Arch. Rational Mech. Anal. (2024) 248:42
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-024-01992-y

A Priori Estimates for Solutions to Landau Equation Under Prodi–Serrin Like Criteria

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Communicated by P.-L. LIONS.

Abstract

In this paper, we introduce Prodi–Serrin like criteria which enable us to provide a priori estimates for the solutions to the spatially homogeneous Landau equation for all classical soft potentials and dimensions $d \geqq 3$. The physical case of Coulomb interaction in dimension $d = 3$ is included in our analysis; this generalizes the work of Silvestre (J Differ Equ 262:3034–3055, 2017). Our approach is quantitative and does not require a preliminary knowledge of elaborate tools for nonlinear parabolic equations.

1. Introduction

1.1. The Spatially Homogeneous Landau Equation

In this work, we are interested in the regularity properties of the solutions to the (spatially homogeneous) Landau equation

$$\partial_t f(t, v) = \mathcal{Q}(f(t, \cdot))(v), \qquad t \geqq 0, \quad v \in \mathbb{R}^d, \tag{1.1a}$$

supplemented with the initial condition

$$f(t = 0, v) = f_{\text{in}}(v), \qquad v \in \mathbb{R}^d, \tag{1.1b}$$

where $\mathcal{Q}$ denotes the (quadratic) Landau collision operator

$$\mathcal{Q}(f)(v) := \nabla_v \cdot \int_{\mathbb{R}^d} |v - v_*|^{\gamma+2}\, \Pi(v - v_*) \Big\{ f_* \nabla f - f \nabla f_* \Big\} \, \mathrm{d}v_* \,, \tag{1.2}$$

with the usual shorthands $f := f(v)$, $f_* := f(v_*)$, and where $\Pi(z)$ is the orthogonal projector onto $z^\perp$:

$$\Pi(z) := \mathrm{Id} - \frac{z \otimes z}{|z|^2}.$$

Published online: 13 May 2024

In the present contribution we obtain *new results* for the Landau equation in the case of so-called very soft potentials

$$-d \leqq \gamma < -2,$$

including the physically relevant Coulomb case $d = 3$, $\gamma = -3$, in which the Landau operator models collisions between charged particle; see Lifschitz-Pitaevskii [43]. We nevertheless point out that the results presented here still hold when $-2 \leqq \gamma < 0$ (so-called moderately soft potentials), and this enables us to get *new proofs of already known results* (see for example Wu [59]; Desvillettes [18] and Alonso et al. [5]). The case of Maxwell molecules ($\gamma = 0$) and hard potentials ($\gamma \in (0, 1]$) is quite different (see for example Desvillettes, Villani [22]), and is not considered in this work.

1.2. Notations

For $k \in \mathbb{R}$ and $p \geqq 1$, we define the Lebesgue space $L^p_k = L^p_k(\mathbb{R}^d)$ through the norm

$$\|f\|_{L^p_k} := \left(\int_{\mathbb{R}^d} \left|f(v)\right|^p \langle v\rangle^k \, \mathrm{d}v\right)^{\frac{1}{p}}, \qquad L^p_k(\mathbb{R}^d) := \left\{f \, : \mathbb{R}^d \to \mathbb{R};\ \|f\|_{L^p_k} < \infty\right\},$$

where $\langle v\rangle := \sqrt{1+|v|^2}$, $v \in \mathbb{R}^d$. For $k = 0$, we simply denote $\|\cdot\|_p$ as the L^p-norm. We also denote the homogeneous Sobolev space $\dot{\mathbb{H}}^m$ through the norm

$$\|f\|^2_{\dot{\mathbb{H}}^m} = \int_{\mathbb{R}^d} |\xi|^{2m} \, |\mathcal{F}[f](\xi)|^2 \, \mathrm{d}\xi, \qquad m \in \mathbb{N}$$

and the weighted homogenous Sobolev space $\dot{\mathbb{H}}^m_k$ through the norm

$$\|f\|^2_{\dot{\mathbb{H}}^m_k} = \int_{\mathbb{R}^d} |\xi|^{2m} \left|\mathcal{F}[\langle\cdot\rangle^{\frac{k}{2}} f](\xi)\right|^2 \mathrm{d}\xi, \qquad m \in \mathbb{N}, \quad k \in \mathbb{R},$$

where $\mathcal{F}[f]$ denotes the Fourier transform of f. Given $k \in \mathbb{R}$ and $f \in L^1_k(\mathbb{R}^d)$, we also define the statistical moments as

$$\boldsymbol{m}_k(f) = \int_{\mathbb{R}^d} f(v)\langle v\rangle^k \mathrm{d}v.$$

In relation to the coefficients of the Landau equation, we introduce, for $\gamma \in [-d, 0)$,

$$\begin{cases} a(z) := \left(a_{i,j}(z)\right)_{i,j} \quad \text{with} \quad a_{i,j}(z) = |z|^{\gamma+2} \left(\delta_{i,j} - \frac{z_i z_j}{|z|^2}\right), \\ b_i(z) := \sum_k \partial_k a_{i,k}(z) = -(d-1)\, z_i \, |z|^{\gamma}, \\ c(z) := -\sum_{k,l} \partial^2_{kl} a_{k,l}(z). \end{cases}$$

Therefore,

$$c(z) = \begin{cases} (d-1)\,(\gamma+d)\,|z|^{\gamma}, & \text{for } \gamma \in (-d, 0), \\ (d-1)(d-2)|\mathbb{S}^{d-1}|\delta_0(z), & \text{for } \gamma = -d. \end{cases}$$

For any $f \in L^1_{2+\gamma}(\mathbb{R}^d)$, we define then the matrix-valued mapping $\mathcal{A}[f]$ given by

$$\mathcal{A}[f] := \big(\mathcal{A}_{ij}[f]\big)_{ij} := \big(a_{ij} * f\big)_{ij}. \tag{1.3}$$

In the same way, we define $\boldsymbol{b}[f] \in \mathbb{R}^d$ and $\boldsymbol{c}_\gamma[f] \in \mathbb{R}$ by

$$\boldsymbol{b}_i[f] := b_i * f, \qquad i = 1, \ldots, d; \qquad \boldsymbol{c}_\gamma[f] := c * f.$$

We emphasize the dependency with respect to the parameter γ in $\boldsymbol{c}_\gamma[f]$ since, in several places, we apply the same definition with $\gamma + 1$ replacing γ. Notice that

$$\boldsymbol{c}_\gamma[f](v) = \begin{cases} (d-1)\,(\gamma+d) \displaystyle\int_{\mathbb{R}^d} |v - v_*|^\gamma f(v_*)\mathrm{d}v_* & \text{for } \gamma \in (-d, 0), \\ c_d\, f(v) & \text{for } \gamma = -d, \end{cases} \tag{1.4}$$

where $c_d := (d-1)(d-2)|\mathbb{S}^{d-1}|$.
With these notations, the Landau equation can then be written alternatively under the form

$$\begin{cases} \partial_t f &= \nabla \cdot \big(\mathcal{A}[f]\,\nabla f - \boldsymbol{b}[f]\,f\big) = \mathrm{Tr}\,\big(\mathcal{A}[f]D^2 f\big) + \boldsymbol{c}_\gamma[f]f, \\ f(t=0) &= f_{\mathrm{in}}\,, \end{cases} \tag{1.5}$$

where $D^2 f$ is the Hessian matrix of f. Notice that, with respect to previous existing works on the field, we adopt here a convention in which the term $\boldsymbol{c}_\gamma[f]$ in (1.5) is nonnegative. With such a convention, $\boldsymbol{c}_\gamma[f] = -\nabla \cdot \boldsymbol{b}[f]$.

1.3. Solutions to the (Spatially Homogeneous) Landau Equation

We discuss here some of the results existing in the literature for the Landau equation in dimension $d = 3$.

In the case of hard potentials ($\gamma \in (0, 1]$), existence, uniqueness, appearance of smoothness and of moments is known Desvillettes, Villani [22]. Exponential convergence towards equilibrium also holds, see Alonso et al. [8] and Desvillettes [20]. The situation for Maxwell molecules ($\gamma = 0$) and moderately soft potentials ($\gamma \in [-2, 0)$), is almost identical in terms of existence, uniqueness and appearance of smoothness. The main difference concerns the statistical moments, which are propagated (and grow at most linearly with time) but are not created (see Wu [59]; Desvillettes [18]; Desvillettes, Villani [22]). Moreover, the speed of convergence towards equilibrium is proved under suitable assumptions on the initial datum to be "stretched exponential" (see Theorem 1.4 of Carrapatoso [15] for the case $-1 < \gamma < 0$ or Section 6 of Alonso et al. [7] for the more general case of Landau-Fermi-Dirac equation, covering in particular the Landau case). A systematic study of the Landau equation for moderately soft potentials, including in particular the “critical case” $\gamma = -2$ has been addressed recently with techniques partly similar to those of the present paper in Alonso et al. [5].

The case of very soft potentials ($\gamma \in [-3, -2)$), which includes the physically relevant case of Coulomb potential ($\gamma = -3$) is different. Up to the appearance of the very recent paper [34], only weak solutions (including H-solutions) were known to exist and uniqueness was an open problem as discussed in Desvillettes [18]. Notice that stretched exponential convergence to equilibrium still holds Carrapatoso et al. [16].

Focusing on Coulomb interactions $\gamma = -3$, the main *a priori* estimate in Lebesgue spaces states that H-solutions f satisfy

$$f \in L^1([0,T]; L^3_{3\gamma}(\mathbb{R}^3)),$$

see Desvillettes [18]. We point out the very recent improvement in terms of moments for Coulomb potential $\gamma = -3$ which shows that $f \in L^1([0,T]; L^3_s(\mathbb{R}^3))$ with $s \geqq -5$, see Ji [36]. Such an estimate is deduced from a careful study of the entropy production of the Landau collision operator and can be interpolated with the energy estimate stating that $f \in L^\infty([0,T]; L^1_2(\mathbb{R}^3))$.

As far as the regularity of solutions is concerned, it is possible to get a bound on the Hausdorff dimension of the times in which the solution might be singular (see Golse et al. [27] for the Coulomb case and Golse et al. [28] for general very soft potentials). Various perturbative results are also available in the literature: we refer to Golding et al. [24] for a recent construction of close-to-equilibrium solutions in the case of Coulomb interactions while local in time solutions for large data and global in time solutions for data close to equilibrium have been recently obtained in Desvillettes et al. [21].

Finally, in [53, Theorem 3.8], conditional results show that if the solution lies in $L^\infty([0,T]; L^p_\kappa(\mathbb{R}^3))$ for $p > \frac{3}{2}$ and κ sufficiently large, then it is bounded and consequently smooth. We refer also to [12] for a local version of a close result.

We end this description of the literature on the Landau equation with very interesting results about a related model, known as the *isotropic Landau equation*, introduced in Ben Porath [38] and investigated later in Gressman et al. [29]; Gualdani and Guillen [31,32]. For such a model, the projection $\Pi(z)$ in the Landau equation is simply replaced with $\Pi(z) = \mathrm{Id}$ and this simplification yields striking results. In particular, there exists $\gamma_* \in (-\frac{5}{2}, -2)$ such that, for $\gamma \in (\gamma_*, -2)$, L^p-norms are *propagated* for $p > \frac{d}{d+\gamma+2}$ and become instantaneously bounded. Therefore, classical solutions of the isotropic Landau equation remain smooth for every finite time (see Theorem 1.1 of Gualdani and Guillen [32]). Similar results have been obtained recently for an isotropic version of the Boltzmann equation; see Snelson [54].

1.4. Link Between the Landau Equation and the Navier–Stokes Equation

Some striking analogies between the spatially homogeneous Landau for very soft potentials and the incompressible $3d$-Navier–Stokes have been observed in recent contributions. More precisely, recall the incompressible $3d$-Navier–Stokes with viscosity $\boldsymbol{\nu} > 0$:

$$\partial_t u + u \cdot \nabla u - \boldsymbol{\nu}\, \Delta u + \nabla p = 0, \qquad \nabla \cdot u = 0, \qquad u(t=0) = u_{\mathrm{in}}, \qquad \nabla \cdot u_{\mathrm{in}} = 0. \tag{1.6}$$

Here $u = u(t,x) \in \mathbb{R}^3$. It appears that the H-solutions constructed in Villani [57] share several common points with Leray solutions to the incompressible $3d$-Navier–Stokes equations Leray [41]; Ożański and Pooley [47]:

- One first evidence is the contribution of the third author about the role of entropy dissipation, showing that H-solutions to Landau equation satisfy (1.13) for any $T > 0$ providing an a priori estimate for solutions f to (1.1). In particular, H-solutions to Landau equation are weak-solutions just like Leray solutions to Navier–Stokes equations are weak solutions and the entropy H plays for the Landau equation a similar role as the energy $\|u(t)\|_2^2$ for the Navier–Stokes equation. Note that in this analogy, $\sqrt{f}$ plays the role of u, since on one hand $\sqrt{f} \in L_t^\infty(L_v^2)$, and on the other hand $\sqrt{f}$ belongs to some weigthed $L_t^2(H_v^1)$ space.
- Based on this estimate, it has been shown in Golse et al. [27] that the Hausdorff dimension of the set of time singularities is at most $\frac{1}{2}$ for solutions to (1.1) in the Coulomb case ($\gamma = -3$, $d = 3$). This result has been extended to soft potentials $-3 < \gamma < -2$ in Golse et al. [28] and is the analogue of the celebrated Caffarelli-Kohn-Nirenberg Theorem Caffarelli et al. [14]; Ożański [48] for the Navier-Stokes equations.
- Finally, a new monotonicity formula for a functional involving entropy, regularity and moments has been obtained in Desvillettes, He and Jiang [21]. This formula entails in particular that, if

$$H(f_{\rm in})\left(\|f_{\rm in} - \mathcal{M}\|_{\dot{\mathbb{H}}^1} + C\right) \leqq \frac{5}{2},$$

then no blow-up can occur for solutions to (1.1). Here, $H(f_{\rm in})$ is the entropy of the initial datum, $\mathcal{M}$ is the associated Maxwellian steady state, and $C > 0$ is some fixed universal constant. Such a result can be compared to the result of Leray [41] for Navier–Stokes equation which shows that, provided $\|u_{\rm in}\|_2 \, \|\nabla u_{\rm in}\|_2 \ll 1$, the Navier–Stokes equations admit global smooth solutions (see also Ożański and Pooley [47]). Regularity of solutions after a given (sufficiently large) time can also be considered as a common feature of both equations.

An important criterion about the existence of global classical solutions to 3d-incompressible Navier–Stokes equation is the celebrated *Prodi–Serrin* criterion which reads as follows (see [11, Theorems 4.8 and 4.9]):

Theorem 1.1. (Prodi–Serrin criterion for 3d-incompressible Navier–Stokes equation) *Consider two Leray-Hopf weak solutions $u, \tilde{u}$ of the Navier–Stokes equation with $u(0) = \tilde{u}(0) = u_{\rm in}$. If*

$$u \in L^r([0,T]; L^q(\mathbb{R}^3)), \qquad with \quad \frac{2}{r} + \frac{3}{q} = 1, \quad q \in (3, \infty], \tag{1.7}$$

then $u = \tilde{u}$ on $[0, T)$. Moreover, for $q = 3$ (and $r = \infty$), there exists a universal constant $\delta > 0$ such that, if

$$\sup_{t \in [0,T]} \|u(t)\|_3 \leqq \delta \boldsymbol{\nu},$$

then $u = \tilde{u}$ on $[0, T)$. Finally, in both cases, u is a strong solution on $[0, T]$ provided $u_{\rm in} \in H^1(\mathbb{R}^3)$ whereas, if $u_{\rm in} \in L^2(\mathbb{R}^3)$, then u is a strong solution solution on $[t_*, T]$ for any $t_* > 0$.

We refer to Bedrossian and Vicol [11]; Lemarié-Rieusset [40] for more details on the notions of Leray-Hopf weak solutions in the above Theorem. The aforementioned result has been proven first in Prodi [50] (with some additional assumption) for $q > 3$ and derived in the present form in Serrin [52], still for $q > 3$. The statement here above for $q = 3$ is a very special case called the *endpoint Prodi–Serrin* criterion. It is somehow possible to remove the smallness assumption on $\|u\|_{L^\infty([0,T];L^3(\mathbb{R}^3))}$ in that case, but the corresponding result reads then a bit differently: the breakthrough result Iskauriaza et al. [35] dealing with the case $r = \infty, q = 3$ shows indeed smoothness and uniqueness in some indirect way. Precisely, result of Iskauriaza et al. [35] rather reads as follows: if u is a classical solution to Navier-Stokes equation whose maximal time of existence T is finite, then

$$\limsup_{t\to T} \|u(t)\|_3 = +\infty.$$

We also refer the reader to Chapters 12 and 15 of Lemarié–Rieusset [40]. Similar criteria have been established for other types of problems, spanning from the study of MHD equations Jia and Zhou [37] to stochastic differential equations Neves and Olivera [46]; Krylov and Röckner [39]; Rööckner and Zhao [51]. We notice that the proof of Theorem 1.1 is fully quantitative whereas the endpoint proof of Iskauriaza et al. [35] is obtained by contradiction and thanks to a compactness argument (we refer to the recent contributions Tao [56]; Palasek [49] for a quantitative approach of the same result).

1.5. Main Results of the Paper

In the present paper, we intend to obtain a result which extends the bounds obtained in Silvestre [53], and which is the equivalent for the Landau equation of the Prodi–Serrin result for the Navier–Stokes equation. As mentioned already, our results cover all dimensions $d \geqq 3$, any soft potentials $\gamma \in [-d, 0)$, and are valid for a suitable range of admissible $L^r_t(L^q_v)$ spaces (with exception of the end-point estimate $r = \infty$, see Section 5.2).

As it is the case for the Navier–Stokes equation, the Prodi–Serrin criteria that we obtain are actually the fundamental conditional assumptions which allow to prove the propagation and appearance of L^p-norms for suitable p, for the solutions to the spatially homogeneous equation. Such propagation/appearance of L^p-norms can be seen as the main result of the present contribution and such conditional results are completely new to our knowledge (for $r < \infty$). We will see that they are also the cornerstone for uniqueness and further smoothness of solutions to (1.1).

For the clarity of presentation, and due to the physical relevance of the Coulomb case, we distinguish the two cases $\gamma = -d$ and $\gamma \in (-d, 0)$, beginning with the Prodi–Serrin criterion for Landau equation in the Coulomb case $\gamma = -d$.

Theorem 1.2. (Prodi–Serrin criterion for Landau equation and Coulomb interaction) *We consider an integer dimension* $d \geqq 3$ *and let* f_{in} *and* $f = f(t, v)$ *define a solution to Eq.* (1.1) *in the sense of Definition* 1.15 *with*

$$\gamma = -d.$$

Assume that the solution f satisfies one of the following conditions:

$$f \in L^1([0,T];\, L^\infty(\mathbb{R}^d)), \tag{1.8a}$$

or

$$\langle\cdot\rangle^d f \in L^r([0,T];\, L^q(\mathbb{R}^d)), \qquad \frac{2}{r}+\frac{d}{q}=2, \quad r \in (1,\infty), \quad q \in \left(\frac{d}{2},\infty\right). \tag{1.8b}$$

Then, the following holds:

(a) ***(Propagation of L^p-norms).*** *Given $p \in (1,\infty)$, if $f_{\rm in} \in L^p(\mathbb{R}^d)$ then*

$$\sup_{t\in[0,T]} \|f(t)\|_p \leqq \boldsymbol{C}_{p,q,T}\, \|f_{\rm in}\|_p$$

where $\boldsymbol{C}_{p,q,T}$ is an explicit *constant depending on p, d, q and the norms $\|f\|_{L^1([0,T];\, L^\infty(\mathbb{R}^d))}$ or $\|\langle\cdot\rangle^d f\|_{L^r([0,T];\, L^q(\mathbb{R}^d))}$ according to the above condition* (1.8).

(b) ***(Appearance of L^p-norms).*** *Given $p \in (1,\infty)$, assume that*

$$f_{\rm in} \in L^1_{\nu_p}(\mathbb{R}^d), \qquad \nu_p = \frac{d^2}{2}\frac{p-1}{p}$$

then

$$\|f(t)\|_p \leqq C_{p,T}(f_{\rm in})\, t^{-\frac{d}{2}\left(1-\frac{1}{p}\right)}, \qquad \forall t \in (0,T]$$

for some explicit *positive constant $C_{p,T}(f_{\rm in})$ which depends on T, p, d, q, the initial datum $f_{\rm in}$ through the quantifies $\varrho_{\rm in}$, $E_{\rm in}$, $H(f_{\rm in})$ defined in Definition* 1.11*, $\|f_{\rm in}\|_{L^1_{\nu_p}}$, and the norms associated to the above condition* (1.8)*.*

Remark 1.3. We emphasise here the different nature of the two Prodi–Serrin criteria in (1.8). The assumption (1.8a) does not require any moment assumption on f whereas the Prodi–Serrin criterion (1.8b) involves the weighted solution $\langle\cdot\rangle^d f$, see Remark 1.6 for more details. We also point out that, under the assumption $f \in L^1([0,T],\, L^\infty(\mathbb{R}^d))$, the conclusion of point (a) still holds for $p=\infty$ (see Proposition 2.1).

The full proof of the above Theorem is presented in Section 2 and is based on several intermediate results. Typically, Proposition 2.1 shows the propagation of L^p-norm under the $L^1_t(L^\infty_v)$ criterion while Proposition 2.2 shows the appearance of L^p-norms under this assumption. The general $L^r_t(L^q_v)$ case is dealt with in Proposition 2.5 and Corollary 2.7 for appearance and propagation of L^p-norms respectively. Note also that constants are explicitly given in those propositions.

An analogue of the above result holds true for general soft-potentials $\gamma \in (-d,0)$.

Theorem 1.4. (Prodi–Serrin criterion for Landau equation with soft potentials) *We consider an integer dimension $d \geqq 3$ and let f_{in} and $f = f(t, v)$ define a solution to Eq.* (1.1) *in the sense of Definition* 1.15 *with*

$$-d < \gamma < 0.$$

Assume that the solution f satisfies either

$$f \in L^1\left([0, T];\ L^{\frac{d}{d+\gamma}}(\mathbb{R}^d)\right) \tag{1.9a}$$

or

$$\langle\cdot\rangle^{|\gamma|} f \in L^r([0, T];\ L^q(\mathbb{R}^d)), \qquad \frac{2}{r} + \frac{d}{q} = 2 + d + \gamma, \tag{1.9b}$$

where $r \in (1, \infty)$, $q \in \left(\max\left(1, \frac{d}{d+\gamma+2}\right), \frac{d}{d+\gamma}\right)$. Then, the following holds:

(a) ***(Propagation of L^p-norms).*** *If $f_{\text{in}} \in L^p(\mathbb{R}^d)$ with $p \in (1, \frac{d}{d+\gamma})$ under condition* (1.9a) *or $p \in (1, \infty)$ under condition* (1.9b)*, then*

$$\sup_{t \in [0,T]} \|f(t)\|_p \leqq \boldsymbol{C}_{p,\gamma,q,T}\, \|f_{\text{in}}\|_p$$

where $\boldsymbol{C}_{p,\gamma,q,T}$ is an explicit *constant depending on p, d, q, γ the initial datum f_{in} through ϱ_{in}, E_{in}, $H(f_{\text{in}})$, and the norms associated to the above conditions* (1.9).

(b) ***(Appearance of L^p-norms).*** *Let $p \in (1, \frac{d}{d+\gamma})$ under condition* (1.9a) *or $p \in (1, \infty)$ under condition* (1.9b)*. Assume that*

$$f_{\text{in}} \in L^1_{\nu_{\gamma,p}}(\mathbb{R}^d), \qquad \nu_{\gamma,p} := \frac{|\gamma| d}{2}\left(1 - \frac{1}{p}\right),$$

then

$$\|f(t)\|_p \leqq C_{p,\gamma,q,T}(f_{\text{in}})\, t^{-\frac{d}{2}(1-\frac{1}{p})}, \qquad \forall t \in (0, T]$$

for some explicit *positive constant $C_{p,\gamma,q,T}(f_{\text{in}})$ which depends on d, γ, T, p, q, the initial datum f_{in} through ϱ_{in}, E_{in}, $H(f_{\text{in}})$, $\|f_{\text{in}}\|_{L^1_{\nu_{\gamma,p}}}$, and the norms associated to the above condition* (1.9).

Remark 1.5. Notice that the link $\frac{2}{r} + \frac{d}{q} = 2 + d + \gamma$, with $r, q > 1$ implies

$$\max\left(1, \frac{d}{d + 2 + \gamma}\right) < q < \frac{d}{d + \gamma},$$

so that this condition is automatically satisfied in Theorem 1.4.

