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Hirzebruch L-polynomials and multiple zeta values

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Abstract We express the coefficients of the Hirzebruch L-polynomials in terms of certain alternating multiple zeta values. In particular, we show that every monomial in the Pontryagin classes appears with a non-zero coefficient, with the expected sign. Similar results hold for the polynomials associated to the $\hat{A}$-genus.

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1 Introduction

The Hirzebruch L-polynomials are certain polynomials with rational coefficients,

$$\begin{aligned} \mathsf{L}_1 &= \frac{1}{3}\mathsf{p}_1, \\ \mathsf{L}_2 &= \frac{1}{45}(7\mathsf{p}_2 - \mathsf{p}_1^2), \\ \mathsf{L}_3 &= \frac{1}{945}(62\mathsf{p}_3 - 13\mathsf{p}_2\mathsf{p}_1 + 2\mathsf{p}_1^3), \\ &\vdots \end{aligned}$$

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featured in the Hirzebruch signature theorem, which expresses the signature $\sigma(M)$ of a smooth compact oriented manifold M^{4k} as

$$\sigma(M) = \langle \mathsf{L}_k, [M] \rangle,$$

where p_i are taken to be the Pontryagin classes of the tangent bundle of M, see [3, Theorem 8.2.2] or [5, Theorem 19.4]. The kth polynomial has the form

$$\mathsf{L}_k = \mathsf{L}_k(\mathsf{p}_1, \ldots, \mathsf{p}_k) = \sum h_{j_1,\ldots,j_r} \mathsf{p}_{j_1} \cdots \mathsf{p}_{j_r},$$

where the sum is over all partitions $(j_1, \ldots, j_r)$ of k, i.e., sequences of integers $j_1 \geq \cdots \geq j_r \geq 1$ such that $j_1 + \cdots + j_r = k$. The purpose of this note is to establish certain properties of the coefficients $h_{j_1,\ldots,j_r}$.

For real numbers $s_1, \ldots, s_r > 1$, we define the series

$$T(s_1, \ldots, s_r) = \sum_{n_1 \geq_2 \cdots \geq_2 n_r \geq 1} \frac{(-1)^{n_1+\cdots+n_r}}{n_1^{s_1} \cdots n_r^{s_r}},$$

where $n \geq_2 m$ means "$n \geq m$ with equality only if n is even". Define the symmetrization of this series by

$$T^{\Sigma}(s_1, \ldots, s_r) = \sum_{\sigma \in \Sigma_r} T(s_{\sigma_1}, \ldots, s_{\sigma_r}),$$

where Σ_r is the symmetric group.

Theorem 1 *The coefficients of the Hirzebruch L-polynomials are given by*

$$h_{j_1,\ldots,j_r} = \frac{(-1)^r}{\alpha_1! \cdots \alpha_k!} \frac{2^{2k}}{\pi^{2k}} T^{\Sigma}(2j_1, \ldots, 2j_r),$$

where α_ℓ counts how many of $j_1, \ldots, j_r$ are equal to ℓ.

It is well-known that h_k is positive for all k. In [8, Appendix A], it is argued that $h_{i,j}$ is always negative and that $h_{i,j,k}$ is always positive (following an argument attributed to Galatius in the case of $h_{i,j}$), and it is asked whether it has been proved in general that $(-1)^{r-1} h_{j_1,\ldots,j_r}$ is positive. We have not been able to locate such a result in the literature, but we can prove it using our formula. It follows from the following result.

Theorem 2 *For all real $s_1, \ldots, s_r > 1$,*

$$T(s_1, \ldots, s_r) < 0.$$

Corollary 3 *The coefficient $h_{j_1,\ldots,j_r}$ in the Hirzebruch L-polynomial L_k is non-zero for every partition $(j_1, \ldots, j_r)$ of k. It is negative if r is even and positive if r is odd.*

It is remarked in [8] that a similar pattern has been observed in the multiplicative sequence of polynomials associated with the $\hat{A}$-genus. The polynomials in question are

$$\begin{aligned}
\hat{\mathsf{A}}_1 &= -\frac{1}{24}\mathsf{p}_1,\\
\hat{\mathsf{A}}_2 &= \frac{1}{5760}\big(-4\mathsf{p}_2+7\mathsf{p}_1^2\big),\\
\hat{\mathsf{A}}_3 &= \frac{1}{967680}\big(-16\mathsf{p}_3+44\mathsf{p}_2\mathsf{p}_1-31\mathsf{p}_1^3\big),\\
&\vdots
\end{aligned}$$

