at the residue characteristics (for de Rham) or on p -power roots of q (for variants of de Rham–Witt), establishing correspondingly weakened versions of the conjectures of [13]; in Remark 3.15, we suggest a possible candidate for a theory without those restrictions. The essence of our construction of q -de Rham cohomology of A over R is to set q to be an element of rank 1 for the Λ -ring structure, and to look at flat Λ -rings B over $R[q]$ equipped with morphisms $A \rightarrow B/(q - 1)$ of Λ -rings over R . If these seem unfamiliar, reassurance should be provided by the observation that $(q - 1)B$ carries q -analogues of divided power operations (Remark 1.4). For the variants of de Rham–Witt cohomology in Sect. 3, the key to giving a characterisation independent of lifts of Frobenius is the factorisation of the tilting equivalence for perfectoid algebras via a category of Λ_p -rings, leading to constructions similar to [4].

I would like to thank Peter Scholze for many helpful comments, in particular about the possibility of a q -analogue of de Rham–Witt cohomology, and Michel Gros for spotting a missing hypothesis. I would also like to thank the anonymous referee for suggesting many improvements.

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1 Comparisons for Λ -rings

We will follow standard notational conventions for Λ -rings. These are commutative rings equipped with operations λ^i resembling alternating powers, in particular satisfying $\lambda^k(a+b) = \sum_{i=0}^k \lambda^i(a)\lambda^{k-i}(b)$, with $\lambda^0(a) = 1$ and $\lambda^1(a) = a$. For background, see [5] and references therein. The Λ -rings we encounter are all torsion-free, in which case [16] shows the Λ -ring structure is equivalent to giving ring endomorphisms Ψ^n for $n \in \mathbb{Z}_{>0}$ with $\Psi^{mn} = \Psi^m \circ \Psi^n$ and $\Psi^p(x) \equiv x^p \pmod{p}$ for all primes p . If we write $\lambda_t(f) := \sum_{i \geq 0} \lambda^i(f)t^i$ and $\Psi_t(f) := \sum_{n \geq 1} \Psi^n(f)t^n$, then the families of operations are related by the formula $\Psi_t = -t \frac{d \log \lambda_{-t}}{dt}$.

We refer to elements x with $\lambda^i(x) = 0$ for all $i > 1$ (or equivalently $\Psi^n(x) = x^n$ for all n) as elements of rank 1.

1.1 The Λ -ring $\mathbb{Z}[q]$

Definition 1.1 Define $\mathbb{Z}[q]$ to be the Λ -ring with operations determined by setting q to be of rank 1.

We now consider the q -analogues $[n]_q := \frac{q^n - 1}{q - 1} \in \mathbb{Z}[q]$ of the integers, with $[n]_q! = [n]_q [n-1]_q \dots [1]_q$, and $\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!}$.

Remark 1.2 To see the importance of regarding $\mathbb{Z}[q]$ as a Λ -ring observe that the binomial expressions

$$\lambda^k(n) = \binom{n}{k}, \quad \lambda^k(-n) = (-1)^k \binom{n+k-1}{k}$$

have as q -analogues the Gaussian binomial theorems

$$\lambda^k([n]_q) = q^{k(k-1)/2} \binom{n}{k}_q, \quad \lambda^k(-[n]_q) = (-1)^k \binom{n+k-1}{k}_q,$$

as well as Adams operations

$$\Psi^i([n]_q) = [n]_{q^i}.$$

For any torsion-free Λ -ring, localisation at a set of elements closed under the Adams operations always yields another Λ -ring, since $\Psi^p(a^{-1}) - a^{-p} = (\Psi^p(a)a^p)^{-1}(a^p - \Psi^p(a))$ is divisible by p .

Lemma 1.3 For the Λ -ring structure on $\mathbb{Z}[x, y]$ with x, y of rank 1, the elements

$$\lambda^n \left(\frac{y-x}{q-1} \right) \in \mathbb{Z}[q, \{(q^n - 1)^{-1}\}_{n \geq 1}, x, y]$$

are given by

$$\begin{aligned}\lambda^k\left(\frac{y-x}{q-1}\right) &= \frac{(y-x)(y-qx)\cdots(y-q^{k-1}x)}{(q-1)^k[k]_q!}, \\ &= \sum_{j=0}^k \frac{q^{j(j-1)/2}(-x)^j y^{k-j}}{[j]_q![k-j]_q!}.\end{aligned}$$

Proof The second expression comes from multiplying out the Gaussian binomial expansions. The easiest way to prove the first is to observe that $\lambda^k(\frac{y-x}{q-1})$ must be a homogeneous polynomial of degree k in x, y , with coefficients in the integral domain $\mathbb{Z}[q, \{(q^n - 1)^{-1}\}_{n \geq 1}]$, and to note that

$$\lambda^k\left(\frac{q^n x - x}{q-1}\right) = \lambda^k([n]_q x) = q^{k(k-1)/2} \binom{n}{k}_q x^k.$$

Thus $\lambda^k(\frac{y-x}{q-1})$ agrees with the homogeneous polynomial above for infinitely many values of $\frac{y}{x}$, so must be equal to it. $\square$

Remark 1.4 Note that as $q \rightarrow 1$, Lemma 1.3 gives $(q-1)^k \lambda^k(\frac{y-x}{q-1}) \rightarrow \frac{(x-y)^k}{k!}$. Indeed, for any rank 1 element x in a Λ -ring we have

$$\lambda_{(q-1)t}\left(\frac{x}{q-1}\right) = \sum_{k \geq 0} \frac{(xt)^k}{[k]_q!},$$

which is just the q -exponential $e_q(xt)$. Multiplicativity and universality then imply that $\lambda_{(q-1)t}(\frac{a}{q-1})$ is a q -deformation of $\exp(at)$ for all a . Thus $(q-1)^k \lambda^k(\frac{a}{q-1})$ is a q -analogue of the k th divided power $(a^k/k!)$. An explicit expression comes recursively from the formula

$$[k]_q(q-1)\lambda^k\left(\frac{a}{q-1}\right) = \sum_{i > 0} \lambda^i(a)\lambda^{k-i}\left(\frac{a}{q-1}\right),$$

obtained by subtracting $\lambda_t(\frac{a}{q-1})$ from each side of the expression $\lambda_{qt}(\frac{a}{q-1}) = \lambda_t(a)\lambda_t(\frac{a}{q-1})$, which arises because q is of rank 1 and $\frac{qa}{q-1} = a + \frac{a}{q-1}$.

Lemma 1.5 For elements x, y of rank 1, the Λ -subring of $\mathbb{Z}[q, \{(q^n - 1)^{-1}\}_{n \geq 1}, x, y]$ generated by $q, x, y, \frac{y-x}{q-1}$ has basis $\lambda^k(\frac{y-x}{q-1})$ as a $\mathbb{Z}[q, x]$ -module.

Proof The Λ -subring clearly contains the $\mathbb{Z}[q, x]$ -module M generated by the elements $\lambda^k(\frac{y-x}{q-1})$, which are also clearly $\mathbb{Z}[q, x]$ -linearly independent. Since $\mathbb{Z}[q, x]$ is a Λ -ring, it suffices to show that M is closed under multiplication.

By Lemma 1.3, we know that

$$\lambda^i\left(\frac{y-x}{q-1}\right)\lambda^j\left(\frac{y-q^i x}{q-1}\right) = \binom{i+j}{i}_q \lambda^{i+j}\left(\frac{y-x}{q-1}\right).$$

We can rewrite $\frac{y-q^i x}{q-1} = \frac{y-x}{q-1} - [i]_q x$, so $\lambda^j(\frac{y-q^i x}{q-1}) - \lambda^j(\frac{y-x}{q-1})$ lies in the $\mathbb{Z}[q, x]$ -module spanned by $\lambda^m(\frac{y-x}{q-1})$ for $m < j$. By induction on j , it thus follows that

$$\lambda^i\left(\frac{y-x}{q-1}\right)\lambda^j\left(\frac{y-q^i x}{q-1}\right) - \lambda^i\left(\frac{y-x}{q-1}\right)\lambda^j\left(\frac{y-x}{q-1}\right) \in M,$$

so the binomial expression above implies $\lambda^i(\frac{y-x}{q-1})\lambda^j(\frac{y-x}{q-1}) \in M$. $\square$

1.2 q -cohomology of Λ -rings

Definition 1.6 Given a Λ -ring R , say that A is a Λ -ring over R if it is a Λ -ring equipped with a morphism $R \rightarrow A$ of Λ -rings. We say that A is a flat Λ -ring over R if A is flat as a module over the commutative ring underlying R .

Definition 1.7 Given a morphism $R \rightarrow A$ of Λ -rings, we define the category $\text{Strat}_{A/R}^q$ to consist of flat Λ -rings B over $R[q]$ equipped with a compatible morphism $f : A \rightarrow B/(q-1)$, such that f admits a lift to B ; a choice of lift is not taken to be part of the data, so need not be preserved by morphisms.

More concisely, $\text{Strat}_{A/R}^q$ is the Grothendieck construction of the set-valued functor

$$(\text{Spec } A)_{\text{strat}}^q : B \mapsto \text{Im}(\text{Hom}_{\Lambda, R}(A, B) \rightarrow \text{Hom}_{\Lambda, R}(A, B/(q-1)))$$

on the category $f\Lambda(R[q])$ of flat Λ -rings over $R[q]$.

Definition 1.8 Given a flat morphism $R \rightarrow A$ of Λ -rings, define $\text{qDR}(A/R)$ to be the cochain complex of $R[q]$ -modules given by taking the homotopy limit (in the sense of [6]) of the functor

$$\begin{aligned} \text{Strat}_{A/R}^q &\rightarrow \text{Ch}(R[q]) \\ B &\mapsto B. \end{aligned}$$

The cochain complex $\text{qDR}(A/R)$ naturally carries $(R[q], \Psi^n)$ -semilinear operations Ψ^n coming from the morphisms $\Psi^n : B \otimes_{R[q], \Psi^n} R[q] \rightarrow B$ of $R[q]$ -modules, for $B \in \text{Strat}_{A/R}^q$.

Equivalently, can we follow the approach of [8, 14] towards the stratified site and de Rham stack by regarding $\text{qDR}(A/R)$ as the quasi-coherent cohomology complex of $(\text{Spec } A)_{\text{strat}}^q$, as follows.

Definition 1.9 Given a category $\mathcal{C}$, write $[\mathcal{C}, \text{Set}]$ and $[\mathcal{C}, \text{Ab}]$ for the categories of functors on $\mathcal{C}$ taking values in sets and abelian groups, respectively. For any functor $X : \mathcal{C} \rightarrow \text{Set}$, we then denote by $\mathbf{RHom}_{[\mathcal{C}, \text{Set}]}(X, -)$ the functor from $[\mathcal{C}, \text{Ab}]$ to cochain complexes given by taking the right-derived functor of the functor

$$\text{Hom}_{[\mathcal{C}, \text{Set}]}(X, -) : [\mathcal{C}, \text{Ab}] \rightarrow \text{Ab}$$

of natural transformations with source X .

For the forgetful functor $\mathcal{O} : f\Lambda(R[q]) \rightarrow \text{Mod}(R[q])$ to the category of $R[q]$ -modules, we then have

$$\text{qDR}(A/R) = \mathbf{R}\text{Hom}_{[f\Lambda(R[q]), \text{Set}]}((\text{Spec } A)_{\text{strat}}^q, \mathcal{O}),$$

with Adams operations $\Psi^n : \mathcal{O} \otimes_{R[q], \Psi^n} R[q] \rightarrow \mathcal{O}$ giving the $(R[q], \Psi^n)$ -semilinear operations Ψ^n on $\text{qDR}(A/R)$.

Remark 1.10 The cochain complex $\text{qDR}(A/R)$ naturally carries much more structure than these Adams operations. Whenever we can factor the functor $\mathcal{O}$ through a model category $\mathcal{C}$ equipped with a forgetful functor to $\text{Ch}(R[q])$ preserving weak equivalences and homotopy limits, we can regard $\text{qDR}(A/R)$ as an object of the homotopy category of $\mathcal{C}$ by taking the defining homotopy limit in $\mathcal{C}$.

The universal such example for $\mathcal{C}$ is given by the model category of cosimplicial Λ -rings over $R[q]$, with weak equivalences being quasi-isomorphisms (i.e. cohomology isomorphisms) and fibrations being surjections; the underlying cochain complex has differential $\sum (-1)^j \partial^j$. That this determines a model structure follows from Kan's transfer theorem [9, Theorem 11.3.2] applied to the cosimplicial Dold–Kan normalisation functor taking values in unbounded chain complexes with the projective model structure; the conditions of that theorem are satisfied because the left adjoint functor sends acyclic cofibrant complexes to cosimplicial Λ -rings which automatically have a contracting homotopy in the form of an extra codegeneracy map.

In particular, $\text{qDR}(A/R)$ naturally underlies a quasi-isomorphism class of cosimplicial Λ -rings over $R[q]$; forgetting the λ -operations gives a cosimplicial commutative $R[q]$ -algebra, and stabilisation then gives an E_∞ -algebra over $R[q]$, all with underlying cochain complex $\text{qDR}(A/R)$.

