




Moduli of non-commutative polarized schemes

K. Behrend¹ · B. Noohi² 

Received: 21 April 2016 / Revised: 17 November 2017 / Published online: 30 November 2017
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Abstract We construct, using geometric invariant theory, a quasi-projective Deligne–Mumford stack of stable graded algebras. We also construct a derived enhancement, which classifies twisted bundles of stable graded A_∞ -algebras. The tangent complex of the derived scheme is given by graded Hochschild cohomology, which we relate to ordinary Hochschild cohomology. We obtain a version of Hilbert stability for non-commutative projective schemes.

Contents

1	Introduction	1376
1.1	Notation and conventions	1377
2	The derived stack of graded algebras	1377
2.1	Gerstenhaber bracket	1377
2.2	Maurer–Cartan equation	1378
2.3	Gauge group: moduli stack	1378
2.3.1	Derived moduli stack of algebras	1380
2.4	Hochschild cohomology: deformation theory	1380
2.5	Truncation	1381
3	Graded Hochschild cohomology	1382
3.1	Hochschild cohomology of a polarized Grothendieck category	1382

Communicated by Jean-Yves Welschinger.

✉ B. Noohi
b.noohi@qmul.ac.uk
K. Behrend
behrend@math.ubc.ca

¹ University of British Columbia, Vancouver, Canada

² Queen Mary University of London, London, UK

3.1.1 Preliminaries	1382
3.1.2 Polarized Grothendieck categories	1384
3.1.3 Reduced Hochschild cohomology	1384
3.1.4 Graded Hochschild cohomology	1385
3.1.5 Heuristic remarks	1388
3.2 Relative Hochschild cohomology (commutative case)	1389
3.2.1 Relative Hochschild cohomology for schemes	1389
3.2.2 Equivariant Hochschild cohomology	1390
3.2.3 Relation to ordinary Hochschild cohomology	1391
3.2.4 A lemma on Hochschild cohomology of quasi-affine schemes	1392
3.2.5 Graded Hochschild cohomology	1393
3.3 The smooth case	1395
3.3.1 Hochschild–Kostant–Rosenberg	1395
3.3.2 Further considerations	1395
4 Stability for graded algebras	1397
4.1 The GIT problem	1397
4.2 The Hilbert–Mumford criterion	1399
4.3 Reformulation using test configurations	1400
4.3.1 Test configurations for A	1400
4.3.2 Admissible test configurations	1401
4.3.3 Standard admissible test configurations	1402
4.3.4 Stability	1402
4.4 Standard stability parameters	1403
4.4.1 Moduli	1405
4.5 Unbounded algebras	1405
4.5.1 Commutative case	1406
4.5.2 Moduli	1406
4.5.3 Discussion	1407
References	1408

1 Introduction

All our graded algebras will be unital and associative, with finite dimensional graded pieces.

We study derived moduli of graded algebras. In the first part of this paper, we construct a differential graded stack X , classifying graded algebras of a fixed dimension $\mathbf{d} = (d_1, d_2, \dots)$. The construction is as a stack quotient of a vector bundle of curved differential graded Lie algebras over a linear space, divided by an algebraic ‘gauge’ group. The construction is infinite-dimensional, but equal to the projective limit of its finite dimensional truncations.

For a graded algebra A , representing the point P of X , the tangent complex of X at P has (shifted) Hochschild cohomology of A , computed with homogeneous cochains of degree 0, for cohomology groups: $H^i(\mathbb{T}_X|_P) = HH_{\text{gr}}^{i+1}(A)$. In other words, the derived deformation theory of a graded algebra is given by its (shifted) graded Hochschild cohomology.

In the second section, we study graded Hochschild cohomology in some detail, and relate it to more familiar invariants. We do this for algebras A ‘coming from geometry’, by which we mean that there exists a \mathbb{C} -linear Grothendieck category \mathcal{C} , an object \mathcal{O} of \mathcal{C} , and an autoequivalence s of \mathcal{C} , satisfying suitable hypotheses, such that $A = \bigoplus_{n \geq 0} \text{Hom}_{\mathcal{C}}(\mathcal{O}, s^n \mathcal{O})$. In the commutative case, this essentially means that $A = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}(n))$, for a projective scheme $(X, \mathcal{O}(1))$.

Our results can be understood as supporting the idea that (under certain hypotheses), the derived deformation theory of the graded algebra A coincides with the derived deformation theory of the triple $(\mathcal{C}, \mathcal{O}, s)$.

In the last part of the paper we define a notion of *stability* for graded algebras and construct a (derived) separated Deligne–Mumford stack \tilde{X}^s classifying stable graded algebras of fixed dimension vector. Our notion of stability comes from geometric invariant theory for the finite-dimensional truncations $\tilde{X}_{\leq q}$ of \tilde{X} . We expect that for many interesting dimension vectors the stack \tilde{X}^s (or at least an interesting substack) will be of finite type.

In the commutative case, our notion of stability coincides with the classical notion of Hilbert stability. Thus our stack \tilde{X}^s extends the classical stack of Hilbert stable projective varieties into the non-commutative world.

We speculate that the stack \tilde{X}^s (or a suitable open substack) is a moduli stack of non-commutative projective schemes in the sense of Artin-Zhang [1].

As an example, 3-dimensional quadratic Artin-Schelter regular algebras are semi-stable, and generically stable [3].

1.1 Notation and conventions

We work over a field of characteristic zero, which we shall denote by \mathbb{C} . Unless specified otherwise, a graded vector space will refer to a \mathbb{Z} -graded vector space. A graded vector space is *locally finite*, if each graded piece is finite dimensional over \mathbb{C} .

All our graded algebras will be unital and associative, locally finite and graded in non-negative degrees. The component in degree zero will be assumed to be equal to the ground field \mathbb{C} . Often we will replace such an algebra A by its graded ideal $A_{>0}$ of elements of positive degree (A can be recovered from $A_{>0}$ in a canonical way).

Our algebraic stacks will have affine diagonal, but we do not require the diagonal to be of finite type, in general.

We follow the Bourbaki convention that set inclusion (proper or not) is denoted by ‘ \subset ’.

2 The derived stack of graded algebras

2.1 Gerstenhaber bracket

Let $V = \bigoplus_{n>0} V_n$ be a locally finite positively graded vector space. For $p \geq 0$, let

$$L^p = \text{Hom}_{\text{gr}}(V^{\otimes(p+1)}, V)$$

be the vector space of $(p + 1)$ -ary multilinear operations on V , which preserve degree. We have

$$L^p = \prod_n \text{Hom}((V^{\otimes(p+1)})_n, V_n),$$

which is a product of finite dimensional vector spaces. Therefore, it is an affine \mathbb{C} -scheme. If V is finite dimensional, it is an affine \mathbb{C} -scheme of finite type.

On $L = \bigoplus_{p \geq 0} L^p$ we introduce a (non-associative) product $\circ : L^p \otimes L^q \rightarrow L^{p+q}$ by the formula

$$(\mu \circ \nu)(a_0, \dots, a_{p+q}) = \sum_{i=0}^p (-1)^{iq} \mu(a_0, \dots, \nu(a_i, \dots, a_{i+q}), \dots, a_{p+q}).$$

We antisymmetrize and obtain the *Gerstenhaber bracket*

$$[\mu, \nu] = \mu \circ \nu - (-1)^{pq} \nu \circ \mu.$$

The pair $(L, [,])$ is a graded Lie algebra (see [5]). It is finite dimensional, if V is finite dimensional.

Augmentation. Sometimes it will be convenient to augment L by putting a copy of \mathbb{C} in degree -1 , i.e., setting $L^{-1} = \mathbb{C}$, and defining the differential $L^{-1} \rightarrow L^0$ to be the map $\mathbb{C} \rightarrow \text{Hom}_{\text{gr}}(V, V)$ given by the tautological graded endomorphism γ of V , which is multiplication by the degree, so $\gamma(\mu) = \text{deg}(\mu)\mu$, for homogeneous elements $\mu \in V$. Define the bracket by $[L^{-1}, L] = 0$. The fact that the augmented object is a differential graded Lie algebra follows from the fact that the tautological endomorphism γ is central in L . (This kind of construction would not work with the identity in place of γ , as the identity is not central.) We denote by \tilde{L} the graded Lie algebra obtained by dividing L in degree 0 by the ideal $\mathbb{C}\gamma$. Note that \tilde{L} is quasi-isomorphic to the augmented L .

2.2 Maurer–Cartan equation

The Maurer–Cartan equation for L is

$$\mu \circ \mu = 0, \quad \mu \in L^1.$$

Thus a Maurer–Cartan element μ is a degree preserving binary operation $\mu : V \otimes V \rightarrow V$, satisfying the equation

$$(\mu \circ \mu)(a, b, c) = 0, \quad \text{for all } a, b, c \in V.$$

Equivalently,

$$\mu(\mu(a, b), c) = \mu(a, \mu(b, c)), \quad \text{for all } a, b, c \in V,$$

i.e., μ is associative. Thus the Maurer–Cartan locus $MC(L)$ of L is the scheme of all associative graded products on V . It is a closed subscheme of the affine scheme L^1 .

2.3 Gauge group: moduli stack

The *gauge group* of L is $G = \prod_n GL(V_n)$. It is an affine group scheme over \mathbb{C} , and it is algebraic, if V is finite dimensional. Its Lie algebra is $L^0 = \text{Hom}_{\text{gr}}(V, V)$. The

gauge group acts on L by conjugation, preserving the Gerstenhaber bracket, and hence the Maurer–Cartan locus. The *moduli stack* of L is the stack quotient

$$X = [MC(L)/G].$$

It classifies graded associative products on V up to change of basis in V . In other words, X classifies graded associative algebras (without unit), whose underlying graded vector space is isomorphic to V . The stack X is an algebraic stack with affine diagonal, although the diagonal is not of finite type, unless V is finite dimensional.

Let $d_i = \dim V_i$, and $\mathbf{d} = (d_1, d_2, \dots)$. The groupoid $X(T)$, for a scheme T , is the category of bundles of graded algebras of rank \mathbf{d} , parametrized by T . Such a bundle of algebras is given by a graded vector bundle $\mathcal{V} = \bigoplus_{n>0} \mathcal{V}_n$ over T , where $\text{rank } \mathcal{V}_n = d_n$, endowed with \mathcal{O}_T -bilinear operations $\mathcal{V}_i \otimes \mathcal{V}_j \rightarrow \mathcal{V}_{i+j}$, satisfying associativity. (We can always add to such a bundle of graded algebras a copy of \mathcal{O}_T in degree 0, and make it into a bundle of unital algebras, in a canonical way.)

Definition 2.1 For a graded algebra, we call the automorphisms ϕ_λ , for $\lambda \in \mathbb{G}_m$, given by $\phi_\lambda(a) = \lambda^{\deg a} a$ on homogeneous elements, **tautological**. The tautological automorphisms define the tautological one-parameter group of automorphisms.

This leads to a modified moduli problem: Denote by Γ the central one-parameter subgroup of G which acts with weight n on V_n , for all n . Let \tilde{G} be G/Γ . The Lie algebra of \tilde{G} is \tilde{L}^0 . The group \tilde{G} acts by conjugation on \tilde{L} , and we call \tilde{G} the gauge group of \tilde{L} . It is an affine group scheme over \mathbb{C} . Consider the quotient stack

$$\tilde{X} = [MC(L)/\tilde{G}],$$

which is again an algebraic stack, the moduli stack of \tilde{L} .

We have a morphism of stacks $X \rightarrow \tilde{X}$, which is a \mathbb{G}_m -gerbe.

The moduli problem solved by \tilde{X} is the following: for a scheme T , the groupoid $\tilde{X}(T)$ is the groupoid of pairs $(\mathfrak{X}, \mathcal{V})$, where \mathfrak{X} is a \mathbb{G}_m -gerbe over T , and $\mathcal{V} = \bigoplus_{n>0} \mathcal{V}_n$ is an \mathfrak{X} -twisted vector bundle on T , where \mathcal{V}_n is n -twisted, and $\text{rank}(\mathcal{V}_n) = d_n$, for all $n > 0$. Moreover, \mathcal{V} is endowed with the structure of graded algebra. We call such pairs $(\mathfrak{X}, \mathcal{V})$ *twisted bundles of graded algebras*.

