CrossMark

Hartogs-type extension for tube-like domains in $\mathbb{C}^2$

**Al Boggess · Roman J. Dwilewicz ·
Zbigniew Slodkowski**

Abstract In this paper we consider the Hartogs-type extension problem for unbounded domains in $\mathbb{C}^2$. An easy necessary condition for a domain to be of Hartogs-type is that there is no a closed (in $\mathbb{C}^2$) complex variety of codimension one in the domain which is given by a holomorphic function smooth up to the boundary. The question is, how far this necessary condition is from the sufficient one? To show how complicated this question is, we give a class of tube-like domains which contain a complex line in the boundary which are either of Hartogs-type or not, depending on how the complex line is positioned with respect to the domain.

Mathematics Subject Classification Primary 32V10; Secondary 32V25 · 32D15

A. Boggess
School of Mathematical and Statistical Sciences, Arizona State University, Tempe, AZ 85287, USA
e-mail: boggess@asu.edu

R. J. Dwilewicz
Department of Mathematics and Statistics, Missouri University of Science and Technology, Rolla,
MO 65409, USA

R. J. Dwilewicz (✉)
Faculty of Mathematics, Cardinal Stefan Wyszyński University, Wóycickiego 1/3,
01-938 Warsaw, Poland
e-mail: romand@mst.edu

Z. Slodkowski
Department of Mathematics, University of Illinois at Chicago, Chicago, IL 60607, USA
e-mail: zbigniew@uic.edu

1 Introduction

In this paper we consider a Hartogs-type extension problem for unbounded domains in $\mathbb{C}^2$. The following classical holomorphic extension result of Hartogs [11] in 1906 is considered by most researchers as a formal beginning of Complex Analysis of several variables.

Theorem (Hartogs [11], 1906) *Let Ω be a bounded domain in $\mathbb{C}^n$, $n \geq 2$, with connected boundary $b\Omega$. Then any holomorphic function in a neighborhood U of the boundary $b\Omega$ can be holomorphically extended to Ω.*

Interestingly, it was not until 2007 that the above theorem was proved in its full generality by Merker and Porten [18] using the same analytic disc method presented by Hartogs in 1906 (Hartogs' proof had some gaps). Later on, other methods of proof of the Hartogs' theorem became popular, they originated from the paper of Fueter [10], then developed by Martinelli [15], independently by Bochner [2], and finally by Ehrenpreis [8]. A very beautiful recollection of results and historical corrections are given in the paper by Range [23]. We refer the reader to these papers [18,23] for very detailed explanation of historical context of the Hartogs Theorem.

The Hartogs extension theorem has been generalized to the case of CR functions which are defined only on the boundary $b\Omega$ and satisfy the tangential Cauchy–Riemann equations (instead of being analytically defined on a neighborhood of the boundary). It was proved by Severi [24] in 1931 when the boundary $b\Omega$ and CR functions are real analytic, by Kneser [12] in 1936 if the boundary is of class C^2 and strictly pseudoconvex, by Fichera [9] in 1957 if the boundary is of class $C^{1+\varepsilon}$, and finally by Martinelli [16] in 1961 when the boundary is of class C^1. Also many other generalizations and variations of the Hartogs theorem appeared, see the survey article by Merker and Porten [17] or [19] and related Coltoiu and Ruppenthal [6]. Also should be mentioned recent articles by Ohsawa [20], Øvrelid and Vassiliadou [21], Porten [22], or earlier articles by Koziarz and Sarkis [13], Laurent-Thiébaut and Leiterer [14].

By a *Hartogs-type domain* we call a domain for which all smooth CR functions on the boundary extend to the interior as holomorphic functions; such extension we call *Hartogs-type extension*.

2 Conjectures and main results

Most of the research on Hartogs-type extension problems have focused on bounded domains, but very little work on this topic has appeared for unbounded ones. For an unbounded domain, Ω, to be of Hartogs-type, an easy necessary condition is that there cannot exist a holomorphic function, f, on Ω which is smooth up to the boundary and whose zero set is contained in Ω (for otherwise, $1/f$ would be a CR function on the boundary which does not holomorphically extend to Ω). However, it is far from clear how close this condition is to being sufficient to ensure such a domain is of Hartogs-type. The goal of this paper is to investigate possible sufficient conditions for an unbounded tube-like domain to be of Hartogs-type. A similar question, for generalized tubes, was considered by the authors in [4].

Fig. 1 Illustration of the conjecture for tube-like domains in $\mathbb{C}^2$
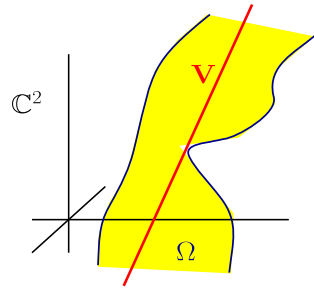
After some basic definitions (in Sect. 3), we consider the case of tube-like domains in $\mathbb{C}^2$ (see Fig. 3) where the projection of the domain along a 2-dimensional real plane is bounded in a transversal real plane. Here, we formulate the following conjecture:

Conjecture for tube-like domains in $\mathbb{C}^2$. *Let $\Omega \subset \mathbb{C}^2$ be a tube-like domain with connected C^1 boundary. Then Ω is a Hartogs-type domain if there is no complex line contained in $\overline{\Omega}$, see Fig. 1.*

The conjecture is still open. It should be mentioned that for non-tube-like domains, the non-existence of complex varieties in the closure of the domain is not a suitable sufficient condition. This will be considered in a forthcoming paper.

The main theorems of this paper are proved in Sects. 4 and 5. We consider tube-like domains Ω along the complex line $\{0\} \times \mathbb{C} \subset \mathbb{C}^2$ with smooth (C^∞) boundary $M = b\Omega$, see Fig. 2. Moreover, the line $\{0\} \times \mathbb{C}$ is contained entirely in the boundary $b\Omega$. For each $w \in \mathbb{C}$, let Ω_w be the cross-section of Ω by $\mathbb{C} \times \{w\}$, and $M_w = b\Omega_w$. Finally let $\eta : \mathbb{C} \longrightarrow \mathbb{R}$ be a continuous function such that for every $w \in \mathbb{C}$, $e^{i\eta(w)}$ is the unit normal vector to $b\Omega_w$ at $(0, w)$ pointing inside Ω_w.

In the theorems below, depending how the domain Ω is positioned with respect to the line $\{0\} \times \mathbb{C}$, in one case the domain is of Hartogs-type in the other case not. Here we formulate non-invariant versions of the main theorems which are easier to visualize. Invariant versions can be found in Theorems 4.6 and 5.4.

In Sect. 4, we give a sufficient condition for a domain not to be of Hartogs type:

Theorem 1 (Corollary 4.5) *Let $\Omega \subset \mathbb{C}^2$ be a domain with smooth (C^∞) boundary $M = b\Omega$ containing the line L, $L = \{0\} \times \mathbb{C} \subset M$. With the setting as above, assume further that there is an entire harmonic function $\chi : \mathbb{C} \longrightarrow \mathbb{R}$ such that*

$$|\eta(w) - \chi(w)| < \frac{\pi}{2} \quad \text{for all} \ \ w \in \mathbb{C}.$$

Then there is a C^∞ smooth CR function on M that cannot be continuously extended to a holomorphic function in Ω, i.e., Ω is not a Hartogs-type domain.

In the theorem, in particular, if the function $\eta(w)$ is bounded on $\mathbb{C}$, then the harmonic function $\chi(w)$ is constant. Consequently, the normal vector $e^{i\eta(w)}$ is contained in an open half-plane. Very roughly (and imprecisely) speaking, the domain Ω is lying on "one side" of the complex line $\{0\} \times \mathbb{C}$.
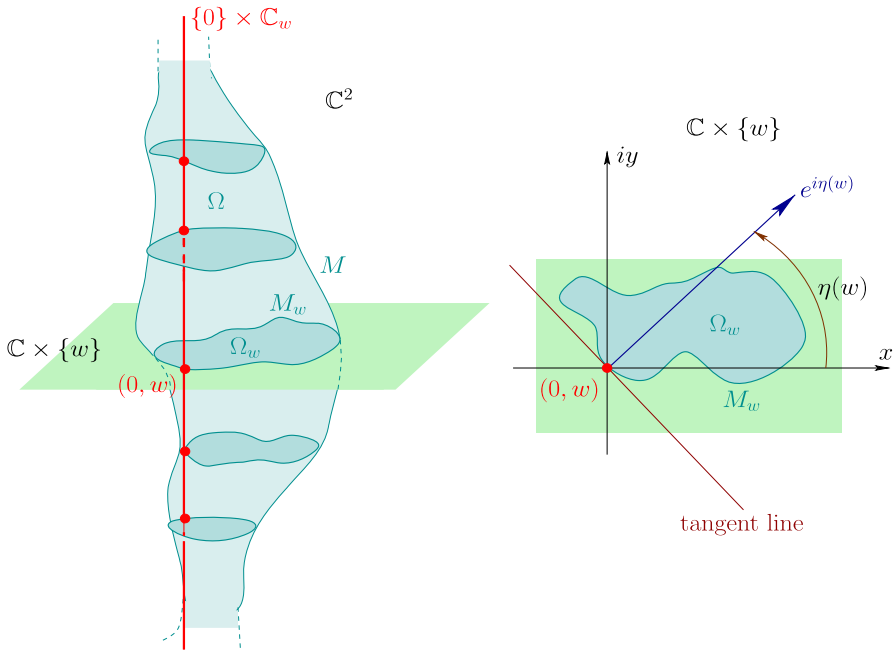
Fig. 2 Domain Ω with the w-axis in the boundary and the level sets Ω_w

The above theorem is an improvement (and generalization) of an example by the authors in [3], where a domain $G \subset \mathbb{C}^2$ was constructed which contained two complex lines in its closure $\overline{G}$ (actually the lines were "mostly" contained in G, only touching the boundary at some points). Also a continuous CR function on the boundary bG was constructed (actually the function was smooth except a small portion of the boundary) that could not be holomorphically extended to G.