Definition 1.11 Given a polynomial ring $R[x]$, recall from [13] that the q -de Rham (or Aomoto–Jackson) cohomology $q\text{-}\Omega_{R[x]/R}^\bullet$ is given by the complex

$$R[x][q] \xrightarrow{\nabla_q} R[x][q]dx, \quad \text{where} \quad \nabla_q(f) = \frac{f(qx) - f(x)}{x(q-1)}dx,$$

so $\nabla_q(x^n) = [n]_q x^{n-1} dx$.

Given a polynomial ring $R[x_1, \dots, x_d]$, the q -de Rham complex $q\text{-}\Omega_{R[x_1, \dots, x_d]/R}^\bullet$ is then set to be

$$q\text{-}\Omega_{R[x_1]/R}^\bullet \otimes_{R[q]} q\text{-}\Omega_{R[x_2]/R}^\bullet \otimes_{R[q]} \dots \otimes_{R[q]} q\text{-}\Omega_{R[x_d]/R}^\bullet,$$

so takes the form

$$R[x_1, \dots, x_d][q] \xrightarrow{\nabla_q} \Omega_{R[x_1, \dots, x_d]/R}^1[q] \xrightarrow{\nabla_q} \dots \xrightarrow{\nabla_q} \Omega_{R[x_1, \dots, x_d]/R}^d[q].$$

Definition 1.12 Given a flat morphism $R \rightarrow A$ of Λ -rings with $X = \text{Spec } A$, define the functor $\tilde{X}_{\text{strat}}^q$ from flat Λ -rings over $R[q]$ to simplicial sets by taking the Čech nerve of $\text{Hom}_{\Lambda, R}(A, B) \rightarrow \text{Hom}_{\Lambda, R}(A, B/(q-1))$, so

$$\begin{aligned}
 (\tilde{X}_{\text{strat}}^q)_n(B) &:= \overbrace{\text{Hom}_{\Lambda, R}(A, B) \times \text{Hom}_{\Lambda, R}(A, B/(q-1)) \cdots \times \text{Hom}_{\Lambda, R}(A, B/(q-1))}^{n+1} \text{Hom}_{\Lambda, R}(A, B) \\
 &= \text{Hom}_{\Lambda, R}(A, \overbrace{B \times_{B/(q-1)} \cdots \times_{B/(q-1)} B}^{n+1}),
 \end{aligned}$$

with simplicial operations

$$\begin{aligned}
 \partial_j(f_0, f_1, \dots, f_n) &:= (f_0, f_1, \dots, f_{j-1}, f_{j+1}, f_{j+2}, \dots, f_n), \\
 \sigma_j(f_0, f_1, \dots, f_n) &:= (f_0, f_1, \dots, f_j, f_j, f_{j+1}, \dots, f_n).
 \end{aligned}$$

Definition 1.13 Given a cosimplicial abelian group $V^\bullet$, we write NV for the Dold–Kan normalisation of V ([15, Lemma 8.3.7] applied the opposite category). This is a cochain complex with $N^r V = V^r \cap_{j < r} \ker \sigma^j$ and differential $d = \sum_{j=0}^{r+1} (-1)^j \partial^j : N^r V \rightarrow N^{r+1} V$.

Lemma 1.14 If, for $X = \text{Spec } A$, the functors $(\tilde{X}_{\text{strat}}^q)_n$ are represented by flat Λ -rings $\Gamma((\tilde{X}_{\text{strat}}^q)_n, \mathcal{O})$ over $R[q]$, then a model for $\text{qDR}(A/R)$ is given by the Dold–Kan normalisation of the cosimplicial module $n \mapsto \Gamma((\tilde{X}_{\text{strat}}^q)_n, \mathcal{O})$.

Proof The set-valued functor $X_{\text{strat}}^q = (\text{Spec } A)_{\text{strat}}^q$ of Definition 1.7 is resolved by the simplicial functor $\tilde{X}_{\text{strat}}^q$ of Definition 1.12. In the notation of Definition 1.9, this implies that the functor $\text{Hom}_{[f\Lambda(R[q]), \text{Set}]}(X_{\text{strat}}^q, -)$ on $[f\Lambda(R[q]), \text{Ab}]$ is resolved by the cochain complex

$$N\text{Hom}_{[f\Lambda(R[q]), \text{Set}]}((\tilde{X}_{\text{strat}}^q)_\bullet, -).$$

Although X_{strat}^q is not representable on the category of flat Λ -rings over $R[q]$, our hypotheses ensure that each functor $(\tilde{X}_{\text{strat}}^q)_n$ is so. Thus the functors $\text{Hom}_{[f\Lambda(R[q]), \text{Set}]}((\tilde{X}_{\text{strat}}^q)_n, -)$ and their direct summands $N^n \text{Hom}_{[f\Lambda(R[q]), \text{Set}]}((\tilde{X}_{\text{strat}}^q)_\bullet, -)$ are exact, and are their own right-derived functors. This implies that the cochain complex of functors above models $\mathbf{R}\text{Hom}_{[f\Lambda(R[q]), \text{Set}]}(X_{\text{strat}}^q, -)$, and the result follows by evaluation at $\mathcal{O}$. $\square$

Proposition 1.15 If R is a Λ -ring and x of rank 1, then $\text{qDR}(R[x]/R)$ can be calculated by Dold–Kan normalisation of the cosimplicial $R[q]$ -module $U^\bullet$ given by setting U^n to be the Λ -subring

$$U^n \subset R[q, \{(q^m - 1)^{-1}\}_{m \geq 1}, x_0, \dots, x_n]$$

generated by q and the elements x_i and $\frac{x_i - x_j}{q-1}$, with cosimplicial operations

$$\partial^j x_i := \begin{cases} x_i & j > i \\ x_{i+1} & j \leq i, \end{cases} \quad \sigma^j x_i := \begin{cases} x_i & j \geq i \\ x_{i-1} & j < i. \end{cases}$$

Proof We verify the conditions of Lemma 1.14 by showing that each U^n is a flat Λ -ring over $R[q]$ representing $(\tilde{X}_{\text{strat}}^q)_n$. Taking $X = \text{Spec } R[x]$, observe that any element of $(\tilde{X}_{\text{strat}}^q)_n(B)$ gives rise to a morphism $f : R[q, x_0, \dots, x_n] \rightarrow B$ of Λ -rings over $R[q]$, with the image of $x_i - x_j$ divisible by $(q-1)$. Flatness of B then gives a unique element $f(x_i - x_j)/(q-1) \in B$, so we have a map f to B from the free Λ -ring L over $R[q, x_0, \dots, x_n]$ generated by elements z_{ij} with $(q-1)z_{ij} = x_i - x_j$.

Since B is flat, it embeds in $B[\{(q^m - 1)^{-1}\}_{m \geq 1}]$ (the only hypothesis we really need) implying that the image of f factors through the image U^n of L in $R[q, \{(q^m - 1)^{-1}\}_{m \geq 1}, x_0, \dots, x_n]$. To see that $(\tilde{X}_{\text{strat}}^q)_n$ is represented by U^n , we only now need to check that U^n is itself flat over $R[q]$, which follows because the argument of Lemma 1.5 gives a basis

$$x_0^{r_0} \lambda^{r_1} \left(\frac{x_1 - x_0}{q-1} \right) \dots \lambda^{r_n} \left(\frac{x_n - x_{n-1}}{q-1} \right)$$

for U^n over $R[q]$. We therefore have $\text{qDR}(R[x]/R) \simeq NU^\bullet$. $\square$

In fact, the proofs of Lemma 1.14 and Proposition 1.15 show that the natural cosimplicial Λ -ring structure on $U^\bullet$ gives a model for the cosimplicial Λ -ring structure on $\text{qDR}(R[x]/R)$ coming from Remark 1.10.

Definition 1.16 Following [13, Proposition 5.4], we denote by $\mathbf{L}\eta_{(q-1)}$ the décalage functor with respect to the derived $(q-1)$ -adic filtration. This is given on complexes $C^\bullet$ of $(q-1)$ -torsion-free $R[q]$ -modules by

$$(\eta_{(q-1)}C)^n := \{c \in (q-1)^n C^n : dc \in (q-1)^{n+1} C^{n+1}\},$$

and is extended to the derived category of $R[q]$ -modules by taking torsion-free resolutions.

Theorem 1.17 *If R is a Λ -ring and if the polynomial ring $R[x_1, \dots, x_n]$ is given the Λ -ring structure for which the elements x_i are of rank 1, then there are $R[q]$ -linear zigzags of quasi-isomorphisms*

$$\begin{aligned} \text{qDR}(R[x_1, \dots, x_n]/R) &\simeq (\Omega_{R[x_1, \dots, x_n]/R}^*[q], (q-1)\nabla_q) \\ \mathbf{L}\eta_{(q-1)}\text{qDR}(R[x_1, \dots, x_n]/R) &\simeq q\text{-}\Omega_{R[x_1, \dots, x_n]/R}^\bullet. \end{aligned}$$

Proof It suffices to prove the first statement, the second following immediately by décalage. We have $(\text{Spec } A \otimes_R A')_{\text{strat}}^q(B) = (\text{Spec } A)_{\text{strat}}^q(B) \times (\text{Spec } A')_{\text{strat}}^q(B)$, and similarly for the simplicial functor $(\text{Spec } A \otimes_R A')_{\text{strat}}^q$ of Definition 1.12. Since coproduct of flat Λ -rings over $R[q]$ is given by $\otimes_{R[q]}$, it follows from Lemma 1.14 and Proposition 1.15 that $\text{qDR}(R[x_1, \dots, x_n]/R)$ can be calculated as the Dold–Kan normalisation of $(U^\bullet)^{\otimes_{R[q]} n}$ (given by the n -fold tensor product $(U^m)^{\otimes_{R[q]} n}$ in cosimplicial level m), for the cosimplicial module $U^\bullet$ of Proposition 1.15.

The proof now proceeds in a similar fashion to the comparison between crystalline and de Rham cohomology in [3]. We consider the cochain complexes $\tilde{\Omega}^\bullet(U^m)$ given by

$$U^m \xrightarrow{(q-1)\nabla_q} \bigoplus_i U^m dx_i \xrightarrow{(q-1)\nabla_q} \bigoplus_{i < j} U^m dx_i \wedge dx_j \xrightarrow{(q-1)\nabla_q} \dots$$

In order to see that this differential takes values in the codomains given, observe that

$$\begin{aligned} (q-1)\nabla_{q,y}\lambda^k\left(\frac{y-x}{q-1}\right) &= y^{-1}\left(\lambda^k\left(\frac{qy-x}{q-1}\right) - \lambda^k\left(\frac{y-x}{q-1}\right)\right)dy \\ &= y^{-1}\left(\lambda^k\left(y + \frac{y-x}{q-1}\right) - \lambda^k\left(\frac{y-x}{q-1}\right)\right)dy \\ &= \lambda^{k-1}\left(\frac{y-x}{q-1}\right)dy, \end{aligned}$$

and similarly

$$(q-1)\nabla_{q,x}\lambda^k\left(\frac{y-x}{q-1}\right) = \sum_{i \geq 1} (-1)^i x^{i-1} \lambda^{k-i}\left(\frac{y-x}{q-1}\right) dx.$$

The first calculation also shows that the inclusion $\tilde{\Omega}^\bullet(U^{m-1}) \hookrightarrow \tilde{\Omega}^\bullet(U^m)$ is a quasi-isomorphism, since for $\omega \in \tilde{\Omega}^\bullet(U^{m-1})$, we have

$$(q-1)\nabla_{q,x_m}\omega\lambda^k\left(\frac{x_m-x_{m-1}}{q-1}\right) = \omega\lambda^{k-1}\left(\frac{x_m-x_{m-1}}{q-1}\right)dx_m$$

for $k \geq 1$, allowing us to define a contracting homotopy

$$\begin{aligned} h\left(\omega\lambda^{k-1}\left(\frac{x_m-x_{m-1}}{q-1}\right)dx_m\right) &:= \omega\lambda^k\left(\frac{x_m-x_{m-1}}{q-1}\right), \\ h\left(\omega\lambda^{k-1}\left(\frac{x_m-x_{m-1}}{q-1}\right)\right) &:= 0. \end{aligned}$$

Since contracting homotopies interact well with tensor products, it also follows that the inclusion $\tilde{\Omega}^\bullet(U^{m-1})^{\otimes_{R[q]}n} \hookrightarrow \tilde{\Omega}^\bullet(U^m)^{\otimes_{R[q]}n}$ is a quasi-isomorphism. By induction on m we deduce that the inclusions $\tilde{\Omega}^\bullet(U^0)^{\otimes_{R[q]}n} \hookrightarrow \tilde{\Omega}^\bullet(U^m)^{\otimes_{R[q]}n}$, and hence their retractions given by diagonals $U^m \rightarrow U^0$, are quasi-isomorphisms. These combine to give a quasi-isomorphism

$$\mathrm{Tot} N(\tilde{\Omega}^\bullet(U^\bullet)^{\otimes_{R[q]}n}) \rightarrow \tilde{\Omega}^\bullet(U^0)^{\otimes_{R[q]}n} = \tilde{\Omega}^\bullet(R[x])^{\otimes_{R[q]}n}$$

on total complexes of normalisations.