Remark 1.6. As in the Coulomb case, one notices the discrepancy between the two sets of assumptions for our Prodi–Serrin like criteria. In particular, the assumption (1.9a) does not require any moment assumption on f whereas the Prodi–Serrin criterion (1.9b) applies to the weighted solution $\langle\cdot\rangle^{|\gamma|} f$ instead of the mere solution f. We point out that, by using a simple interpolation argument, we can reformulate the Prodi–Serrin assumption with an *inequality* linking the two parameters (r, q). Namely, if one assumes that $\gamma + 2 < 0$ and

$$f \in L^{r_0}([0,T],\ L^{q_0}(\mathbb{R}^d)) \qquad \frac{2}{r_0} + \frac{d}{q_0} < d + \gamma + 2$$

then given $\max\left(1, \frac{d}{d+\gamma+2}\right) < q < \infty$, assuming

$$f_{\text{in}} \in L^1_\mu(\mathbb{R}^d), \qquad \mu = |\gamma| \frac{q(q_0 - 1)}{q_0 - q}$$

one can find $r > 1$ such that

$$\langle\cdot\rangle^{|\gamma|} f \in L^r([0,T],\ L^q(\mathbb{R}^d)), \qquad \frac{2}{r} + \frac{d}{q} = d + \gamma + 2.$$

We also refer to Golding et al. [24] for consideration on another kind of inequality relating the parameters r and q.

The above result is similar to its analogue for the Coulomb case but we point out here that, dealing with soft-potentials $\gamma \in (-d, 0)$ leads to additional technical difficulties due to the fact that the lowest order term $\boldsymbol{c}_\gamma[f]$ in (1.1) is a convolution. To overcome this difficulty, we give here a general estimate involving such a term, which is a variant of estimates introduced in Gualdani and Guillen [30], who named it ε-Poincaré inequality. For moderately soft potentials, the ε-Poincaré inequality has been already used to show the appearance of L^p-norms in the contributions Alonso et al. [7]; Alonso et al. [5]. We propose here a version, possibly sharp, of such a functional inequality, which involves suitable L^q-space estimates.

Proposition 1.7. (General ε-Poincaré inequality) *Assume that $d \in \mathbb{N}$, $d \geqq 3$, $-d \leqq \gamma < 0$, and*

$$\max\left(1, \frac{d}{d+2+\gamma}\right) < q < \frac{d}{d+\gamma}, \qquad \left(\frac{d}{d+\gamma} = \infty \quad \textit{if} \quad \gamma = -d\right).$$

Then there exists $C_0 > 0$ depending only on d, γ, q such that, for any $\varepsilon > 0$ (and suitable functions ϕ and $g \geqq 0$),

$$\int_{\mathbb{R}^d} \phi^2 \boldsymbol{c}_\gamma[g] \mathrm{d}v \leqq \varepsilon \int_{\mathbb{R}^d} \left|\nabla\left(\langle v\rangle^{\frac{\gamma}{2}} \phi(v)\right)\right|^2 \mathrm{d}v + C_0 \left(\|g\|_1 + \varepsilon^{-\frac{s}{1-s}} \left\|\langle\cdot\rangle^{|\gamma|} g\right\|_q^{\frac{1}{1-s}}\right) \int_{\mathbb{R}^d} \phi^2 \langle v\rangle^\gamma \mathrm{d}v, \tag{1.10}$$

where $s \in (0,1)$ is given by $s = \dfrac{d - q(d+\gamma)}{2q}$.

Remark 1.8. With the above notations, if one sets $r = \frac{1}{1-s}$, one notices that $\frac{2}{r} + \frac{d}{q} = d + \gamma + 2$. When applied to the solution $g = f(t)$ to the Landau equation (1.1), one sees therefore that the integrability in time of $\left\| \langle \cdot \rangle^{|\gamma|} f(t) \right\|_q^{\frac{1}{1-s}}$ exactly corresponds to our Prodi–Serrin criterion (1.9b).

The proof of this fundamental functional inequality is given in Section 3.2. This is the main tool in the proof of Theorem 1.4, under assumption (1.9b). The proof of Theorem 1.4 is then given in Section 3, in which Proposition 3.2 shows the propagation of L^p-norm under condition (1.9a) while Proposition 3.4 shows the appearance of L^p-norms under this same condition. The general $L^r_t(L^q_v)$ case (1.9b) is dealt with in Proposition 3.6 and Corollary 3.7, for appearance and propagation of L^p-norms respectively. The constants in Theorem 1.4 are made explicit in those results.

1.6. Relevance of Our Main Results

We chose to present the main results of the paper as conditional theorems ensuring the propagation and appearance of L^p-norms for the solutions to the Landau equation. Besides their specific interest, such propagation and appearance results are the fundamental tools which allow to deduce both *uniqueness* and *regularity* of solutions to the Landau equation. Precisely, as already observed in Chern and Gualdani [13] in the Coulomb case, suitable bounds on L^p-norms for solutions to Landau equation imply uniqueness of the solution. While the result is stated in Chern and Gualdani [13] as a consequence of $L^\infty([0,T], L^p(\mathbb{R}^d))$ norms with $p > \frac{3}{2}$ in the Coulomb case $\gamma = -3, d = 3$, our main results Theorem 1.2 and 1.4 can be seen as suitable conditional result yielding such bounds valid for any dimension $d \geqq 3$ and for any choice $\gamma \in [-d, 0)$. Then, as in Chern and Gualdani [13], such bounds yield uniqueness of suitable strong solutions. This question is addressed in Section 4 and the main result of this section can then be stated as follows:

Theorem 1.9. *We consider an integer dimension $d \geqq 3$. Let $\gamma \in [-d, -2)$ and assume that $2 > \frac{d}{d+\gamma+2}$. Let $f = f(t, v)$, $g = g(t, v)$ define two solutions to Eq. (1.5) in the sense of Definition 1.15, with*

$$f(0, \cdot) = g(0, \cdot) = f_{\text{in}} \in L^2_{k+2|\gamma|}(\mathbb{R}^d) \cap L^1_N(\mathbb{R}^d),$$

where $k > d$, and N is defined in Proposition 4.2. If f, g satisfy one of the two assumptions (1.9a) or (1.9b) if $\gamma \in (-d, 0)$, or (1.8a) or (1.8b) in the Coulomb case $\gamma = -d$, then

$$f(t) = g(t) \qquad \forall t \in [0, T].$$

Remark 1.10. We restrict our uniqueness result to the case of very soft potentials $-d \leqq \gamma < -2$ only, since the case of moderate soft potentials $\gamma \in [-2, 0)$ is well-understood (see Alexandre et al. [1]; Wu [59] and the recent contribution Alonso et al. [5]). Notice that because of the condition $2 > \frac{d}{d+\gamma+2}$, our analysis is restricted in the Coulomb case to the physical dimension $d = 3$.

The above result can be seen as a generalisation and an improvement on existing conditional uniqueness results for Landau equation. As mentioned already, the uniqueness of weak solutions with bounded mass, momentum, energy, and entropy which further lie in $L^1([0,T]; L^\infty(\mathbb{R}^d))$ (that is, under condition (1.8a)) has been established in the Coulomb case in Fournier [23] via probabilistic arguments, whereas the results of Chern and Gualdani [13] provide uniqueness of solutions to (1.1) under some $L^\infty([0,T],\ L^p(\mathbb{R}^d))$ bound (and decay) with $p > \frac{3}{2}$. ***Our uniqueness result generalises these two results and extends them to general soft potentials*** $\gamma \in [-d, 0)$, $d \geqq 3$, ***and the whole range*** $L^r([0,T]; L^q(\mathbb{R}^d))$, $\frac{2}{r} + \frac{d}{q} = 2 + d + \gamma$, ***with*** $r \in [1, \infty)$.

Besides the above uniqueness result, our two main results also allow to deduce suitable *unique continuation and regularity results* for the solutions to Landau equation. Even though we do not elaborate on this point in the present contribution, we mention here that, due to the parabolic nature of the Landau equation, the appearance of L^p-bounds is of paramount important in order to deduce regularity estimates:

(1) Elaborating on the pioneering work of De Giorgi [17] (suitably adapted to spatially homogeneous kinetic equations in Alonso [3]), it is a well-established fact that the appearance of (uniform in time) L^p-bounds (for p large enough) yields the *appearance of* L^∞*-bounds* for solutions to Landau equation. Such a $L^p - L^\infty$ De Giorgi's argument has been already introduced in the study of Landau equations in Alonso et al. [7]; Golding et al. [24]; Alonso et al. [5]. More specifically, one can deduce from Theorems 1.2 or 1.4 the following: if $f = f(t, v)$ is a solution to the Landau equation (1.1) satisfying (1.8) or (1.9b) (according to $\gamma = -d$ or $-d < \gamma < 0$), and suitable L^1-moments estimates, then for any $t_\star \in (0, T)$ there exists $K(t_\star, T) > 0$ such that

$$\sup_{t \in [t_*, T]} \|f(t, \cdot)\|_\infty \leqq K(t_*, T). \tag{1.11}$$

A full proof of such an estimate is given in the case of moderately soft potentials $\gamma \in [-2, 0)$ in Alonso et al. [5] and in Alonso et al. [6] for general $-d \leqq \gamma < -2$. This appearance of L^∞-bounds (uniform in time on $[t_*, T]$) is close to results presented in Silvestre [53]. While the approach of the latter is based upon general results regarding strong solutions to parabolic equations, the approach in Alonso et al. [6]; Alonso et al. [5] is, as said, based upon an adaptation of the approach of De Giorgi [17] to spatially homogeneous kinetic equations introduced in Alonso [3].

(2) Suitable L^∞-bounds like (1.11) can then be combined with parabolic regularising effect to deduce the smoothness of the solution. We refer to [28, Proposition A.1] for a full proof of the fact that, in dimension $d = 3$ and for any $\gamma \in [-3, -2)$, for any weak solution $f = f(t)$ to (1.1) in the sense of Definition 1.15, it holds

$$f \in L^\infty([t_*, T];\ L^\infty(\mathbb{R}^d)) \implies f \in \mathscr{C}^\infty((t_*, T] \times \mathbb{R}^d).$$

Therefore, our main results Theorems 1.2 and 1.4, provide conditional results (in the form of the assumptions (1.8) or (1.9)) ensuring the smoothness of the solution to the Landau equation. We recall also that such smoothness provides also *unique continuation* criteria of the solution. We refer to the recent contribution Snelson and Solomon [55] for more details about this question.

The above two points illustrate the importance of devising sufficient conditions yielding the appearance and/or propagation of L^p-norms for solutions to the Landau equation and, on this basis, we strongly believe that the Prodi–Serrin criteria (1.8) (in the case $\gamma = -d$) or (1.9) (in the case $\gamma \in (-d, 0)$) provide a significant contribution to the conditional uniqueness and regularity of Landau equation, yielding an analogue of the Theorem 1.1 for Navier-Stokes equation.

We end this description mentioning that the method and ideas developed in this paper can be adapted to derive Prodi–Serrin criterion for the spatially homogeneous Boltzmann equation with soft potentials without cut-off assumptions. We refer to the work in progress Alonso et al. [9] for further details.

1.7. About the Class of Solutions

Our main results Theorems 1.2 and 1.4 have to be interpreted as practical criteria to get *a priori* estimates for weak solutions to the Landau equation in an explicit and quantitative way. In this context, most of the computations that we are providing may be seen as formal, especially since they use one or several integrations by parts. Those estimates can however be fully justified under some extra assumption and, in particular, we strongly believe that our formal estimates are actually valid for *any weak solutions* to the Landau equation, which satisfy the Prodi–Serrin criteria that we stated. We define here weak solutions in the following way:

Definition 1.11. Given $d \in \mathbb{N}$, $d \geqq 3$, $\gamma \in [-d, 0)$ and $T > 0$, let $f_{\text{in}} \geqq 0$ lie in $L^1_2(\mathbb{R}^d) \cap L\log L(\mathbb{R}^d)$, that is f_{in} admits finite mass, energy and entropy as defined respectively by

$$\varrho_{\text{in}} := \int_{\mathbb{R}^d} f_{\text{in}}(v)\mathrm{d}v > 0, \qquad E_{\text{in}} := \int_{\mathbb{R}^d} f_{\text{in}}(v)|v|^2\mathrm{d}v,$$

and

$$H(f_{\text{in}}) := \int_{\mathbb{R}^d} f_{\text{in}}(v)\log f_{\text{in}}(v)\mathrm{d}v.$$

We say that a family $f = f(t, v) \geqq 0$ is a weak solution to (1.1) with initial condition $f(0, v) = f_{\text{in}}(v)$ if the following hold:

(1) $f \in \mathcal{C}([0, T); \mathcal{D}'(\mathbb{R}^d))$ and

$$\int_{\mathbb{R}^d} f(t, v)\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix}\mathrm{d}v = \int_{\mathbb{R}^d} f_{\text{in}}(v)\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix}\mathrm{d}v\,. \tag{1.12}$$

(2) One has

$$H(f(t)) \leqq H(f_{\text{in}}) \qquad \forall t \in [0, T]\,, \qquad \text{and}$$
$$\int_0^T \mathrm{d}t\int_{\mathbb{R}^d}\left|\nabla\sqrt{f(t, v)}\right|^2\langle v\rangle^{\gamma}\mathrm{d}v < \infty\,. \tag{1.13}$$

(3) For any $\varphi = \varphi(t, v) \in \mathscr{C}_c^2([0, T) \times \mathbb{R}^d)$,

$$
\begin{aligned}
&-\int_0^T \mathrm{d}t \int_{\mathbb{R}^d} f(t, v)\partial_t\varphi(t, v)\mathrm{d}v - \int_{\mathbb{R}^d} f_{\mathrm{in}}(v)\varphi(0, v)\mathrm{d}v \\
&= \frac{1}{2}\sum_{i,j=1}^d \int_0^T \mathrm{d}t \int_{\mathbb{R}^d\times\mathbb{R}^d} f(t, v) f(t, w) a_{i,j}(v - w) \\
&\left[\partial^2_{v_i,v_j}\varphi(t, v) + \partial^2_{w_i,w_j}\varphi(t, w)\right]\mathrm{d}v\mathrm{d}w \\
&+ \sum_{i=1}^d \int_0^T \mathrm{d}t \int_{\mathbb{R}^d\times\mathbb{R}^d} f(t, v) f(t, w)\boldsymbol{b}_i(v - w) \\
&\left[\partial_{v_i}\varphi(t, v) - \partial_{w_i}\varphi(t, w)\right]\mathrm{d}v\mathrm{d}w. \qquad (1.14)
\end{aligned}
$$

Remark 1.12. Notice that slightly weaker solutions to Landau equation have been shown to exist in Villani [57] for initial data considered in Def. 1.11 (and called *H-solutions*) which are not assumed to satisfy estimate (1.13). It has been shown later (when $d = 3$) in Desvillettes [18] that such solutions (when they are built thanks to a suitable approximation process) do satisfy (1.13) and are in fact weak solutions in the sense of Def. 1.11. Subsequently, it has been shown in Gualdani, Zamponi [33] that such solutions satisfy the estimates

$$
\mathcal{A}[f] \in L^\infty\left([0, T], L^{\frac{d}{d-2}}_{\mathrm{loc}}(\mathbb{R}^d)\right), \qquad \boldsymbol{b}[f] \in L^\infty([0, T];\ L^{\frac{d}{2}}_{\mathrm{loc}}(\mathbb{R}^d))
$$

and, for any test-function $\phi \in L^\infty((0, T)\ ,\ \mathbb{W}^{1,\infty}_c(\mathbb{R}^d))$, it holds that

$$
\begin{aligned}
&\int_0^T \mathrm{d}t \int_{\mathbb{R}^d} \partial_t f(t, v)\phi(t, v)\mathrm{d}v \\
&\quad + \int_0^T \mathrm{d}t \int_{\mathbb{R}^d} (\mathcal{A}[f(t)](v)\nabla f(t, v) - f\boldsymbol{b}[f(t)](v)) \cdot \nabla\phi(t, v)\mathrm{d}v = 0. \qquad (1.15)
\end{aligned}
$$

Here, $\mathbb{W}^{1,\infty}_c(\mathbb{R}^d)$ denotes the collection of all bounded and compactly supported functions ϕ such that $\nabla\phi \in L^\infty(\mathbb{R}^d)$, and all terms can be well defined.

Remark 1.13. We point out that weak solutions $f = f(t)$ as defined in Definition 1.11 are such that there exists a constant $K_0 > 0$, depending on $H(f_{\mathrm{in}})$ and $\| f_{\mathrm{in}}\|_{L^1_2}$ such that

$$
\sum_{i,j} \mathcal{A}_{i,j}[f(t)](v)\,\xi_i\,\xi_j \geqq K_0\langle v\rangle^\gamma\,|\xi|^2, \qquad \forall\, v,\ \xi \in \mathbb{R}^d, \qquad t \geqq 0.
$$

We refer to Lemma A.3 in Appendix A for a full proof of this estimate.

To avoid technical complications and keep the presentation as simple as possible, we rather consider in the paper a class of solutions which already enjoy a sufficient regularity to justify the computations in the next sections. Such local-in-time solutions were recently constructed in Desvillettes et al. [21] in dimension $d = 3$, and can be as expressed as follows:

Theorem 1.14. (Desvillettes et al. [21]) *Let* $f_{\text{in}} \in L\log L(\mathbb{R}^3) \cap L^1_{55}(\mathbb{R}^3) \cap \dot{\mathbb{H}}^1(\mathbb{R}^3)$ *be nonnegative. There exists some explicit* $T_\star > 0$ *depending on* $\boldsymbol{m}_{55}(f_{\text{in}})$, $\|f_{\text{in}}\|_{\dot{\mathbb{H}}^1}$ *and* $H(f_{\text{in}})$, ϱ_{in}, E_{in} *such that* (1.1) *admits a unique solution* f *on the time interval* $[0, T_\star]$ *such that*

$$f \in \mathscr{C}([0, T_\star], \dot{\mathbb{H}}^1(\mathbb{R}^3)) \cap L^2\left([0, T_\star],\ \dot{\mathbb{H}}^2_\gamma(\mathbb{R}^3)\right).$$

Motivated by the above existence and uniqueness result, we define the following notion of solutions, which we will use through out the sequel:

Definition 1.15. In all the sequel, given $d \in \mathbb{N}$, $d \geqq 3$, $\gamma \in [-d, 0)$ and $T > 0$, we will say that an initial datum $f_{\text{in}} \geqq 0$ and a family $f = f(t, v) \geqq 0$ define a solution to (1.1) if $f = f(t, v)$ is a weak solution to (1.1) in the sense of Definition 1.11 that further satisfies

$$f \in \mathscr{C}([0, T], \dot{\mathbb{H}}^1(\mathbb{R}^d)) \cap L^2\left([0, T],\ \dot{\mathbb{H}}^2_\gamma\right).$$

Remark 1.16. The computations in the next section will consist in choosing formally ϕ as a power of f as test function in the identity (1.15). Under the above additional regularity, we point out that this can be made rigorous simply by choosing, as in Chern and Gualdani [13], for the test function ϕ in (1.15) a suitable truncation of a power of f using a cutoff function $\eta_R(v) = \eta\left(R^{-1}v\right)$, $R > 1$, where η is a smooth cutoff function identically equal to 1 on the unit ball of $\mathbb{R}^d$ and vanishing on the ball centered at the origin and with radius 2. The computations are then valid for such ϕ and all the estimates turn out to be independent of R so, letting $R \to \infty$, we justify our computations.

1.8. Organization of the Paper

The proof of Theorem 1.2 is given in Section 2 where the two different criteria (1.8a) and (1.8b) are treated separately as well as propagation (point (*a*)) and appearance (point (*b*)) of L^p-norms.

Section 3 gives the full proof of Theorem 1.4 with again the various cases (*a*), (*b*) under conditions (1.9a) or (1.9b) treated separately. In particular, the full proof of the ε-Poincaré Proposition 1.7 is given in Section 3.2.

The stability and uniqueness of solutions to the Landau equation is discussed in Section 4, which culminates with the proof of Theorem 1.9.

A final Section 5 is devoted to additional features of the Landau equation. Namely, we show in Section 5.1 how the Prodi–Serrin criteria in Theorem 1.4 applies to the case of moderate soft potentials $-2 \leqq \gamma < 0$. It is well-known that, for such potentials, propagation/appearance of L^p-norms occurs as well as the uniqueness and regularity of the solution (see Alexandre et al. [1]; Wu [59]; Alonso et al. [5]) but we believe that Theorem 1.4 sheds a new light on this case, showing that Prodi–Serrin criteria (1.9) are met in this case. We also discuss the end-point case $r = \infty$, $q = \frac{d}{d+\gamma+2}$ of the Prodi–Serrin criterion (1.8b)–(1.9b) in Section 5.2, showing an analogue of the end-point case $r = \infty$, $q = 3$ in Theorem 1.1 for Navier–Stokes equations. We point out here that the role of viscosity $\boldsymbol{\nu}$ in

Theorem 1.1 is played by the coercivity constant K_0 in Remark 1.13 (see Theorem 5.2 for details).

In Appendix A we recall some known results about weak solutions to the Landau equation. Some of them are found in the literature only in dimension $d = 3$, so that we also give some sketches of proof to extend them to arbitrary dimension $d \geqq 3$.

Finally, Appendix B gives some technical results regarding the evolution of weighted L^p-norms which are useful for the proof of the uniqueness result Theorem 1.9.

2. Prodi–Serrin like Criteria for Coulomb Interactions

In this section, we focus on the Landau equation for charged particles associated to Coulomb interactions. In this situation, we recall from (1.4)–(1.5) that

$$\begin{cases} \partial_t f &= \nabla \cdot \big(\mathcal{A}[f]\,\nabla f - \boldsymbol{b}[f]\,f\big) = \operatorname{Tr}\big(\mathcal{A}[f]D^2 f\big) + c_d f^2, \\ f(t=0) &= f_{\rm in}\,, \end{cases} \tag{2.1}$$

where $D^2 f$ is the Hessian matrix of f and $c_d := (d-1)(d-2)|\mathbb{S}^{d-1}|$.

We start with a simple proposition which shows the propagation of L^p norms (including the case when $p = \infty$) under one of the end point of Prodi–Serrin condition.

Proposition 2.1. *Let $f_{\rm in}$ and $f = f(t, v)$ define a solution to Eq. (2.1) in the sense of Definition 1.15. Assume that*

$$f \in L^1([0, T]; L^\infty(\mathbb{R}^d))$$

for some $T > 0$, and $f_{\rm in} \in L^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$. Then f also lies in $L^\infty([0, T]; L^p(\mathbb{R}^d))$. More precisely, the following estimate holds:

$$\|f\|_{L^\infty([0,T],L^p(\mathbb{R}^d))} \leqq \|f_{\rm in}\|_p \ \exp\left(c_d \int_0^T \|f(t)\|_\infty \mathrm{d}t\right).$$

Proof. Using the multiplicator f^{p-1} for $p \in [1, \infty[$ in Eq. (2.1), we see that, using the nonnegativity of the matrix $\mathcal{A}[f]$, and the identity $-\nabla \cdot \boldsymbol{b}[f] = \boldsymbol{c}[f] = c_d\, f$,

$$\begin{aligned} \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} f^p \mathrm{d}v &= -(p-1)\int_{\mathbb{R}^d} f^{p-2}\nabla f \cdot \mathcal{A}[f]\,\nabla f \mathrm{d}v \\ &\quad + (p-1)\int_{\mathbb{R}^d} f^{p-1}\,\boldsymbol{b}[f]\cdot\nabla f \mathrm{d}v \\ &\leqq c_d \frac{(p-1)}{p}\int_{\mathbb{R}^d} f^{p+1}\mathrm{d}v \leqq c_d \frac{(p-1)}{p}\|f\|_\infty \int_{\mathbb{R}^d} f^p \mathrm{d}v\,. \end{aligned}$$

Thanks to Grönwall's lemma, we end up with, for $t \in [0, T]$,

$$\int_{\mathbb{R}^d} f^p(t, v)\,\mathrm{d}v \leqq \int_{\mathbb{R}^d} f_{\rm in}^p(v)\,\mathrm{d}v \ \exp\left(c_d(p-1)\int_0^t \|f(s)\|_\infty\,\mathrm{d}s\right), \qquad \forall t \in [0, T]$$

and consequently,

$$\|f\|_{L^\infty([0,T],L^p(\mathbb{R}^d))} \leqq \|f_{\rm in}\|_p \ \exp\left(c_d \int_0^T \|f(s)\|_\infty \,\mathrm{d}s\right).$$

This concludes the proof when $p \in [1, \infty[$. Passing to the limit when $p \to \infty$ in the above estimate shows that the proposition also holds when $p = \infty$. □

We then show the appearance when $t > 0$ of L^p norms for finite $1 < p < \infty$ (notice that the result now excludes the case when $p = \infty$) under one of the end point of Prodi–Serrin condition, assuming that the initial datum has some moments in L^1.

