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On K3 double planes covering Enriques surfaces

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Abstract

The moduli space of nodal Enriques surfaces is irreducible of dimension 9. A nodal Enriques surface is shown to be the quotient of a double cover of the plane by a lift of the Cremona involution. We also show that this gives a straightforward proof of the known description of the automorphism group for the generic such surface.

1 Introduction

The goal of this note is to give a simple geometric construction of nodal Enriques surfaces and to show that the known structure of the automorphism group for generic nodal Enriques surfaces can be deduced from it in a straightforward fashion. As a side result we show that these surfaces always have a Nikulin involution. We also determine the transcendental and Picard lattices in the generic nodal case.

1.1 A geometric construction

K3 and Enriques surfaces form a well studied class of compact complex surfaces.[1] Let us recall that a K3 surface is a simply connected surface with trivial canonical bundle, while an Enriques surface has a K3 surface as a double covering.[2] The corresponding involution is called an *Enriques involution*.

It is well known (see e.g., [2] for the complex case) that a general Enriques surface does not contain smooth rational curves of self-intersection -2, also called *nodal*

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[1] In this note we stay entirely within the realm of *complex surfaces*.

[2] Enriques surfaces, contrary to K3 surfaces, are always projective.

To the memory of Wolf Barth.

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curves. Since the moduli space of Enriques surfaces has dimension 10, we expect that nodal Enriques surfaces have 9 moduli. From an abstract moduli point of view the notion of a *general nodal Enriques surface* can be made precise and this leads naturally to an irreducible 9-dimensional moduli space (see Sect. 2.4). The main goal of this paper is to give a simple geometric construction for the corresponding Enriques surfaces.

Although this can be extracted from the results in [6] (see, Remark 3.5), the simple direct construction we propose has not been observed before. Our main result is as follows. It uses the standard plane Cremona transformation c with points of indeterminacy p_0, p_1, p_2 and the blow up P of the plane in these points.

Theorem 1. *Let $X \to \mathbf{P}^2$ be a (minimal desingularization of a) double cover branched in a sextic curve which is invariant under c and which has only ordinary double points at the points p_0, p_1, p_2. Then X is a K3 surface and c lifts on X to a fixed point free involution with quotient a nodal Enriques surface. This construction depends on 9 effective parameters and its general member is a "general nodal Enriques surface".*

2. *A K3 surface X which is the double cover of a general nodal Enriques surface admits a second involution such that the resulting quotient of X is a degree 6 del Pezzo surface $P \subset \mathbf{P}^6$. The Enriques involution on X induces an involution on P, the projective plane $\mathbf{P}^2$ blown up in three general points p_0, p_1, p_2; this in turn is induced by the standard Cremona transformation c. The branch locus of the resulting cover $X \to \mathbf{P}^2$ is a sextic curve, invariant under c and having ordinary double points at the three points p_0, p_1, p_2.*

We call the double cover $X \to P$ a *del Pezzo double cover* and the corresponding involution a *del Pezzo involution*. There is a second way to lift the Cremona involution to the K3 double cover X and, surprisingly, this is a Nikulin involution. A *Nikulin involution* on a K3 surface is a symplectic involution, i.e., one leaving non-zero holomorphic two forms invariant. They were studied by Morrison [11] as a sequel to Nikulin's study of finite groups of automorphisms of K3 surfaces. Nikulin involutions are also central in the work of van Geemen and Sarti [9]. Every Nikulin involution has eight fixed points leading to a quotient surface with eight nodes; the blow-up of these nodes yields a K3 surface.

To give the proof of the above theorem, we recall in Sect. 2 the moduli description of K3 and Enriques surfaces based on [16]. It is relatively easy to show the first part of the above theorem. We do this next, in Sect. 3, cf. Theorems 3.4 and 3.6. The proof of part (2) – this is Theorem 4.8 – can only be given after considerable work based on a precise lattice theoretic description of the nodal classes on general nodal Enriques surfaces. We have gathered the relevant lattice theory in Appendix A. Although well known among experts, we found it useful to give a concise and self-contained exposition of these results. For technical details we refer to Nikulin's investigations [12,14] and the monographs [6,7].

1.2 Automorphisms

For the generic Enriques surface without nodal curves, the automorphism group is known to be equal to the subgroup of isometries of the Enriques lattice $M = H^2(Y, \mathbf{Z})/(\text{torsion})$ which induce the identity on $M/2M$ and which, in addition, preserve each of the two connected components of the "light cone" $\{x \in M \otimes \mathbf{R} \mid x \cdot x > 0\}$. Here the "dot" stands for the cup-product pairing. For the complex case see [2], while one should consult [5–7] for positive characteristic. Note that this group is infinite.

In the latter works, one can also find a characteristic free description of the automorphism group of the generic nodal Enriques surface. However, in positive characteristic one cannot use the period map for the covering K3-surface, making these results less explicit. For instance, our definition of a "generic nodal" Enriques surface is very explicit in terms of the periods of the universal cover. Secondly, without using transcendental methods it is harder to see when the representation of the automorphism group on the Néron-Severi lattice is faithful. Our precise genericity assumption ensures this faithfulness. In the nodal situation the main result of [5] states that the image of the automorphism group in the isometry group of the Néron-Severi lattice is always contained in the group made explicit in Corollary 5.8 and generically coincides with it. The original proof [5] of this result used computer calculations. Allcock [1] found a simpler group-theoretic proof.

Our geometric construction also gives a relatively easy way to determine the automorphism group of the generic[3] nodal Enriques surface. See Corollaries 5.6, 5.8 for precise statements. Also in this situation the automorphism group is infinite. See Remark 5.9. Our simplification uses in a crucial way the explicit description of the nodal curves on a generic nodal Enriques surface which we deduce from the geometric model we use.

1.3 Notational conventions

A *lattice* is a free $\mathbf{Z}$-module of finite rank equipped with a symmetric bilinear integral pairing denoted by a dot. If L is a lattice, and F the field $\mathbf{Q}$, $\mathbf{R}$ or $\mathbf{C}$, the vector space $L_F = L \otimes F$ inherits this product, likewise denoted by a dot. A sublattice $M \subset L$ is called *primitive* if L/M is torsion free. A non-zero element $x \in L$ is primitive if the lattice it generates is primitive, that is, x is not a non-trivial multiple of some $y \in L$.

The *dual* L^* of the lattice L consists of the group of $y \in L_{\mathbf{Q}} = L \otimes \mathbf{Q}$ for which all products $y \cdot x$, $x \in L$ are integral. Obviously $L \subset L^*$ is a subgroup and the *discriminant group* of L is the abelian group

$$A_L = L^*/L.$$

For a non-degenerate *even* lattice L, the pairing on L induces a $\mathbf{Q}/2\mathbf{Z}$-valued quadratic form δ_L on A_L, the *discriminant form*. The pair (A_L, δ_L) is called the *discriminant*,

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[3] "Generic" has the precise meaning given in Proposition 5.5. In the moduli space of the general nodal Enriques surfaces these belong to the complement of countably many subvarieties.

but we shall be sloppy in that we use this terminology for the group A_L as well a for the form δ_L.

We shall use the following standard lattices:

- the lattice $\langle n \rangle$ which is $\mathbf{Z}$ equipped with the quadratic form $x \mapsto nx^2$;
- the hyperbolic lattice $U = \mathbf{Z}f \oplus \mathbf{Z}g$, $f^2 = g^2 = 0$, $f \cdot g = 1$;
- the root lattices E_n (see, Sect. A.1).

Furthermore, we use the following conventions:

- If L is a lattice, $nL \subset L$ is the sublattice consisting of elements nx, $x \in L$, while $L(n)$ denotes the lattice with a new product given by n times the product on L.
- The orthogonal direct sum of lattices is denoted by $\perp$.
- The group of isometries of a lattice L is denoted by $O(L)$.
- If $S \subset L_\mathbf{C}$, its isotropy group within the isometry group is denoted

$$O(G)_S = \{g \in O(G) \mid g_\mathbf{C}(S) = S\}.$$

- A *root* is an element $\mathbf{r} \in L$ with $\mathbf{r}^2 = -2$ defining a *reflection*

$$s_\mathbf{r} : L \to L, \quad s_\mathbf{r}(x) = x + (x \cdot \mathbf{r})\mathbf{r} \in O(L).$$

- $\Delta(L) = \{\mathbf{r} \in L \mid \mathbf{r}^2 = -2\}$, the collection of roots in L.
- L_{root} = lattice generated by $\Delta(L)$.
- $W(L)$ = {the group generated by the reflections}, *the Weyl group* of L. This is a normal subgroup of $O(L)$.
- If M is *hyperbolic*, that is, has signature $(1, n)$, the "light cone" $\{x \in M \otimes \mathbf{R} \mid x^2 > 0\}$ has two connected components. Choose one and call it C_M. Then $-C_M$ is the other component. We set

$$O^\uparrow(M) = \{g \in O(M) \mid gC_M = C_M\},$$

a group of index 2 in $O(M)$. The Weyl group $W(M)$ is a normal subgroup of this subgroup (each reflection preserves the components of the light cone).
- $A(M) = O^\uparrow(M)/W(M)$.
- If G is a group of isometries of a lattice L, the automorphism induced by $g \in G$ on L/nL is denoted by g_n and we set

$$G(n) := \{g \in G \mid g_n = \text{id} : L/nL \to L/nL\}.$$

2 Preliminaries

2.1 K3 surfaces and Enriques surfaces

As recalled in the introduction, a *K3 surface* is a simply connected surface X whose canonical bundle K_X is trivial. A nowhere zero section of K_X defines a holomorphic 2–form ω_X, unique up to a multiplicative scalar. The intersection pairing on $H^2(X, \mathbf{Z})$

makes this free abelian group of rank 22 into a lattice whose signature is (3,19). It is unimodular and even, that is, for all $x \in H^2(X, \mathbf{Z})$ the intersection number $x^2 = x \cdot x$ is even. As recalled in Appendix A.4, this characterizes the lattice up to isometry as

$$L = U \perp (U \perp E_8(-1)) \perp (U \perp E_8(-1)), \tag{1}$$

which we call the *K3–lattice*. A choice of an isometry $\varphi : H^2(X, \mathbf{Z}) \xrightarrow{\sim} L$ is called a *marking* for X.

We employ the following notation:

$$S_X \subset H^2(X, \mathbf{Z}) : \text{ the } \textit{Picard lattice} \text{ or } \textit{Néron- Severi lattice}$$
$$T_X = S_X^{\perp}, \textit{ the transcendental lattice.}$$

The transcendental lattice can be characterized as the smallest sublattice T of $H^2(X, \mathbf{Z})$ such that $T_{\mathbf{C}}$ contains $\mathbf{C}\omega_X$. When a marking is given, we frequently confuse S_X and T_X with their images in L.

An *Enriques surface* Y is a surface with a K3 surface X as an unramified double cover $u : X \to Y$. The corresponding involution ι_{Enr} is called an *Enriques involution* and one has $Y = X/\langle \iota_{\mathrm{Enr}} \rangle$. The morphism u induces an isomorphism

$$u^* : H^2(Y, \mathbf{Z})/\text{torsion} \xrightarrow{\sim} H^2(X, \mathbf{Z})^{\iota_{\mathrm{Enr}}=\mathrm{id}},$$

which is not an isometry; one has $u^*(x) \cdot u^*(y) = 2x \cdot y$. This can be described abstractly by introducing the *Enriques involution* on the level of lattices:

$$\sigma (v, (u, e), (u', e')) = (-v, (u', e'), (u, e)), \tag{2}$$

One can always choose a marking $\varphi : H^2(X, \mathbf{Z}) \xrightarrow{\sim} L$ such that ι_{Enr}^* corresponds to σ (see, e.g., [3, Lemma 19.1]). Such a marking is called a *marking for the Enriques surface Y* or for the pair $(X, \iota_{\mathrm{Enr}})$. It follows that such a marking induces an isometry

$$H^2(Y, \mathbf{Z})/\text{torsion} \simeq L^{\sigma=\mathrm{id}}(^1/_2) = M,$$

where

$$M = U \perp E_8(-1). \tag{3}$$

is called the *Enriques lattice*. All classes in this lattice correspond to classes of divisors.