In Sect. 5 we give a sufficient condition for a domain Ω containing a complex line as above to be of Hartogs-type, namely

Theorem 2 (Corollary 5.2) *Let $\Omega \subset \mathbb{C}^2$ be a domain with smooth (C^∞) boundary $M = b\Omega$ containing the line L, $L = \{0\} \times \mathbb{C} \subset M$. With the notation as above, assume additionally that M_w is a smooth Jordan curve for every $w \in \mathbb{C}$, and that*

$$\lim_{w \to \infty} \operatorname{diam}(M_w) = 0. \tag{1}$$

If there does not exist an entire harmonic function $\chi : \mathbb{C} \longrightarrow \mathbb{R}$ such that

$$|\eta(w) - \chi(w)| \leq \frac{\pi}{2} \quad for \;\; w \in \mathbb{C}, \tag{2}$$

then every smooth CR function on M continuously extends to a holomorphic function in Ω, i.e., Ω is a Hartogs-type domain.

Again, if the function $\eta(w)$ is bounded on $\mathbb{C}$, then non-existence of a harmonic function satisfying (2) means that the normal vector $e^{i\eta(w)}$ rotates more than the angle π when w varies. Very roughly (and imprecisely) speaking, the domain Ω "goes around" the complex line $\{0\} \times \mathbb{C}$.

In the above theorems we see that the difference between the sufficient condition and the necessary condition is very small. The theorems also show that it will be hard to formulate a necessary and sufficient condition for a domain to be of Hartogs-type.

3 Basic definitions and notation

CR manifolds Let M be a real embedded submanifold of class C^1 in $\mathbb{C}^n$. Let $T_p M$ be the real tangent space to M at p. We can consider $T_p M$ as a real vector subspace of the complex space $T_p \mathbb{C}^n$. We say that M is a *Cauchy–Riemann* (CR) *manifold* if

$$\dim_{\mathbb{C}}(H_p M) = \dim_{\mathbb{C}}(T_p M \cap \sqrt{-1} T_p M) \equiv \text{constant on } M.$$

This constant is called *CR dimension* of M and denoted by $\dim_{CR} M$. If $\dim_{CR} M = 0$, then M is called a *totally real manifold*.

CR functions By CR functions we mean functions $f : M \longrightarrow \mathbb{C}$ of class C^1 that satisfy the tangential Cauchy–Riemann equations,

$$\overline{L} f = 0 \ \ \text{for any} \ \ \overline{L} = a_1 \frac{\partial}{\partial \overline{z}_1} + \cdots + a_n \frac{\partial}{\partial \overline{z}_n} \ \ \text{tangent to } M.$$

Obviously, the restriction $f = F|_M$ of a holomorphic function $F : U \longrightarrow \mathbb{C}, M \subset U$, to M is CR, but in general, not all CR functions arise as restrictions of holomorphic functions.

Smoothness By *smooth* (manifold or function) we shall mean of class C^∞, unless otherwise specified. Usually the regularity conditions can be weakened, for instance, to consider CR functions in the distributional sense (e.g., continuous CR functions), however, it is not our intention here to formulate and prove results under the weakest regularity hypothesis.

In what follows, by Ω we will denote a domain (open, connected) in $\mathbb{C}^2$ ($n = 2$) with connected boundary $b\Omega$ and also $\mathbb{C}^2 \backslash \overline{\Omega}$ is connected. Usually we assume that the boundary $b\Omega$ is smooth, unless otherwise specified.

Tubes and tube-like domains Let P be a 2-dimensional real subspace in $\mathbb{C}^2$ and $\pi : \mathbb{C}^2 \longrightarrow P$ be a projection, an $\mathbb{R}$-linear mapping. By a *tube* (see Fig. 3) we will mean $T = \pi^{-1}(U)$, where $U \subset P$ is a bounded domain with smooth boundary. A *tube-like domain* is an unbounded domain $\Omega \subset \mathbb{C}^2$ with smooth boundary such that $\pi(\Omega) \subset P$ is bounded.

If $\pi^{-1}(0)$ is a totally real plane, we say that the *tube or tube-like domain is along a totally real plane*. If $\pi^{-1}(0)$ is a complex line, we say that the *tube or tube-like domain is along a complex line*.
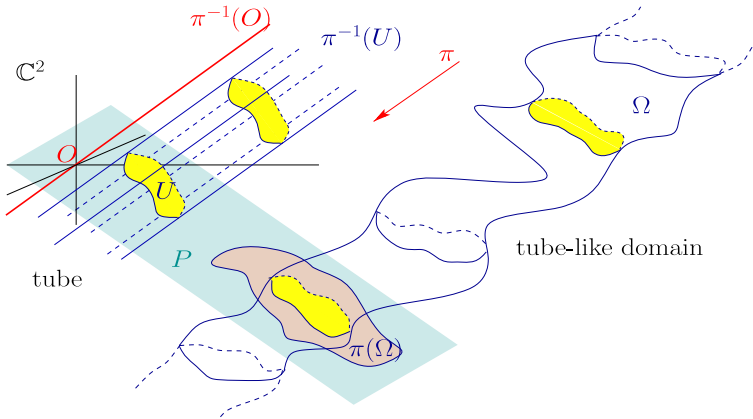
Fig. 3 Tube and tube-like domains

4 Non Hartogs-type domains with a complex line in the boundary

4.1 Formulation of the theorem

We start with some basic observation and establish convenient notation. We consider now any domain $\Omega \subset \mathbb{C}^2$ with smooth connected boundary $M = b\Omega$ containing a complex line L. Choosing a suitable coordinate system we can assume

$$M \supseteq L = \{z = 0\} = \{(0, w) : w \in \mathbb{C}\}.$$

For each fixed $w \in \mathbb{C}$, denote

$$\Omega_w := \{z \in \mathbb{C} : (z, w) \in \Omega\}, \quad M_w := \{z \in \mathbb{C} : (z, w) \in M\};$$

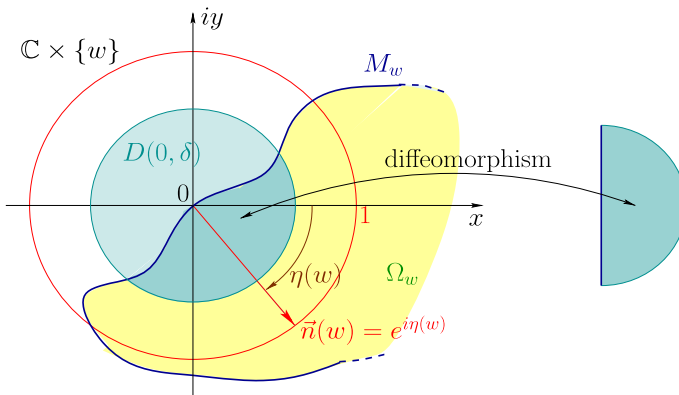for illustration of these and following concepts, see Figs. 2 and 4.

Fig. 4 Cross-section Ω_w and normal vector $\vec{n}(w)$

In general, the whole intersection $M_w = M \cap (\mathbb{C} \times \{w\})$ does not have to be a smooth curve. However, since the real tangent planes $T_{(0,w)}(M)$ and $T_{(0,w)}(\mathbb{C} \times \{w\})$ intersect transversally (as the former contains $\{0\} \times \mathbb{C}$), a neighborhood of 0 in M_w is a smooth simple (open) arc and, for some $\delta > 0$, the intersection $D(0, \delta) \cap (M_w \cup \Omega_w)$ is diffeomorphic to a half-disc $\{(x, y) : x \geq 0, \ x^2 + y^2 < 1\}$. Due to this, there is a unique unit normal vector $\vec{n}(w)$ to M_w at 0 that points inside Ω_w. Letting w vary, the above transversality argument also implies that the function

$$w \longrightarrow \vec{n}(w) \ : \ \mathbb{C} \longrightarrow S^1 \subset \mathbb{C}$$

is continuous. We omit details of the proof.

Finally (by monodromy) there is a continuous function $\eta : \mathbb{C} \longrightarrow \mathbb{R}$ such that

$$\vec{n}(w) = e^{i\eta(w)}, \qquad w \in \mathbb{C}.$$

($\eta(\cdot)$ is unique up to addition of a constant multiple of 2π; we fix one choice.)

Theorem 4.1 *Let $\Omega \subset \mathbb{C}^2$ be a domain with smooth boundary $M = b\Omega$ containing the line L, $L = \{0\} \times \mathbb{C} \subset M$, as above. Assuming the above set-up and notation, suppose further that $-\frac{\pi}{2} < \eta(w) < \frac{\pi}{2}$ for all $w \in \mathbb{C}$. Then there is a C^∞-smooth CR function $\varphi : M \longrightarrow \mathbb{C}$ that does not extend continuously to a holomorphic function in Ω.*

Remark 4.2 Actually in the above theorem, the domain Ω need not be tube-like. However, the line $L = \{0\} \times \mathbb{C}$ is contained in the boundary.

4.2 Proof of Theorem 4.1

Special holomorphic function For the proof we need to construct a special holomorphic function. Denote by P the semi-infinite strip

$$P = \{\zeta \in \mathbb{C} : \ \mathrm{Re}\,\zeta > \pi, \ \ |\mathrm{Im}\,\zeta| < \pi\} \tag{3}$$

(see Fig. 5), and by S the enlarged semi-infinite strip

$$S = \{\zeta \in \mathbb{C} : \ \mathrm{Re}\,\zeta > 0, \ \ |\mathrm{Im}\,\zeta| < 2\pi\}. \tag{4}$$
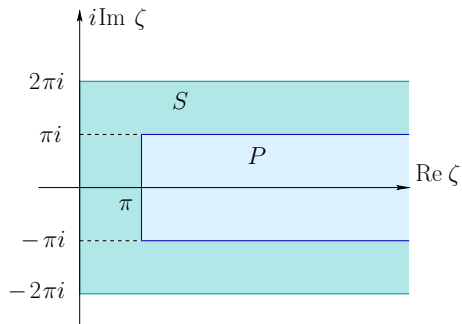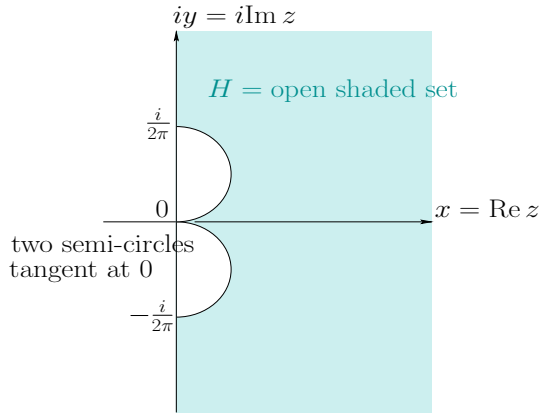
Fig. 5 Semi-infinite strips

Fig. 6 Set H

Define

$$H = \left\{ z \in \mathbb{C} : \frac{1}{z} \in S \right\},$$

illustrated in Fig. 6. Under the map $z \longrightarrow 1/z$, angles are preserved and half-lines and intervals are mapped to circular arcs.

