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Free involutions on $S^1 \times S^n$

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Abstract Topological free involutions on $S^1 \times S^n$ are classified up to conjugation. We prove that this is the same as classifying quotient manifolds up to homeomorphism. There are exactly four possible homotopy types of such quotients, and surgery theory is used to classify all manifolds within each homotopy type.

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1 Introduction and statements of the main results

Classification of free (topological or PL) involutions on S^n, $n \geq 5$, constituted a natural and important problem in topology of manifolds. Its successful solution (cf. [3,15,24]) was one of the first impressive applications of surgery theory and was achieved by (topological, PL) classification of quotient manifolds, i.e. manifolds homotopy equivalent to the standard RP^n. A natural generalization is the classification of free involutions on $S^1 \times S^n$, but this is considerably more challenging and difficult. The first indication of this is reflected in the fact that in this case there are (in each dimension) four distinct homotopy types of quotient manifolds. But there are also deeper reasons for these difficulties:

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A. Computation of surgery groups for the infinite dihedral group $\mathbb{Z}/2 * \mathbb{Z}/2$. This group is the fundamental group of quotient manifolds in the homotopy type of the connected sum $RP^n \# RP^n$.
B. Understanding the relation between conjugacy classes of involutions and homeomorphism types of quotient manifolds (note that the manifolds $S^1 \times S^n$ are not simply connected).

The problem addressed in A was solved in [5,6] and the classification of manifolds homotopy equivalent to $RP^n \# RP^n$ was achieved in [2], Theorem 2. (Cf. [11] for a discussion of the case $n = 4$).

The purpose of this paper is to give a complete classification of free topological involutions on $S^1 \times S^n$, $n \geq 3$. The simplest involutions have the form $\alpha \times \beta$, a product of involutions on S^1 and S^n, at least one of them free. Examples are: $(t, x) \mapsto (-t, x)$, $(t, x) \mapsto (-t, r(x))$, where r is reflection in a hyperplane, $(t, x) \mapsto (t, -x)$ and $(t, x) \mapsto (\bar{t}, -x)$ with $\bar{t}$ the complex conjugate of $t \in S^1$. We will refer to these as the four *standard involutions,* and it is not difficult to see that every product of linear involutions is conjugate to one of these. The quotients are $S^1 \times S^n$, $S^1 \tilde{\times} S^n$ (the non–trivial S^n–bundle over S^1), $S^1 \times RP^n$ and $RP^{n+1} \# RP^{n+1}$, respectively.

It is known that the four standard involutions on $S^1 \times S^2$ represent all involutions up to conjugacy, see [23, Theorem C]. This result can also be recovered by the methods of the present paper. For $S^1 \times S^1$, elementary considerations show that there are only two involutions up to conjugacy; in this case the four standard involutions reduce to just two—the first and third are conjugate, as are the second and fourth. Hence, we will concentrate on the case $n \geqslant 3$ in this paper.

Our main goal is to classify all involutions up to topological conjugacy. A weaker, but still interesting result, and one much easier to state, is the classification up to *homotopical* conjugacy: we say that two involutions τ_1 and τ_2 on a space X are homotopically conjugate if there exists a homotopy equivalence $f : X \to X$ such that $\tau_1 \circ f = f \circ \tau_2$.

Theorem 1.1 *Any free involution τ on $S^1 \times S^n$ is homotopically conjugate to one of the four standard involutions.*

This gives a preliminary classification, and it will be convenient to say that an involution is of *type* $S^1 \times S^n$, $S^1 \tilde{\times} S^n$, $S^1 \times RP^n$ or $RP^{n+1} \# RP^{n+1}$ if this is the homotopy type of its quotient. Note that the type is determined completely by the fundamental group of the quotient, plus orientability in the case $\pi_1 \approx \mathbb{Z}$. However, two involutions may be of different types even if they are homotopic as *maps*—e. g. $(t, x) \mapsto (-t, x)$ and $(t, x) \mapsto (t, -x)$ if n odd. But in some cases the homotopy class does determine the type: the fourth type is distinguished from the others by the induced action on $H_1(S^1 \times S^n; \mathbb{Z})$, and in each dimension one of the first two types can also be distinguished by this action.

The topological classification now proceeds case by case according to type. In fact, the classification reduces to the classification of quotient manifolds up to homeomorphism, by the following result:

Theorem 1.2 *Two free involutions on $S^1 \times S^n$ are topologically conjugate if and only if the two quotients are homeomorphic.*

We use surgery theory to determine all homeomorphism classes of manifolds homotopy equivalent to the four standard quotients. Note that this is valid also for $n = 3$, since all fundamental groups are "good" in the sense of [8].

The surgery theory is simplest for the the homotopy types $S^1 \times S^n$ and $S^1 \tilde{\times} S^n$ with infinite cyclic fundamental group, and the result is:

Theorem 1.3 *Every free involution on $S^1 \times S^n$ of type $S^1 \times S^n$ or $S^1 \tilde{\times} S^n$ is topologically conjugate to the standard involution of the same type.*

The case $RP^{n+1} \# RP^{n+1}$ is in many ways the most interesting one, but it was treated in detail in [2] and [11], so except for a few comments in Sect. 3.2 we refer to these papers for precise statements.

Our main calculations concern the case $S^1 \times RP^n$, and this takes up most of Sect. 3. Let $\mathbb{N}$ be the set of non-negative integers $\{0, 1, 2, \ldots\}$. The resulting classification for this type is:

Theorem 1.4 *There is a one–one correspondence between the set of conjugacy classes of free involutions on $S^1 \times S^n$ of type $S^1 \times RP^n$ and*

$$\begin{cases} (\mathbb{Z}/2)^{\left[\frac{n-1}{2}\right]+\left[\frac{n-2}{4}\right]} & \textit{when } n \not\equiv 3 \textit{ mod } 4 \\ \mathbb{N} \times (\mathbb{Z}/2)^{\frac{3n-9}{4}} & \textit{when } n \equiv 3 \textit{ mod } 4. \end{cases}$$

Our proofs actually allow slightly sharper statements:

Addendum *Except for involutions of type $RP^{n+1} \# RP^{n+1}$ with $n \equiv 2$ mod 4, the classification is the same whether we consider conjugation by arbitrary or just orientation preserving homeomorphisms.*

The particular form of the classification derives from the algebraic structure of surgery theory, due to Ranicki [17, Theorem 18.5].

Remark 1.5 Farrell–Hsiang's splitting theorem [7] implies that all quotient manifolds in Theorem 1.4 can be described as mapping tori of homeomorphisms of homotopy projective spaces. Using the algebraic structure mentioned above, they are generated by mapping tori of two simple types:

- The mapping tori of the identity maps of all fake projective spaces, coming from involutions of the type $\mathrm{id}_{S^1} \times \tau$, with τ a free involution on S^n. In Theorem 1.4 they correspond to the factors

$$\begin{cases} (\mathbb{Z}/2)^{\left[\frac{n-1}{2}\right]} & \text{when } n \not\equiv 3 \text{ mod } 4 \\ \mathbb{N} \times (\mathbb{Z}/2)^{\frac{n-3}{2}} & \text{when } n \equiv 3 \text{ mod } 4. \end{cases}$$

- The remaining factors in Theorem 1.4 correspond to mapping tori of homeomorphisms of the standard RP^n, where one needs all concordance classes of homeomorphisms of RP^n. (Orientation preserving if n is odd. See also Remark 1.6.)

Although this observation says something about the structure of the actual involutions, it does not go very far towards describing them as maps. This is a natural, but presumably very difficult, open problem.

Remark 1.6 One can apply the results obtained in the proof of Theorem 1.4 to give a short computation of concordance classes of homeomorphisms of RP^n, $n \geq 3$, cf. [14]. Namely, the group of concordance classes of homeomorphisms of RP^n, $n \geq 3$, i.e. $\pi_0(\widetilde{Top}(RP^n))$ is given by

$$\pi_0(\widetilde{Top}(RP^n)) \cong \begin{cases} (\mathbb{Z}/2)^{\left[\frac{n-2}{4}\right]} & \text{when } n \text{ is even} \\ (\mathbb{Z}/2)^{\left[\frac{n+1}{4}\right]} & \text{when } n \text{ is odd.} \end{cases}$$

Here is an outline of the organization of the paper. In Sect. 2 we show that the only homotopy types that occur as quotients of free involutions on $S^1 \times S^n$ are the quotients by the four standard involutions. This is one of the main technical parts of the paper. The methods belong to standard homotopy theory, but providing all the necessary details requires a substantial amount of work. Section 2 contains the setup and all the necessary calculations for n odd. The even case is similar, but the details are different since the non-orientability gives rise to non-trivial local coefficient systems. We choose to defer the discussion of this case to an appendix.

Section 3 contains the surgery classification, mainly devoted to the case $S^1 \times RP^n$. The main ingredients are as follows:

- The topological structure set is easily computed, using a splitting result which reduces to the cases RP^n and $I \times RP^n$.
- The group $\pi_0(\mathrm{Aut}(S^1 \times RP^n))$ of homotopy classes of homotopy equivalences of $S^1 \times RP^n$ is determined. To achieve this we calculate $\pi_1(\mathrm{Aut}(RP^n))$, thereby correcting an incorrect statement in the literature ([25, Erratum, n even]). The main tool is the Federer–Schultz spectral sequence.
- To pass from homotopy structures to homeomorphism classes of manifold, one needs to determine the action of the homotopy equivalence group on the structure set. This has, in general, been a very difficult problem, but using recent results of Ranicki [18], we completely determine the action of $\pi_0(\mathrm{Aut}(S^1 \times RP^n))$ on the topological structure set $\mathscr{S}(S^1 \times RP^n)$.