These can be treated similarly. Let us write

$$\hat{\mathsf{A}}_k = \sum a_{j_1,\dots,j_r}\mathsf{p}_{j_1}\cdots\mathsf{p}_{j_r},$$

where the sum is over all partitions $(j_1,\dots,j_r)$ of k. Consider the series

$$S(s_1,\dots,s_r) = \sum_{n_1\geq\cdots\geq n_r\geq 1}\frac{1}{n_1^{s_1}\cdots n_r^{s_r}},$$

and its symmetrization

$$S^{\Sigma}(s_1,\dots,s_r) = \sum_{\sigma\in\Sigma_r} S(s_{\sigma_1},\dots,s_{\sigma_r}).$$

Theorem 4 *The coefficients of the $\hat{A}$-polynomials are given by*

$$a_{j_1,\dots,j_r} = \frac{(-1)^r}{\alpha_1!\cdots\alpha_k!}\frac{1}{(2\pi)^{2k}}S^{\Sigma}(2j_1,\dots,2j_r).$$

In particular, the coefficient $a_{j_1,\dots,j_r}$ is negative if r is odd and positive if r is even.

2 Proofs

The first step in our proof is to establish a formula that expresses the coefficient $h_{j_1,\dots,j_r}$ as a linear combination of products $h_{k_1}\cdots h_{k_\ell}$. This generalizes the formulas for $h_{i,j}$ and $h_{i,j,k}$ found in [8]. In the appendix of [2], recursive formulas for computing $h_{j_1,\dots,j_r}$ in terms of products $h_{k_1}\cdots h_{k_\ell}$ are given. Here we give an explicit closed formula. The result holds for arbitrary multiplicative sequences of polynomials (see [3, §1]).

Theorem 5 *Let $K_0, K_1, K_2, \dots$ be a multiplicative sequence of polynomials with*

$$K_k = \sum \lambda_{j_1,\dots,j_r}\mathsf{p}_{j_1}\cdots\mathsf{p}_{j_r}.$$

The coefficients satisfy the relation

$$\lambda_{j_1,\dots,j_r} = \frac{1}{\alpha_1!\cdots\alpha_k!}\sum_{\mathcal{P}}(-1)^{r-\ell}c_{\mathcal{P}}\,\lambda_{k_1}\cdots\lambda_{k_\ell}, \tag{1}$$

where α_i counts how many of $j_1,\dots,j_r$ are equal to i, the sum is over all partitions $\mathcal{P}=\{P_1,\dots,P_\ell\}$ of the set $\{1,2,\dots,r\}$,

$$c_{\mathcal{P}} = \big(|P_1|-1\big)!\cdots\big(|P_\ell|-1\big)!,$$

and

$$k_m = \sum_{i\in P_m} j_i.$$

Proof A multiplicative sequence of polynomials is determined by its characteristic power series

$$Q(z) = \sum_{k=0}^{\infty} b_k z^k,$$

where $b_k = \lambda_{1,\dots,1}$ is the coefficient of p_1^k in K_k. Indeed, if we, as in [3], formally interpret the coefficients b_k as elementary symmetric functions in $\beta_1',\dots,\beta_m'$ $(m\geq k)$, so that

$$1+b_1z+b_2z^2+\cdots+b_mz^m = (1+\beta_1'z)\cdots(1+\beta_m'z),$$

then the coefficient $\lambda_{j_1,\dots,j_r}$ is the *monomial symmetric function* in $\beta_1',\dots,\beta_m'$ (see [3, Lemma 1.4.1]).