Lemma 4.3 *There is a function* $f : \mathbb{C} \longrightarrow \mathbb{C}$ *such that*

- $f(0) = 0$,
- f *is holomorphic in* $\mathbb{C} \backslash \{0\}$ *with an essential singularity at* 0,
- *for every* $n \geq 0$ *we have* $\lim_{\mathbb{C} \backslash H \ni z \to 0} f^{(n)}(z) = 0$.

For construction and proof, see the Appendix.
   Define $F : \mathbb{C}^2 \backslash \Omega \longrightarrow \mathbb{C}$ by

$$F(z, w) := f(z), \qquad (z, w) \in \mathbb{C}^2 \backslash \Omega.$$

Lemma 4.4 *The function* F *is smooth on* $\mathbb{C}^2 \backslash \Omega$.

Postponing for a moment the proof of the lemma, define now $\varphi : M \longrightarrow \mathbb{C}$ as $\varphi := F\big|_M$. Then φ is a C^∞ function on $M = b\Omega$. Clearly φ satisfies the CR condition at the points of $M \backslash (\{0\} \times \mathbb{C}) \subset \mathbb{C}^2 \backslash (\{0\} \times \mathbb{C})$, as F is holomorphic there. Since $\varphi\big|_{(\{0\} \times \mathbb{C})} = 0$, and $T^{\mathbb{C}}_{(0, w_0)}(M) = \{0\} \times \mathbb{C}$, φ satisfies trivially the CR condition at the points of $\{0\} \times \mathbb{C}$, and so is a CR function on M. Suppose φ can be continued to Ω, i.e., there is a function $\Phi \in C(\overline{\Omega}) \cap H(\Omega)$ such that $\Phi\big|_M = \varphi$. Consider an $r > 0$ small enough so that $D(0, r) \cap M_0$, where $D(0, r) = \{z \in \mathbb{C}; \ |z| < r\}$ and $M_0 = M_{w=0}$, is a simple smooth arc, and $D(0, r) \cap \Omega_0$ is connected. Then $\Phi(z, 0)$ and $f(z) = F(z, 0)$ are two functions that are continuous on $\Omega_0 \cup (M_0 \backslash \{0\})$, holomorphic on Ω_0, and equal on $M_0 \backslash \{0\}$. Thus $f(z) = \Phi(z, 0)$, $z \in \Omega_0$. But then

$$\lim_{\Omega_0 \ni z \to 0} f(z) = \lim_{z \to 0} \Phi(z, 0) = 0,$$

Fig. 7 Sectors and angles
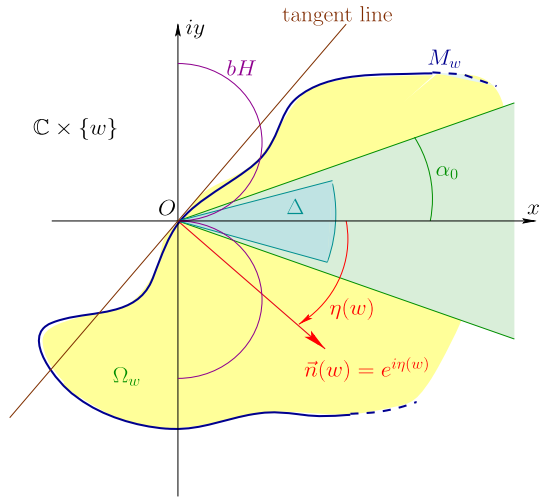
which implies that f has removable singularity at 0, contrary to its construction. Thus an extension is impossible. The proof of the theorem will be complete when Lemma 4.4 will be proved.

Proof of Lemma 4.4 It is enough to show that F is smooth on $(\mathbb{C}^2 \backslash \Omega) \cap (\mathbb{C} \times \overline{D}(0, R))$ for every $R > 0$. Fix R.

Let

$$\alpha_0 := \min_{|w| \leq R} \min \left(\left| \frac{\pi}{2} - \eta(w) \right|, \left| \frac{\pi}{2} + \eta(w) \right| \right).$$

Then $\alpha_0 > 0$ and $-\frac{\pi}{2} + \alpha_0 \leq \eta(w) \leq \frac{\pi}{2} - \alpha_0$, $|w| \leq R$. This means that the smallest unoriented angle between the tangent line to M_w at 0 (for $|w| \leq R$) and the positive real axis is α_0, see Fig. 7. For a fixed $\alpha \in (0, \alpha_0)$ there is a $\rho_0 > 0$ such that the circular sector

$$\Delta := \{ \rho e^{i\theta} : 0 < \rho < \rho_0, \ |\theta| < \alpha \}$$

is contained in Ω_w for all $|w| \leq R$. Such Δ must contain a "germ" of the cusp of H, i.e., there is $\delta > 0$ such that $\Delta \supset H \cap D(0, \delta)$, and so

$$\mathbb{C} \backslash \Omega_w \subset (\mathbb{C} \backslash H) \cup (\mathbb{C} \backslash D(0, \delta)), \quad |w| \leq R,$$

i.e,

$$(\mathbb{C}^2 \backslash \Omega) \cap (\mathbb{C} \times D(0, R)) \subset \left[(\mathbb{C} \backslash H) \times \overline{D}(0, R) \right] \cup \left[(\mathbb{C} \backslash D(0, \delta)) \times \overline{D}(0, R) \right].$$

Denote these three sets as Set(1), Set(2), Set(3), i.e.,

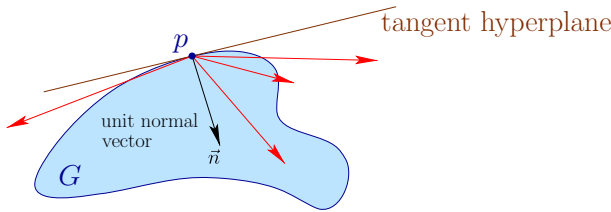$$\text{Set}(1) \subset \text{Set}(2) \cup \text{Set}(3).$$

Fig. 8 Vectors pointing inside the domain

We refer now to Lemma 4.3. Clearly the derivatives of all orders of F exist and are continuous at points of Set(2) and Set(3) and therefore for Set(1). By the lemma, their restrictions to Set(2)\ ({0} × $\mathbb{C}$) have all limits 0 at the point $(0, w_0)$, $(|w_0| \leq R)$, and so the same holds for Set(1)\ ({0} × $\mathbb{C}$). Therefore the derivatives of F are continuous up to the boundary and the lemma is established. $\square$

4.3 Invariant formulation of Theorem 4.1

Theorem 4.1 can be reformulated in a more invariant way, namely, instead of the inequalities $-\frac{\pi}{2} < \eta(w) < \frac{\pi}{2}$ we use a condition that involves an entire harmonic function. We have the following

Corollary 4.5 *Let Ω be a domain in $\mathbb{C}^2$ with smooth boundary M such that $\{0\} \times \mathbb{C} \subset M$. Let $\eta : \mathbb{C} \longrightarrow \mathbb{R}$ be a continuous function such that for every $w \in \mathbb{C}$, $e^{i\eta(w)}$ is the unit normal vector to $b\Omega_w$ at $(0, w)$ pointing inside Ω_w. Assume further that there is an entire harmonic function $\chi : \mathbb{C} \longrightarrow \mathbb{R}$ such that*

$$|\chi(w) - \eta(w)| < \frac{\pi}{2} \quad \text{for all} \quad w \in \mathbb{C}. \tag{5}$$

Then there is a smooth CR function on M that cannot be continuously extended to a holomorphic function in Ω.

Before we give a proof of the Corollary, we take a closer look at the notion of a vector pointing inside a domain. Let $G \subset \mathbb{C}$ be a domain with smooth boundary bG and let $\vec{v}$ be a vector with the initial point $p \in bG$. We say that $\vec{v}$ is *pointing inside the domain* if it lies in the open half-space determined by the tangent hyperplane to bG at p and containing the unit normal vector to bG at p pointing inside the domain (see Fig. 8). Obviously, this notion can be generalized for higher dimensions.