Now, the cosimplicial module $\tilde{\Omega}^r(U^\bullet)$ is given by the cosimplicial (i.e. levelwise) tensor product of $U^\bullet$ with the cosimplicial $\mathbb{Z}$ -module

$$j \mapsto \bigoplus_{0 \leq i_1 < i_2 < \dots < i_r \leq j} \mathbb{Z} dx_{i_1} \wedge \dots \wedge dx_{i_r},$$

with operations induced by those in Proposition 1.15. For $r > 0$, this cosimplicial $\mathbb{Z}$ -module is contractible, via the extra codegeneracy map given by

$$\sigma^{-1}(dx_{i_1} \wedge \cdots \wedge dx_{i_r}) = \begin{cases} dx_{i_1-1} \wedge \cdots \wedge dx_{i_r-1} & i_1 > 0, \\ 0 & i_1 = 0. \end{cases}$$

The Eilenberg–Zilber theorem ([15, §8.5] applied to the opposite category) ensures that the normalisation of a cosimplicial tensor product is quasi-isomorphic to the tensor product of the normalisations. Tensoring with a complex which has an extra codegeneracy map always produces an acyclic complex, so $\tilde{\Omega}^r(U^\bullet)$ and its tensor powers are all acyclic for $r > 0$.

The brutal truncation maps

$$\mathrm{Tot} N(\tilde{\Omega}^\bullet(U^\bullet)^{\otimes_{R[q]} n}) \rightarrow N(U^\bullet)^{\otimes_{R[q]} n} \simeq \mathrm{qDR}(R[x_1, \dots, x_n]/R)$$

are therefore quasi-isomorphisms of flat cochain complexes over $R[q]$, so

$$\mathrm{qDR}(R[x_1, \dots, x_n]/R) \simeq \tilde{\Omega}^\bullet(R[x])^{\otimes_{R[q]} n},$$

and we just observe that $\tilde{\Omega}^\bullet(R[x]) = (\Omega_{R[x]/R}^*[q], (q-1)\nabla_q)$. $\square$

Remark 1.18 Note that Theorem 1.17 and Remark 1.10 together imply that $q\text{-}\Omega_{R[x_1, \dots, x_n]/R}^\bullet$ naturally underlies the décalage of a cosimplicial Λ -ring over $R[q]$. Even the underlying cosimplicial commutative ring structure carries more information than an E_∞ -structure when $\mathbb{Q} \not\subseteq R$.

1.3 Completed q -cohomology

Definition 1.19 Given a morphism $R \rightarrow A$ of Λ -rings, we define the category $\hat{\mathrm{Strat}}_{A/R}^q \subset \mathrm{Strat}_{A/R}^q$ to consist of those objects which are $(q-1)$ -adically complete.

Equivalently, $\hat{\mathrm{Strat}}_{A/R}^q$ is the Grothendieck construction of the functor

$$(\widehat{\mathrm{Spec} A})_{\mathrm{strat}}^q : B \mapsto \mathrm{Im}(\mathrm{Hom}_{\Lambda, R}(A, B) \rightarrow \mathrm{Hom}_{\Lambda, R}(A, B/(q-1)))$$

on the category of flat $(q-1)$ -adically complete Λ -rings over $R[q]$.

Definition 1.20 Given a flat morphism $R \rightarrow A$ of Λ -rings, define $\widehat{\mathrm{qDR}}(A/R)$ to be the cochain complex of $R[[q-1]]$ -modules given by taking the homotopy limit of the functor

$$\begin{aligned} \hat{\mathrm{Strat}}_{A/R}^q &\rightarrow \mathrm{Ch}(R[[q-1]]) \\ B &\mapsto B. \end{aligned}$$

The following is immediate:

Lemma 1.21 *Given a flat morphism $R \rightarrow A$ of Λ -rings, the complex $\widehat{\mathrm{qDR}}(A/R)$ is the derived $(q-1)$ -adic completion of $\mathrm{qDR}(A/R)$.*

Definition 1.22 As in [13, §3], given a formally étale map $\square : R[x_1, \dots, x_d] \rightarrow A$, define $\widehat{q\text{-}\Omega}_{A/R, \square}^\bullet$ to be the complex

$$A[[q-1]] \xrightarrow{\nabla_q} \Omega_{A/R}^1[[q-1]] \xrightarrow{\nabla_q} \cdots \xrightarrow{\nabla_q} \Omega_{A/R}^d[[q-1]],$$

where ∇_q is defined as follows. First note that the $R[[q-1]]$ -linear ring endomorphisms γ_i of $R[x_1, \dots, x_d][[q-1]]$ given by $\gamma_i(x_j) = q^{\delta_{ij}}x_j$ extend uniquely to endomorphisms of $A[[q-1]]$ which are the identity modulo $(q-1)$, then set

$$\nabla_q(f) := \sum_i \frac{\gamma_i(f) - f}{(q-1)x_i} dx_i.$$

Note that $\widehat{q\text{-}\Omega}_{R[x_1, \dots, x_d]/R}^\bullet$ is just the $(q-1)$ -adic completion of $q\text{-}\Omega_{R[x_1, \dots, x_d]/R}^\bullet$.

Theorem 1.23 *If R is a flat Λ -ring over $\mathbb{Z}$ and $\square : R[x_1, \dots, x_d] \rightarrow A$ is a formally étale map of Λ -rings, the elements x_i having rank 1, then there are zigzags of $R[[q]]$ -linear quasi-isomorphisms*

$$\widehat{\mathrm{qDR}}(A/R) \simeq (\Omega_{A/R}^*[[q-1]], (q-1)\nabla_q), \quad L\eta_{(q-1)}\widehat{\mathrm{qDR}}(A/R) \simeq \widehat{q\text{-}\Omega}_{A/R, \square}^\bullet.$$

The induced quasi-isomorphisms

$$\widehat{\mathrm{qDR}}(A/R) \otimes_{R[[q-1]]}^L R \simeq (\Omega_{A/R}^*, 0), \quad (L\eta_{(q-1)}\widehat{\mathrm{qDR}}(A/R)) \otimes_{R[[q-1]]}^L R \simeq \Omega_{A/R}^\bullet$$

are independent of the choice of framing.

Proof Since the framing $\square$ is formally étale, for any $(q-1)$ -adically complete commutative $R[q]$ -algebra B , any commutative square

$$\begin{array}{ccc} R[x_1, \dots, x_d] & \xrightarrow{\quad} & B \\ \square \downarrow & \nearrow \text{dashed} & \downarrow \\ A & \xrightarrow{\quad} & B/(q-1). \end{array}$$

of R -algebra homomorphisms admits a unique dashed arrow as shown.

For any $(q-1)$ -adically complete flat Λ -ring B over R , we then have the same property for Λ -ring homomorphisms over R instead of R -algebra homomorphisms: the diagram above gives a unique dashed R -algebra homomorphism, and uniqueness of lifts ensures that it commutes with Adams operations, so is a Λ -ring homomorphism (R being flat over $\mathbb{Z}$). Similarly (taking $B = A[[q-1]]$) uniqueness of lifts ensures that the operations γ_i are Λ -ring endomorphisms of $A[[q-1]]$.

We can now proceed as in the proof of Theorem 1.17. The complex $\widehat{\text{qDR}}(A/R)$ can be realised as the cochain complex underlying a cosimplicial Λ -ring $\hat{U}(A)$, representing the functor $\tilde{X}_{\text{strat}}^q$ of Definition 1.12 for $X = \text{Spec } A$, restricted to $(q-1)$ -adically complete Λ -rings B . By the consequences of formal étaleness, we have

$$\begin{aligned} & \text{Hom}_{\Lambda, R}(A, B) \times_{\text{Hom}_{\Lambda, R}(A, B/(q-1))} \text{Hom}_{\Lambda, R}(A, B) \\ & \cong \text{Hom}_{\Lambda, R}(A, B) \times_{\text{Hom}_{\Lambda, R}(R[x_1, \dots, x_d], B/(q-1))} \text{Hom}_{\Lambda, R}(R[x_1, \dots, x_d], B), \end{aligned}$$

giving $(\tilde{X}_{\text{strat}}^q)_n \cong \text{Hom}_{\Lambda, R}(A, B) \times_{\text{Hom}_{\Lambda, R}(R[x_1, \dots, x_d], B)} (\tilde{Y}_{\text{strat}}^q)_n$ for each n , where $Y = \text{Spec } R[x_1, \dots, x_d]$ and the fibre product is given via the projection of $(\tilde{Y}_{\text{strat}}^q)_n$ onto the first factor.

In particular, this means that $\hat{U}(A)^n$ is the $(q-1)$ -adic completion of

$$A \otimes_{R[x_1, \dots, x_d]} (U(R[x_1])^n \otimes_{R[q]} \dots \otimes_{R[q]} U(R[x_d])^n),$$

where each $U(R[x_i])$ is a copy of the cosimplicial ring U from Proposition 1.15. This isomorphism respects the cosimplicial operations; note that ∂^0 is not linear for the left multiplication by A , but is still determined via formal étaleness of the framing.

We now define a cosimplicial cochain complex $\tilde{\Omega}^\bullet(\hat{U}(A))$ by setting $\tilde{\Omega}^\bullet(\hat{U}(A)^n)$ to be the $(q-1)$ -adic completion of

$$\begin{aligned} & (A \otimes_{R[x_1, \dots, x_d]} (\tilde{\Omega}^*(U(R[x_1])^n) \otimes_{R[q]} \dots \otimes_{R[q]} \tilde{\Omega}^*(U(R[x_d])^n)), (q-1)\nabla_q) \\ & \cong (\hat{U}(A)^n \otimes_{A^{\otimes(n+1)}} (\Omega_{A/R}^*)^{\otimes(n+1)}, (q-1)\nabla_q). \end{aligned}$$

where each $\tilde{\Omega}^\bullet(U(R[x_i]))$ is a copy of the complex $\tilde{\Omega}^\bullet(U^n)$ from the proof of Theorem 1.17. Compatibility of this construction with the cosimplicial operations follows because the γ_i are Λ -ring homomorphisms.

The calculations contributing to the proof of Theorem 1.17 are still valid after base change, with contracting homotopies giving quasi-isomorphisms

$$(\Omega_{A/R}^*[[q]], (q-1)\nabla_q) \leftarrow \text{Tot } N\tilde{\Omega}^\bullet(\hat{U}(A)^\bullet) \rightarrow N\hat{U}(A)^\bullet.$$

Reduction of this modulo $(q-1)^2$, or of its décalage modulo $(q-1)$ (cf. [4, Proposition 6.12]), replaces ∇_q with d throughout, removing any dependence on co-ordinates. $\square$

As in [13, Definition 7.3], there is a notion of q -connection $\nabla_q = (\nabla_{1,q}, \dots, \nabla_{d,q})$ on a finite projective $A[[q-1]]$ -module M , in the form of commuting $R[[q-1]]$ -linear operators $\nabla_{i,q}$ on M , with each $\nabla_{i,q}$ satisfying $\nabla_{i,q}(av) = \nabla_{q,x_i}(a)v + \gamma_i(a)\nabla_{i,q}(v)$ for $a \in A, v \in M$.

Definition 1.24 Given a flat morphism $R \rightarrow A$ of Λ -rings with $X := \text{Spec } A$, denote the forgetful functor $(B, f) \mapsto B$ from $\hat{\text{Strat}}_{A/R}^q$ to rings by $\mathcal{O}_{\hat{X}^q, \text{strat}}$.

There is then a notion of $\mathcal{O}_{\hat{X}^q, \text{strat}}$ -modules in the category of functors from $\hat{\text{Strat}}_{A/R}^q$ to abelian groups; we will simply refer to these as $\mathcal{O}_{\hat{X}^q, \text{strat}}$ -modules. Given a property

P of modules, we will say that an $\mathcal{O}_{\hat{X}^q, \text{strat}}$ -module $\mathcal{F}$ has the property P if for each $(B, f) \in \hat{\text{Strat}}_{A/R}^q$, the B -module $\mathcal{F}(B, f)$ has property P .

We say that an $\mathcal{O}_{\hat{X}^q, \text{strat}}$ -module $\mathcal{F}$ is Cartesian if for each morphism $(B, f) \rightarrow (B', f')$ in $\hat{\text{Strat}}_{A/R}^q$, the map $\mathcal{F}(B, f) \otimes_B B' \rightarrow \mathcal{F}(B', f')$ is an isomorphism.

Given an $\mathcal{O}_{\hat{X}^q, \text{strat}}$ -module $\mathcal{F}$, we define $\Gamma(\hat{X}_{\text{strat}}^q, \mathcal{F}) := \varprojlim_{\hat{\text{Strat}}_{A/R}^q} \mathcal{F}$.

In [13, Conjecture 7.5], Scholze predicted that the category of q -connections on finite projective $A[[q-1]]$ -module is independent of co-ordinates on A . The following proposition gives the weaker statement that the category depends only on the Λ -ring structure on A .