Note that λ_k equals the power sum $\sum_i(\beta_i')^k$. The product $\lambda_{k_1}\cdots\lambda_{k_\ell}$ is then the *power sum symmetric function* evaluated at β_i', and the claim follows from a general formula that expresses the monomial symmetric functions in terms of power sum symmetric functions, see Theorem 8 below. □

The characteristic series of the Hirzebruch L-polynomials is

$$\frac{\sqrt{z}}{\tanh\sqrt{z}} = 1+\sum_{k=1}^{\infty} b_k z^k,$$

where

$$b_k = (-1)^{k-1}\frac{2^{2k}}{(2k)!}B_k,$$

and B_k are the Bernoulli numbers,

$$B_1 = \frac{1}{6}, \ \ B_2 = \frac{1}{30}, \ \ B_3 = \frac{1}{42}, \ldots,$$

see [3, §1.5].

As is well-known, the leading coefficient h_k of p_k in L_k is given by

$$h_k = \frac{2^{2k}\left(2^{2k-1} - 1\right)}{(2k)!} B_k, \tag{2}$$

see [3, p.12]. In [8], the formula

$$h_k = \zeta(2k) \frac{2^{2k} - 2}{\pi^{2k}},$$

involving the Riemann zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

is used to argue that $h_{i,j} < 0$ and $h_{i,j,k} > 0$. From this point on, our argument will depart from that of [8]. A key observation is that we can express h_k in terms of the alternating zeta function,

$$\zeta^*(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s},$$

instead of the Riemann zeta function. It is well-known, and easily seen, that

$$\zeta^*(s) = \left(1 - 2^{1-s}\right)\zeta(s).$$

Moreover, the following holds for all positive integers k,

$$\zeta^*(2k) = \frac{\pi^{2k}(2^{2k-1} - 1)}{(2k)!} B_k. \tag{3}$$

Combining this with (2), we see that

$$h_k = \frac{2^{2k}}{\pi^{2k}} \zeta^*(2k). \tag{4}$$

From (1) we get

$$h_{j_1,\ldots,j_r} = \frac{1}{\alpha_1! \cdots \alpha_k!} \frac{2^{2k}}{\pi^{2k}} \sum_{\mathcal{P}} (-1)^{r-\ell} c_{\mathcal{P}} \, \zeta^*(2k_1) \cdots \zeta^*(2k_\ell), \tag{5}$$

where the notation is as in Theorem 5.

The next observation is that the sum in the right hand side bears a striking resemblance with the right hand side of Hoffman's formula [4, Theorem 2.2] (proved anew in Theorem 7 below), which relates multiple zeta values and products of zeta values—the only difference is that ζ^* appears instead of ζ. The second step in the proof is then to find a Hoffman-like formula for ζ^*. Here is what we were led to write down: Define, for real numbers $s_1, \ldots, s_r > 1$,

$$T(s_1, \ldots, s_r) = \sum_{n_1 \geq_2 \cdots \geq_2 n_r \geq 1} \frac{(-1)^{n_1+\cdots+n_r}}{n_1^{s_1} \cdots n_r^{s_r}},$$

where $n \geq_2 m$ means "$n \geq m$ with equality only if n is even". Then symmetrize, and define

$$T^{\Sigma}(s_1, \ldots, s_r) = \sum_{\sigma \in \Sigma_r} T(s_{\sigma_1}, \ldots, s_{\sigma_r}).$$

Here is our Hoffman-like formula. Together with (5) it implies Theorem 1.

Theorem 6 *The following equality holds for all real $s_1, \ldots, s_r > 1$,*

$$\sum_{\mathcal{P}} (-1)^{r-\ell} c_{\mathcal{P}} \, \zeta^*(\underline{s}, \mathcal{P}) = (-1)^r T^{\Sigma}(s_1, \ldots, s_r), \tag{6}$$

where the sum is over all partitions $\mathcal{P} = \{P_1, \ldots, P_\ell\}$ of $\{1, 2, \ldots, r\}$ and

$$c_{\mathcal{P}} = (|P_1| - 1)! \cdots (|P_\ell| - 1)!, \quad \zeta^*(\underline{s}, \mathcal{P}) = \zeta^*\left(\sum_{i \in P_1} s_i\right) \cdots \zeta^*\left(\sum_{i \in P_\ell} s_i\right).$$

Proof This will follow by specialization of Theorem 10 below. □

Next we turn to the proof of Theorem 2, which says that

$$T(s_1, \ldots, s_r) < 0$$

for all real $s_1, \ldots, s_r > 1$.