For a review of twisted sheaves, see [9]. Our terminology is as follows: a quasi-coherent \mathfrak{X} -twisted sheaf \mathcal{F} on T is a quasi-coherent sheaf on \mathfrak{X} . Such a sheaf decomposes naturally into a direct sum $\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n$, where on \mathcal{F}_n the natural inertia action is equal to the n -th power of the linear action given by the $\mathcal{O}_{\mathfrak{X}}$ -module structure on \mathcal{F} . If $\mathcal{F} = \mathcal{F}_n$, we refer to \mathcal{F} as n -twisted. If all \mathcal{F}_n are vector bundles, we call \mathcal{F} a twisted vector bundle.

If the components of the dimension vector \mathbf{d} of V are *strongly coprime*, by which we mean that there exists a k such that $(d_1, 2d_2, \dots, kd_k) = 1$, then the gerbe $X \rightarrow \tilde{X}$ is trivial. In this case, the universal twisted bundle of algebras can be represented by a bundle of algebras. It can be constructed by twisting the given action of G on V by the character $\chi : G \rightarrow \mathbb{G}_m$, defined by $\chi(g_1, g_2, \dots) = \det(g_1)^{r_1} \det(g_2)^{r_2} \dots$, where the r_i are such that $\sum i r_i d_i = 1$. The point is that this twist does not affect the action on L , but it changes the action on V in such a way that it factors through \tilde{G} .

2.3.1 Derived moduli stack of algebras

One of the simplest kinds of derived moduli stacks is given by a *bundle of curved differential graded Lie algebras* on a smooth algebraic stack (see [2], for the definitions). In the present case, the construction is as follows.

We start with the affine scheme L^1 , and construct over it a bundle of curved differential graded Lie algebras: the underlying graded vector bundle $\mathcal{L} = \bigoplus_{p \geq 2} \mathcal{L}^p$ is the trivial bundle over L^1 , with fibre L^p in degree p , for $p \geq 2$. The curvature map $L^1 \rightarrow L^2$, given by $x \mapsto x \circ x$, gives rise to a global section f of \mathcal{L}^2 , the *curvature* of our bundle of curved differential graded Lie algebras. The twisted differential $d^\mu : \mathcal{L}^i \rightarrow \mathcal{L}^{i+1}$ is given by $d^\mu = [\mu, \cdot]$, in the fibre over $\mu \in L^1$. The Lie bracket on \mathcal{L} is constant over L^1 , induced from the Gerstenhaber bracket in each fibre.

Then we notice that the gauge group action on L^1 lifts to an action on all of \mathcal{L} , preserving the structure of bundle of curved differential graded Lie algebras. Thus, this structure descends to the quotient stack $M = [L^1/G]$, giving rise to a bundle of curved differential graded Lie algebras over M . From now on, let us reserve the notation $(\mathcal{L}, f, d^\mu, [\cdot, \cdot])$ for the descendant bundle on M . (If V is finite dimensional, each \mathcal{L}^p is a bundle of finite rank.)

Our moduli stack X is now realized as the closed substack $X \subset M$, cut out by the vanishing of the curvature f of \mathcal{L} .

In [2], it is explained how a bundle of curved differential graded Lie algebras (M, \mathcal{L}) gives rise to a differential graded stack, which we shall denote by (M, \mathcal{R}_M) . In fact, the curved differential graded Lie algebra structure on \mathcal{L} defines a differential graded co-algebra structure on $\text{Sym } \mathcal{L}[1]$, which dualizes to a differential graded algebra structure on $\mathcal{R}_M = (\text{Sym } \mathcal{L}[1])^\vee$.

We also get a functor on differential graded schemes: if (T, \mathcal{R}_T) is a differential graded scheme, we associate to it the set of pairs (\mathcal{V}, μ) , where \mathcal{V} is a graded vector bundle of dimension \mathbf{d} over T , and μ is a global Maurer–Cartan element of the sheaf of differential graded Lie algebras

$$\text{Hom}_{\mathcal{O}_T}(\mathcal{V}^{\otimes \bullet + 1}, \mathcal{V}) \otimes_{\mathcal{O}_T} \mathcal{R}_T.$$

This is the same thing as the structure of a graded A_∞ -algebra on $\mathcal{V} \otimes_{\mathcal{O}_T} \mathcal{R}_T$.

We also have a bundle of curved differential graded Lie algebras over $M = [L^1/\tilde{G}]$, giving rise to a differential graded stack $(\tilde{M}, \tilde{\mathcal{R}}_{\tilde{M}})$, whose underlying classical stack is \tilde{X} . This gives rise to the derived stack of twisted bundles of graded A_∞ -algebras.

2.4 Hochschild cohomology: deformation theory

Let us consider a point of X , represented by the Maurer–Cartan element $\mu \in L^1$. The derived stack (M, \mathcal{L}) gives rise to a complex of vector bundles on X , the *tangent complex*, which governs deformations and obstructions of morphisms from square zero extensions of differential graded schemes. (For details, see [2]). At the point μ , this complex is our original graded Lie algebra L , endowed with the twisted differential

$d^\mu = [\mu, \cdot]$. This differential is the *Hochschild differential* of the associative algebra (V, μ) . It makes $(L, d^\mu, [, \cdot])$ into a differential graded Lie algebra.

Explicitly, for $\alpha \in L^p, \alpha : V^{\otimes p+1} \rightarrow V$, we have

$$(d^\mu \alpha)(a_0, \dots, a_{p+1}) = \alpha(a_0, \dots, a_p) a_{p+1} + (-1)^p a_0 \alpha(a_1, \dots, a_{p+1}) - (-1)^p \sum_{i=0}^p (-1)^i \alpha(\dots, a_i a_{i+1}, \dots),$$

where we have written μ as concatenation.

The cohomology spaces

$$H^p(L, d^\mu) = HH_{\text{gr}}^{p+1}(V, \mu)$$

are the *graded* Hochschild cohomology spaces of the graded associative algebra (V, μ) , computed with Hochschild cochains which are homogeneous of degree zero. Graded deformations/obstructions of the graded algebra (V, μ) are given by $H^1(L, d^\mu)$ and $H^2(L, d^\mu)$, respectively.

Explicitly, if $\alpha : V^{\otimes 2} \rightarrow V$ is a 1-cocycle with respect to d^μ (a Hochschild 2-cocycle), then

$$\alpha(a, b) c - \alpha(a, bc) + \alpha(ab, c) - a \alpha(b, c) = 0. \tag{1}$$

The corresponding infinitesimal deformation of (V, μ) is given by $V_\epsilon = V \oplus \epsilon V$ with multiplication $*$, which is determined on $V \subset V_\epsilon$ by

$$a * b = ab + \epsilon \alpha(a, b).$$

Associativity of $*$ follows from the cocycle condition (1).

If $\beta : V \rightarrow V$ is a 0-cochain (a Hochschild 1-cochain), then $\text{id} + \beta : V_\epsilon \rightarrow V_\epsilon$ defines an isomorphism from $*_\alpha$ to $*_{\alpha+d^\mu \beta}$.

The infinitesimal deformation given by α extends to $\mathbb{C}[\epsilon]/\epsilon^3$, if and only if the *primary obstruction* $\alpha \circ \alpha$ vanishes in $H^2(L, d^\mu) = HH_{\text{gr}}^3(V, \mu)$.

2.5 Truncation

Let $V \rightarrow V_{\leq q}$ be the truncation of V into degrees less than or equal to q , for a positive integer q . We will always consider $V_{\leq q}$ as a quotient of V . Repeating the above constructions with V replaced by $V_{\leq q}$, we obtain a finite dimensional graded Lie algebra $L_{\leq q} = \bigoplus L_{\leq q}^p$, together with an epimorphism of graded Lie algebras $L \rightarrow L_{\leq q}$. Let $X_{\leq q}$ and $\tilde{X}_{\leq q}$ denote the corresponding moduli stacks, which are algebraic stacks of finite type, whose diagonal is affine of finite type. We have

$$X = \varprojlim_q X_{\leq q}, \quad \tilde{X} = \varprojlim_q \tilde{X}_{\leq q}. \tag{2}$$

Let us write $M = [L^1/G]$ and $\tilde{M} = [L^1/\tilde{G}]$, etc. Then we have also $\tau_q : M \rightarrow M_{\leq q}$, and a morphism of bundles of curved differential graded Lie algebras

$$\mathcal{L} \rightarrow \tau_q^* \mathcal{L}_{\leq q},$$

for every q . Then

$$\mathcal{L} = \varprojlim_q \tau_q^* \mathcal{L}_{\leq q}, \tag{3}$$

(and a similar fact with tildes), as bundles of curved differential graded Lie algebras.

To state the compatibility with truncations on the level of deformation theory, let $A = (V, \mu)$ be an algebra giving rise to a point of X . Then we have

$$H^p(L, d^\mu) = \varprojlim_q H^p(L_{\leq q}, d^\mu),$$

as a direct consequence of (3). Thus, we also have

$$HH_{\text{gr}}^p(A, A) = \varprojlim_q HH_{\text{gr}}^p(A_{\leq q}, A_{\leq q}).$$

Remark 2.2 The projective system L_q is a projective system of \mathbb{C} -vector spaces, and all transition maps are obviously surjective. The reason to insist that we think of $V \rightarrow V_{\leq q}$ as a quotient map is only to prove that the Hochschild boundary commutes with the maps of the projective system. A simple argument with \varprojlim^1 proves that taking cohomology commutes with the projective limit.

3 Graded Hochschild cohomology

In this section we will relate graded Hochschild cohomology to more familiar invariants. We will do this for certain graded rings S which ‘come from geometry’. By this we mean that S is the homogeneous coordinate ring of a ‘sufficiently amply polarized’ non-commutative projective scheme (\mathcal{C}, A, s) in the sense of [1]. Since our hypotheses are going to diverge slightly from [1], we will call our triples (\mathcal{C}, A, s) *polarized Grothendieck categories*, rather than non-commutative projective schemes.

We will define the concept of *reduced Hochschild cohomology* for such a triple. We apologize for this abuse of established terminology.

3.1 Hochschild cohomology of a polarized Grothendieck category

3.1.1 Preliminaries

We summarize a result from [11], which allows us to write down a relatively small complex which computes the Hochschild cohomology of a Grothendieck category.

Let \mathcal{C} be a \mathbb{C} -linear Grothendieck category, and $A : u \rightarrow \mathcal{C}$ a \mathbb{C} -linear functor from a \mathbb{C} -linear category u . This situation gives rise to the Yoneda functor $\mathcal{C} \rightarrow \text{Mod}(u)$,

where $\text{Mod}(\mathfrak{u})$ is the category of right \mathfrak{u} -modules, i.e., the category of \mathbb{C} -linear functors $\mathfrak{u}^{\text{op}} \rightarrow (\mathbb{C}\text{-vector spaces})$.

We will need to assume that $\mathcal{C} \rightarrow \text{Mod}(\mathfrak{u})$ is fully faithful and has an exact left adjoint. By the Gabriel-Popescu theorem, for this it suffices that $\{A(u)\}_{u \in \text{ob } \mathfrak{u}}$ is a generating family for \mathcal{C} , and that $A : \mathfrak{u} \rightarrow \mathcal{C}$ is fully faithful. By Theorem 1.2 of [10], this latter condition can be weakened to

- (i) $A : \mathfrak{u} \rightarrow \mathcal{C}$ is faithful,
- (ii) for objects u, v in \mathfrak{u} , and a morphism $f : A(u) \rightarrow A(v)$ in \mathcal{C} , there always exists a family of morphisms $u_i \rightarrow u$ in \mathfrak{u} , such that $\coprod_i A(u_i) \rightarrow A(u)$ is an epimorphism in \mathcal{C} , and $f|_{A(u_i)} \in \mathfrak{u}$, for all i .