Proposition 1.25 *Under the conditions of Theorem 1.23, with $X := \text{Spec } A$, the category of finite projective $A[[q-1]]$ -modules (M, ∇) with q -connection is equivalent to the category of those finite projective $\mathcal{O}_{\hat{X}^q, \text{strat}}$ -modules $\mathcal{N}$ for which the map*

$$\Gamma(\hat{X}_{\text{strat}}^q, \mathcal{N}/(q-1)) \otimes_A (\mathcal{O}_{\hat{X}^q, \text{strat}}/(q-1)) \rightarrow \mathcal{N}/(q-1)$$

is an isomorphism.

Proof The restriction on $\mathcal{N}/(q-1)$ ensures that it is Cartesian; this also implies that $\mathcal{N}$ is Cartesian, because finite projective modules are flat and $(q-1)$ -adically complete.

Now, the cosimplicial Λ -ring $\hat{U}(A)$ realising $\widehat{\text{qDR}}(A/R)$ in the proof of Theorem 1.23 admits a natural map $A \rightarrow \hat{U}(A)/(q-1)$ from the constant cosimplicial diagram. Thus $\hat{U}(A)$ defines a cosimplicial diagram in $\hat{\text{Strat}}_{A/R}^q$. Since the functor $\tilde{X}_{\text{strat}}^q$ of Definition 1.12 resolves X_{strat}^q , it follows that the functor $\hat{U}(A) : \Delta \rightarrow \hat{\text{Strat}}_{A/R}^q$ from the simplex category is initial in the sense of [11, §IX.3].

In particular, this means that the category of Cartesian $\mathcal{O}_{\hat{X}^q, \text{strat}}$ -modules $\mathcal{N}$ is equivalent to the category of Cartesian cosimplicial $\hat{U}(A)$ -modules N , where the Cartesian condition amounts to saying that the maps $N^m \otimes_{\hat{U}(A)^m, \partial^i} \hat{U}(A)^{m+1} \rightarrow N^{m+1}$ are all isomorphisms. Setting $M = N^0$, Cartesian $\hat{U}(A)$ -modules are equivalent to $\hat{U}(A)^0 = A[[q-1]]$ -modules M with isomorphisms $\Delta : (\partial^1)^* M \cong (\partial^0)^* M$ satisfying the cocycle condition $\partial^1 \Delta = (\partial^0 \Delta) \circ (\partial^2 \Delta) : (\partial^2 \partial^0)^* M \rightarrow (\partial^0 \partial^0)^* M$.

The map Δ is determined by its restriction to M , so using the basis for U^1 from Lemma 1.5, and taking $v \in M$, we have

$$\Delta(v) = \sum_{\underline{k} \in \mathbb{N}_0^d} \partial^0(\Delta_{\underline{k}}(v)) \lambda^{k_1} \left(\frac{\partial^1 x_1 - \partial^0 x_1}{q-1} \right) \cdots \lambda^{k_d} \left(\frac{\partial^1 x_d - \partial^0 x_d}{q-1} \right)$$

for $R[[q-1]]$ -linear endomorphisms $\Delta_{\underline{k}}$ of M . Since $\lambda_i(a+b) = \lambda_i(a)\lambda_i(b)$, the cocycle condition becomes $\Delta_{\underline{j}+\underline{k}} = \Delta_{\underline{j}} \circ \Delta_{\underline{k}}$, meaning Δ is determined by the operators Δ_{e_i} at the basis vectors, which must moreover commute.

Linearity of Δ with respect to $\hat{U}(A)^1$ then reduces to the condition that $\Delta(av) = \partial^1(a)\Delta(v)$ for $a \in A$, $v \in M$. Writing A for $\partial^0 A$ and $h_i^{[k]} := \lambda^k \left(\frac{\partial^1 x_i - \partial^0 x_i}{q-1} \right)$, the ideal

$J := (h_i^{[\geq 2]}, h_i h_j)_{i \neq j}$ satisfies $U^1 = A \oplus \bigoplus_i A h_i \oplus J$. The proof of Theorem 1.23 gives $\partial^1(a) \equiv a + (q-1) \sum_i \nabla_{q, x_i}(a) h_i \pmod{J}$, and in U^1/J we have $[h_i]^2 \equiv x_i [h_i]$. Comparing coefficients of h_i in the equation $\Delta(av) \equiv \partial^1(a) \Delta(v) \pmod{J}$ then gives

$$\begin{aligned} \Delta_{e_i}(av) &= (q-1) \nabla_{q, x_i}(a)v + a \Delta_{e_i}(v) + (q-1) x_i \nabla_{q, x_i}(a) \Delta_{e_i}(v) \\ &= (q-1) \nabla_{q, x_i}(a)v + \gamma_i(a) \Delta_{e_i}(v). \end{aligned}$$

Finally, note that the condition that $\mathcal{N}/(q-1)$ be the pullback of an A -module (necessarily $\Gamma(\hat{X}_{\text{strat}}^q, \mathcal{N}/(q-1))$) is equivalent to saying that $\partial_N^0 \equiv \partial_N^1 \pmod{q-1}$, or that $(q-1)$ divides $\Delta_{\underline{k}}$ whenever $\underline{k} \neq 0$. In particular, $(q-1)$ divides Δ_{e_i} , and setting $\nabla_{i,q} := (q-1)^{-1} \Delta_{e_i}$ gives a q -connection $(\nabla_{i,q})_{1 \leq i \leq d}$ on $M = N^0$ uniquely determining Δ .

The inverse construction is given by $\Delta_{\underline{k}} = (q-1)^{\sum k_i} \nabla_{1,q}^{k_1} \circ \dots \circ \nabla_{d,q}^{k_d}$. $\square$

2 Comparisons for Λ_P -rings

Since very few étale maps $R[x_1, \dots, x_d] \rightarrow A$ give rise to Λ -ring structures on A , Theorem 1.23 is fairly limited in its scope for applications. We now show how the construction of $\widehat{\text{qDR}}$ and the comparison quasi-isomorphism survive when we weaken the Λ -ring structure by discarding Adams operations at invertible primes.

2.1 q -cohomology for Λ_P -rings

Our earlier constructions for Λ -rings all carry over to Λ_P -rings, as follows.

Definition 2.1 Given a set P of primes, we define a Λ_P -ring A to be a $\Lambda_{\mathbb{Z}, P}$ -ring in the sense of [5]. This means that it is a coalgebra in commutative rings for the comonad given by the functor $W^{(P)}$ of P -typical Witt vectors. When a commutative ring A is flat over $\mathbb{Z}$, giving a Λ_P -ring structure on A is equivalent to giving commuting Adams operations Ψ^p for all $p \in P$, with $\Psi^p(a) \equiv a^p \pmod{p}$ for all a .

Thus when P is the set of all primes, a Λ_P -ring is just a Λ -ring; a $\Lambda_{\emptyset}$ -ring is just a commutative ring; for a single prime p , we write $\Lambda_p := \Lambda_{\{p\}}$, and note that a Λ_p -ring is a δ -ring in the sense of [10].

Definition 2.2 Given a Λ_P -ring R , say that A is a Λ_P -ring over R if it is a Λ_P -ring equipped with a morphism $R \rightarrow A$ of Λ_P -rings. We say that A is a flat Λ_P -ring over R if A is flat as a module over the commutative ring underlying R .

Definition 2.3 Given a morphism $R \rightarrow A$ of Λ_P -rings, we define the category $\text{Strat}_{A/R}^{q,P}$ to consist of flat Λ_P -rings B over $R[q]$ equipped with a compatible morphism $A \rightarrow B/(q-1)$, such that the map $A \rightarrow B/(q-1)$ admits a lift to B . We define the category $\hat{\text{Strat}}_{A/R}^{q,P} \subset \text{Strat}_{A/R}^q$ to consist of those objects which are $(q-1)$ -adically complete.

More concisely, $\text{Strat}_{A/R}^{q,P}$ (resp. $\widehat{\text{Strat}}_{A/R}^{q,P}$) is the Grothendieck construction of the functor $(\text{Spec } A)_{\text{strat}}^{q,P}$ (resp. $(\widehat{\text{Spec } A})_{\text{strat}}^{q,P}$) given by

$$B \mapsto \text{Im}(\text{Hom}_{\Lambda_P, R}(A, B) \rightarrow \text{Hom}_{\Lambda_P, R}(A, B/(q-1)))$$

on the category of flat Λ_P -rings (resp. $(q-1)$ -adically complete flat Λ_P -rings) over $R[q]$.

Definition 2.4 Given a flat morphism $R \rightarrow A$ of Λ_P -rings, define $\text{qDR}_P(A/R)$ to be the cochain complex of $R[q]$ -modules given by taking the homotopy limit of the functor

$$\begin{aligned} \text{Strat}_{A/R}^{q,P} &\rightarrow \text{Ch}(R[q]) \\ B &\mapsto B. \end{aligned}$$

Define $\widehat{\text{qDR}}_P(A/R)$ to be the cochain complex of $R[[q-1]]$ -modules given by the corresponding homotopy limit over $\widehat{\text{Strat}}_{A/R}^{q,P}$.

For $p \in P$, the cochain complex $\text{qDR}_P(A/R)$ naturally carries $(R[q], \Psi^p)$ -semilinear operations Ψ^p coming from the morphisms $\Psi^p : B \otimes_{R[q], \Psi^p} R[q] \rightarrow B$ of $R[q]$ -modules, for $B \in \text{Strat}_{A/R}^{q,P}$.

Thus when P is the set of all primes, we have $\text{qDR}_P(A/R) = \text{qDR}(A/R)$. At the other extreme, for A smooth, $\widehat{\text{qDR}}_\emptyset(A/R)$ is the Rees construction of the Hodge filtration on the infinitesimal cohomology complex [8] of A over R , with formal variable $(q-1)$. In more detail, there is a decreasing filtration F of $\mathcal{O}_{\text{inf}}$ given by powers of the augmentation ideal of $\mathcal{O}_{\text{inf}} \rightarrow \mathcal{O}_{\text{Zar}}$ (with $F^\nu \mathcal{O}_{\text{inf}} = \mathcal{O}_{\text{inf}}$ for $\nu \leq 0$), and then

$$\widehat{\text{qDR}}_\emptyset(A/R) \simeq \prod_{v \in \mathbb{Z}} (q-1)^{-v} \mathbf{R}\Gamma(\text{Spec } A, F^\nu \mathcal{O}_{\text{inf}}).$$

Lemma 2.5 For a set P of primes, the forgetful functor from Λ -rings to Λ_P -rings has a right adjoint $W^{(\notin P)}$. There is a canonical ghost component morphism

$$W^{(\notin P)}(B) \rightarrow \prod_{\substack{n \in \mathbb{N}: \\ (n, p) = 1 \ \forall p \in P}} B,$$

which is an isomorphism when P contains all the residue characteristics of B .

Proof Existence of a right adjoint follows from the comonadic definitions of Λ -rings and Λ_P -rings. The ghost component morphism is given by taking the Adams operations Ψ^n coming from the Λ -ring structure on $W^{(\notin P)}(B)$, followed by projection to B . When P contains all the residue characteristics of B , a Λ -ring structure is the same as a Λ_P -ring structure with compatible commuting Adams operations for all primes not in P , leading to the description above. $\square$

Note that the big Witt vector functor W on commutative rings thus factorises as $W = W^{(\notin P)} \circ W^{(P)}$, for $W^{(P)}$ the P -typical Witt vectors.

Proposition 2.6 *Given a morphism $R \rightarrow A$ of Λ -rings, and a set P of primes, there are natural maps*

$$\mathrm{qDR}_P(A/R) \rightarrow \mathrm{qDR}(A/R), \quad \widehat{\mathrm{qDR}}_P(A/R) \rightarrow \widehat{\mathrm{qDR}}(A/R),$$

and the latter map is a quasi-isomorphism when P contains all the residue characteristics of A .

Proof We have functors

$$\begin{aligned} (\mathrm{Spec} A)_{\mathrm{strat}}^q \circ W^{(\notin P)} : B &\mapsto \mathrm{Im}(\mathrm{Hom}_{\Lambda, R}(A, W^{(\notin P)} B) \rightarrow \mathrm{Hom}_{\Lambda, R}(A, (W^{(\notin P)} B)/(q-1))) \\ (\mathrm{Spec} A)_{\mathrm{strat}}^{q, P} : B &\mapsto \mathrm{Im}(\mathrm{Hom}_{\Lambda_P, R}(A, B) \rightarrow \mathrm{Hom}_{\Lambda_P, R}(A, B/(q-1))) \end{aligned}$$

on the category of flat Λ_P -rings over $R[q]$. There is an obvious map

$$(W^{(\notin P)} B)/(q-1) \rightarrow W^{(\notin P)}(B/(q-1)),$$

and hence a natural transformation $(\mathrm{Spec} A)_{\mathrm{strat}}^q \circ W^{(\notin P)} \rightarrow (\mathrm{Spec} A)_{\mathrm{strat}}^{q, P}$, which induces the morphism $\mathrm{qDR}_P(A/R) \rightarrow \mathrm{qDR}(A/R)$ on cohomology.