Proposition 2.2. *Let $f_{\rm in}$ and $f = f(t, v)$ define a solution to Eq. (2.1) in the sense of Definition 1.15. Given $p \in (1, \infty)$, assume that*

$$f_{\rm in} \in L^1_{\nu_p}(\mathbb{R}^d), \qquad \nu_p = \frac{d^2}{2}\frac{p-1}{p},$$

and that

$$f \in L^1([0, T]; L^\infty(\mathbb{R}^d))$$

for some $T > 0$. Then, the solution $f = f(t, v)$ satisfies the following estimate, for any $t \in (0, T]$:

$$\|f(t)\|_p \leqq C_p\left[\sup_{\tau\in[0,T]} \boldsymbol{m}_{\nu_p}(\tau)\right] t^{-\frac{d}{2}(1-\frac{1}{p})}, \tag{2.2}$$

where

$$C_p = C_p(T) := \left[\frac{4\,K_0}{dpC_{Sob}}\right]^{-\frac{d}{2}(1-\frac{1}{p})}$$
$$\exp\left(\frac{d^2}{p^2}K_0(p-1)\,T + c_d\left(1-\frac{1}{p}\right)\int_0^T \|f(\tau)\|_\infty\, d\tau\right),$$

K_0 is the constant (depending on $\varrho_{\rm in}$, $E_{\rm in}$ and $H(f_{\rm in})$) appearing in Remark 1.13, and $C_{\rm Sob}$ is the constant appearing in the Sobolev embedding $\dot{\mathbb{H}}^1(\mathbb{R}^d) \hookrightarrow L^{\frac{2d}{d-2}}(\mathbb{R}^d)$ as defined in (2.5).

Remark 2.3. Recall that, under the assumption $f_{\rm in} \in L^1_{\nu_p}(\mathbb{R}^d)$, one has

$$\sup_{\tau\in[0,T]} \boldsymbol{m}_{\nu_p}(\tau) \leqq C_{d,p}(1+T),$$

where $C_{d,p}$ is a positive constant depending only on d, p and $\boldsymbol{m}_{\nu_p}(0) = \|f_{\rm in}\|_{L^1_{\nu_p}}$ (see Proposition A.8).

Proof. Proceeding as in the proof of Proposition 2.1, we write

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} f^p(t,v)\mathrm{d}v &= -(p-1)\int_{\mathbb{R}^d} f^{p-2}\nabla f\cdot\mathcal{A}[f]\,\nabla f\mathrm{d}v\\ &\quad+(p-1)\int_{\mathbb{R}^d} f^{p-1}\,\boldsymbol{b}[f]\cdot\nabla f\mathrm{d}v\\ &= -\frac{4\,(p-1)}{p^2}\int_{\mathbb{R}^d}\nabla f^{\frac{p}{2}}\cdot\mathcal{A}[f]\,\nabla f^{\frac{p}{2}}\mathrm{d}v\\ &\quad+c_d\,\frac{p-1}{p}\int_{\mathbb{R}^d} f^{p+1}(t,v)\mathrm{d}v\,,\end{aligned}\tag{2.3}$$

where we notice that

$$\nabla f^{\frac{p}{2}}\cdot\mathcal{A}[f]\,\nabla f^{\frac{p}{2}}=\frac{p^2}{4}f^{p-2}\mathcal{A}[f]\nabla f\cdot\nabla f.$$

Using the uniform ellipticity of the diffusion matrix $\mathcal{A}[f]$ (recall Proposition 1.13), we deduce that

$$\begin{aligned}&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} f^p(t,v)\mathrm{d}v+\frac{4\,K_0\,(p-1)}{p^2}\int_{\mathbb{R}^d}\langle\cdot\rangle^{-d}\left|\nabla f^{\frac{p}{2}}(t,v)\right|^2\\ &\leqq c_d\,\frac{p-1}{p}\int_{\mathbb{R}^d} f^{p+1}(t,v)\mathrm{d}v.\end{aligned}$$

Multiplying this inequality by p and using the elementary inequality

$$\left|\nabla\left[\langle\cdot\rangle^{-\frac{d}{2}}f^{\frac{p}{2}}\right]\right|^2\leqq 2\langle\cdot\rangle^{-d}\left|\nabla f^{\frac{p}{2}}\right|^2+\frac{d^2}{2}\,\langle\cdot\rangle^{-d-2}\,f^p,$$

this turns into

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} f^p(t,v)\mathrm{d}v+\frac{2\,K_0\,(p-1)}{p}\int_{\mathbb{R}^d}\left|\nabla\left[\langle v\rangle^{-\frac{d}{2}}\,f^{\frac{p}{2}}(t,v)\right]\right|^2\mathrm{d}v\\ &\leqq c_d\,(p-1)\int_{\mathbb{R}^d} f^{p+1}(t,v)\mathrm{d}v+\frac{d^2\,K_0\,(p-1)}{p}\int_{\mathbb{R}^d}\langle v\rangle^{-d-2}\,f^p(t,v)\mathrm{d}v\end{aligned}\tag{2.4}$$

At this point, we combine Sobolev's inequality with a suitable interpolation inequality with weights to conclude. Namely, recall that Sobolev's inequality reads (with C_{Sob} the best constant in the Sobolev embedding $\dot{\mathbb{H}}^1(\mathbb{R}^d)\hookrightarrow L^{\frac{2d}{d-2}}(\mathbb{R}^d)$) as

$$\left\|\langle\cdot\rangle^{-\frac{d}{p}}f\right\|^p_{\frac{pd}{d-2}}=\left\|\langle\cdot\rangle^{-\frac{d}{2}}f^{\frac{p}{2}}\right\|^2_{\frac{2d}{d-2}}\leqq C_{\mathrm{Sob}}\int_{\mathbb{R}^d}\left|\nabla\left[\langle v\rangle^{-\frac{d}{2}}f^{\frac{p}{2}}\right]\right|^2\mathrm{d}v\,.\tag{2.5}$$

Now, we use the standard Hölder interpolation inequality (with weights), which holds for all functions g for which the norms are

$$\|\langle\cdot\rangle^a g\|_r\leqq\|\langle\cdot\rangle^{a_1}g\|^{\theta}_{r_1}\,\|\langle\cdot\rangle^{a_2}g\|^{1-\theta}_{r_2}\,,\tag{2.6}$$

with $r,r_1,r_2\geqq 1$, $a,a_1,a_2\in\mathbb{R}$,

$$\frac{1}{r}=\frac{\theta}{r_1}+\frac{1-\theta}{r_2},\quad a=\theta\,a_1+(1-\theta)a_2,\quad\theta\in(0,1).$$

Using inequality (2.6) for $g := f(t,\cdot)$ with

$$a := 0,\ \ a_1 := \nu_p,\ \ a_2 := -\frac{d}{p}, \qquad r := p,\ \ r_1 := 1,$$
$$r_2 := \frac{dp}{d-2}, \qquad \theta := \frac{2}{dp+2-d},$$

we get

$$\begin{aligned}\int_{\mathbb{R}^d} f^p(t,v)\mathrm{d}v &\leqq \boldsymbol{m}_{a_1}(t)^{p\theta}\ \left\|\langle\cdot\rangle^{-\frac{d}{p}} f(t,\cdot)\right\|_{\frac{dp}{d-2}}^{p(1-\theta)} \\ &\leqq \boldsymbol{m}_{a_1}(t)^{p\theta} C_{\mathrm{Sob}}^{1-\theta}\left(\int_{\mathbb{R}^d}\left|\nabla\left[\langle v\rangle^{-\frac{d}{2}} f^{\frac{p}{2}}(t,v)\right]\right|^2 \mathrm{d}v\right)^{1-\theta},\end{aligned}$$

where we used (2.5) in the last inequality. Recalling now the expression of a_1 and θ, we use this estimate under the form

$$\int_{\mathbb{R}^d}\left|\nabla\left[\langle v\rangle^{-\frac{d}{2}} f^{\frac{p}{2}}(t,v)\right]\right|^2 \mathrm{d}v \geqq C_{\mathrm{Sob}}^{-1}\boldsymbol{m}_{\nu_p}(t)^{-\frac{2p}{d(p-1)}}\left(\int_{\mathbb{R}^d} f^p(t,v)\mathrm{d}v\right)^{1+\frac{2}{d(p-1)}}. \tag{2.7}$$

Plugging this in (2.4), we end up with

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} f^p(t,v)\mathrm{d}v + \frac{2\,K_0\,(p-1)}{p\,C_{\mathrm{Sob}}}\boldsymbol{m}_{\nu_p}(t)^{-\frac{2p}{d(p-1)}}\left(\int_{\mathbb{R}^d} f^p(t,v)\mathrm{d}v\right)^{1+\frac{2}{d(p-1)}} \\ &\leqq \left[c_d\,(p-1)\|f(t)\|_{L^\infty(\mathbb{R}^d)} + \frac{d^2\,K_0\,(p-1)}{p}\right]\int_{\mathbb{R}^d} f^p(t,v)\mathrm{d}v,\end{aligned}$$

after having dropped the weight $\langle\cdot\rangle^{-d-2}$. Defining

$$y(t) := \left(\int_{\mathbb{R}^d} f^p(t,v)\,\mathrm{d}v\right)\exp\left(-(p-1)\int_0^t\left[c_d\|f(s)\|_{L^\infty(\mathbb{R}^d)} + \frac{d^2\,K_0}{p}\right]\mathrm{d}s\right),$$

we see that $y(t)$ satisfies the differential inequality

$$y'(t) \leqq -\frac{2\,K_0\,(p-1)}{p\,C_{\mathrm{Sob}}}\boldsymbol{m}_{\nu_p}(t)^{-\frac{2p}{d(p-1)}}\ y(t)^{1+\frac{2}{dp-d}},$$

which can be solved (for $t \in [0,T]$) as

$$y(t) \leqq \left[\frac{4\,K_0}{dp\,C_{\mathrm{Sob}}}\left(\sup_{\tau\in[0,T]}\boldsymbol{m}_{\nu_p}(\tau)\right)^{-\frac{2p}{d(p-1)}} t\right]^{-\frac{d(p-1)}{2}}.$$

Recalling the definition of y, we get the desired estimate.

$\square$

Remark 2.4. Following the dependency with respect to p in the proof, we can prove that, if $f_{\text{in}} \in L^1_{\frac{d^2}{2}}(\mathbb{R}^d)$ then, for all $t_*, T > 0$ and $\kappa > 0$ small enough, the following estimate holds:

$$\sup_{t \in [t_*, T]} \int_{\mathbb{R}^d} \exp\left(\kappa f(t, v)^{\frac{2}{d}}\right) \mathrm{d}v \leqq C.$$

Here $C > 0$ is a constant depending only on $t_*, T, \varrho_{\text{in}}, E_{\text{in}}, H(f_{\text{in}}), d, \|f_{in}\|_{L^1_{\frac{d^2}{2}}(\mathbb{R}^d)}$, and $\|f\|_{L^1([0,T], L^\infty(\mathbb{R}^d))}$.

We now turn to the general case, in which the Prodi–Serrin condition appears as a weighted $L^r_t(L^q_v)$ norm assumed to be finite. More precisely, we have

Proposition 2.5. *Let f_{in} and $f = f(t, v)$ define a solution to Eq.* (2.1) *in the sense of Definition* 1.15*. Assume that*

$$\langle \cdot \rangle^d f \in L^r([0, T];\ L^q(\mathbb{R}^d)) \quad \textit{with} \quad \frac{2}{r} + \frac{d}{q} = 2 \tag{2.8}$$

for some $T > 0$, where $r \in (1, \infty)$, $q \in (\frac{d}{2}, \infty)$. Given $p \in (1, \infty)$, assume that

$$f_{\text{in}} \in L^1_{\nu_p}(\mathbb{R}^d), \qquad \nu_p := \frac{d^2}{2}\left(1 - \frac{1}{p}\right),$$

then the solution $f = f(t, v)$ satisfies, for any $t \in (0, T]$

$$\|f(t)\|_p \leqq K_{p,q} \left[\sup_{\tau \in [0,T]} \boldsymbol{m}_{\nu_p}(\tau)\right] t^{-\frac{d}{2}(1-\frac{1}{p})}, \tag{2.9}$$

where

$$\begin{aligned} K_{p,q} = K_{p,q}(T) &:= \left[\frac{2K_0}{dp\, C_{Sob}}\right]^{-\frac{d}{2}\left(1-\frac{1}{p}\right)} \\ &\quad \exp\left(\frac{d^2}{p^2} K_0 (p-1)\, T + \frac{1}{p} C_{p,q} \int_0^T \left\| \langle \cdot \rangle^{|\gamma|} f(\tau) \right\|_q^r d\tau\right), \end{aligned} \tag{2.10}$$

$$C_{p,q} := \left(1 - \frac{d}{2q}\right) \left[c_d\, (p-1)\, C_{Sob,q}\right]^{\frac{2q}{2q-d}} \left(\frac{K_0\, (p-1)}{p} \frac{2q}{d}\right)^{\frac{-d}{2q-d}},$$

and where K_0 is the constant (depending on ϱ_{in}, E_{in} and $H(f_{\text{in}})$) appearing in Remark 1.13*, C_{Sob} is the Sobolev constant defined in* (2.5)*, and $C_{\text{Sob},q}$ is the Sobolev constant relative to the embedding $\dot{\mathbb{H}}^{\frac{d}{2q}}(\mathbb{R}^d) \hookrightarrow L^{\frac{2q}{q-1}}(\mathbb{R}^d)$.*

Proof. We start from inequality (2.4) obtained in the proof of Proposition 2.2 and we estimate the term

$$\int_{\mathbb{R}^d} f^{p+1}(t, v) \mathrm{d}v = \int_{\mathbb{R}^d} \left[\langle v \rangle^d f(t, v)\right] \left[\langle v \rangle^{-\frac{d}{2}} f^{\frac{p}{2}}(t, v)\right]^2 \mathrm{d}v$$

thanks to Hölder's inequality. Denoting $q' = \frac{q}{q-1}$, where q is the exponent in the Prodi–Serrin condition, we obtain

$$\int_{\mathbb{R}^d} f^{p+1}(t,v)\mathrm{d}v \leqq \left\| \langle \cdot \rangle^d f(t) \right\|_q \left\| \langle \cdot \rangle^{-\frac{d}{2}} f^{\frac{p}{2}}(t) \right\|_{2q'}^2 .$$

Now using the inequality $\| \cdot \|_{2q'}^2 \leqq C_{\mathrm{Sob},q} \| \cdot \|_{\dot{\mathbb{H}}^{\frac{d}{2q}}}^2$ resulting from the Sobolev embedding $\dot{\mathbb{H}}^{\frac{d}{2q}}(\mathbb{R}^d) \hookrightarrow L^{\frac{2q}{q-1}}(\mathbb{R}^d)$) (see Bahouri et al. [10], Theorem 1.38), we first deduce that

$$\int_{\mathbb{R}^d} f^{p+1}(t,v)\mathrm{d}v \leqq C_{\mathrm{Sob},q} \left\| \langle \cdot \rangle^d f(t) \right\|_q \left\| \langle \cdot \rangle^{-\frac{d}{2}} f^{\frac{p}{2}}(t) \right\|_{\dot{\mathbb{H}}^{\frac{d}{2q}}}^2 .$$

Observing now that, since $q > \frac{d}{2}$, $0 < \frac{d}{2q} < 1$, we can invoke the interpolation inequality $\dot{\mathbb{H}}^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \subset \dot{\mathbb{H}}^{\frac{d}{2q}}(\mathbb{R}^d)$ (see Bahouri et al. [10], Proposition 1.32) to deduce that

$$\left\| \langle \cdot \rangle^{-\frac{d}{2}} f^{\frac{p}{2}}(t) \right\|_{\dot{\mathbb{H}}^{\frac{d}{2q}}}^2 \leqq \left\| \langle \cdot \rangle^{-\frac{d}{2}} f^{\frac{p}{2}}(t) \right\|_2^{2-\frac{d}{q}} \left\| \nabla \left[\langle \cdot \rangle^{-\frac{d}{2}} f^{\frac{p}{2}}(t) \right] \right\|_2^{\frac{d}{q}} .$$

Plugging this in inequality (2.4), we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} f^p(t,v)\mathrm{d}v + \frac{2\,K_0\,(p-1)}{p} \int_{\mathbb{R}^d} \left| \nabla \left[\langle v \rangle^{-\frac{d}{2}} f^{\frac{p}{2}}(t,v) \right] \right|^2 \mathrm{d}v \\ &\quad \leqq c_d\,(p-1)\,C_{\mathrm{Sob},q} \left\| \langle \cdot \rangle^d f(t) \right\|_q \left\| \langle \cdot \rangle^{-\frac{d}{2}} f^{\frac{p}{2}}(t) \right\|_2^{2-\frac{d}{q}} \left\| \nabla \left[\langle \cdot \rangle^{-\frac{d}{2}} f^{\frac{p}{2}}(t) \right] \right\|_2^{\frac{d}{q}} \\ &\qquad + \frac{d^2\,K_0\,(p-1)}{p} \int_{\mathbb{R}^d} \langle v \rangle^{-d-2} f^p(t,v)\mathrm{d}v. \end{aligned} \tag{2.11}$$

We now resort to Young's inequality (for $x, y \geqq 0,\ \varepsilon > 0$) in the form

$$x\,y \leqq \left(\frac{2q}{d} \right)^{-1} (\varepsilon x)^{\frac{2q}{d}} + \left(\frac{2q}{2q-d} \right)^{-1} \left(\frac{y}{\varepsilon} \right)^{\frac{2q}{2q-d}} ,$$

with the choice

$$x := \left\| \nabla \left[\langle \cdot \rangle^{-\frac{d}{2}} f^{\frac{p}{2}}(t) \right] \right\|_2^{\frac{d}{q}} , \qquad y := c_d\,(p-1)\,C_{\mathrm{Sob},q} \left\| \langle \cdot \rangle^d f(t) \right\|_q \left\| \langle \cdot \rangle^{-\frac{d}{2}} f^{\frac{p}{2}}(t) \right\|_2^{2-\frac{d}{q}} ,$$

and

$$\varepsilon := \left[\frac{K_0\,(p-1)}{p} \frac{2q}{d} \right]^{\frac{d}{2q}} .$$

Inserting this into (2.11), we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} f^p(t,v)\mathrm{d}v + \frac{K_0\,(p-1)}{p} \int_{\mathbb{R}^d} \left| \nabla \left[\langle v \rangle^{-\frac{d}{2}} f^{\frac{p}{2}}(t,v) \right] \right|^2 \mathrm{d}v \\ &\quad \leqq \left(1 - \frac{d}{2q} \right) \left[c_d\,(p-1)\,C_{\mathrm{Sob},q}\,\varepsilon^{-1} \right]^{\frac{2q}{2q-d}} \left\| \langle \cdot \rangle^d f(t) \right\|_q^{\frac{2q}{2q-d}} \left\| \langle \cdot \rangle^{-\frac{d}{2}} f^{\frac{p}{2}}(t) \right\|_2^2 \end{aligned}$$

$$+\frac{d^2 K_0 (p-1)}{p}\int_{\mathbb{R}^d}\langle v\rangle^{-d-2} f^p(t,v)\mathrm{d}v.$$

Observing that $\left\|\langle\cdot\rangle^{-\frac{d}{2}} f^{\frac{p}{2}}(t)\right\|_2^2 = \int_{\mathbb{R}^d}\langle v\rangle^{-d} f^p(t,v)\mathrm{d}v$, we can deduce from the above that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} f^p(t,v)\mathrm{d}v + \frac{K_0(p-1)}{p}\int_{\mathbb{R}^d}\left|\nabla\left[\langle v\rangle^{-\frac{d}{2}} f^{\frac{p}{2}}(t,v)\right]\right|^2\mathrm{d}v\\ &\quad\leqq \left[C_{p,q}\,\|\langle\cdot\rangle^d f(t)\|_q^{\frac{2q}{2q-d}} + \frac{d^2 K_0(p-1)}{p}\right]\int_{\mathbb{R}^d}\langle v\rangle^{-d} f^p(t,v)\mathrm{d}v, \end{aligned}\tag{2.12}$$

where

$$C_{p,q} := \left(1-\frac{d}{2q}\right)\left[c_d\,(p-1)\,C_{\mathrm{Sob},q}\right]^{\frac{2q}{2q-d}}\left(\frac{K_0(p-1)}{p}\frac{2q}{d}\right)^{\frac{-d}{2q-d}}. \tag{2.13}$$

We now plug estimate (2.7) in this inequality to deduce (neglecting the negative weight in the last integral):

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} f^p(t,v)\mathrm{d}v + \frac{K_0(p-1)}{p\,C_{\mathrm{Sob}}}\boldsymbol{m}_{\nu_p}(t)^{-\frac{2p}{d(p-1)}}\left(\int_{\mathbb{R}^d} f^p(t,v)\mathrm{d}v\right)^{1+\frac{2}{d(p-1)}}\\ &\leqq \left[C_{p,q}\,\|\langle\cdot\rangle^d f(t)\|_q^{\frac{2q}{2q-d}} + \frac{d^2 K_0(p-1)}{p}\right]\int_{\mathbb{R}^d} f^p(t,v)\mathrm{d}v.\end{aligned}$$

We now proceed as at the end of the proof of Proposition 2.2, introducing

$$y(t) := \int_{\mathbb{R}^d} f^p(t,v)\,\mathrm{d}v\,\exp\left(-\int_0^t\left[C_{p,q}\,\|\langle\cdot\rangle^d f(s,\cdot)\|_q^{\frac{2q}{2q-d}} + \frac{d^2 K_0(p-1)}{p}\right]\mathrm{d}s\right),$$

and checking that

$$y'(t) \leqq -\frac{K_0(p-1)}{p\,C_{\mathrm{Sob}}}\boldsymbol{m}_{\nu_p}(t)^{-\frac{2p}{d(p-1)}}\,y(t)^{1+\frac{2}{d(p-1)}},$$

so that (for $t\in[0,T]$)

$$y(t) \leqq \left[\frac{2K_0}{dp\,C_{\mathrm{Sob}}}\left(\sup_{\tau\in[0,T]}\boldsymbol{m}_{\nu_p}(\tau)\right)^{-\frac{2p}{d(p-1)}} t\right]^{-\frac{d(p-1)}{2}}.$$

Recalling the definition of y, we get estimate (2.9) after observing that $r=\frac{2q}{2q-d}$.
□

Remark 2.6. In both Propositions 2.2 and 2.5, whenever $p \leqq \frac{d^2}{d^2-4}$, we observe that $\frac{d^2}{2}\left(1-\frac{1}{p}\right)\leqq 2$, so that no extra moment (beyond the kinetic energy) for the initial datum is required in this case.

One can also deduce from the proof above the following propagation result:

Corollary 2.7. *Let f_{in} and $f = f(t, v)$ define a solution to Eq. (2.1) in the sense of Definition 1.15. Assume that (2.8) holds for some $T > 0$, where $r \in (1, \infty)$, $q \in (\frac{d}{2}, \infty)$. Given $p \in (1, \infty)$, assume that $f_{\text{in}} \in L^p(\mathbb{R}^d)$. Then, the solution $f = f(t, v)$ satisfies the estimate*

$$\|f\|_{L^\infty([0,T];\,L^p(\mathbb{R}^d))} \leqq \|f_{\text{in}}\|_p \exp\left(\frac{C_{p,q}}{p}\int_0^T \|\langle\cdot\rangle^d f(t)\|_q^r \,\mathrm{d}t + \frac{d^2 K_0\,(p-1)}{p^2}T\right),$$

where $C_{p,q}$ is defined in (2.10).

Proof. We start from inequality (2.12) for the evolution of the $L^p(\mathbb{R}^d)$-norm of f (note that this inequality does not use the assumption that L^1-moments are initially finite). Then, dropping the weight $\langle\cdot\rangle^{-d}$ and the nonnegative term involving the integral of the gradient on the right-hand-side, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} f^p(t, v)\mathrm{d}v \leqq \left[C_{p,q}\,\|\langle\cdot\rangle^d f(t,\cdot)\|_q^r + \frac{d^2 K_0\,(p-1)}{p}\right]\int_{\mathbb{R}^d} f^p(t, v)\mathrm{d}v,$$

where $C_{p,q}$ is given by (2.10). According to Gronwall's Lemma, one deduces that, for all $t \in [0, T]$,

$$\int_{\mathbb{R}^d} f^p(t, v)\mathrm{d}v \leqq \left(\int_{\mathbb{R}^d} f_{\text{in}}^p(v)\mathrm{d}v\right)\exp\left(\int_0^t \left[C_{p,q}\,\|\langle\cdot\rangle^d f(\tau)\|_q^r + \frac{d^2 K_0\,(p-1)}{p}\right]\mathrm{d}\tau\right),$$

from which the result easily follows. □

Proof of Theorem 1.2. It is a direct consequence of Proposition 2.1 and Proposition 2.2 when the $L^1_t(L^\infty_v)$ condition is considered, and of Proposition 2.5 and Corollary 2.7 when the $L^r_t(L^q_v)$ condition is considered. □

3. Prodi–Serrin like Criteria for Soft Potentials $-d < \gamma < 0$

We present here an analysis, similar to the one of the previous section, for the Landau equation with soft potentials $-d < \gamma < 0$. The main difference between the Coulomb case treated in Section 2 and the present one is the nature of the lower order term $\boldsymbol{c}_\gamma[f]$ which is now a convolution operator. The analysis of this term requires then the specific use of the Hardy-Littlewood-Sobolev inequality, which we recall here for the sake of completeness.