2.2 Involutions on K3 surfaces and on the K3 lattice

Forgetting the lattice structure, an involution τ on a free finite rank $\mathbf{Z}$-module K can be characterized by its *involution invariant*, the triple of non-negative integers $\vec{t}(\tau) = (t_+, t_-, t_{\mathrm{sw}})$ defined by the ranks of the three submodules in any splitting $K = K_+ \oplus K_- \oplus K_{\mathrm{sw}}$ for which $s|K_{\pm} = \mathrm{id}_{\pm}$ and $K_{\mathrm{sw}} = U^{\oplus t}$ where U has rank 2 with basis $\{u, \tau(u)\}$. This triple $\vec{t}(\tau)$ is indeed independent of how the splitting is achieved [17]. We determine these invariants for the following three involutions on L:

1. The Enriques involution; it clearly has invariant $(0, 2, 10)$.
2. The *Nikulin involution* on lattice level, defined as

$$\nu(v, u, e, u', e') = (v, u, e', u', e). \tag{4}$$

   We have $\vec{t}(\nu) = (6, 0, 8)$.
3. The *del Pezzo involution* given as follows. Fix a root $\mathbf{r} \in E_8(-1) \subset L$, the first copy of $E_8(-1)$, set

$$\mu(v, u, e, u', e') = (-v, u', -s_{\mathbf{r}}e, u, -s_{\sigma(\mathbf{r})}e'). $$

   This involution has invariant $(0, 14, 4)$. Indeed, it preserves each of the two summands $E_8(-1)$ and on such a summand the reflection $s_{\mathbf{r}}$ interchanges two vectors x and $x - \mathbf{r}$ where x is any vector such that $x \cdot \mathbf{r} = -1$, while it leaves the orthogonal complement of the plane spanned by these two vectors invariant. Hence $-s_{\mathbf{r}}$ as well as $-s_{\sigma(\mathbf{r})}$ has invariants $(0, 6, 1)$ and μ has the stated invariants.

For later use we observe that

$$\sigma \circ \mu = \mu \circ \sigma = s'_{\sigma(\mathbf{r})} \circ s'_{\mathbf{r}} \circ \nu, \tag{5}$$

where $s'_{\mathbf{r}} = -s_{\mathbf{r}}$ on the first copy of $E_8(-1)$ and the identity on its orthogonal complement, and likewise for $s'_{\sigma(\mathbf{r})}$.

The above lattice type involutions come from geometry. The "geometric" Enriques involution, as we have seen, is fixed point free. Nikulin involutions on a marked K3 surface inducing the one on L have 8 isolated fixed points and the quotients, after blowing up the fixed points, are K3 surfaces again. See [13], [11] and [9, Sect. 1.2]. The geometry of the del Pezzo involution will be explained in Sect. 3.2.

Remark 2.1 Involutions on K3 surfaces have been classified [14, Sect. 4], [20]. Apart from the Enriques and the Nikulin type involutions, all other involutions have rational surfaces as quotient. A double cover of the projective plane ramified in a smooth sextic curve is an example of the last type of involution.

2.3 Moduli of K3 and Enriques surfaces

Let X be a K3 surface and ω_X a non-vanishing holomorphic 2-form on it. A marking $\varphi : H^2(X, \mathbf{Z}) \to L$ induces an isometry $\varphi_{\mathbf{C}} : H^2(X, \mathbf{C}) \xrightarrow{\sim} L_{\mathbf{C}}$. The line $[\varphi(\omega_X)] \in \mathbf{P}(L_{\mathbf{C}})$ defines the *period point* of (X, φ); it belongs to the period domain

$$\widetilde{D} := \{[\omega] \in \mathbf{P}(L \otimes \mathbf{C}) \mid \omega \cdot \omega = 0, \quad \omega \cdot \bar{\omega} > 0\}.$$

The group $O(L)$ acts on $\widetilde{D}$.

Moduli of Enriques surfaces $Y = X/\langle \iota_{\mathrm{Enr}} \rangle$ can be described by means of the K3-surface X as follows. Choose a marking $\varphi : H^2(X, \mathbf{Z}) \xrightarrow{\sim} L$ for $(H^2(X, \mathbf{Z}), \iota_{\mathrm{Enr}})$ and set

$$L^+ = L^{(\sigma=\text{id})} = M(2) = U(2) \perp E_8(-2) \tag{6}$$
$$L^- = L^{(\sigma=-\text{id})} = U \perp M(2). \tag{7}$$

Since $\iota^*_{\text{Enr}}\omega_X = -\omega_X$, the period point of (X, φ) then belongs to $\widetilde{D} \cap \mathbf{P}(L^-_\mathbf{C})$. In fact it belongs to an open subset thereof obtained by deleting the points on hyperplanes $H_\mathbf{r} \subset \mathbf{P}(L^-_\mathbf{C})$, orthogonal to roots $\mathbf{r} \in L^-$:

$$D = \widetilde{D} \cap \left(\mathbf{P}(L^-_\mathbf{C}) - \bigcup_{\mathbf{r}\in\Delta(L^-)} H_\mathbf{r} \right) \tag{8}$$

The group

$$O(L, \sigma) := \{g \in O(L) \mid g\sigma = \sigma g\}$$

acts on D in a discrete fashion and the quotient

$$\mathcal{M} := O(L, \sigma)\backslash D \tag{9}$$

is the moduli space of Enriques surfaces. Note that $\dim \mathcal{M} = 10$. The class of $[\omega_X]$ in it depends on Y and not on the chosen marking. It is called the *moduli point* of Y.

2.4 Moduli of nodal Enriques surfaces

Let $Y = X/\iota_{\text{Enr}}$ be an Enriques surface. A nodal curve C on Y corresponds to a pair of disjoint nodal curves $\{B, \iota_{\text{Enr}}(B)\}$ on X. Let us see how this is reflected on the level of lattices. For clarity, we write the K3-lattice as

$$L = U \perp M' \perp \sigma(M'),$$

where $M' \simeq M$ is the first sublattice between parentheses in (1). Via the marking, the homomorphism $u^* : H^2(Y) \rightarrow H^2(X)$ corresponds to the homomorphism

$$M \rightarrow U \perp M' \perp \sigma(M')$$
$$x \mapsto (0, x, \sigma(x)).$$

In particular, if $\mathbf{r}^+ \in M$ is the class of a nodal curve on Y, we see that $u^*(\mathbf{r}^+) = \mathbf{r}+\sigma(\mathbf{r}) \in M' \perp \sigma(M')$. This leads to the following terminology for the corresponding roots in L.

Definition 2.2 A pair of roots $\{\mathbf{r}, \sigma(\mathbf{r})\}$ in L for which $\mathbf{r} \cdot \sigma(\mathbf{r}) = 0$ is called a σ-*adapted pair of roots*. If $\mathbf{r} + \sigma(\mathbf{r}) \in M' \perp \sigma(M')$, then $\{\mathbf{r}, \sigma(\mathbf{r})\}$ is called a *strongly σ-adapted pair of roots*.

The corresponding two nodal hyperplanes $H_{\mathbf{r}} = \mathbf{r}^{\perp}$ and $H_{\sigma(\mathbf{r})} = \sigma(\mathbf{r})^{\perp}$ meet $\mathbf{P}(L^{-})$ in the same hyperplane which we denote

$$H_c \subset \mathbf{P}(L^{-}), \ c = \mathbf{r} - \sigma(\mathbf{r}).$$

We set

$$D_c = D \cap H_c$$
$$\mathcal{M}_c = O(L, \sigma)_c \backslash D_c.$$

Remark 2.3 Observe that c can be written in many ways as a difference $c = \mathbf{s} - \sigma(\mathbf{s})$ of a pair of σ-adapted roots which are not necessarily strongly σ-adapted. For an Enriques surface $Y = X/\iota_{\mathrm{Enr}}$ with period point on D_c such pairs $\{\mathbf{s}, \sigma(\mathbf{s})\}$ consist of algebraic classes on X, but they are not necessarily nodal classes. The classes $\mathbf{s} + \sigma(\mathbf{s})$ correspond to roots in the Enriques lattice. In fact, below, in Sect. 4.2 we determine these roots.

By Corollary A.9 the group $O(L, \sigma)$ acts transitively on *strongly* σ-adapted pairs of roots and so the isomorphism class of $\mathcal{M}_c$ does not depend on c. More precisely, if $c' = \gamma(c)$ for some $\gamma \in O(L, \sigma)$, we have $\gamma(D_c) = D_{c'}$ and hence γ induces an isomorphism between $\mathcal{M}_c$ and $\mathcal{M}_{c'}$. The isomorphism class of this hypersurface is the moduli space of *"nodal" Enriques surfaces*. We denote it $\mathcal{M}_{\mathrm{root}}$.

On the other hand, suppose we have a strongly σ-adapted pair $\{\mathbf{s}, \sigma(\mathbf{s})\}$ of roots for which $c' = \mathbf{s} - \sigma(\mathbf{s})$ is different from c. If the period point of Y does not belong to the corresponding hyperplane $H_{c'}$, the class $\mathbf{s}$ is not an algebraic class on X and in particular does come from a nodal class on Y. Accordingly, we set:

$$D_c^o = D_c \ - \bigcup_{c' \neq c} H_{c'} \cap D_c.$$
$$\mathcal{M}_{\mathrm{root}}^{\mathrm{gen}} = \text{(the isomorphism class of) } O(L, \sigma)_c \backslash D_c^o.$$

Points in $\mathcal{M}_{\mathrm{root}}^{\mathrm{gen}}$ correspond to Enriques surfaces $Y = X/\iota_{\mathrm{Enr}}$ whose only nodal classes correspond to strongly σ-adapted pairs $\{\mathbf{r}, \sigma(\mathbf{r})\}$ of roots in the Picard lattice of X with $\mathbf{r} - \sigma(\mathbf{r}) = c$.

3 Del Pezzo double planes

3.1 The standard Cremona transformation and invariant sextics

Consider the projective plane $\mathbf{P}^2$ with homogeneous coordinates (x_0, x_1, x_2) and let $p_0 = (1 : 0 : 0)$, $p_1 = (0 : 1 : 0)$, $p_2 = (0 : 0 : 1)$. The *standard Cremona transformation* centered at p_0, p_1, p_2 is the birational involution given by

$$(x_0, x_1, x_2) \mapsto (x_1 x_2, x_0 x_2, x_0 x_1).$$

Let $\pi : P \to \mathbf{P}^2$ the plane blown up in the three points p_0, p_1, p_2 with exceptional curves E_i mapping to $p_i, i = 0, 1, 2$ and let L_i be the proper transform of the coordinate axis $\{x_i = 0\}$.

We now consider sextic curves C having an ordinary double point in each of the points p_0, p_1, p_2 and left invariant by the Cremona transformation. Such a sextic curve is given by a polynomial of the shape

$$
\begin{aligned}
\phi = {}& a_{222} x_0^2 x_1^2 x_2^2 \\
& + a_{420}(x_0^4 x_1^2 + x_1^2 x_2^4) + a_{240}(x_0^2 x_1^4 + x_0^2 x_2^4) + a_{042}(x_1^4 x_2^2 + x_0^4 x_2^2) \\
& + a_{330}(x_0^3 x_1^3 + x_0 x_1 x_2^4) + a_{303}(x_0^3 x_2^3 + x_0 x_1^4 x_2) + a_{033}(x_1^3 x_2^3 + x_0^4 x_1 x_2) \\
& + a_{213}(x_0^2 x_1 x_2^3 + x_0^2 x_1^3 x_2) + a_{123}(x_0 x_1^2 x_2^3 + x_0^3 x_1^2 x_2) \\
& + a_{132}(x_0 x_1^3 x_2^2 + x_0^3 x_1 x_2^2).
\end{aligned}
\tag{10}
$$

Lemma 3.1   (i) *For a general choice of the coefficients a_{ijk} in (10) the proper transform $\tilde{C}$ of the sextic $C = \{\phi = 0\}$ on P is smooth and is disjoint from the fixed point locus of the involution c.*
  (ii) *The family of sextics defined by (10) depends on nine effective parameters.*

Proof   (i) A generic member of the family (10) is a sextic which has three ordinary double points at the points p_0, p_1, p_2 and is smooth elsewhere. To see this it suffices to give one member of the family for which this is true. Such a member is the sextic with equation

$$
x_0^2 x_1^2 x_2^2 + (x_0^4 x_1^2 + x_1^2 x_2^4) + (x_0^2 x_1^4 + x_0^2 x_2^4) + (x_1^4 x_2^2 + x_0^4 x_2^2) = 0.
$$

This can be easily checked; for instance the coordinates of the four fixed points $(1 : 1 : 1)$, $(-1 : 1 : 1)$, $(1 : -1 : 1)$, $(1 : 1 : -1)$ of the Cremona involution don't satisfy the above equation.