When P contains all the residue characteristics of A , the map $(W^{(\notin P)} B)/(q-1) \rightarrow W^{(\notin P)}(B/(q-1))$ is just

$$\prod_{\substack{n \in \mathbb{N}: \\ (n, p)=1 \forall p \in P}} B/(q^n - 1) \rightarrow \prod_{\substack{n \in \mathbb{N}: \\ (n, p)=1 \forall p \in P}} B/(q - 1),$$

since the morphism $R[q] \rightarrow W^{(\notin P)} B$ is given by Adams operations, with $\Psi^n(q-1) = q^n - 1$.

We have $(q^n - 1) = (q - 1)[n]_q$, and $[n]_q$ is a unit in $\mathbb{Z}[\frac{1}{n}][[q - 1]]$, hence a unit in B when n is coprime to the residue characteristics. Thus the map $(W^{(\notin P)} B)/(q - 1) \rightarrow W^{(\notin P)}(B/(q - 1))$ gives an isomorphism whenever B is $(q - 1)$ -adically complete and admits a map from A , so the transformation $(\mathrm{Spec} A)_{\mathrm{strat}}^q \circ W^{(\notin P)} \rightarrow (\mathrm{Spec} A)_{\mathrm{strat}}^{q, P}$ is a natural isomorphism on the category of flat $(q - 1)$ -adically complete Λ_P -rings over $R[q]$, and hence $\widehat{\mathrm{qDR}}_P(A/R) \xrightarrow{\sim} \widehat{\mathrm{qDR}}(A/R)$. $\square$

Remark 2.7 Remark 1.10 shows that $\mathrm{qDR}(A/R)$ can naturally be promoted to a cosimplicial Λ -ring, and the same reasoning promotes $\mathrm{qDR}_P(A/R)$ to a cosimplicial Λ_P -ring. The proof of Proposition 2.6 then ensures that the map $\mathrm{qDR}_P(A/R) \rightarrow \mathrm{qDR}(A/R)$ is naturally a morphism of cosimplicial Λ_P -rings,

Over $\mathbb{Z}[\{\frac{1}{p} : p \in P\}]$, every Λ_P -ring can be canonically made into a Λ -ring, by setting all the additional Adams operations to be the identity. However, this observation is of limited use in establishing functoriality of q -de Rham cohomology, because the resulting Λ -ring structure will not satisfy the conditions of Theorem 1.23. We now give a more general result which does allow for meaningful comparisons.

Theorem 2.8 *If R is a flat Λ - p -ring over $\mathbb{Z}$ and $\square : R[x_1, \dots, x_d] \rightarrow A$ is a formally étale map of Λ - p -rings, the elements x_i having rank 1, then there are zigzags of $R[[q-1]]$ -linear quasi-isomorphisms*

$$\widehat{\mathrm{qDR}}_P(A/R) \simeq (\Omega_{A/R}^*[[q-1]], (q-1)\nabla_q), \quad L\eta_{(q-1)}\widehat{\mathrm{qDR}}_P(A/R) \simeq \widehat{q\text{-}\Omega}_{A/R, \square}^\bullet.$$

whenever P contains all the residue characteristics of A .

Proof The key observation to make is that formally étale maps have a unique lifting property with respect to nilpotent extensions of flat Λ - p -rings, because the Adams operations must also lift uniquely. In particular, this means that the operations γ_i featuring in the definition of q -de Rham cohomology are necessarily endomorphisms of A as a Λ - p -ring.

Similarly to Theorem 1.23, $\widehat{\mathrm{qDR}}_P(A/R)$ is calculated using a cosimplicial Λ - p -ring given in level n by the $(q-1)$ -adic completion $\hat{U}_{P,A}^\bullet$ of the Λ - p -ring over $R[q]$ generated by $A^{\otimes_R(n+1)}[q]$ and $(q-1)^{-1} \ker(A^{\otimes_R(n+1)} \rightarrow A)[q]$. The observation above shows that $\hat{U}_{P,A}^n \cong \hat{U}_{P,R[x_1, \dots, x_d]}^n \hat{\otimes}_{R[x_1, \dots, x_d]} A$, changing base along $\square$ applied to the first factor.

As in Proposition 2.6, $\hat{U}_{P,R[x_1, \dots, x_d]}^\bullet$ is just the $(q-1)$ -adic completion of the complex $U^\bullet$ from Proposition 1.15. Further application of the key observation above then allows us to adapt the constructions of Theorem 1.17, giving the desired quasi-isomorphisms. $\square$

2.2 Cartier isomorphisms in mixed characteristic

In [13, Conjecture 7.1], Scholze predicted that $\widehat{q\text{-}\Omega}_{A/R, \square}^\bullet$ is a functorial invariant of the R -algebra A , independent of the choice of framing, so extends to all smooth schemes. Theorem 2.8 shows that $\widehat{q\text{-}\Omega}_{A/R, \square}^\bullet$ is functorial invariant of the Λ - p -ring A over R .

The only setting in which Theorem 2.8 leads to results close to Scholze's conjecture is when $R = W^{(p)}(k)$, the p -typical Witt vectors of a perfect field of characteristic p , and $A = \varprojlim_n A_n$ is a formal deformation of a smooth k -algebra A_0 . Then any formally étale morphism $W^{(p)}(k)[x_1, \dots, x_d] \rightarrow A$ of topological rings gives rise to a unique compatible lift Ψ^p of absolute Frobenius on A with $\Psi^p(x_i) = x_i^p$, so gives A the structure of a topological Λ - p -ring. The framing still affects the choice of Λ - p -ring structure, but at least such a structure is guaranteed to exist, giving rise to a complex $\mathrm{qDR}_P(A/R)^{\wedge p} := \mathbf{R} \varprojlim_n \mathrm{qDR}_P(A/R) \otimes_R^{\mathbf{L}} R_n$ depending only on the choice of Ψ^p , where $R_n = W_n^{(p)}(k)$.

Our constructions now allow us to globalise the quasi-isomorphism

$$\left(\widehat{q\text{-}\Omega}_{A/R, \square}^\bullet\right)^{\wedge p} / [p]_q \simeq (\Omega_{A/R}^*)^{\wedge p} [[q-1]] / [p]_q$$

of [13, Proposition 3.4], where $\Omega_{A/R}^*$ denotes the complex $A \xrightarrow{0} \Omega_{A/R}^1 \xrightarrow{0} \Omega_{A/R}^2 \xrightarrow{0} \dots$

Lemma 2.9 *Under the quasi-isomorphism $\widehat{\mathrm{qDR}}_p(A/R) \simeq (\Omega_{A/R}^*[q-1], (q-1)\nabla_q)$ from Theorem 2.8, the semilinear Adams operation Ψ^p on $\widehat{\mathrm{qDR}}_p(A/R)$ described in Definition 1.8 corresponds to the operation on $\Omega_{A/R}^*[q-1]$ given by setting*

$$\Psi^p(\mathrm{ad}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_m}) := \Psi^p(a)x_{i_1}^{p-1} \cdots x_{i_m}^{p-1} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_m}.$$

for $a \in A[q-1]$.

Proof Just observe that this expression defines a chain map on $(\Omega_{A/R}^*[q-1], (q-1)\nabla_q)$ (for instance $\Psi^p((q-1)\nabla_q x_i) = (q^p-1)\Psi^p(\mathrm{d}x) = (q-1)\nabla_q x_i^p$), and that the quasi-isomorphisms in the proof of Theorem 1.23 commute with these operations. $\square$

As in [13, §4], we refer to formal schemes over $W^{(p)}(k)$ as smooth if they are flat deformations of smooth schemes over k . We refer to morphisms of such schemes as étale if they are flat deformations of étale morphisms over k .

Proposition 2.10 *Take a smooth formal scheme $\mathfrak{X}$ over $R = W^{(p)}(k)$ equipped with a lift Ψ^p of Frobenius which étale locally admits co-ordinates $\{x_i\}_i$ as above with $\Psi^p(x_i) = x_i^p$. Then there is a global quasi-isomorphism*

$$C_q^{-1} : (\Omega_{\mathfrak{X}/R}^*)^{\wedge p} [q-1] / [p]_q \rightarrow \left(L\eta_{(q-1)} \widehat{\mathrm{qDR}}_p(\mathcal{O}_{\mathfrak{X}}/R) \right)^{\wedge p} / [p]_q$$

in the derived category of étale sheaves on $\mathfrak{X}$.

Proof The unique lifting property of formally étale morphisms ensures that each affine formal scheme $\mathfrak{U}$ étale over $\mathfrak{X}$ has a unique lift $\Psi^p|_{\mathfrak{U}}$ of Frobenius compatible with the given operation Ψ^p on $\mathfrak{X}$. Functoriality of the construction qDR_p for rings with Frobenius lifts thus gives us an étale presheaf $\widehat{\mathrm{qDR}}_p(\mathcal{O}_{\mathfrak{X}}/R)^{\wedge p}$ of complexes on $\mathfrak{X}$. As in Definition 1.8, the Adams operation Ψ^p on $\mathcal{O}_{\mathfrak{X}}$ then extends to $(R[q-1], \Psi^p)$ -semilinear maps

$$\begin{aligned} \Psi^p : \mathrm{qDR}_p(\mathcal{O}_{\mathfrak{X}}/R)^{\wedge p} &\rightarrow \mathrm{qDR}_p(\mathcal{O}_{\mathfrak{X}}/R)^{\wedge p} \\ \mathrm{qDR}_p(\mathcal{O}_{\mathfrak{X}}/R)^{\wedge p} / (q-1) &\rightarrow \mathrm{qDR}_p(\mathcal{O}_{\mathfrak{X}}/R)^{\wedge p} / (q^p-1), \end{aligned}$$

and thus, denoting good truncation by τ ,

$$(q-1)^i \Psi^p : \tau^{\leq i}(\mathrm{qDR}_p(\mathcal{O}_{\mathfrak{X}}/R)^{\wedge p} / (q-1)) \rightarrow \left(L\eta_{(q-1)} \widehat{\mathrm{qDR}}_p(\mathcal{O}_{\mathfrak{X}}/R) \right)^{\wedge p} / [p]_q;$$

the left-hand side is quasi-isomorphic to $\bigoplus_{j \leq i} (\Omega_{\mathcal{O}_{\mathfrak{X}}/R}^j)^{\wedge p} [-j]$ by Theorem 1.23.

Extending the construction $R[q]$ -linearly and restricting to top summands therefore gives us the global map C_q^{-1} . For a local choice of framing, Lemma 2.9 gives equivalences

$$(q-1)^i \Psi^p \simeq \sum_{j \leq i} (q-1)^{i-j} (\tilde{C}^{-1})^j$$

for Scholze's locally defined lifts $(\tilde{C}^{-1})^j : (\Omega_{A/R}^j)^{\wedge p}[-j] \rightarrow (\widehat{q\text{-}\Omega}_{A/R, \square}^\bullet)^{\wedge p}/[p]_q$ of the Cartier quasi-isomorphism. The local calculation of [13, Proposition 3.4] then ensures that C_q^{-1} is a quasi-isomorphism. $\square$

3 Functoriality via analogues of de Rham–Witt cohomology

In order to obtain a cohomology theory for smooth commutative rings rather than for Λ_p -rings, we now consider q -analogues of de Rham–Witt cohomology. Our starting point is to observe that if we allow roots of q , we can extend the Jackson differential to fractional powers of x by the formula

$$\nabla_q(x^{m/n}) = \frac{q^{m/n} - 1}{q - 1} x^{m/n} d \log x,$$

where $d \log x = x^{-1} dx$, so terms such as $[n]_{q^{1/n}} x^{m/n}$ have integral derivative, where $[n]_{q^{1/n}} = \frac{q^{1/n} - 1}{q^{1/n} - 1}$.

3.1 Motivation

Definition 3.1 Given a Λ_p -ring B , define $\Psi^{1/p^\infty} B$ to be the smallest Λ_p -ring which is equipped with a morphism from B and for which the Adams operations are automorphisms.

In the case $P = \{p\}$, the Λ_p -ring $\Psi^{1/p^\infty} B$ is thus the colimit of the diagram

$$B \xrightarrow{\Psi^p} B \xrightarrow{\Psi^p} B \xrightarrow{\Psi^p} \dots$$

By Remark 2.7, $\widehat{q\text{DR}}_p(A/R)$ naturally underlies a cosimplicial Λ_p -ring, so applying Ψ^{1/p^∞} levelwise gives another cosimplicial Λ_p -ring. For the Adams operation Ψ^p of Definition 2.3, the underlying cochain complex is just $\Psi^{1/p^\infty} \widehat{q\text{DR}}_p(A/R) := \varinjlim_{\Psi^p} \widehat{q\text{DR}}_p(A/R)$. As an immediate consequence of Lemma 2.9, we have:

Lemma 3.2 *If R is a flat Λ_p -ring over $\mathbb{Z}_{(p)}$ with Ψ^p an isomorphism, then $\Psi^{1/p^\infty} \widehat{q\text{DR}}_p(R[x]/R)$ is quasi-isomorphic to the complex*

$$(R[x^{1/p^\infty}, q^{1/p^\infty}] \xrightarrow{(q-1)\nabla_q} (x^{1/p^\infty})R[x^{1/p^\infty}, q^{1/p^\infty}] d \log x)^{\wedge (q-1)},$$

so the décalage $L\eta_{(q-1)} \Psi^{1/p^\infty} \widehat{q\text{DR}}_p(R[x]/R)$ and the complex

$$\begin{aligned} \{a \in R[x^{1/p^\infty}, q^{1/p^\infty}] : \nabla_q a \in R[x^{1/p^\infty}, q^{1/p^\infty}] d \log x\} \\ \xrightarrow{\nabla_q} (x^{1/p^\infty})R[x^{1/p^\infty}, q^{1/p^\infty}] d \log x. \end{aligned}$$

are quasi-isomorphic after $(q - 1)$ -adic completion.