Proposition 3.1. (Hardy-Littlewood-Sobolev inequality) *Let $d \in \mathbb{N}$, $d \geqq 1$, $1 < r, p < \infty$ and $0 < \lambda < d$ with*

$$\frac{1}{p} + \frac{\lambda}{d} + \frac{1}{r} = 2.$$

Then there exists $C_{\text{HLS}} > 0$ (depending on d, p, λ) such that the estimate

$$\int_{\mathbb{R}^{2d}} g(x)|x-y|^{-\lambda}h(y)\mathrm{d}x\mathrm{d}y \leqq C_{\text{HLS}}\,\|g\|_p\,\|h\|_r \tag{3.1}$$

holds for any smooth $g, h : \mathbb{R}^d \to \mathbb{R}$.

3.1. Appearance of L^p-norms - the $L^1_t(L^{\frac{d}{d+\gamma}})$ case

As in the Coulomb case, we are already in position to prove the propagation of L^p norms for $p < \frac{d}{d+\gamma}$ under one of the end point of Prodi–Serrin condition

Proposition 3.2. *We consider $d \in \mathbb{N}$, $d \geqq 3$ and $\gamma \in (-d, 0)$. Let f_{in} and $f = f(t, v)$ define a solution to Eq. (2.1) in the sense of Definition 1.15. Assume that*

$$f \in L^1\left([0, T]\,;\; L^{\frac{d}{d+\gamma}}(\mathbb{R}^d)\right) \tag{3.2}$$

for some $T > 0$ and that $f_{\text{in}} \in L^p(\mathbb{R}^d)$ for some $p \in (1, \frac{d}{d+\gamma})$. Then f also lies in $L^\infty([0, T]; L^p(\mathbb{R}^d))$. More precisely, the following estimate holds:

$$\|f\|_{L^\infty([0,T],L^p(\mathbb{R}^d))} \leqq \|f_{\text{in}}\|_p \;\exp\left(C_{d,\gamma,p} \int_0^T \|f(t)\|_{\frac{d}{d+\gamma}}\,\mathrm{d}t\right).$$

Here $C_{d,\gamma,p} > 0$ is an explicit constant depending only on d, γ, p.

Proof. Since only propagation of L^p-norms are involved here, we simply notice that

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} f^p\mathrm{d}v &= -(p-1)\int_{\mathbb{R}^d} f^{p-2}\nabla f\cdot\mathcal{A}[f]\,\nabla f\mathrm{d}v + (p-1)\int_{\mathbb{R}^d} f^{p-1}\,\boldsymbol{b}[f]\cdot\nabla f\mathrm{d}v\\ &= -(p-1)\int_{\mathbb{R}^d} f^{p-2}\nabla f\cdot\mathcal{A}[f]\,\nabla f\mathrm{d}v + \frac{p-1}{p}\int_{\mathbb{R}^d} f^p\boldsymbol{c}_\gamma[f]\,\mathrm{d}v\,,\end{aligned} \tag{3.3}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} f^p(t,v)\mathrm{d}v \leqq (p-1)\int_{\mathbb{R}^d}\boldsymbol{c}_\gamma[f(t)](v)f^p(t,v)\mathrm{d}v,$$

and we only need to estimate this last integral. Using Hardy-Littlewood-Sobolev inequality (3.1) with $g = f(t,\cdot) \in L^p$, $h = f^p(t,\cdot)$ and $\lambda = -\gamma$, we observe that $p < \frac{d}{d+\gamma}$ implies

$$\frac{1}{r} = 2 + \frac{\gamma}{d} - \frac{1}{p} < 1,$$

and conclude that

$$\int_{\mathbb{R}^d}\boldsymbol{c}_\gamma[f(t)](v)f^p(t,v)\mathrm{d}v \leqq C_{d,\gamma,p}\|f(t)\|_p\|f^p(t)\|_r = C_{d,\gamma,p}\|f(t)\|_p\|f(t)\|_{pr}^p,$$

where $C_{d,\gamma,p} := (d-1)(d+\gamma)\,C_{\text{HLS}}$ is depending only on d, γ, p. Writing $q_0 = \frac{d}{d+\gamma}$ and recalling that $p < \frac{d}{d+\gamma}$, we use an interpolation based on Hölder's inequality, namely,

$$\|f(t)\|_{pr}^p \leqq \|f(t)\|_p^{p\theta}\,\|f(t)\|_{q_0}^{p(1-\theta)},$$

with

$$\frac{1}{pr} = \frac{\theta}{p} + \frac{1-\theta}{q_0}, \qquad \theta = \frac{q_0 - pr}{r(q_0-p)}.$$

Observing that $\frac{1}{r} = \frac{1}{q_0} + 1 - \frac{1}{p}$, one actually has $\theta = 1 - \frac{1}{p}$, so that $p(1-\theta) = 1$ and

$$\int_{\mathbb{R}^d} \boldsymbol{c}_\gamma[f(t)](v) f^p(t,v)\mathrm{d}v \leqq C_{d,\gamma,p} \|f(t)\|_p^p \|f(t)\|_{q_0} = \Lambda_0(t) \int_{\mathbb{R}^d} f^p(t,v)\mathrm{d}v, \tag{3.4}$$

with

$$\Lambda_0(t) := C_{d,\gamma,p} \|f(t)\|_{q_0} \in L^1([0,T])$$

by assumption. Thanks to Grönwall's lemma, we obtain that, for $t \in [0,T]$,

$$\int_{\mathbb{R}^d} f^p(t,v)\,\mathrm{d}v \leqq \left(\int_{\mathbb{R}^d} f_{\mathrm{in}}^p(v)\,\mathrm{d}v\right) \exp\left(C_{d,\gamma,p}(p-1) \int_0^t \|f(s)\|_{q_0}\,\mathrm{d}s\right), \quad \forall t \in [0,T]$$

which gives the desired result. □

In order to show the *appearance* of L^p-norms, we need to investigate more carefully the evolution of L^p-norms for solutions to the Landau equation (1.5). We use in the sequel the notation, for $k \in \mathbb{R}$ and $p \in (1,\infty)$,

$$\mathbb{M}_{k,p}(t) := \int_{\mathbb{R}^d} f(t,v)^p \langle v\rangle^k \mathrm{d}v, \qquad \mathbb{D}_{k,p}(t) := \int_{\mathbb{R}^d} \left|\nabla\left(\langle v\rangle^{\frac{k}{2}} f^{\frac{p}{2}}(t,v)\right)\right|^2 \mathrm{d}v,$$

together with the shorthand notation $\mathbb{M}_p(t) := \mathbb{M}_{0,p}(t)$.

Lemma 3.3. *We consider $d \in \mathbb{N}$, $d \geqq 3$ and $\gamma \in [-d,0)$, and assume that f_{in} and $f = f(t,v)$ define a solution to Eq. (1.1) in the sense of Definition 1.15. Then, for all $p \in (1,\infty)$, if*

$$f_{\mathrm{in}} \in L^1_{\nu_{\gamma,p}}(\mathbb{R}^d), \qquad \nu_{\gamma,p} := \frac{|\gamma| d}{2}\left(1 - \frac{1}{p}\right), \tag{3.5}$$

it holds that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}_p(t) + \frac{2K(p)}{C_{\mathrm{Sob}}}\boldsymbol{m}_{\nu_{\gamma,p}}(t)^{-\frac{2p}{d(p-1)}}\mathbb{M}_p(t)^{1+\frac{2}{d(p-1)}} &\leqq K(p)\gamma^2\mathbb{M}_p(t)\\ +(p-1)\int_{\mathbb{R}^d} \boldsymbol{c}_\gamma[f(t)](v)\, f^p(t,v)\mathrm{d}v,&\end{aligned} \tag{3.6}$$

where $K(p) := \frac{p-1}{p}K_0$, K_0 being the constant appearing in Remark 1.13 and C_{Sob} is the Sobolev constant appearing in the Sobolev embedding $\dot{\mathbb{H}}^1(\mathbb{R}^d) \hookrightarrow L^{\frac{2d}{d-2}}(\mathbb{R}^d)$ (see (2.5)).

Proof. As in the proof of the previous proposition, we start with

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} f^p\mathrm{d}v &= -(p-1)\int_{\mathbb{R}^d} f^{p-2}\nabla f\cdot\mathcal{A}[f]\nabla f\mathrm{d}v + (p-1)\int_{\mathbb{R}^d} f^{p-1}\boldsymbol{b}[f]\cdot\nabla f\mathrm{d}v\\ &= -(p-1)\int_{\mathbb{R}^d} f^{p-2}\nabla f\cdot\mathcal{A}[f]\nabla f\mathrm{d}v + \frac{p-1}{p}\int_{\mathbb{R}^d} f^p\boldsymbol{c}_\gamma[f]\,\mathrm{d}v,\end{aligned} \tag{3.7}$$

where we recall that $\nabla \cdot \boldsymbol{b}[f] = -\boldsymbol{c}_\gamma[f]$. Now, as previously observed,

$$(p-1)\int_{\mathbb{R}^d} f^{p-2}\mathcal{A}[f]\nabla f \cdot \nabla f \mathrm{d}v \geqq \frac{4K_0(p-1)}{p^2}\int_{\mathbb{R}^d} \langle v\rangle^\gamma \left|\nabla(f^{\frac{p}{2}})\right|^2 \mathrm{d}v.$$

Moreover, writing

$$\nabla\left(\langle v\rangle^{\frac{\gamma}{2}} f^{\frac{p}{2}}\right) = \langle v\rangle^{\frac{\gamma}{2}}\nabla(f^{\frac{p}{2}}) + \frac{\gamma}{2} v \langle v\rangle^{\frac{\gamma}{2}-2} f^{\frac{p}{2}},$$

which implies

$$\langle v\rangle^\gamma \left|\nabla(f^{\frac{p}{2}})\right|^2 \geqq \frac{1}{2}\left|\nabla\left(\langle v\rangle^{\frac{\gamma}{2}} f^{\frac{p}{2}}\right)\right|^2 - \frac{\gamma^2}{4}\langle v\rangle^{\gamma-2} f^p, \tag{3.8}$$

we observe that

$$(p-1)\int_{\mathbb{R}^d} f^{p-2}\mathcal{A}[f]\nabla f \cdot \nabla f \mathrm{d}v \geqq \frac{2K_0(p-1)}{p^2}\int_{\mathbb{R}^d} \left|\nabla\left(\langle v\rangle^{\frac{\gamma}{2}} f^{\frac{p}{2}}\right)\right|^2 \mathrm{d}v$$
$$-\frac{K_0(p-1)\gamma^2}{p^2}\int_{\mathbb{R}^d} \langle v\rangle^{\gamma-2} f^p \mathrm{d}v.$$

Inserting this into (3.7) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}_p(t) + 2K(p)\mathbb{D}_{\gamma,p}(t) \leqq K(p)\gamma^2\mathbb{M}_p(t) + (p-1)\int_{\mathbb{R}^d} \boldsymbol{c}_\gamma[f(t)](v)\, f^p(t,v)\mathrm{d}v, \tag{3.9}$$

where we simply observe that $\langle v\rangle^{\gamma-2} \leqq 1$.

Now, as in the proof of Proposition 2.2, we combine a Sobolev inequality with a suitable interpolation inequality with weights to estimate $\mathbb{D}_{\gamma,p}(t)$ in terms of $\boldsymbol{m}_{\nu_{\gamma,p}}(t)$ and $\mathbb{M}_p(t)$. Namely, one can reformulate the Sobolev embedding inequality (2.5) as

$$\left\|\langle\cdot\rangle^{\frac{\gamma}{p}} f\right\|^p_{\frac{pd}{d-2}} = \left\|\langle\cdot\rangle^{\frac{\gamma}{2}} f^{\frac{p}{2}}\right\|^2_{\frac{2d}{d-2}} \leqq C_{\mathrm{Sob}}\int_{\mathbb{R}^d} \left|\nabla\left[\langle v\rangle^{\frac{\gamma}{2}} f^{\frac{p}{2}}\right]\right|^2 \mathrm{d}v = C_{\mathrm{Sob}}\mathbb{D}_{\gamma,p}(t). \tag{3.10}$$

We now use the interpolation inequality (2.6) with the choice

$$a = 0, \quad a_1 = \nu_{\gamma,p}, \quad a_2 = \frac{\gamma}{p}, \qquad r := p,$$
$$r_1 := 1, \quad r_2 := \frac{dp}{d-2}, \qquad \theta := \frac{2}{dp+2-d}.$$

We can reformulate (2.7) as

$$\int_{\mathbb{R}^d} \left|\nabla\left[\langle v\rangle^{\frac{\gamma}{2}} f^{\frac{p}{2}}(t,v)\right]\right|^2 \mathrm{d}v \geqq C_{\mathrm{Sob}}^{-1}\boldsymbol{m}_{\nu_{\gamma,p}}(t)^{-\frac{2p}{d(p-1)}}\left(\int_{\mathbb{R}^d} f^p(t,v)\mathrm{d}v\right)^{1+\frac{2}{d(p-1)}}. \tag{3.11}$$

Plugging this into (3.9), we get the result. □

Thanks to the above evolution, we can now show the analogue of Proposition 2.2 and prove the *appearance* of L^p-norms, for $p < \frac{d}{d+\gamma}$ under the endpoint Prodi–Serrin criterion (3.2)

Proposition 3.4. *We consider $d \in \mathbb{N}$, $d \geqq 3$ and $\gamma \in (-d, 0)$, and assume that f_{in} and $f = f(t, v)$ define a solution to Eq. (1.1) in the sense of Definition 1.15. Assume that*

$$f \in L^1\left([0, T]\,;\; L^{\frac{d}{d+\gamma}}(\mathbb{R}^d)\right)$$

for some $T > 0$. Assume moreover that

$$f_{\text{in}} \in L^1_{\nu_{\gamma,p}}(\mathbb{R}^d), \qquad p \in \left(1, \frac{d}{d+\gamma}\right),$$

with $\nu_{\gamma,p}$ defined by (3.5). Then, for any $t \in (0, T]$,

$$\|f(t)\|_p \leqq C_{\gamma,p}\, t^{-\frac{d}{2}(1-\frac{1}{p})} \sup_{\tau \in [0,T]} \boldsymbol{m}_{\nu_{\gamma,p}}(\tau)\,, \tag{3.12}$$

where

$$C_{\gamma,p} = C_{\gamma,p}(T) = \left[\frac{4\,K_0}{dpC_{\text{Sob}}}\right]^{-\frac{d}{2}(1-\frac{1}{p})}$$
$$\exp\left(\frac{\gamma^2}{p^2}K_0(p-1)T + C_{d,\gamma,p}\left(1 - \frac{1}{p}\right)\int_0^T \|f(\tau)\|_{\frac{d}{d+\gamma}}\,\mathrm{d}\tau\right),$$

K_0 is the constant (depending on ϱ_{in}, E_{in} and $H(f_{\text{in}})$) appearing in Remark 1.13, C_{Sob} is the constant appearing in the Sobolev embedding $\dot{\mathbb{H}}^1(\mathbb{R}^d) \hookrightarrow L^{\frac{2d}{d-2}}(\mathbb{R}^d)$ (see (2.5)), and $C_{d,\gamma.p}$ is the positive constant (depending on d, γ and p) appearing in Proposition 3.2.

Proof. We already observed (see (3.4)) that, under assumption (3.2),

$$\int_{\mathbb{R}^d} \boldsymbol{c}_\gamma[f(t)](v) f^p(t, v)\,\mathrm{d}v \leqq \Lambda_0(t)\,\|f(t)\|_p^p = \Lambda_0(t)\mathbb{M}_p(t),$$

where $\Lambda_0(t) := C_{d,\gamma,p}\|f(t)\|_{\frac{d}{d+\gamma}}$ and $1 < p < \frac{d}{d+\gamma}$. Inserting this in the evolution of $\mathbb{M}_p(t)$ given in (3.6), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}_p(t) + \frac{2K(p)}{C_{\text{Sob}}}\boldsymbol{m}_{\nu_{\gamma,p}}(t)^{-\frac{2p}{d(p-1)}}\,\mathbb{M}_p(t)^{1+\frac{2}{d(p-1)}}$$
$$\leqq \Big(K(p)\gamma^2 + (p-1)\Lambda_0(t)\Big)\,\mathbb{M}_p(t), \tag{3.13}$$

with $\Lambda_0 \in L^1([0, T])$. As in the proof of Proposition 2.2, we can now define

$$y(t) := \mathbb{M}_p(t)\,\exp\left(-\int_0^t \Big[K(p)\gamma^2 + (p-1)\Lambda_0(s)\Big]\,\mathrm{d}s\right),$$

and we see that $y(t)$ satisfies the differential inequality

$$y'(t) \leqq -\frac{2\,K(p)}{C_{\mathrm{Sob}}}\,\boldsymbol{m}_{\nu_{\gamma,p}}(t)^{-\frac{2p}{d(p-1)}}\;y(t)^{1+\frac{2}{dp-d}},$$

where we used that $y(t) \leqq \mathbb{M}_p(t)$. This inequality can be solved (for $t \in [0,T]$) as

$$y(t) \leqq \left[\frac{4\,K(p)}{d(p-1)\,C_{\mathrm{Sob}}}\left(\sup_{\tau\in[0,T]} \boldsymbol{m}_{\nu_{\gamma,p}}(\tau)\right)^{-\frac{2p}{d(p-1)}} t\right]^{-\frac{d(p-1)}{2}}.$$

This proves the result. □

Remark 3.5. The aforementioned criterion (3.2) is always met in the case of moderately soft potentials $\gamma \in [-2,0)$, see Section 5.1.

3.2. Proof of the ε-Poincaré Inequality

For the Prodi–Serrin criteria with $r > 1$, by virtue of the proof of the above Proposition, the crucial point for the appearance of L^p-moments lies in a suitable estimate for

$$\int_{\mathbb{R}^d} \boldsymbol{c}_\gamma[f(t)](v)\,f^p(t,v)\mathrm{d}v.$$

To deal with such a term, one resorts to the ε-Poincaré inequality stated in Proposition 1.7. We first give the proof of this fundamental result

Proof of Proposition 1.7. We start with the case $-d < \gamma < 0$. For a given $g, \phi \geqq 0$, we define

$$I[\phi] := \int_{\mathbb{R}^d} \phi^2 \boldsymbol{c}_\gamma[g]\mathrm{d}v = (d-1)\,(\gamma+d)\int_{\mathbb{R}^d\times\mathbb{R}^d} |v-v_*|^\gamma \phi^2(v)g(v_*)\mathrm{d}v\mathrm{d}v_*.$$

For any $v, v_* \in \mathbb{R}^d$, if $|v-v_*| < \frac{1}{2}\langle v\rangle$, then $\langle v\rangle \leqq 2\langle v_*\rangle$. We deduce from this (see) [2, Eq. (2.5)], that

$$|v-v_*|^\gamma \leqq 2^{-\gamma}\langle v\rangle^\gamma\left(\mathbf{1}_{\left\{|v-v_*|\geqq\frac{\langle v\rangle}{2}\right\}} + \langle v_*\rangle^{-\gamma}|v-v_*|^\gamma\mathbf{1}_{\left\{|v-v_*|<\frac{\langle v\rangle}{2}\right\}}\right).$$

Therefore,

$$I[\phi] \leqq 2^{-\gamma}(d-1)\,(\gamma+d)\,(I_1+I_2)\,, \tag{3.14}$$

with

$$I_1 := \int_{\mathbb{R}^d}\langle v\rangle^\gamma\phi^2(v)\mathrm{d}v\int_{|v-v_*|\geqq\frac{\langle v\rangle}{2}} g(v_*)\mathrm{d}v_*,$$

$$\text{and}\qquad I_2 := \int_{\mathbb{R}^d}\langle v_*\rangle^{-\gamma}g(v_*)\mathrm{d}v_*\int_{|v-v_*|<\frac{1}{2}\langle v\rangle}|v-v_*|^\gamma\langle v\rangle^\gamma\phi^2(v)\mathrm{d}v.$$

Introducing

$$F = \langle\cdot\rangle^{-\gamma} g, \qquad \psi = \langle\cdot\rangle^{\frac{\gamma}{2}}\phi,$$

one checks that

$$I_1 \leqq \|g\|_1 \|\psi^2\|_1,$$

while

$$I_2 \leqq \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - v_*|^\gamma F(v_*)\psi^2(v)\mathrm{d}v\mathrm{d}v_*.$$

We estimate then I_2 thanks to Hardy-Littlewood-Sobolev inequality (3.1) to get

$$I_2 \leqq C\,\|F\|_q\,\|\psi^2\|_r = C\,\|F\|_q\,\|\psi\|_{2r}^2, \qquad \frac{1}{r} = 2 - \frac{1}{q} + \frac{\gamma}{d} \quad (1 < q, r < \infty),$$

where C depends only on γ, d, q.

We recall that we apply this with $\frac{d}{d+2+\gamma} < q < \frac{d}{d+\gamma}$. We write, for some $s \in (0, 1)$,

$$q = \frac{d}{d + \gamma + 2s}, \qquad r = \frac{d}{d - 2s}. \tag{3.15}$$

Notice that $2r = \frac{2d}{d-2s}$. According for example to Theorem 1.38 of Bahouri et al. [10], $\dot{\mathbb{H}}^s$ is continuously embedded in L^{2r}, that is (for $C > 0$ depending on d, s),

$$I_2 \leqq C\|F\|_q\|\psi\|_{\dot{\mathbb{H}}^s}^2.$$

Moreover, since

$$\|\psi\|_{\dot{\mathbb{H}}^s} \leqq \|\psi\|_{\dot{\mathbb{H}}^1}^s\,\|\psi\|_2^{1-s},$$

(see for example) [10, Proposition 1.32], one has that

$$I_2 \leqq C\|F\|_q\|\psi\|_{\dot{\mathbb{H}}^1}^{2s}\,\|\psi\|_2^{2-2s}.$$

Thanks to Young's inequality, there is $\tilde{C} > 0$ depending only on d, γ, s such that, for any $\delta > 0$,

$$2^{-\gamma}(d-1)\,(\gamma+d)I_2 \leqq \delta\,\|\psi\|_{\dot{\mathbb{H}}^1}^2 + \tilde{C}\|F\|_q^{\frac{1}{1-s}}\,\delta^{-\frac{s}{1-s}}\|\psi\|_2^2\,.$$

Plugging this inequality into the estimate for I_1, we see that

$$I[\phi] \leqq \delta\,\|\psi\|_{\dot{\mathbb{H}}^1}^2 + C\left(\|g\|_1 + \delta^{-\frac{s}{1-s}}\|F\|_q^{\frac{1}{1-s}}\right)\|\psi\|_2^2,$$

which is exactly the desired estimate, since $F = \langle\cdot\rangle^{|\gamma|}g$.