Section 4 completes the argument by addressing the relation between the classification of quotients and equivalence classes of involutions. For free actions on simply connected manifolds there is always a one–one correspondence between homeomorphism types of quotients and conjugacy classes of actions, but in the non-simply connected case this is not necessarily true. (For counterexamples in the two-dimensional case, see [1, Theorems 1.1 and 1.3].) The last ingredient of the proof is to show that this, never the less, holds in the situation studied here. Hence the classification of involutions up to conjugacy is the same as the classification of quotient manifolds up to homeomorphism. (Theorem 1.2.) A simpler version of the same argument applied to homotopy equivalences combined with the classification in Sect. 2 gives Theorem 1.1.

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2 Homotopy classification

If $\tau : S^1 \times S^n \to S^1 \times S^n$ is a free involution, we let $Q = S^1 \times S^n/\tau$ be the quotient manifold and $p : S^1 \times S^n \to Q$ the projection. Q is orientable if and only if τ preserves orientation.

The main result of this section is

Theorem 2.1 *Let $n \geqslant 2$ an let τ be a free involution on $S^1 \times S^n$. Then the quotient belongs to one of the four homotopy types $S^1 \times S^n$, $S^1 \tilde{\times} S^n$, $S^1 \times RP^n$ and $RP^{n+1} \# RP^{n+1}$ realized by the standard involutions.*

We start with the following easy observations:

Lemma 2.2 *The Euler characteristic $\chi(Q)$ is zero.*

Proof This is because $\chi(S^1 \times S^n) = 2\chi(Q)$ is zero. □

Lemma 2.3 *If $n \geqslant 2$, the fundamental group of Q is either $\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}/2$ or the infinite dihedral group $D_\infty \cong \mathbb{Z}/2 * \mathbb{Z}/2$.*

Proof By covering space theory $\pi_1(S^1 \times S^n) \cong \mathbb{Z}$ is a subgroup of index two in $\pi_1(Q)$, hence it is also normal. If $\pi_1(Q)$ is abelian, it is then either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/2$. If it is nonabelian, it has to be generated by a generator t of $\mathbb{Z}$ and an element x not in $\mathbb{Z}$, and we must have $xtx^{-1} = t^{-1}$. It remains to show that $x^2 = 1$.

Since x^2 maps to the identity in $\mathbb{Z}/2$, we have $x^2 = t^m$, for some integer m. But then

$$t^m = x \cdot x^2 \cdot x^{-1} = xt^m x^{-1} = (xtx^{-1})^m = t^{-m},$$

hence $m = 0$. □

In general, we cannot assume that τ is a product of involutions, but the induced homomorphism in homology is ($n \geqslant 2$). If τ is any self-map of $S^1 \times S^n$, the induced map $\tau_* : H_i(S^1 \times S^n) \to H_i(S^1 \times S^n)$, $i = 1, n, n+1$ is multiplication by an integer d_i. Then $d_{n+1} = d_1 d_n$ and the Lefschetz number of τ is

$$L(\tau) = 1 - d_1 + (-1)^n d_n + (-1)^{n+1} d_1 d_n = (1 - d_1)(1 + (-1)^n d_n).$$

If τ is a free involution, it follows that $d_1 = 1$ or $d_n = (-1)^{n+1}$. Now recall the transfer homomorphism $\mathrm{tr} : \mathrm{H}_*(\mathrm{Q}) \to \mathrm{H}_*(\mathrm{S}^1 \times \mathrm{S}^n)$. It has the property that $p_* \circ \mathrm{tr}$ is multiplication by 2 and $\mathrm{tr} \circ \mathrm{p}_* = 1 + \tau_*$. It follows that up to elements of order two, $H_i(Q)$ is a direct summand of $H_i(S^1 \times S^n)$, and nontrivial — hence isomorphic to $\mathbb{Z}$—if and only if $d_i = 1$.

Example 2.4 In the case $\pi_1(Q) = \mathbb{Z}/2 * \mathbb{Z}/2$ we must have $d_1 = -1$, and hence $d_n = (-1)^{n+1}$. Thus Q is orientable if and only if $1 = d_1 d_n = (-1)^n$, i. e. n is *even.*

To prove Theorem 2.1 we consider separately each of the three fundamental groups given by Lemma 2.3.

Case 1 $\pi_1(Q) \cong \mathbb{Z}$.

This is the simplest case. Let $\rho : Q \to S^1$ be a map inducing an isomorphism on π_1. Then ρ classifies the universal covering of Q, hence the homotopy fiber is S^n. Thus Q has the homotopy type of the total space of an S^n-fibration over S^1, and there are only two such homotopy types — $S^1 \times S^n$ and $S^1 \tilde{\times} S^n$.

Case 2 $\pi_1(Q) \cong \mathbb{Z} \oplus \mathbb{Z}/2$.

This time let $\rho : Q \to S^1$ be a map inducing a *surjection* on π_1. Now ρ has homotopy fiber equivalent to the quotient $\overline{Q}$ of a free involution on $\mathbb{R} \times S^n$.

Lemma 2.5 *Let τ be a free involution on a finite dimensional space X which has the homotopy type of S^n. Then the quotient space $Y = X/\tau$ is homotopy equivalent to RP^n.*

Remark 2.6 This is a generalization of the well–known fact that the quotient of a free involution on S^n has the homotopy type of RP^n. (See e. g. [16, Theorem IV], [24, 14E].) Thomas and Wall give a proof for n odd [22, Theorem 2.2 and Corollary 2.4]. Note that finite dimensionality is necessary: an example is $S^\infty \times S^n$ with the antipodal action on the first factor.

Proof of Lemma 2.5 There is a fibration (up to homotopy) $X \to Y \to RP^\infty$, and since $X \simeq S^n$, the usual argument gives a Gysin sequence

$$\cdots \to H_k(Y) \to H_k(RP^\infty) \xrightarrow{\mu_k} H_{k-n-1}(RP^\infty, H_n(X)) \to H_{k-1}(Y) \to \cdots .$$

The third term may have twisted coefficients, in which case the homology has a $\mathbb{Z}/2$ in every even dimension and a zero in every odd dimension. (See Appendix.) Y has finite dimension m, say, hence it has no homology in degrees above m. It follows that μ_k must be an isomorphism for k large. This can only happen if the coefficients are twisted for n even and untwisted for n odd. It is now easy to see that Y has the same homology as RP^n and that the map $Y \to RP^\infty$ induces the "same" map in homology as $RP^n \subset RP^\infty$. In fact, μ_k is cap product with a "Thom class" in $H^{n-1}(RP^\infty, H^n(X))$, hence must be an isomorphism whenever $k - n - 1 > 0$.

Obviously Y and RP^n also have isomorphic *homotopy* groups in all dimensions. By obstruction theory we see that there is a map $RP^n \xrightarrow{f} Y$, unique up to homotopy on RP^{n-1} and inducing an isomorphism on π_i, $i < n$. All we have to do is to show that we can choose the map on the top cell of RP^n such that $f_* : \pi_n(RP^n) \to \pi_n(Y) \cong \mathbb{Z}$ is an isomorphism.

With chosen generators for the two π_n's, f_* is multiplication by an integer, and we want to show that we can realize $f_* = 1$. The different choices of extensions of f from RP^{n-1} to RP^n are parametrized by $\pi_n(Y)$, and an element represented by $h : S^n \to Y$ changes f to the composition

$$RP^n \xrightarrow{\nabla} RP^n \vee S^n \xrightarrow{f \vee h} Y ,$$

where $\triangledown$ pinches the boundary of an embedded n–disk to a point. This has the effect of adding an *even* number to f_*, as is seen from the diagram:

$$\begin{array}{ccccc} S^n & \longrightarrow & S^n \cup_{S^0 \times *} S^o \times S^n & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ RP^n & \xrightarrow{\triangledown} & RP^n \vee S^n & \xrightarrow{f \vee h} & Y , \end{array}$$

where the vertical maps are double coverings and the upper horizontal maps are the unique base-point preserving liftings.

We now finish the proof of Lemma 2.5 by showing that

$f_* : \pi_n(RP^n) \to \pi_n(Y)$ *is always multiplication by an* odd *number.*

In fact, the map $Y \to RP^\infty$ factors (non–uniquely) through RP^n, since all obstructions to this lie in vanishing groups $(H^k(Y, \pi_{k-1}(S^n)))$. The composite $RP^n \to Y \to RP^n$ is an isomorphism on π_1, and any such self–map of RP^n induces multiplication by an odd number on π_n. □

It follows that Q must be of the homotopy type of an RP^n–fibration over S^1. If n is even, this fibration must be trivial, since $\pi_0(\mathrm{Aut}(RP^n)) = 0$, where $\mathrm{Aut}(M)$ denotes the space of homotopy self–equivalences of M. If n is odd, $\pi_0(\mathrm{Aut}(RP^n))$ has two elements, represented by the identity and reflection in an RP^{n-1}. Hence Q must be of the homotopy type of either $S^1 \times RP^n$ or a twisted product $S^1 \tilde{\times} RP^n$. But it is not difficult to see that the latter does not have a double covering homotopy equivalent to $S^1 \times S^n$. In fact, the double coverings are classified by the index two subgroups of $\pi_1(Q) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. There are three such subgroups — two isomorphic to $\mathbb{Z}$ and one isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2$. The first two are mapped to each other by an isomorphism of $\pi_1(Q)$ induced by a homotopy equivalence of Q. This lifts to a homeomorphism of the double coverings, which then both have to be $S^1 \tilde{\times} S^n$. The last subgroup corresponds to the product of the double covering of S^1 and the identity map of RP^n.