There exist (Section 1.10 in [7]) functorial injective resolutions for the objects of \mathfrak{u} . This means we have a 2-commutative diagram

$$\begin{array}{ccc}
 \mathfrak{u} & \xrightarrow{A} & \mathcal{C} \\
 & \searrow E & \downarrow \\
 & & C^\bullet(\mathcal{C})
 \end{array} \tag{4}$$

where $C^\bullet(\mathcal{C})$ denotes the differential graded category of complexes in \mathcal{C} . For every $u \in \mathfrak{u}$, the homomorphism of complexes $A(u) \rightarrow E(u)$ (given by the natural transformation ‘ \Rightarrow ’ in the diagram) is an injective resolution.

Denote by \tilde{E} the \mathfrak{u} -bimodule defined by the functor $E : \mathfrak{u} \rightarrow C^\bullet(\mathcal{C})$. We have

$$\tilde{E}(u, v) = \text{Hom}_{\mathcal{C}}^\bullet(E(u), E(v)) = \text{RHom}_{\mathcal{C}}(A(u), A(v)).$$

We shall consider the Hochschild cochain complex $C^\bullet(\mathfrak{u}, \tilde{E})$, see [11, (2.4)]. It is the product total complex of the double complex whose p -th column is given by

$$\prod_{u_0, \dots, u_p} \text{Hom}_{\mathbb{C}} \left(\text{Hom}_{\mathfrak{u}}(u_{p-1}, u_p) \otimes \dots \otimes \text{Hom}_{\mathfrak{u}}(u_0, u_1), \text{Hom}_{\mathcal{C}}^\bullet(E(u_0), E(u_p)) \right).$$

Proposition 3.1 [11, Lemma 5.4.2] *The complex $C^\bullet(\mathfrak{u}, \tilde{E})$ computes the Hochschild cohomology of \mathcal{C} as abelian category, and therefore governs the deformation theory of \mathcal{C} as abelian category.*

We will apply this result in the situation where $\{A(-n)\}_{n \in \mathbb{N}}$ is a family of objects of \mathcal{C} , such that for every N , the family $\{A(-n)\}_{n < N}$ generates \mathcal{C} . We let \mathfrak{u} be the category whose objects are the negative integers, and whose morphisms are given by

$$\mathfrak{u}(-m, -n) = \begin{cases} \text{Hom}_{\mathcal{C}}(A(-m), A(-n)) & \text{if } -m \leq -n \\ 0 & \text{if } -m > -n \end{cases}.$$

By construction, u comes with a faithful (although not necessarily full) functor $A : u \rightarrow \mathcal{C}$, which satisfies Condition 3.1.1, above. The Hochschild complex $C^\bullet(u, \tilde{E})$ is given by

$$\prod_{-n_0 \leq \dots \leq -n_p} \text{Hom}_{\mathbb{C}} \left(\text{Hom} (A(-n_{p-1}), A(-n_p)) \otimes \dots \otimes \text{Hom} (A(-n_0), A(-n_1)), \text{Hom}_{\mathcal{C}}^\bullet (E(-n_0), E(-n_p)) \right). \tag{5}$$

3.1.2 Polarized Grothendieck categories

Let \mathcal{C} be a \mathbb{C} -linear Grothendieck category. A **polarization** of \mathcal{C} is a pair (s, A) , where s is an auto-equivalence of \mathcal{C} , and A is an object of \mathcal{C} , such that

- (i) for every N , the family $(A(n))_{n < N}$ generates \mathcal{C} ,
- (ii) $\text{Ext}_{\mathcal{C}}^i(A, A(n)) = 0$, if $n > 0$, and $i > 0$,

where we have written $s^n A = A(n)$.

In addition, we will make the assumption that $\text{Hom}_{\mathcal{C}}(A, A) = \mathbb{C}$.

For example, the Grothendieck category of quasi-coherent \mathcal{O}_X -modules on a projective \mathbb{C} -scheme X is polarized by $(\mathcal{F} \mapsto \mathcal{F}(1), \mathcal{O}_X)$, if $\mathcal{O}_X(1)$ is ‘sufficiently ample’. It satisfies the additional assumption, if X is connected.

For another example, if (\mathcal{C}, A, s) is a finite-dimensional non-commutative projective scheme in the sense of [1], by which we mean that it satisfies the conditions (H1), (H2), (H3), (H4), and (H5) of [1], and has finite cohomological dimension, then (s, A) is a polarization of \mathcal{C} , if we replace s by a sufficiently large power.

As explained in [1, Proposition 4.2], we may, and shall, assume that s is an automorphism of \mathcal{C} , rather than an autoequivalence.

We choose functorial injective resolutions for the objects $A(-n)$, $n \in \mathbb{N}$, and use the complex $C^\bullet(u, \tilde{E})$, defined as above (5), to compute the Hochschild cohomology of \mathcal{C} .

3.1.3 Reduced Hochschild cohomology

Let \tilde{E}^* be the same u -bimodule as \tilde{E} , except that we set $\tilde{E}(-1, -n)$ equal to zero:

$$\tilde{E}^*(-m, -n) = \begin{cases} 0 & \text{if } -m = -1, \\ \tilde{E}(-m, -n) & \text{otherwise.} \end{cases}$$

By the definition of u , we have that \tilde{E}^* is a bi-submodule of \tilde{E} . Let \bar{E} be the quotient bimodule

$$0 \longrightarrow \tilde{E}^* \longrightarrow \tilde{E} \longrightarrow \bar{E} \longrightarrow 0.$$

Again, by the definition of u , we have for all p that the p -th column of $C^\bullet(u, \bar{E})$ is a single copy of $\tilde{E}(-1, -1) = \text{Hom}_{\mathcal{C}}^\bullet(E(-1), E(-1))$. The Hochschild differential is trivial, and therefore $C^\bullet(u, \bar{E})$ is quasi-isomorphic to $\tilde{E}(-1, -1) = \text{RHom}_{\mathcal{C}}(A, A)$.

We call the cohomology of $C^\bullet(u, \tilde{E}^*)$ the **reduced** Hochschild cohomology of \mathcal{C} , with respect to the base object A , notation $\overline{HH}^*(\mathcal{C}, A)$.

There is a short exact sequence of complexes

$$0 \longrightarrow C^\bullet(u, \tilde{E}^*) \longrightarrow C^\bullet(u, \tilde{E}) \longrightarrow C^\bullet(u, \bar{E}) \longrightarrow 0, \tag{6}$$

which gives rise to a long exact sequence in cohomology

$$\longrightarrow \overline{HH}^*(\mathcal{C}, A) \longrightarrow HH^*(\mathcal{C}) \longrightarrow \text{Ext}_{\mathcal{C}}^*(A, A) \xrightarrow{+1} . \tag{7}$$

Remark 3.2 As $HH^*(\mathcal{C})$ governs deformations of the abelian category \mathcal{C} , and $\text{Ext}_{\mathcal{C}}^*(A, A)$ governs deformations of the object A within \mathcal{C} (see [14]), the sequence (7) suggests that $\overline{HH}^*(\mathcal{C}, A)$ governs the deformations of the pair (\mathcal{C}, A) . This motivates our terminology. We apologize for the somewhat ad-hoc definition, which is motivated by its convenience for what follows.

3.1.4 Graded Hochschild cohomology

Define the unital graded \mathbb{C} -algebra

$$S = \bigoplus_{n \geq 0} \text{Hom}_{\mathcal{C}}(A, A(n)),$$

and the *graded* differential graded S -bimodule

$$M^\bullet = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}^\bullet(E, E(n)).$$

The grading coming from the autoequivalence s will be called the *projective* grading and will be denoted using lower indices, in contrast to the cohomological grading, which is indicated with superscripts.

We have the Hochschild complex of S with values in M^\bullet

$$C^\bullet(S, M^\bullet)$$

and the subcomplex

$$C_{\text{gr}}^\bullet(S, M^\bullet)$$

of projective degree 0 cochains. These are the cochains which preserve the projective degree.

Lemma 3.3 *We have a short exact sequence of complexes of \mathbb{C} -vector spaces*

$$0 \longrightarrow C_{\text{gr}}^\bullet(S, M^\bullet) \longrightarrow C^\bullet(u, \tilde{E}) \xrightarrow{1-s^{-1}} C^\bullet(u, \tilde{E}) \longrightarrow 0. \tag{8}$$

Proof During this proof we will disregard the vertical degree (the coefficient degree) and consider only the horizontal degree (the Hochschild degree). Thus, $C^p(u, \tilde{E})$ will denote the p -th column (5) of $C^\bullet(u, \tilde{E})$. The same applies to $C_{\text{gr}}^\bullet(S, M^\bullet)$.

A p -cochain $\chi \in C_{\text{gr}}^p(S, M^\bullet)$, is a family $(\chi_{\ell_1, \dots, \ell_p})_{\ell_1, \dots, \ell_p \geq 0}$, where

$$\chi_{\ell_1, \dots, \ell_p} : S_{\ell_p} \otimes \dots \otimes S_{\ell_1} \longrightarrow M_{\ell_1 + \dots + \ell_p}^\bullet$$

is a multilinear map. We associate to χ the p -cochain $\psi \in C^p(u, \tilde{E})$ given by the family $(\psi_{n_0, \dots, n_p})_{n_0 \geq \dots \geq n_p \geq 1}$, where

$$\begin{aligned} \psi_{n_0, \dots, n_p} : \text{Hom}_{\mathcal{C}}(A(-n_{p-1}), A(-n_p)) \otimes \dots \otimes \text{Hom}_{\mathcal{C}}(A(-n_0), A(-n_1)) \\ \longrightarrow \text{Hom}_{\mathcal{C}}^\bullet(E(-n_0), E(-n_p)) \end{aligned}$$

is the multilinear operation given by

$$\psi_{n_0, \dots, n_p}(\alpha_p, \dots, \alpha_1) = s^{-n_0} \chi_{n_0 - n_1, \dots, n_{p-1} - n_p}(s^{n_{p-1}} \alpha_p, \dots, s^{n_0} \alpha_1)$$

Sending χ to ψ defines the injective arrow in (8).

The functor s^{-1} restricts to a fully faithful functor $s^{-1} : u \rightarrow u$, and defines an endomorphism of the diagram (4), and so induces an endomorphism s^{-1} of $C^\bullet(u, \tilde{E})$. Given a p -cochain $\psi \in C^p(u, \tilde{E})$, the p -cochain $s^{-1}\psi \in C^p(u, \tilde{E})$ is given by

$$(s^{-1}\psi)_{n_0, \dots, n_p}(\alpha_p, \dots, \alpha_1) = s(\psi_{n_0+1, \dots, n_p+1}(s^{-1}\alpha_p, \dots, s^{-1}\alpha_1)).$$

So the condition $(1 - s^{-1})\psi = 0$ is equivalent to

$$s(\psi_{n_0+1, \dots, n_p+1}(s^{-1}\alpha_p, \dots, s^{-1}\alpha_1)) = \psi_{n_0, \dots, n_p}(\alpha_p, \dots, \alpha_1).$$

Such a ψ is the image of $\chi \in C_{\text{gr}}^p(S, M^\bullet)$, with

$$\chi_{\ell_1, \dots, \ell_p}(\alpha_p, \dots, \alpha_1) = s^{n_0}(\psi_{n_0, \dots, n_p}(s^{-n_{p-1}} \alpha_p, \dots, s^{-n_0} \alpha_1)),$$

where, for $i = 0, \dots, p$, we have used the abbreviation $n_i = n + \sum_{j>i} \ell_j$, for an arbitrary $n \geq 1$. This proves that (8) is exact in the middle.