Thus in level 0 (resp. level 1), $L\eta_{(q-1)}\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}(R[x]/R)$ is spanned by elements of the form $[p^n]_{q^{1/p^n}}x^{m/p^n}$ (resp. $x^{m/p^n}d \log x$), so setting $q^{1/p^\infty} = 1$ gives a complex whose p -adic completion is the p -typical de Rham–Witt complex.

Lemma 3.3 *Let R and A be flat p -adically complete Λ_p -algebras over $\mathbb{Z}_p$, with Ψ^p an isomorphism on R . For elements x_i of rank 1, take a map $\square : R[x_1, \dots, x_d]^{1/p} \rightarrow A$ of Λ_p -rings which is a flat p -adic deformation of an étale map. Then the map*

$$\left(R[q^{1/p^\infty}] \otimes_{R[q]} L\eta_{(q-1)}\widehat{\mathrm{qDR}}_p(A/R) \right)^{\wedge p} \rightarrow L\eta_{(q-1)} \left(\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}_p(A/R) \right)^{\wedge p}$$

is a quasi-isomorphism.

Proof The map $\Psi^p : A \otimes_{R[x_1, \dots, x_d]} R[x_1^{1/p}, \dots, x_d^{1/p}] \rightarrow A$ becomes an isomorphism on p -adic completion, because $\square$ is flat and we have an isomorphism modulo p . Thus

$$\Psi^{1/p^\infty} A \cong A[x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}]^{\wedge p} := (A \otimes_{R[x_1, \dots, x_d]} R[x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}])^{\wedge p}$$

Combined with the calculation of Lemma 2.9, this gives us a quasi-isomorphism between $(\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}_p(A/R))^{\wedge p}$ and the $(p, q - 1)$ -adic completion of

$$\left(\bigoplus_I \bigoplus_\alpha A[[q - 1]]x_1^{\alpha_1} \dots x_d^{\alpha_d} dx^I [-|I|], (q - 1)\nabla_q \right),$$

where I ranges over finite subsets of $\{1, \dots, d\}$ and α ranges over elements of $p^{-\infty}\mathbb{Z}^d$ with $0 \leq \alpha_i < 1$ if $i \notin I$ and $-1 < \alpha_i \leq 0$ if $i \in I$.

We then observe that the contributions to the décalage $\eta_{(q-1)}$ from terms with $\alpha \neq 0$ must be acyclic, via a contracting homotopy defined by the restriction to $\eta_{(q-1)}$ of the q -integration map

$$fx_1^{\alpha_1} \dots x_d^{\alpha_d} dx^I \mapsto fx_1^{\alpha_1} \dots x_d^{\alpha_d} \sum_{i \in I} \pm x_i [\alpha_i]_q^{-1} dx^{(I \setminus i)},$$

where $[\frac{m}{p^n}]_q^{-1} = [m]_{q^{1/p^n}}^{-1} [p^n]_{q^{1/p^n}}$ for m coprime to p , noting that $[m]_{q^{1/p^n}}$ is a unit in $\mathbb{Z}[q^{1/p^\infty}]^{\wedge(p, q-1)}$. $\square$

Remark 3.4 The endomorphism given on $\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}_p(A/R)$ by

$$a \mapsto \Psi^{1/n}([n]_q a) = [n]_{q^{1/n}} \Psi^{1/n} a$$

descends to an endomorphism of $H^0(\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}_p(A/R)/(q - 1))$, which we may denote by V_n because it mimics Verschiebung in the sense that $\Psi^n V_n = n \cdot \mathrm{id}$ (since

$[n]_q \equiv n \pmod{(q-1)}$). For A smooth over $\mathbb{Z}$, we then have

$$\begin{aligned} H^0 \left(\Psi^{1/P^\infty} \widehat{qDR}_P(A/\mathbb{Z})/(q-1) \right) / (V_p : p \in P) &\cong A[q^{1/P^\infty}] / ([p]_{q^{1/p}} : p \in P) \\ &\cong A[\zeta_{P^\infty}], \end{aligned}$$

for ζ_n a primitive n th root of unity.

By adjunction, this gives an injective map

$$H^0 \left(\Psi^{1/P^\infty} \widehat{qDR}_P(A/\mathbb{Z})/(q-1) \right) \hookrightarrow W^{(P)} A[\zeta_{P^\infty}]$$

of Λ_P -rings, which becomes an isomorphism on completing $\Psi^{1/P^\infty} \widehat{qDR}(A/\mathbb{Z})$ with respect to the system $\{([n]_{q^{1/n}})\}_{n \in P^\infty}$ of ideals, where we write P^∞ for the set of integers whose prime factors are all in P . This implies that the cokernel is annihilated by all elements of $(q^{1/P^\infty} - 1)$, so leads us to consider almost mathematics as in [7].

3.2 Almost isomorphisms

From now on, we consider only the case $P = \{p\}$. Combined with Lemma 3.3, Remark 3.4 allows us to regard $L\eta_{(q-1)} \Psi^{1/p^\infty} \widehat{qDR}_p(A/\mathbb{Z}_p)^{\wedge p}$ as being almost a q^{1/p^∞} -analogue of p -typical de Rham–Witt cohomology.

The ideal $(q^{1/p^\infty} - 1)^{\wedge(p, q-1)} = \ker(\mathbb{Z}[q^{1/p^\infty}]^{\wedge(p, q-1)} \rightarrow \mathbb{Z}_p)$ is equal to the p -adic completion of its square, since we may write it as the kernel $W^{(p)}(\mathfrak{m})$ of $W^{(p)}(\mathbb{F}_p[q^{1/p^\infty}]^{\wedge(q-1)}) \rightarrow W^{(p)}(\mathbb{F}_p)$, for the idempotent maximal ideal $\mathfrak{m} = ((q-1)^{1/p^\infty})^{\wedge(q-1)}$ in $\mathbb{F}_p[q^{1/p^\infty}]^{\wedge(q-1)}$. If we set h^{1/p^n} to be the Teichmüller element

$$[q^{1/p^n} - 1] = \lim_{r \rightarrow \infty} (q^{1/p^{nr}} - 1)^{p^r} \in \mathbb{Z}[q^{1/p^\infty}]^{\wedge(p, q-1)},$$

then $W^{(p)}(\mathfrak{m}) = (h^{1/p^\infty})^{\wedge(p, h)}$. Although $W^{(p)}(\mathfrak{m})/p^n$ is not maximal in $\mathbb{Z}[h^{1/p^\infty}]^{\wedge(h)}/p^n$, it is idempotent and flat, so gives a basic setup in the sense of [7, 2.1.1]. We thus regard the pair $(\mathbb{Z}[q^{1/p^\infty}]^{\wedge(p, q-1)}, W^{(p)}(\mathfrak{m}))$ as an inverse system of basic setups for almost ring theory.

We then follow the terminology and notation of [7], studying p -adically complete $(\mathbb{Z}[q^{1/p^\infty}]^{\wedge(p, q-1)})^a$ -modules (almost $\mathbb{Z}[q^{1/p^\infty}]^{\wedge(p, q-1)}$ -modules) given by localising at almost isomorphisms, the maps whose kernel and cokernel are $W^{(p)}(\mathfrak{m})$ -torsion.

Definition 3.5 The obvious functor $(-)^a$ from modules to almost modules has a right adjoint $(-)_*$, given by $N_* := \operatorname{Hom}_{\mathbb{Z}[q^{1/p^\infty}]^{\wedge(p, q-1)}}(W^{(p)}(\mathfrak{m}), N)$, the module of almost elements.

Since the counit $(M_*)^a \rightarrow M$ of the adjunction is an (almost) isomorphism, we may also regard almost modules as a full subcategory of the category of modules, consisting of those M for which the natural map $M \rightarrow (M^a)_*$ is an isomorphism. We can define p -adically complete $(\mathbb{Z}[q^{1/p^\infty}]^{\wedge(p, q-1)})^a$ -algebras similarly, forming a full subcategory of $\mathbb{Z}[q^{1/p^\infty}]^{\wedge(p, q-1)}$ -algebras.

3.3 Perfectoid algebras

We now relate Scholze's perfectoid algebras to a class of Λ_p -rings, by factorising the tilting equivalence. For simplicity, we work over $\mathbb{Z}[\zeta_{p^\infty}]^{\wedge p}$, although Lemma 3.8 has natural analogues over the ring $K^o \subset K$ of power-bounded elements of any perfectoid field K in the sense of [12].

Definition 3.6 Define Fontaine's period ring functor $\mathcal{A}_{\text{inf}}$ from commutative rings to Λ_p -rings by $\mathcal{A}_{\text{inf}}(C) := \varprojlim_{\Psi^p} W^{(p)}(C)$.

Definition 3.7 Define a perfectoid Λ_p -ring to be a flat p -adically complete Λ_p -algebra over $\mathbb{Z}_p$, on which the Adams operation Ψ^p is an isomorphism.

By analogy with [2, Notation 1.4], we say that a perfectoid Λ_p -ring over $\mathbb{Z}[q^{1/p^\infty}]^{\wedge(p,q-1)}$ is integral if the morphism $B \rightarrow B_*$ of Definition 3.5 is an isomorphism.

Lemma 3.8 *We have equivalences of categories*

$$\begin{array}{ccc}
 \text{perfectoid almost } \mathbb{Z}[\zeta_{p^\infty}]^{\wedge p}\text{-algebras} & & \\
 \downarrow \scriptstyle -/[p]_{q^{1/p}} & \begin{array}{c} \uparrow \\ \downarrow \end{array} \scriptstyle \mathcal{A}_{\text{inf}}(-)_* & \\
 \text{integral perfectoid } \Lambda_p\text{-rings over } \mathbb{Z}[q^{1/p^\infty}]^{\wedge(p,q-1)} & & \\
 \downarrow \scriptstyle -/p & \begin{array}{c} \uparrow \\ \downarrow \end{array} \scriptstyle W^{(p)}(-)_* & \\
 \text{perfectoid almost } \mathbb{F}_p[q^{1/p^\infty}]^{\wedge(q-1)}\text{-algebras.} & &
 \end{array}$$

Proof A perfectoid Λ_p -ring B is a deformation of the perfect $\mathbb{F}_p$ -algebra B/p . As in [12, Proposition 5.13], a perfect $\mathbb{F}_p$ -algebra C has a unique deformation $W^{(p)}(C)$ over $\mathbb{Z}_p$, to which Frobenius must lift uniquely; this shows that $W^{(p)}$ gives an equivalence between perfect $\mathbb{F}_p$ -algebras and perfectoid Λ_p -rings. To obtain the bottom equivalence of the diagram, we will show that the functor $W^{(p)}$ commutes with the respective functors $C \mapsto C_*$ of almost elements, then appeal to the tilting equivalence.

Because the idempotent ideals of the basic setups in each of our three categories are generated by the rank 1 elements $h^{p^{-n}}$ constructed before Definition 3.5, we can write $C_* = \bigcap_n h^{-p^{-n}} C$ in each setting. For a Teichmüller element $[c] \in W^{(p)}(C)$, the standard isomorphism $W^{(p)}(C) \cong C^{\mathbb{N}_0}$ of sets gives an isomorphism $[c]W^{(p)}(C) \cong \prod_{m \geq 0} c^{p^m} C$. Thus the natural map $W^{(p)}(C)_* \rightarrow W^{(p)}(C_*)$ of Λ_p -rings is an isomorphism, since

$$W^{(p)}(C)_* \cong \bigcap_{n \geq 0} \prod_{m \geq 0} h^{-p^{m-n}} C \cong \prod_{m \geq 0} C_* \cong W^{(p)}(C_*),$$

and taking inverse limits with respect to Ψ^p gives $\mathcal{A}_{\text{inf}}(C)_* \cong \mathcal{A}_{\text{inf}}(C_*)$ as well.