Proof of Corollary 4.5 Let $\lambda : \mathbb{C} \longrightarrow \mathbb{R}$ be a harmonic conjugate of χ. We use now the biholomorphic map K of $\mathbb{C}^2$ onto itself,

$$K(z, w) = (z \exp(-\lambda(w) - i\chi(w)), \ w),$$

to reduce the general case to that of Theorem 4.1. Namely, the hypersurface $M_1 := K(M)$ and domain $\Omega_1 := K(\Omega)$ satisfy the assumptions of Theorem 4.1 due to

condition (5), so there is a CR function $\varphi_1 \in C^\infty(M_1)$ which cannot be continuously extended to a holomorphic function on Ω_1. Let $\varphi := \varphi_1 \circ (K|_M)^{-1}$. It is clear then that φ cannot be continuously extended to a holomorphic function in Ω. □

Observe that condition (5) can be given the following invariant form

(⋆) *Let Ω be a domain with smooth boundary $M = b\Omega$ and let V be a closed complex variety contained in M. There exists a nowhere vanishing holomorphic vector field $X : V \longrightarrow T^{\mathbb{C}}(\mathbb{C}^2)$ (complex tangent bundle) such that for every $p \in V$, the vector $X(p)$ points into the Ω-side of $T_p^{\mathbb{C}}(\mathbb{C}^2) \backslash T_p^{\mathbb{C}}(M)$.*

In case $V = \{0\} \times \mathbb{C}$, conditions (⋆) and (5) are equivalent, because for $p = (0, w)$, the complex tangent space $T_p^{\mathbb{C}}(M)$ contains $\{0\} \times \mathbb{C}$. If X satisfying condition (⋆) exists, then $X(0, w) = \langle f(w), g(w) \rangle$, where $f(w) \neq 0$ for all $w \in \mathbb{C}$, because $f(w) = 0$ would imply $X(0, w) \in T_{(0,w)}^{\mathbb{C}}(M)$. Furthermore, the condition (⋆) implies that $|\measuredangle(f(w), e^{i\eta(w)})| < \frac{\pi}{2}$ for all w. This means that $\log f(w)$ has a continuous branch $\lambda(w) + i\chi(w)$ on $\mathbb{C}$, such that $|\chi(w) - \eta(w)| < \frac{\pi}{2}$. Since χ is harmonic, this implies condition (5).

Conversely, if (5) holds, let λ be an entire harmonic function such that $\lambda + i\chi$ is holomorphic. Let

$$X(0, w) = \langle e^{\lambda(w) + i\chi(w)}, 0 \rangle.$$

Then $X(0, w)$ is a nonvanishing holomorphic field on $V = \{0\} \times \mathbb{C}$ satisfying condition (⋆).

Therefore, Corollary 4.5 can be reformulated as

Theorem 4.6 *Let Ω be a domain in $\mathbb{C}^2$ with smooth boundary $M = b\Omega$ that contains a complex line $V \subset M$. Moreover, assume that there exists a nowhere vanishing holomorphic vector field $X : V \longrightarrow T^{\mathbb{C}}(\mathbb{C}^2)$ such that for every $p \in V$, the vector $X(p)$ points into the Ω-side of $T_p^{\mathbb{C}}(\mathbb{C}^2) \backslash T_p^{\mathbb{C}}(M)$. Then there is a smooth CR function on M that cannot be continuously extended to a holomorphic function in Ω.*

5 Hartogs-type domains with a complex line in the boundary

The main goal of this section is to investigate whether the condition (5) is necessary for failure of the Hartogs extension phenomenon in the class of domains Ω containing a complex line in its boundary $M = b\Omega$. The following theorem and its corollary suggest that it is "close" to being such, but there is a gap. The class of domains considered in this section is very similar to those from Sect. 4. We need to make an additional assumption that the tubes are shrinking at infinity, because otherwise there might be other obstacles to extension. We use the notation and meanings of $\eta = \eta(w)$, Ω_w, M_w as in the previous section.

Theorem 5.1 *Let Ω be a domain in $\mathbb{C}^2$ with smooth boundary $M = b\Omega$ and $\{0\} \times \mathbb{C} \subset M$. Assume that M_w is a smooth Jordan curve for every $w \in \mathbb{C}$, and that*
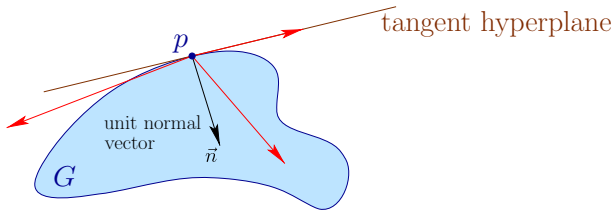
$$\lim_{w \to \infty} \text{diam}(M_w) = 0. \tag{6}$$

Fig. 9 Vectors pointing into the closed side of the domain

If there exists a smooth CR function on M that cannot be holomorphically continued to Ω, *then there is an entire harmonic function* $\chi : \mathbb{C} \longrightarrow \mathbb{R}$ *such that*

$$|\eta(w) - \chi(w)| \leq \frac{\pi}{2} \quad for \quad w \in \mathbb{C}. \tag{7}$$

The proof of the theorem is given in the rest of this section. As an immediate consequence of the theorem, we obtain

Corollary 5.2 *Let* Ω *be a domain as in the above theorem. Assume additionally that there does not exist an entire harmonic function* $\chi : \mathbb{C} \longrightarrow \mathbb{R}$ *such that*

$$|\eta(w) - \chi(w)| \leq \frac{\pi}{2} \quad for \quad w \in \mathbb{C}. \tag{8}$$

Then every smooth CR function on M continuously extends to a holomorphic function in Ω, *i.e.,* Ω *is a Hartogs-type domain.*

Remark 5.3 The discrepancy between the necessary condition (7) and (5) is small: the first says $\|\chi - \eta\|_\infty \leq \frac{\pi}{2}$, the second

$$\forall_w \ |\chi(z) - \eta(z)| < \frac{\pi}{2} \quad \left(\text{not needed that } \|\chi - \eta\|_\infty < \frac{\pi}{2}\right).$$

As in the previous section, the above corollary can be formulated more invariantly. Let $G \subset \mathbb{C}$ be a domain with smooth boundary bG and let $\vec{v}$ be a vector with the initial point $p \in bG$. We say that $\vec{v}$ is *pointing into the closed side of the domain* if $\vec{v} \neq \vec{0}$ and it lies in the closed half-space determined by the tangent hyperplane to bG at p and the unit normal vector to bG at p pointing inside the domain (see Fig. 9). Obviously, this notion can be generalized to higher dimensions.

Theorem 5.4 *Let* $\Omega \subset \mathbb{C}^2$ *be a domain as in Theorem 5.1 with boundary* $M = b\Omega$ *containing a complex line V. Moreover assume that there does not exist a nowhere vanishing holomorphic vector field* $X : V \longrightarrow T^{\mathbb{C}}(\mathbb{C}^2)$ *such that for every* $p \in V$ *the vector* $X(p)$ *points into the closed* Ω-side of $T_p^{\mathbb{C}}(\mathbb{C}^2) \backslash T_p^{\mathbb{C}}(M)$. *Then every smooth CR function on M continuously extends to a holomorphic function in* Ω, *i.e.,* Ω *is a Hartogs-type domain.*
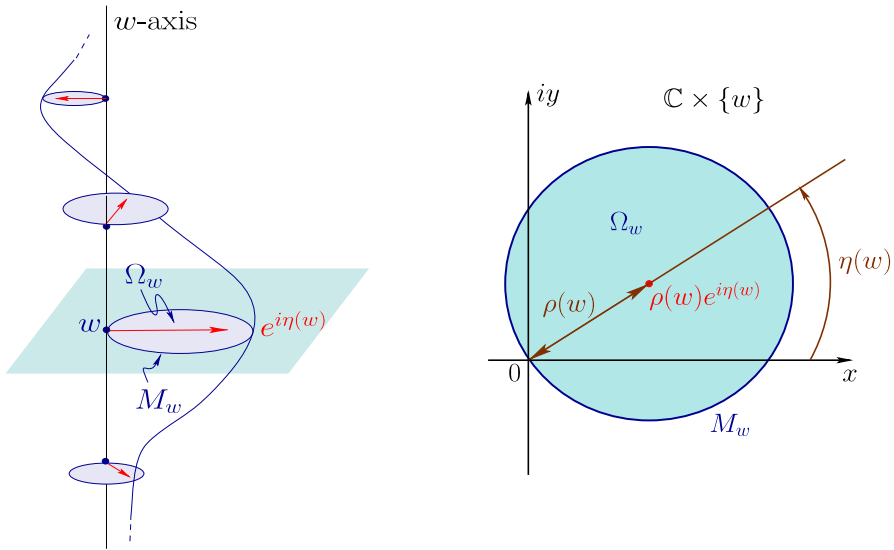
Fig. 10 Tube-like domain along the w-axis with discs as cross-sections

5.1 Example—rotating discs

Here we give an example of a class of domains, described in an easy geometric way (illustrated in Fig. 10), which satisfy the conditions of the theorem.

Let $\rho : \mathbb{C} \longrightarrow (0, 1)$ be a smooth function with the property $\lim_{w \to \infty} \rho(w) = 0$, and let $\eta : \mathbb{C} \longrightarrow \mathbb{R}$ be a smooth function with compact support such that $\eta(w) \geq 0$ and $\max_w \eta(w) > \pi$. Define

$$\Omega_w := D(\rho(w)e^{i\eta(w)}, \rho(w)) = \{z \in \mathbb{C} : |z - \rho(w) e^{i\eta(w)}| < \rho(w)\}.$$

Then every entire harmonic function χ such that $\|\eta - \chi\|_\infty \leq \frac{\pi}{2}$ would have to be bounded, and so constant, and clearly such constant does not exist. Hence Corollary 5.2 applies.

5.2 Some properties of holomorphic and plurisubharmonic functions

We proceed now with the proof of Theorem 5.1, which will take a couple of steps. First we prove some lemmas.

Lemma 5.5 *Let* $F : (\mathbb{C}\backslash\{0\}) \times \mathbb{C} \longrightarrow \mathbb{C}$ *be a holomorphic function that does not have an entire extension to* $\mathbb{C}^2$*. Let* E *be the set of all* w *such that* $z \to F(z, w)$ *extends to an entire function in* $\mathbb{C}$*. Then* E *is a closed and discrete subset of* $\mathbb{C}$*, at most countable. Define for* $(z, w) \in (\mathbb{C}\backslash\{0\}) \times (\mathbb{C}\backslash E)$,

$$\Psi(z, w) = \sup \{|F(tz, w)| : 0 < t \leq 1\}.$$

Let

$$W := \left\{(z, w) \in (\mathbb{C}\backslash\{0\}) \times (\mathbb{C}\backslash E) : \begin{array}{l} \textit{there is a neighborhood } V \textit{ of } (z, w) \\ \textit{such that } \Psi|_V \textit{ is bounded from above} \end{array}\right\}$$

and $\mathscr{S} := [\mathbb{C} \times (\mathbb{C}\backslash E)]\backslash W$. Then W is a pseudoconvex set and $\mathscr{S}$ is a pseudoconcave, relatively closed subset of $\mathbb{C} \times (\mathbb{C}\backslash E)$, whose fibers $\mathscr{S}_w$ are $\mathbb{R}_+$-cones, $\mathscr{S}_w \neq \{0\}$, for every $w \in \mathbb{C}\backslash E$.