To prove that $(1 - s^{-1})$ is surjective, note that given ϕ , the equation $\phi = (1 - s^{-1})\psi$ is equivalent to $s^{-1}\psi = \psi - \phi$, which is a recursive equation for the components of ψ in terms of those ψ_{n_0, \dots, n_p} with $n_p = 1$. □

Sequences (6) and (8) exhibit two subcomplexes of $C^\bullet(u, \tilde{E})$. In the intersection of $C^\bullet_{\text{gr}}(S, M^\bullet)$ and $C^\bullet(u, \tilde{E}^*)$ inside $C^\bullet(u, \tilde{E})$, there is $C^\bullet_{\text{gr}}(S, S_{>0})$, giving rise to the commutative diagram of complexes with exact rows and columns

$$\begin{CD}
 C^\bullet_{\text{gr}}(S, S_{>0}) @>>> C^\bullet(u, \tilde{E}^*) @>>> R \\
 @VVV @VVV @VV\beta V \\
 C^\bullet_{\text{gr}}(S, M^\bullet) @>>> C^\bullet(u, \tilde{E}) @>1-s^{-1}>> C^\bullet(u, \tilde{E}) \\
 @VVV @VVV @. \\
 Q @>\alpha>> C^\bullet(u, \bar{E})
 \end{CD} \tag{9}$$

Lemma 3.4 *Both α and β are quasi-isomorphisms.*

Proof In fact, the two claims are equivalent, so let us prove the one for α . Let us start by noting that in $C^\bullet_{\text{gr}}(S, M^\bullet)$, we can replace M^\bullet by

$$M_{\geq 0}^\bullet = \bigoplus_{n \geq 0} \text{Hom}_{\mathcal{C}}^\bullet(E, E(n)).$$

Consider the monomorphism of S -bimodules $S_{>0} \rightarrow M_{\geq 0}^\bullet$. By the second condition that we require of polarizations, the quotient of $M_{\geq 0}^\bullet$ modulo $S_{>0}$ is quasi-isomorphic to the bimodule $M_0 = \text{Hom}_{\mathcal{C}}^\bullet(E, E)$, which exists entirely in projective degree 0. It follows that Q is quasi-isomorphic to $C^\bullet_{\text{gr}}(S, M_0)$. But for every p , we have $C^p_{\text{gr}}(S, M_0) = M_0$. It follows that $C^\bullet_{\text{gr}}(S, M_0)$ is, in fact, quasi-isomorphic to $\text{Hom}_{\mathcal{C}}^\bullet(E, E) = R \text{Hom}_{\mathcal{C}}(A, A)$. The same is true for $C^\bullet(u, \bar{E})$.

We have used the fact that graded Hochschild cohomology of S is invariant under quasi-isomorphisms of the coefficient bimodule. This can be reduced to the case of Hochschild cohomology of the category u via Lemma 3.3, which applies to any u -bimodule. □

Corollary 3.5 *There is a distinguished triangle of complexes of \mathbb{C} -vector spaces*

$$C^\bullet_{\text{gr}}(S_{>0}, S_{>0}) \longrightarrow C^\bullet(u, \tilde{E}^*) \xrightarrow{1-s^{-1}} C^\bullet(u, \tilde{E}) \xrightarrow{+1} ,$$

and hence a long exact sequence in cohomology

$$\longrightarrow HH^*_{\text{gr}}(S_{>0}, S_{>0}) \longrightarrow \overline{HH}^*(\mathcal{C}, A) \xrightarrow{1-s^{-1}} HH^*(\mathcal{C}) \xrightarrow{+1} . \tag{10}$$

Proof This is where we use the connectedness assumption that $\text{Hom}_{\mathcal{C}}(A, A) = \mathbb{C}$. By this assumption, the normalized graded Hochschild complex of S with values in $S_{>0}$ is $C^\bullet(S_{>0}, S_{>0})$. □

Thus Diagram (9) gives rise to a diagram of long exact sequences in cohomology:

$$\begin{array}{ccccccc}
 HH_{\text{gr}}^*(S_{>0}, S_{>0}) & \longrightarrow & \overline{HH}^*(\mathcal{C}, A) & \xrightarrow{1-s^{-1}} & HH^*(\mathcal{C}) & \xrightarrow{+1} & \\
 \downarrow & & \downarrow & & \parallel & & \\
 HH_{\text{gr}}^*(S, M^\bullet) & \longrightarrow & HH^*(\mathcal{C}) & \xrightarrow{1-s^{-1}} & HH^*(\mathcal{C}) & \xrightarrow{+1} & \\
 \downarrow & & \downarrow & & & & \\
 \text{Ext}_{\mathcal{C}}^*(A, A) & \xlongequal{\quad} & \text{Ext}_{\mathcal{C}}^*(A, A) & & & & \\
 \downarrow +1 & & \downarrow +1 & & & &
 \end{array} \tag{11}$$

3.1.5 Heuristic remarks

Unfortunately, this result about

$$(L, d^\mu)[-1] = C_{\text{gr}}^\bullet(S_{>0}, S_{>0}),$$

with notation $S_{>0} = (V, \mu)$, is only about the tangent complex of our derived stack as a complex, disregarding the structure of deformation functor, i.e. the L_∞ -structure. We would like to make a few heuristic remarks, which may explain the provenance of Diagram (11).

There is an octahedron of deformation functors

$$\begin{array}{ccccc}
 & & \text{Def}_{\mathcal{C}}(A) \xlongequal{\quad} \text{Def}_{\mathcal{C}}(A) & & \\
 & & \downarrow & & \downarrow \\
 \text{Def}_{\mathcal{C}}(s) & \longrightarrow & \text{Def}(\mathcal{C}, s, A) & \longrightarrow & \text{Def}(\mathcal{C}, A) \\
 \parallel & & \downarrow & & \downarrow \\
 \text{Def}_{\mathcal{C}}(s) & \longrightarrow & \text{Def}(\mathcal{C}, s) & \longrightarrow & \text{Def}(\mathcal{C})
 \end{array}$$

Then there is an isomorphism $\text{Def}(\mathcal{C}) = \text{Def}_{\mathcal{C}}(s)[1]$, so we can rewrite this as

$$\begin{array}{ccccc}
 \text{Def}_{\mathcal{C}}(A) \xlongequal{\quad} \text{Def}_{\mathcal{C}}(A) & & & & \\
 \downarrow & & \downarrow & & \\
 \text{Def}(\mathcal{C}, s, A) & \longrightarrow & \text{Def}(\mathcal{C}, A) & \longrightarrow & \text{Def}(\mathcal{C}) \\
 \downarrow & & \downarrow & & \parallel \\
 \text{Def}(\mathcal{C}, s) & \longrightarrow & \text{Def}(\mathcal{C}) & \longrightarrow & \text{Def}(\mathcal{C})
 \end{array}$$

and as

$$\begin{array}{ccccc}
 \text{Def}(\mathcal{C}, s, A) & \longrightarrow & \text{Def}(\mathcal{C}, A) & \longrightarrow & \text{Def}(\mathcal{C}) \\
 \downarrow & & \downarrow & & \parallel \\
 \text{Def}(\mathcal{C}, s) & \longrightarrow & \text{Def}(\mathcal{C}) & \longrightarrow & \text{Def}(\mathcal{C}) \\
 \downarrow & & \downarrow & & \\
 \text{Def}_{\mathcal{C}}(A)[1] & \xlongequal{\quad} & \text{Def}_{\mathcal{C}}(A)[1] & &
 \end{array}$$

We believe that this latter diagram is, in fact, (11), and this justifies our suspicion that $C_{\text{gr}}^{\bullet}(S_{>0}, S_{>0})[+1]$ governs the deformation theory of the triple (\mathcal{C}, A, s) . From Sect. 2, we know that $C_{\text{gr}}^{\bullet}(S_{>0}, S_{>0})[+1]$ governs the deformation theory of (non-unital) graded algebras. This is consistent with the Artin-Zhang philosophy that graded algebras are just triples (\mathcal{C}, A, s) in disguise.

3.2 Relative Hochschild cohomology (commutative case)

In the commutative case, we can interpret graded Hochschild cohomology of a graded ring as *reduced equivariant* Hochschild cohomology. We will introduce this concept, and prove results analogous to the non-commutative case.

3.2.1 Relative Hochschild cohomology for schemes

Let X be a separated scheme and $X \rightarrow Y$ a separated morphism of algebraic stacks. Consider the diagonal morphism

$$\Delta : X \longrightarrow X \times_Y X,$$

which is a closed immersion of schemes. As for any closed immersion of schemes, the derived category object $L\Delta^* \Delta_* \mathcal{O}_X$ splits off $\mathcal{H}^0(L\Delta^* \Delta_* \mathcal{O}_X) = \Delta^* \Delta_* \mathcal{O}_X = \mathcal{O}_X$, and we write $(L\Delta^* \Delta_* \mathcal{O}_X)^{\text{red}}$ for the complement $\tau_{<0}(L\Delta^* \Delta_* \mathcal{O}_X)$.

For a sheaf of \mathcal{O}_X -modules \mathcal{F} , we define the **relative** Hochschild cohomology of X over Y with values in \mathcal{F} to be

$$HH_Y^*(X, \mathcal{F}) = R\text{Hom}(L\Delta^* \Delta_* \mathcal{O}_X, \mathcal{F}).$$

We also call

$$\overline{HH}_Y^*(X, \mathcal{F}) = R\text{Hom}((L\Delta^* \Delta_* \mathcal{O}_X)^{\text{red}}, \mathcal{F})$$

the **reduced** Hochschild cohomology of X over Y with values in \mathcal{F} . For $\mathcal{F} = \mathcal{O}_X$, we use the usual abbreviations

$$HH_Y^*(X) = HH_Y^*(X, \mathcal{O}_X), \quad \overline{HH}_Y^*(X) = \overline{HH}_Y^*(X, \mathcal{O}_X).$$

We have

$$\begin{aligned}
 HH_Y^*(X, \mathcal{F}) &= \overline{HH}_Y^*(X, \mathcal{F}) \oplus H^*(X, \mathcal{F}), \\
 HH_Y^*(X) &= \overline{HH}_Y^*(X) \oplus H^*(X, \mathcal{O}_X).
 \end{aligned}$$

3.2.2 Equivariant Hochschild cohomology

If G is a reductive algebraic group, $\pi : P \rightarrow X$ is a principal G -bundle, and $X \rightarrow BG$ the associated classifying morphism, then we write HH_G^* for HH_{BG}^* , and \overline{HH}_G^* for \overline{HH}_{BG}^* , and speak of *equivariant* (reduced) Hochschild cohomology.

Proposition 3.6 *For any quasi-coherent sheaf of \mathcal{O}_X -modules \mathcal{F} , There is a natural G -action on $HH^*(P, \pi^* \mathcal{F})$, and we have canonical isomorphisms*

$$HH_G^*(X, \mathcal{F}) = HH^*(P, \pi^* \mathcal{F})^G, \quad \overline{HH}_G^*(X, \mathcal{F}) = \overline{HH}^*(P, \pi^* \mathcal{F})^G.$$

In particular,

$$HH_G^*(X) = HH^*(P)^G, \quad \overline{HH}_G^*(X) = \overline{HH}^*(P)^G.$$

Proof Consider the cartesian diagram

$$\begin{array}{ccc}
 P & \xrightarrow{\Delta'} & P \times P \\
 \downarrow \pi & & \downarrow \tilde{\pi} \\
 X & \xrightarrow{\Delta} & X \times_{BG} X
 \end{array}$$

and write $\mathcal{A} = \pi_* \mathcal{O}_P$, so that P is the relative spectrum of the \mathcal{O}_X -algebra \mathcal{A} over X . By flat base change, we have

$$\pi^* L\Delta^* \Delta_* \mathcal{O}_X = L\Delta'^* \Delta'_* \mathcal{O}_P,$$

and therefore

$$R\text{Hom}(\pi^* L\Delta^* \Delta_* \mathcal{O}_X, \pi^* \mathcal{F}) = R\text{Hom}(L\Delta'^* \Delta'_* \mathcal{O}_P, \pi^* \mathcal{F}),$$

and by adjunction

$$HH_G^*(X, \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{F}) = HH^*(P, \pi^* \mathcal{F}).$$

We have a G -action on \mathcal{A} , and the invariants are $\mathcal{A}^G = \mathcal{O}_X$. We get an induced action on $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{F}$ with invariants \mathcal{F} , and an induced action on $HH_G^*(X, \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{F})$ with invariants $HH_G^*(X, \mathcal{F})$. This proves the claim for usual Hochschild cohomology. The proof goes through also in the reduced case. □

3.2.3 Relation to ordinary Hochschild cohomology

We specialize to the case $G = \mathbb{G}_m$.