(ii) We first show that if a projective transformation T preserves a generic member C of the family (10), $T^{12} = 1$ and so such T form a finite group G. Indeed, T permutes the singularities of C and hence T^6 fixes each of them. But then T^6 permutes each of the two tangents to C at p_j, $j = 0, 1, 2$ and so $(T^6)^2$ fixes each of these 6 lines and hence must be the identity. Next, observe that there are 10 coefficients in the equation (10) and hence the family depends on 9 parameters. Since G is finite, this shows that there are nine moduli.   □

The Cremona transformation induces a biregular involution $\iota_{\mathrm{Cr}} : P \to P$ with $\iota_{\mathrm{Cr}}(E_i) = L_i$, $i = 0, 1, 2$, the proper transform of the coordinate axis $\{x_i = 0\}$. The corresponding divisor classes in the Picard group will be denoted by the corresponding lower case letters. The class of the proper transform L of a line will be denoted by ℓ. We have:

Lemma 3.2   (i) $\iota_{\mathrm{Cr}}^* e_i = l_i = \ell - (e_0 + e_1 + e_2) + e_i$, $i = 0, 1, 2$ *and* $\iota_{\mathrm{Cr}}^* \ell = 2\ell - (e_0 + e_1 + e_2)$.
  (ii) ι_{Cr}^* *preserves the class* $-k_P = 3\ell - (e_0 + e_1 + e_2)$ *of the anti-canonical bundle; this class is (very) ample.*

Proof (i) The first three equalities are clear. The last equality follows since ι_{Cr} transforms a line into a conic through the three points p_0, p_1, p_2.

(ii) This follows from (i). For the ampleness of the anti-canonical class, see e.g., [7, Theorem 8.2.3]. □

3.2 Del Pezzo double covers admit Enriques involutions

The class of the proper transform in P of a sextic having double points in p_0, p_1, p_2 in the Picard group equals $6\ell - 2(e_0 + e_1 + e_2) = -2k_P$ and so is 2-divisible. We thus may form the double cover $q : X \to P$ of P branched in the curve $\tilde{C}$.

Definition 3.3 A double cover $q : X \to P$ of P branched in the proper transform $\tilde{C}$ of a sextic (10), is called a *del Pezzo double cover* and the corresponding involution ι_{delP}, a *del Pezzo involution*. If $\tilde{C}$ is smooth, we speak of a smooth del Pezzo double cover.

Theorem 3.4 *Let $q : X \to P$ be a smooth del Pezzo cover. The surface X is a K3 surface. The involution ι_{Cr} lifts to a fixed point free involution $\iota_{Enr} : X \to X$ commuting with ι_{delP}. The involution $\iota_{Nik} = \iota_{Enr} \circ \iota_{delP}$ is a Nikulin involution.*

Proof Set $\mathscr{F} = \mathbb{O}(-K_P)$, the bundle for which $\mathscr{F}^{\otimes 2} = \mathbb{O}_P(\tilde{C})$. Hence $K_X = q^*(K_P \otimes \mathscr{F}) = \mathbb{O}_X$. As a minimal resolution of a plane double cover, X is simply connected and so X is a K3 surface.

The surface X is canonically embedded in the total space F of $\mathscr{F}$ as the zero set of the section $p^* f - s^2$ where $p : F \to X$ is the bundle projection, f a section of $\mathscr{F}^{\otimes 2}$ vanishing along $\tilde{C}$ and s the tautological section of $p^*\mathscr{F}$. The involution ι_{Cr} acts naturally on $\mathscr{F}$ and so ι_{Cr} can be lifted to X in two ways. Since ι_{Cr} acts fixed point freely on $\tilde{C}$, the two involutions on X lifting ι_{Cr} act without fixed point on $B = q^{-1}\tilde{C}$. Over the 4 fixed points of ι_{Cr} one has 8 points in X all of which are either fixed or not fixed. In the first case the involution preserves the canonical two-form; in the second case it sends it to its negative. So either we get a Nikulin involution or a fixed point free involution, i.e., an Enriques involution. □

On X the inverse images $q^{-1}E_i$, $i = 0, 1, 2$ of the 3 exceptional curves E_i are nodal curves. Their classes in the Picard group will be denoted $\tilde{e}_i$. The inverse images $q^{-1}L_i$, $i = 0, 1, 2$ of the three rational curves L_i also are nodal curves with classes, say $\tilde{\ell}_i$. Then $\iota_{Enr}^*(\tilde{e}_i) = \tilde{\ell}_i$. Let $\tilde{\ell}$ be the class of $q^{-1}(L)$. The intersection behaviour of these classes is as follows:

$$\tilde{e}_i \cdot \tilde{e}_j = \tilde{\ell}_i \cdot \tilde{\ell}_j = \delta_{ij} \cdot (-2)$$
$$\tilde{e}_i \cdot \tilde{\ell}_j = (1 - \delta_{ij}) \cdot 2$$
$$\tilde{\ell} \cdot \tilde{\ell}_i = 2. \tag{11}$$

We say that the six divisors $\tilde{e}_i$, $\tilde{\ell}_j$ form a *doubly bonded hexagon* of nodal curves. Its classes are said to form a *doubly bonded hexagon of roots*. The intersection graph can indeed be depicted as

$$(12)$$
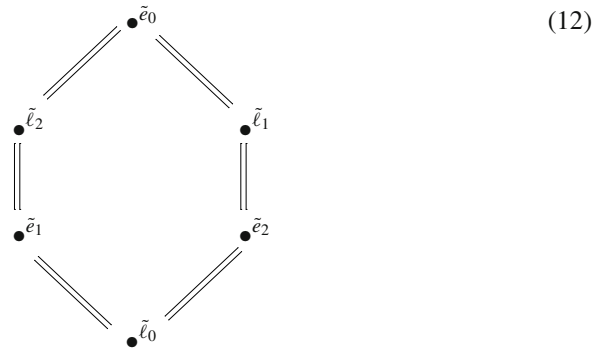
where the vertices represent classes of selfintersection -2 and where adjacent vertices have inner product 2. We also note that the sum of the divisors in the doubly bonded hexagon is ample, since we have:

$$-q^*k_P = 3\tilde{\ell} - \sum_{i=0}^{2} \tilde{e}_i = \sum_{i=0}^{2}\left(\tilde{\ell}_i + \tilde{e}_i\right) = \sum_{i=0}^{2}\left(\tilde{e}_i + \iota_{\text{Enr}}^* \tilde{e}_i\right). \qquad (13)$$

To motivate the terminology "del Pezzo involution" for ι_{delP} we explain the geometry behind this. The anticanonical system $|-K_P|$ corresponds to the six-dimensional system of cubics passing through p_0, p_1 and p_2 which gives an embedding $j : P \hookrightarrow \mathbf{P}^6$ as a surface of degree 6. This is the standard anticanonical embedding of the del Pezzo surface P. Note that, taking as a basis for $H^0(\mathbb{O}_P(-K_P))$ the representing cubics

$$c_0 = x_0 x_1 x_2 \; c_1 = x_0 x_1^2 \; c_2 = x_0 x_2^2 \; c_3 = x_1 x_0^2 \; c_4 = x_1 x_2^2 \; c_5 = x_2 x_0^2 \; c_6 = x_2 x_1^2,$$

the branch locus of q has equation

$$\begin{aligned}
\phi = \; & a_{222} c_0^2 \\
& + a_{420}(c_3^2 + c_4^2) + a_{240}(c_1^2 + c_2^2) + a_{042}(c_6^2 + c_5^2) \\
& + a_{330}(c_1 c_3 + c_2 c_4) + a_{303}(c_2 c_5 + c_1 c_6) + a_{033}(c_4 c_6 + c_3 c_5) \\
& + a_{213} c_0(c_1 + c_2) + a_{123} c_0(c_3 + c_4) + a_{132} c_0(c_5 + c_6).
\end{aligned} \qquad (14)$$

On the other hand, the system $|-q^*K_P|$ has dimension 7 since the corresponding ample line bundle $q^*\mathbb{O}_P(K_P^{-1})$ on the K3 surface X has degree 12 and Riemann–Roch and Kodaira vanishing then give that the dimension of its space of sections equals $6 + 2 = 8$. See e.g., [3, Ch. VIII.3]. The system $|-q^*K_P|$ embeds X in $\mathbf{P}^7$ as a degree 12 surface; the subsystem of divisors of the form q^*D, D a divisor of $|-K_P|$ defines a rational map $\tilde{q} : \mathbf{P}^7 \dashrightarrow \mathbf{P}^6$ which maps X in a 2-1 fashion onto $j(P) \subset \mathbf{P}^6$. Indeed, there exists $y \in H^0(\mathbb{O}_X(-q^*K_P))$ such that $y^2 = \phi$ with ϕ as in (14). If $\mathbf{P}^7$ has homogeneous coordinates $(c_0 : \cdots : c_6 : y)$, the del Pezzo involution is induced from $(c_0 : \cdots : c_6 : y) \mapsto (c_0 : \cdots : c_6 : -y)$ and the map $\tilde{q}$ which is projection

from the point $(0 : \cdots : 1)$ induces q. The Cremona involution is induced from the involution on $\mathbf{P}^6$ given by

$$\underline{c} = (c_0 : c_1 : c_2 : c_3 : c_4 : c_5 : c_6) \mapsto \iota_{\mathrm{Cr}}(\underline{c}) = (c_0 : c_2 : c_1 : c_4 : c_3 : c_6 : c_5)$$

which lifts to the involutions $(\underline{c} : y) \mapsto (\iota_{\mathrm{Cr}}(\underline{c}) : \pm y)$ on $\mathbf{P}^7$; these induce the Nikulin involution, respectively the Enriques involution. This is summarized by the following diagrams.
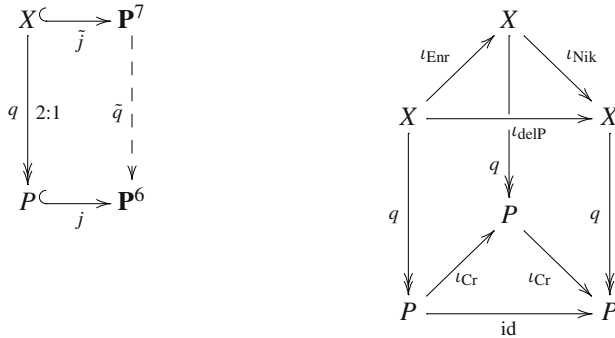
Remark 3.5 From this diagram it follows that the Enriques surface $Y = X/\iota_{\mathrm{Enr}}$ has an extra involution, say ι, induced by ι_{delP} (or ι_{Nik}) as depicted below.
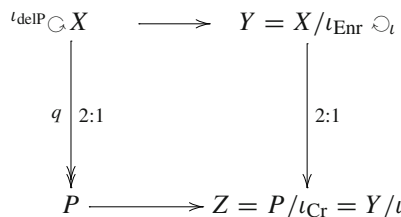
The quotient $Z = P/\iota_{\mathrm{Cr}}$ has 4 nodes since ι_{Cr} has 4 isolated fixed points on P. The linear forms $c_0, c_1+c_2, c_3+c_4, c_5+c_6$ on $\mathbf{P}^6$ are ι-invariant and the induced projection $\mathbf{P}^6 \ni (c_0 : c_1 : c_2 : c_3 : c_4 : c_5 : c_6) \to (c_0 : c_1+c_2 : c_3+c_4 : c_5+c_6) \in \mathbf{P}^3$ realizes the surface Z as the Cayley 4-nodal cubic surface. In [5] this induced involution ι is called the "Kantor involution". Clearly, from the Kantor involution one can obtain the del Pezzo involution, but in the literature we could not find an explicit construction of Enriques surfaces using the del Pezzo double cover.