Proof Representing $F(z, w)$ by a Laurent series in z, $z \neq 0$,

$$F(z, w) = \sum_{n=0}^{\infty} a_n(w)z^n + \sum_{n=1}^{\infty} b_n(w)z^{-n},$$

where $a_n, b_n : \mathbb{C} \longrightarrow \mathbb{C}$ are entire functions, we obtain that

$$E = \{w \in \mathbb{C} : b_n(w) = 0, \ n \geq 1\}.$$

Thus E is closed. If E is not discrete, all $b_n \equiv 0$ and F has entire extension to $\mathbb{C}^2$, a contradiction.

To show that W is pseudoconvex, we apply Lemma 6.2 from the Appendix to the set

$$U := (\mathbb{C}\backslash\{0\}) \times (\mathbb{C}\backslash E)$$

and the family of plurisubharmonic functions

$$\Psi_t(z, w) := |F(tz, w)|, \quad (z, w) \in U, \quad 0 < t \leq 1.$$

We denote

$$\Psi(z, w) := \sup\{\Psi_t(z, w) : 0 < t \leq 1\}.$$

Then Lemma 6.2 implies that W is relatively pseudoconvex in $(\mathbb{C}\backslash\{0\}) \times (\mathbb{C}\backslash E)$. Since the latter set is pseudoconvex, so is W, and $\mathscr{S}$ is pseudoconcave by definition. The fibers $\mathscr{S}_w$ are cones because it follows immediately from the definition of $\mathscr{S}$, namely if there is no neighborhood V of $(z, w) \in \mathscr{S}$ such that $\Psi|_V$ is bounded then the same property has the point (tz, w) for $t > 0$ (Fig. 11).

Suppose that $\mathscr{S}_{w_0} = \{0\}$ for some $w_0 \in \mathbb{C}\backslash E$. This would mean that $S^1 \times \{w_0\} \subset W$, and then Corollary 6.4 would imply that there is a constant $M < +\infty$ such that $\Psi_t(z, w_0) \leq M$ for every $|z| = 1$ and $0 < t \leq 1$. It follows that $|F(tz, w_0)| \leq M$ for $z \in S^1$, $t \in (0, 1]$, and so $z \to F(z, w_0) : \mathbb{C}\backslash\{0\} \longrightarrow \mathbb{C}$ is bounded on a deleted neighborhood of 0 (i.e., 0 is removed from the neighborhood), and the singularity is removable. Consequently, $w_0 \in E$, a contradiction. $\qquad\square$
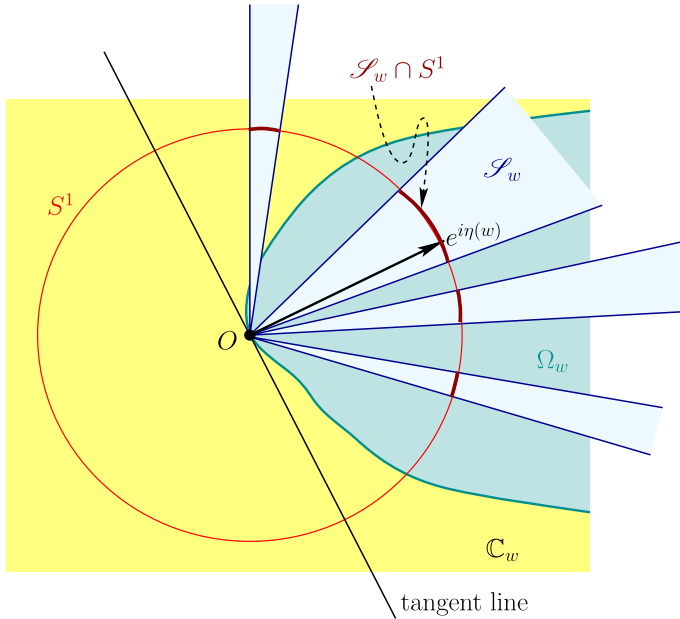
Fig. 11 Set $\mathscr{S}_w$

Lemma 5.6 *In the set-up of Lemma 5.5, assume further that there is a continuous function $\eta : \mathbb{C} \longrightarrow \mathbb{R}$ such that for every $w \in \mathbb{C} \backslash E$,*

$$\mathscr{S}_w \subset \{z \in \mathbb{C} \backslash \{0\} : |\mathrm{Arg}(ze^{-i\eta(w)})| \leq \pi/2\} \cup \{0\}, \qquad (9)$$

where $\mathrm{Arg}(\cdot)$ is defined in $\mathbb{C} \backslash \{0\}$ with $\mathrm{Arg}(1) = 0$ and takes values from the interval $(-\pi, \pi]$. Then there is an entire harmonic function $\chi : \mathbb{C} \longrightarrow \mathbb{R}$ such that $|\eta(w) - \chi(w)| \leq \frac{\pi}{2}$ for $w \in \mathbb{C}$.

Proof Denote

$$\mathscr{A}(z, w) = \eta(w) + \mathrm{Arg}(ze^{-i\eta(w)}) \qquad \text{for} \quad (z, w) \in (\mathbb{C} \backslash \{0\}) \times \mathbb{C}. \qquad (10)$$

The function $\mathscr{A}$ is a continuous branch of $\arg(z)$ on the set

$$(z, w) \in [\mathbb{C} \times \mathbb{C}] \backslash \{(te^{i\eta(w)}, w) : t \in (-\infty, 0]\}$$

and so pluriharmonic on its domain of definition. In particular, for fixed $z \in \mathbb{C} \backslash \{0\}$ the function $\mathscr{A} = \mathscr{A}(z, w)$ is independent of w.

Define

$$\begin{aligned} \chi_0(w) &= \max\{\mathscr{A}(z, w) : z \in \mathscr{S}_w\} \\ \mu_0(w) &= \min\{\mathscr{A}(z, w) : z \in \mathscr{S}_w\} \end{aligned} \qquad w \in \mathbb{C} \backslash E.$$

Because $\mathscr{S}_w$ being a cone, we can write

$$\chi_0(w) = \max\{\mathscr{A}(z, w) : z \in \mathscr{S}_w \cap S^1\}$$
$$\mu_0(w) = \min\{\mathscr{A}(z, w) : z \in \mathscr{S}_w \cap S^1\} \qquad S^1 = \{z \in \mathbb{C} : |z| = 1\}. \qquad (11)$$

We note that for each $w \in \mathbb{C}\backslash E$, the cone $\mathscr{S}_w$ is closed and the set $\mathscr{S}_w \cap S^1$ is compact, see Fig. 11. Consequently the maximum and minimum actually exist and for each $w \in \mathbb{C}\backslash E$ there are $z^{\max}(w) \in S^1$ and $z^{\min}(w) \in S^1$ such that

$$\chi_0(w) = \mathscr{A}(z^{\max}(w), w) \qquad \text{and} \qquad \mu_0(w) = \mathscr{A}(z^{\min}(w), w).$$

Now we prove that $\chi_0 = \chi_0(w)$ is an upper-semi-continuous (usc) in $\mathbb{C}\backslash E$; the proof that μ_0 is a lower-semi-continuous (lsc) is analogous. Let's calculate $\limsup_{w \to w_0} \chi_0(w)$, namely take any sequence w_n converging to w_0. Without any loss of generality, we can assume that $z^{\max}(w_n)$ converges to $z_0^{\max}(w_0)$. If

$$|\mathrm{Arg}(z_0^{\max}(w_0) e^{-i\eta(w_0)})| < \pi,$$

the function χ_0 is continuous at w_0. If

$$\mathrm{Arg}(z_0^{\max}(w_0) e^{-i\eta(w_0)}) = \pi,$$

we see, using (10) and (11), that

$$\chi_0(w_0) = \eta(w_0) + \mathrm{Arg}(z_0^{\max}(w_0) e^{-i\eta(w_0)}) = \eta(w_0) + \pi$$

but

$$\limsup_{w_n \to w_0} \chi_0(w_n) = \eta(w_0) + \pi \quad \text{or} \quad \limsup_{w_n \to w_0} \chi_0(w_n) = \eta(w_0) - \pi$$

which gives

$$\limsup_{w_n \to w_0} \chi_0(w_n) \le \chi_0(w_0),$$

and because of the arbitrary choice of the sequence $\{w_n\}$ it implies that

$$\limsup_{w \to w_0} \chi_0(w_0) \le \chi_0(w_0),$$

which proves that χ_0 is upper-semi-continuous.

To show that χ_0 is subharmonic in $\mathbb{C}\backslash E$, we prove the following

Sublemma *Let $\chi : U \longrightarrow [-\infty, +\infty)$, U open in $\mathbb{C}$, be an upper semi-continuous function. Suppose χ is not subharmonic. Then there is a point $a \in U$, a harmonic function $h : D(a, r) \longrightarrow \mathbb{R}$, $\varepsilon > 0$, where $\overline{D(a, r)} \subset U$, such that*

$(\chi + h)(a) = 0$ *and for* $|z - a| < r$ *we have the inequality* $(\chi + h)(z) \leq -\varepsilon |z - a|^2$.