We now turn to the case of when $\gamma = -d$. One notices that

$$\int_{\mathbb{R}^d}\phi^2\boldsymbol{c}_{-d}[g]\mathrm{d}v = c_d\int_{\mathbb{R}^d}\phi^2(v)g(v)\mathrm{d}v = c_d\int_{\mathbb{R}^d}F\psi^2\mathrm{d}v.$$

Thus, a simple use of Hölder's inequality yields

$$\int_{\mathbb{R}^d}\phi^2\boldsymbol{c}_{-d}[g]\mathrm{d}v \leqq c_d\|F\|_q\|\psi\|_{2r}^2, \qquad \frac{1}{r} + \frac{1}{q} = 1.$$

Writing for $s \in (0, 1)$ that $q = \frac{d}{2s}$, we see that $r = \frac{d}{d-2s}$, and we proceed identically as to the case $-d < \gamma < 0$. $\square$

3.3. Appearance of L^p-Norms–General $L^r_t(L^q)$ Case

We now turn to the general Prodi–Serrin criterion in the $L^r_t(L^q_v)$ case, with $r > 1$. In such a situation, L^p-norms appear for *any* choice of $p > 1$. This contrasts with the previous case in which appearance of L^p norms occurred only for $p < \frac{d}{d+\gamma}$. The following is the analogue of Proposition 2.5 for $-d < \gamma < 0$:

Proposition 3.6. *Let f_{in} and $f = f(t, v)$ define a solution to Eq. (1.5) in the sense of Definition 1.15 with $-d < \gamma < 0$. Assume that*

$$\langle\cdot\rangle^{|\gamma|} f \in L^r([0,T];\, L^q(\mathbb{R}^d)) \quad \text{with} \quad \frac{2}{r} + \frac{d}{q} = 2 + d + \gamma \tag{3.16}$$

for some $T > 0$, where $r \in (1, \infty)$, $q \in \left(\max\left(1, \frac{d}{2+d+\gamma}\right), \frac{d}{d+\gamma}\right)$. Given $p \in (1, \infty)$, assume that

$$f_{\mathrm{in}} \in L^1_{\nu_{\gamma,p}}(\mathbb{R}^d), \qquad \nu_{\gamma,p} := \frac{|\gamma| d}{2}\left(1 - \frac{1}{p}\right).$$

Then, the solution $f = f(t, v)$ satisfies, for any $t \in (0, T]$,

$$\|f(t)\|_p \leqq K_{p,\gamma,q}\, t^{-\frac{d}{2}(1-\frac{1}{p})} \sup_{\tau \in [0,T]} \boldsymbol{m}_{\nu_{\gamma,p}}(\tau)\,, \tag{3.17}$$

where

$$K_{p,\gamma,q} := \left[\frac{2K_0}{dp\, C_{\mathrm{Sob}}}\right]^{-\frac{d}{2}(1-\frac{1}{p})} \exp\left(\frac{p-1}{p}\left[\frac{\gamma^2}{p} K_0 + C_0\|f_{\mathrm{in}}\|_1\right] T\right)$$
$$\exp\left(\frac{p-1}{p}\, C_0 \left(\frac{K_0}{p}\right)^{r-1} \int_0^T \left\|\langle\cdot\rangle^{|\gamma|} f(\tau)\right\|_q^r \mathrm{d}\tau\right),$$

and where K_0 is the constant (depending on ϱ_{in}, E_{in} and $H(f_{\mathrm{in}})$) appearing in Remark 1.13, C_{Sob} is the constant appearing in the Sobolev embedding $\dot{\mathbb{H}}^1(\mathbb{R}^d) \hookrightarrow L^{\frac{2d}{d-2}}(\mathbb{R}^d)$ (see (2.5)), and C_0 is the positive constant (depending on d, γ, q) appearing in the ε-Poincaré inequality (1.10).

Proof. The Prodi–Serrin condition (3.16) amounts to

$$\int_0^T \|f(t)\|^r_{L^q_{q|\gamma|}} \mathrm{d}t < \infty, \qquad \frac{d}{q} + \frac{2}{r} = d + 2 + \gamma, \qquad 1 < r < \infty.$$

Notice that the assumption on q implies that $\frac{d}{d+2+\gamma} < q < \frac{d}{d+\gamma}$, and therefore, for some $s \in (0, 1)$,

$$q = \frac{d}{d + 2s + \gamma},$$

and the relation between r and s is $r = \frac{1}{1-s}$.

We recall the evolution of L^p-norms given by (3.9):

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}_p(t)+2K(p)\mathbb{D}_{\gamma,p}(t)\leqq K(p)\gamma^2\mathbb{M}_p(t)+(p-1)\int_{\mathbb{R}^d}\boldsymbol{c}_\gamma[f(t)](v)\,f^p(t,v)\mathrm{d}v.$$

We also estimate the last integral thanks to an application of the ε-Poincaré inequality (1.10) with $g=f(t,\cdot)$, $\phi=f(t,\cdot)^{\frac{p}{2}}$. This yields

$$\int_{\mathbb{R}^d}\boldsymbol{c}_\gamma[f(t)](v)\,f^p(t,v)\mathrm{d}v\ \leqq\varepsilon\,\mathbb{D}_{\gamma,p}(t)$$
$$+C_0\left(\|f(t)\|_1+\varepsilon^{-\frac{s}{1-s}}\|\langle\cdot\rangle^{|\gamma|}f(t)\|_q^{\frac{1}{1-s}}\right)\int_{\mathbb{R}^d}f(t,v)^p\langle v\rangle^\gamma\mathrm{d}v.$$

Choosing ε in such a way that $(p-1)\,\varepsilon=K(p)$, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}_p(t)+K(p)\mathbb{D}_{\gamma,p}(t)\leqq K(p)\gamma^2\mathbb{M}_p(t)+(p-1)\left(C_0\|f_{\mathrm{in}}\|_1+\boldsymbol{\Lambda}(t)\right)$$
$$\int_{\mathbb{R}^d}f(t,v)^p\langle v\rangle^\gamma\mathrm{d}v,$$

with

$$\boldsymbol{\Lambda}(t):=C_0\varepsilon^{-\frac{s}{s-1}}\|\langle\cdot\rangle^{|\gamma|}f(t)\|_q^{\frac{1}{1-s}}=C_0\left(\frac{K_0}{p}\right)^{r-1}\|\langle\cdot\rangle^{|\gamma|}f(t)\|_q^r\in L^1([0,T])$$

(the last point is precisely the Prodi–Serrin assumption (3.16)). Dropping the weight $\langle v\rangle^\gamma$ in the last integral, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}_p(t)+K(p)\mathbb{D}_{\gamma,p}(t)\leqq\left[K(p)\gamma^2+(p-1)C_0\|f_{\mathrm{in}}\|_1+(p-1)\boldsymbol{\Lambda}(t)\right]\mathbb{M}_p(t).\quad(3.18)$$

At this point, using again, (3.11) as in the proof of Proposition 3.3, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}_p(t)+\frac{K(p)}{C_{\mathrm{Sob}}}\boldsymbol{m}_{\nu_{\gamma,p}}(t)^{-\frac{2p}{d(p-1)}}\mathbb{M}_p(t)^{1+\frac{2}{d(p-1)}}$$
$$\leqq\left[K(p)\gamma^2+(p-1)C_0\|f_{\mathrm{in}}\|_1+(p-1)\boldsymbol{\Lambda}(t)\right]\mathbb{M}_p(t).$$

The conclusion follows exactly as in the proof of Proposition 3.4 (which corresponds to the case $s=0$). □

As far as propagation of L^p-norms is concerned, we can also deduce the following:

Corollary 3.7. *Let f_{in} and $f=f(t,v)$ define a solution to Eq. (1.5) in the sense of Definition 1.15 with $-d<\gamma<0$. Assume that (3.16) holds true for some $T>0$, where $r\in(1,\infty)$, $q\in\left(\max\left(1,\frac{d}{2+d+\gamma}\right),\frac{d}{d+\gamma}\right)$. Given $p\in(1,\infty)$, assume that $f_{\mathrm{in}}\in L^p(\mathbb{R}^d)$. Then, the solution $f=f(t,v)$ belongs to $L^\infty([0,T],\,L^p(\mathbb{R}^d))$. More precisely,*

$$\|f\|_{L^\infty([0,T];L^p(\mathbb{R}^d))}\leqq\|f_{\mathrm{in}}\|_p\exp\left(\frac{p-1}{p}\left[\frac{\gamma^2}{p}K_0+C_0\|f_{\mathrm{in}}\|_1\right]T\right)$$

$$\exp\left(\frac{p-1}{p}C_0\left(\frac{K_0}{p}\right)^{r-1}\int_0^T\left\|\langle\cdot\rangle^{|\gamma|}f(\tau)\right\|_q^r\mathrm{d}\tau\right),\tag{3.19}$$

where K_0 is the constant (depending on $\varrho_{\rm in}$, $E_{\rm in}$ and $H(f_{\rm in})$) appearing in Remark 1.13, and C_0 is the positive constant (depending on d,γ,q) appearing in the ε-Poincaré inequality (1.10).

Proof. Coming back to (3.18) (notice that no assumption on the finiteness of L^1-moment is needed to get this inequality), dropping the nonnegative term $K(p)\mathbb{D}_p(t)$ on the right-hand-side, we deduce from Grönwall's Lemma that

$$\mathbb{M}_p(t)\leqq\exp\left\{\left(K(p)\gamma^2+(p-1)C_0\|f_{\rm in}\|_1\right)t+(p-1)\int_0^t\mathbf{\Lambda}(\tau)\mathrm{d}\tau\right\}\mathbb{M}_p(0),$$

from which the result follows. □

Proof of Theorem 1.4. It is a direct consequence of Proposition 3.2 and Proposition 3.4 when the $L^1_t(L^\infty_v)$ condition is considered, and of Proposition 3.6 and Corollary 3.7 when the $L^r_t(L^q_v)$ condition is considered. □

4. Stability and Uniqueness of Solution

We adapt here the strategy proposed in Chern and Gualdani [13] to deduce uniqueness of solutions. Notice that our proof covers all cases $\gamma\in[-d,0)$ and $d\geqq 3$ whereas the uniqueness result in Chern and Gualdani [13] is given in dimension $d=3$ for the Coulomb case $\gamma=-3$.

Our strategy is inspired by the work Chern and Gualdani [13] in the sense that we deduce the stability from suitable $L^\infty([0,T];\ L^p_k(\mathbb{R}^d))$ estimates on the solution to (1.5). We show first that such uniform in time estimates, which are variants of the estimates obtained in Sections 2 and 3, hold true under our Prodi–Serrin conditions. The result is a consequence of the study of the evolution of weighted L^p-norms provided in Appendix B. More precisely, we show

Proposition 4.1. *Let $f_{\rm in}$ and $f=f(t,v)$ define a solution to Eq.* (1.5) *in the sense of Definition 1.15 on* $[0,T]$ *(for some $T>0$) with $d\in\mathbb{N}$, $d\geqq 3$, $-d\leqq\gamma<0$. Let $k\geqq 0$. We consider the following two alternative assumptions (with the convention $\frac{d}{d+\gamma}=\infty$ if $\gamma=-d$):*

Hyp. 1. *$f_{\rm in}\in L^p_k(\mathbb{R}^d)$ for some $1<p<\frac{d}{d+\gamma}$ and*

$$f\in L^1\left([0,T];\ L^{\frac{d}{d+\gamma}}(\mathbb{R}^d)\right)\quad .$$

Hyp. 2. *$f_{\rm in}\in L^p_k(\mathbb{R}^d)$ for some $p>1$ and*

$$\langle\cdot\rangle^{|\gamma|}f\in L^r([0,T];\ L^q(\mathbb{R}^d))\quad\text{with}\quad\frac{2}{r}+\frac{d}{q}=2+d+\gamma,$$

where $r\in(1,\infty)$, $q\in\left(\max\left(1,\frac{d}{2+d+\gamma}\right),\frac{d}{d+\gamma}\right)$. Moreover, in the Coulomb case in dimension $d=3=-\gamma$, we assume $f_{\rm in}\in L^1_s(\mathbb{R}^d)$ for some $s>2$.

Then, under **Hyp. 1** *or* **Hyp. 2***, and for any $t \in [0, T]$, one has*

$$\int_{\mathbb{R}^d} f^p(t,v)\langle v\rangle^k \mathrm{d}v + K_0\left(1-\frac{1}{p}\right)\int_0^t \mathrm{d}\tau \int_{\mathbb{R}^d}\left|\nabla\left(\langle v\rangle^{\frac{k+\gamma}{2}} f^{\frac{p}{2}}(\tau,v)\right)\right|^2 \mathrm{d}v \leqq \boldsymbol{C}(T, f_{\mathrm{in}}), \quad (4.1)$$

where $\boldsymbol{C}(T, f_{\mathrm{in}})$ is an explicit positive constant depending on d, γ, p, k, T, K_0, $\|f_{\mathrm{in}}\|_{L^1_2}$, $H(f_{\mathrm{in}})$, $\|f_{\mathrm{in}}\|_{L^p_k}$ and the Prodi–Serrin norms $\|f\|_{L^1_t(L^{\frac{d}{d+\gamma}})}$ or $\left\|\langle\cdot\rangle^{|\gamma|} f\right\|_{L^r_t(L^q)}$ (and q), according to the considered case (and on $\|f_{\mathrm{in}}\|_{L^1_s}$ $s > 2$ in the Coulomb case in dimension $d = 3$).

If one moreover assumes in Assumptions **Hyp. 1** *or* **Hyp. 2** *that $p > \frac{d}{d+\gamma+2}$ and*

$$f_{\mathrm{in}} \in L^1_\nu(\mathbb{R}^d), \qquad \nu > \overline{\nu}$$

with

$$\overline{\nu} := \frac{d(p-1)|\gamma| - k(\gamma+2)^-}{d(p-1) - p(\gamma+2)^-}\mathbf{1}_{k\leqq p|\gamma|} + \frac{(d(p-1)-k)|\gamma|}{d(p-1)+p\gamma}\mathbf{1}_{k>p|\gamma|},$$

where $a^- = -a\mathbf{1}_{a<0}$, then, for any smooth ϕ, the following version of the ε-Poincaré inequality is valid for any $t \in [0, T]$ and any $\varepsilon > 0$:

$$\int_{\mathbb{R}^d}\phi^2(v)\boldsymbol{c}_\gamma[f(t)]\mathrm{d}v \leqq \varepsilon\int_{\mathbb{R}^d}\left|\nabla\left(\langle v\rangle^{\frac{\gamma}{2}}\phi(v)\right)\right|^2 \mathrm{d}v + C_k(T, f_{\mathrm{in}}, \varepsilon)\int_{\mathbb{R}^d}\phi^2(v)\langle v\rangle^\gamma \mathrm{d}v. \quad (4.2)$$

Here $C_k(T, f_{\mathrm{in}}, \varepsilon)$ is an explicit positive constant depending on $\|f_{\mathrm{in}}\|_{L^p_k}$, $\|f_{\mathrm{in}}\|_{L^1_{\max(2,\nu)}}$, ε, d, γ, p, k, T, K_0, and the Prodi–Serrin norms $\|f\|_{L^1_t(L^{\frac{d}{d+\gamma}})}$ or $\left\|\langle\cdot\rangle^{|\gamma|} f\right\|_{L^r_t(L^q)}$ (and q), according to the considered case.

Proof. The proof follows the line of the corresponding results in Proposition 3.4 and Theorem 3.6, whose proofs are modified in order to handle weights. Given $k \geqq 0$, we use the notations of Appendix B and deduce the evolution of the weighted L^p-norms from (B.1), which holds for any $p > 1$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}_{k,p}(t) + 2K(p)\mathbb{D}_{k+\gamma,p}(t) \leqq K(p)(k+\gamma)^2\mathbb{M}_{k+\gamma,p}(t)$$
$$+ C_{k,\gamma,p}\sum_{i=0}^{2}\int_{\mathbb{R}^d}\langle v\rangle^{k-i}\boldsymbol{c}_{\gamma+i}[f(t)](v)\, f^p(t,v)\mathrm{d}v. \quad (4.3)$$

Here $C_{k,\gamma,p}$ depends on d, k, γ, p (the explicit value of the constant is given in Appendix B).

We divide the proof according to the two cases $\gamma = -d$ or $-d < \gamma < 0$.

1) The Coulomb case. Let us begin with the Coulomb case $\gamma = -d$. Thanks to (B.5), we deduce now from (4.3) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}_{k,p}(t) + 2K(p)\mathbb{D}_{k-d,p}(t) \leqq \boldsymbol{c}_{k,p}\mathbb{M}_{k,p}(t) + \boldsymbol{C}_{k,p}$$

$$\int_{\mathbb{R}^d} \langle v\rangle^{k-1}\overline{\boldsymbol{c}}_{1-d}[f(t)](v) f^p(t,v)\mathrm{d}v$$
$$+C_{k,p}\int_{\mathbb{R}^d} \langle v\rangle^k f^{p+1}(t,v)\mathrm{d}v, \tag{4.4}$$

where $\boldsymbol{C}_{k,p}$, $\boldsymbol{c}_{k,p}$, $C_{k,p}$ are explicit positive constants depending only on p, d, k, K_0, $\|f_{\text{in}}\|_{L^1}$ and where, for all $\alpha \in (-d, 0)$

$$\overline{\boldsymbol{c}}_\alpha[f(t)](v) := (d-1)(d+\alpha)\int_{|v-v_*|\leqq 1} |v-v_*|^\alpha f(t,v_*)\mathrm{d}v_*.$$

We estimate the last two integrals in (4.4) in different ways according to the Prodi–Serrin conditions that we consider.

• ***Hyp. 1.*** First, one has $\int_{\mathbb{R}^d}\langle v\rangle^k f^{p+1}(t,v)\mathrm{d}v \leqq \|f(t)\|_\infty \mathbb{M}_{k,p}(t)$. Second,

$$\overline{\boldsymbol{c}}_{1-d}[f(t)](v) = (d-1)\int_{|v-v_*|\leqq 1} |v-v_*|^{1-d} f(t,v_*)\mathrm{d}v_* \leqq (d-1)\|f(t)\|_\infty |\mathbb{S}^{d-1}| \tag{4.5}$$

so that

$$\int_{\mathbb{R}^d}\langle v\rangle^{k-1}\overline{\boldsymbol{c}}_{1-d}[f(t)](v) f^p(t,v)\mathrm{d}v \leqq (d-1)|\mathbb{S}^{d-1}|\|f(t)\|_\infty \mathbb{M}_{k-1,p}(t).$$

Therefore, (4.4) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}_{k,p}(t) + 2K(p)\mathbb{D}_{k-d,p}(t) \leqq \boldsymbol{C}(k,p,d)\|f(t)\|_\infty \mathbb{M}_{k,p}(t)$$

and, since $\|f(\cdot)\|_\infty \in L^1([0,T])$ by assumption, we deduce from Grönwall's Lemma the bound

$$\mathbb{M}_{k,p}(t) + 2K(p)\int_0^t \mathbb{D}_{k-d,p}(s)\exp\left(\boldsymbol{C}(k,p,d)\int_s^t \|f(\tau)\|_\infty \mathrm{d}\tau\right) ds$$
$$\leqq \mathbb{M}_{k,p}(0)\exp\left(\boldsymbol{C}(k,p,d)\int_0^t \|f(\tau)\|_\infty \mathrm{d}\tau\right)$$

which gives the result.

• ***Hyp. 2.*** In this second case, we observe that the term $\int_{\mathbb{R}^d}\langle v\rangle^k f^{p+1}(t,v)\mathrm{d}v$ is identical (up to a constant) to $\int_{\mathbb{R}^d}\langle v\rangle^k \boldsymbol{c}_{-d}[f(t)](v) f^p(t,v)\mathrm{d}v$ and is estimated using the Coulomb version of the ε-Poincaré inequality (1.10), with $\phi = \langle\cdot\rangle^{\frac{k}{2}} f^{\frac{p}{2}}$ and $g = f$, leading to

$$C_{k,p}\int_{\mathbb{R}^d}\langle v\rangle^k f^{p+1}(t,v)\mathrm{d}v \leqq \varepsilon\mathbb{D}_{k-d,p}(t) + C_\varepsilon \|\langle\cdot\rangle^d f(t)\|_q^{\frac{2q}{2q-d}}\mathbb{M}_{k-d,p}(t), \tag{4.6}$$

where $q > \frac{d}{2}$ is given by $\frac{2}{r} + \frac{d}{q} = 2$ (recall we consider here $\gamma = -d$) and $r = \frac{2q}{2q-d}$.

To estimate the term

$$\int_{\mathbb{R}^d} \langle v\rangle^{k-1}\overline{\boldsymbol{c}}_{1-d}[f(t)](v) f^p(t,v)\mathrm{d}v,$$

we have to additionally distinguish the two cases $d=3$, $d \geqq 4$. We give here only the proof in the case $d=3$ and refer to [6, Proof of Prop. 1.8, $d \geqq 4$] for the other case. We point out already that the use that we will make of Proposition 4.1 in the subsequent stability result is actually restricted to the case $d=3$.

For $d=3$, one has $\overline{\boldsymbol{c}}_{1-d}[f(t)] = \overline{\boldsymbol{c}}_{-2}[f(t)] \leqq \boldsymbol{c}_{-2}[f(t)]$ and, resorting to the critical ε-Poincaré inequality in Alonso et al. [5] (see Proposition A.10 in Appendix A), one has the following: for any $\alpha > 2$ and any $\varepsilon > 0$,

$$\begin{aligned}
&\int_{\mathbb{R}^3} \langle v\rangle^{k-1}\overline{\boldsymbol{c}}_{-2}[f(t)](v)\, f^p(t,v)\mathrm{d}v \\
&\quad \leqq \varepsilon\, \mathbb{D}_{k-3,p}(t) + C_0 \exp\left(\varepsilon^{-\frac{\alpha}{\alpha-2}}\, \boldsymbol{m}_\alpha(f(t))^{\frac{\alpha}{\alpha-2}}\right)\int_{\mathbb{R}^3} f(t,v)^p\langle v\rangle^{k-3}\mathrm{d}v
\end{aligned}$$

for some $C_0 > 0$ depending only on ϱ_{in}, E_{in} and $H(f_{\mathrm{in}})$. In particular, assuming $f_{\mathrm{in}} \in L^1_\alpha(\mathbb{R}^3)$ and $\alpha > 2$, one deduces that there exists $C_T > 0$ depending only on $\|f_{\mathrm{in}}\|_{L^1_\alpha}$, $H(f_{\mathrm{in}})$ such that

$$\begin{aligned}
\boldsymbol{C}_{k,p}\int_{\mathbb{R}^d} \langle v\rangle^{k-1}\overline{\boldsymbol{c}}_{1-d}[f(t)](v)\, f^p(t,v)\mathrm{d}v &\leqq \varepsilon\, \mathbb{D}_{k-d,p}(t) \\
&+ \exp\left(C_T\left(1+\varepsilon^{-\frac{\alpha}{\alpha-2}}\right)\right)\mathbb{M}_{k-d,p}(t) \quad (d=3).
\end{aligned} \tag{4.7}$$

Plugging (4.7), (4.6) into (4.4) and choosing $\varepsilon = \varepsilon_0 > 0$ small enough, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}_{k,p}(t) + K(p)\mathbb{D}_{k-d,p}(t) \leqq \Lambda(t)\,\mathbb{M}_{k,p}(t), \qquad (d=3)$$

where we introduced, for the specific choice of $\varepsilon = \varepsilon_0$,

$$\Lambda(t) = \exp\left(C_T(1+\varepsilon_0^{-\frac{\alpha}{\alpha-2}})\right) + C_{\varepsilon_0}\|\langle\cdot\rangle^d f(t)\|_q^{\frac{2q}{2q-d}} + \boldsymbol{c}_{k,p}.$$

By assumption, $\Lambda \in L^1([0,T])$ and one concludes as before by a Grönwall argument.

2) The case $-d < \gamma < 0$. Combining Eq. (4.3) with Lemma B.2 (and modifying the name of the constant appearing in this lemma), we deduce then that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}_{k,p}(t) + 2K(p)\mathbb{D}_{k+\gamma,p}(t) \leqq \boldsymbol{c}_{k,p}\mathbb{M}_{k,p}(t) + \boldsymbol{C}_{k,p}\int_{\mathbb{R}^d}\langle v\rangle^k\overline{\boldsymbol{c}}_\gamma[f(t)](v)f^p(t,v)\mathrm{d}v\,, \tag{4.8}$$

where $\boldsymbol{C}_{k,p}$, $\boldsymbol{c}_{k,p}$ are explicit positive constants depending only on k, p, d, γ, K_0. As before, we estimate the last integral in different ways according to the Prodi–Serrin conditions that we consider.

