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Erratum

The Bando-Calabi-Futaki character as an obstruction to semistability

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Abstract. The purpose of this note is to correct some errors in (b) and (c) of Main Theorem in [MN], and to make an additional observation. In particular, we shall give a first nonvanishing example of the obstruction $Obstr_{\mathrm{asymp}}(M, L)$ introduced in [M2].

1. Corrections

In this note, we use the same notation as in [MN]. Fix once for all a compact connected n-dimensional complex manifold M with an integral Kähler class η, a linear algebraic group G defined over $\mathbb{C}$ acting biregularly on M, and an ample G-linearized holomorphic line bundle L over M such that $c_1(L) = \eta$. For an ample G-linearized holomorphic line bundle N over M and a positive integer $m \in \mathbb{Z}$ such that N^m is very ample, we put

$$V_{N^m} := H^0(M, N^m)^*, \quad W_{N^m} := \left\{S^d(V_{N^m})\right\}^{\otimes n+1},$$

where $d = d(N^m)$ is the degree of $M \subset \mathbb{P}(V_{N^m})$. For the dual $V_{N^m}^*$ of the complex vector space V_{N^m}, we naturally have an algebraic group homomorphism

$$\nu_{N^m} : G \to \mathrm{GL}(V_{N^m}^*).$$

Define a positive integer $\beta = \beta(N^m)$ by setting $\beta := \dim_{\mathbb{C}} V_{N^m}$. Then ν_{N^m} induces an algebraic character $\det \nu_{N^m} : G \to \mathrm{GL}(\wedge^\beta V_{N^m}^*) \cong \mathbb{C}^*$, and let $(\det \nu_{N^m})_* : \mathfrak{g} \to \mathbb{C}$ be the associated Lie algebra character. For a suitable finite

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unramified cover $\iota : \tilde{G} \to G$ of G, the composite $(\det \nu_{N^m}) \circ \iota$ is expressible as a multiple $(\chi_{N^m})^\beta$ for some algebraic character $\chi_{N^m} : \tilde{G} \to \mathbb{C}^*$ of $\tilde{G}$. In particular,

$$(\det \nu_{N^m})_* = \beta \cdot (\chi_{N^m})_*, \tag{1}$$

where $(\chi_{N^m})_* : \mathfrak{g} \to \mathbb{C}$ denotes the Lie algebra character associated with χ_{N^m}. We now define an algebraic group homomorphism $\tilde{\nu}_{N^m} : \tilde{G} \to \mathrm{SL}(V^*_{N^m})$ by

$$\tilde{\nu}_{N^m}(\tilde{g}) := \chi_{N^m}(\tilde{g})^{-1}(\nu_{N^m} \circ \iota)(\tilde{g}), \qquad \text{for all } \tilde{g} \in \tilde{G}. \tag{2}$$

Let $\mu_{N^m} : G \to \mathrm{GL}(W_{N^m})$ and $\tilde{\mu}_{N^m} : \tilde{G} \to \mathrm{SL}(W_{N^m})$ be the algebraic group homomorphisms induced naturally by ν_{N^m} and $\tilde{\nu}_{N^m}$, respectively. Then by (2) and the definition of W_{N^m}, we obtain

$$\tilde{\mu}_{N^m}(\tilde{g}) := \chi_{N^m}(\tilde{g})^{(n+1)d}(\mu_{N^m} \circ \iota)(\tilde{g}), \qquad \text{for all } \tilde{g} \in \tilde{G}. \tag{3}$$

In [MN], the factors $\chi_{N^m}(\tilde{g})^{-1}$ and $\chi_{N^m}(\tilde{g})^{-(n+1)d}$ in the expression of $\tilde{\nu}_{N^m}(\tilde{g})$ and $\tilde{\mu}_{N^m}(\tilde{g})$, respectively, were disregarded by mistake. Let us first consider the case $N = K_M^{-1}$. Then in view of (1) above, (c) of Main Theorem in [MN] should be replaced by

Theorem A. *Assume that $\eta = c_1(M) > 0$ with $L = K_M^{-1}$. Let $G = \mathrm{Aut}^0(M)$, where G acts naturally on M. If M is Chow-semistable with respect to K_M^{-m} for some $0 < m \in \mathbb{Z}$, then F belongs to $\mathbb{Q}\,(\det \nu_{K_M^{-m}})_*$.*

Similarly, in view of Remark 3.6 in [MN], we next consider the case $N = K_M \otimes L^k$ with $m = 1$. Then (b) of Main Theorem in [MN] should be replaced by the following:

Theorem B. *Assume, for some positive integer λ, the following conditions are satisfied for all integers k with $\lambda \le k \le \lambda + 2n + 1$:*

(a) $K_M \otimes L^k$ is very ample,

(b) M is Chow-semistable with respect to $K_M \otimes L^k$.