Case 3 $\pi_1(Q) \cong \mathbb{Z}/2 * \mathbb{Z}/2$.

Let $\rho : Q \to RP^\infty \vee RP^\infty$ be a π_1–isomorphism. The two copies of RP^∞ determine two generators of $\pi_1(Q)$ and we let $g' : S^1 \vee S^1 \to Q$ be the wedge of maps representing these two generators.

The groups $\pi_i(Q)$ are trivial for $1 < i < n$, $\mathbb{Z}$ for $i = n$ and $\mathbb{Z}/2$ for $i = n + 1$. Using the obvious cell structure on $RP^n \vee RP^n$ with $S^1 \vee S^1$ as 1-skeleton, it then follows by obstruction theory that g' extends to a map $g : RP^n \vee RP^n \to Q$ which is an isomorphism on π_i for $i < n$. Moreover, $g|RP^{n-1} \vee RP^{n-1}$ is unique up to homotopy and $\rho \circ g$ is homotopic to the natural inclusion.

We obtain $RP^{n+1} \# RP^{n+1}$ from $RP^n \vee RP^n$ by adding an $(n + 1)$-cell which is attached by the map $S^n \to S^n \vee S^n \to RP^n \vee RP^n$. Here the first arrow is the usual pinch map and the second is the wedge of the two canonical double covers. We want to show that g can be chosen such that it extends to a map $RP^{n+1} \vee RP^{n+1} \to Q$ which is a homotopy equivalence.

The group $\pi_n(Q) \oplus \pi_n(Q) \cong \mathbb{Z} \oplus \mathbb{Z}$ acts transitively on the set of homotopy classes of extensions of $g|RP^{n-1} \vee RP^{n-1}$ to $RP^n \vee RP^n$. To describe the effect of the action, let g_1, g_2 be the restrictions of g to the two RP^n's. Then $g_{i*} : \pi_n(RP^n) \to \pi_n(Q)$ is multiplication by an integer a_i. *Case 3* now follows immediately from the following Lemma:

Lemma 2.7 1. *The numbers a_1 and a_2 are always odd.*
2. *g can be extended to $RP^{n+1} \# RP^{n+1}$ if and only if $a_1 + a_2 = 0$, and the extension is a homotopy equivalence if and only if $a_1 = -a_2 = \pm 1$.*
3. *The action of an element $(k_1, k_2) \in \pi_n(Q) \oplus \pi_n(Q)$ replaces (a_1, a_2) by $(a_1 + 2k_1, a_2 + 2k_2)$.*

Proof of Lemma 2.7 The proof of (3) is exactly like a similar statement in the discussion of Case 2, applied to each summand. To prove (2), observe that the extension exists if and only if the composite map

$$S^n \to S^n \vee S^n \to RP^n \vee RP^n \xrightarrow{a_1 \vee a_2} Q$$

is trivial. But this composition represents $a_1 + a_2$ in $\pi_n(Q)$. The second part of (2) follows from the observation that both inclusions $RP^n \subset RP^{n+1} \# RP^{n+1}$ induce isomorphisms on π_n. Therefore $a_1 = -a_2 = \pm 1$ if and only if $g_* : \pi_i(RP^{n+1} \# RP^{n+1}) \to \pi_i(Q)$ is iso for $i \leq n$, and since both universal coverings are $\mathbb{R} \times S^n$, it must be an isomorphism for all i.

It remains to prove (1). We start with a cohomology calculation. ρ is trivially n–connected, hence iso on H_i, $i < n$. This determines the remaining homology of Q as follows:

If n is *even*, Q is orientable, and the rest is given by Poincaré duality: $H_{n+1}(Q) \cong \mathbb{Z}$ and $H_n(Q) = 0$. This means that we cannot detect anything using H_n.

If n is *odd*, however, Q is nonorientable; hence we know that $H_{n+1}(Q) = 0$ and $H_n(Q) \cong \mathbb{Z}/2 \oplus$ a free summand. But the rank of this summand must be 1 since the Euler characteristic is 0. Thus $H_n(Q) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. Moreover, the map $\rho_* : H_n(Q) \to H_n(RP^\infty \vee RP^\infty)$ is onto, since the composition $H_n(RP^n \vee RP^n) \to H_n(Q) \to H_n(RP^\infty \vee RP^\infty)$ is.

We also immediately get the ring structure on the mod 2 cohomology $H^*(Q, \mathbb{Z}/2)$. In fact, it follows from the calculations above that both the maps in the composition

$$H^i(RP^\infty \vee RP^\infty, \mathbb{Z}/2) \to H^i(Q, \mathbb{Z}/2) \to H^i(RP^n \vee RP^n, \mathbb{Z}/2)$$

are isomorphisms for all $i \leq n$. This fact and Poincaré duality (for the top dimension) means that the cohomology ring is given as

$$H^*(Q, \mathbb{Z}/2) \cong \mathbb{Z}/2[x, y]/(xy, x^{n+1} - y^{n+1}). \tag{2.1}$$

The elements x and y are images under ρ_* of the generators for the mod 2 cohomology rings of the two copies of RP^∞.

We now finish the proof of Lemma 2.7 (1), and hence also of Theorem 2.1. Assume first n odd. It turns out that we can get rid of some extraneous torsion by dualizing, and this doesn't change the map we are interested in, $g_{i*} : \pi_n(RP^n) \to \pi_n(Q)$. So, let $A^* = Hom(A, \mathbb{Z})$, for an abelian group A, and consider the diagram

$$\begin{array}{ccc}
\mathbb{Z} \cong \pi_n(Q)^* & \xrightarrow{a_i} & \pi_n(RP^n)^* \cong \mathbb{Z} \\
\uparrow {\scriptstyle h_Q{}^*} & & \uparrow {\scriptstyle h_P{}^*} \\
\mathbb{Z} \cong H_n(Q)^* & \xrightarrow{(g_{i*})^*} & H_n(RP^n)^* \cong \mathbb{Z} \\
\uparrow \cong & & \uparrow \cong \\
\mathbb{Z} \cong H^n(Q) & \xrightarrow{g_i{}^*} & H^n(RP^n) \cong \mathbb{Z} \\
\downarrow {\scriptstyle \gamma} & & \downarrow \\
(\mathbb{Z}/2)^2 \cong H^n(Q, \mathbb{Z}/2) & \xrightarrow{g_i{}^*} & H^n(RP^n, \mathbb{Z}/2) \cong \mathbb{Z}/2
\end{array} \tag{2.2}$$

where h_P and h_Q are Hurewicz maps and the bottom vertical maps are reductions of coefficients. We know that h_P is multiplication by 2, so (1) will follow if we can prove

(i) $h_Q{}^*$ is multiplication by an even number, and
(ii) $g_i{}^* \circ \gamma$ is surjective.

To prove (i), we go back to $h_Q : \pi_n(Q) \to H_n(Q)$ and compare with the Hurewicz map for the universal covering $\mathbb{R} \times S^n$:

$$\begin{array}{ccc}
\pi_n(Q) & \xleftarrow{\cong} & \pi_n(\mathbb{R} \times S^n) \\
\downarrow {\scriptstyle h_Q} & & \downarrow \cong \\
\mathbb{Z} \oplus \mathbb{Z}/2 \cong H_n(Q) & \xleftarrow{p_*} & H_n(\mathbb{R} \times S^n) \cong \mathbb{Z}
\end{array} .$$

The element $p_*(1)$ maps to 0 in $H_n(RP^\infty \vee RP^\infty)$ since $\rho \circ p$ factors through the universal covering of $RP^\infty \vee RP^\infty$, which is contractible. But $\rho_* : H_n(Q) \to H_n(RP^\infty \vee RP^\infty)$ is surjective, so this means that as an element in $\mathbb{Z} \oplus \mathbb{Z}/2$, $p_*(1)$ must have the form $(2k, l)$. (i) follows from this.

(ii) uses the cohomology calculation (1). The image of γ must be nontrivial, and there are just three nontrivial elements in $H^n(Q, \mathbb{Z}/2)$ — $x_1{}^n$, $x_2{}^n$ and $x_1{}^n + x_2{}^n$. Since n is odd, $\mathrm{Sq}^1(x_1{}^n) = \mathrm{Sq}^1(x_2{}^n) \neq 0$ and $\mathrm{Sq}^1(x_1{}^n + x_2{}^n) = 0$, Hence the image of γ must contain $x_1{}^n + x_2{}^n$. But $g_i{}^*(x_1{}^n + x_2{}^n)$ is the generator of $H^*(RP^n, \mathbb{Z}/2)$ for both indexes i.