Hyp. 1. According to Peetre's inequality, for all $v, v_* \in \mathbb{R}^d$ and $s \in \mathbb{R}$, $\langle v\rangle^s \leqq 2^{\frac{|s|}{2}}\langle v-v_*\rangle^{|s|}\langle v_*\rangle^s$, so that (for some $C_s > 0$ depending on s)

$$\begin{aligned}&\int_{\mathbb{R}^d}\langle v\rangle^k \overline{\boldsymbol{c}}_\gamma[f(t)](v) f^p(t,v)\mathrm{d}v\\ &\leqq C_s\int_{|v-v_*|\leqq 1}|v-v_*|^\gamma\langle v_*\rangle^s f(t,v_*)\langle v\rangle^{k-s} f^p(t,v)\mathrm{d}v\mathrm{d}v_*\end{aligned}$$

for any $0 \leqq s \leqq k$. Using the Hardy-Littlewood-Sobolev inequality (3.1) with $g = \langle\cdot\rangle^s f(t,\cdot)$, $h(\cdot) = \langle\cdot\rangle^{k-s} f^p(t,\cdot)$, so that

$$\frac{1}{r} = 2 + \frac{\gamma}{d} - \frac{1}{p} < 1,$$

and we conclude that (for some $C_{d,\gamma,p,s} > 0$ depending on d, γ, p, s)

$$\begin{aligned}\int_{\mathbb{R}^d}\langle v\rangle^k \overline{\boldsymbol{c}}_\gamma[f(t)](v) f^p(t,v)\mathrm{d}v &\leqq C_{d,\gamma,p,s}\|\langle\cdot\rangle^s f(t)\|_p\|\langle\cdot\rangle^{k-s} f^p(t)\|_r\\ &= C_{d,\gamma,p,s}\|\langle\cdot\rangle^s f(t)\|_p\|\langle\cdot\rangle^{\frac{k-s}{p}} f(t)\|_{pr}^p.\end{aligned}$$

Writing $q_0 = \frac{d}{d+\gamma}$ and recalling that $p < \frac{d}{d+\gamma}$, we use an interpolation based on Hölder's inequality, namely

$$\|\langle\cdot\rangle^{\frac{k-s}{p}} f(t)\|_{pr}^p \leqq \|\langle\cdot\rangle^a f(t)\|_p^{p\theta}\|\langle\cdot\rangle^b f(t)\|_{q_0}^{p(1-\theta)},$$

with

$$\frac{1}{pr} = \frac{\theta}{p} + \frac{1-\theta}{q_0}, \qquad \theta = \frac{q_0 - pr}{r(q_0-p)}, \qquad \frac{k-s}{p} = a\theta + b(1-\theta).$$

Observing that $\frac{1}{r} = \frac{1}{q_0} + 1 - \frac{1}{p}$, one actually has $\theta = 1 - \frac{1}{p}$, so that $p(1-\theta) = 1$ and

$$\int_{\mathbb{R}^d}\langle v\rangle^k \overline{\boldsymbol{c}}_\gamma[f(t)](v) f^p(t,v)\mathrm{d}v \leqq C_{d,\gamma,p,s}\,\|\langle\cdot\rangle^s f(t)\|_p\|\langle\cdot\rangle^a f(t)\|_p^{p-1}\|\langle\cdot\rangle^b f(t)\|_{q_0}.$$

Choosing $s = a = \frac{k}{p}$ and observing that $b = 0$ and $\|\langle\cdot\rangle^s f(t)\|_p^p = \mathbb{M}_{k,p}(t)$, we see that

$$\int_{\mathbb{R}^d}\langle v\rangle^k \overline{\boldsymbol{c}}_\gamma[f(t)](v) f^p(t,v)\mathrm{d}v \leqq \Lambda_0(t)\mathbb{M}_{k,p}(t), \tag{4.9}$$

with

$$\Lambda_0(t) = C_{d,\gamma,p,s}\,\|f(t)\|_{q_0} \in L^1([0,T]).$$

Coming back to (4.8), we end up with the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}_{k,p}(t) + 2K(p)\mathbb{D}_{k+\gamma,p}(t) \leqq \left(\boldsymbol{c}_{k,p} + \boldsymbol{C}_{k,p}\Lambda_0(t)\right)\mathbb{M}_{k,p}(t),$$

so that assuming that $f_{\text{in}} \in L^p_k(\mathbb{R}^d)$ and remembering that $\boldsymbol{c}_p + \boldsymbol{C}_p\Lambda_0(t) \in L^1([0,T])$, we deduce from Grönwall's Lemma the bound

$$\mathbb{M}_{k,p}(t) + 2K(p)\int_0^t \mathbb{D}_{k+\gamma,p}(s)\exp\left(\int_s^t \left(\boldsymbol{c}_{k,p} + \boldsymbol{C}_{k,p}\Lambda_0(\tau)\right)\mathrm{d}\tau\right)\mathrm{d}s$$
$$\leqq \mathbb{M}_{k,p}(0)\exp\left(\int_0^t \left(\boldsymbol{c}_{k,p} + \boldsymbol{C}_{k,p}\Lambda_0(\tau)\right)\mathrm{d}\tau\right),$$

which gives the result.

Hyp. 2. In this second case, observing that $\bar{\boldsymbol{c}}_\gamma[f(t)] \leqq \boldsymbol{c}_\gamma[f(t)]$, we apply the ε-Poincaré inequality (1.10) with $g = f(t,\cdot)$, $\phi = \langle\cdot\rangle^{\frac{k}{2}} f^{\frac{p}{2}}(t,\cdot)$, and we deduce that, when $\varepsilon > 0$,

$$\int_{\mathbb{R}^d} \langle v\rangle^k \bar{\boldsymbol{c}}_\gamma[f(t)](v)\, f^p(t,v)\mathrm{d}v \leqq \varepsilon\,\mathbb{D}_{k+\gamma,p}(t)$$
$$+C\left(\|f(t)\|_1 + \varepsilon^{-\frac{s}{1-s}}\|f(t)\|_{L^q_{q|\gamma|}}^{\frac{1}{1-s}}\right)\int_{\mathbb{R}^d} f(t,v)^p\langle v\rangle^{\gamma+k}\mathrm{d}v,$$

for some positive constant $C > 0$, and where $s = \frac{d-q(d+\gamma)}{2q} \in (0,1)$. Proceeding now as in the proof of Theorem 3.6, choosing ε in such a way that $\boldsymbol{C}_{k,p}\varepsilon = K(p)$, we deduce from (4.8) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}_{k,p}(t) + K(p)\mathbb{D}_{k+\gamma,p}(t) \leqq \left(\boldsymbol{c}_{k,p} + \boldsymbol{C}_{k,p}\Lambda(t)\right)\mathbb{M}_{k,p}(t)\,, \tag{4.10}$$

with (using $C_\varepsilon > 0$ for emphasizing the dependence w.r.t ε)

$$\Lambda(t) := C\|f_{\text{in}}\|_1 + C_\varepsilon\|\langle\cdot\rangle^{|\gamma|}f(t)\|_q^{\frac{1}{1-s}} = C\|f_{\text{in}}\|_1 + C_\varepsilon\|\langle\cdot\rangle^{|\gamma|}f(t)\|_q^r \in L^1([0,T]),$$

(the last point is precisely the assumption of the Proposition). As before, we conclude thanks to Grönwall's Lemma and integration of (4.10). This proves (4.1) in the case $-d < \gamma < 0$ under the two possible assumptions ***Hyp. 1*** or ***Hyp. 2***.

Let us now show how the $L^\infty([0,T], L^p_k(\mathbb{R}^d))$ estimate (4.1) is enough to deduce (4.2). Let $p > \frac{d}{d+\gamma+2}$ be fixed – with the restriction $p < \frac{d}{d+\gamma}$ when we work under ***Hyp. 1***.

Applying the ε-Poincaré inequality (1.10) with $g = f(t,\cdot)$, we deduce as above that (for any suitable ϕ)

$$\int_{\mathbb{R}^d} \boldsymbol{c}_\gamma[f(t)](v)\,\phi^2(v)\mathrm{d}v \leqq \varepsilon\int_{\mathbb{R}^d}\left|\nabla\left(\langle v\rangle^{\frac{\gamma}{2}}\phi(v)\right)\right|^2\mathrm{d}v$$
$$+C\left(\|f(t)\|_1 + \varepsilon^{-\frac{s}{1-s}}\|\langle\cdot\rangle^{|\gamma|}f(t)\|_{\bar{q}}^{\frac{1}{1-s}}\right)\int_{\mathbb{R}^d}\phi^2(v)\langle v\rangle^\gamma\mathrm{d}v,$$

is valid for any $\max\left(1, \frac{d}{d+\gamma+2}\right) < \bar{q} < \frac{d}{d+\gamma}$ with $s = \frac{d-\bar{q}(d+\gamma)}{2\bar{q}}$. Using simple interpolation with $\max\left(1, \frac{d}{d+\gamma+2}\right) < \bar{q} < p$ one has

$$\|\langle\cdot\rangle^{|\gamma|}f(t)\|_{\bar{q}} \leqq \|\langle\cdot\rangle^m f(t)\|_1^{1-\theta_0}\|\langle\cdot\rangle^{\frac{k}{p}}f(t)\|_p^{\theta_0},$$

with

$$\theta_0 = \frac{p}{\bar{q}}\frac{\bar{q}-1}{p-1} \qquad \text{and} \quad m = m(\bar{q}) = \frac{p|\gamma| - \theta_0 k}{p(1-\theta_0)}.$$

Since $m = m(\bar{q}) = \frac{\bar{q}(p-1)|\gamma| - k(\bar{q}-1)}{p-\bar{q}}$, the mapping $\bar{q} \mapsto m(\bar{q})$ is nondecreasing if $p|\gamma| \geqq k$ and nonincreasing if $k \geqq p|\gamma|$. Thus, recalling that $\max\left(1, \frac{d}{d+\gamma+2}\right) < \bar{q} < \frac{d}{d+\gamma}$, one has

$$\overline{\nu} = \inf_{\bar{q}} m(\bar{q}) = \begin{cases} \dfrac{d(p-1)|\gamma| - k(\gamma+2)^-}{d(p-1) - p(\gamma+2)^-} & \text{if } k \leqq p|\gamma|, \\[2ex] \dfrac{(d(p-1)-k)|\gamma|}{d(p-1)+p\gamma} & \text{if } k \geqq p|\gamma|, \end{cases} \tag{4.11}$$

where $a^- := -a\, 1_{a<0}$.

Thus, assuming $f_{\text{in}} \in L^1_\nu(\mathbb{R}^d)$ with $\nu > \overline{\nu}$, one can choose $\max\left(1, \frac{d}{d+\gamma+2}\right) < \bar{q} < \frac{d}{d+\gamma}$ with $m = m(\bar{q}) \leqq \nu$ so that

$$\sup_{t\in[0,T]} \|\langle\cdot\rangle^m f(t)\|_1 < \infty.$$

Then, thanks to (4.1),

$$\sup_{t\in[0,T]} \|\langle\cdot\rangle^{|\gamma|} f(t)\|_{\bar{q}}^{\frac{1}{1-s}} \leqq C_T(f_{\text{in}}),$$

where $C_T(f_{\text{in}})$ depends on $\boldsymbol{C}_T(f_{\text{in}})$ appearing in (4.1) and $\|f_{\text{in}}\|_{L^1_\nu}$. This proves the result. □

With this, we can deduce the following stability result for solutions to Landau equation note that the assumptions on the two considered initial data are not identical:

Proposition 4.2. *Let d be an integer, $d \geqq 3$, and γ be such that $-d \leqq \gamma < -2$ and $\frac{d}{d+\gamma+2} < 2$. Let $f_{\text{in}}, g_{\text{in}}$ and $f = f(t,v)$, $g = g(t,v)$ define two solutions to Eq. (1.5) in the sense of Definition 1.15 with $f(0,\cdot) = f_{\text{in}}$, $g(0,\cdot) = g_{\text{in}}$. We consider one of the following two assumptions (with the convention $\frac{d}{d+\gamma} = \infty$ when $\gamma = -d$):*

Hyp. 1: *there exists $T > 0$ such that $f, g \in L^1([0,T];\ L^{\frac{d}{d+\gamma}}(\mathbb{R}^d))$ and $\frac{d}{d+\gamma} > 2$;*
Hyp. 2: *there exists $T > 0$ such that*

$$\langle\cdot\rangle^{|\gamma|} f;\ \langle\cdot\rangle^{|\gamma|} g \in L^r([0,T];\ L^q(\mathbb{R}^d)), \qquad \textit{with} \quad \frac{2}{r} + \frac{d}{q} = 2 + d + \gamma,$$

where $r \in (1,\infty)$, $q \in \left(\frac{d}{2+d+\gamma}, \frac{d}{d+\gamma}\right)$.

Given $k > d$, assume moreover that

$$f_{\rm in} \in L^2_k(\mathbb{R}^d) \cap L^1_N(\mathbb{R}^d), \quad g_{\rm in} \in L^2_{k+2|\gamma|}(\mathbb{R}^d)$$

with $N > \max\left(\frac{d}{2}, 2, \frac{(d-k)|\gamma|}{d+2\gamma}, \frac{(d-k)|\gamma|+2k}{d+2(\gamma+2)}\right)$. Then, there exists $\boldsymbol{C}_T(f_{\rm in}, g_{\rm in}) > 0$ depending on $d, \gamma, T, k, K_0, \|f_{\rm in}\|_{L^1_N}, \|g_{\rm in}\|_{L^1_2}, H(f_{\rm in}), H(g_{\rm in}), \|f_{\rm in}\|_{L^2_k}, \|g_{\rm in}\|_{L^2_{k+2|\gamma|}}$, and the Prodi–Serrin norms $\|f, g\|_{L^1_t(L^{\frac{d}{d+\gamma}})}$ or $\left\|\langle\cdot\rangle^{|\gamma|} f, \langle\cdot\rangle^{|\gamma|} g\right\|_{L^r_t(L^q)}$ (and q), such that

$$\|f(t) - g(t)\|^2_{L^2_k} \leqq \boldsymbol{C}_T \|f_{\rm in} - g_{\rm in}\|^2_{L^2_k} \qquad \forall t \in [0, T]. \tag{4.12}$$

Remark 4.3. Notice that the result still applies if f and g do not satisfy the *same* hypothesis: it applies if f satisfies **Hyp. 1** and g satisfies **Hyp. 2** or the opposite. We stated the result in the above way for simplicity.

Remark 4.4. We point out that, in the Coulomb case $\gamma = -d$, the restriction $\frac{d}{d+\gamma+2} < 2$ enforces $d = 3$.

Proof. We consider two solutions f, g with initial data $f(0) = f_{\rm in}$ and $g(0) = g_{\rm in}$, and write $h = f - g$. We notice first that, since $f_{\rm in}, g_{\rm in} \in L^2_k(\mathbb{R}^d)$, estimate (4.1) holds true for both f, g with $p = 2$, where we recall that, under Assumption ***Hyp. 1***, we assume $2 < \frac{d}{d+\gamma}$. It ensures that

$$f, g \in L^\infty([0, T]; \; L^2_k(\mathbb{R}^d)).$$

This means that for all $t \in [0, T]$, $h(t, \cdot) \in L^\infty([0, T]; \; L^2_k(\mathbb{R}^d))$. Furthermore, since $g_{\rm in} \in L^2_{k+2|\gamma|}(\mathbb{R}^d)$, we deduce from Proposition 4.1 that

$$\sup_{t \in [0,T]} \int_{\mathbb{R}^d} g^2(t, v)\langle v\rangle^{k+2|\gamma|} \mathrm{d}v + \frac{K_0}{2} \int_0^T \mathrm{d}\tau \int_{\mathbb{R}^d} \left|\nabla\left(\langle v\rangle^{\frac{k+|\gamma|}{2}} g(\tau, v)\right)\right|^2 \mathrm{d}v < \infty. \tag{4.13}$$

We then investigate the evolution of

$$\mathscr{J}(t) := \int_{\mathbb{R}^d} \langle v\rangle^k h^2(t, v) \mathrm{d}v.$$

One that has

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{J}(t) &= 2\int_{\mathbb{R}^d} \langle v\rangle^k h(t, v)\, \partial_t h(t, v) \mathrm{d}v \\
&= -2\int_{\mathbb{R}^d} \mathcal{A}[f]\nabla h \cdot \nabla\left(h\langle v\rangle^k\right) \mathrm{d}v - 2\int_{\mathbb{R}^d} \mathcal{A}[h]\nabla g \cdot \nabla\left(h\langle v\rangle^k\right) \mathrm{d}v \\
&\quad + 2\int_{\mathbb{R}^d} h\, \boldsymbol{b}[f] \cdot \nabla\left(h\langle v\rangle^k\right) \mathrm{d}v + 2\int_{\mathbb{R}^d} g\boldsymbol{b}[h] \cdot \nabla\left(h\langle v\rangle^k\right) \mathrm{d}v \\
&= \mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3 + \mathscr{I}_4.
\end{aligned}$$

As in the proof of Lemma B.1, one has $\mathscr{I}_1 = \mathscr{I}_{1,1} + \mathscr{I}_{1,2}$, with

$$\mathscr{I}_{1,1} = -2\int_{\mathbb{R}^d} \langle v\rangle^k \mathcal{A}[f]\nabla h \cdot \nabla h \mathrm{d}v, \qquad \mathscr{I}_{1,2} = -2k\int_{\mathbb{R}^d} \langle v\rangle^{k-2} h\mathcal{A}[f]\nabla h \cdot v \mathrm{d}v.$$

From the diffusive properties of $\mathcal{A}[f]$,

$$\int_{\mathbb{R}^d} \langle v\rangle^k \mathcal{A}[f]\nabla h \cdot \nabla h \mathrm{d}v \geqq K_0 \int_{\mathbb{R}^d} \langle v\rangle^{k+\gamma} |\nabla h|^2 \mathrm{d}v.$$

Now,

$$\mathscr{I}_{1,2} = k \int_{\mathbb{R}^d} h^2(t,v)\nabla \cdot \left[\mathcal{A}[f]\langle v\rangle^{k-2} v\right] \mathrm{d}v.$$

Therefore, using the reasoning and notations of Lemma B.1

$$\mathscr{I}_{1,2} = k \int_{\mathbb{R}^d} h^2 \langle v\rangle^{k-2} \left(\boldsymbol{b}[f] \cdot v\right) \mathrm{d}v + k \int_{\mathbb{R}^d} \langle v\rangle^{k-4} h^2 \mathrm{Trace}\left(\mathcal{A}[f] \cdot \boldsymbol{A}(v)\right) \mathrm{d}v.$$

Using several integration by parts and using that $\nabla \cdot \boldsymbol{b}[f] = -\boldsymbol{c}_\gamma[f]$, one also shows that

$$\mathscr{I}_3 = \int_{\mathbb{R}^d} \langle v\rangle^k h^2 \boldsymbol{c}_\gamma[f] \mathrm{d}v + k \int_{\mathbb{R}^d} \langle v\rangle^{k-2} h^2 \left(\boldsymbol{b}[f] \cdot v\right) \mathrm{d}v$$

therefore

$$\begin{aligned}\mathscr{I}_{1,2} + \mathscr{I}_3 = k \int_{\mathbb{R}^d} \langle v\rangle^{k-4} h^2 \mathrm{Trace}\left(\mathcal{A}[f] \cdot \boldsymbol{A}(v)\right) \mathrm{d}v + \int_{\mathbb{R}^d} \langle v\rangle^k h^2 \boldsymbol{c}_\gamma[f] \mathrm{d}v \\ +2k \int_{\mathbb{R}^d} \langle v\rangle^{k-2} h^2 \left(\boldsymbol{b}[f] \cdot v\right) \mathrm{d}v.\end{aligned}$$

Using the estimate of $\mathrm{Trace}\left(\mathcal{A}[f] \cdot \boldsymbol{A}(v)\right)$ in Lemma B.1 and estimate (B.4) for estimating the integral involving $\boldsymbol{b}[f]$, we deduce that

$$\begin{aligned}\mathscr{I}_{1,2} + \mathscr{I}_3 \leqq C_{d,k,\gamma} \left(\int_{\mathbb{R}^d} \langle v\rangle^{k-2} h^2 \boldsymbol{c}_{\gamma+2}[f] \mathrm{d}v + \int_{\mathbb{R}^d} \langle v\rangle^{k-1} h^2 \boldsymbol{c}_{\gamma+1}[f] \mathrm{d}v \right) \\ + \int_{\mathbb{R}^d} \langle v\rangle^k h^2 \boldsymbol{c}_\gamma[f] \mathrm{d}v.\end{aligned}$$

Using Lemma B.2, we deduce in the case $-d < \gamma < -2$ that there exist $C = C(d,k,\gamma)$, $c = c(d,k,\gamma) > 0$ (depending on $\|f_{\mathrm{in}}\|_{L^1_2(\mathbb{R}^d)}$) such that

$$\mathscr{I}_{1,2} + \mathscr{I}_3 \leqq C \int_{\mathbb{R}^d} \langle v\rangle^k h^2 \overline{\boldsymbol{c}}_\gamma[f] \mathrm{d}v + c \int_{\mathbb{R}^d} \langle v\rangle^k h^2 \mathrm{d}v.$$

Thus, observing that $\overline{\boldsymbol{c}}_\gamma[f] \leqq \boldsymbol{c}_\gamma[f]$, we get

$$\mathscr{I}_1 + \mathscr{I}_3 \leqq -2K_0 \int_{\mathbb{R}^d} \langle v\rangle^{k+\gamma} |\nabla h|^2 \mathrm{d}v + C \int_{\mathbb{R}^d} \langle v\rangle^k h^2 \boldsymbol{c}_\gamma[f](v) \mathrm{d}v + c \int_{\mathbb{R}^d} \langle v\rangle^k h^2 \mathrm{d}v. \tag{4.14}$$

In the case $\gamma = -d$, Lemma B.2 implies that there exist $C = C(d,k)$, $\tilde{C} = \tilde{C}(d,k)$, $c = c(d,k) > 0$ such that

$$\mathscr{I}_{1,2} + \mathscr{I}_3 \leqq C \int_{\mathbb{R}^d} \langle v\rangle^k h^2 \boldsymbol{c}_{-d}[f] \mathrm{d}v + \tilde{C} \int_{\mathbb{R}^d} \langle v\rangle^{k-1} h^2 \overline{\boldsymbol{c}}_{1-d}[f] \mathrm{d}v + c \int_{\mathbb{R}^d} \langle v\rangle^{k-1} h^2 \mathrm{d}v.$$

Note that (4.5) holds when Assumption ***Hyp. 1*** is in force and implies that

$$\int_{\mathbb{R}^d} \langle v\rangle^{k-1} h^2 \overline{\boldsymbol{c}}_{1-d}[f]\mathrm{d}v \leqq (d-1)|\mathbb{S}^{d-1}|\|f(t)\|_\infty \int_{\mathbb{R}^d} \langle v\rangle^k h^2 \mathrm{d}v.$$

Now, under Assumption ***Hyp. 2***, since $d=3$ (see Remark 4.4), we have $\overline{\boldsymbol{c}}_{1-d}[f(t)] \leqq \boldsymbol{c}_{-2}[f(t)]$ and we deduce from Proposition A.10 that there exist $C_0>0$ such that, for any $\varepsilon>0$ ($N>2$ being defined in the statement of the theorem),

$$\int_{\mathbb{R}^d} \langle v\rangle^{k-1} \overline{\boldsymbol{c}}_{1-d}[f(t)](v)\, h^2(t,v)\mathrm{d}v \leqq \varepsilon \int_{\mathbb{R}^d} \left|\nabla\left(\langle v\rangle^{\frac{k-d}{2}} h(t,v)\right)\right|^2 \mathrm{d}v$$
$$+C_0 \exp\left(\varepsilon^{-\frac{N}{N-2}}\, \boldsymbol{m}_N(f(t))^{\frac{N}{N-2}}\right) \int_{\mathbb{R}^d} h(t,v)^2 \langle v\rangle^{k-d}\mathrm{d}v,$$

Thus, in the case $\gamma=-d$, for any $\varepsilon>0$,

$$\mathscr{I}_1+\mathscr{I}_3 \leqq -2K_0 \int_{\mathbb{R}^d} \langle v\rangle^{k+\gamma} |\nabla h|^2 \mathrm{d}v + \varepsilon \tilde{C} \int_{\mathbb{R}^d} \left|\nabla\left(\langle v\rangle^{\frac{k+\gamma}{2}} h(t,v)\right)\right|^2 \mathrm{d}v$$
$$+C \int_{\mathbb{R}^d} \langle v\rangle^k h^2 \boldsymbol{c}_\gamma[f](v)\mathrm{d}v + \Gamma_\varepsilon(t) \int_{\mathbb{R}^d} \langle v\rangle^k h^2 \mathrm{d}v, \tag{4.15}$$

where $\Gamma_\varepsilon \in L^1([0,T])$.

Finally, gathering (4.14) and (4.15), we obtain for $-d \leqq \gamma < -2$ and for any $\varepsilon>0$,

$$\mathscr{I}_1+\mathscr{I}_3 \leqq -2K_0 \int_{\mathbb{R}^d} \langle v\rangle^{k+\gamma} |\nabla h|^2 \mathrm{d}v + \varepsilon \tilde{C} \int_{\mathbb{R}^d} \left|\nabla\left(\langle v\rangle^{\frac{k+\gamma}{2}} h(t,v)\right)\right|^2 \mathrm{d}v$$
$$+C \int_{\mathbb{R}^d} \langle v\rangle^k h^2 \boldsymbol{c}_\gamma[f](v)\mathrm{d}v + \Gamma_\varepsilon(t) \int_{\mathbb{R}^d} \langle v\rangle^k h^2 \mathrm{d}v, \tag{4.16}$$

with $\Gamma_\varepsilon \in L^1([0,T])$.