3.3 Del Pezzo double covers give general nodal Enriques surfaces

Let $q : X \to P$ be a del Pezzo double cover. We ask for an explicit isometric embedding of $q^*S_P \subset S_X$, the sublattice of the Picard lattice of X coming from P, into the K3 lattice $L = U \perp (U \perp E_8(-1)) \perp (U \perp E_8(-1))$. To that end, let

$\{f, g\}$ be the standard basis of the second copy of U and $\mathbf{r}$ any root in the first copy of $E_8(-1)$. We claim:

Theorem 3.6 *Let $q : X \to P$ be a del Pezzo double cover with corresponding Enriques surface $Y = X/\iota_{\mathrm{Enr}}$. Then*

(i) *There is a marking for $(X, \iota_{\mathrm{Enr}})$, say $\varphi : H^2(X) \xrightarrow{\sim} L$ such that*

$$\begin{aligned}
\varphi(\tilde{e}_0) &= \mathbf{r}, \\
\varphi(\tilde{e}_1) &= g + \sigma(g) - \sigma(\mathbf{r}) \\
\varphi(\tilde{e}_2) &= f + \sigma(f) - \sigma(\mathbf{r}) \\
\varphi(\tilde{\ell}) &= f + g + \sigma(f + g) - \sigma(\mathbf{r}).
\end{aligned}$$

(ii) *The classes in the doubly bonded hexagon (11) are embedded in the lattice L in such a way that σ coincides with the action of ι_{Enr}^*. In particular*

$$\begin{aligned}
\iota_{\mathrm{Enr}}^* \tilde{e}_0 &= \tilde{\ell} - \tilde{e}_1 - \tilde{e}_2 \mapsto_\varphi \sigma(\mathbf{r}) \\
\iota_{\mathrm{Enr}}^* \tilde{e}_1 &= \tilde{\ell} - \tilde{e}_0 - \tilde{e}_2 \mapsto_\varphi f + \sigma(f) - \mathbf{r} \\
\iota_{\mathrm{Enr}}^* \tilde{e}_2 &= \tilde{\ell} - \tilde{e}_0 - \tilde{e}_1 \mapsto_\varphi g + \sigma(g) - \mathbf{r}.
\end{aligned}$$

(iii) *The class*

$$\begin{aligned}
-q^* k_P = 3\tilde{\ell} - \tilde{e}_0 - \tilde{e}_1 - \tilde{e}_2 &\mapsto_\varphi 2(f + g) + 2\sigma(f + g) - \mathbf{r} - \sigma(\mathbf{r}) \\
&= \varphi(\tilde{\ell}) + \sigma(\varphi(\tilde{\ell}))
\end{aligned}$$

is the class of an ample divisor.

(iv) *The period point of X belongs to the hyperplane H_c, where $c = \mathbf{r} - \sigma(\mathbf{r})$ and Y is a nodal Enriques surface. For a general choice of the curve (10) the surface Y is a general nodal Enriques surface, i.e., its moduli point belongs to $\mathcal{M}_{\mathrm{root}}^{\mathrm{gen}}$.*

Proof We first consider the sublattice of L^+ spanned by $f^+ = f + \sigma(f)$ and $g^+ = g + \sigma(g)$. It is isometric to $U(2)$, the first summand of $M(2) = U(2) \perp E_8(-2)$. The sublattice of $H^2(X, \mathbf{Z})^{\iota_{\mathrm{Enr}}^* = \mathrm{id}}$ spanned by $\tilde{\ell} - \tilde{e}_1$ and $\tilde{\ell} - \tilde{e}_2$ is likewise isometric to $U(2)$. The correspondence $\tilde{\ell} - \tilde{e}_1 \mapsto f^+, \tilde{\ell} - \tilde{e}_2 \mapsto g^+$ can be extended to give an isometry

$$\varphi^+ : H^2(X, \mathbf{Z})^{\iota_{\mathrm{Enr}}^* = \mathrm{id}} \xrightarrow{\sim} L^+$$

identifying the two summands $U(2)$. Moreover, since any two roots in $E_8(-1)$ are conjugate under an isometry (see, Appendix A.1), we can choose the isometry in such a way that $\tilde{e}_0 + \iota_{\mathrm{Enr}}^*(\tilde{e}_0)$ maps to $\mathbf{r}^+ = \mathbf{r} + \sigma(\mathbf{r})$. We now use Cor. A.7 from the Appendix which states that there is a "matching isometry"

$$\varphi^- : H^2(X, \mathbf{Z})^{\iota_{\mathrm{Enr}}^* = -\mathrm{id}} \xrightarrow{\sim} L^- = U \perp L^+$$

such that the pair (φ^+, φ^-) extends to an isometry $\varphi : H^2(X, \mathbf{Z}) \xrightarrow{\sim} L$ compatible with the Enriques involution. By definition, $\varphi^-(\tilde{e}_0 - \iota^*_{\text{Enr}}(\tilde{e}_0)) = \mathbf{r} - \sigma(\mathbf{r})$ and the required properties for φ are a consequence.

(iii) Lemma 3.2 implies that the class $-q^* k_P$ is ample.

The first assertion in (iv) is a consequence of the existence of this particular marking. Since by Lemma 3.1 the family we constructed has nine effective parameters, and $\mathcal{M}_c$ has dimension 9, the general member of the family is a general nodal Enriques surface.   □

3.4 Comparison of involutions

The three involutions on a del Pezzo double plane, the del Pezzo involution, the Enriques involution and their composition, the Nikulin involution can be compared to their lattice theoretic counterparts.

Corollary 3.7 *Let $q : X \to P$ be a smooth del Pezzo double plane with corresponding del Pezzo involution ι_{delP}, Enriques involution ι_{Enr} and Nikulin involution $\iota_{\text{Nik}} = \iota_{\text{delP}} \circ \iota_{\text{Enr}}$. The marking φ of Theorem 3.6 intertwines ι^*_{delP}, respectively ι^*_{Enr} and the lattice variants μ, σ of the del Pezzo and Enriques involutions, while the marking $s'_{\sigma(\mathbf{r})} \circ s'_{\mathbf{r}} \circ \varphi$ intertwines ι^*_{Nik} and ν.*

Proof The marking intertwines ι^*_{Enr} and σ by construction. Observe that the classes in S_X invariant under ι^*_{delP} under the marking all belong to the primitive lattice generated by $\mathbf{r}, \sigma(\mathbf{r})$, $f + \sigma(f)$ and $g + \sigma(g)$. This implies first of all that, as for μ, the two copies of U are switched (note that $f - \sigma(f)$ and $g - \sigma(g)$ are mapped to their opposites). In the first copy of $E_8(-1)$ the involution ι^*_{delP} must act as $-s_{\mathbf{r}}$ (with only $\mathbf{Z}\mathbf{r}$ as invariant sublattice) and, similarly, in the second copy ι^*_{delP} must act as $-s_{\sigma(\mathbf{r})}$. This implies that the marking indeed intertwines ι^*_{delP} and μ. Since $\iota_{\text{Nik}} = \iota_{\text{delP}} \circ \iota_{\text{Enr}}$, by Eq. (5), the marking $s'_{\sigma(\mathbf{r})} \circ s'_{\mathbf{r}} \circ \varphi$ then automatically intertwines ν and ι^*_{Nik}.   □

4 The K3 cover of a general nodal Enriques surface as a del Pezzo double plane

4.1 Nodal curves and the ample cone

Let (X, φ) be any marked K3 surface with period point $[\omega] \in L_{\mathbf{C}}$. Divisors on a K3 surface X give classes in S_X, the Néron–Severi lattice; effective or ample divisors give effective, respectively ample classes in S_X. By Riemann–Roch, if $D^2 \geq -2$ either D or $-D$ is effective and so in particular, a root $\mathbf{s} \in S_X$ is either effective or $-\mathbf{s}$ is effective, giving rise to the partitioning:

$$\Delta(S_X) = \Delta^+ \cup -\Delta^+, \quad \Delta^+ = \{\text{effective roots}\}.$$

Nodal classes are the effective roots defined by the (-2)-curves, these are precisely the indecomposable roots. The lattice S_X has signature $(1, s)$ and so we can apply

the considerations of Appendix A.5 to S_X. The cone $\{x \in S_X \otimes \mathbf{R} \mid x^2 > 0\}$ has two connected components, only one of which contains effective classes. It is called the *positive cone* C_X. The ample classes form a subcone which can be characterized lattice theoretically as a "fundamental chamber" with respect to the above partitioning of the roots. In this case the fundamental chamber is the *ample cone*,

$$C_X^{\text{amp}} = \{x \in C_X \mid x \cdot \mathbf{r} > 0 \text{ for all } \mathbf{r} \in \Delta^+\}.$$

The ample cone is completely determined by the nodal classes since these correspond to the indecomposable roots in Δ^+. The elements in $C(\Delta^+) \cap S_X$ are precisely the ample classes. Conversely, if $h \in S_X$ is an ample class, we can find back the effective roots

$$\Delta^+ = \{\mathbf{r} \in \Delta(S_X) \mid h \cdot \mathbf{r} > 0\},$$

and hence the ample cone.

On an Enriques surface $Y = X/\langle \iota_{\text{Enr}} \rangle$ all cohomology classes are classes of divisors:

$$S_Y = H^2(Y, \mathbf{Z})/\text{torsion}.$$

Any class $d \in S_Y$ with $d^2 \geq 0$ has the property that either d or $-d$ is effective. All effective such classes belong to the closure of the positive cone $C_Y \subset S_Y \otimes \mathbf{R}$. As on a K3 surface, on an Enriques surface an indecomposable root is the class of a (-2)-curve. However, if D is a divisor with $D^2 = -2$, it is not true in general that $\pm D$ is effective. Classes of nodal curves on Y correspond to pairs of disjoint nodal curves $\{C, \iota_{\text{Enr}}(C)\}$ on X. So, to study nodal curves, we have to get back to the associated K3 surface. Using this remark, one sees [2, Lemma 2.4] that the natural unramified double cover $u : X \to Y$ induces a bijection

$$C_Y^{\text{amp}} := \text{the ample cone in } S_Y \overset{u^*}{=} \{x \in C_X^{\text{amp}} \mid \iota_{\text{Enr}}^* x = x\}. \tag{15}$$

4.2 Nodal classes on general nodal Enriques surfaces

Now let $Y = X/\langle \iota_{\text{Enr}} \rangle$ be a general nodal Enriques surface with marking $\varphi : H^2(X, \mathbf{Z}) \overset{\sim}{\to} L$. The nodal classes on $Y = X/\iota_{\text{Enr}}$ correspond to strongly σ-adapted pairs of irreducible roots $\{\mathbf{s}, \sigma(\mathbf{s})\}$ in the lattice $\varphi(S_X) \subset L$ (cf. Definition 2.2).

Next, introduce

$$\Delta_- = \{r_- \in L^- \mid r_-^2 = -4, \exists r_+ \in L^+ \text{ with } r_+^2 = -4 \text{ and } r_- + r_+ \in 2L\} \tag{16}$$

and similarly for $\Delta_+ \subset L^+$. Referring to Remark 2.3, observe that writing $r_+ + r_- = 2\mathbf{r}$, we have that $\{\mathbf{r}, \sigma(\mathbf{r})\}$ is a σ-adapted pair of roots in the lattice L with $r_\pm = \mathbf{r} \pm \sigma(\mathbf{r})$. For given $c \in \Delta_-$ we want to determine

$$\Delta(c) := \{\sigma\text{-adapted roots } \mathbf{s} \in L \text{ with } \mathbf{s} - \sigma(\mathbf{s}) = \pm c\}, \tag{17}$$

as well as the associated set

$$\Delta^+(c) = \{s + \sigma(s) \in L^+(^1/2) = M \mid s \in \Delta(c)\}, \textit{ the set of c-nodal roots.} \quad (18)$$

From Lemma A.9, if $c = \mathbf{r} - \sigma(\mathbf{r})$, the nodal class $\mathbf{r}$ can be chosen at will.

Lemma 4.1 *Assume that $c = \mathbf{r} - \sigma(\mathbf{r})$ with $\mathbf{r} \in E_8(-1)$, a root in the first summand $E_8(-1) \subset L$. Fix a standard basis $\{f, g\}$ for the second copy of U in L. The set $\Delta(c) \subset L$ consists of roots $\{\pm s, \pm \sigma(s)\}$ with*

$$\mathbf{s} = x(f + \sigma(f)) + y(g + \sigma(g)) - (1 - z)\sigma(\mathbf{r}) + z\mathbf{r} + e' + \sigma(e'),$$
$$e' \in \mathbf{r}^\perp \cap E_8(-1) = E_7(-1),$$

such that $xy - z(z - 1) + \frac{1}{2}(e')^2 = 0$. In particular, if $x = y = \pm 1$ and $z = 0$ then e' is a root and $\mathbf{s}$ is orthogonal to $\mathbf{r}$.