Proposition 3.7 *Let X be a separated scheme and $X \rightarrow B\mathbb{G}_m$ a morphism. Denote the diagonal $X \rightarrow X \times X$ by Δ , and the diagonal $X \rightarrow X \times_{B\mathbb{G}_m} X$ by $\tilde{\Delta}$. Then in $D(\mathcal{O}_X)$ there are distinguished triangles*

$$L\Delta^* \Delta_* \mathcal{O}_X \xrightarrow{(t-1)} L\Delta^* \Delta_* \mathcal{O}_X \longrightarrow L\tilde{\Delta}^* \tilde{\Delta}_* \mathcal{O}_X \xrightarrow{+1}, \tag{12}$$

and

$$L\Delta^* \Delta_* \mathcal{O}_X \xrightarrow{(t-1)} (L\Delta^* \Delta_* \mathcal{O}_X)^{\text{red}} \longrightarrow (L\tilde{\Delta}^* \tilde{\Delta}_* \mathcal{O}_X)^{\text{red}} \xrightarrow{+1} .$$

Proof Let X be a scheme, and

$$\mathbb{G}_m \times X \xrightarrow{\iota} \tilde{R} \xrightarrow{\pi} R$$

a central extension of groupoids over X . The example which will concern us is given by $\tilde{R} = X \times_{B\mathbb{G}_m} X$, and $R = X \times X$. Denote the identity sections of \tilde{R} and R by $\tilde{\Delta}$ and Δ , respectively, and assume that Δ is a closed immersion. Then ι is a closed immersion, as it is a pullback of Δ . Let us denote the identity of $\mathbb{G}_m \times X$ by e , and let t be the standard coordinate on \mathbb{G}_m .

We have a short exact sequence of sheaves of \mathcal{O} -modules on $\mathbb{G}_m \times X$

$$0 \longrightarrow \mathcal{O}_{\mathbb{G}_m \times X} \xrightarrow{(t-1)} \mathcal{O}_{\mathbb{G}_m \times X} \longrightarrow e_* \mathcal{O}_X \longrightarrow 0 .$$

Applying ι_* we get the short exact sequence

$$0 \longrightarrow \iota_* \mathcal{O}_{\mathbb{G}_m \times X} \xrightarrow{(t-1)} \iota_* \mathcal{O}_{\mathbb{G}_m \times X} \longrightarrow \iota_* e_* \mathcal{O}_X \longrightarrow 0 .$$

We have $\iota_* \mathcal{O}_{\mathbb{G}_m \times X} = \pi^* \Delta_* \mathcal{O}_X$, and $\iota_* e_* \mathcal{O}_X = \tilde{\Delta}^* \mathcal{O}_X$, by the cartesian diagram

$$\begin{array}{ccc} \mathbb{G}_m \times X & \xrightarrow{\iota} & \tilde{R} \\ \downarrow & \nearrow \tilde{\Delta} & \downarrow \pi \\ X & \xrightarrow{\Delta} & R \end{array}$$

So we can rewrite our exact sequence as

$$0 \longrightarrow \pi^* \Delta_* \mathcal{O}_X \xrightarrow{(t-1)} \pi^* \Delta_* \mathcal{O}_X \longrightarrow \tilde{\Delta}_* \mathcal{O}_X \longrightarrow 0 .$$

Now we apply $L\tilde{\Delta}^*$ to this exact sequence of \mathcal{O} -modules on \tilde{R} , to obtain the distinguished triangle (12). □

Corollary 3.8 *There are long exact sequences*

$$\longrightarrow HH_{\mathbb{G}_m}^*(X) \longrightarrow HH^*(X) \longrightarrow HH^*(X) \xrightarrow{+1} ,$$

and

$$\longrightarrow \overline{HH}_{\mathbb{G}_m}^*(X) \longrightarrow \overline{HH}^*(X) \longrightarrow HH^*(X) \xrightarrow{+1} .$$

3.2.4 A lemma on Hochschild cohomology of quasi-affine schemes

If X is quasi-affine, say $X \subset V = \text{Spec } A$, we can apply the usual tilde construction to the Hochschild complex $C_{\bullet}(A)$ of A . We obtain a complex of quasi-coherent sheaves $C_{\bullet}(A) \sim|_X$ on X , whose component in degree p is the free \mathcal{O}_X -module

$$C_p(A) \sim|_X = \mathcal{O}_X \otimes_{\mathbb{C}} A^{\otimes p}.$$

Removing the degree 0 part from $C_{\bullet}(A)$ gives the reduced Hochschild complex $\overline{C}_{\bullet}(A)$, and the associated complex of quasi-coherent sheaves $\overline{C}_{\bullet}(A) \sim|_X$, which is obtained from $C_{\bullet}(A) \sim|_X$ by removing the component in degree 0.

Lemma 3.9 *In the derived category of X , the complex $C_{\bullet}(A) \sim|_X$ represents the object $L\Delta^* \Delta_* \mathcal{O}_X$, where $\Delta : X \rightarrow X \times X$ is the absolute diagonal. Moreover, the complex $\overline{C}_{\bullet}(A) \sim|_X$ represents $(L\Delta^* \Delta_* \mathcal{O}_X)^{\text{red}}$.*

Proof This follows from [13], where it is proved that on a quasi-projective scheme the complex of non-quasi-coherent sheaves \mathcal{C}_{\bullet}^X , which sheafifies the Hochschild complex, represents the derived category object $L\Delta^* \Delta_* \mathcal{O}_X$.

Then we have a canonical quasi-isomorphism

$$C_{\bullet}(A) \sim|_X \xrightarrow{\sim} \mathcal{C}_{\bullet}^X,$$

because Hochschild homology commutes with localization. □

Now suppose $\mathcal{F} = \widetilde{M}|_X$, for an A -module M .

Lemma 3.10 *We have spectral sequences*

$$\begin{aligned} E_2^{pq} &= HH^p(A, H^q(X, \mathcal{F})) \implies HH^{p+q}(X, \mathcal{F}), \\ E_2^{pq} &= \overline{HH}^p(A, H^q(X, \mathcal{F})) \implies \overline{HH}^{p+q}(X, \mathcal{F}). \end{aligned}$$

Proof By the previous lemma, the derived category object $R\mathcal{H}om(L\Delta^* \Delta_* \mathcal{O}_X, \mathcal{F})$ can be represented by the complex $C^{\bullet}(A, \mathcal{F})$ of sheaves on X , whose degree p component is

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X \otimes_{\mathbb{C}} A^{\otimes p}, \mathcal{F}) = \text{Hom}_{\mathbb{C}}(A^{\otimes p}, \mathcal{F}),$$

i.e., an infinite product of copies of \mathcal{F} . It follows that Hochschild cohomology of X with values in \mathcal{F} is equal to hypercohomology

$$HH^*(X, \mathcal{F}) = \mathbb{H}^*(X, C^\bullet(A, \mathcal{F})).$$

This hypercohomology can be computed using a finite affine Čech cover \mathcal{U} of X , because an infinite product of quasi-coherent sheaves is acyclic over an affine scheme (even though not quasi-coherent in itself). Thus

$$\begin{aligned} \mathbb{H}^*(X, C^\bullet(A, \mathcal{F})) &= \text{tot} \left(\check{C}^\bullet(\mathcal{U}, C^\bullet(A, \mathcal{F})) \right) \\ &= \text{tot} \left(C^\bullet(A, \check{C}^\bullet(\mathcal{U}, \mathcal{F})) \right). \end{aligned}$$

We now consider the double complex. Computing cohomology in the Čech direction gives us $C^\bullet(A, H^q(X, \mathcal{F}))$, because infinite products are exact in the category of A -modules. Next, computing cohomology in the Hochschild direction gives us $HH^p(A, H^q(X, \mathcal{F}))$, by definition. Thus the desired spectral sequence is the standard E_2 spectral sequence of our double complex.

The proof is the same for the reduced case. □

3.2.5 Graded Hochschild cohomology

Let A be a locally finite commutative graded \mathbb{C} -algebra, such that

- (i) A is graded in non-negative degrees: $A = A_{\geq 0}$,
- (ii) A is connected: $A_0 = \mathbb{C}$,
- (iii) A is generated in degree 1.

Let $V = \text{Spec } A$, and $Y = V \setminus \{0\}$, where $0 \in V$ is the vertex defined by the homogeneous maximal ideal $A_{>0}$. Moreover, let $X = Y/\mathbb{G}_m = \mathbb{P}\text{roj } A$, and denote the quotient map by $\pi : Y \rightarrow X$. Assume further that

- (iv) for all $n > 0$, the homomorphism $A_n \rightarrow \Gamma(X, \mathcal{O}_X(n))$ is bijective,
- (v) for all $q > 0$ and $n > 0$, we have $H^q(X, \mathcal{O}_X(n)) = 0$.

Let us remark that

$$H^q(Y, \mathcal{O}_Y) = H^q(X, \pi_* \mathcal{O}_Y) = \bigoplus_n H^q(X, \mathcal{O}_X(n)).$$

For example, if X is a connected projective scheme, and $\mathcal{O}_X(1)$ is a sufficiently ample line bundle, then $A = \bigoplus_i \Gamma(X, \mathcal{O}_X(i))$ satisfies our assumptions.

Theorem 3.11 *We have*

$$\overline{HH}_{\mathbb{G}_m}^*(X) = HH_{\text{gr}}^*(A_{>0}, A_{>0}).$$

Proof By Proposition 3.6, we have

$$\overline{HH}_{\mathbb{G}_m}^*(X) = \overline{HH}_{\text{gr}}^*(Y).$$

We can then use Lemma 3.10 to determine $\overline{HH}_{\text{gr}}^*(Y)$. In fact, \mathbb{G}_m acts on the relevant spectral sequence, and we get an induced spectral sequence of invariants

$$E_2^{p,q} = \overline{HH}_{\text{gr}}^p(A, H^q(Y, \mathcal{O}_Y)) \implies \overline{HH}_{\text{gr}}^{p+q}(Y). \tag{13}$$

To deal with the E_2 -term, notice that, passing to the normalized complex, we have

$$HH^p(A, H^q(Y, \mathcal{O}_Y)) = HH^p(A_{>0}, H^q(Y, \mathcal{O}_Y)),$$

This implies also

$$\overline{HH}_{\text{gr}}^p(A, H^q(Y, \mathcal{O}_Y)) = \overline{HH}_{\text{gr}}^p(A_{>0}, H^q(Y, \mathcal{O}_Y)).$$

For $q > 0$ and $p > 0$, we have

$$HH_{\text{gr}}^p(A_{>0}, H^q(Y, \mathcal{O}_Y)) = 0,$$

because there are no graded cochains in the relevant degrees (and taking invariants commutes with computing Hochschild cohomology). So the E_2 -term of the spectral sequence (13) is entirely contained in the row $q = 0$. We deduce that

$$\overline{HH}_{\text{gr}}^*(Y) = \overline{HH}_{\text{gr}}^*(A_{>0}, H^0(Y, \mathcal{O}_Y))^{\text{gr}}.$$

We have

$$C_{\text{gr}}^p(A_{>0}, H^0(Y, \mathcal{O}_Y)) = C_{\text{gr}}^p(A_{>0}, A_{>0}),$$

and we conclude that $\overline{HH}_{\text{gr}}^*(Y) = \overline{HH}_{\text{gr}}^*(A_{>0}, A_{>0})$. □

Remark 3.12 This argument would fail for non-reduced Hochschild cohomology, because the corresponding spectral sequence would also contain the non-vanishing $n = 0$ column. This is the reason for working with reduced Hochschild cohomology. In fact, for Hochschild cohomology, we have

$$HH_{\mathbb{G}_m}^*(X) = HH_{\text{gr}}^*(A_{>0}, A_{>0}) \oplus H^*(X, \mathcal{O}_X).$$

Corollary 3.13 *There is a long exact cohomology sequence*

$$\longrightarrow HH_{\text{gr}}^*(A_{>0}, A_{>0}) \longrightarrow \overline{HH}^*(X) \longrightarrow HH^*(X) \xrightarrow{+1} \dots \tag{14}$$

This sequence is also the sequence (10), for $S = A$.