Then F belongs to $\Sigma_{k=\lambda}^{\lambda+2n+1}\,\mathbb{Q}\,(\det \nu_{K_M \otimes L^k})_*$.

In Theorem A, recall that $F = -2\pi(n+1)^{-1}\mathcal{C}\{c_1^{n+1}, K_M^{-1}\} = \alpha_m(\det \nu_{K_M^{-m}})_*$ for some $\alpha_m \in \mathbb{Q}$, where $\alpha_m \neq 0$ as we see later. Then by (a) of Main Theorem in [MN], we obtain

Corollary C. *In* Theorem A *above, we assume in addition that (M, K_M^{-1}) is semistable in the sense of Tian. Then $(\det \nu_{K_M^{-m}})_* = 0$, i.e., $G = \mathrm{Aut}^0(M)$ acts on $H^0(M, K_M^{-m})$ as a subgroup of $\mathrm{SL}(H^0(M, K_M^{-m}))$.*

It is very plausible that the existence of Kähler-Einstein metrics for Fano manifolds implies a very strong stability condition for polarized algebraic manifolds (M, K_M^{-1}). Hence, in view of Corollary C above, we pose the following:

Conjecture D. *If a Fano manifold M admits a Kähler-Einstein metric, then $\mathrm{Aut}^0(M)$ acts naturally on $H^0(M, K_M^{-m})$ as a subgroup of $\mathrm{SL}(H^0(M, K_M^{-m}))$ for all $m \gg 1$.*

2. An additional observation

In this section, we make the following additional observation. Let m be a positive integer such that L^m is very ample and let $[\hat{M}] \in \mathbb{P}(W_{L^m})$ be the Chow point of the irreducible reduced effective algebraic cycle M on $\mathbb{P}(V_{L^m})$, where $[\hat{M}]$ is the natural image of $0 \neq \hat{M} \in W_{L^m}$ in $\mathbb{P}(W_{L^m})$. Let G be the maximal connected linear algebraic subgroup of $\mathrm{Aut}^0(M)$, so that $\mathrm{Aut}^0(M)/G$ is an Abelian variety. Then G is known to be an algebraic subgroup of $\mathrm{PGL}(V^*_{L^m})$ by a natural inclusion:

$$G \hookrightarrow \mathrm{PGL}(V^*_{L^m}). \tag{4}$$

On the other hand, the G-linearization of L allows us to regard G as an algebraic subgroup of $\mathrm{GL}(V^*_{L^m})$, and the corresponding inclusion

$$j : G \hookrightarrow \mathrm{GL}(V^*_{L^m})$$

is written as ν_{L^m} by the notation in Section 1. Next, let $\tilde{G}$ be the identity component of the isotropy subgroup of $\mathrm{SL}(V^*_{L^m})$ at $[\hat{M}]$ for the natural $\mathrm{SL}(V^*_{L^m})$-action on $\mathbb{P}(W_{L^m})$. Let $\varphi : \mathrm{SL}(V^*_{L^m}) \to \mathrm{PGL}(V^*_{L^m})$ be the natural isogeny. Then by the definition of $\tilde{G}$, we see that $\tilde{G}$ and the identity component of $\varphi^{-1}(G)$ coincide for (4) above. Hence the natural inclusion

$$\tilde{j} : \tilde{G} \hookrightarrow \mathrm{SL}(V^*_{L^m})$$

is expressible as $\tilde{\nu}_{L^m}$ by the notation in Section 1, because the representations $j \circ \varphi$ and $\tilde{j}$ of G differ only by a character of $\tilde{G}$.

Assume now that M is Chow-semistable with respect to L^m. Let $(\tilde{\mu}_{L^m |\mathbb{C}\cdot\hat{M}})_* : \mathfrak{g} \to \mathbb{C}$ denote the Lie algebra character associated with the isotropy representation

$$\tilde{G} \times \mathbb{C}\cdot\hat{M} \to \mathbb{C}\cdot\hat{M}, \qquad (\tilde{g}, \ell) \mapsto \{\tilde{\mu}_{L^m}(\tilde{g})\}(\ell)$$

of $\tilde{G}$ at $[\hat{M}]$. Similarly, we have the Lie algebra character $(\mu_{L^m |\mathbb{C}\cdot\hat{M}})_* : \mathfrak{g} \to \mathbb{C}$. By virtue of the G-linearization of L, we see from (3.5) in [MN] that the equalities $(\mu_{L^m |\mathbb{C}\cdot\hat{M}})_* = 2\,\mathcal{C}\{c_1^{n+1}; L^m\} = 2\,m^{n+1}\mathcal{C}\{c_1^{n+1}; L\}$ hold, while the Chow-semistability implies that the isotropy representation is trivial, i.e., $(\tilde{\mu}_{L^m |\mathbb{C}\cdot\hat{M}})_* = 0$. Now by (2) and (3), we have $(\tilde{\mu}_{L^m})_* = (\mu_{L^m})_* + \beta(L^m)^{-1}(n+1)d(L^m)(\det \nu_{L^m})_*$. Hence, in view of the identity $d(L^m) = m^n c_1(L)^n[M]$, we see that