Proof of Theorem 2 The proof is in principle not more difficult than the proof that $\zeta^*(s)$ is positive; to see this one simply arranges the sum as

$$\zeta^*(s) = \left(1 - \frac{1}{2^s}\right) + \left(\frac{1}{3^s} - \frac{1}{4^s}\right) + \cdots + \left(\frac{1}{(2k-1)^s} - \frac{1}{(2k)^s}\right) + \cdots,$$

and notes that the summands are positive. Since the series is absolutely convergent, we are free to rearrange as we please, as the reader will recall from elementary analysis.

Towards the general case, introduce for $k \geq 1$ the auxiliary series

$$T_{2k}(s_1, \ldots, s_r) = \sum_{n_1 \geq_2 \cdots \geq_2 n_r \geq 2k} \frac{(-1)^{n_1+\cdots+n_r}}{n_1^{s_1} \cdots n_r^{s_r}}.$$

Then one can argue using the following two equalities, whose verification we leave to the reader:

$$T(s_1, \ldots, s_r) = \sum_{k \geq 1} \left(-\frac{1}{(2k-1)^{s_r}} + \frac{1}{(2k)^{s_r}} \right) T_{2k}(s_1, \ldots, s_{r-1}),$$

$$T_{2k}(s_1, \ldots, s_r)$$
$$= \sum_{\ell \geq k} \sum_{j=1}^{r} \frac{1}{(2\ell)^{s_r} \cdots (2\ell)^{s_{j+1}}} \left(\frac{1}{(2\ell)^{s_j}} - \frac{1}{(2\ell+1)^{s_j}} \right) T_{2\ell+2}(s_1, \ldots, s_{j-1}).$$

Here $T_{2\ell+2}(s_1, \ldots, s_{j-1})$ should be interpreted as 1 for $j = 1$. The second equality may be used to show that $T_{2k}(s_1, \ldots, s_r)$ is positive by induction on r. The first equality then shows that $T(s_1, \ldots, s_r)$ is negative. □

Finally, we turn to the proof of Theorem 4. The argument turns out to be easier in this case. Recall that the $\hat{A}$-genus has characteristic series

$$Q(z) = \frac{\sqrt{z}/2}{\sinh(\sqrt{z}/2)}.$$

Let us write

$$\hat{\mathsf{A}}_k = \sum a_{j_1,\ldots,j_r} \mathsf{p}_{j_1} \cdots \mathsf{p}_{j_r},$$

where the sum is over all partitions $(j_1, \ldots, j_r)$ of k. By using the Cauchy formula (see [3, p.11]) one can calculate the coefficient a_k of p_k in $\hat{\mathsf{A}}_k$. The result is

$$a_k = \frac{(-1)^k}{2(2k)!} B_k = -\frac{1}{(2\pi)^{2k}} \zeta(2k).$$

It follows that

$$a_{k_1} \cdots a_{k_\ell} = \frac{(-1)^\ell}{(2\pi)^{2k}} \zeta(2k_1) \cdots \zeta(2k_\ell),$$

for every partition $(k_1, \ldots, k_\ell)$ of k. Theorem 5 then yields

$$a_{j_1,\ldots,j_r} = \frac{1}{\alpha_1! \cdots \alpha_k!} \sum_{\mathcal{P}} (-1)^{r-\ell} c_{\mathcal{P}} a_{k_1} \cdots a_{k_\ell}$$
$$= \frac{1}{\alpha_1! \cdots \alpha_k!} \frac{(-1)^r}{(2\pi)^{2k}} \sum_{\mathcal{P}} c_{\mathcal{P}} \zeta(2k_1) \cdots \zeta(2k_\ell).$$

The terms in the sum are clearly positive, so we see already from this expression that $(-1)^r a_{j_1,\ldots,j_r} > 0$ for all partitions $(j_1, \ldots, j_r)$ of k. However, more can be said; the sum in the right hand side now not only resembles but is *equal* to the right hand side of another formula of Hoffman [4, Theorem 2.1]. In our notation this formula says that

$$S^{\Sigma}(s_1, \ldots, s_r) = \sum_{\mathcal{P}} c_{\mathcal{P}} \zeta(\underline{s}, \mathcal{P}).$$

This proves Theorem 4.