Proof of the Sublemma If χ is not subharmonic, then there is a $z_0 \in U$ and $R > 0$ and a harmonic function F on the disc $\overline{D(z_0, R)}$ such that

$$\chi(z_0) > F(z_0) \quad \text{and} \quad \chi(z) < F(z) \quad \text{for} \quad |z - z_0| = R$$

Define $\phi(z) = \chi(z) - F(z)$. Since ϕ is upper semicontinuous, there exists $m > 0$ such that

$$\phi(z_0) > 0 > -m > \phi(z) \quad \text{for} \quad |z - z_0| = R$$

Therefore, there exists $\varepsilon > 0$ such that

$$\phi(z) + \varepsilon |z - z_0|^2 \leq -\frac{m}{2} < 0 \quad \text{for} \quad |z - z_0| = R.$$

The function $\psi(z) = \phi(z) + \varepsilon |z - z_0|^2$ is upper semicontinuous on $\overline{D(z_0, R)}$ and thus assumes its maximum at a point a which must lie on the inside of $D(z_0, R)$ since $\psi < 0$ on the boundary of this disc and $\psi(z_0) > 0$. We therefore have

$$\phi(a) + \varepsilon |a - z_0|^2 = \psi(a) \geq \psi(z_0) = \phi(z_0) > 0$$

Define

$$\alpha(z) = (\phi(z) + \varepsilon |z - z_0|^2) - (\phi(a) + \varepsilon |a - z_0|^2 + \varepsilon |z - a|^2) = \psi(z) - \psi(a) - \varepsilon |z - a|^2.$$

Since $\psi(z) \leq \psi(a)$, we have

$$\alpha(z) \leq -\varepsilon |z - a|^2 \quad \text{on} \quad \overline{D(z_0, R)} \tag{12}$$

Since a belongs to the inside of $D(z_0, R)$, there is an $r > 0$, such that (12) holds as well for $z \in \overline{D(a, r)}$. Also observe that we can write

$$\alpha(z) = \phi(z) + \ell(z)$$

where $\ell(z)$ is an affine linear function of z and hence is harmonic. We now let $h(z) = \ell(z) - F(z)$. Clearly h is harmonic, and from the definitions we have

$$\alpha(z) = \chi(z) + h(z)$$

Note that α satisfies the desired estimate in view of (12) and note that $\alpha(a) = 0$ in view of its definition. Thus h satisfies the properties stated in the sublemma. $\square$

End of the proof of Lemma 5.6 Now coming back to the proof of the lemma, we show that χ_0 is subharmonic in $\mathbb{C} \backslash E$. Suppose not. Then, using the Sublemma, there is a

point $w_0 \in \mathbb{C} \backslash E$ and a smooth subharmonic function $\alpha(w)$ in $\{|w - w_0| < r\}, r > 0$, such that

$$\chi_0(w_0) + \alpha(w_0) = 0,$$
$$\chi_0(w) + \alpha(w) \leq -\varepsilon |w - w_0|^2$$

for some $\varepsilon > 0$ and for $|w - w_0| < r$.

Let now

$$\varphi(z, w) = \mathscr{A}(z, w) + \alpha(w) \quad \text{for} \quad (z, w) \in U = \mathbb{C} \times \{|w - w_0| < r\}.$$

Take $z_0 \neq 0, z_0 \in \mathscr{S}_{w_0}$, such that $\chi_0(w_0) = \mathscr{A}(z_0, w_0)$. Then

$$\varphi(z_0, w_0) = \mathscr{A}(z_0, w_0) + \alpha(w_0) = \chi_0(w_0) + \alpha(w_0) = 0,$$
$$\varphi(z, w) = \mathscr{A}(z, w) + \alpha(w) \leq \chi_0(w) + \alpha(w) \leq -\varepsilon |w - w_0|^2$$

if $(z, w) \in \mathscr{S} \cap U$. Since U is a neighborhood of (z_0, w_0) and $\varphi(\cdot, \cdot)$ a pluri-subharmonic function in U, we obtained a contradiction with local maximum principle for pseudoconcave sets, due essentially to Wermer [26] (or see [25], Theorem 2.1(iv) for the exact form used above). Thus χ_0 is subharmonic in $\mathbb{C} \backslash E$.

A symmetrical reflection of the above proof shows that μ_0 defined in (11) is super-harmonic in $\mathbb{C} \backslash E$. In addition, assumption (9) implies

$$\chi_0(w) \leq \eta(w) + \frac{\pi}{2}; \quad \mu_0(w) \geq \eta(w) - \frac{\pi}{2} \quad \text{for} \quad w \in \mathbb{C} \backslash E. \tag{13}$$

Then $(\chi_0 - \mu_0)(w) \leq \pi, \ w \in \mathbb{C} \backslash E$, is uniformly bounded from the above (on $\mathbb{C} \backslash E$) subharmonic function, which can be extended to $\mathbb{C}$ to be subharmonic at points of E also, and so is constant, say $\chi_0 - \mu_0 = c_0$. As $\chi_0 = \mu_0 + c_0$ is both sub- and super-harmonic, it is harmonic. Since χ_0 is continuous at points of E, inequalities (13) imply that χ_0 is locally bounded (above and below) in neighborhoods of points of E, and so has entire harmonic extension $\chi : \mathbb{C} \longrightarrow \mathbb{R}$ satisfying $|\chi(w) - \eta(w)| \leq \frac{\pi}{2}$, $w \in \mathbb{C}$. The lemma is proved.   □

5.3 End of the proof of Theorem 5.1

Proof Consider a smooth CR function $f : M \longrightarrow \mathbb{C}$ that cannot be continuously extended to a holomorphic function in Ω. We can decompose $f = F_+|_M - F_-|_M$, where $F_+ \in C^\infty(\overline{\Omega}) \cap \text{Hol}(\Omega)$, $F_- \in C^\infty(\mathbb{C}^2 \backslash \Omega) \cap \text{Hol}(\mathbb{C}^2 \backslash \overline{\Omega})$. Since $\lim_{w \to \infty}$ (diam M_w) = 0, the projection map

$$\pi \big|_{\overline{\Omega} \backslash (\{0\} \times \mathbb{C})} \ : \ \overline{\Omega} \backslash (\{0\} \times \mathbb{C}) \longrightarrow \mathbb{C} \backslash \{0\}$$

is proper. Consequently, all leaves of the foliation $(\mathbb{C} \backslash \{0\}) \times \mathbb{C}$ by the complex lines $\{z\} \times \mathbb{C}, z \neq 0$, intersect compactly $\overline{\Omega} = M \cup \Omega$, and the intersection is empty for large $|z|$. To extend the function F_- we apply the foliation method derived in [7], p. 561,

and applied in [3], Sec. 3, p. 121. First we extend F_- smoothly (C^∞) to a function $\widetilde{F}_-$ on Ω with support close to $b\Omega \cap [(\mathbb{C}\backslash\{0\}) \times \mathbb{C}]$, then define the $(0, 1)$-form

$$\omega = \begin{cases} \overline{\partial}\widetilde{F}_- & \text{on } \Omega \\ 0 & \text{on } \mathbb{C}^2\backslash\Omega. \end{cases}$$

Obviously ω is a smooth $\overline{\partial}$-closed form on $(\mathbb{C}\backslash\{0\}) \times \mathbb{C}$. Next we solve the $\overline{\partial}$-problem

$$\overline{\partial}u = \omega \quad \text{on} \quad (\mathbb{C}\backslash\{0\}) \times \mathbb{C}$$

in such a way that u vanishes outside $\overline{\Omega}$ because the fibers $\{z\} \times \mathbb{C}$ do not intersect $\overline{\Omega}$ for $|z|$ large; see details in [7], pp. 561–563. Therefore the function $F = \widetilde{F}_- - u$ has the properties that $F\big|_{(\mathbb{C}\backslash\{0\})\times\mathbb{C}}$ is holomorphic and $F\big|_{\mathbb{C}^2\backslash\Omega}$ is continuous. Observe that F is not entire, for otherwise $F_+ - F\big|_\Omega$ would be a holomorphic extension of f to Ω, a contradiction.

Thus F satisfies the assumption of Lemma 5.5. Let E, $\mathscr{S}$ be sets with properties stated there. Let $\vec{n}(w)$, $w \in \mathbb{C}$, denote the unit normal vector at $0 \in M_w$ to M_w, pointing inside the Jordan domain Ω_w. Then (by monodromy) there is a continuous function $\eta : \mathbb{C} \longrightarrow \mathbb{R}$ such that $\vec{n}(w) = e^{i\eta(w)}$. We will show now that this function η satisfies condition (9), which is equivalent to

$$\left\{ z \in \mathbb{C} : \ |\text{Arg}(ze^{-i\eta(w)})| > \frac{\pi}{2} \right\} \cap \mathscr{S}_w = \emptyset \quad for \ \ w \in \mathbb{C}.$$

Consider (z_0, w_0), $z_0 \neq 0$, with $\sphericalangle(z_0, e^{i\eta(w_0)}) > \frac{\pi}{2}$. By smoothness of M, there is $1 \geq t_0 > 0$, $\varepsilon > 0$, such that

$$Q = \{(tz, w) : \ 0 \leq t \leq t_0, \ \ |z - z_0| \leq \varepsilon, \ \ |w - w_0| \leq \varepsilon\} \subset \mathbb{C}^2\backslash\Omega.$$

Since $F(z, w)$ is continuous on $\mathbb{C}^2\backslash\Omega$, it is bounded on Q, so $z_0 \notin \mathscr{S}_{w_0}$.

We are allowed now to apply Lemma 5.6, and so obtain a harmonic function $\chi :$ $\mathbb{C} \longrightarrow \mathbb{R}$ with $|\chi(w) - \eta(w)| \leq \frac{\pi}{2}$ for all $w \in \mathbb{C}$, which completes the proof of the theorem. $\qquad\qquad\square$

6 Appendix

The Appendix has two parts: proof of Lemma 4.3 and proof of some properties of supremum of plurisubharmonic functions.

6.1 Proof of Lemma 4.3

In this section we will use the sets P and S that were defined in (3) and (4). For the proof of Lemma 4.3 we need the following fact

Lemma 6.1 *There is an entire function* $g : \mathbb{C} \longrightarrow \mathbb{C}$ $(g \not\equiv 0)$ *such that for every* $k, N = 0, 1, 2, \ldots,$ *we have*

$$\lim_{\mathbb{C} \backslash S \ni \zeta \to \infty} \zeta^N g^{(k)}(\zeta) = 0.$$

Proof of Lemma 6.1 We split the proof into few steps.