One now looks at $\mathscr{I}_2$, which is more delicate. One has again $\mathscr{I}_2=\mathscr{I}_{2,1}+\mathscr{I}_{2,2}$, with

$$\mathscr{I}_{2,1} = -2\int_{\mathbb{R}^d} \langle v\rangle^k \mathcal{A}[h]\nabla g \cdot \nabla h\, \mathrm{d}v, \qquad \mathscr{I}_{2,2} = -2k \int_{\mathbb{R}^d} \langle v\rangle^{k-2}\, h\, \mathcal{A}[h]\nabla g \cdot v \mathrm{d}v.$$

Writing $\langle v\rangle^{k-2} h\mathcal{A}[h]\nabla g\cdot v = \left(\langle v\rangle^{\frac{k-2}{2}} h\, v\right)\cdot\left(\langle v\rangle^{\frac{k-2}{2}}\mathcal{A}[h]\nabla g\right)$ and using Young's inequality, we deduce that

$$|\mathscr{I}_{2,2}| \leqq k\int_{\mathbb{R}^d} \langle v\rangle^k h^2(t,v)\mathrm{d}v + k\int_{\mathbb{R}^d} \langle v\rangle^k\, |\mathcal{A}[h]|^2\, |\nabla g|^2 \mathrm{d}v.$$

Similarly, with $\langle v\rangle^k \mathcal{A}[h]\nabla g\cdot\nabla h = \left(\langle v\rangle^{\frac{k+\gamma}{2}}\nabla h\right)\cdot\left(\langle v\rangle^{\frac{k-\gamma}{2}}\mathcal{A}[h]\nabla g\right)$, we deduce from Young's inequality that, for any $\delta>0$,

$$|\mathscr{I}_{2,1}| \leqq \delta\int_{\mathbb{R}^d} \langle v\rangle^{k+\gamma}\, |\nabla h|^2 \mathrm{d}v + \frac{1}{\delta}\int_{\mathbb{R}^d} \langle v\rangle^{k-\gamma} |\mathcal{A}[h]|^2\, |\nabla g|^2 \mathrm{d}v.$$

We deduce that there exists $C_k > 0$ such that, for any $\delta \in]0, 1]$,

$$|\mathscr{I}_2| \leqq \delta \int_{\mathbb{R}^d} \langle v\rangle^{k+\gamma} |\nabla h|^2 \mathrm{d}v + \frac{C_k}{\delta}\left(\int_{\mathbb{R}^d} \langle v\rangle^k h^2(t,v)\mathrm{d}v + \|\mathcal{A}[h]\|_\infty^2 \int_{\mathbb{R}^d} \langle v\rangle^{k-\gamma} |\nabla g|^2 \mathrm{d}v\right).$$

Now, from Lemma A.9, Eq. (A.7), there is $C > 0$ such that, when $q > \frac{d}{d+2+\gamma}$ and $m > d\left(1 - \frac{1}{q}\right)$,

$$\|\mathcal{A}[h]\|_\infty \leqq C \|\langle\cdot\rangle^m h\|_q.$$

Applying this with $q = 2$, $m = \frac{k}{2}$, and this shows that, if $k > d$,

$$\|\mathcal{A}[h]\|_\infty^2 \leqq C^2 \int_{\mathbb{R}^d} \langle v\rangle^k h^2(t,v)\mathrm{d}v,$$

and we deduce therefore that, for any $\delta \in]0, 1]$,

$$|\mathscr{I}_2| \leqq \delta \int_{\mathbb{R}^d} \langle v\rangle^{k+\gamma} |\nabla h|^2 \mathrm{d}v + \Xi_\delta(t) \int_{\mathbb{R}^d} \langle v\rangle^k h^2 \mathrm{d}v, \tag{4.17}$$

where

$$\Xi_\delta(t) = \delta^{-1} C_k \left[1 + C^2 \int_{\mathbb{R}^d} \langle v\rangle^{k-\gamma} |\nabla g(t,v)|^2 \mathrm{d}v\right].$$

Notice that (4.13) implies that for any $\delta \in]0, 1]$,

$$\Xi_\delta \in L^1([0,T]).$$

One proceeds in that exactly same way for $\mathscr{I}_4 = \mathscr{I}_{4,1} + \mathscr{I}_{4,2}$, with

$$\mathscr{I}_{4,1} = 2\int_{\mathbb{R}^d} \langle v\rangle^k g(t,v)\boldsymbol{b}[h]\cdot\nabla h \mathrm{d}v, \quad \mathscr{I}_{4,2} = 2k\int_{\mathbb{R}^d} \langle v\rangle^{k-2} h(t,v)g(t,v)\boldsymbol{b}[h]\cdot v\mathrm{d}v.$$

To estimate $|\boldsymbol{b}[h]|$, we need to distinguish between the case $2 > \frac{d}{d+\gamma+1}$ for which can apply (A.8) to estimate $\|\boldsymbol{b}[h]\|_\infty$ in terms of $\|\langle\cdot\rangle^m h\|_2$ ($m > \frac{d}{2}$), and the case $2 \leqq \frac{d}{d+\gamma+1}$ for which we resort to (A.9) or (A.10). We focus here on this second case, the case $\boldsymbol{b}[h] \in L^\infty(\mathbb{R}^d)$ being simpler (one can follow the line of the previous estimate for $\|\mathcal{A}[h]\|_\infty$, we leave it to the reader). We treat in details first the case $2 < \frac{d}{d+\gamma+1}$. Clearly,

$$\left|\mathscr{I}_{4,1}\right| \leqq 2\int_{\mathbb{R}^d} \left(\langle v\rangle^{\frac{k+\gamma}{2}} |\nabla h|\right)\left(g\langle v\rangle^{\frac{k-\gamma}{2}}\right)|\boldsymbol{b}[h]|\mathrm{d}v.$$

Let p_γ^* be defined as in (A.9) with the choice of $q = 2$, that is $p_\gamma^* = \frac{2d}{d-2(1+\gamma+d)} > 2$. Let p be such that $\frac{1}{p_\gamma^*} + \frac{1}{2} + \frac{1}{p} = 1$, that is $p = \frac{d}{1+\gamma+d}$. One deduces from Hölder's inequality that

$$\left|\mathscr{I}_{4,1}\right| \leqq 2\|\boldsymbol{b}[h]\|_{p_\gamma^*} \left\|\langle\cdot\rangle^{\frac{k+\gamma}{2}} |\nabla h|\right\|_2 \left\|\langle\cdot\rangle^{\frac{k-\gamma}{2}} g\right\|_p.$$

Now, a simple use of Young's inequality yields, for any $\delta > 0$,

$$\left|\mathscr{I}_{4,1}\right| \leqq \delta \int_{\mathbb{R}^d} \langle v\rangle^{k+\gamma} \left|\nabla h(t,v)\right|^2 \mathrm{d}v + \frac{1}{\delta} \left\|\boldsymbol{b}[h]\right\|_{p_\gamma^*}^2 \left\|\langle\cdot\rangle^{\frac{k-\gamma}{2}} g(t)\right\|_p^2 .$$

We now use (A.9) to deduce that $\|\boldsymbol{b}[h]\|_{p_\gamma^*}^2 \leqq C(f_{\mathrm{in}}, g_{\mathrm{in}})\|h\|_2^2$, where $C(f_{\mathrm{in}}, g_{\mathrm{in}})$ depends on d, γ and $\|f_{\mathrm{in}}\|_1$, $\|g_{\mathrm{in}}\|_1$. It remains only to estimate

$$\left\|\langle\cdot\rangle^{\frac{k-\gamma}{2}} g(t)\right\|_p^2, \qquad p = \frac{d}{1+\gamma+d}.$$

One checks easily that $p < \frac{2d}{d-2}$ under the assumption $2 > \frac{d}{d+\gamma+2}$, while $p > 2$ clearly holds by assumption. Then, by interpolation and Sobolev embedding, there exist $C > 0$ and $\alpha \in (0,1)$ such that

$$\left\|\langle\cdot\rangle^{\frac{k-\gamma}{2}} g(t)\right\|_p \leqq C \left\|\langle\cdot\rangle^{\frac{k-\gamma}{2}} g(t)\right\|_2^{1-\alpha} \left\|\nabla\left(\langle\cdot\rangle^{\frac{k-\gamma}{2}} g(t)\right)\right\|_2^{\alpha}.$$

Thanks to Young's inequality, we deduce that

$$\left\|\langle\cdot\rangle^{\frac{k-\gamma}{2}} g(t)\right\|_p^2 \leqq C_0 \left(\left\|\langle\cdot\rangle^{\frac{k-\gamma}{2}} g(t)\right\|_2^2 + \int_{\mathbb{R}^d} \left|\nabla\left(\langle v\rangle^{\frac{k-\gamma}{2}} g(t,v)\right)\right|^2 \mathrm{d}v\right).$$

We showed that, for any $\delta > 0$,

$$\left|\mathscr{I}_{4,1}\right| \leqq \delta \int_{\mathbb{R}^d} \langle v\rangle^{k+\gamma} \left|\nabla h(t,v)\right|^2 \mathrm{d}v + \Theta_\delta(t) \int_{\mathbb{R}^d} \langle v\rangle^{k+\gamma} h^2(t,v)\mathrm{d}v,$$

where

$$\Theta_\delta(t) = \delta^{-1} C_{d,\gamma,k} \left(\left\|\langle\cdot\rangle^{\frac{k-\gamma}{2}} g(t)\right\|_2^2 + \int_{\mathbb{R}^d} \left|\nabla\left(\langle v\rangle^{\frac{k-\gamma}{2}} g(t,v)\right)\right|^2 \mathrm{d}v\right),$$

for some positive constant $C_{d,\gamma,k}$ depending on d, γ, k and $\|f_{\mathrm{in}}\|_1$, $\|g_{\mathrm{in}}\|_1$. Notice that, according to (4.13), one has $\Theta_\delta \in L^1([0,T])$. In the same way,

$$\begin{aligned}
\left|\mathscr{I}_{4,2}\right| &\leqq 2k\|\boldsymbol{b}[h]\|_{p_\gamma^*} \left\|\langle\cdot\rangle^{\frac{k-1}{2}} h\right\|_2 \left\|\langle\cdot\rangle^{\frac{k-1}{2}} g\right\|_p \\
&\leqq k \int_{\mathbb{R}^d} \langle v\rangle^{k-1} h^2(t,v)\,\mathrm{d}v + k\|\boldsymbol{b}[h]\|_{p_\gamma^*}^2 \left\|\langle\cdot\rangle^{\frac{k-1}{2}} g\right\|_p^2 .
\end{aligned}$$

As before, this yields then the estimate

$$\left|\mathscr{I}_{4,2}\right| \leqq \Theta(t) \int_{\mathbb{R}^d} \langle v\rangle^k h^2(t,v)\,\mathrm{d}v$$

where

$$\Theta(t) = C_{d,\gamma,k}\left(1 + \left\|\langle\cdot\rangle^{\frac{k-1}{2}} g(t)\right\|_2^2 + \int_{\mathbb{R}^d} \left|\nabla\left(\langle v\rangle^{\frac{k-1}{2}} g(t,v)\right)\right|^2 \mathrm{d}v\right)$$

for some positive constant $C_{d,\gamma,k} > 0$ depending on d, γ, k and $\|f_{\text{in}}\|_{L^1_2}$. In particular, $\Theta \in L^1([0,T])$. Combining these two estimates, for any $\delta > 0$, there is $\Lambda_\delta \in L^1([0,T])$ such that

$$|\mathscr{I}_4| \leqq \delta \int_{\mathbb{R}^d} \langle v\rangle^{k+\gamma} |\nabla h(t,v)|^2 \mathrm{d}v + \Lambda_\delta(t) \int_{\mathbb{R}^d} \langle v\rangle^k h^2 \mathrm{d}v\,. \tag{4.18}$$

Choosing $\delta = \delta_0 = \min(1, \frac{K_0}{2})$ in (4.17)-(4.18) and combining this with (4.16), one sees that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \langle v\rangle^k h^2(t,v)\mathrm{d}v + K_0 \int_{\mathbb{R}^d} \langle v\rangle^{k+\gamma} |\nabla h(t,v)|^2 \mathrm{d}v \\
&\quad\leqq \varepsilon \tilde{C} \int_{\mathbb{R}^d} \left|\nabla\left(\langle v\rangle^{\frac{k+\gamma}{2}} h(t,v)\right)\right|^2 \mathrm{d}v \\
&\quad+\boldsymbol{\Xi}_\varepsilon(t) \int_{\mathbb{R}^d} \langle v\rangle^k h^2(t,v)\mathrm{d}v + C\int_{\mathbb{R}^d} \langle v\rangle^k h^2(t,v) \bar{\boldsymbol{c}}_\gamma[f(t)](v)\mathrm{d}v
\end{aligned}$$

with

$$\boldsymbol{\Xi}_\varepsilon(t) = \Lambda_{\delta_0}(t) + \Xi_{\delta_0}(t) + \Gamma_\varepsilon(t), \qquad \boldsymbol{\Xi}_\varepsilon \in L^1([0,T]).$$

Since $f_{\text{in}} \in L^1_N(\mathbb{R}^d)$, we may now use the ε-Poincaré inequality (4.2) with $\phi^2 = \langle\cdot\rangle^k h^2$. Notice that the definition of $\overline{\nu}$ in (4.11) with $p = 2$ yields $\bar{\nu} = \frac{d|\gamma|+k\gamma+2k}{d+2(\gamma+2)}$ if $k \leqq 2|\gamma|$ while $\bar{\nu} = \frac{(d-k)|\gamma|}{d+2\gamma}$ if $k \geqq 2|\gamma|$. We deduce that, for any $\varepsilon > 0$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \langle v\rangle^k h^2(t,v)\mathrm{d}v + K_0 \int_{\mathbb{R}^d} \langle v\rangle^{k+\gamma} |\nabla h(t,v)|^2 \mathrm{d}v \\
&\quad\leqq C\varepsilon \int_{\mathbb{R}^d} \left|\nabla\left(\langle v\rangle^{\frac{k+\gamma}{2}} h(t,v)\right)\right|^2 \mathrm{d}v + C_T(f_{\text{in}},\varepsilon)\int_{\mathbb{R}^d} \langle v\rangle^{k+\gamma} h^2(t,v)\mathrm{d}v \\
&\quad+\boldsymbol{\Xi}_\varepsilon(t) \int_{\mathbb{R}^d} \langle v\rangle^k h^2(t,v)\mathrm{d}v,
\end{aligned}$$

where $C_T(f_{\text{in}},\varepsilon) > 0$ is a positive constant depending only on $\varepsilon, T, \|f_{\text{in}}\|_{L^2_k}, \gamma$. Recalling that there are $c_0, c_1 > 0$ such that

$$\int_{\mathbb{R}^d} \langle v\rangle^{k+\gamma} |\nabla h|^2 \mathrm{d}v \geqq c_0 \int_{\mathbb{R}^d} \left|\nabla\left(\langle v\rangle^{\frac{k+\gamma}{2}} h\right)\right|^2 \mathrm{d}v - c_1 \int_{\mathbb{R}^d} \langle v\rangle^{k+\gamma-2} h^2 \mathrm{d}v,$$

one chooses then $\varepsilon = \varepsilon_0 > 0$ such that $C\varepsilon_0 = \frac{1}{2}K_0 c_0$ to deduce that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \langle v\rangle^k h^2(t,v)\mathrm{d}v + \frac{1}{2}K_0 c_0 \int_{\mathbb{R}^d} \left|\nabla\left(\langle v\rangle^{\frac{k+\gamma}{2}} h(t,v)\right)\right|^2 \mathrm{d}v \\
&\quad\leqq \boldsymbol{\Lambda}(t) \int_{\mathbb{R}^d} \langle v\rangle^k h^2(t,v)\mathrm{d}v,
\end{aligned}$$

with

$$\boldsymbol{\Lambda}(t) = \boldsymbol{\Xi}_{\varepsilon_0}(t) + \frac{1}{2}K_0\, c_1 + C_T(f_{\text{in}},\varepsilon_0), \qquad \boldsymbol{\Lambda} \in L^1([0,T]).$$

We conclude by a Gronwall argument that, for any $t \in [0, T]$,

$$\int_{\mathbb{R}^d} \langle v\rangle^k h^2(t,v)\mathrm{d}v \leqq \boldsymbol{C}_T \int_{\mathbb{R}^d} \langle v\rangle^k h^2(0,v)\mathrm{d}v, \qquad \boldsymbol{C}_T := \exp\left(\int_0^T \boldsymbol{\Lambda}(t)\mathrm{d}t\right),$$

which gives the result.

The case $\frac{d}{d+\gamma+1} = 2$ is treated in the same way using now the weighted estimate (A.10) for $q = 2$. Given $\frac{2d}{d+2} < \ell < 2$ one defines $\bar{p}_\gamma^* = \frac{2\ell}{2-\ell}$ and for p such that $\frac{1}{\bar{p}_\gamma^*} + \frac{1}{2} + \frac{1}{p} = 1$, that is $p = \ell' = \frac{\ell}{\ell-1}$, we get, as before, for any $\delta > 0$,

$$\begin{aligned} \left|\mathscr{I}_{4,1}\right| &\leqq \delta \int_{\mathbb{R}^d} \langle v\rangle^{k+\gamma} \left|\nabla h(t,v)\right|^2 \mathrm{d}v + \frac{1}{\delta} \|\boldsymbol{b}[h]\|_{\bar{p}_\gamma^*}^2 \left\|\langle\cdot\rangle^{\frac{k-\gamma}{2}} g(t)\right\|_p^2 \\ &\leqq \delta \int_{\mathbb{R}^d} \langle v\rangle^{k+\gamma} \left|\nabla h(t,v)\right|^2 \mathrm{d}v + \frac{C}{\delta} \|\langle\cdot\rangle^m h\|_2^2 \left\|\langle\cdot\rangle^{\frac{k-\gamma}{2}} g(t)\right\|_p^2 \end{aligned}$$

for $m > |1+\gamma| = \frac{d}{2}$, where we used (A.10) in the case $q = 2$. The estimate for $\left\|\langle\cdot\rangle^{\frac{k-\gamma}{2}} g(t)\right\|_p^2$ with $p = \frac{\ell}{\ell-1}$ is made as before since $2 < p < \frac{2d}{d-2}$ (because $\frac{2d}{d+2} < \ell < 2$). Details are left to the reader. □

As a consequence, we immediately deduce our main uniqueness result as stated in the Introduction.

Proof of Theorem 1.9. The proof is a simple consequence of the above stability inequality and the fact that $g_{\text{in}} = f_{\text{in}}$. □

5. Further Comments

In this last section we give an informal discussion of some additional features of our contribution. Specifically, we explain how the results of the previous sections allow to recover well-known results in the case of moderately soft potentials $-2 \leqq \gamma < 0$, and then we discuss the endpoint case of Prodi–Serrin criterion for which $r = \infty$.

5.1. About the Moderate Soft Potential Case

We illustrate here how Prodi–Serrin's criterion is satisfied by any weak solution constructed with an approximation procedure, as in Desvillettes [18] to the Landau equation, in the moderate soft potential case

$$-2 \leqq \gamma < 0.$$

It is already a well-known fact that Landau equation is globally well-posed in this case (weak solutions are global and unique under suitable assumptions) and that apperance/propagation of L^p-norm occurs in this case. Moreover, as shown in the recent contribution Alonso et al. [5], such appearance of L^p-bounds implies the appearance of pointwise bounds for solutions to (1.5). The key point which makes the Prodi–Serrin criterion satisfied is the *a priori* estimate related to the weighted Fisher information obtained by the third author in Alonso et al. [18], see Corollary A.6 which, for $-2 \leqq \gamma < 0$ is strong enough to ensure one of the Prodi–Serrin criterion to hold.

Proposition 5.1. *Let $f_{\rm in}$ and $f = f(t,v)$ a weak solution to equation (1.1) in the sense of Definition 1.15. Then the following holds:*

(1) Assume that $-2 < \gamma < 0$ and $f_{\rm in} \in L^1_s(\mathbb{R}^d)$ with $s = \frac{\gamma^2}{2(\gamma+2)}$, then

$$f \in L^1([0,T],\, L^{\frac{d}{d+\gamma}}(\mathbb{R}^d)), \qquad \forall T > 0.$$

(2) For $-2 \leqq \gamma < 0$, assume that there is some $\delta > 0$ such that $f_{\rm in} \in L^1_{|\gamma|+\delta}(\mathbb{R}^d)$. Then, there is a pair (r,q) with $r > 1$, $1 < q < \frac{d}{d+\gamma}$ and $\frac{2}{r} + \frac{d}{q} = d + \gamma + 2$ such that

$$\langle \cdot \rangle^{|\gamma|} f \in L^r([0,T],\, L^q(\mathbb{R}^d)), \qquad \forall T > 0.$$

Proof. Recall that Corollary A.6 ensures that the weak solution f satisfies

$$\int_0^T \mathrm{d}t \int_{\mathbb{R}^d} \left| \nabla \left(\langle v \rangle^{\frac{\gamma}{2}} \sqrt{f(t,v)} \right) \right|^2 \mathrm{d}v < \infty . \tag{5.1}$$

We begin with Assumption 1 (and $-2 < \gamma < 0$). Thanks to Sobolev inequality

$$\|h\|^2_{\frac{2d}{d-2}} \leqq C_{\rm Sob} \|\nabla h\|_2^2, \qquad \forall h \in \dot{\mathbb{H}}^1(\mathbb{R}^d)$$

(and since $\gamma > -2$), we can use the interpolation (Hölder's) inequality

$$\|f(t)\|_{\frac{d}{d+\gamma}} = \|\sqrt{f(t)}\|^2_{\frac{2d}{d+\gamma}} \leqq \left\| \langle \cdot \rangle^a \sqrt{f(t)} \right\|_2^{2(1-\alpha)} \, \left\| \langle \cdot \rangle^{\frac{\gamma}{2}} \sqrt{f(t)} \right\|^{2\alpha}_{\frac{2d}{d-2}}$$

with $\frac{d+\gamma}{2d} = \frac{1-\alpha}{2} + \frac{d-2}{2d}\alpha$, $a(1-\alpha) + \frac{\gamma}{2}\alpha = 0$, which means $\alpha = \frac{|\gamma|}{2}$ and $a = \frac{\gamma^2}{2(2+\gamma)}$. This, combined with the above Sobolev inequality, implies that

$$\|f(t)\|_{\frac{d}{d+\gamma}} \leqq C_{\rm Sob}^{\frac{|\gamma|}{2}} \left[\sup_{t \in [0,T]} \boldsymbol{m}_{\frac{|\gamma|^2}{2+\gamma}}(f(t)) \right]^{2-|\gamma|} \, \left\| \nabla \left(\langle \cdot \rangle^{\frac{|\gamma|}{2}} \sqrt{f(t)} \right) \right\|_2^{|\gamma|} .$$

Since $|\gamma| < 2$, we see from (5.1) that $\|f(\cdot)\|_{\frac{d}{d+\gamma}} \in L^1([0,T])$ and this proves the first point.

Now, for the second Prodi–Serrin criterion, we still resort to (5.1) but now with the weighted interpolation inequality

$$\|\langle \cdot \rangle^{|\gamma|} f(t)\|_q \leqq \|\langle \cdot \rangle^{\beta} f(t)\|_1^{1-\theta} \, \left\| \langle \cdot \rangle^{\gamma} f(t) \right\|^{\theta}_{\frac{d}{d-2}} \leqq \|\langle \cdot \rangle^{\beta} f(t)\|_1^{1-\theta} \, \left\| \langle \cdot \rangle^{\frac{\gamma}{2}} \sqrt{f(t)} \right\|^{2\theta}_{\frac{2d}{d-2}}$$

with

$$\frac{1}{q} = 1 - \theta + \frac{d-2}{d}\theta = 1 - \frac{2}{d}\theta, \qquad |\gamma| = \beta(1-\theta) + \gamma\theta.$$

We assume that $1 < q < \frac{d}{d+\gamma}$ so that $0 < \theta < \frac{|\gamma|}{2} \leqq 1$. We then choose

$$\frac{1}{r} = 1 + \frac{\gamma}{2} + \theta < 1,$$

so that

$$\|\langle\cdot\rangle^{|\gamma|} f(t)\|_q^r \leqq C_{\mathrm{Sob}}^{r\theta}\, \boldsymbol{m}_\beta(f(t))^{r(1-\theta)} \left\| \nabla\left(\langle\cdot\rangle^{\frac{|\gamma|}{2}} \sqrt{f(t)} \right) \right\|_2^{2r\theta},$$

while

$$\frac{d}{q} + \frac{2}{r} = \frac{d}{q} + 2 + \gamma + 2\theta = d + \gamma + 2,$$

which makes (r, q) an admissible Prodi–Serrin pair. Now, since $2r\theta \leqq 2$, one sees that

$$\sup_{t\in[0,T]} \boldsymbol{m}_\beta(f(t)) < \infty \implies \|\langle\cdot\rangle^{|\gamma|} f(t)\|_q \in L^r([0,T]).$$

One notices now that $\beta = |\gamma| \frac{1+\frac{d}{2}(1-q^{-1})}{1+\frac{d}{2}(1-q^{-1})}$. We see that if $\gamma > -2$, we can get $\beta = 2$ by choosing $q > 1$ sufficiently close to 1. If $\gamma = -2$, we get (for some given $\delta > 0$) $\beta = 2 + \delta$ by also choosing $q > 1$ sufficiently close to 1. This proves the result. □

Remark 5.2. For $-2 < \gamma < 0$, the second point here above means that Prodi–Serrin's criterion $L_t^r(L^q)$ with $r > 1$ is satisfied for some pair (r, q) for *any admissible* initial datum $f_{\mathrm{in}} \in L_2^1(\mathbb{R}^d)$ whereas, for $\gamma = -2$, some additional moment assumption $f_{\mathrm{in}} \in L_{2+\delta}^1(\mathbb{R}^d)$ is needed. Notice that, arguing as in Alonso et al. [5] and resorting to the de la Vallée-Poussin criterion, this additional assumption can be removed. Even if the first point of the Proposition seems to yield a non optimal moment assumption on f_{in}, we still believe it provides interesting information on the link between the Prodi–Serrin criterion in the endpoint case $r = 1, q = \frac{d}{d+\gamma}$, and the Fisher information estimate of Corollary A.6.