If n is even, we have to use homology with local coefficients. $RP^\infty \vee RP^\infty$ has a local coefficient system which is twisted by the nontrivial character on both summands RP^∞, and the other spaces involved come with maps to $RP^\infty \vee RP^\infty$ and hence also with induced coefficient systems. With these coefficient systems the homology/cohomology calculations go just as before. For more details, see the Appendix, Sect. 5. □

3 Surgery calculations

To find all manifolds in each of the homotopy types given by Theorem 2.1 we use surgery theory. In fact, all the surgery obstruction groups involved are completely known, and all the terms in the surgery exact sequences are easily computable. Moreover, all fundamental groups are "small" in the sense of [8, p. 99], such that topological surgery also works in dimension four. Hence, with modifications which will be explicitly pointed out, all the results in this section will be valid for $n \geqslant 3$. Note that all the Whitehead groups involved are trivial, so we will write just L for the surgery groups $L^s = L^h$, and write $\mathscr{S}$ for the structure set $\mathscr{S}^s = \mathscr{S}^h$. We recall that the structure set $\mathscr{S}(M)$ of a closed topological manifold is defined to be the set of equivalence classes of "s-triangulations": simple homotopy equivalences $h : (X, \partial X) \to (M, \partial M)$ where X is a topological manifold, where we require that the restriction $h : \partial X \to \partial M$ to the boundary is a homeomorphism. We may also assume $h^{-1}(\partial M) = \partial X$.

Two such s-triangulations $h_1 : (X_1, \partial X_1) \to (M, \partial M)$, $h_2 : (X_2, \partial X_2) \to (M, \partial M)$ are equivalent if there is a homeomorphism $f : (X_1, \partial X_1) \to (X_2, \partial X_2)$ such that $(h_2 \circ f)|\partial X_1 = h_1|\partial X_1$ and $h_2 \circ f$ and h_1 are homotopic rel ∂X_1.

Most of this section will be a detailed study of the case $\pi_1(Q) \cong \mathbb{Z} \times \mathbb{Z}/2$. The case $\pi_1(Q) \cong \mathbb{Z}$ is easy, and for the case $\pi_1(Q) \cong D_\infty$ we will refer to [2] and [11].

3.1 $Q \simeq S^1 \times S^n$ or $S^1 \tilde{\times} S^n$

The relevant part of the surgery exact sequence ([24, Chap. 10]) is

$$[\Sigma(Q_+), G/Top] \xrightarrow{\theta'} L_{n+2}(\mathbb{Z}, \omega) \to \mathscr{S}(Q) \to [Q, G/Top] \xrightarrow{\theta} L_{n+1}(\mathbb{Z}, \omega),$$

where $\omega : \mathbb{Z} \to \{\pm 1\}$ is the orientation character. A case by case check ($n \equiv 0, 1, 2, 3 \bmod 4$) reveals that θ is a bijection and θ' is a surjection for all $n \geqslant 3$. Therefore $\mathscr{S}(Q)$ has only one element, and Q is unique up to homeomorphism.

3.2 $Q \simeq RP^{n+1} \# RP^{n+1}$

This case was considered in [11] and a complete answer appeared in [2, Theorem 2]. Here we only indicate some of the most interesting qualitative results.

For $n \equiv 0$ or $1 \bmod 4$ there are only finitely many homeomorphism types, all of which are obtained as connected sums of fake projective spaces. If $n \equiv 3 \bmod 4$ there is an infinite number of distinct such connected sums, and when $n \equiv 2$ or $3 \bmod 4$ there are also infinitely many mutually non–homeomorphic examples that can *not* be split as connected sums. These manifolds exist because of the appearence of large $UNil$–groups in these dimensions. For more precise statements and calculations, see [2]. (Note, however, that surprisingly little is known about the geometry of these manifolds, in particular for $n = 3$. See Remarks in [11, p. 251]).

3.3 $Q \simeq S^1 \times RP^n$

This will take up the rest of this rather long section, and the aim is to prove the following result, which after Sect. 4 will turn out to be just another version of Theorem 1.4:

Theorem 3.1 *There is a one–one correspondence between the set of homeomorphism classes of manifolds homotopy equivalent to $S^1 \times RP^n$ and*

$$\begin{cases} (\mathbb{Z}/2)^{\left[\frac{n-1}{2}\right]+\left[\frac{n-2}{4}\right]} & \textit{when } n \not\equiv 3 \textit{ mod } 4 \\ \mathbb{N} \times (\mathbb{Z}/2)^{\frac{3n-9}{4}} & \textit{when } n \equiv 3 \textit{ mod } 4. \end{cases}$$

We will make essential use of the fact that the topological surgery sequence has the structure of a natural sequence of abelian groups if we identify $[M^m/\partial M^m, G/Top]$ with $h_m(M, \underline{\mathscr{L}})$, where $\underline{\mathscr{L}}$ is the surgery spectrum. (Note that if M is non–orientable, this homology has twisted coefficients.) The geometric meaning of this algebraic structure is still quite mysterious, but [18] goes a long way towards explaining the functoriality — at least with respect to homotopy equivalences. The neutral element of $\mathscr{S}(M)$ is represented by the identity map on M.

For $S^1 \times RP^n$ the surgery exact sequence is as follows:

$$\begin{aligned} \cdots \to [\Sigma(S^1 \times RP^n_+), G/Top] &\xrightarrow{\theta'} L_{n+2}(\mathbb{Z} \oplus \mathbb{Z}/2, \omega) \\ \to \mathscr{S}(S^1 \times RP^n) \to [S^1 \times RP^n, G/Top] &\xrightarrow{\theta} L_{n+1}(\mathbb{Z} \oplus \mathbb{Z}/2, \omega) \end{aligned} \quad (3.1)$$

The orientation character ω is nontrivial (isomorphism on $\mathbb{Z}/2$) if n is even, and trivial if n is odd. The L–groups split ([20, Theorem 5.1]) as

$$L_*(\mathbb{Z} \oplus \mathbb{Z}/2, \omega) \cong L_*(\mathbb{Z}/2, \omega) \times L_{*-1}(\mathbb{Z}/2, \omega),$$

compatibly with splittings

$$[S^1 \times RP^n, G/Top] \cong [\Sigma RP^n_+, G/Top] \times [RP^n, G/Top]$$

and

$$[\Sigma(S^1 \times RP^n)_+, G/Top] \cong [\Sigma^2 RP^n_+, G/Top] \times [\Sigma RP^n_+, G/Top]$$

of the sets of normal invariants. In fact, these splittings extend to a splitting of the whole surgery sequence (3.1):

Lemma 3.2 *There is a splitting*

$$\mathscr{S}(S^1 \times RP^n) \approx \mathscr{S}(I \times RP^n) \times \mathscr{S}(RP^n),$$

and the surgery sequence (3.1) splits as a product of the surgery sequences for $I \times RP^n$ and RP^n.

Remark 3.3 When $n = 3$, $\mathscr{S}(RP^n)$ has to be interpreted as in [13] and [10]. Then the lemma and its proof are valid also in this dimension.

Proof of Lemma 3.2 The idea is to construct a homomorphism

$$\psi : \mathscr{S}(I \times RP^n) \times \mathscr{S}(RP^n) \to \mathscr{S}(S^1 \times RP^n),$$

compatible with the isomorphisms above. Then we have a map from the product of the exact surgery sequences for $I \times RP^n$ and RP to the sequence for $S^1 \times RP^n$, and the result follows by the five–lemma.

To construct ψ it suffices, using the abelian group structure, to construct the restrictions to each factor. On $\mathscr{S}(RP^n)$ it is product with the identity map on S^1. If $f : W \to I \times RP^n$ represents an element in $\mathscr{S}(I \times RP^n)$, define $\overline{W}$ by identifying $f^{-1}(0, x)$ with $f^{-1}(1, x)$ for every $x \in RP^n$. (Recall that f is a homeomorphism between $\partial W = f^{-1}(\partial(I \times RP^n))$ and $\partial(I \times RP^n)$.) Then $\psi([f])$ is represented by the obvious induced map $\overline{f} : \overline{W} \to \overline{I \times RP^n} = S^1 \times RP^n$.

That ψ is compatible with the other isomorphisms, follows since these can be defined using analogous constructions. □

Remark 3.4 A topological version of Farrell–Hsiang's splitting theorem [7, Theorem 1.3] provides a splitting map $\rho : \mathscr{S}(S^1 \times RP^n) \to \mathscr{S}(RP^n)$.

Observe that the surgery sequence for $I \times RP^n$ is part of the sequence for RP^n ("forget the last three terms"):

$$\begin{aligned} &\cdots [\Sigma^2(RP^n_+), G/Top\,] \xrightarrow{\theta''} L_{n+2}(\mathbb{Z}/2, \omega) \to \\ &\to \mathscr{S}(I \times RP^n) \to [\Sigma(RP^n_+), G/Top\,] \xrightarrow{\theta'} L_{n+1}(\mathbb{Z}/2, \omega) \to \\ &\to \mathscr{S}(RP^n) \to [RP^n, G/Top\,] \xrightarrow{\theta} L_n(\mathbb{Z}/2, \omega). \end{aligned} \tag{3.2}$$

The L–groups occurring here are as follows (for $n \equiv 0, 1, 2, 3 \bmod 4$) [24]:

$$\begin{aligned} L_n(\mathbb{Z}/2, \omega) &= \mathbb{Z}/2,\ 0,\ \mathbb{Z}/2,\ \mathbb{Z}/2 \\ L_{n+1}(\mathbb{Z}/2, \omega) &= 0,\ \mathbb{Z}/2,\ 0,\ \mathbb{Z}^2 \\ L_{n+2}(\mathbb{Z}/2, \omega) &= \mathbb{Z}/2,\ \mathbb{Z}/2,\ \mathbb{Z}/2,\ 0. \end{aligned}$$

The usual cohomology calculation gives

$$\begin{aligned} [RP^n, G/Top\,] &\cong (\mathbb{Z}/2)^{\left[\frac{n}{2}\right]} \\ [\Sigma(RP^n_+), G/Top\,] &\cong (\mathbb{Z}/2)^{\left[\frac{n+3}{4}\right]} \times K_n, \end{aligned}$$

where $K_n = \mathbb{Z}$ if $n \equiv 3 \bmod 4$ and $K_n = 0$ otherwise.