1. Assertion *There is an entire analytic function* $\tilde{g} : \mathbb{C} \longrightarrow \mathbb{C}, \tilde{g} = \tilde{g}(\zeta) \not\equiv 0$, *with the property:*

for any $A > 0$ and $N = 1, 2, \ldots$, we have $\lim_{\substack{\zeta \to \infty \\ \mathrm{Re}\,\zeta \leq A}} \zeta^N \tilde{g}(\zeta) = 0.$

To establish this Assertion, we consider a C^∞-smooth real function $\psi(t)$ with support in $[0, 1]$ and let

$$\tilde{g}(\zeta) = \int_{-\infty}^{\infty} e^{t\zeta} \psi(t)\, dt = \int_0^1 e^{t\zeta} \psi(t)\, dt.$$

Since

$$\sup_t \left| e^{t\zeta} \psi(t) \right| \leq \sup_t \left[e^{t(\mathrm{Re}\,\zeta)} |\psi(t)| \right] \leq \max(e^{\mathrm{Re}\,\zeta}, 1)\, \|\psi\|_\infty,$$

the integral is convergent and $\tilde{g}(\zeta)$ is well-defined for every $\zeta \in \mathbb{C}$.

It is not hard to see that $\tilde{g}(\zeta)$ is holomorphic. By repeated integration by parts (note the compact support of ψ), we have

$$\tilde{g}(\zeta) = \int_{-\infty}^{\infty} e^{t\zeta} \psi(t)\, dt = \frac{(-1)^N}{\zeta^N} \int_{-\infty}^{\infty} e^{t\zeta}\, \psi^{(N)}(t)\, dt \quad \text{for} \quad \zeta \neq 0,$$

so

$$|\tilde{g}(\zeta)| \leq \frac{1}{|\zeta|^N} \int_0^1 |e^{t\zeta}|\, |\psi^{(N)}(t)|\, dt.$$

Fix any $A > 0$. For $\mathrm{Re}\,\zeta \leq A$ we have

$$\mathrm{Re}(t\zeta) \leq A, \qquad t \in [0, 1],$$

and so

$$|\tilde{g}(\zeta)| \le \frac{1}{|\zeta|^N}\, e^A\, \|\psi^{(N)}\|_\infty \longrightarrow 0 \quad \text{for} \quad \operatorname{Re}\zeta \le A \quad \text{as} \quad |\zeta| \to \infty,$$

which establishes the Assertion. It should be mentioned that the same argument handles derivatives of $\tilde{g}$.

2. Approximation of a branch of $\zeta \longrightarrow i\sqrt{\zeta - 2\pi}$ by entire functions

Consider now a holomorphic function $h_0 : \mathbb{C}\setminus[2\pi, +\infty) \longrightarrow \mathbb{C}$ that is a continuous branch of the multi-valued function $i\sqrt{\zeta - 2\pi}$ defined as follows:

$$h_0(\zeta) = \rho^{1/2} e^{i(\theta+\pi)/2} \quad \text{for} \quad \zeta \in \mathbb{C}\setminus[2\pi, \infty), \zeta = 2\pi + \rho e^{i\theta}, \rho > 0, \, 0 < \theta < 2\pi.$$

Then for $\zeta \in \mathbb{C}\setminus[2\pi, \infty)$,

$$\operatorname{Re} h_0(\zeta) < 0, \quad |h_0(\zeta)| = |\zeta - 2\pi|^{1/2}.$$

Let

$$E := \mathbb{C}\setminus P \subset \mathbb{C}\setminus[2\pi, \infty).$$

Observe that E is closed in $\mathbb{C}$ and its complement in $\overline{\mathbb{C}}$ is connected and locally connected, as $\overline{\mathbb{C}}\setminus E = P \cup \{\infty\}$. Therefore $h_0\big|_E$ can be uniformly approximated (on E) by entire functions (Theorem 3 in Brown et al. [5], see also Arakeljan [1]). In particular, there is an entire function

$$h : \mathbb{C} \longrightarrow \mathbb{C} \quad \text{such that} \quad \sup_{\zeta \in E} |h_0(\zeta) - h(\zeta)| \le 1.$$

It follows that

$$\begin{aligned} \operatorname{Re} h(\zeta) &\le 1 & &\text{for } \zeta \in \mathbb{C}\setminus P, \\ \sqrt{|\zeta - 2\pi|} - 1 &\le |h(\zeta)| \le \sqrt{|\zeta - 2\pi|} + 1 & &\text{for } \zeta \in \mathbb{C}\setminus P, \end{aligned}$$

and so

$$\sqrt{|\zeta|} \le 4|h(\zeta)| \quad \text{if} \quad \zeta \in \mathbb{C}\setminus P \quad \text{and} \quad |\zeta| \ge 4\pi. \tag{14}$$

Consequently,

$$\lim_{\substack{\zeta \to \infty \\ \zeta \in \mathbb{C}\setminus P}} h(\zeta) = \infty. \tag{15}$$

3. Definition of the function g

With $A = 1$ in the Assertion, let $g(\zeta) = \tilde{g}(h(\zeta))$, $\zeta \in \mathbb{C}$. Then g is an entire function. Furthermore, using (14) and (15), for any $N = 1, 2, \ldots$ the estimate

$$\left|\zeta^N g(\zeta)\right| = \left(\frac{\sqrt{|\zeta|}}{|h(\zeta)|}\right)^{2N} |h(\zeta)|^{2N}\, |\tilde{g}(h(\zeta))| \le 4^{2N} |h(\zeta)|^{2N}\, |\tilde{g}(h(\zeta))|$$

for $|\zeta| \geq 4\pi$, $\zeta \in \mathbb{C} \backslash P$. Since $\operatorname{Re} h(\zeta) \leq A = 1$, if $\zeta \in \mathbb{C} \backslash P$, we have that

$$\lim_{\substack{\zeta \to \infty \\ \zeta \in \mathbb{C} \backslash P}} h(\zeta)^{2N} \tilde{g}(h(\zeta)) = 0,$$

by the Assertion. Thus

$$\lim_{\substack{\zeta \to \infty \\ \zeta \in \mathbb{C} \backslash P}} \zeta^{N} g(\zeta) = 0 \quad \text{for} \quad N = 1, 2, \ldots,$$

which proves Lemma 6.1 for the function g, i.e., when $k = 0$. It remains to prove for derivatives of g.

4. Properties of derivatives of g

It remains to observe that all derivatives of g have the same property in the slightly smaller set $\mathbb{C} \backslash S$. Fix $N = 1, 2, \ldots$. If we denote

$$M = \sup_{\zeta \in \mathbb{C} \backslash P} |\zeta^{N+1} g(\zeta)|,$$

then we have for every $\zeta_0 \in \mathbb{C} \backslash S$

$$\{|\sigma - \zeta_0| \leq 1\} \subset \mathbb{C} \backslash P,$$

and so

$$\max_{|\sigma - \zeta_0| \leq 1} |g(\sigma)| \leq M \sup_{|\sigma - \zeta_0| = 1} \frac{1}{|\sigma|^{N+1}} \leq M \left(\frac{1}{|\zeta_0| - 1} \right)^{N+1} \quad \text{if} \quad |\zeta_0| > 1,$$

and

$$|\zeta_0|^{N} \sup_{|\sigma - \zeta_0| = 1} |g(\sigma)| \leq \frac{M}{|\zeta_0| - 1} \left(\frac{|\zeta_0|}{|\zeta_0| - 1} \right)^{N} \leq \frac{2^{N} M}{|\zeta_0| - 1} \quad \text{if} \quad |\zeta_0| \geq 2.$$

Using the Cauchy formula

$$g^{(k)}(\zeta_0) = \frac{k!}{2\pi i} \int_{|\sigma - \zeta_0| = 1} \frac{g(\sigma)}{(\sigma - \zeta_0)^{k+1}} \, d\sigma, \quad k \geq 0,$$

we obtain, for $|\zeta_0| \geq 2$ and $\zeta_0 \in \mathbb{C} \backslash S$,

$$|\zeta_0^{N} g^{(k)}(\zeta_0)| \leq (k! \, 2^{N} M) \frac{1}{|\zeta_0| - 1}$$

and so

$$\lim_{\substack{\zeta \to \infty \\ \zeta \in \mathbb{C} \setminus P}} \zeta^N g^{(k)}(\zeta) = 0.$$

Lemma 6.1 is proved.   □

Proof of Lemma 4.3 Define the function $f : \mathbb{C} \longrightarrow \mathbb{C}$ by

$$f(z) = \begin{cases} 0 & \text{for } z = 0 \\ g(1/z) & \text{for } z \neq 0 \end{cases}$$

Then $f|_{\mathbb{C} \setminus \{0\}}$ is holomorphic, and f has an essential singularity at 0. To see this, suppose that f has a removable singularity or a pole, then $\lim_{z \to 0} f(z)$ exists either as a finite number or as infinity. Then $\lim_{\zeta \to \infty} g(\zeta)$ exists in the same sense and so $f(z)$ is a polynomial, perhaps a constant one. This however is not possible by the way g was constructed.

To see that $f(z)$ and $f^{(n)}(z)$ approach 0 as $z \to 0$ within $\mathbb{C} \setminus H$ we use the following fact that can be shown by induction:

For every natural number $n = 1, 2, \dots$ there are complex polynomials

$$P_{1,n}, P_{2,n}, \dots, P_{n,n},$$

all of degree $\leq 2n$, such that for $z \neq 0$

$$f^{(n)}(z) = \sum_{k=1}^{n} P_{k,n}\left(\frac{1}{z}\right) g^{(k)}\left(\frac{1}{z}\right).$$

Now, if $z \to 0$ within $\mathbb{C} \setminus H$, then $1/z \to \infty$, $1/z \in \mathbb{C} \setminus S$, and by Lemma 6.1, we have

$$\left(\frac{1}{z}\right)^{2n} g^{(k)}\left(\frac{1}{z}\right) \to 0.$$

Thus $f^{(n)}(z) \to 0$ for $z \to 0$, $z \in \mathbb{C} \setminus H$, what we wanted to prove.   □

6.2 Properties of supremum of plurisubharmonic functions

Lemma 6.2 *Let $U \subset \mathbb{C}^2$ be an open subset, A a set of parameters, and let $\Psi_\alpha : U \longrightarrow [-\infty, +\infty)$ be a plurisubharmonic function on U for each $\alpha \in A$. Denote*

$$\Psi(z, w) = \sup\{\Psi_\alpha(z, w) : \alpha \in A\}, \qquad (z, w) \in U,$$

and let

$$W := \left\{ (z, w) \in U : \begin{array}{l} \text{for some neighborhood } V \text{ of } (z, w) \\ \text{with } V \subset U, \ \Psi|_V \text{ is bounded from above} \end{array} \right\}.$$

Then W is relatively pseudoconvex in U.