3 Combinatorics of infinite sums

The proofs of Theorems 5 and 6, as well as of Hoffman's formula, share the same combinatorial underpinnings; this is the topic of the present section.

Recall that a *partition* of a set S is a set of non-empty disjoint subsets,

$$\pi = \{\pi_1, \ldots, \pi_r\},$$

such that $S = \pi_1 \cup \cdots \cup \pi_r$. Write $\ell(\pi) = r$ for the *length* of π. The set of partitions Π_S is partially ordered by refinement, $\pi = \{\pi_1 \ldots, \pi_r\} \leq \rho = \{\rho_1, \ldots, \rho_\ell\}$ if and only if there is a partition $\mathcal{P} = \{P_1, \ldots, P_\ell\}$ of the set $\{1, 2, \ldots, r\}$ such that

$$\rho_i = \bigcup_{j \in P_i} \pi_j, \quad 1 \leq i \leq \ell. \tag{7}$$

We will write $\rho = \mathcal{P}(\pi)$ if (7) holds. Note that for every $\rho \geq \pi$ there is a unique partition $\mathcal{P}$ such that $\rho = \mathcal{P}(\pi)$.

We will consider certain formal power series in indeterminates a_n for $a \in S$ and positive integers n. For a subset $T \subseteq S$, write

$$f_T(n) = \prod_{a \in T} a_n.$$

For a partition $\pi = \{\pi_1, \ldots, \pi_r\}$ of S, consider the formal power series

$$p_\pi = \sum_{n_1,\ldots,n_r} f_{\pi_1}(n_1) \cdots f_{\pi_r}(n_r),$$
$$m_\pi = \sum_{\substack{n_1,\ldots,n_r \\ distinct}} f_{\pi_1}(n_1) \cdots f_{\pi_r}(n_r).$$

It is then immediate that

$$p_\pi = \sum_{\rho \geq \pi} m_\rho.$$

By applying the Möbius inversion formula (see e.g. [6, Proposition 3.7.2]), we get

$$m_\pi = \sum_{\rho \geq \pi} \mu(\pi, \rho) p_\rho. \tag{8}$$

The Möbius function of Π_S is given by

$$\mu(\pi, \rho) = (-1)^{\ell(\pi)-\ell(\rho)}(b_1 - 1)! \dots (b_{\ell(\rho)} - 1)!,$$

where the number

$$b_i = b_i(\pi, \rho) = |P_i|$$

counts how many 'π-blocks' ρ_i consists of, see e.g. [6, Example 3.10.4].

Let us first note that this gives a neat proof of Hoffman's formula (though we would be surprised if this has not been noticed before). Recall that the multiple zeta function is defined by

$$\zeta(s_1, \dots, s_r) = \sum_{n_1 > \dots > n_r \geq 1} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}},$$

for real $s_1, \dots, s_r > 1$.

Theorem 7 (Hoffman [4, Theorem 2.2])

$$\sum_{\sigma \in \Sigma_r} \zeta(s_{\sigma_1}, \dots, s_{\sigma_r}) = \sum_{\mathcal{P}} (-1)^{r-\ell} c_{\mathcal{P}} \zeta(\underline{s}, \mathcal{P}),$$

where the sum is over all partitions $\mathcal{P} = \{P_1, \dots, P_\ell\}$ *of* $\{1, 2, \dots, r\}$ *and*

$$c_{\mathcal{P}} = (|P_1| - 1)! \cdots (|P_\ell| - 1)!, \quad \zeta(\underline{s}, \mathcal{P}) = \zeta\left(\sum_{i \in P_1} s_i\right) \cdots \zeta\left(\sum_{i \in P_\ell} s_i\right).$$

Proof Take $S = \{1, 2, \dots, r\}$ and substitute a_n by $\frac{1}{n^{s_a}}$ for $a \in S$ in (8). □

Secondly, we will use (8) to express the monomial symmetric functions in terms of power sum symmetric functions. We refer to [7, Chapter 7] for a pleasant introduction to symmetric functions. For an integer partition $I = (i_1, \dots, i_r) \vdash k$, recall that the *power sum symmetric function* p_I is the formal power series in indeterminates $x_1, x_2, \dots$ defined by $p_I = p_{i_1} \cdots p_{i_r}$, where

$$p_j = \sum_i x_i^j.$$

The *monomial symmetric function* m_I is defined as the sum of all pairwise distinct monomials of the form $x_{\sigma_1}^{i_1} \cdots x_{\sigma_r}^{i_r}$.