3.3 The smooth case

3.3.1 Hochschild–Kostant–Rosenberg

We return to the case of a separated scheme X , with a separated morphism $X \rightarrow Y$ to an algebraic stack Y , and assume that $X \rightarrow Y$ is smooth, of relative dimension d . The usual proof of the Hochschild-Kostant-Rosenberg theorem goes through and gives

$$L\Delta^* \Delta_* \mathcal{O}_X = \bigoplus_{j=0}^d \Omega_{X/Y}^j[j],$$

$$(L\Delta^* \Delta_* \mathcal{O}_X)^{\text{red}} = \bigoplus_{j=1}^d \Omega_{X/Y}^j[j].$$

Corollary 3.14 *For relative Hochschild cohomology, we have*

$$HH_Y^q(X) = \bigoplus_{j=0}^d H^{q-j}(X, \Lambda^j T_{X/Y})$$

$$\overline{HH}_Y^q(X) = \bigoplus_{j=1}^d H^{q-j}(X, \Lambda^j T_{X/Y}).$$

In particular, consider the case $Y = B\mathbb{G}_m$, and X smooth. The bundles $\Lambda^j T_{X/B\mathbb{G}_m}$ can be related to the $\Lambda^j T_X$ by considering the short exact sequence of vector bundles on X

$$0 \longrightarrow \mathcal{O}_X \longrightarrow T_{X/B\mathbb{G}_m} \longrightarrow T_X \longrightarrow 0$$

(the Euler sequence), which induces, for every $j > 0$, another short exact sequence of vector bundles

$$0 \longrightarrow \Lambda^{j-1} T_X \longrightarrow \Lambda^j T_{X/B\mathbb{G}_m} \longrightarrow \Lambda^j T_X \longrightarrow 0.$$

If A is a graded ring as in Theorem 3.11, and $X = \mathbb{P}\text{roj } A$ is smooth of dimension d , then for $q > 0$ we have

$$HH_{\text{gr}}^q(A_{>0}, A_{>0}) = \bigoplus_{j=1}^{d+1} H^{q-j}(X, \Lambda^j T_{X/B\mathbb{G}_m}).$$

3.3.2 Further considerations

We consider the case that $(X, \mathcal{O}_X(1))$ is a smooth projective connected scheme of dimension d . The polarization $\mathcal{O}_X(1)$ gives rise to the morphism $X \rightarrow B\mathbb{G}_m$. Assume

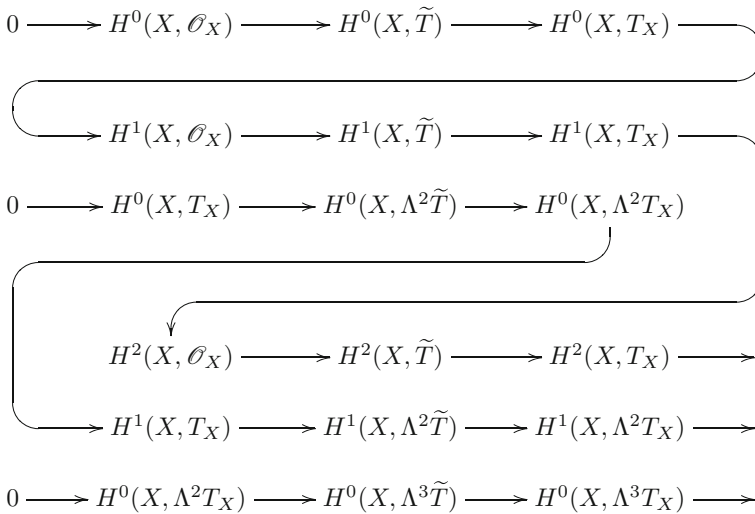


Fig. 1 .

that $\mathcal{O}_X(1)$ is sufficiently ample, so that the hypotheses of Theorem 3.11 are satisfied. Let A be the homogeneous coordinate ring of X . Then A defines a point of the derived moduli scheme of algebras constructed in Sect. 2. The tangent complex at X of the derived scheme is

$$HH_{gr}^*(A_{>0}, A_{>0})[1] = \bigoplus_{j=1}^{d+1} H^*(X, \Lambda^j T_{X/B\mathbb{G}_m})[1 - j].$$

Therefore, the virtual dimension of the derived scheme at the point X is

$$1 - \chi(X, \mathcal{O}_X) = (-1)^{1+\dim X} p_a(X),$$

i.e., the arithmetic genus up to sign.

In this case, the beginning of the long exact sequence (14), or (10), is a direct sum of long exact sequences as in Fig. 1, where we have written \tilde{T} for $T_{X/B\mathbb{G}_m}$. The left column contains $HH^*(X)[-1]$, the middle column $\overline{HH}_{\mathbb{G}_m}^*(X) = HH_{gr}^*(A_{>0}, A_{>0})$, and the right column $\overline{HH}^*(X)$.

Thus, the infinitesimal non-commutative polarized automorphisms of X are given by

$$\overline{HH}_{\mathbb{G}_m}^1(X) = H^0(X, \tilde{T}).$$

This is equal to the classical, commutative infinitesimal automorphisms of the pair $(X, \mathcal{O}_X(1))$. It is an extension of the kernel of $H^0(X, T_X) \rightarrow H^1(X, \mathcal{O}_X)$ by $\mathbb{C} = H^0(X, \mathcal{O}_X)$.

The infinitesimal non-commutative polarized deformations of X are given by $\overline{HH}^2_{\mathbb{G}_m}(X)$. This splits up into two direct summands

$$\overline{HH}^2_{\mathbb{G}_m}(X) = H^1(X, \tilde{T}) \oplus H^0(X, \Lambda^2 \tilde{T}).$$

There is the classical, commutative part $H^1(X, \tilde{T})$. This is an extension of the kernel of $H^1(X, T_X) \rightarrow H^2(X, \mathcal{O}_X)$, i.e., the infinitesimal deformations of X lifting to the pair $(X, \mathcal{O}_X(1))$, by the cokernel of $H^0(X, T_X) \rightarrow H^1(X, \mathcal{O}_X)$, i.e., the infinitesimal deformations of $\mathcal{O}_X(1)$, modulo those that come from infinitesimal automorphisms of X .

Then there is the non-commutative part $H^0(X, \Lambda^2 \tilde{T})$. This is an extension of the kernel of $H^0(X, \Lambda^2 T_X) \rightarrow H^1(X, T_X)$ by $H^0(X, T_X)$. The subspace $H^0(X, T_X)$ corresponds to non-commutative deformations of the graded sheaf of algebras $\bigoplus_n \mathcal{O}(n)$ coming from automorphisms of X , via the ‘twisted coordinate ring construction’. The quotient space consists of non-commutative deformations of the structure sheaf (given by $H^0(X, \Lambda^2 T_X)$, which map to zero in $H^1(X, T_X)$).

The infinitesimal non-commutative polarized obstructions of X are given by $\overline{HH}^3_{\mathbb{G}_m}(X)$, and split up into three parts. The classical, commutative part $H^2(X, \tilde{T})$, and two non-classical parts $H^1(X, \Lambda^2 \tilde{T})$ and $H^0(X, \Lambda^3 \tilde{T})$. In particular, they contain $H^0(X, \Lambda^2 T_X)$ as a subspace.

Remark 3.15 For the obstruction theory to be perfect at X , i.e., for the higher obstructions to vanish, we could require

$$H^i(X, \Lambda^j T_X) = 0, \quad \text{for all } i + j \geq 3.$$

For X a curve this is always true. This leads to the speculation that there may be interesting moduli spaces of non-commutative polarized curves, which admit virtual fundamental classes.

For surfaces, this would give the three conditions

$$H^1(X, \Lambda^2 T_X) = 0, \quad H^2(X, T_X) = 0, \quad H^2(X, \Lambda^2 T_X) = 0.$$

4 Stability for graded algebras

In this section we study the geometric invariant theory quotient associated to the action of G on L^1 (notation from Sect. 2). Because of (2) we restrict to the case of truncated algebras. Then both L^1 and G are of finite type, and we are in a classical geometric invariant theory context.

4.1 The GIT problem

Here we construct quasi-projective moduli schemes of finite graded stable algebras. We start by setting up a Geometric Invariant Theory problem.

Let q be a positive integer, $\mathbf{d} = (d_1, \dots, d_q)$ a vector of positive integers, and

$$V = \bigoplus_{i=1}^q V_i$$

a finite-dimensional graded vector space of dimension \mathbf{d} . Let $G = \prod_{i=1}^q \text{GL}(V_i)$. We write elements of V as $x = (x_1, \dots, x_q)$ and elements of G as $g = (g_1, \dots, g_q)$.

Let $R = \text{Hom}_{\text{gr}}(V^{\otimes 2}, V)$, with elements written as $\mu = (\mu_{ij})_{ij}$, where $\mu_{ij} : V_i \otimes V_j \rightarrow V_{i+j}$. Note that $\mu_{ij} \neq 0$ only if $i, j \geq 1$ and $i + j \leq q$. Consider the left action of G on R by conjugation. More precisely, for $g \in G$ and $\mu \in R$, we have

$$(g * \mu)_{ij} = g_{i+j} \circ \mu_{ij} \circ (g_i^{-1} \otimes g_j^{-1}).$$

Remark 4.1 This is not a space of quiver representations. So we cannot directly quote results for moduli of quiver representations. Although similar techniques do apply.

There are two canonical one-parameter subgroups of G . The anti-diagonal $\Delta^{-1} : \mathbb{G}_m \rightarrow G$ acts by scalar multiplication (i.e., by weight 1) on R , and hence destabilizes every point of R . The other, $\Gamma : \mathbb{G}_m \rightarrow G$ given by $\Gamma(t) = (t, t^2, \dots, t^q)$ acts trivially on R , prompting us to pass from G to $\tilde{G} = G/\Gamma$.

Definition 4.2 We call a vector of integers $\theta = (\theta_1, \dots, \theta_q)$ a **stability parameter** if

(i)
$$\sum_{i=1}^q \theta_i d_i < 0,$$

(ii)
$$\sum_{i=1}^q i \theta_i d_i = 0.$$

Any stability parameter defines a character χ_θ of \tilde{G} by

$$\chi_\theta(g_1, \dots, g_q) = \prod_{i=1}^q \det(g_i)^{\theta_i}.$$

The second condition on θ says that θ factors through $G \rightarrow \tilde{G}$, and the first condition implies that $\langle \chi, \Delta^{-1} \rangle > 0$.

We then linearize the action of \tilde{G} on R by taking the trivial line bundle on R , and lifting the action to $R \times \mathbb{C}$ by the formula $g * (\mu, t) = (g * \mu, \chi(g)^{-1}t)$. Then the GIT quotient of R by \tilde{G} is

$$R // \tilde{G} = \mathbb{P}\text{roj} \bigoplus_{n=0}^{\infty} \Gamma(R)_{\chi^n},$$

where

$$\Gamma(R)_{\chi^n}^{\tilde{G}} = \{f \in \Gamma(R) \mid f(g * \mu) = \chi(g)^n f(\mu)\}$$

are the twisted invariants. Note that the condition $\langle \chi, \Delta^{-1} \rangle > 0$ implies that $\bigoplus_n \Gamma(R)_{\chi^n}^{\tilde{G}}$ is non-negatively graded.

The GIT quotient is a projective scheme, because the affine quotient $\text{Spec } \Gamma(R)_{\chi^n}^{\tilde{G}}$ is reduced to a point.