Proof A root $\mathbf{s}$ such that $\mathbf{s} - \sigma(\mathbf{s}) = \mathbf{r} - \sigma(\mathbf{r})$ must be of the form $-\sigma(\mathbf{r}) + \ell + \sigma(\ell)$ with $\ell = u + e \in U \oplus E_8(-1)$. Since $\mathbf{s}^2 = -2$ we must have $u^2 + e^2 - \mathbf{r} \cdot e = 0$ which implies that $\mathbf{r} \cdot e$ is even and we apply Lemma A.2 iv) to write $e = z\mathbf{r} + e'$, $e' \in \mathbf{r}^\perp \cap E_8(-1) = E_7(-1)$. Then we find the desired expression. □

    Convention: For $x \in L$ the vector $x + \sigma(x) \in M = L^+\left(\frac{1}{2}\right)$ is denoted x^+.

Corollary 4.2   (i) *The lattice $N(c)$ generated by $\Delta(c)$ is isometric to $U(2) \perp \mathbf{Z}\mathbf{r} \perp \mathbf{Z}\sigma(\mathbf{r}) \perp E_7(-2)$.*

  (ii) *Roots from the set $\Delta^+(c)$ all belong to $\mathbf{r}^+ + 2R$ with $R = \mathbf{Z}\mathbf{r}^+ \perp \mathbf{r}^{+\perp}$.*

 (iii) *The lattice N generated by $\Delta^+(c)$ in M is isometric to*

$$N = 2\mathbf{r}^{+\perp} \perp \mathbf{Z}\mathbf{r}^+. \quad (19)$$

    Moreover, the set of roots in N coincides with $\Delta^+(c)$.

Proof   (i) The roots $f + \sigma(f) - \sigma(\mathbf{r}), g + \sigma(g) - \sigma(\mathbf{r}), \mathbf{r}, \sigma(\mathbf{r})$ in $\Delta(c)$ generate $\mathbf{Z}(f + \sigma(f)) \oplus \mathbf{Z}(g + \sigma(g)) \perp \mathbf{Z}\mathbf{r} \perp \mathbf{Z}\sigma(\mathbf{r})$, a lattice isometric to $U(2) \perp \mathbf{Z}\mathbf{r} \perp \mathbf{Z}\sigma(\mathbf{r})$. Since for any root $e' \in \mathbf{r}^\perp \cap E_8(-1)$ the root $f + \sigma(f) + g + \sigma(g) - \sigma(\mathbf{r}) + e' + \sigma(e')$ belongs to $\Delta(c)$, the lattice $N(c)$ also must contain $e' + \sigma(e')$ for any $e' \in E_7(-1)$, since this lattice is generated by its roots. The last set is a lattice isometric to $E_7(-2)$.

  (ii) We have

$$\mathbf{s}^+ \in \Delta^+(c)(^1/2) \iff \exists x, y, z \in \mathbf{Z}, e' \in E_7(-1)$$
$$\text{with } \frac{1}{2}(e')^2 - z(z - 1) + xy = 0$$
$$\text{s.t. } \mathbf{s}^+ = 2xf^+ + 2yg^+ + (2z - 1)\mathbf{r}^+ + 2e'^+.$$

    If follows that $\mathbf{s}^+ \in \mathbf{r}^+ + 2R$.

 (iii) Is a direct consequence of (i) and (ii).   □

Lemma 4.3 *Fix a symmetric set of roots* $\mathbf{r}_i \in \mathbf{r}^\perp \cap E_8(-1) = E_7(-1)$, $i = 1, \ldots, 7$, *(see Definition A.3) for example those from Example A.4 and put:*

$$
\left.
\begin{aligned}
\delta_0 &= \mathbf{r}, \\
\delta_1 &= f + \sigma(f) - \sigma(\mathbf{r}) \\
\delta_2 &= g + \sigma(g) - \sigma(\mathbf{r}) \\
\delta_j &= f + g + \sigma(f + g) - \sigma(\mathbf{r}) + \mathbf{r}_{j-2} + \sigma(\mathbf{r}_{j-2}), \ j = 3, \ldots, 9.
\end{aligned}
\right\}
\tag{20}
$$

Introduce the sets of roots

$$
\left.
\begin{aligned}
\Sigma &= \{\delta_j, \ j = 0, \ldots, 9\} \subset L \\
\Sigma^+ &= \{d_j = \delta_j + \sigma(\delta_j), \ j = 0, \ldots, 9\} \subset L^+(^1/_2).
\end{aligned}
\right\}.
\tag{21}
$$

(i) *The roots from each of the sets* $\pm\Sigma$, $\pm\sigma(\Sigma)$ *belong to* $\Delta(c)$ *and form a maximal orthogonal set of roots in* $\Delta(c)$.
(ii) *Two distinct roots* $\mathbf{d}, \mathbf{d}' \in \Sigma^+$ *satisfy* $\mathbf{d} \cdot \mathbf{d}' = 2$.

Proof While (ii) is clear, for (i) we first observe that all ten roots $\delta_j \in \Sigma$ are mutually orthogonal and $\delta_j - \sigma(\delta_j) = c$ and so $\Sigma \subset \Delta(c)$ and then also $-\sigma(\Sigma) \subset \Delta(c)$. That $\mathbf{d} \in \Sigma^+$ is a root is clear, as well as their intersection behaviour. A small calculation shows that these $\mathbf{d}$ are independent. Since rank $M = 10$ this shows that Σ (and also $-\sigma(\Sigma)$) forms a maximal orthogonal set of roots in $\Delta(c)$. $\qquad\square$

Remark 4.4 Any three roots from the set Σ together with their images under σ form a doubly bonded hexagon of roots.

For the general nodal Enriques surface we observe that the definition is just so that the following holds:

Proposition 4.5 *Choose* $c \in L^-$ *as above. For a general nodal Enriques surface the nodal curves correspond precisely to the irreducible roots in* $\Delta^+(c)$. *These generate the lattice* N *given in Eq. (19).*

Remark 4.6 In [15] Nikulin defines a certain invariant for Enriques surfaces as follows. Consider the sublattice $K \subset L^-(\frac{1}{2})$ generated by Δ_- (see (16)) and consider the natural homomorphism of groups

$$
\xi : K \bmod 2K \to M \bmod 2M
$$

which sends the class of r_- to the class of a corresponding r_+ with $r_- + r_+ \in 2L$. The pair $(K, \mathrm{Ker}\,\xi)$ is Nikulin's *root invariant* of the Enriques surface. The root invariant is such that the rank of K is precisely the codimension of the corresponding moduli space in $\mathcal{M}$. Note that for a generic Enriques surface the root invariant is $(0, 0)$. For a generic nodal Enriques surface $K = \langle -2 \rangle$ and since ξ is injective its root-invariant is $(\langle -2 \rangle, 0)$. In our situation the root invariant is completely determined by the so-called *nodality class*[4]

---

[4] Allcock [1] calls this "nodal class". For obvious reasons for us this is not an appropriate notion.

$$c \bmod 2M = \mathbf{r}^+ \bmod 2M \in V := M/2M \simeq \mathbf{F}^{10}. \tag{22}$$

We suppose now that Y is an Enriques surface whose moduli point belongs to $\mathcal{M}_{\mathrm{root}}^{\mathrm{gen}}$. We adapt the marking φ in such a way that the sum h of the roots in the hexagon (12) is an ample class (see, (13) and Theorem 3.6(iii))

$$h = \varphi(\tilde{e}_0 + \tilde{e}_1 + \tilde{e}_1 + \tilde{\ell}_0 + \tilde{\ell}_1 + \tilde{\ell}_2) = 2f^+ + 2g^+ - \mathbf{r}^+ \in C_Y^{\mathrm{amp}}$$
$$\tilde{h} = f + g + \sigma(f + g) - \sigma(\mathbf{r}) \in C_X^{\mathrm{amp}}.$$

Since by (15) both assertions are equivalent, we explain the construction only for the classes on Y.

Lemma 4.7 (i) *One has $h \cdot \mathbf{s}^+ \neq 0$ for all $\mathbf{s}^+ \in \Delta^+(c)(^1/_2)$ and there is a marking for Y for which h is ample.*

(ii) *The roots $\mathbf{s}^+ \in M$ with $\mathbf{s}^+ \cdot h = 2$ are nodal and so this holds in particular for $\mathbf{r}^+$, $2f^+ - \mathbf{r}^+$ and $2g^+ - \mathbf{r}^+$.*

Proof (i) By Lemma 4.2 we have $\mathbf{s}^+ = \mathbf{r} + 2m$ with $m \in M$ and so $h \cdot \mathbf{s}^+ \equiv 2 \bmod 4$ and hence can never be 0. So h is not on a hyperplane orthogonal to one of these roots. Since the only possible nodal classes on Y belong to $\Delta^+(c)$, the ample cone C_Y^{amp} corresponds to a connected component C_M^+ of $M_{\mathbf{R}} - \bigcup_{\mathbf{s} \in \Delta^+(c)} H_{\mathbf{s}}$. Choose the marking such that $h \in C_M^+$.

(ii) Since $\mathbf{s}^+ \cdot h > 0$, the root $\mathbf{s}^+$ is effective. If it is not nodal, it would be a sum of nodal classes, but since $\mathbf{s}^+ \cdot h = 2$ this is impossible. □

In what follows we shall use the above marking to identify $(H^2(X, \mathbf{Z}), \iota_{\mathrm{Enr}}^*)$ and (L, σ). We are now ready to prove our second main result:

Theorem 4.8 *Let $Y = X/\iota_{\mathrm{Enr}}$ be a general nodal Enriques surface. Then X is a del Pezzo double cover with branch locus coming from a sextic curve $C \subset \mathbf{P}^2$. Moreover,*

(i) *the Enriques involution descends to a biregular transformation induced by a standard Cremona transformation of $\mathbf{P}^2$. The curve C is invariant under the Cremona transformation and has three ordinary double points in the fundamental points of the Cremona transformation;*

(ii) *the composition of the covering involution and the Enriques involution is a Nikulin involution. In particular, the double cover of any generic nodal Enriques surface admits a Nikulin involution.*

With regards to the Néron-Severi and transcendental lattices, one has $L^+ + \mathbf{Z}c \subset S_X$ and $T_X \subset c^\perp \cap L^-$ with equality if the Picard number of X equals 11. In that case we have

$$T_X \simeq U \perp U(2) \perp E_7(-2), \text{ and } S_X \simeq U \perp \langle -4 \rangle \perp E_8(-2).$$

Proof (i) We first observe that the del Pezzo involution μ preserves the ample class $\tilde{h} = \delta_1 + \delta_2 + \sigma(\delta_0) = f + g + \sigma(f + g) - \sigma(\mathbf{r})$. By the Torelli theorem (e.g., [3, Theorem 11.1]) this implies that there is an involution ι_{delP} on X inducing μ.

The involution μ in addition fixes the nodal classes $\delta_0, \delta_1, \delta_2 \in L$ given by (20). Let $q : X \to P' = X/\iota_{\text{delP}}$ be the natural quotient map. Since P' can at most have double point singularities which would give rise to extra nodal curves on X, we see that P' is smooth. Moreover, from Remark 2.1 we infer that P' must be a rational surface. To see that it is $\mathbf{P}^2$ blown up in three points, we first notice that there an isomorphism $\mathbf{Q}$-vector spaces

$$q^* : H^2(P', \mathbf{Q}) \xrightarrow{\sim} H^2(X, \mathbf{Q})^{\iota^*_{\text{delP}}=\text{id}}.$$

Let $\ell, e_0, e_2, e_3 \in H^2(P', \mathbf{Q})$ such that $q^*\ell = \tilde{h}, q^*e_0 = \delta_0, q^*e_1 = \delta_1, q^*e_2 = \delta_2$. Since q^* multiplies the intersection form with 2 we see that $\ell^2 = 1$ and that the $e_j, j = 0, 1, 2$, are classes of exceptional curves on P'. Hence $H^2(P', \mathbf{Q}) = \mathbf{Q}\ell \perp \mathbf{Q}e_0 \perp \mathbf{Q}e_1 \perp \mathbf{Q}e_2$ which establishes our claim that $P' = P$, the projective plane blown up in three points p_0, p_1, p_2. But then $q : X \to P$ is a double cover branched in the proper transform $\tilde{C}$ of a sextic curve $C \subset \mathbf{P}^2$ having three double points at $p_j, j = 0, 1, 2$. Since the period point of X belongs to $\mathcal{M}_{\text{root}}^{\text{gen}}$, the sextic curve has no other singular points.

Next, recall that the class of $-q^*K_P$ corresponds to $\tilde{h} + \sigma(\tilde{h})$ which is fixed under the Enriques involution σ. So ι_{Enr} descends to an involution c on P which on $\mathbf{P}^2$ is the Cremona involution. Since σ and μ commute, c leaves the branch locus invariant. Hence X is a del Pezzo double cover.