The surgery obstruction maps θ and θ' have been computed, e. g. in [15, Chap. IV]. López de Medrano treats mainly the PL case, but the results are essentially the same in the topological case, as explained in [15, IV.6]. Note, however, that each $\mathbb{Z}/4$ in his Theorem IV.3.4 is to be replaced by $(\mathbb{Z}/2)^2$ in the topological case.

The result is:

- θ is surjective, and
- $\operatorname{coker}\theta' \cong K_n$.

More precisely: The K_n–factor of $[\Sigma(RP^n_+), G/Top]$ (for $n \equiv 3 \bmod 4$) maps isomorphically onto one of the $\mathbb{Z}$-factors of $L_{n+1}(\mathbb{Z}/2, \omega)$, and the cokernel, which is naturally identified with a subgroup of $\mathscr{S}(RP^n)$, is again isomorphic to K_n. In fact, we claim that the K_n–subgroup splits off $\mathscr{S}(RP^n)$ as a direct summand.

A natural candidate for a splitting is the Browder–Livesay desuspension invariant [3, p. 75], [15, I.2.2]. This defines a splitting of *sets*, but since the group structure on $\mathscr{S}(RP^n)$ is still rather mysterious, it is not clear that it is a homomorphism. However, Siebenmann periodicity ([21, p. 277, Theorem C.4.], [9, Corollary 3.2]) gives a diagram

$$\begin{array}{ccc} K_n & \longrightarrow & \mathscr{S}(RP^n) \\ \cong\downarrow & & \downarrow \\ K_{n+4} & \longrightarrow & \mathscr{S}(RP^n \times I^4), \end{array}$$

hence it suffices to construct a splitting of the inclusion $K_{n+4} \to \mathscr{S}(RP^n \times I^4)$. But the group structure on $\mathscr{S}(RP^n \times I^4)$ is well known ("glue along $RP^n \times I^3 \times \{1\}$ and $RP^n \times I^3 \times \{0\}$"), and it is easy to see that the natural analogue of the Browder–Livesay invariant defined on $\mathscr{S}(RP^n \times I^4)$ is a homomorphism. There are two points here worthwhile noting. First, the Browder–Livesay invariant is essentially defined in [15] for manifolds with boundary, see Definition in I.2.1, p. 14, where only for simplicity it is assumed that the boundary is empty (last line on p. 14 of [15]). Second, the additivity of the Browder–Livesay invariant with respect to the "gluing along the boundary" group structure on $\mathscr{S}(RP^n \times I^4)$ boils down to the use of Mayer-Vietoris long exact sequence when computing homology groups of the resulting "sum" manifold in $\mathscr{S}(RP^n \times I^4)$.

Denote the resulting splitting $\mathscr{S}(RP^n) \to K_n$ by BL. Since $[RP^n, G/Top]$ is a product of $\mathbb{Z}/2$'s, this determines $\mathscr{S}(RP^n)$ completely as an abelian group. Note that if n is even, then $[n/2] - 1 = [(n-1)/2]$, and if $n \equiv 1 \bmod 4$, $[n/2] = (n-1)/2$. If $n \equiv 3 \bmod 4$, $[n/2] - 1 = (n-3)/2$. Thus the result can conveniently be stated as follows:

$$\mathscr{S}(RP^n) \cong \begin{cases} (\mathbb{Z}/2)^{\left[\frac{n-1}{2}\right]} & \text{if } n \not\equiv 3 \bmod 4 \\ \mathbb{Z} \times (\mathbb{Z}/2)^{\frac{n-3}{2}} & \text{if } n \equiv 3 \bmod 4. \end{cases} \tag{3.3}$$

Clearly, every element in $\mathscr{S}(RP^n)$ gives rise to an involution on $S^1 \times S^n$ of the form $\mathrm{id}_{S^1} \times \tau$, where τ is a free involution on S^n.

We can also determine the last surgery obstruction map, θ''; we claim that it is always surjective:

When n is even, $L_{n+2} \cong \mathbb{Z}/2$ is detected by the Arf invariant and can be easily realized by the product formula.

If $n \equiv 3 \mod 4$, $L_{n+2} = 0$, so there is nothing to prove. In the remaining case, $n \equiv 1 \mod 4$, the obvious maps $\mathbb{Z}/2 \cong L_2(1) \xrightarrow{\times \mathbb{Z}} L_3(\mathbb{Z}) \to L_3(\mathbb{Z}/2)$ are all isomorphisms ([24, Chap. 13A]). The non–trivial element can be realized on a neighborhood of $\{0\} \times RP^1 \subset D^2 \times RP^n$.

This also determines $\mathscr{S}(I \times RP^n)$, and we record that

$$\mathscr{S}(I \times RP^n) \cong (\mathbb{Z}/2)^{\left[\frac{n+2}{4}\right]}. \tag{3.4}$$

(The exponent is $\left[\frac{n+3}{4}\right]$ for $n \not\equiv 1 \mod 4$ and $\left[\frac{n+3}{4}\right] - 1$ for $n \equiv 1 \mod 4$, and one easily checks that both can also be written as $\left[\frac{n+2}{4}\right]$.)

By Lemma 3.2 we have now determined $\mathscr{S}(S^1 \times RP^n)$:

$$\mathscr{S}(S^1 \times RP^n) \cong \begin{cases} (\mathbb{Z}/2)^{\left[\frac{n-1}{2}\right]+\left[\frac{n+2}{4}\right]} & \text{if } n \not\equiv 3 \mod 4 \\ \mathbb{Z} \times (\mathbb{Z}/2)^{\frac{3n-5}{4}} & \text{if } n \equiv 3 \mod 4 \end{cases} \tag{3.5}$$

We remark that as a consequence of the s–cobordism theorem an element in $\mathscr{S}(I \times RP^n) \subset \mathscr{S}(S^1 \times RP^n)$ can be represented by the mapping torus of a (degree one) homeomorphism $h : RP^n \to RP^n$, together with a homotopy between h and the identity.

Our goal is to determine the set of *homeomorphism classes* of manifolds homotopy equivalent to $S^1 \times RP^n$. This is the quotient of the structure set $\mathscr{S}(S^1 \times RP^n)$ by the group $\pi_0(\mathrm{Aut}(S^1 \times RP^n))$ of homotopy classes of homotopy equivalences of $S^1 \times RP^n$, acting by post–composition. We now compute this group.

Proposition 3.5 *Let $n \geqslant 2$. Then*

$$\pi_0(\mathrm{Aut}(S^1 \times RP^n)) \cong \begin{cases} (\mathbb{Z}/2)^3 & \text{if } n \text{ is even} \\ (\mathbb{Z}/2)^3 \times \mathbb{Z}/4 & \text{if } n \equiv 1 \bmod 4 \\ (\mathbb{Z}/2)^5 & \text{if } n \equiv 3 \bmod 4. \end{cases}$$

In each case there is a $\mathbb{Z}/2$–summand generated by a tangential homotopy equivalence with non–trivial normal invariant. The other summands can be represented by homeomorphisms.

Proof We begin by observing that there is a decomposition

$$\pi_0(\mathrm{Aut}(S^1 \times RP^n)) \cong \mathbb{Z}/2 \times \pi_0(\mathrm{Aut}(RP^n)) \times \pi_1(\mathrm{Aut}(RP^n)).$$

The first two factors represent the product maps. Here we first have

$$\pi_0(\mathrm{Aut}(RP^n)) = \mathbb{Z}/2 \quad \text{for } n \text{ odd and } 0 \text{ for } n \text{ even.}$$

(The generator is given by the reflection

$$r_0([x_0, \dots x_n]) = [-x_0, x_1, \dots, x_n] = [x_0, -x_1, \dots, -x_n],$$

and if n is even, this can be deformed linearly to the identity.)

The group $\pi_1(\mathrm{Aut}(RP^n))$ can be computed using the methods in [19]. (The base-point in $\mathrm{Aut}(RP^n)$ is the identity map.) Let $\mathrm{Aut}_{\mathbb{Z}/2}(S^n)$ be the space of homotopy equivalences of S^n which are equivariant with respect to the antipodal action of $\mathbb{Z}/2$. Taking quotients defines a double covering projection $\mathrm{Aut}_{\mathbb{Z}/2}(S^n) \to \mathrm{Aut}(RP^n)$, which is trivial if n is even and non–trivial (over each component) if n is odd. To see this, observe that the preimage of the identity map of RP^n consists of the identity and the antipodal maps of S^n. These are not even homotopic for n even, but equivariantly (in fact, linearly) isotopic for n odd.