Let $R^s \subset R^{ss} \subset R$ be the open subsets of stable and semi-stable points in R , respectively. Then $[R^s/\tilde{G}]$ is a separated Deligne–Mumford stack with quasi-projective coarse moduli space $R^s // \tilde{G}$. Moreover, $R^{ss} // \tilde{G} = R // \tilde{G}$ is a projective scheme, containing $R^s // \tilde{G}$ as an open subscheme. If $R^s = R^{ss}$, then $[R^s/\tilde{G}]$ is a proper Deligne–Mumford stack with projective coarse moduli space $R // \tilde{G}$.

When we need to specify the stability parameter, we call points of R^s (R^{ss}) θ -(semi)-stable.

4.2 The Hilbert–Mumford criterion

We recall the Hilbert–Mumford criterion (see Proposition 2.5 in [8]):

Proposition 4.3 [Hilbert–Mumford numerical criterion] The point $\mu \in R$ is (semi)-stable (with respect to the linearization given by χ) if and only if for every non-trivial one-parameter subgroup λ of \tilde{G} , such that $\lim_{t \rightarrow 0} \lambda(t) * \mu$ exists in R , we have $\langle \chi, \lambda \rangle > 0$ (≥ 0).

Proposition 4.4 The point $\mu \in R$ is θ -(semi)-stable, if and only if for all descending filtrations $V = V^{(0)} \supset V^{(1)} \supset \dots$, compatible with the lower grading, and satisfying the conditions

- (i) For n sufficiently large, $V^{(n)} = 0$, but $V^{(1)} \neq 0$,
- (ii) $(V^{(k)})$ does not dominate the tautological filtration, where to dominate the tautological filtration means that $V^{(k)} \supset V_{\geq k}$, for all k ,
- (iii) $\mu(V^{(i)}, V^{(j)}) \subset V^{(i+j)}$, for all i, j ,

we have

$$\sum_{i=1}^q \theta_i w_i > 0 \quad (\geq 0).$$

Here $w_i = \sum_{m \geq 1} \dim V_i^{(m)}$ is the **weight function** of the filtration $V^{(k)}$.

Proof A one-parameter subgroup of \tilde{G} is the same thing as a one-parameter subgroup of G , up to translation by Γ . One-parameter subgroups of G are the same thing as gradings on each of the V_i , which we denote by upper indices $V_i = \bigoplus_m V_i^m$. The upper grading $V = \bigoplus_{i,m} V_i^m$ gives rise to the same one-parameter subgroup of \tilde{G} as the upper grading $V = \bigoplus_{i,m} V_i^{m+i}$. Thus we call the upper gradings $\bigoplus V_i^m$ and

$\bigoplus V_i^{m+i}$ equivalent. The upper grading defined by $V = V^0$, as well as all equivalent upper gradings are called *trivial*, as they correspond to the trivial cocharacter of \tilde{G} . In each equivalence class there is a unique upper grading such that no weights are negative, but there exists a non-zero space V_i^m with $m < i$. Let us call such an upper grading *standard*.

Now let $\mu \in R$ be given. Let λ be a one-parameter subgroup of G , corresponding to the double grading $V = \bigoplus V_i^m$ on V . Then $\lim_{t \rightarrow 0} \lambda(t) * \mu$ exists in R , if and only if none of the λ -weights of μ are negative. This is equivalent to μ preserving the descending filtration given by $V^{\geq n} = \bigoplus_{m \geq n} V^m$, by which we mean that $\mu(V^{\geq m}, V^{\geq n}) \subset V^{\geq m+n}$. Note that this condition is preserved under equivalence of upper gradings, even though the upper filtration itself changes in the equivalence class.

Now suppose that $\mu \in R$ preserves the filtration $V^{\geq n}$, given by λ . Then

$$\langle \chi, \lambda \rangle = \sum_{i=1}^q \theta_i \sum_m m \dim V_i^m.$$

Note that for standard upper gradings, we have $V \subset V^{\geq 0}$, and hence

$$\sum_m m \dim V_i^m = \sum_{m \geq 1} \dim V_i^{\geq m},$$

so that we have

$$\langle \chi, \lambda \rangle = \sum_{i=1}^q \theta_i \sum_{m \geq 1} \dim V_i^{\geq m}.$$

Thus we conclude that $\mu \in R$ is stable if and only if for every descending filtration $V = V^{(0)} \supset V^{(1)} \supset \dots$ (compatible with the lower grading), satisfying

- (i) (non-trivial) $V^{(1)} \neq 0$, but $V^{(n)} = 0$, for $n \gg 0$,
- (ii) (standard) there exists a k , such that $V^{(k)} \not\supset V_{\geq k}$,
- (iii) $\mu(V^{(m)}, V^{(n)}) \subset V^{(m+n)}$,

we have $\sum_{i=1}^q \theta_i \sum_{m \geq 1} \dim V_i^{(m)} > 0$ (≥ 0). □

4.3 Reformulation using test configurations

Suppose now that A is an associative and unital graded algebra, which is locally finite and connected, with $A_0 = \mathbb{C}$.

4.3.1 Test configurations for A

Definition 4.5 A **test configuration** for A is a bundle of graded unital algebras \mathcal{B} (as defined in Sect. 2) over the affine line \mathbb{A}^1 , together with a \mathbb{G}_m -action on the bundle

\mathcal{B} , lifting the natural action on \mathbb{A}^1 , such that the restriction of \mathcal{B} to $\mathbb{G}_m \subset \mathbb{A}^1$ is \mathbb{G}_m -equivariantly isomorphic to the constant bundle with fibre A .

Two test configurations for A are **equivalent**, if one can be obtained from the other by multiplying the \mathbb{G}_m -action by a suitable power of the tautological action. A test configuration for A is **trivial**, if it is equivalent to a \mathbb{G}_m -equivariantly constant test configuration.

The special fibre $\mathcal{B}|_0$ of a test configuration is a graded algebra with the same Hilbert function as A , endowed with a \mathbb{G}_m -action. The weight of the \mathbb{G}_m -action on the graded piece of degree i of $\mathcal{B}|_0$ is denoted w_i , and the function

$$F(i) = \frac{w_i}{i \dim A_i},$$

defined for $i > 0$, is the **Futaki function** of the test configuration \mathcal{B} . (It takes values in $\mathbb{Q} \cup \{\infty\}$.) The Futaki functions of two equivalent test configurations differ by a constant integer. The Futaki function of a trivial test configuration is a constant integer.

A test configuration for A , together with a \mathbb{G}_m -equivariant trivialization of its restriction to $\mathbb{G}_m \subset \mathbb{A}^1$, is the same thing as a doubly graded $\mathbb{C}[t]$ -subalgebra

$$B = \bigoplus_{k \in \mathbb{Z}} B^{(k)} t^{-k} \subset A[t, t^{-1}],$$

such that every $B_i \subset A_i[t, t^{-1}]$ is a finitely generated $\mathbb{C}[t]$ -submodule of rank $\dim A_i$. The test configuration is trivial, if and only if there exists an $\ell \in \mathbb{Z}$, such that

$$B_i^{(k)} = \begin{cases} A_i & \text{if } k \leq i\ell \\ 0 & \text{if } k > i\ell \end{cases}$$

For our purposes it will not be important to distinguish between a test configuration and one with \mathbb{G}_m -equivariant trivialization over $\mathbb{G}_m \subset \mathbb{A}^1$, and so we will identify test configurations with doubly graded $\mathbb{C}[t]$ -algebras $B \subset A[t, t^{-1}]$ such that $\text{rk } B_i = \dim A_i$, for all i .

By definition, **generators** of a test configuration \mathcal{B} for A are generators for the algebra of global sections $B = \Gamma(\mathbb{A}^1, \mathcal{B})$ as $\mathbb{C}[t]$ -algebra.

Remark 4.6 If A admits a finitely generated test configuration, then A is finitely generated, itself.

4.3.2 Admissible test configurations

Definition 4.7 A test configuration B is called **admissible**, if it is equivalent to a test configuration which can be written as

$$A[t] \subset B \subset A[t, t^{-1}].$$

Let us suppose B is an admissible test configuration written in this way. We have

$$w_i = \sum_{k>0} \dim B_i^{(k)}.$$

Moreover,

- (i) every $B^{(k)}$ for $k > 0$ is a two-sided ideal in A ,
- (ii) $A \supset B^{(1)} \supset B^{(2)} \supset \dots$,
- (iii) $B^{(k)} B^{(\ell)} \subset B^{(k+\ell)}$, for all $k, \ell > 0$,
- (iv) for every $i > 0$, there exists an $\ell > 0$, such that $B_i^{(k)} = 0$, for all $k \geq \ell$.

Definition 4.8 We call a sequence of two-sided ideals $(I^{(k)})_{k>0}$ in A satisfying these conditions an **admissible family of ideals** in A .

An admissible family of ideals $(I^{(k)})_{k>0}$ defines a test configuration by

$$B = \bigoplus_{k \in \mathbb{Z}} I^{(k)} t^{-k},$$

where we set $I^{(k)} = A$, for all $k \leq 0$. The special fibre of this test configuration is

$$B/tB = \bigoplus_{k \geq 0} I^{(k)} / I^{(k+1)}.$$

Remark 4.9 If A is finitely generated, then every test configuration for A is admissible.

4.3.3 Standard admissible test configurations

If a test configuration is admissible, there is a unique equivalent test configuration with the properties

- (i) $A[t] \subset B$,
- (ii) $\bigoplus_{k \in \mathbb{Z}} A_{\geq k} t^{-k} \subsetneq B$.

Such a test configuration is called **standard admissible**.

A test configuration is standard admissible if and only if the corresponding admissible family of ideals does not contain the **tautological** admissible family given by $I^{(k)} = A_{\geq k}$. Such an admissible family of ideals is called **standard admissible**.

4.3.4 Stability

Now let us return to the setup of 4.1. Suppose that $\mu \in R = L^1$ is a Maurer–Cartan element, so that $A = (V, \mu)$ is a graded algebra with $A_i = 0$, for $i > q$.

Proposition 4.10 *The Maurer–Cartan element μ is θ -(semi)-stable if and only if, for every non-trivial test configuration for A , the weights w_i satisfy*

$$\sum_i \theta_i w_i > 0 \quad (\geq 0).$$

Proof As $A_i = 0$ for $i \gg 0$, all test configurations for A are admissible. Because of $\sum_i i\theta_i d_i = 0$, the stability condition $\sum_i \theta_i w_i > 0$ (≥ 0) is independent of the choice of a test configuration within its equivalence class. So to test the condition of this proposition it is sufficient to use standard admissible test configurations. To conclude, we remark that non-trivial standard admissible test configurations correspond exactly to the filtrations of V , which are tested in Proposition 4.4. \square

This proposition motivates the following definition.

Definition 4.11 A finite graded algebra A is θ -(semi)-stable, if

- (i) $\sum_i \theta_i \dim A_i < 0$,
- (ii) $\sum_i i\theta_i \dim A_i = 0$,
- (iii) for every non-trivial test configuration for A , the weights satisfy

$$\sum_i \theta_i w_i > 0 \quad (\geq 0).$$

Proposition 4.12 *To test (semi)-stability of A , it suffices to check standard admissible families of ideals in A .*

4.4 Standard stability parameters

We fix a dimension vector (d_1, \dots, d_q) and a stability parameter θ , as above, and study θ -stability of finite graded algebras A with $\dim A_i = d_i$.

Let us remark that there is no a priori reason to expect complete moduli of stable algebras:

Remark 4.13 We can eliminate θ_1 from the stability condition. The stability parameter condition becomes

$$\sum_{i=2}^q (i - 1)d_i\theta_i > 0,$$

and as stability condition we obtain

$$\sum_{i=2}^q (d_1 w_i - i d_i w_1)\theta_i > 0 \quad (\geq 0),$$

or

$$\sum_{i=2}^q (F(i) - F(1))i d_i\theta_i > 0 \quad (\geq 0).$$

We see that no matter the choice of stability parameter θ , an admissible sequence of ideals with constant Futaki function will always violate stability. The Futaki function being constant means that

$$w_k = kd_k \frac{w_1}{d_1}, \quad \text{for all } k \geq 1.$$

There is no a priori reason why d_1 should not divide w_1 , and so there is no divisibility condition on the dimension vector (d_1, d_2, \dots) which would exclude the possibility of strictly semi-stable objects. Therefore, there is no such condition that would ensure a projective coarse moduli space of stable algebras.