(ii) This is a direct consequence of Theorem 3.4. The statement about S_X and T_X for general nodal X are clear.

If the Picard number equals 11, one has $T_X = c^\perp \subset L^-$ which is clearly isometric to $U \perp U(2) \perp E_7(-2)$. Since $S_X = T_X^\perp$, by [12, Sect. 1.5] the discriminant form of S_X is the same as the one for $T_X(-1)$. By Example A.12 we have that $S_X \simeq U \perp \langle -4 \rangle \perp E_8(-2)$. $\qquad\square$

Remark 4.9 For any general nodal Enriques surface one can determine the nodal curves D with low degree with respect to the ample class h as follows. As observed, all (-2)-curves D with $\deg D = 2$ are nodal. These are $\mathbf{r}^+, d_1 = 2f^+ - \mathbf{r}^+$ and $d_2 = 2g^+ - \mathbf{r}^+$. Next, consider degree 6. Among those are linear combinations of the preceding nodal curves, found using nodal reflections. We get the following 6 reducible classes of degree 6:

$$2f^+ + \mathbf{r}^+, 2g^+ + \mathbf{r}^+, 4f^+ - \mathbf{r}^+, 4g^+ - \mathbf{r}^+, 4f^+ + 2g^+ - 3\mathbf{r}^+, 4g^+ + 2f^+ - 3\mathbf{r}^+.$$

The remaining classes of degree 6 are all of the form $2f^+ + 2g^+ - \mathbf{r}^+ + 2e^+$ with e^+ some root of $E_7(-1)$ and hence must be nodal. In particular, this holds for the 7 classes $d_j, j = 3, \ldots, 9$ defined by (20) and (21).

5 Automorphisms

5.1 Automorphisms of the Enriques lattice

Reduction modulo 2

The product on M defines on the $\mathbf{F}_2$-vector space $V := M/2M$ an induced $\mathbf{F}_2$-bilinear form b upon setting $b(\bar{x}, \bar{y}) = x \cdot y \bmod 2$. The associated quadratic form $q(\bar{x}) = \frac{1}{2}x \cdot x \bmod 2$ satisfies $q(\bar{x} + \bar{y}) + q(\bar{x}) + q(\bar{y}) = b(\bar{x}, \bar{y})$. By [2, 1.4] there is a surjective homomorphism

$$r_2 : O^{\uparrow}(M) \twoheadrightarrow O(V, q).$$

The kernel of this homomorphism by definition equals $O^{\uparrow}(M)(2)$. Roots $\mathbf{s} \in M$ give classes $\bar{\mathbf{s}} \in V$ with $q(\bar{\mathbf{s}}) = 1$. These give the transvections $t_{\bar{\mathbf{s}}} \in O(V, q)$, i.e., $t_{\bar{\mathbf{s}}}(x) = x + b(x, \bar{\mathbf{s}})\bar{\mathbf{s}}$.

Lemma 5.1 $t_{\bar{\mathbf{s}}}$ *acts trivially on* $\bar{\mathbf{s}}^{\perp}$ *and conversely, if* $t \in O(V, q)$, $t \neq \mathrm{id}$ *acts trivially on* $\bar{\mathbf{s}}^{\perp}$ *then* $t = t_{\bar{\mathbf{s}}}$.

To see this, since r_2 is onto and since all roots in M are conjugate under $O^{\uparrow}(M)$ we may assume that $\bar{\mathbf{s}} = \bar{f} + \bar{g}$, where $\{\bar{f}, \bar{g}\}$ is the image in V of a basis of the direct summand U of $M = U \perp E_8(-1)$. Suppose $t(\bar{f}) = \xi \bar{f} + \eta \bar{g} + z$, where z is the class of an element in $U^{\perp}$. From $b(t(x), t(y)) = b(x, y)$ one deduces that $z = 0$ and then it easily follows that t switches $\bar{f}$ and $\bar{g}$ and so $t = t_{\bar{\mathbf{s}}}$.

The group $\Gamma(c)$

Note that the reflection $s_{\mathbf{r}^+}$ of M does not belong to $O^{\uparrow}(M)(2)$. We show that it only depends on $c = \mathbf{r} - \sigma(\mathbf{r}) \in L^-$. Indeed, $s_{\mathbf{r}} \circ s_{\sigma(\mathbf{r})} \in O(L, \sigma)$ sends c to $-c$ while it induces the identity on $c^{\perp}$. Since in M we have $s_{\mathbf{r}} \circ s_{\sigma(\mathbf{r})} = s_{\mathbf{r}^+}$, its action is indeed determined by c. Consider the extended group

$$\Gamma(c) := \text{subgroup of } O^{\uparrow}(M) \text{ generated by } O^{\uparrow}(M)(2) \text{ and } s_{\mathbf{r}^+}.$$

We let

$$r^{\pm} : O(L, \sigma) \to O(L^{\pm})$$

be the natural restrictions. We then have:

Lemma 5.2 *Introduce the abelian group*

$$A_c = \{\mathrm{id}, -\mathrm{id}, s_{\mathbf{r}} \circ s_{\sigma(\mathbf{r})}, -s_{\mathbf{r}} \circ s_{\sigma(\mathbf{r})}\} \subset O(L, \sigma). \tag{23}$$

Under the restriction $r^+ : O(L, \sigma) \to O(L^+) = O(M)$ *the group*

$$\{g \in O(L, \sigma) \mid r^-(g) \in r^-(A_c)\}$$

maps isomorphically to $\Gamma(c) \subset O(M)$.

Proof Let $g \in O(L, \sigma)$ such that $r^-(g) \in r^-(A_c)$. Replacing g by $\pm s_{\mathbf{r}} \circ s_{\sigma(\mathbf{r})} g$ if necessary, we may assume that $r^-(g) = \mathrm{id}$. We apply the results of Sect. A.3 as follows. We first observe that g induces the identity on $A_{L^-} \simeq A_{L^+}$ which means precisely that $r^+(g) \in O(L^+)(2)$. For the converse, one uses Corollary A.7. $\square$

We next translate this in terms of M. Recall that the class

$$\bar{\mathbf{r}} = \mathbf{r}^+ + 2M \in V$$

by definition is the nodality class determined by the root $\bar{\mathbf{r}}^+$. It depends only on $c = \mathbf{r} - \sigma(\mathbf{r})$. Observe that the mod 2 reduction of $s_{\mathbf{r}^+}$ is the transvection $t_{\bar{r}} \in O(V, q)$ and so, by Lemma 5.1 we have

$$\Gamma(c) = \{u \in O^\uparrow(M) \mid u|\bar{\mathbf{r}}^\perp = \mathrm{id}\}. \tag{24}$$

The Reye lattice

Note that since $\mathbf{r}^+$ is a root, $q(\bar{\mathbf{r}}) = 1$, while $b(\bar{\mathbf{r}}, \bar{\mathbf{r}}) = 0$. So $\bar{\mathbf{r}}^\perp \subset V$ is a 9-dimensional subspace containing $\bar{\mathbf{r}}$. This leads to the following notion, introduced by Cossec and Dolgachev [5].

Definition 5.3 Let $\pi : M \to (V, q)$ be the projection. The lattice

$$R_{10} = \pi^{-1}\bar{\mathbf{r}}^\perp = \mathbf{Z}\mathbf{r}^+ \perp \mathbf{r}^{+\perp} \subset M$$

is called the *Reye lattice*.

From now on we suppose, as before, that $\mathbf{r}$ is a root in the first copy of $E_8(-1)$ giving $\mathbf{r}^+$ in the corresponding copy of $E_8(-1) \subset M$.

We need the following result.

Lemma 5.4 *The group $\Gamma(c) \subset O^\uparrow(M)$ stabilizes the set of roots $\Delta^+(c)$ in the sublattice $N \subset M$ it generates. Moreover, it preserves the Reye lattice R_{10}.*

Proof For simplicity we write R instead of R_{10}. Suppose that $g \equiv \mathrm{id} \bmod 2M$. Any c-nodal root $\mathbf{s}^+$ belongs to $\mathbf{r}^+ + 2R$ and so $g(\mathbf{s}^+) \in \mathbf{r}^+ + 2R$. Clearly $s_{\mathbf{r}^+}$ preserves the set of c-nodal roots. It follows that any $g \in \Gamma(c)$ fixes $\bar{\mathbf{r}} \in V$ and hence preserves $\bar{\mathbf{r}}^\perp$. But this means precisely that g preserves the Reye lattice. $\square$

5.2 Automorphisms of the generic nodal Enriques surface

Fix a marking for $(X, \iota_{\mathrm{Enr}})$ and identify $(C_X^{\mathrm{amp}})^{\iota_{\mathrm{Enr}}^* = \mathrm{id}} = C_Y^{\mathrm{amp}}$ (see (15)) with its image in $M \otimes \mathbf{R}$ under the marking. The marking induces a representation of $\mathrm{Aut}(Y)$ on the lattice M and we set

$$\Gamma_Y = \mathrm{Im}\left(\mathrm{Aut}(Y) \to O^\uparrow(M)\right), \quad O^\uparrow(Y) := O^\uparrow(M)_{C_Y^{\mathrm{amp}}}$$

The marking sends the holomorphic 2 form on X to the corresponding line $[\omega_X]$. Any automorphisms g of X induces an automorphism of the transcendental lattice T_X and preserves the plane $P_X \subset T_X \otimes \mathbf{R}$ spanned by the real and imaginary parts of ω_X. The intersection form is positive definite on this plane and it follows that g has finite order on T_X since T_X is the smallest sublattice of $H^2(X, \mathbf{Z})$ with $P_X \subset T_X \otimes \mathbf{R}$. Hence g acts on the complex line $\mathbf{C}\omega_X$ as multiplication by an ℓ-th root of unity ϵ_ℓ, which is at the same time an eigenvalue for $g^*|T_X$. Since the characteristic polynomial of this action has degree equal to rank $T_X \leq 12$ we must have $\varphi(\ell) \leq 12$, where φ denotes the Euler function. This leads to the number field

$$k = \mathbf{Q}[\eta_\ell]_{\ell \text{ with } \varphi(\ell) \leq 12}, \quad \eta_\ell \text{ a primitive } \ell\text{-th root of unity}.$$

Proposition 5.5 *Let $Y = X/\iota_{\mathrm{Enr}}$ a general nodal Enriques surface, say with period point in D_c^0, $c = \mathbf{r} - \sigma(\mathbf{r})$, $\mathbf{r}$ a σ-separated root in L. Suppose that X has Picard number 11 and that, in addition, the period point is* not *contained in any proper linear subspace of $\mathbf{P}(c^\perp \cap L^-) \subset \mathbf{P}(L^-)$ defined over the field k. Then $\mathrm{Aut}(Y)$ maps injectively to its image $\Gamma_Y \subset O^\uparrow(M)$. Moreover, $\Gamma_Y = \Gamma(c) \cap O^\uparrow(Y)$.*

Proof The global Torelli theorem for K3 surfaces [3, Theorem 11.1] implies that

$$\mathrm{Aut}(Y) = \{g \in O(L, \sigma) \mid g_{\mathbf{C}}[\omega_X] = [\omega_X] \text{ and } g_{\mathbf{R}}(C_X^{\mathrm{amp}}) = C_X^{\mathrm{amp}}\}/\{1, \sigma\} \subset O^\uparrow(Y). \tag{25}$$

The argument of [2, Lemma 2.9] shows that the assumption on the period point implies that $g \in \mathrm{Aut}(Y)$ acts on T_X as $\pm\mathrm{id}$. Since the Picard number of X is 11, Theorem 4.8 tells us that $T_X = c^\perp \cap L^-$. Since g must also preserve c up to sign, we deduce that in that case $g|L^- \in r^-(A_c)$. So, by Lemma 5.2 $g \in \Gamma(c)$. Conversely, every $g \in \Gamma(c)$ induces $\pm\mathrm{id}$ on T_X and in particularly it preserves $[\omega_X]$. If $g_{\mathbf{R}}$ also preserves the ample cone of Y, it sends an ample invariant class in C_X^{ample} to an ample class. But $g_{\mathbf{R}}$ permutes the chambers of the Weyl group and so if $g_{\mathbf{R}}C_X^{\mathrm{ample}} \cap C_X^{\mathrm{ample}} \neq \varnothing$, the entire ample cone C_X^{ample} must be preserved. So by (25) we have $g \in \mathrm{Aut}(Y)$. It follows that the image of $\mathrm{Aut}(Y)$ in $O^\uparrow(Y)$ under the inclusion (25) equals $\Gamma(c) \cap O^\uparrow(Y)$. $\square$

We now make explicit the condition that C_Y^{amp} be preserved. Set

$$W_{\mathrm{nodal}} := \text{ Weyl group of } \Delta(N^+(c)).$$

This is the group generated by the reflections $s_{\mathbf{s}}$, $\mathbf{s} \in \Delta^+(c)$. Since C_Y^{amp} is a fundamental domain for the action of W_{nodal} on the "light cone" C_M, we have

Corollary 5.6

$$\Gamma(c) = \Gamma_Y \ltimes W_{\mathrm{nodal}} \implies \mathrm{Aut}(Y) \simeq \Gamma_Y = \Gamma(c)/W_{\mathrm{nodal}}.$$

Remark If we replace c by $c' = \gamma(c)$ for some $\gamma \in O(L, \sigma)$, then $\Gamma(c') = \gamma\Gamma(c)\gamma^{-1}$ and $W(N(^+(c'))) = \gamma W(N^+(c))\gamma^{-1}$, and so the above isomorphism for $\mathrm{Aut}(Y)$ is canonical up to $O(L, \sigma)$-conjugacy.