Theorem 8 *For every $k \geq 1$ and every integer partition $I = (i_1, \dots, i_r)$ of k,*

$$m_I = \frac{1}{\alpha_1! \cdots \alpha_k!} \sum_{\mathcal{P}} (-1)^{r-\ell} c_{\mathcal{P}}\, p_J,$$

where the sum is over all partitions $\mathcal{P} = \{P_1, \dots, P_\ell\}$ of the set $\{1, 2, \dots, r\}$, the number α_j counts how many of $i_1, \dots, i_r$ are equal to j, and $J = (j_1, \dots, j_\ell)$ is given by

$$j_u = \sum_{v \in P_u} i_v, \quad 1 \leq u \leq \ell.$$

Proof Let S be any set with k elements. Perform the substitution $a_n = x_n$ for each $a \in S$ in the equality (8) and note that this takes p_π to p_I and m_π to $\alpha_1! \cdots \alpha_k!\, m_I$, where $I = \big(|\pi_1|, \dots, |\pi_r|\big)$ is the integer partition underlying the set partition π (assuming, as we may, $|\pi_1| \geq \cdots \geq |\pi_r|$). □

Next, we turn to the result that will specialize to our formula in Theorem 6. Consider the following alternating version of p_π:

$$\overline{p}_\pi = \sum_{n_1, \dots, n_r} (-1)^{n_1 + \cdots + n_r} f_{\pi_1}(n_1) \cdots f_{\pi_r}(n_r).$$

For an *ordered* partition $\widetilde{\pi} = (\pi_1, \dots, \pi_r)$ we define

$$T_{\widetilde{\pi}} = \sum_{n_1 \geq_2 \cdots \geq_2 n_r} (-1)^{n_1 + \cdots + n_r} f_{\pi_1}(n_1) \cdots f_{\pi_r}(n_r).$$

Then for an unordered partition $\pi = \{\pi_1, \dots, \pi_r\}$, define

$$T_\pi^\Sigma = \sum_{\widetilde{\pi}} T_{\widetilde{\pi}},$$

where the sum is over the $r!$ ordered partitions $\widetilde{\pi}$ whose underlying unordered partition is π.

Lemma 9 *For every partition $\pi \in \Pi_S$,*

$$\sum_{\rho \geq \pi} (-1)^{\ell(\rho)} \ell(\rho)! = (-1)^{\ell(\pi)}.$$

Proof We may without loss of generality assume that π is the minimal element, because the poset $\{\rho \in \Pi_S \mid \rho \geq \pi\}$ is isomorphic to the poset Π_π of partitions of the set π. Then $n = \ell(\pi)$ is the number of elements of S. The number of partitions of length k in Π_S is equal to the Stirling number of the second kind $S(n,k)$, see e.g. [6, Example 3.10.4]. Thus,

$$\sum_{\rho\in\Pi_S}(-1)^{\ell(\rho)}\ell(\rho)! = \sum_{k=1}^{n}(-1)^k S(n,k)k!. \tag{9}$$

By plugging in $x=-1$ in the well-known identity

$$\sum_{k=1}^{n} S(n,k)(x)_k = x^n,$$

where $(x)_k = x(x-1)(x-2)\cdots(x-k+1)$, we see that (9) equals $(-1)^n$. □

Theorem 10 *For every partition π,*

$$(-1)^{\ell(\pi)}T_\pi^\Sigma = \sum_{\rho\geq\pi}(-1)^{\ell(\rho)}\mu(\pi,\rho)\overline{p}_\rho. \tag{10}$$