For certain stability parameters, stability implies generated in degree 1:

Proposition 4.14 *Suppose that $\theta_1 < 0$ and $\theta_i \geq 0$, for all $i > 1$. Then θ -stable algebras are generated in degree 1. If, in addition, $\theta_i > 0$, for all $i > 1$, then θ -semi-stable algebras are generated in degree 1.*

Proof Write $I = A_{\geq 1}$, and consider the admissible sequence of ideals of powers of I , given by $I^{(k)} = I^k$, for all $k \geq 1$. Assume that *not* $I^k = A_{\geq k}$, for all k . Then (I^k) is properly contained in the tautological filtration, and hence does not dominate it. Thus (I^k) is standard admissible.

If A is θ -stable, then $(-\theta_1)w_1 < \sum_{i>1} \theta_i w_i$. This implies $(-\theta_1)d_1 < \sum_{i>1} \theta_i w_i$, and hence $\sum_{i>1} i\theta_i d_i < \sum_{i>1} \theta_i w_i$. This is a contradiction, because $w_i \leq id_i$, for all i . Thus $I^k = A_{\geq k}$, for all k , which implies that A is generated in degree 1.

To prove the additional claim, assume that V is θ -semi-stable. Then we can still conclude that $\sum_{i>1} i\theta_i d_i \leq \sum_{i>1} \theta_i w_i$. Thus, from $w_i \leq id_i$, and the fact that none of the θ_i vanish, we conclude that $w_i = id_i$, for all $i > 1$. Again, we reach a contradiction, proving that A is generated in degree 1.

Remark 4.15 We have, in both cases, proved that any admissible sequence of ideals which is contained in the tautological one, and satisfies $I_1^{(1)} = A_1$, is necessarily the tautological sequence.

Proposition 4.16 *Suppose that we have $\theta_i \leq 0$, for all $i < q$. Then every θ -stable algebra has no non-zero ideal I , which vanishes in degree q . If $\theta_i < 0$, for all $i < q$, we can reach the same conclusion for θ -semi-stable algebras.*

Proof In fact, if we assume that $I^{(k)}$ is an admissible sequence of ideals which vanishes in degree q , we can conclude that $I^{(k)} = 0$, for all $k \geq 1$, under either of the two assumptions.

Remark 4.17 If $\theta_q = 0$, there are no stable algebras.

Remark 4.18 If we want the assumptions of both Propositions 4.14 and 4.16 to hold, we need to have $\theta_1 < 0$, and $\theta_q > 0$, as well as $\theta_i = 0$, for all $1 < i < q$. For the conclusions to hold, we need to assume stability, not just semi-stability.

Definition 4.19 The stability parameter θ is **standard**, if θ_1 and θ_q are the only non-zero θ_i .

For a standard stability parameter θ , the stability condition reads

$$\theta_q w_q > (-\theta_1)w_1 \quad (\geq).$$

This is equivalent to

$$F(q) > F(1) \ (\geq),$$

which is independent of the sizes of θ_1 and θ_q .

When not specified otherwise, we always work with a standard stability condition, and make the following definition.

Definition 4.20 Let A be a finite graded algebra, graded in the interval $[0, q]$. Then A is called **(semi)-stable**, if for every non-trivial test configuration for A , the Futaki function satisfies $F(q) > F(1) (\geq)$. It suffices to check admissible families of ideals, or standard admissible sequences of ideals.

Corollary 4.21 *Suppose A is stable. Then A is generated in degree 1, and has no non-trivial two-sided ideals which vanish in degree q .*

4.4.1 Moduli

Consider the dimension vector $\mathbf{d} = (d_1, \dots, d_q)$, and the associated stack of twisted bundles of graded algebras of dimension \mathbf{d} , which we called $\tilde{X}_{\leq q}$ in Sect. 2. Let $\tilde{X}_{\leq q}^s$ be the open substack of stable algebras. It is a closed substack of the quotient stack $[R^s/\tilde{G}]$, and it is a separated Deligne–Mumford stack with quasi-projective coarse moduli space, which is a closed subscheme of $R^s // \tilde{G}$. The \mathbb{C} -points of this coarse moduli space correspond in a one-to-one fashion to isomorphism classes of stable algebras of dimension \mathbf{d} .

4.5 Unbounded algebras

For simplicity, we will only consider stability, not semi-stability. In view of Corollary 4.21, we will only consider algebras generated in degree 1.

Proposition 4.22 *Fix an integer $q > 1$, and let A be a graded algebra, finitely generated in degree 1. The following are equivalent*

- (i) *For every test configuration \mathcal{B} for A , whose truncation $\mathcal{B}_{\leq q}$ is a non-trivial test configuration for $A_{\leq q}$, the Futaki function satisfies $F(q) > F(1)$.*
- (ii) *For every non-trivial test configuration for A generated in degrees $\leq q$, the Futaki function satisfies $F(q) > F(1)$.*
- (iii) *For every non-trivial test configuration for A generated in degree 1, the Futaki function satisfies $F(q) > F(1)$.*
- (iv) *For every filtration $A_1 \supseteq V^{(1)} \supset \dots \supset V^{(r)} \supseteq 0$ of A_1 by vector subspaces, the admissible sequence of ideals generated by $\{V^{(k)}\}$ has a Futaki function which satisfies $F(q) > F(1)$.*
- (v) *The truncation $A_{\leq q}$ is stable.*

Proof The fact that (i) implies (ii), follows because if a test configuration \mathcal{B} for A is generated in degrees $\leq q$, and is non-trivial, then also its truncation $\mathcal{B}_{\leq q}$ is non-trivial.

Obviously, (ii) implies (iii).

Next we claim that (iii) implies (iv). Here we will use that A is generated in degree 1. The admissible sequence of ideals generated by the filtration $\{V^{(k)}\}$ is the smallest admissible sequence of ideals $\{I^{(k)}\}$, with $I_1^{(k)} = V_1^{(k)}$, for all $k > 0$. The corresponding test configuration B is generated as $\mathbb{C}[t]$ -algebra by $\bigoplus_{k \geq 0} V^{(k)}t^{-k} \subset A_1[t, t^{-1}]$ inside $A[t, t^{-1}]$, if we set $V^{(0)} = A_1$. It is generated in degree 1. Thus, (iii) implies (iv).

Now let us assume that (iv) is satisfied. To prove (v), i.e., that $A_{\leq q}$ is stable, it suffices to check all standard admissible test configurations for $A_{\leq q}$. Among these, it suffices to check those that are generated in degree 1, because adding generators in higher degree can only increase $F(q)$, without affecting $F(1)$. But non-trivial standard admissible test configurations generated in degree 1 are all generated by a filtration $\{V^{(k)}\}$ as in (iv).

Finally, the fact that (v) implies (i) is, again, trivial. □

Remark 4.23 For a given dimension d_1 of A_1 , in Condition (iv), we can further reduce to considering only flags whose dimensions $(\dim V^{(1)}, \dim V^{(2)}, \dots)$ come from a finite list of integer sequences, but as we currently have no use for this fact, we will not prove it here.

Definition 4.24 A connected graded algebra, finitely generated in degree 1, is called q -stable, if any of the equivalent conditions in Proposition 4.22 is satisfied. It is called stable, if there exists and $N > 0$, such that it is q -stable for all $q \geq N$.

4.5.1 Commutative case

Suppose that $(Y, \mathcal{O}_Y(1))$ is a connected projective \mathbb{C} -scheme, such that $H^i(Y, \mathcal{O}(1)) = 0$, for all $i > 0$. Let A be the homogeneous coordinate ring of $(Y, \mathcal{O}_Y(1))$. This is the image of $\text{Sym } \Gamma(Y, \mathcal{O}(1)) \rightarrow \bigoplus_{i \geq 0} \Gamma(Y, \mathcal{O}(n))$, and is a connected graded algebra, generated in degree 1.

Proposition 4.25 *The polarized scheme $(Y, \mathcal{O}_Y(1))$ is Hilbert stable if and only if A is stable in the sense of Definition 4.24.*

Proof For the definition of Hilbert stability (more precisely, Hilbert stability with respect to $r = 1$), see [12]. By definition, the Hilbert stability of $(Y, \mathcal{O}_Y(1))$ is tested against all filtrations of $A_1 = \Gamma(Y, \mathcal{O}_Y(1))$, exactly as in Proposition 4.22 (iv). This immediately implies the result.

4.5.2 Moduli

Return to the moduli stack

$$\tilde{X} = \varprojlim_q \tilde{X}_{\leq q}.$$

We have now defined open substacks $\tilde{X}_{\leq q}^s \subset \tilde{X}_{\leq q}$ of stable algebras. We let \tilde{X}_q^s be the preimage of $\tilde{X}_{\leq q}^s$ in \tilde{X} . This is the substack of q -stable algebras. Hence we have

in \tilde{X} a family of open substacks \tilde{X}_q^s , parametrized by $q \in \mathbb{N}$. A point in \tilde{X} represents a stable algebra if and only if it is in almost all open substacks $\tilde{X}_q^s \subset \tilde{X}$. The locus of stable algebras in \tilde{X} is

$$\tilde{X}^s = \bigcup_{N \in \mathbb{N}} \bigcap_{q \geq N} \tilde{X}_q^s.$$

We see no obvious reason why \tilde{X}^s should be an open substack of \tilde{X} .

4.5.3 Discussion

We have, for every $N \in \mathbb{N}$ a diagram

$$\begin{array}{ccc} \bigcap_{q \geq N} \tilde{X}_q^s & \hookrightarrow & \tilde{X}_N^s \twoheadrightarrow \tilde{X}_{\leq N}^s \\ \downarrow & & \\ \tilde{X}^s & & \end{array} \tag{15}$$

and we find it reasonable, that there should exist dimension vectors \mathbf{d} and integers N , for which all arrows in (15) are isomorphisms, so that $\tilde{X}^s = \tilde{X}_{\leq N}^s$, and \tilde{X}^s is a finite type, separated, (in fact quasi-projective) Deligne–Mumford stack.

In the commutative case (where \mathbf{d} is a numerical polynomial), the category of graded algebras generated in degree 1, with fixed Hilbert polynomial, is bounded. This is due to results of Macaulay–Gotzmann–Gasharov (see [4,6]) on persistence and the Macaulay bound. Furthermore, it can be shown that stability is also a bounded condition, namely it can be checked in the N -truncation, for a sufficiently large N (depending on \mathbf{d}).

From this it follows that, in the commutative case, the corresponding claim $\tilde{X}^s = \tilde{X}_{\leq N}^s$, holds. This, in particular, implies that the commutative analogue of the stack \tilde{X}^s can be realized as a moduli stack of projective polarized schemes (by taking $\mathbb{P}\text{roj}$).

Lack of suitable persistence theorems and flattening stratifications currently keep us from generalizing this result to the non-commutative case. But it stands to reason, by analogy with the commutative case, that for certain dimension vectors \mathbf{d} , the stack \tilde{X}^s , or an open substack, is indeed a moduli stack for stable non-commutative projective schemes, where we propose the following definition for the latter.

Definition 4.26 Call a sufficiently ample (meaning that $H^i(\mathcal{C}, \mathcal{O}(n))$ vanishes for $i > 0$ and $n > 0$) non-commutative projective scheme $(\mathcal{C}, \mathcal{O}, s)$ **stable**, if $\bigoplus_{n>0} \Gamma(\mathcal{C}, \mathcal{O}(n))$ is a stable graded algebra.

Further evidence for this expectation is provided by the deformation theory arguments from Sect. 2, which indicate that the derived deformation theory of a non-commutative projective scheme coincides with that of its algebra of homogeneous coordinates.

Acknowledgements We started this project at the 10th Lisbon Summer Lectures in Geometry, 2009. We thank Gustavo Granja for the hospitality during our visit. This work was supported by a Grant from the Royal Society under the International Exchange Scheme IE111640.

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