Following [5,7] the group Γ_Y can be described more explicit. We give a simplified argument. For simplicity we write R for the Reye lattice instead of R_{10}. Recall that by Lemma A.5(ii) we have $O^\uparrow(R) = O^\uparrow(M)_R$. We need an auxiliary result.

Lemma 5.7 W_{nodal} *is contained in* $O^\uparrow(R) = O^\uparrow(M)_R$ *and it is the smallest normal subgroup therein generated by* $s_{\mathbf{r}^+}$.

Proof Lemma 5.4 tells us that $\Gamma(c)$ and hence W_{nodal} preserves the Reye lattice so that $W_{\text{nodal}} \subset O^\uparrow(R)$.

Next we consider the conjugacy class of $s_{\mathbf{r}^+}$. Within M all roots are conjugate. We want to show that a c-nodal root $\mathbf{s} = \mathbf{r}^+ + 2y$, $y \in R$ is conjugate to $\mathbf{r}^+$ by an element in $O(M)$ that preserves the Reye lattice. So pick $u \in O(M)$ with $u(\mathbf{s}) = \mathbf{r}^+$. The Reye lattice can be characterized as

$$R = \{x \in M \mid x \cdot \mathbf{r}^+ \equiv 0 \bmod 2\}.$$

Consequently, if $x \in R$ we find $u(x) \cdot \mathbf{r}^+ = u(x) \cdot u(\mathbf{s}) = x \cdot \mathbf{s} \equiv x \cdot \mathbf{r}^+ \equiv 0 \bmod 2$ and so $u(x) \in R$. So, indeed, all c-nodal roots of M are conjugate within $O^\uparrow(M)_R$. Since $u s_{\mathbf{s}} u^{-1} = s_{u(\mathbf{s})}$, applied to c-nodal roots $\mathbf{s}$, this shows that W_{nodal} is the smallest normal subgroup of $O^\uparrow(R)$ containing the reflection $s_{\mathbf{r}^+}$. $\qquad\square$

From this result, Corollary 5.6, the characterisation (24) of $\Gamma(c)$, and Lemma A.5(iii) we deduce:

Corollary 5.8 [5,7] *Let* $\Gamma_{2,4,6}$ *be the T-shaped graph of Fig. 2 and let* $\text{Cox}(T_{2,4,6})$ *be its Coxeter group, i.e., the group generated by the reflections defined by its nodes.*
There is an isomorphism

$$\Gamma_Y \simeq \{u \in \text{Cox}(T_{2,4,6}) \mid u|\bar{\mathbf{r}}^\perp = \text{id}\}.$$

Remark 5.9 1. The Coxeter group $\text{Cox}(T_{2,4,6})$, as any hyperbolic Coxeter group containing an isotropic vector, is infinite, as we see as follows. Referring to Fig. 2, the obvious subdiagram $\widetilde{E}_7$ contains a primitive isotropic vector which was called f in the course of the proof of Lemma A.2. For $y \in f^\perp$ the Eichler–Siegel transformation

$$\psi_{f,y} : R \longrightarrow R, \quad R = R_{10}$$
$$x \mapsto x + (x \cdot y)f - (x \cdot f)y - (x \cdot f)(^1/_2 y^2)f.$$

belongs to $\text{Cox}(T_{2,4,6})_f$, hence preserves $f^\perp$. Moreover, since $\psi_{f,y} = \psi_{f,y+f}$ we may write $\psi_{f,\bar{y}}$, $\bar{y} \in f^\perp/\mathbf{Z}f$. One easily verifies

$$\psi_{f,\bar{y}_1} \circ \psi_{f\bar{y}_2} = \psi_{f,\bar{y}_1+\bar{y}_2}$$

and so $\text{Cox}(T_{2,4,6})$ contains a free abelian group of rank equal to the rank of $f^\perp/\mathbf{Z}f$, i.e., of rank 8.

2. The group $\Gamma_Y \subset \text{Cox}(T_{2,4,6})$ obviously contains the congruence subgroup $\text{Cox}(T_{2,4,6})(2)$. The index of the latter in Γ_Y is 2^8. This was first observed in [5]. See also [8].

A Some lattice theory

A.1 The E_n-lattices

We use the description of the root lattices $E_n(-1)$ as given in [7] which differs from the classical description but turns out to be better adapted to calculations. Consider $\mathbf{R}^{n+1}$ equipped with the standard Lorentz inner product given in coordinates (ranging from 0 to n) by $(x_0, x_1, \ldots, x_n) \cdot (y_0, y_1, \ldots, y_n) = x_0 y_0 - (\sum_{j=1}^{n} x_j y_j)$ and let $E^{1,n}$ be the lattice $\mathbf{Z}^{n+1} \subset \mathbf{R}^{n+1}$ with the induced Lorentz product, and put

$$k = (-3, 1, \ldots, 1) \in E^{1,n}, \quad E_n(-1) = k^{\perp} \subset E^{1,n}.$$

For $n = 6, 7, 8$ this gives the "negative" of the root lattices E_n. For $n = 9$ the form has a one-dimensional radical spanned by k and for $n = 10$ the lattice has signature $(1, 9)$. Below we show that the latter is isometric to the Enriques lattice M. The above description of E_n makes it easy to deal with roots. See e.g., [7, Section 8.2.3]. For instance, we have:

Lemma *With $\{e_0, \ldots, e_n\}$ the standard basis for $\mathbf{R}^{n+1}$, a root basis for $E_n(-1)$ is given by the roots $\alpha_0 = (1, -1, -1, -1, 0, \ldots, 0)$ and $\alpha_j = e_{j+1} - e_j$, $j = 1, \ldots, n-1$.*

We shall use a basis of roots in $E_7(-1)$ and $E_8(-1)$. The Dynkin diagrams together with their extensions are as follows.
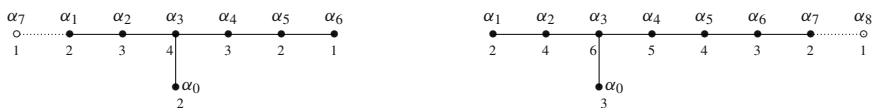
With the numbers m_j below the nodes α_j, the vector $\sum_{j=0}^{n} m_j \alpha_j$, $n = 7, 8$ spans the isotropic subspace of the extended lattices $\widetilde{E_n(-1)}$ and so, if we delete α_n from this sum, the resulting vector

$$\gamma_n = \sum_{j=0}^{n-1} m_j \alpha_j \in E_n(-1) \tag{26}$$

is a root orthogonal to all roots from the Dynkin diagram except the one connecting to α_n. Using this, we see that the Enriques lattice is isometric to $E_{10}(-1)$ (Fig. 1). Indeed, the leftmost part of the diagram is $\widetilde{E_8}(-1)$ and $\{\gamma_8 + \alpha_8, \gamma_8 + \alpha_8 + \alpha_9\}$ span a hyperbolic plane orthogonal to the $E_8(-1)$ formed by $\{\alpha_0, \ldots, \alpha_7\}$.
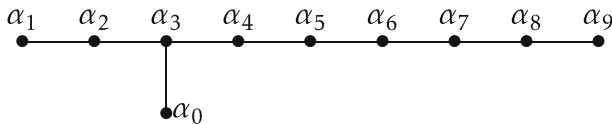
Fig. 1 Enriques lattice $E_{10}(-1)$

Using the preceding expression for the roots, a small calculation shows that for $n \neq 9$

$$y = \frac{1}{n-9}(-3, 1, 1, \ldots, 1, -n + 10) \in E_n(-1) \otimes \mathbf{Q}$$

generates the discriminant group. Since $y^2 = \frac{n-10}{9-n} \equiv \frac{8-n}{9-n}$ mod $2\mathbf{Z}$, it follows that we have:

Lemma A.1 *For $n \neq 9$ the discriminant of $E_n(-1)$ equals $(\mathbf{Z}/(9-n)\mathbf{Z}, \frac{8-n}{9-n}$ mod $2\mathbf{Z})$, and hence $E_8(-1)$ and $E_{10}(-1)$ are unimodular, $A_{E_7(-1)} = (\mathbf{Z}/2\mathbf{Z}, \frac{1}{2})$ and $A_{E_6(-1)} = (\mathbf{Z}/3\mathbf{Z}, \frac{2}{3})$.*

We assemble some further useful facts:

Lemma A.2   (i) *For $n = 7, 8$ we have $O(E_n(-1)) = W(E_n(-1))$; for $n = 10$ we have that $O^{\uparrow}(E_{10}(-1)) = W(E_{10}(-1))$.*
 (ii) *The orthogonal complement of a root in $E_8(-1)$ is isometric to $E_7(-1)$.*
(iii) *Given a root $\mathbf{r} \in E_8(-1)$, the sublattice*

$$\mathbf{Zr} \perp \mathbf{r}^{\perp} = \mathbf{Zr} \perp E_7(-1)$$

consists of those $e \in E_8(-1)$ for which $e \cdot \mathbf{r}$ is even.
 (iv) *For $n = 6, 7, 8$ the Weyl group $W(E_n(-1))$ acts transitively on the set of roots.*[5]

Proof For (i) and (iv) see [7, Section 8.2.3]. Assertion (ii) and (iii) are clear for the special root γ_8 (cf. (26)) and since all roots are conjugate under the isometry group, this implies the assertion for all roots.   □

We furthermore need special sets of roots in $E_n(-1)$ called symmetric:

Definition A.3 A set of k roots $b_1, \ldots, b_k \in E_n(-1)$ for which $b_i \cdot b_j = -1, i \neq j$ is called *a symmetric set* of k roots.

---

[5] For $E_{10}(-1)$, the Enriques lattice, this is also true, see the proof of A.8.

Example A.4 The following roots

$$
\begin{aligned}
\mathbf{r}_1' &= (2, 0, -1, -1, -1, -1, -1, -1) \\
\mathbf{r}_2' &= (0, 1, -1, 0, 0, 0, 0, 0) \\
\mathbf{r}_3' &= (0, 1, 0, -1, 0, 0, 0, 0) \\
\mathbf{r}_4' &= (0, 1, 0, 0, -1, 0, 0, 0) \\
\mathbf{r}_5' &= (0, 1, 0, 0, 0, -1, 0, 0) \\
\mathbf{r}_6' &= (0, 1, 0, 0, 0, 0, -1, 0) \\
\mathbf{r}_7' &= (0, 1, 0, 0, 0, 0, 0, -1)
\end{aligned}
$$

form a symmetric set of roots in $E_7(-1)$.

A.2 The Reye lattice

Consider the T-shaped graph $T_{2,4,6}$ formed by the leftmost 10 nodes of the graph of Fig. 2. Here $\beta_i^2 = -2$, $i = 0, \ldots, 9$ and $\mathbf{r}^+$ is a root with $\mathbf{r}^+ \cdot \beta_9 = -2$, as indicated by the dotted double line. We show the following.