Comments The lemma (which is true in $\mathbb{C}^n$) is presumably well-known, but we could not find a reference. Note that the function Ψ does not have to be upper semi continuous. We base the proof on the well-known characterization of pseudoconvex domains in terms of Hartogs figures. Recall the definition of a (compact) Hartogs figure in $\mathbb{C}^2$. Let $(z_0, w_0) \in \mathbb{C}^2$ and $\vec{a}, \vec{b} \in \mathbb{C}^2$ be two vectors linearly independent over $\mathbb{C}$. Then by a Hartogs figure with center at (z_0, w_0) and frame $\vec{a}, \vec{b}$ we mean the compact set

$$\begin{aligned} K = \{ (z_0, w_0) + u\vec{a} : \ &u \in \mathbb{C}, \ |u| \leq 1 \} \cup \{ (z_0, w_0) + e^{i\theta}\vec{a} + v\vec{b} : \\ &\theta \in \mathbb{R}, \ v \in \mathbb{C}, \ |v| \leq 1 \}, \end{aligned}$$

and a filled Hartogs figure is the compact set

$$\widehat{K} := \{ (z_0, w_0) + u\vec{a} + v\vec{b} : \ u, v, \in \mathbb{C}, \ |u|, |v| \leq 1 \}.$$

We note that $\widehat{K}$ is a bi-disc; the notation $\widehat{K}$ is justified as this bi-disk is actually a polynomial hull of K.

    We use a characterization of relatively pseudoconvex sets given in the following

Lemma 6.3 *Let $W \subset U \subset \mathbb{C}^2$ be open sets. Then W is relatively pseudoconvex in U if and only if for every Hartogs figure $K \subset W$, such that $\widehat{K} \subset U$, the bi-disc $\widehat{K}$ must be contained in W as well.*

Proof of Lemma 6.2 Consider any Hartogs figure $K \subset W$ with $\widehat{K} \subset U$. It has some center (z_0, w_0) and frame $\vec{a}, \vec{b}$. Since $K \subset W$, by definition K has a covering $\{V_j\}_{j \in J}$, by open neighborhoods, such that $V_j \subset W$, $j \in J$, and there are finite constants M_j such that

$$\sup \Psi|_{V_j} \leq M_j.$$

Select a finite covering $V_{j_1}, V_{j_2}, \ldots, V_{j_n}$ of K, and let

$$V := V_{j_1} \cup V_{j_2} \cup \cdots \cup V_{j_n}, \quad M := \max(M_{j_1}, M_{j_2}, \ldots, M_{j_n}).$$

Then we have

$$K \subset V \subset W$$

and

$$\sup \Psi_\alpha|_V \leq M \qquad \text{for} \quad \alpha \in A.$$

Consider now another Hartogs figure K_ε, also with center (z_0, w_0), where $\varepsilon > 0$ is yet to be chosen, and the frame

$$\vec{a}_\varepsilon = (1 + \varepsilon)\vec{a}, \qquad \vec{b}_\varepsilon = (1 + \varepsilon)\vec{b}.$$

Clearly, if $\varepsilon > 0$ is small enough,

$$K_\varepsilon \subset V \qquad \text{and} \qquad \widehat{K}_\varepsilon \subset U.$$

Fix such ε. Since $K_\varepsilon \subset V$, we obtain for every $\alpha \in A$

$$\max(\Psi_\alpha\big|_{K_\varepsilon}) \le \sup \Psi_\alpha\big|_V \le M,$$

(note that Ψ_α itself is upper semi continuous, so the maximum exists). Since $\widehat{K}_\varepsilon \subset U$, Ψ_α is a plurisubharmonic function on a neighborhood of $\widehat{K}_\varepsilon$. Since $\widehat{K}_\varepsilon$ is a polynomial hull of K_ε, or simply by looking at the way complex discs fill $\widehat{K}_\varepsilon$, we conclude

$$\max(\Psi_\alpha\big|_{\widehat{K}_\varepsilon}) = \max(\Psi_\alpha\big|_{K_\varepsilon}) \le M, \qquad \text{forall} \quad \alpha.$$

Hence

$$\sup(\Psi\big|_{\widehat{K}_\varepsilon}) \le M < +\infty. \tag{16}$$

Observe finally that by the way it is constructed, the bi-disc $\widehat{K}_\varepsilon$ is a neighborhood of $\widehat{K}$, i.e., $\widehat{K} \subset \mathrm{Int}(\widehat{K}_\varepsilon)$. By (16), $\mathrm{Int}(\widehat{K}_\varepsilon) \subset W$, and so $\widehat{K} \subset W$. The lemma is proved. $\square$

The finite covering argument used in the proof implies also the following

Corollary 6.4 *In the setting of Lemma 6.2, the function Ψ is uniformly bounded from the above on every compact subset of W.*

Observe that this statement is not completely trivial as Ψ is not shown to be upper semi continuous.

References

1. Arakeljan, N.U.: Uniform approximation on closed sets by entire functions. Izv. Akad. Nauk SSSR Ser. Mat. (Russian) **28**, 1187–1206 (1964)
2. Bochner, S.: Analytic and meromorphic continuation by means of Green's formula. Ann. Math. **44**, 652–673 (1943)
3. Boggess, A., Dwilewicz, R., Slodkowski, Z.: Hartogs phenomenon on unbounded domains—conjectures and examples. In: CRM Proceedings and Lecture Notes, vol. 55, pp. 117–134. Amer. Math. Soc, Providence (2012)
4. Boggess, A., Dwilewicz, R., Słodkowski, Z.: Hartogs extension for generalized tubes in $\mathbb{C}^n$. J. Math. Anal. Appl. **402**(2), 574–578 (2013)
5. Brown, L., Gauthier, P.M., Seidel, W.: Complex approximation for vector-valued functions with an application to boundary behaviour. Trans. Am. Math. Soc. **191**, 149–163 (1974)
6. Coltoiu, M., Ruppenthal, J.: On Hartogs' extension theorem on $(n - 1)$-complete complex spaces. J. Reine Angew. Math. **637**, 41–47 (2009)

7. Dwilewicz, R.: Holomorphic extensions in complex fiber bundles. J. Math. Anal. Appl. **322**, 556–565 (2006)

8. Ehrenpreis, L.: A new proof and an extension of Hartogs' theorem. Bull. Am. Math. Soc. **67**, 507–509 (1961)

9. Fichera, G.: Caratterizazione della traccia, sulla frontiera di un campo, di una funzione analitica di più variabili complesse. Rend. Acc. Naz. Lincei VII **23**, 706–715 (1957)

10. Fueter, R.: Über einen Hartogs'schen Satz. Commun. Math. Helv. **12**, 75–80 (1939)

11. Hartogs, F.: Zür Theorie der analytischen Functionen mehrerer unabhangiger Veränderlichen insbesondere über die Darstellung derselben durch Reihen, welche nach Potenzen einer Veränderlichen fortschreiten. Math. Ann. **62**, 1–88 (1906)

12. Kneser, H.: Die Randwerte einer analytischen Funktion zweier Veränderlichen. Monatsh. für Math. u. Phys. **43**, 364–380 (1936)

13. Koziarz, V., Sarkis, F.: Problème du bord dans les variétés q-convexes et phénomène de Hartogs–Bochner. Math. Ann. **321**(3), 569–585 (2001)

14. Laurent-Thiébaut, C., Leiterer, J.: On the Hartogs–Bochner extension phenomenon for differential forms. Math. Ann. **284**(1), 103–119 (1989)

15. Martinelli, E.: Sopra una dimostrazione di R. Fueter per un teorema di Hartogs. Commun. Math. Helv. **15**, 340–349 (1942/43)

16. Martinelli, E.: Sulla determinazione di una funzione analitica di pi? variabili complesse in un campo, assegnatane la traccia sulla frontiera. Ann. Mat. Pura e Appl. **55**, 191–202 (1961)

17. Merker, J., Porten, E.: Holomorphic extension of CR functions, envelopes of holomorphy, and removable singularities. Int. Math. Res. Surv. **1**, 1–287 (2006)

18. Merker, J., Porten, E.: A Morse-theoretical proof of the Hartogs extension theorem. J. Geom. Anal. **17**, 513–546 (2007)

19. Merker, J., Porten, E.: The Hartogs extension theorem on $(n-1)$-complete complex spaces. J. Reine Angew. Math. **637**, 239 (2009)

20. Ohsawa, T.: Hartogs type extension theorems on some domains in Kähler manifolds. Ann. Polon. Math. **106**, 243–254 (2012)

21. Øvrelid, N., Vassiliadou, S.: Hartogs extension theorems on Stein spaces. J. Geom. Anal. **20**(4), 817–836 (2010)

22. Porten, E.: The Hartogs phenomenon on weakly pseudoconcave hypersurfaces. Math. Ann. **354**(2), 659–683 (2012)

23. Range, R.M.: Extension phenomena in multidimensional complex analysis: correction of the historical record. Math. Intelligencer **24**(2), 4–12 (2002)

24. Severi, F.: Risoluzione generale del problema di Dirichlet per le funzioni biarmoniche. Rend. Reale Accad. Lincei **23**, 795–804 (1931)

25. Słodkowski, Z.: Analytic set-valued functions and spectra. Math. Ann. **256**, 363–386 (1981)

26. Wermer, J.: Maximum modulus algebras and singularity sets. Proc. R. Soc. Edinburgh Sect. A **86**(3–4), 327–331 (1980)