Next, we observe that since $B := \mathcal{A}_{\text{inf}}(C)$ is a perfectoid Λ_p -ring for any flat p -adically complete $\mathbb{Z}_p$ -algebra C , we must have $B \cong W^{(p)}(B/p)$. Comparing rank

1 elements then gives a monoid isomorphism $(B/p) \cong \varprojlim_{x \mapsto x^p} C$, from which it follows that

$$\mathbb{F}_p \otimes_{\mathbb{Z}_p} \mathcal{A}_{\text{inf}}(C) \cong \varprojlim_{\Phi} (C/p) = C^b$$

whenever C is perfectoid. Since tilting gives an equivalence of almost algebras by [12, Theorem 5.2], this completes the proof. $\square$

3.4 Functoriality of q -de Rham cohomology

Since $(\Psi^{1/p^\infty} \widehat{\text{qDR}}_p(A/\mathbb{Z}_p))^{\wedge p}$ is represented by a cosimplicial perfectoid Λ_p -ring over $\mathbb{Z}[q^{1/p^\infty}]^{\wedge(p, q-1)}$ for any flat Λ_p -ring A over $\mathbb{Z}_p$, it corresponds under Lemma 3.8 to a cosimplicial perfectoid $(\mathbb{Z}[\zeta_{p^\infty}]^{\wedge p})^a$ -algebra, representing the following functor:

Lemma 3.9 *For a perfectoid $(\mathbb{Z}[\zeta_{p^\infty}]^{\wedge p})^a$ -algebra C , and a Λ_p -ring A over $\mathbb{Z}_p$ with $X = \text{Spec } A$, there is a canonical isomorphism*

$$X_{\text{strat}}^{q,p}(\mathcal{A}_{\text{inf}}(C)_*) \cong \text{Im} \left(\varprojlim_{\Psi^p} X(C_*) \rightarrow X(C_*) \right),$$

for the ring C_* of almost elements.

Proof By definition, $X_{\text{strat}}^{q,p}(\mathcal{A}_{\text{inf}}(C)_*)$ is the image of

$$\text{Hom}_{\Lambda_p}(A, \mathcal{A}_{\text{inf}}(C)_*) \rightarrow \text{Hom}_{\Lambda_p}(A, (\mathcal{A}_{\text{inf}}(C)_*)/(q-1)).$$

Since right adjoints commute with limits and $\mathcal{A}_{\text{inf}} = \varprojlim_{\Psi^p} W^{(p)}$, we may rewrite the first term as $\varprojlim_{\Psi^p} \text{Hom}_{\Lambda_p}(A, W^{(p)}(C_*)) = \varprojlim_{\Psi^p} X(C_*)$.

Setting $B := \varprojlim_{\Psi^p} W^{(p)}(C)_*$, observe that because $[p^n]_{q^{1/p^n}}(q^{1/p^n} - 1) = (q-1)$, we have $\bigcap_n [p^n]_{q^{1/p^n}} B = (q-1)B$, any element on the left defining an almost element of $(q-1)B$, hence a genuine element since $B = B_*$ is flat. Then note that since the projection map $\theta : B \rightarrow C_*$ has kernel $([p]_{q^{1/p}})$, the map $\theta \circ \Psi^{p^{n-1}}$ has kernel $([p]_{q^{1/p^n}})$, and so $B \rightarrow W^{(p)}(C)_*$ has kernel $\bigcap_n [p^n]_{q^{1/p^n}} B$. Thus

$$\text{Hom}_{\Lambda_p} \left(A, \left(\varprojlim_{\Psi^p} W^{(p)}(C)_* \right) / (q-1) \right) \hookrightarrow \text{Hom}_{\Lambda_p}(A, W^{(p)}(C)_*) = X(C_*).$$

$\square$

In fact, the tilting equivalence gives $\varprojlim_{\Psi^p} X(C_*) \cong X(C_*^b)$, so the only dependence of $X_{\text{strat}}^{q,p}(\mathcal{A}_{\text{inf}}(C)_*)$, and hence $((\Psi^{1/p^\infty} \widehat{\text{qDR}}_p(A/\mathbb{Z}_p))^{\wedge p})^a$, on the Frobenius lift Ψ^p is in determining the image of $X(C_*^b) \rightarrow X(C_*)$ as C varies.

Although the map $X(C_*^b) \rightarrow X(C_*)$ is not surjective, it is almost so in a precise sense, which we now use to establish independence of Ψ^p , showing that, up to faithfully flat descent, $\widehat{\mathrm{qDR}}_p(A/\mathbb{Z}_p)^{\wedge p}/[p]_{q^{1/p}}$ is the best possible perfectoid approximation to $A[\zeta_{p^\infty}]^{\wedge p}$.

Definition 3.10 Given a functor X from $(\mathbb{Z}[\zeta_{p^\infty}]^{\wedge p})^a$ -algebras to sets and a functor $\mathcal{A}$ from perfectoid $(\mathbb{Z}[\zeta_{p^\infty}]^{\wedge p})^a$ -algebras to abelian groups, we write

$$\mathbf{R}\Gamma_{\mathrm{Pfd}}(X, \mathcal{A}) := \mathbf{R}\mathrm{Hom}_{[\mathrm{Pfd}((\mathbb{Z}_p[\zeta_{p^\infty}]^{\wedge p})^a), \mathrm{Set}]}(X, \mathcal{A}),$$

where $\mathrm{Pfd}(S^a)$ denotes the category of perfectoid almost S -algebras, and $\mathbf{R}\mathrm{Hom}_{[\mathcal{C}, \mathrm{Set}]}(-, -)$ is as in Definition 1.9.

When X is representable by a $(\mathbb{Z}[\zeta_{p^\infty}]^{\wedge p})^a$ -algebra C , we simply denote $\mathbf{R}\Gamma_{\mathrm{Pfd}}(X, \mathcal{A})$ by $\mathbf{R}\Gamma_{\mathrm{Pfd}}(C, \mathcal{A})$ — when C is perfectoid, this will just be $\mathcal{A}(C)$.

Thus $\mathbf{R}\Gamma_{\mathrm{Pfd}}(C, \mathcal{A})$ is the homotopy limit of the functor $\mathcal{A}$ (regarded as taking values in cochain complexes) on the category of perfectoid $(\mathbb{Z}[\zeta_{p^\infty}]^{\wedge p})^a$ -algebras equipped with a map from C . This is closely related to the pushforward from the pro-étale site of the generic fibre, whose décalage for $\mathcal{A} = \mathcal{A}_{\mathrm{inf}}$ is the complex $A\Omega$ of [4, Definition 9.1].

Theorem 3.11 *If R is a p -adically complete Λ_p -ring over $\mathbb{Z}_p$, and A a formal R -deformation of a smooth ring over (R/p) , then the complex*

$$\mathbf{R}\Gamma_{\mathrm{Pfd}}((A[\zeta_{p^\infty}] \otimes_R \Psi^{1/p^\infty} R)^{\wedge p}, \mathcal{A}_{\mathrm{inf}})$$

of $(\Psi^{1/p^\infty} R[q])^{\wedge(p, q-1)}$ -modules is almost quasi-isomorphic to $(\Psi^{1/p^\infty} \widehat{\mathrm{qDR}}_p(A/R))^{\wedge p}$ for any Λ_p -ring structure on A coming from a framing over R as in Theorem 2.8.

Proof Since passage to almost modules is an exact functor, it follows from the definition of qDR_p that the cochain complex $((\Psi^{1/p^\infty} \widehat{\mathrm{qDR}}_p(A/R))^{\wedge p})^a$ is given by $\mathbf{R}\mathrm{Hom}_{[f\hat{\Lambda}_p(R[[q-1]]), \mathrm{Set}]}(X_{\mathrm{strat}}^{q,p}, ((\Psi^{1/p^\infty} \mathcal{O})^{\wedge p})^a)$ in the notation of Definition 1.9, where $f\hat{\Lambda}_p(R[[q-1]])$ denotes the category of flat $(p, q-1)$ -adically complete Λ_p -algebras over $R[[q-1]]$.

Now note that $C \mapsto ((\Psi^{1/p^\infty} C)^{\wedge p})_*$ is left adjoint to the inclusion functor $i : \mathrm{Pfd}\Lambda_p(R[[q-1]]) \rightarrow f\hat{\Lambda}_p(R[[q-1]])$ from the category of integral perfectoid Λ_p -rings over $\Psi^{1/p^\infty}(R[[q-1]])^{\wedge p}$. Thus $i^* : \mathrm{Ch}([f\hat{\Lambda}_p(R[[q-1]]), \mathrm{Ab}]) \rightarrow \mathrm{Ch}([\mathrm{Pfd}\Lambda_p(R[[q-1]]), \mathrm{Ab}])$ has exact right adjoint $\mathcal{F} \mapsto (\mathcal{F} \circ (\Psi^{1/p^\infty})^{\wedge p})_*$. We therefore have

$$\mathbf{R}\mathrm{Hom}_{[f\hat{\Lambda}_p(R[[q-1]]), \mathrm{Set}]}(X_{\mathrm{strat}}^{q,p}, ((\Psi^{1/p^\infty} \mathcal{O})^{\wedge p})^a) \simeq \mathbf{R}\mathrm{Hom}_{[\mathrm{Pfd}\Lambda_p(R[[q-1]]), \mathrm{Set}]}(i^* X_{\mathrm{strat}}^{q,p}, \mathcal{O}^a).$$

It thus follows that the cochain complex $((\Psi^{1/p^\infty} \widehat{\mathrm{qDR}}_p(A/R))^{\wedge p})^a$ is the homotopy limit of the functor $(B, x, y) \mapsto B^a$ on the category of triples (B, x, y) for integral perfectoid Λ_p -rings B over $\mathbb{Z}[q^{1/p^\infty}]^{\wedge(p, q-1)}$ and

$$(x, y) \in X_{\text{strat}}^{q,p}(B) \times_{Y_{\text{strat}}^{q,p}(B)} Y(B),$$

where $X = \text{Spec } A$ and $Y = \text{Spec } R$.

By Lemma 3.8, such Λ_p -rings B are uniquely of the form $\mathcal{A}_{\text{inf}}(C_*)$ for $C \in \text{Pfd}((\mathbb{Z}_p[\zeta_{p^\infty}]^{\wedge p})^a)$, so this homotopy limit becomes

$$\left(\left(\Psi^{1/p^\infty} \widehat{\text{qDR}}_p(A/R) \right)^{\wedge p} \right)^a \simeq \mathbf{R}\Gamma_{\text{Pfd}}((X_{\text{strat}}^{q,p} \times_{Y_{\text{strat}}^{q,p}} Y) \circ (\mathcal{A}_{\text{inf}})_*, (\mathcal{A}_{\text{inf}}))^a.$$

Writing $X^\infty(C) := \text{Im}(\varprojlim_{\Psi^p} X(C_*) \rightarrow X(C_*))$, Lemma 3.9 then combines with the description above to give

$$\begin{aligned} \left(\widehat{\text{qDR}}_p(A/R)^{\wedge p} \right)^a &\simeq \mathbf{R}\Gamma_{\text{Pfd}} \left(X^\infty \times_{Y^\infty} \varprojlim_{\Psi^p} Y, (\mathcal{A}_{\text{inf}}) \right)^a, \\ &\simeq \mathbf{R}\Gamma_{\text{Pfd}} \left(X^\infty \times_Y \varprojlim_{\Psi^p} Y, (\mathcal{A}_{\text{inf}}) \right)^a. \end{aligned}$$

We now introduce a Grothendieck topology on the category $[\text{Pfd}(\mathbb{Z}[\zeta_{p^\infty}]^{\wedge p})^a, \text{Set}]$ by taking covering morphisms to be those maps $C \rightarrow C'$ of perfectoid algebras which are almost faithfully flat modulo p . Since $C^\flat = \varprojlim_{\Phi} (C/p)$, the functor $\mathcal{A}_{\text{inf}}$ satisfies descent with respect to these coverings, so the map

$$\mathbf{R}\Gamma_{\text{Pfd}} \left(\left(X^\infty \times_Y \varprojlim_{\Psi^p} Y \right)^\sharp, \mathcal{A}_{\text{inf}} \right)^a \rightarrow \mathbf{R}\Gamma_{\text{Pfd}} \left(X^\infty \times_Y \varprojlim_{\Psi^p} Y, \mathcal{A}_{\text{inf}} \right)^a$$

is a quasi-isomorphism, where $(-)^{\sharp}$ denotes sheafification.

In other words, the calculation of $(\widehat{\text{qDR}}_p(A/R)^{\wedge p})^a$ is not affected if we tweak the definition of X^∞ by taking the image sheaf instead of the image presheaf. We then have

$$(X^\infty)^\sharp(C) = \bigcup_{C \rightarrow C'} \text{Im} \left(X(C_*) \times_{X(C'_*)} \varprojlim_{\Psi^p} X(C'_*) \rightarrow X(C_*) \right),$$

where $C \rightarrow C'$ runs over all covering morphisms.