Thus, for n even there is an isomorphism

$$\pi_1(\mathrm{Aut}_{\mathbb{Z}/2}(S^n)) \cong \pi_1(\mathrm{Aut}(RP^n)),$$

whereas for n odd there is an exact sequence

$$0 \to \pi_1(\mathrm{Aut}_{\mathbb{Z}/2}(S^n)) \to \pi_1(\mathrm{Aut}(RP^n)) \to \mathbb{Z}/2 \to 0. \tag{3.6}$$

Let Γ_n be the space of sections of the fibration $\rho : S^n \times_{\mathbb{Z}/2} S^n \to RP^n$ (standard diagonal action), induced by projection on the first factor. Then there is a homeomorphism $\mathrm{Aut}_{\mathbb{Z}/2}(S^n) \approx \Gamma_n$ defined by taking graphs and quotients by the $\mathbb{Z}/2$–actions. The fibration ρ has simple fibers (S^n), so we can apply the spectral sequence in [19]. This is a homology type, second quadrant spectral sequence with $E^2_{p,q} = H^{-p}(RP^n, \pi_q(S^n))$, converging to $\pi_{q+p}(\Gamma_n)$.

The E^2–term has local coefficients coming from the monodromy in the fibration ρ. When $q = n$ this is the same as the orientation system of RP^n — hence $E^2_{p,n} \cong H_{n+p}(RP^n, \mathbb{Z})$ with *trivial* coefficients. In particular, $E^2_{-n+1,n} \cong \mathbb{Z}/2$ and $E^2_{-n+2,n} \cong 0$. When $q = n + 1$ and $n \geqslant 3$, the coefficients are automatically trivial, since $\pi_{n+1}(S^n) \cong \mathbb{Z}/2$. Therefore $E^2_{-n,n+1} \cong \mathbb{Z}/2$. If $n = 2$, the monodromy is induced by the antipodal map on S^2, but this is trivial on $\pi_3(S^2) \cong \mathbb{Z}$. One way to see this is to use that two times a generator of $\pi_3(S^2)$ is represented by the Whitehead product of the identity with itself. Hence the map induced on $\pi_3(S^2)$ by a map of degree d on S^2 is multiplication by d^2. It follows that also $E^2_{-2,3} = H^2(RP^2, \mathbb{Z}) \cong \mathbb{Z}/2$.

Thus, in total degree $p + q = 1$ we have only two non–trivial groups, $E^2_{-n,n+1}$ and $E^2_{-n+1,n}$, both isomorphic to $\mathbb{Z}/2$. There are no differentials involving these groups, hence also $E^\infty_{-n,n+1} \cong E^\infty_{1-n,n} \cong \mathbb{Z}/2$. The resulting exact sequence

$$0 \to H^n(RP^n, \pi_{n+1}(S^n)) \to \pi_1(\Gamma_n) \to H^{n-1}(RP^n, \pi_n(S^n)) \to 0$$

splits, since the generator of $\pi_1(SO_{n+1})$ defines an element in $\pi_1(\mathrm{Aut}_{\mathbb{Z}/2}(S^n))$ of order two which maps to the generator of $H^{n-1}(RP^n, \pi_n(S^n)) \cong \mathbb{Z}/2$. (Using the standard cell structure of RP^n, the image is represented by the cellular cocycle mapping the unique $(n-1)$–cell to a homotopy equivalence of S^n, by the construction in [19, p. 52].) It follows that this factor can be represented by a one–parameter family of homeomorphisms — hence the associated homotopy equivalence of $S^1 \times RP^n$ can also be represented by a homeomorphism.

The other factor, however, can not. It corresponds to (one–parameter families of) maps that restrict to the inclusion on RP^{n-1}, and the nontrivial element can be constructed as the composition

$$S^1 \times RP^n \to (S^1 \times RP^n) \vee S^{n+1} \xrightarrow{\text{pr}\vee\eta} RP^n ,$$

where η is the generator of $\pi_{n+1}(RP^n)$. It follows e. g. by the method of [12, pp. 31–32] that the associated homotopy equivalence h_η of $S^1 \times RP^n$ has non–trivial normal invariant; we only need to observe that h_η is a *tangential* homotopy equivalence. But this follows since $\eta^*\tau(RP^n) = \eta^*\tau(S^n)$ is stably trivial.

It remains to examine the exact sequence (3.6). Write $n = 2m - 1$, and think of RP^n as a quotient of the unit sphere in $\mathbb{C}^m$. Consider the family $g_t(z_1, \ldots, z_m) = (e^{\pi it} z_1, \ldots, e^{\pi it} z_m)$, $t \in [0, 1]$. This is a path of equivariant homeomorphisms of S^n, but the path is closed only in $\text{Aut}(RP^n)$. Hence it maps to the nontrivial element in $\mathbb{Z}/2$ to the right.

The image ρ of the generator of $\pi_1(SO_{n+1})$ can be represented by $(t, z) \mapsto e^{2\pi it} z$, $t \in [0, 1]$ in any one of the coordinates $z = z_i$. Hence $\rho^m = [g]^2$ in $\pi_1(\text{Aut}(RP^n))$. Since $\rho^2 = 1$, it follows that if m is odd, $[g]$ has order 4 and $\rho = [g]^2$, and if m is even, it has order 2. Since $\pi_1(\text{Aut}(RP^n))$ is abelian, the result follows. □

Remark 3.6 (i) Since homeomorphisms are tangential, it follows that *all* homotopy self–equivalences of $S^1 \times RP^n$ are tangential.

(ii) The obvious map $SO(n+1) \to \text{Aut}(RP^n)$ clearly factors through $PSO(n+1) = SO(n+1)/\text{center}$. Then the main calculation in the above proof can be formulated as

$$\pi_1(\text{Aut}(RP^n)) \cong \pi_1(PSO(n+1)) \times \mathbb{Z}/2 ,$$

— the last factor being represented by h_η.

To compute the action of $\pi_0(\text{Aut}(S^1 \times RP^n))$ on the structure set, we use the result in Theorem 2.3 of [18]. A homotopy self–equivalence h of a manifold M induces an automorphism $h_* : \mathscr{S}(M) \to \mathscr{S}(M)$ (by functoriality). Let $g : N \to M$ represent an element in $\mathscr{S}(M)$. Then Ranicki shows that the composition hg represents the element $[h] + h_*([g]) \in \mathscr{S}(M)$.

The homomorphism h_* can be computed from the induced commutative diagram of exact sequences (3.1), which we now know can be written in the following form:

$$\begin{array}{ccccccc} 0 \longrightarrow & K_n & \longrightarrow & \mathscr{S}(S^1 \times RP^n) & \xrightarrow{\eta} & h_{n+1}(S^1 \times RP^n, \underline{\mathscr{L}}) \\ & \downarrow h_* & & \downarrow h_* & & \downarrow h_* \\ 0 \longrightarrow & K_n & \longrightarrow & \mathscr{S}(S^1 \times RP^n) & \xrightarrow{\eta} & h_{n+1}(S^1 \times RP^n, \underline{\mathscr{L}}). \end{array}$$

For the next result it will be convenient to introduce the homomorphism $\mu : \pi_0(\mathrm{Aut}(S^1 \times RP^n)) \to \{\pm 1\}$ defined for n odd as follows:

When n is odd, $H_n(S^1 \times RP^n, \mathbb{Z}) \cong \mathbb{Z}$. Hence, if $h \in \mathrm{Aut}(S^1 \times RP^n)$, the homomorphism induced on H_n by h is multiplication by an integer in $\{\pm 1\}$, which we denote by $\mu(h)$. It is clear that μ is a homomorphism.

Let $r_0 : RP^n \to RP^n$, n-odd, be an orientation reversing reflection and put $r = I_{S^1} \times r_0$. Then $\mu(r) = -1$, and $-1 \mapsto r$ defines a splitting of μ. Note that μ maps all the other generators of $\pi_0(\mathrm{Aut}(S^1 \times RP^n))$ identified in Prop. 3.5 to 1.

Lemma 3.7 *The map $h_* : h_{n+1}(S^1 \times RP^n, \underline{\mathscr{L}}) \to h_{n+1}(S^1 \times RP^n, \underline{\mathscr{L}})$ is the identity for every homotopy equivalence h. Hence $h_* : \mathscr{S}(S^1 \times RP^n) \to \mathscr{S}(S^1 \times RP^n)$ is always the identity if $n \not\equiv 3 \bmod 4$.*

If $n \equiv 3 \bmod 4$, then $h_* : K_n \to K_n$ is multiplication by $\mu(h)$.

This determines $h_* : \mathscr{S}(S^1 \times RP^n) \to \mathscr{S}(S^1 \times RP^n)$ completely.

Proof It is easy to see that

$$h^* : [S^1 \times RP^n, G/Top] \to [S^1 \times RP^n, G/Top]$$

is the identity homomoprhism for every h. For the first part of the Lemma, it then suffices to observe that via the identification

$$h_{n+1}(S^1 \times RP^n, \underline{\mathscr{L}}) = [S^1 \times RP^n, G/Top]$$

we have $h_* = (h^*)^{-1}$. For a general homotopy equivalence $h : M \to N$, this is true if h_* preserves L–theory fundamental classes, i. e. $h_*([M]_L) = [N]_L$. (See [18, Lemma 2.5].) This is not always the case, but it follows from the characterization of fundamental classes in [17, Prop. 16.16] that it holds for *tangential* homotopy equivalences — hence for all $h \in \mathrm{Aut}(S^1 \times RP^n)$, by remark 3.6.