Lemma A.5 (i) *The Reye lattice $R_{10} := E_7(-1) \perp \langle -2 \rangle \perp U$ is isometric to $T_{2,4,6}$. In this isometry the summand $\langle -2 \rangle$ corresponds to $\mathbf{r}^+$.*

(ii) *Every $u \in O^{\uparrow}(R_{10})$ extends uniquely as an isometry of M. Hence $O^{\uparrow}(M)_{R_{10}} = O^{\uparrow}(R_{10})$.*

(iii) *Let $Cox_{2,4,6}$ be the Coxeter group defined by the diagram $T_{2,4,6}$, i.e. the group generated by the reflections in the roots of $T_{2,4,6}$. We have*

$$
W(R_{10}) = O^{\uparrow}(R_{10}) = Cox_{2,4,6} \ltimes N(s_{\mathbf{r}}),
$$

where $N(s_{\mathbf{r}})$ is the minimal normal subgroup containing $s_{\mathbf{r}^+}$.

Proof (i) $T_{2,4,6}$ has an obvious Dynkin subdiagram of type $\widetilde{E}_7(-1)$ with β_7 in the role of α_7. Hence $f = \gamma_7 + \beta_7$ is isotropic in $\widetilde{E}_7(-1)$ but $f \cdot \beta_8 = 1$, $f \cdot \beta_9 = 0$. Hence f, $f + \beta_8 + \beta_9$ span a hyperbolic plane orthogonal to $E_7(-1)$ and to the root $\mathbf{r}^+ = f + \beta_9$. Hence $T_{2,4,6}$ is indeed isometric to $E_7(-1) \perp U \perp \langle -2 \rangle$.

(ii) Since the generating reflections from $Cox_{2,4,6}$ come from roots in M this is obvious.

(iii) Let P be the lattice spanned by $\mathbf{r}^+$ and β_9. It is left invariant by the two reflections $s_{\mathbf{r}^+}$ and s_{β_9}. Observe $t = s_{\mathbf{r}^+} \circ s_{\beta_9}$ preserves the plane P and in the basis $\{\mathbf{r}^+, \beta\}$
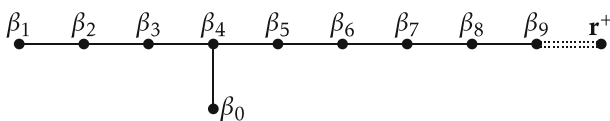

Fig. 2 Reye lattice $T_{2,4,6}$ and Coxeter graph of $W(R_{10})$

the automorphism $t|P$ has matrix

$$\begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix},$$

which has infinite order. Hence the natural division of the nodes in the extended Dynkin diagram into the 10 nodes of $T_{2,3,6}$ and $\mathbf{r}^+$ is such that $s_{\mathbf{r}^+} \circ s_{\beta_j}$ either has order 2 or ∞. We then apply [19, Lemma 1]. $\qquad\square$

A.3 On isometries of L and M

We investigate the K3 lattice L and the behaviour of its isometries with respect to the Enriques involution σ. The sublattices of L on which $\sigma = \pm\mathrm{id}$ are denoted $L^\pm$. Recall that $L^+ \simeq M(2)$ and $L^- \simeq U \perp M(2)$ where $M = U \perp E_8(-1)$. We consider the question as to whether a pair of isometries of $L^\pm$ extend to give an isometry on L. To answer this question, introduce the isometry

$$\psi : L^- \overset{\sim}{\to} U \perp L^+$$
$$(v, (u, e), (-u, -e)) \mapsto (v, (u, e), (u, e))$$

inducing the group isomorphism $\bar{\psi} : A_{L^-} \overset{\sim}{\to} A_{L^+} = (\mathbf{Z}/2\mathbf{Z})^{10}$ between the discriminant groups. The following result is well known ([12, Cor. 1.5.2.]):

Proposition A.6 *A given pair of isometries $\gamma^\pm \in O(L^\pm)$ extend to an isometry $\gamma : L \to L$ if and only if for the induced isometries $\bar{\gamma}^\pm$ of the discriminants one has $\bar{\psi} \circ \bar{\gamma}^- = \bar{\gamma}^+ \circ \bar{\psi}$.*

Corollary A.7 *Let $\gamma^+ \in O(L^+)$ be given and define $\gamma^- \in O(L^-)$ by*

$$\gamma^-(v, y, -y) = (v, \gamma^+(y), -\gamma^+(y)), \quad (v, y, -y) \in L^-.$$

With $r^\pm : O(L, \sigma) \to O(L^\pm)$ the obvious restriction maps, there is a unique isometry γ of L for which $r^\pm\gamma = \gamma^\pm$ on $L^\pm$. Moreover γ commutes with σ.

Proof This follows immediately from the preceding discussion, except that we need to see that γ commutes with σ. This follows since it does so on the sublattice $2L \subset L$ (write $2x = (x + \sigma(x)) + (x - \sigma(x))$). $\qquad\square$

We finish with a property of isometries of the Enriques lattice M.

Lemma A.8 *The group $O(M)$ acts transitively on roots.*

Proof We apply the analog of Witt's theorem [12, Theorem 1.14.4]. This theorem states that one can primitively embed a lattice T of signature (t^+, t^-) in a unimodular lattice S of signature (s^+, s^-) in a unique way up to isometries of S, provided that the conditions $s^+ - t^+ > 0$, $s^- - t^- > 0$, rank S − rank $T \geq$ rank $A_T + 2$ hold. In our case $S = M$, $T = \mathbf{Z}r_+$, $t^+ = 0$, $t^- = 1$, $s^+ = 1$, $s^- = 9$, $A_T = \mathbf{Z}/2\mathbf{Z}$ and the condition is fulfilled. $\qquad\square$

Recalling Definition 2.2 and using Corollary A.7 we then deduce:

Corollary A.9 *Strongly σ-adapted pairs of roots in L are conjugate under the action of $O(L, \sigma)$.*

A.4 On even indefinite lattices

As is well known (see e.g., [18, Chapter 5]), even indefinite unimodular lattices are uniquely determined by their signature. The building blocks are U, E_8 and $E_8(-1)$. The K3 lattice, having signature $(3, 19)$ thus equals $U^3 \perp E_8(-1)^2$. For non-unimodular lattices L, there is an extra invariant, the discriminant form δ_L. Nikulin has given criteria ensuring uniqueness for even indefinite lattices L with given signature and discriminant form. It uses the "signature mod 8" or *Arf-invariant* $\tau(q)$ of a quadratic form on a finite groups. This is an integer modulo 8 with the property that $\tau(q_L) \equiv$ index of L (mod 8). We quote the result [12, Corollary 1.13.3]:

Theorem A.10 *There is a unique indefinite even non-degenerate lattice L with signature (t_+, t_-) and discriminant form (A_L, δ_L) whenever*

- *the signature $t_+ - t_-$ of L and of δ_L are the same modulo 8;*
- *rank $L \geq 2 + |A_L|$.*

Before we can apply this, we need to introduce some $\mathbf{Q}/2\mathbf{Z}$-valued quadratic forms on finite abelian groups.

Notation A.11 1. For a prime p and a an integer prime to p, we let q_{a,p^k} be the group $\mathbf{Z}/p^k\mathbf{Z}$ with form $x \mapsto \frac{a}{p^k}x^2$.

2. The group $\mathbf{Z}/2^k\mathbf{Z} \oplus \mathbf{Z}/2^k\mathbf{Z}$ equipped with the quadratic form $(\bar{x}, \bar{y}) \mapsto \frac{1}{2^{k-1}}xy$ with $x, y \in \mathbf{Z}$ representatives for $\bar{x}, \bar{y}$, gives the discriminant form for the lattice $U(2^k)$ and will be denoted u_{2^k}. It has Arf invariant 0.

3. The group $\mathbf{Z}/2^k\mathbf{Z} \oplus \mathbf{Z}/2^k\mathbf{Z}$ equipped with the quadratic form $(x, y) \mapsto \frac{1}{2^{k-1}}(x^2 + xy + y^2)$ is denoted v_{2^k}. It has Arf-invariant 1.

Observe that it follows immediately that a $\mathbf{Q}/2\mathbf{Z}$-valued form on the group $(\mathbf{Z}/2^k\mathbf{Z})^8$ of rank 8 has Arf-invariant 0 precisely for $u_{2^k}^4$ or $q_{\pm 1, 2^k}^8$.

Next we observe that, given a lattice L of rank r with discriminant form q_{a,p^k}, the modified lattice $L(p^s)$ has discriminant group $\mathbf{Z}/\mathbf{Z}p^{k+s} \oplus (\mathbf{Z}/\mathbf{Z}p^s)^{r-1}$. This follows easily from the theory of elementary divisors applied to the Gram matrix for the lattice $L(p^s)$. It is more complicated to find the induced quadratic form on the discriminant. For $p = 2$ the Arf-invariant facilitates the calculation; can can use e.g. that the Arf-invariant is zero if and only if the form represents 0. We have seen (Lemma A.1) that the discriminant form of $E_7(-1)$ is $q_{1,2}$, a form with Arf-invariant 1 (mod 8), confirming that $1 \equiv -7$ mod 8. There is a unique form on $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ with Arf-invariant 0, namely u_2. Using this, it follows that $E_7(-2)$ has discriminant $q_{1,2^2} \oplus u_2^3$. Finally, note that the discriminant form of $E_8(-2)$ represents zero: the quadratic form vanishes on the vector $(1, -1, 0, 0, 0, 0, 0, 0, 0) \in (k^\perp)^* = E^{1,8}/\mathbf{Z}k$ and hence on its class in $q_{E_8(2)}$. It follows that $q_{E_8(2)} = u_2^4$.

Example A.12 Consider $L_1 = U \perp U(2) \perp E_7(-2)$. This lattice has signature $(2, 9)$ and discriminant $u_2 \oplus q_{1,2^2} \oplus u_2^3$ and satisfies the conditions of the theorem. Consider now the form $q_{-1,2^2} \oplus u_2^4 = -q_{L_1}$. It also satisfies the conditions of the theorem, this time for signature $(1, 10)$. So there is a unique even lattice L_2 of this signature and with $q_{L_2} = -q_{L_1}$. To determine it, note that we just saw that $E_8(-2)$ has Arf-invariant 0 so that its discriminant is u_2^4. It follows that $L_2 = U \perp \langle -4 \rangle \perp E_8(-2)$.

A.5 Hyperbolic lattices

We recall here some well known properties of groups generated by reflections in the vector space $N_{\mathbf{R}}$ associated to an even hyperbolic lattice N, i.e., a lattice whose signature is $(1, n)$. See e.g., [4], [14, Sect. 1.1].

Recall that the half cone C_N is a connected component of the "light cone" of elements $x \in N_{\mathbf{R}}$ with $x \cdot x > 0$. The reflections $s_{\mathbf{r}}$, $\mathbf{r}$ a root in N preserve the cone C_N and generate a normal subgroup of $O^{\uparrow}(N)$, the Weyl group $W(N) \subset O^{\uparrow}(N)$ with quotient group

$$A(N) = O^{\uparrow}(N)/W(N).$$

A fundamental domain for the action on C_N can be obtained as follows. Choose a partition $\Delta(N) = \Delta^+ \cup -\Delta^+$ for the set $\Delta(N)$ of roots in N into "positive" and "negative" roots. Such a partition can be assumed to be stable under taking sums: a root which is a sum of roots from Δ^+ belongs to Δ^+. Roots which cannot be written as such a sum are called *indecomposable*. $W(N)$ is generated by reflections in the indecomposable roots. For two different indecomposable roots $\mathbf{r}$, $\mathbf{s}$ we have $\mathbf{r} \cdot \mathbf{s} \geq 0$.

The roots $\mathbf{r} \in \Delta(N)$ define hyperplanes $H_{\mathbf{r}} = \mathbf{r}^{\perp} \in N \otimes \mathbf{R}$ which divide C_N in "chambers", by definition the components of the complements of their union in $N_{\mathbf{R}}$. The fundamental domain is then the closure of the *fundamental chamber*

$$C(\Delta^+) = \{x \in C_N \mid x \cdot \mathbf{r} > 0 \text{ for all } \mathbf{r} \in \Delta^+\}.$$

A fundamental domain is thus determined by a partition into positive and negative roots. Conversely, given any chamber C gives such a partition: take an element y in its interior, then the roots in N having positive intersection with y are the positive roots in a partition and C is the intersection of the positive half spaces defined by these roots. Consequently we have

$$W(N) \rtimes \{\gamma \in O^{\uparrow}(N) \mid \gamma C(\Delta^+) = C(\Delta^+)\} = O^{\uparrow}(N) \implies$$
$$\{\gamma \in O^{\uparrow}(N) \mid \gamma C(\Delta^+) = C(\Delta^+)\} \simeq A(N).$$

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