Proof By Möbius inversion, the equality is equivalent to

$$(-1)^{\ell(\pi)}\overline{p}_\pi = \sum_{\rho\geq\pi}(-1)^{\ell(\rho)}T_\rho^\Sigma, \tag{11}$$

and we proceed to prove (11). It is clear that both sides can be written as linear combinations of series of the form

$$m_{\nu,e} = \sum_{\substack{n_1,\ldots,n_m\ distinct\\ n_i\equiv_2 e_i}} f_{\nu_1}(n_1)\cdots f_{\nu_m}(n_m),$$

for various $\nu=\{\nu_1,\ldots,\nu_m\}\geq\pi$, where e is an assignment of a parity $e_i\in\{0,1\}$ to each ν_i. For example, if $\nu=\{\{a,b\},\{c\}\}$ and e assigns 1 to $\{a,b\}$ and 0 to $\{c\}$, then

$$m_{\nu,e} = \sum_{n_1\ odd,\ n_2\ even} a_{n_1}b_{n_1}c_{n_2}.$$

The question is with what coefficients $m_{\nu,e}$ will appear in the respective sides of (11). For the left hand side this is not difficult: $m_{\nu,e}$ appears in $(-1)^{\ell(\pi)}\overline{p}_\pi$ with coefficient

$$\operatorname{sgn}(\pi,\nu,e) = (-1)^{v_1(e_1-1)+\cdots+v_m(e_m-1)}, \tag{12}$$

where v_i is the number of π-blocks in ν_i.

The right hand side requires a little more effort—and notation. It is clear that $m_{\nu,e}$ appears in T_ρ^Σ only if $\nu\geq\rho$ and e assigns an even value to ν_i whenever ν_i consists of more than one ρ-block. This can be reformulated as saying that $\nu\geq\rho\geq e(\nu)$, where $e(\nu)\leq\nu$ is the partition that keeps ν_i intact if e_i is odd and splits ν_i completely if e_i is even. Or more precisely, $e(\nu)$ is the smallest element below ν that contains ν_i whenever e_i is odd. Since we symmetrize, there will be repetitions; for $\nu \geq \rho \geq e(\nu)$, the term involving $m_{\nu,e}$ will be repeated $b_{\rho,\nu} = b_1! \cdots b_m!$ times in T_ρ^Σ, where b_i is the number of ρ-blocks in ν_i. Thus, the coefficient of $m_{\nu,e}$ in $(-1)^{\ell(\rho)} T_\rho^\Sigma$ is $\mathrm{sgn}(\rho, \nu, e) b_{\rho,\nu}$. It follows that the coefficient of $m_{\nu,e}$ in $\sum_{\rho \geq \pi} (-1)^{\ell(\rho)} T_\rho^\Sigma$ is

$$\sum_{\nu \geq \rho \geq e(\nu) \vee \pi} \mathrm{sgn}(\rho, \nu, e) b_{\rho,\nu} = \sum_{\nu \geq \rho \geq e(\nu) \vee \pi} (-1)^{b_1(e_1 - 1) + \cdots + b_m(e_m - 1)} b_1! \cdots b_m!, \tag{13}$$

where $e(\nu) \vee \pi$ is the least upper bound of $e(\nu)$ and π.

Put $\pi_{(i)} = \{\pi_j \in \pi : \pi_j \subseteq \nu_i\}$ and $\nu_{(i)} = \{\nu_i\}$. We then have an isomorphism of posets $[e(\nu) \vee \pi, \nu] \cong \prod_{i:e_i=0} [\pi_{(i)}, \nu_{(i)}]$. Under this isomorphism $\rho \in [e(\nu) \vee \pi, \nu]$ is sent to $\rho_{(i)} = \{\rho_j \in \rho : \rho_j \subseteq \nu_i\} \in [\pi_{(i)}, \nu_{(i)}]$. Note that b_i is the length of $\rho_{(i)}$. We now find that the sum (13) factors as a product

$$\prod_{i:e_i=0} \sum_{\nu_{(i)} \geq \rho_{(i)} \geq \pi_{(i)}} (-1)^{b_i} b_i!.$$

By Lemma 9, this is equal to

$$\prod_{i:e_i=0} (-1)^{\nu_i}.$$

This shows that (13) equals (12), and the theorem is proved. □

To prove Theorem 6, take $S = \{1, 2, \ldots, r\}$ and substitute a_n by $\frac{1}{n^{s_a}}$ in (10).

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