When $n \equiv 3 \bmod 4$, $K_n \cong \mathbb{Z}$. Hence, since $\mathrm{im}\,\eta$ is torsion, h_* is determined by the two components $h_* : K_n \to K_n$ and $h_* : h_{n+1}(S^1 \times RP^n, \underline{\mathscr{L}}) \to h_{n+1}(S^1 \times RP^n, \underline{\mathscr{L}})$. K_n comes from the surgery sequence of RP^n, and the homomorphism $K_n \to \mathscr{S}(S^1 \times RP^n)$ factors as

$$K_n \to \mathscr{S}(RP^n) \xrightarrow{S^1 \times (-)} \mathscr{S}(S^1 \times RP^n).$$

Both of these maps are split injections, and if $h : S^1 \times RP^n \to S^1 \times RP^n$ is a homotopy equivalence, $h_* : K_n \to K_n$ is can be described as the composition in the following diagram:

$$\begin{array}{ccccc} K_n & \longrightarrow & \mathscr{S}(RP^n) & \longrightarrow & \mathscr{S}(S^1 \times RP^n) \\ \downarrow h_* & & \downarrow & & \downarrow h_* \\ K_n & \xleftarrow{BL} & \mathscr{S}(RP^n) & \xleftarrow{\rho} & \mathscr{S}(S^1 \times RP^n). \end{array} \tag{3.7}$$

(The dashed arrow is defined by the diagram.) The splitting map ρ maps a class $[f]$ to $f^{-1}(t \times RP^n) \xrightarrow{f} RP^n$ for a suitable representative $f : N \to S^1 \times RP^n$ and $t \in S^1$, and BL is the Browder–Livesay invariant defined earlier. From the proof of Prop. 3.5 we see that every h can be chosen such that h preserves some $t \times RP^n$ and is either the identity or the reflection r_0 there, depending on the value of the invariant is $\mu(h)$.

Now recall that BL was defined as the composition of the Siebenmann periodicity map and a Browder–Livesay map on $\mathscr{S}(RP^n \times I^4)$. The Siebenmann periodicity map $\mathscr{S}(M) \to \mathscr{S}(M \times I^4)$ commutes with composition by homeomorphisms of M, so computing $r_{0*} : K_n \to K_n$ is the same as computing $(r_0 \times \mathrm{id}_{I^4})_* : K_{n+4} \to K_{n+4}$. But $r_0 \times \mathrm{id}_{I^4}$ reverses orientation on $RP^n \times I^4$, hence it multiplies the Browder–Livesay invariant by -1. (Cf. the discussion preceeding Theorem 4 in [2].) Thus $r_{0*} : K_n \to K_n$ is multiplication by -1. □

It follows that for $n \not\equiv 3 \bmod 4$, the action of the homotopy equivalence group on $\mathscr{S}(S^1 \times RP^n)$ reduces to translation by the structure represented by h_η. For $n \equiv 3 \bmod 4$, we also have the homeomorphism r acting by multiplication by -1 (only non–trivial on $K_n \cong \mathbb{Z}$). The set of homeomorphism classes of manifolds homotopy equivalent to $S^1 \times RP^n$ is in one–one correspondence with the elements of the quotient of $\mathscr{S}(S^1 \times RP^n)$ under this action. Note that the structure represented by h_η lies in the subgroup $\mathscr{S}(I \times RP^n)$. It follows that the elements in $\mathscr{S}(RP^n)$ represent unique homeomorphism classes, but g and $g + h_\eta$ represent the same, if $g \in \mathscr{S}(I \times RP^n)$. Hence the quotient under the action is isomorphic to a set which can conveniently be written

$$\begin{cases} (\mathbb{Z}/2)^{\left[\frac{n-1}{2}\right]+\left[\frac{n-2}{4}\right]} & \text{for } n \not\equiv 3 \bmod 4 \\ \mathbb{N} \times (\mathbb{Z}/2)^{\frac{3n-9}{4}} & \text{for } n \equiv 3 \bmod 4, \end{cases} \tag{3.8}$$

with $\mathbb{N}$ the set of numbers $\{0, 1, 2, \dots\}$. This ends the proof of Theorem 3.1.

Remark 3.8 It may be simpler to think of the exponents ℓ in (3.8) as follows:

$$\text{If n} = \begin{cases} 4m \\ 4m+1 \\ 4m+2 \\ 4m+3 \end{cases} \quad \text{then} \quad \ell = \begin{cases} 3m-2 \\ 3m-1 \\ 3m \\ 3m. \end{cases}$$

4 Classification up to conjugacy

In this section, we prove Theorem 1.2 and an analogous statement about homotopical conjugacy, which will complete the proof of Theorem 1.1. Note that for $n < 3$ these results are true by the low-dimensional classification theorems ([1, Theorem 1.3][23, Theorem C]). Hence we may assume $n \geqslant 3$ in the following proof, although only the last part requires this assumption.

We start with Theorem 1.2. One way is trivial: topologically conjugate involutions clearly have homeomorphic quotients.

To go the other way; let τ_1 and τ_2 be two free involutions, and assume that $f : Q_{\tau_1} \to Q_{\tau_2}$ is a homeomorphism. Let $p_i : S^1 \times S^n \to Q_{\tau_i},\ i = 1, 2$ be the two projections. Choose a basepoint $x_0 \in S^1 \times S^n$ and let $q_1 = p_1(x_0), q_2 = f(q_1)$ and x_1 one of the two points in $p_2^{-1}(q_2)$. Then there is a (unique) lifting $F : S^1 \times S^n \to S^1 \times S^n$ such that $F(x_0) = x_1$, if and only if

$$f_* p_{1*}(\pi_1(S^1 \times S^n, x_0)) \subseteq p_{2*}(\pi_1(S^1 \times S^n, x_1)). \tag{4.1}$$

Assume F is such a lifting. Then obviously $\tau_2 \circ F$ and $F \circ \tau_1$ are also liftings of f, both distinct from F. But since there are only two possible liftings of f, we must have $\tau_2 \circ F = F \circ \tau_1$. Hence τ_1 and τ_2 are conjugate. Thus we only need to prove that we always can arrange for (4.1) to hold.

In fact, when $\pi_1(Q_\tau) \cong \mathbb{Z}$ or D_∞, this is automatically true, since in these groups there is a *unique* infinite cyclic subgroup of index two.

In the remaining case, $\pi_1(Q_\tau) \cong \mathbb{Z} \oplus \mathbb{Z}/2$, there are two infinite cyclic subgroups of index 2, generated by (1,0) and (1,1). When n is *even*, they correspond to two distinct double covers — $S^1 \times S^n$ and $S^1 \tilde{\times} S^n$ (mapping torus of the antipodal map on S^n). Note that these are uniquely determined up to homeomorphism by 3.3. But any homeomorphism f will lift to a homeomorphism from $S^1 \times S^n$ to *some* double covering of Q_{τ_2}, hence necessarily to $S^1 \times S^n$.

When n is odd, (4.1) may not be a priori satisfied, but we claim that if not, there is a homeomorphism g of Q_{τ_2} such that it is satisfied by gf, i. e. such that g_* maps the two infinite cyclic index two subgroups of $\pi_1(Q_\tau)$ to each other.

Let $n = 2m - 1$ and think of RP^{2m-1} as given by complex homogeneous coordinates $[z_1, \ldots, z_m]$, and define $g_0 : S^1 \times RP^{2m-1} \to S^1 \times RP^{2m-1}$ by

$$g_0(t, [z_1, \ldots, z_m]) = (t, [\sqrt{t}\, z_1, \ldots, \sqrt{t}\, z_m]).$$

Then g_{0*} maps the two index 2 cyclic subgroups of $\pi_1(S^1 \times RP^n)$ onto each other. Now choose a homotopy equivalence $h : Q_{\tau_2} \to S^1 \times RP^n$ representing an element of the structure set $\mathscr{S}(S^1 \times RP^n)$. It follows from the calculations in Sect. 3.4 that post–composition with g_0 is the identity on this set. Thus there exists a homeomorphism g of Q_{τ_2} such that $g_0 \circ h \simeq h \circ g$. The composed homeomomorphism gf now satisfies (4.1). This completes the proof of Theorem 1.2.

Now observe that the same argument can be used to prove that two involutions are homotopically conjugate if and only if the quotients are homotopy equivalent. In fact, this case is simpler, since now the existence of a *homotopy equivalence* g is obvious.

(In all dimensions, not just $n \geqslant 3$.) Theorem 1.1 follows from this observation and Theorem 2.1.

Proof of the Addendum If τ_1 and τ_2 are conjugate, one may ask if there is an *orientation preserving* homeomorphism $F : S^1 \times S^n \to S^1 \times S^n$ such that $F\tau_1 = \tau_2 F$. The answer is clearly 'yes' if there is an orientation reversing homeomorphism F' such that $F'\tau_2 = \tau_2 F'$. This is trivially the case if τ_2 (hence also τ_1) is itself orientation reversing, i. e. if the quotient manifold is non–orientable. In the orientable case we can use the technique above to construct F' if we can find an orientation reversing self–homeomorphism of the quotient manifold which induces a map on $\pi_1(Q_{\tau_2})$ preserving the image of $\pi_1(S^1 \times S^n)$. Each standard quotient manifold has such self–homeomorphisms, and we can try to lift these to Q_{τ_2} using the method used to produce g from g_0 in the preceding argument. That is, we need an orientation reversing self homeomorphism g_o of the standard quotient Q such that post–composition by g_0 induces the identity map on $\mathscr{S}(Q)$ (and inducing a map on $\pi_1(Q)$ preserving the image of $\pi_1(S^1 \times S^n)$).