$$\frac{(\det \nu_{L^m})_*}{m\,\beta(L^m)} = \frac{\{(\tilde{\mu}_{L^m |\mathbb{C}\cdot\hat{M}})_* - (\mu_{L^m |\mathbb{C}\cdot\hat{M}})_*\}}{m\,(n+1)\,d(L^m)} = \frac{-2\,\mathcal{C}\{c_1^{n+1}, L\}}{(n+1)\,c_1(L)^n[M]}.$$

Here the right-hand side is a Lie algebra character of $\mathfrak{g}$ independent of the choice of m, and will be denoted by $\mathcal{F}_0$. Thus, we obtain

Theorem E. *If M is Chow-semistable with respect to L^m, then $\{m\,\beta(L^m)\}^{-1}$ $(\det \nu_{L^m})_*$ coincides with $\mathcal{F}_0$.*

By this theorem applied to the case $L = K_M^{-1}$, we immediately obtain Theorem A and the nonvanishing of α_m above.

3. An example of (M, L) such that $Obstr_{\text{asymp}}(M, L) \neq 0$

In this section, we assume for simplicity that L is very ample. Consider the infinitesimal action $(\nu_L)_* : \mathfrak{g} \to \mathrm{End}(V_L^*)$ of $\mathfrak{g}$ on V_L^* induced by the G-linearization of L. Put

$$\rho_m := (\nu_L)_* - m^{-1}(\chi_{L^m})_* \,\mathrm{id}_{V_L^*} = (\nu_L)_* - \{m\,\beta(L^m)\}^{-1}(\det \nu_{L^m})_* \,\mathrm{id}_{V_L^*}\,.$$

As seen from Theorem E above, vanishing of the obstruction $Obstr_{\text{asymp}}(M, L)$ to asymptotic Chow-semistability for (M, L) introduced in [M2] is expressible as the stability

$$\rho_{m_0} = \rho_{m_0+1} = \rho_{m_0+2} = \dots$$

of the actions ρ_m, $m \gg 1$, or equivalently, as the coincidence of $\{\,m\,\beta(L^m)\,\}^{-1}$ $(\det \nu_{L^m})_*$ with $\mathcal{F}_0$ for all sufficiently large integers m.

We shall now give an example of (M, L) such that $Obstr_{\text{asymp}}(M, L) \neq 0$. Let M be $\mathbb{P}^2(\mathbb{C})$ blown up at a point, and let $L = K_M^{-1} > 0$. Then there exists an extremal Kähler metric ω on M in the anticanonical class, and M as a toric variety admits an almost homogeneous action of $T := \mathbb{C}^* \times \mathbb{C}^*$. Let $\mathfrak{t}_{\mathbb{R}}$ be the Lie algebra of the maximal compact subgroup $T_{\mathbb{R}} := S^1 \times S^1$ of T. Let m be an arbitrary positive integer. Then for a suitable choice of a basis for $\mathfrak{t}_{\mathbb{R}}^*$, the image of the moment map $\mu : M \to \mathfrak{t}_{\mathbb{R}}^*$ for the Kähler metric $m\omega$ is a compact convex polygon P with integral vertices $(-m, m)$, $(0, m)$, $(2m, -m)$, $(-m, -m)$ (see for instance [M1]). Let b_m be the barycenter of the polygon P and let b'_m be the barycenter of the set of all integral points in P. Then by setting $q(m) := 2(m+1)(2m+1)^{-1}$, we see that

$$b_m = (m/12, -m/6) \quad \text{and} \quad b'_m = q(m)\,(m/12, -m/6).$$

Note that $\{\,m\,\beta(L^m)\,\}^{-1}(\det \nu_{L^m})_* = \mathcal{F}_0$ if and only if $b'_m = b_m$. By $q(m) > 1$, we obtain $b_m \neq b'_m$ for all positive integers m. Thus, we obtain $Obstr_{\text{asymp}} \neq 0$ in this case.

Added in proof. After the completion of this note, we hear that the vanishing of the obstruction $Obstr_{\text{asymp}}(M, L)$ is characterized by Futaki [F1] as the vanishing of a series of integral invariants (including the Bando-Calabi-Futaki character F) by virtue of the equivariant Riemann-Roch theorem. □

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