Now, $\varprojlim_{\Psi^p} X$ is represented by the perfectoid algebra $(\Psi^{1/p^\infty} A)^{\wedge p}$, which is isomorphic to $A[x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}]^{\wedge p}$ as in the proof of Lemma 3.3. This allows us to appeal to André's results [1, §2.5] as generalised in [2, Theorem 2.3]. For any morphism $f : A \rightarrow C$, there exists a covering morphism $C \rightarrow C_i$ such that $f(x_i)$ has arbitrary p -power roots in C_i . Setting $C' := C_1 \otimes_C \dots \otimes_C C_d$, this means that the composite $A \xrightarrow{f} C \rightarrow C'$ extends to a map $(\Psi^{1/p^\infty} A)^{\wedge p} \rightarrow C'$, so $f \in (X^\infty)^\sharp(C)$. We have thus shown that $(X^\infty)^\sharp = X$, giving the required equivalence

$$\left(\left(\Psi^{1/p^\infty} \widehat{\mathrm{qDR}}_p(A/R) \right)^{\wedge p} \right)^a \simeq \mathbf{R}\Gamma_{\mathrm{Pfd}} \left(X \times_Y \varprojlim_{\Psi^p} Y, (\mathcal{A}_{\mathrm{inf}})_* \right)^a.$$

Finally, compatibility of these equivalences with the $(\Psi^{1/p^\infty} R[q])^{\wedge(p,q-1)}$ -module structures is given by functoriality, multiplicativity and the identification $(\Psi^{1/p^\infty} R[q])^{\wedge(p,q-1)} \simeq (\Psi^{1/p^\infty} \widehat{\mathrm{qDR}}_p(R/R))^{\wedge p}$. $\square$

Remark 3.12 Corresponding to the cohomology theory $((\Psi^{1/p^\infty} \widehat{\mathrm{qDR}}_p(A/R))^{\wedge p})^a$, it is natural to consider q -connections on finite projective modules M over

$$\begin{aligned} & \eta_{(q-1)}^0 \left(\left(\Psi^{1/p^\infty} \left(\Omega_{A/R}^* [q-1], (q-1)\nabla_q \right) \right)^{\wedge p, a} \right) \\ &= \left\{ a \in \left(\Psi^{1/p^\infty} (A[q-1]) \right)^{\wedge(p,q-1), a} : \nabla_q a \in (\Psi^{1/p^\infty} (\Omega_A^1 [q-1]))^{\wedge(p,q-1), a} \right\} \\ &= \left(\left(\sum_n [p^n]_{q^{1/p^n}} \Psi^{1/p^n} A[q^{1/p^\infty}] \right)^{\wedge(p,q-1)} \right)^a. \end{aligned}$$

It follows from the proof of Proposition 1.25 that these are equivalent, for $X = \mathrm{Spec} A$, to finite projective almost $(\Psi^{1/p^\infty} \mathcal{O}_{\hat{X}^q, \mathrm{strat}})^{\wedge p}$ -modules $\mathcal{N}$ for which $\mathcal{N}/(q-1)$ is the pullback of the almost $\mathrm{H}^0((\Psi^{1/p^\infty} \widehat{\mathrm{qDR}}_p(A/R))^{\wedge p}/(q-1))$ -module $\Gamma(\hat{X}_{\mathrm{strat}}^q, \mathcal{N}/(q-1)) =: M_0$.

Up to almost isomorphism, these correspond via the proof of Theorem 3.11 to those finite projective $\mathcal{A}_{\mathrm{inf}}$ -modules N on the site of integral perfectoid algebras C over $A[\zeta_{p^\infty}]^{\wedge p} \otimes_R \Psi^{1/p^\infty} R$ for which there exists a $W^{(p)}(A[\zeta_{p^\infty}]^{\wedge p})$ -module M_0 with $W^{(p)}(C)$ -linear isomorphisms

$$N(C) \otimes_{\mathcal{A}_{\mathrm{inf}}(C)} W^{(p)}(C) \cong M_0 \otimes_{W^{(p)}(A[\zeta_{p^\infty}]^{\wedge p})} W^{(p)}(C),$$

functorial in C .

This establishes a weakened form of [13, Conjecture 7.5] on co-ordinate independence of the category of q -connections, giving the statement for almost $(\sum_n [p^n]_{q^{1/p^n}} \Psi^{1/p^n} A[q^{1/p^\infty} - 1])^{\wedge p}$ -modules rather than $A[q-1]$ -modules.

The following gives a slight partial refinement of [4, Theorem 1.17]:

Corollary 3.13 *If R is a p -adically complete Λ_p -ring over $\mathbb{Z}_p$, and A a formal R -deformation of a smooth ring over (R/p) , then the q -de Rham cohomology complex $(q\text{-}\Omega_{A/R, \square}^\bullet \otimes_{R[q]} (\Psi^{1/p^\infty} R)[q^{1/p^\infty}])^{\wedge p}$ is, up to almost quasi-isomorphism, independent of a choice of co-ordinates $\square$. As such, it is naturally an invariant of the commutative p -adically complete $(\Psi^{1/p^\infty} R)[\zeta_{p^\infty}]^{\wedge p}$ -algebra $(A[\zeta_{p^\infty}] \otimes_R \Psi^{1/p^\infty} R)^{\wedge p}$.*

Proof Since

$$\Psi^{1/p^\infty} \mathrm{qDR}_p(A/R) = \Psi^{1/p^\infty} \mathrm{qDR}_p((A \otimes_R \Psi^{1/p^\infty} R)/\Psi^{1/p^\infty} R),$$

Theorem 2.8 combines with Lemma 3.3 to give

$$(q\text{-}\Omega_{A/R, \square}^\bullet \otimes_{R[q]} (\Psi^{1/p^\infty} R)[q^{1/p^\infty}])^{\wedge p} \simeq L\eta_{(q-1)} \left(\left(\Psi^{1/p^\infty} \widehat{\text{qDR}}_p(A/R) \right)^{\wedge p} \right),$$

and by Theorem 3.11, we know that this depends only on $(A[\zeta_{p^\infty}] \otimes_R \Psi^{1/p^\infty} R)^{\wedge p}$ up to almost quasi-isomorphism. $\square$

Remark 3.14 The almost quasi-isomorphism in Corollary 3.13 should be a genuine quasi-isomorphism when we impose some conditions on the base ring R . By [4, Lemma 8.11], it would suffice to verify that $H^*((\Psi^{1/p^\infty}(\Omega_{A/R}^*[q-1]), (q-1)\nabla_q))^{\wedge p}$ and its quotient by $(q-1)$ have no $(q^{1/p^\infty} - 1)$ -torsion, which should follow for R smooth by an argument similar to [4, Proposition 8.9].

Remark 3.15 (Eliminating roots of q) The key feature of the comparison results in this section is that, up to faithfully flat descent, the functor $X_{\text{strat}}^{q,p}$ does not depend on Adams operations when restricted to the category of integral perfectoid Λ_p -rings B over $\mathbb{Z}[q]$, since the proof of Theorem 3.11 gives $(X_{\text{strat}}^{q,p})^\sharp(B) \cong X(B/[p]_{q^{1/p}})$. We can extend the latter functor to more general Λ_p -rings over $\mathbb{Z}[q]$ by setting

$$X^{q,p}(B) := X(B/(\Psi^p)^{-1}([p]_q B)),$$

which does not depend on any Adams operations on X .

When $\mathcal{O}_X$ has a Λ_p -ring structure, there is then a natural map $\alpha : X_{\text{strat}}^{q,p} \rightarrow X^{q,p}$ because $\Psi^p((q-1)B) \subset [p]_q B$. This induces a transformation

$$\alpha^* : \mathbf{RHom}_{[f\hat{\Lambda}_p(R[[q-1]]], \text{Set}}(X^{q,p}, \mathcal{O}) \rightarrow \widehat{\text{qDR}}_p(A/R)^{\wedge p}$$

for $X = \text{Spec } A$. But for integral perfectoid Λ_p -rings B , we know that $X^{q,p}(B) = (X_{\text{strat}}^{q,p})^\sharp(B)$, so by adjunction, as in the proof of Theorem 3.11, α^* becomes an almost quasi-isomorphism on applying a form of completed stabilisation $\Psi^{1/p^\infty}(-)^{\wedge p}$. Thus $H^*(X^{q,p}, \mathcal{O})$ might be a candidate for the co-ordinate independent q -de Rham cohomology theory proposed in [13]. It naturally carries an Adams operation Ψ^p , which would correspond to the operation ϕ_p of [13, Conjecture 6.1].

Any $a \in A$ defines an element of $H^0(X^{q,p}, \mathcal{O}/(\Psi^p)^{-1}([p]_q \mathcal{O}))$ so $\Psi^p(a) \in H^0(X^{q,p}, \mathcal{O}/[p]_q)$ and applying the connecting homomorphism associated to $[p]_q : \mathcal{O} \rightarrow \mathcal{O}$ gives an element $\beta_{[p]_q} \Psi^p(a) \in H^1(X^{q,p}, \mathcal{O})$ whose image under $H^1(\alpha^*)$ is

$$[p]_q^{-1} \Psi^p((q-1)\nabla_q a) = (q-1)\Psi^p(\nabla_q a).$$

Moreover, to $a \in A$ we may associate elements $a_n \in H^0(X^{q,p}, \mathcal{O}/[p^n]_q)$ for $n \geq 1$, determined by the property that $a_n \equiv \Psi^{p^i} a^{p^{n-i}} \pmod{[p]_{q^{p^{i-1}}}}$ for $1 \leq i \leq n$, and these give rise to elements $\beta_{[p^n]_q} a_n \in H^1(X^{q,p}, \mathcal{O})$. Explicitly, if we define operations ε_i on $\mathcal{O}$ by $\varepsilon_0 = \text{id}$ and $\varepsilon_{i+1}(a) := (a^{p^{i+1}} - \Psi^p(a^{p^i}))/p^{i+1}$, then for a local lift $\tilde{a} \in \mathcal{O}$ of $a \in \mathcal{O}/(\Psi^p)^{-1}([p]_q)$, we have

$$a_n = \sum_{i=0}^{n-1} [p^i]_{q^{p^{n-i}}} \Psi^{p^{n-i}}(\varepsilon_i \tilde{a}) + [p^n]_q \mathcal{O},$$

so

$$H^1(\alpha^*)(\beta_{[p^n]_q} a_n) = (q-1) \sum_{i=0}^{n-1} \Psi^{p^{n-i}} (\nabla_q \varepsilon_i \tilde{a}).$$

In particular, for $A = R[x]$ these include all the elements $(q-1)[m]_{q^{p^n}} x^{p^n m-1} dx$, since $\varepsilon_i(x^m) = 0$ for all $i > 0$, x^m having rank 1. This suggests that in general the image of $H^1(\alpha^*)$ might be $(q-1)H^1 \widehat{qDR}_p(A/R)^{\wedge p}$, tying in well with $(q-1)$ -adic décalage. Explicit descriptions for much of the functoriality from Corollary 3.13 can also be inferred from this analysis, since it implies that the transformations $\sum_{i=0}^{n-1} \Psi^{p^{n-i}} \circ \nabla_q \circ \varepsilon_i : A \rightarrow H^1(q\text{-}\Omega_{A/R, \square}^\bullet)$ are all natural in A .

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References

1. André, Y.: La conjecture du facteur direct. *Publ. Math. Inst. Hautes Études Sci.* **127**, 71–93 (2018). [arXiv:1609.00345](#) [math.AG]
2. Bhatt, B.: On the direct summand conjecture and its derived variant. *Invent. Math.* **212**(2), 297–317 (2018). [arXiv:1608.08882](#) [math.AG]
3. Bhatt, B., de Jong, A.J.: Crystalline cohomology and de Rham cohomology. [arXiv:1110.5001](#) [math.AG] (2011)
4. Bhatt, B., Morrow, M., Scholze, P.: Integral p -adic Hodge theory. [arXiv:1602.03148](#) [math.NT] (2016)
5. Borger, J.: The basic geometry of Witt vectors, I: the affine case. *Algebra Number Theory* **5**(2), 231–285 (2011)
6. Bousfield, A.K., Kan, D.M.: *Homotopy Limits, Completions and Localizations*. Lecture Notes in Mathematics, vol. 304. Springer, Berlin (1972)
7. Gabber, O., Ramero, L.: *Almost Ring Theory*, Lecture Notes in Mathematics, vol. 1800. Springer, Berlin. [arXiv:math/0201175v3](#) [math.AG] (2003)
8. Grothendieck, A.: Crystals and the de Rham cohomology of schemes. In: *Dix Exposés sur la Cohomologie des Schémas*, pp. 306–358. North-Holland, Amsterdam (1968)
9. Hirschhorn, P.S.: *Model Categories and Their Localizations*, Mathematical Surveys and Monographs, vol. 99. American Mathematical Society, Providence (2003)
10. Joyal, A.: δ -anneaux et vecteurs de Witt. *C. R. Math. Rep. Acad. Sci. Canada* **7**(3), 177–182 (1985)
11. MacLane, S.: *Categories for the Working Mathematician*, Graduate Texts in Mathematics, vol. 5. Springer, New York (1971)
12. Scholze, P.: Perfectoid spaces. *Publ. Math. Inst. Hautes Études Sci.* **116**, 245–313 (2012)
13. Scholze, P.: Canonical q -deformations in arithmetic geometry. *Ann. Fac. Sci. Toulouse Math. (6)* **26**(5), 1163–1192 (2017). [arXiv:1606.01796v1](#) [math.AG]
14. Simpson, C.: Homotopy over the complex numbers and generalized de Rham cohomology. In: *Moduli of Vector Bundles* (Sanda, 1994; Kyoto, 1994), Lecture Notes in Pure and Applied Mathematics, vol. 179, pp. 229–263. Dekker, New York (1996)
15. Weibel, C.A.: *An Introduction to Homological Algebra*. Cambridge University Press, Cambridge (1994)
16. Wilkerson, C.: Lambda-rings, binomial domains, and vector bundles over $CP(\infty)$. *Commun. Algebra* **10**(3), 311–328 (1982)

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