The homeomorphisms $g_0(t, x) = (\bar{t}, x)$ will do in the cases $Q = S^1 \times S^n$ (trivially) and $S^1 \times RP^n$ (by the calculations in Sect. 3). The case $RP^{n+1} \# RP^{n+1}$ (n even) is decided by the results of [2] (proof of Theorem 2). If $n \equiv 0 \mod 4$ we can use reflection in $RP^n \# RP^n$, but if $n \equiv 2 \mod 4$ there is no such homeomorphism.

It follows that except for involutions of type $RP^{4n+3} \# RP^{4n+3}$, the classification up to conjugation is the same as classification up to *orientable* conjugation. □

5 Appendix

5.1 Homology with local coefficients

In this appendix, we supply the local homology calculations necessary to complete the proof of Lemma 2.4 for n even.

On $RP^\infty \vee RP^\infty$ we let $\mathscr{L}$ be the local coefficient system of groups isomorphic to $\mathbb{Z}$ but nontrivial over each wedge summand RP^∞. When considering spaces *over* $RP^\infty \vee RP^\infty$ we also generically let $\mathscr{L}$ denote the induced coefficient system. Note that $\mathscr{L} \otimes \mathscr{L} = \mathbb{Z}$ — the constant coefficient system.

Example 5.1 We can consider RP^n as a space over $RP^\infty \vee RP^\infty$ in two natural ways, but $\mathscr{L}$ is the same in both cases. For n even this is the orientation system on RP^n. Hence twisted Poincaré duality gives e. g.:

$$H_*(RP^{2m}, \mathscr{L}) \cong H^{2m-*}(RP^{2m}, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } * = 2m \\ \mathbb{Z}/2 & \text{if } * \text{ is even, } 0 \leq * < 2m \\ 0 & \text{otherwise.} \end{cases}$$

Just as in ordinary homology, a k–connected map $X \to Y$ over $RP^\infty \vee RP^\infty$ will induce isomorphisms in homology and cohomology with coefficients in $\mathscr{L}$ in degrees less than k, and if X is the k–skeleton of Y, the map on H_k is onto. Hence it follows that

$$H_*(RP^\infty, \mathscr{L}) \cong \begin{cases} \mathbb{Z}/2 & \text{if } * \text{ is even } \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

and $H_{2m}(RP^n, \mathscr{L}) \to H_{2m}(RP^\infty, \mathscr{L})$ is surjective. These results can, of course, also be obtained using the standard $\mathbb{Z}/2$–equivariant cell structure on the inclusion $S^{2m} \subset S^\infty$.

The homology and cohomology of $RP^\infty \vee RP^\infty$ with coefficients in $\mathscr{L}$ can be computed by a similar trick. We first observe that the natural map $RP^{2m} \# RP^{2m} \to RP^\infty \vee RP^\infty$ is $(2m-1)$–connected and that $\mathscr{L}$ is the orientation system for $RP^{2m} \# RP^{2m}$. Then, for $k < 2m - 1$ we have

$$H_k(RP^\infty \vee RP^\infty, \mathscr{L}) \cong H_k(RP^{2m} \# RP^{2m}, \mathscr{L}) \cong H^{2m-k}(RP^{2m} \# RP^{2m}, \mathbb{Z}).$$

Hence

$$H_k(RP^\infty \vee RP^\infty, \mathscr{L}) \cong \begin{cases} \mathbb{Z}/2 & \text{for } k = 0 \\ \mathbb{Z} & \text{for } k = 1 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{for } k \text{ even } > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $H_{2m}(RP^{2m}, \mathscr{L}) \to H_{2m}(RP^\infty \vee RP^\infty, \mathscr{L})$ is surjective. Dually, the cohomology is

$$H^k(RP^\infty \vee RP^\infty, \mathscr{L}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2 & \text{for } k = 1 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{for } k \text{ odd } > 1 \\ 0 & \text{otherwise.} \end{cases}$$

In general, the relation between homology and cohomology is given by the following

Proposition 5.2 (Universal coefficient Theorem) *For each n there is a functorial, split exact sequence*

$$0 \to Ext(H_{n-1}(X, \mathscr{L}), \mathbb{Z}) \to H^n(X, \mathscr{L}) \to Hom(H_n(X, \mathscr{L}), \mathbb{Z}) \to 0.$$

Proof Let $\pi = \pi_1(X)$ and set $CL_* = C_*(\tilde{X}) \otimes_{\mathbb{Z}\pi} \mathscr{L}$, where $C_*(\tilde{X})$ is the singular complex of the universal covering of X. Then, by definition, $H_n(X, \mathscr{L}) = H_n(CL_*)$, and there is a universal coefficient sequence

$$0 \to Ext(H_{n-1}(CL_*), \mathbb{Z}) \to H^n(Hom(CL_*, \mathbb{Z})) \to Hom(H_n(CL_*), \mathbb{Z}) \to 0.$$

Here Hom means $Hom_{\mathbb{Z}}$. Since $H^n(X, \mathscr{L}) = H_{-n}(Hom_{\mathbb{Z}\pi}(C_*(\tilde{X}), \mathscr{L}))$, then to prove the proposition we only need to verify that $Hom_{\mathbb{Z}\pi}(C_*(\tilde{X}), \mathscr{L}) \cong Hom(CL_*, \mathbb{Z})$. As observed above, $\mathscr{L} \otimes \mathscr{L} \cong \mathbb{Z}$, hence there is a canonical isomorphism $\mathscr{L} \cong \mathscr{L}^*$. But then we have

$$Hom_{\mathbb{Z}}(C_*(\tilde{X}) \otimes_{\mathbb{Z}\pi} \mathscr{L}, \mathbb{Z}) \cong Hom_{\mathbb{Z}\pi}(C_*(\tilde{X}), \mathscr{L}^*) \cong Hom_{\mathbb{Z}\pi}(C_*(\tilde{X}), \mathscr{L}),$$

which is just what we want. □

The exact sequence $0 \to \mathscr{L} \xrightarrow{\cdot 2} \mathscr{L} \to \mathbb{Z}/2 \to 0$ gives rise to a Bockstein homomorphism $\beta : H^k(X, \mathbb{Z}/2) \to H^{k+1}(X, \mathscr{L})$. Reducing coefficients mod 2 again produces a cohomology operation $Sq^1_- : H^k(X, \mathbb{Z}/2) \to H^{k+1}(X, \mathbb{Z}/2)$. Note that in general $Sq^1_- \neq Sq^1$. For example,

$$Sq^1 : H^k(RP^\infty, \mathbb{Z}/2) \to H^{k+1}(RP^\infty, \mathbb{Z}/2)$$

is an isomorphism for k odd and trivial for k even, but

$$Sq^1_- : H^k(RP^\infty, \mathbb{Z}/2) \to H^{k+1}(RP^\infty, \mathbb{Z}/2)$$

is an isomorphism for k even and trivial for k odd. This follows from the Bockstein sequence of $0 \to \mathscr{L} \xrightarrow{\cdot 2} \mathscr{L} \to \mathbb{Z}/2 \to 0$ and the calculation of $H^*(RP^n, \mathscr{L})$ above.

We now have all the ingredients necessary to complete the proof of Lemma 2.4 (i) if n is even. We want to use a version of diagram (2) in Sect. 2 were (co)homology now is taken with coefficients in $\mathscr{L}$ instead of $\mathbb{Z}$, so we need to compute $H_n(Q, \mathscr{L})$ and $H^n(Q, \mathscr{L})$ and the relevant homomorphisms.

First we note that Q and $RP^\infty \vee RP^\infty$ must have isomorphic homology and cohomology in low dimensions (less than $2m$). Since Q now is orientable, Poincaré duality gives

$$H_n(Q, \mathscr{L}) \cong H^1(Q, \mathscr{L}) \cong H^1(RP^\infty \vee RP^\infty, \mathscr{L}) \cong \mathbb{Z} \oplus \mathbb{Z}/2.$$

Hence, by the universal coefficient theorem:

$$H^n(Q, \mathscr{L}) \cong H_n(Q, \mathscr{L})^* \cong \mathbb{Z}.$$

The Hurewicz homomorphism $h_Q : \pi_n(Q) \to H_n(Q, \mathscr{L})$ is defined by $h_Q([f]) = f_*([S^n])$. One has to be a little careful in order to make this definition functorial, since it involves lifting f to the universal covering of Q, but different choices give the same h_Q up to sign, so they don't affect the argument.

To prove that the class $x_1{}^n + x_2{}^n \in H^n(Q, \mathbb{Z}/2)$ is in the image of γ, we use the operation Sq^1_- instead of Sq^1. Then the proof goes exactly as before.

Remark 5.3 One little piece of warning: $H^*(X, \mathscr{L})$ does not have a product, hence there is no "Cartan formula" for Sq^1_-. But $H^*(X, \mathscr{L})$ is a module over $H^*(X, \mathbb{Z})$. So is $H^*(X, \mathbb{Z}/2)$, and the Bockstein sequence is a sequence of $H^*(X, \mathbb{Z})$–module homomorphisms. Therefore Sq^1_- is also a $H^*(X, \mathbb{Z})$–module homomorphism. Note, however, that it can not be a homomorphism of $H^*(X, \mathbb{Z}/2)$–modules, as the calculation for RP^∞ shows.

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