5.2. About the Critical Endpoint $r = \infty$, $q = \frac{d}{d+\gamma+2}$

We consider $d \in \mathbb{N}$, $d \geqq 3$, and $\gamma \in [-d, -2)$. In the endpoint case for which $r = \infty$ and $q = \frac{d}{d+\gamma+2}$, it is actually possible to obtain a result similar to that of Theorem 1.1. Indeed, under some smallness assumption on

$$\|\langle\cdot\rangle^{|\gamma|} f\|_{L^\infty([0,T],\, L^{\frac{d}{d+\gamma+2}}(\mathbb{R}^d))},$$

the appearance and propagation of L^p-norms which is the cornerstone of our analysis still holds true.

Proposition 5.3. *We consider $d \in \mathbb{N}$, $d \geqq 3$ and $\gamma \in [-d, -2)$. Let $f_{\rm in}$ and $f = f(t, v)$ be a weak solution to equation (1.1) in the sense of Definition 1.15 with $f_{\rm in} \in L^1_s(\mathbb{R}^d)$ $(s > 2)$. Additionally, assume that there exists $T > 0$ such that*

$$\langle\cdot\rangle^{|\gamma|} f \in L^\infty([0, T];\ L^{\frac{d}{d+\gamma+2}}(\mathbb{R}^d)).$$

Then, given $k \geqq 0$, $p > 1$, there exists some explicit $\delta > 0$ *(depending on d, γ, k and p) such that, if*

$$\sup_{t\in(0,T)} \left\| \langle\cdot\rangle^{|\gamma|} f(t, \cdot) \right\|_{\frac{d}{d+\gamma+2}} \leqq \delta K_0\,, \tag{5.2}$$

and

$$f_{\rm in} \in L^p_k(\mathbb{R}^d),$$

one has, for any $t \in [0, T]$,

$$\int_{\mathbb{R}^d} f^p(t, v)\langle v\rangle^k \mathrm{d}v + C_{\delta,p,k} K_0 \int_0^t \mathrm{d}\tau \int_{\mathbb{R}^d} \left|\nabla\left(\langle v\rangle^{\frac{k+\gamma}{2}} f(\tau, v)^{\frac{p}{2}}\right)\right|^2 \mathrm{d}v \leqq \boldsymbol{C}(T, f_{\rm in}, \delta)\,, \tag{5.3}$$

where $\boldsymbol{C}(T, f_{\rm in}, \delta)$ is an explicit positive constant depending on d, γ, T, p, k, K_0, δ, $\|f_{\rm in}\|_{L^p_k}$, $\|f_{\rm in}\|_{L^1_s}$ and the Prodi–Serrin norms $\|f\|_{L^\infty_t(L_v^{\frac{d}{d+\gamma+2}})}$, while $C_{\delta,p,k}$ is an explicit constant depending only on p, δ and k.

Proof. We set $q_0 = \frac{d}{d+2+\gamma}$ and recall the evolution of L^p norms given in (4.8):

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}_{k,p}(t) + 2K(p)\mathbb{D}_{k+\gamma,p}(t)\\ &\leqq \boldsymbol{c}_{k,p}\mathbb{M}_{k,p}(t) + \boldsymbol{C}_{k,p}\int_{\mathbb{R}^d} \langle v\rangle^k \overline{\boldsymbol{c}}_\gamma[f(t)](v) f^p(t, v)\mathrm{d}v.\end{aligned} \tag{5.4}$$

As in the proof of Proposition 3.6, the key point is to evaluate the singular term

$$\int_{\mathbb{R}^d} \langle\cdot\rangle^k\, \overline{\boldsymbol{c}}_\gamma[f(t)] f^p(t, v)\mathrm{d}v$$

in terms of $\|\langle\cdot\rangle^{|\gamma|} f(t, \cdot)\|_{q_0}$. We argue as in the proof of the ε-Poincaré inequality to estimate

$$\int_{\mathbb{R}^d} \overline{\boldsymbol{c}}_\gamma[F]\psi^2\mathrm{d}v$$

with

$$F = \langle\cdot\rangle^{-\gamma} f(t, v), \qquad \psi = \langle\cdot\rangle^{\frac{\gamma+k}{2}} f^{\frac{p}{2}}.$$

Indeed, since $\alpha_{1,\gamma} \leqq \langle v_*\rangle^\gamma \leqq \alpha_{2,\gamma}\langle v\rangle^\gamma$ when $|v - v_*| \leqq 1$ (for some constants $\alpha_{1,\gamma}, \alpha_{2,\gamma} > 0$), the estimate on $\int_{\mathbb{R}^d} \overline{\boldsymbol{c}}_\gamma[F]\psi^2\mathrm{d}v$ yields an estimate on $\int_{\mathbb{R}^d} \langle\cdot\rangle^k \overline{\boldsymbol{c}}_\gamma[f(t, \cdot)]\, f^p(t, v)\mathrm{d}v$.

We can use Hardy-Littlewood-Sobolev inequality (3.1) to get

$$\int_{\mathbb{R}^d} \overline{\boldsymbol{c}}_\gamma[F]\psi^2\mathrm{d}v \leqq C_{\rm HLS}\|F\|_{q_0}\|\psi^2\|_r, \qquad \frac{1}{r} = 2 - \frac{1}{q_0} + \frac{\gamma}{d} = \frac{d-2}{d}.$$

This yields

$$\int_{\mathbb{R}^d} \overline{\boldsymbol{c}}_\gamma[F]\psi^2 \mathrm{d}v \leqq C_{\mathrm{HLS}} \|F\|_{q_0} \|\psi\|_{2r}^2 \leqq C_{d,\gamma} \|F\|_{q_0} \|\nabla\psi\|_2^2,$$

since $\frac{2d}{d-2}$ is the critical exponent for the Sobolev embedding and $C_{d,\gamma} = C_{\mathrm{HLS}} C_{\mathrm{Sob}}$. Moreover, since $\psi = \langle\cdot\rangle^{\frac{\gamma+k}{2}} f^{\frac{p}{2}}$, we deduce from (5.4) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}_{k,p}(t) + 2K(p)\mathbb{D}_{k+\gamma,p}(t) \leqq \boldsymbol{C}_{k,p,d,\gamma} \|F(t)\|_{q_0} \mathbb{D}_{k+\gamma,p}(t) + \boldsymbol{c}_{k,p}\, \mathbb{M}_{k,p}(t),$$

for some positive constant $\boldsymbol{C}_{k,p,d,\gamma}$ depending on k, p, d, γ, related to $\boldsymbol{C}_{k,p} C_{d,\gamma}$ and the constants $\alpha_{1,\gamma}$, $\alpha_{2,\gamma}$. Therefore, assuming that

$$\sup_{t\in[0,T]} \|F(t)\|_{q_0} \leqq \delta\, K_0 \qquad \text{with} \qquad \delta < \frac{2(p-1)}{p\boldsymbol{C}_{k,p,d,\gamma}}, \tag{5.5}$$

one obtains, recalling that $K(p) = K_0 \frac{p-1}{p}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}_{k,p}(t) + \left(2\frac{p-1}{p} - \delta\, \boldsymbol{C}_{k,p,d,\gamma}\right) K_0\, \mathbb{D}_{k+\gamma,p}(t) \leqq \boldsymbol{c}_{k,p}\, \mathbb{M}_{k,p}(t),$$

which allows the to conclude exactly as in the proof of Proposition 4.1.

We now briefly explain how to modify the above proof in the case $\gamma = -d$, when $d = 3$. We first observe that $\mathbb{M}_{k,p}$ now satisfies the differential inequality

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}_{k,p}(t) + 2K(p)\mathbb{D}_{k-3,p}(t) \leqq \boldsymbol{c}_{k,p}\, \mathbb{M}_{k,p}(t) + C_{k,p} \\
&\int_{\mathbb{R}^3} \langle v\rangle^{k-1}\, \overline{\boldsymbol{c}}_{-2}[f(t)](v) f^p(t,v)\mathrm{d}v \\
&\quad + C_{k,p} \int_{\mathbb{R}^3} \langle v\rangle^k\, f^{p+1}(t,v)\mathrm{d}v\,.
\end{aligned}$$

Then, we recall (see Proposition A.10) the estimate

$$\int_{\mathbb{R}^3} \langle v\rangle^{k-1}\, \overline{\boldsymbol{c}}_{-2}[f(t)](v) f^p(t,v)\mathrm{d}v \leqq \boldsymbol{\varepsilon}\, \mathbb{D}_{k-3,p}(t) + C(\boldsymbol{\varepsilon}, \boldsymbol{m}_\eta(t))\, \mathbb{M}_{k,p}(t),$$

which holds for any $\boldsymbol{\varepsilon} > 0$ and any $\eta > 2$. Finally, we notice that

$$\begin{aligned}
\int_{\mathbb{R}^3} \langle v\rangle^k\, f^{p+1}(t,v)\mathrm{d}v &= \int_{\mathbb{R}^3} \langle v\rangle^3\, f(t,v)\left[\langle v\rangle^{\frac{k-3}{2}} f^{\frac{p}{2}}(t,v)\right]^2 \mathrm{d}v \\
&\leqq \|\langle\cdot\rangle^3\, f(t)\|_{\frac{3}{2}}\, \|\langle\cdot\rangle^{\frac{k-3}{2}}\, f^{\frac{p}{2}}(t)\|_6^2 \leqq C_{\mathrm{Sob}}\, \delta\, K_0\, \mathbb{D}_{k-3,p}(t).
\end{aligned}$$

This proves the result. $\square$

Remark 5.4. Notice that the above assumption (5.2) is not optimal. Indeed, with the notations of the proof, and decomposing further,

$$F = F\mathbf{1}_{F>R} + F\mathbf{1}_{F\leqq R}.$$

After reproducing the aforementioned computations, it is possible to replace (5.2) with an estimate of the type

$$\limsup_{R\to\infty} \sup_{t\in[0,T)} \|F(t)\mathbf{1}_{F(t)>R}\|_{\frac{d}{d+2+\gamma}} \leqq \delta K_0 , \qquad F(t,v) = \langle v\rangle^{|\gamma|} f(t,v) , \quad (5.6)$$

which is more general than some mere *uniform integrability* of the family $(F^{q_0}(t))_{t\in[0,T]}$ since we do not assume the above lim sup to vanish. In particular, the conclusion of Proposition 5.3 holds true for instance if

$$\langle\cdot\rangle^{|\gamma|} f(t,\cdot) \in L^{\infty}((0,T);\ L^{\frac{d}{d+\gamma+2}}(\log L)^{\alpha}(\mathbb{R}^d)).$$

We do not elaborate more in this direction.

Finally, for the Navier–Stokes equation, a sharper endpoint estimate than Theorem 1.1 has been derived in Iskauriaza et al. [35] by a compactness argument. This result has been made quantitative very recently in Tao [56]; Palasek [49] and appears interesting to inquire about the Landau equation counterpart.

Acknowledgements. B.L. was partially supported by PRIN2022 (project ID: BEMMLZ) *Stochastic control and games and the role of information.* He also gratefully acknowledges the financial support from the Italian Ministry of Education, University and Research (MIUR), Dipartimenti di Eccellenza grant 2022–2027 as well as the support from the de Castro Statistics Initiative, Collegio Carlo Alberto (Torino). **Note added to the paper**. After the completion of this work and the publication of a previous version of our manuscript as a preprint in Alonso et al. [6], an important contribution by Guillen and Silvestre [34] appeared as a preprint, presenting a proof of the fact that classical solutions to (1.1) do not blow up and can be defined globally. A recent significant contribution by Golding et al. [26] also appeared after the submission of the paper. These two works provide new insights on the problem discussed in this paper.

Funding Open access funding provided by Università degli Studi di Torino within the CRUI-CARE Agreement.

Data availability No data was used for the research described in the article.

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Appendix A: Facts About Solutions to the Landau Equation

We collect several mathematical results about the solutions to the Landau equation in the range of parameters we are dealing with here: $-d \leqq \gamma < 0$. These results are meant to serve as a mathematical tool-box for the core of the paper.

Definition A.1. Let $f_{\text{in}} \in L^1_2(\mathbb{R}^d) \cap L \log L(\mathbb{R}^d)$ be given and nonnegative with

$$\varrho_{\text{in}} := \int_{\mathbb{R}^d} f_{\text{in}}(v)\mathrm{d}v > 0, \qquad E_{\text{in}} := \int_{\mathbb{R}^d} f_{\text{in}}(v)|v|^2\mathrm{d}v,$$

and

$$H(f_{\text{in}}) := \int_{\mathbb{R}^d} f_{\text{in}}(v) \log f_{\text{in}}(v)\mathrm{d}v.$$

We say that $f \in \mathcal{Y}_0(f_{\text{in}})$ if $f \in L^1_2(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} f(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \mathrm{d}v = \int_{\mathbb{R}^d} f_{\text{in}}(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \mathrm{d}v,$$

and $H(f) \leqq H(f_{\text{in}})$.

We observe that at the formal level, a solution $f = f(t, v)$ of the Landau equation with initial datum $f_{\text{in}} \in L^1_2(\mathbb{R}^d) \cap L \log L(\mathbb{R}^d)$ satisfies

$$f(t) \in \mathcal{Y}_0(f_{\text{in}}) \qquad \forall t \geqq 0.$$

For general estimates regarding the class of functions $\mathcal{Y}_0(f_{\text{in}})$ and the following lemma, we refer to Alexandre et al. [1]:

Lemma A.2. *Let $0 \leqq f_{\text{in}} \in L^1_2(\mathbb{R}^d) \cap L \log L(\mathbb{R}^d)$ be fixed as in Definition A.1. Then, the following hold:*

(1) For any $f \in \mathcal{Y}_0(f_{\text{in}})$, it holds that

$$\int_{|v|\leqq R} f\,\mathrm{d}v \geqq \varrho_{\text{in}} \left(1 - \frac{E_{\text{in}}}{\varrho_{\text{in}} R^2}\right) \qquad \forall R > 0. \tag{A.1}$$

(2) For any $\delta > 0$ there exists $\eta(\delta) > 0$ depending only on $\|f_{\text{in}}\|_{L^1_2}$ and $H(f_{\text{in}})$ such that for any $f \in \mathcal{Y}_0(f_{\text{in}})$ and measurable set $A \subset \mathbb{R}^d$

$$|A| \leqq \eta(\delta) \Longrightarrow \int_A f\,\mathrm{d}v \leqq \delta\,. \tag{A.2}$$

The aforementioned estimates are the key for the proof of the coercivity estimate for the matrix $\mathcal{A}[f]$ given in Remark 1.13. Namely, one has

Lemma A.3. *Let $0 \leqq f_{\text{in}} \in L^1_2(\mathbb{R}^d) \cap L \log L(\mathbb{R}^d)$ be fixed. Then, there exists a constant $K_0 > 0$, depending on d, γ, ρ_{in}, E_{in}, $H(f_{\text{in}})$ such that*

$$\sum_{i,j} \mathcal{A}_{i,j}[f](v)\, \xi_i\, \xi_j \geqq K_0 \langle v\rangle^{\gamma}\, |\xi|^2, \qquad \forall\, v,\, \xi \in \mathbb{R}^d,$$

holds for any $f \in \mathcal{Y}_0(f_{\text{in}})$.

Proof. We show here how the conclusion of Lemma A.2 allows to deduce the result. We follow the proof of [1, Proposition 2.3] given in dimension $d = 3$ and explain where some changes have to be made to treat general dimension $d \geqq 3$. First, one observes that it is enough to show the result for $\xi \in \mathbb{R}^d$ with $|\xi| = 1$. We fix such ξ and, for $\theta \in (0, \frac{\pi}{2})$, we introduce

$$D_{\theta,\xi}(v) = \left\{ v_* \in \mathbb{R}^d,\ |\xi \cdot (v - v_*)| \geqq |v - v_*| \cos\theta \right\}.$$

Given $v, v_* \in \mathbb{R}^d$, $v \neq v_*$ we set for simplicity $u = v - v_*$, $\widehat{u} = \frac{u}{|u|}$ and notice that

$$\sum_{i,j} a_{i,j}(v - v_*)\xi_i\xi_j = |u|^{\gamma+2} \sum_{i,j} \left(\delta_{i,j} - \widehat{u}_i\widehat{u}_j\right)\xi_i\xi_j = |u|^{\gamma+2}\left(1 - (\widehat{u}\cdot\xi)^2\right).$$

Therefore, if $v \in \mathbb{R}^d$ it holds that

$$\sum_{i,j} a_{i,j}(v - v_*)\xi_i\xi_j \geqq |u|^{\gamma+2}\sin^2\theta, \qquad \forall v_* \notin D_{\theta,\xi}(v).$$

For any $f \in \mathcal{Y}_0(f_{\text{in}})$ and any $v \in \mathbb{R}^d$, we have then, for any $\theta \in \left(0, \frac{\pi}{2}\right)$ and any $R > 0$

$$\sum_{i,j} \mathcal{A}_{i,j}[f](v)\,\xi_i\,\xi_j \geqq \sum_{i,j} \int_{B_R \setminus D_{\theta,\xi}(v)} f(v_*)a_{i,j}(v - v_*)\,\xi_i\,\xi_j \mathrm{d}v_*$$

where $B_R = \{v_* \in \mathbb{R}^d\,,\ |v_*| \leqq R\}$. Thus

$$\sum_{i,j} \mathcal{A}_{i,j}[f](v)\,\xi_i\,\xi_j \geqq \sin^2\theta \int_{B_R \setminus D_{\theta,\xi}} f(v_*)|u|^{\gamma+2}\mathrm{d}v_*. \tag{A.3}$$

We now fix $R > \sqrt{\frac{2E_{\text{in}}}{\varrho_{\text{in}}}}$ so that (A.1) reads $\int_{B_R} f(v_*)\mathrm{d}v_* \geqq \frac{\varrho_{\text{in}}}{2}$ and

$$\int_{B_R \setminus D_{\theta,\xi}(v)} f(v_*)\mathrm{d}v_* \geqq \frac{1}{2}\varrho_{\text{in}} - \int_{B_R \cap D_{\theta,\xi}(v)} f(v_*)\mathrm{d}v_*. \tag{A.4}$$

One argues as in Desvillettes, Villani [22] to estimate $\left|B_R \cap D_{\theta,\xi}(v)\right|$. This is the only place in the proof in which dimension d plays a role, the proof in Desvillettes, Villani [22] being given in $d = 3$. Notice that $B_R \cap D_{\theta,\xi}(v)$ is the intersection of a cone with a ball. It can be estimated from above by the intersection of the cylinder with the ball B_R where the cylinder is obtained choosing $u = v - v_*$ parallel to ξ. The radius of the basis of the cylinder is then $r := (R + |v|)\tan\theta$ and its maximal length is the diameter of the ball. Thus, $|B_R \cap D_{\theta,\xi}(v)| \leqq c_d R\, r^{d-1}$, with c_d depending only on $|\mathbb{S}^{d-2}|$, that is,

$$\left|B_R \cap D_{\theta,\xi}(v)\right| \leqq c_d R\,(R + |v|)^{d-1}\tan^{d-1}\theta. \tag{A.5}$$

We distinguish between two cases.

• *First case:* $|v| > 2R$. In such a case, for $v_* \in B_R$, one has $\frac{1}{2}|v| \leqq |v - v_*| \leqq \frac{3}{2}|v|$ and there exists $C_\gamma > 0$ such that $|v - v_*|^{\gamma+2} \geqq C_\gamma |v|^{\gamma+2}$. According to (A.3)–(A.4), it holds

$$\sum_{i,j} \mathcal{A}_{i,j}[f](v)\,\xi_i\,\xi_j \geqq C_\gamma |v|^{\gamma+2}\sin^2\theta\left(\frac{1}{2}\varrho_{\text{in}} - \int_{B_R \cap D_{\theta,\xi}(v)} f(v_*)\mathrm{d}v_*\right) \tag{A.6}$$

where, according to (A.5), $|B_R \cap D_{\theta,\xi}(v)| \leqq c_d R \left(\frac{3}{2}|v|\right)^{d-1} \tan^{d-1}\theta$. With the notations of (A.2), given $\delta = \frac{1}{4}\varrho_{\text{in}}$, we choose θ (depending on v) small enough so that

$$c_d R \left(\frac{3}{2}|v|\right)^{d-1} \tan^{d-1}\theta = \eta(\delta), \qquad \text{which entails} \quad \int_{B_R \cap D_{\theta,\xi}(v)} f(v_*)\mathrm{d}v_* \leqq \frac{1}{4}\varrho_{\text{in}},$$

according to (A.2) (notice that such a choice of θ is always possible up to increasing R). Then, (A.6) yields

$$\sum_{i,j} \mathcal{A}_{i,j}[f](v)\,\xi_i\,\xi_j \geqq \frac{C_\gamma}{4}\varrho_{\text{in}}|v|^{\gamma+2}\sin^2\theta.$$

Notice that $\tan\theta = \frac{2}{3|v|}\left(\frac{\eta(\delta)}{c_d R}\right)^{\frac{1}{d-1}}$, so that $\sin^2\theta = C_{R,d}|v|^{-2}\cos^2\theta$ for some explicit $C_{R,d}$ depending only on d, R and ρ_{in}, E_{in}, $H(f_{\text{in}})$. Therefore

$$\sum_{i,j} \mathcal{A}_{i,j}[f](v)\,\xi_i\,\xi_j \geqq \frac{C_\gamma}{4}\rho_{\text{in}}C_{R,d}|v|^{\gamma}\cos^2\theta, \qquad |v| > 2R,$$

with $\cos^2\theta > \frac{1}{2}$ since we choose θ small enough (up to having picked R larger). Thus, there is $K > 0$ (depending on d, γ, ρ_{in}, E_{in}, $H(f_{\text{in}})$) such that

$$\sum_{i,j} \mathcal{A}_{i,j}[f](v)\,\xi_i\,\xi_j \geqq K\langle v\rangle^{\gamma}, \qquad \forall |v| > 2R.$$

• *Second case:* $|v| < 2R$. In such a case, arguing exactly as in [1, Prop. 1.3], we conclude that

$$\inf_{|v|<2R}\sum_{i,j} \mathcal{A}_{i,j}[f](v)\,\xi_i\,\xi_j = \tilde{K} > 0$$

which of course implies the result with $K_0 = \min(K, \tilde{K})$. The details of this second case are left to the reader and are readily adapted from the corresponding result in Alexandre et al. [1]; Desvillettes, Villani [22]. □

Remark A.4. From the above construction, the constant K_0 depends on R (and therefore on ϱ_{in}, E_{in}) and on the construction of the mapping η in (A.2) so that K_0 depends also on $H(f_{\text{in}})$.

The following property of the dissipation of entropy for Landau operator was observed in Desvillettes [18,19]:

Proposition A.5. *Let* $0 \leqq f_{\text{in}} \in L^1_2(\mathbb{R}^d) \cap L\log L(\mathbb{R}^d)$ *be fixed as in Definition A.1. Then, there is a positive constant* $C_0(\gamma)$ *depending only on* d, γ, ρ_{in}, E_{in}, $H(f_{\text{in}})$ *such that, for all* $f \in \mathcal{Y}_0(f_{\text{in}})$,

$$\int_{\mathbb{R}^d} \left|\nabla\sqrt{f(v)}\right|^2 \langle v\rangle^{\gamma}\mathrm{d}v \leqq C_0(\gamma)\big(1 + \mathscr{D}(f)\big).$$

where $\mathscr{D}(f)$ *denotes the dissipation of entropy functional defined as*

$$\mathscr{D}(f) = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} |v - v_*|^{\gamma+2} f\,f_*\,|\Pi(v - v_*)\,(\nabla\log f - \nabla\log f_*)|^2\,\mathrm{d}v\mathrm{d}v_* \geqq 0.$$

In particular, since formally the entropy of solutions to the Landau equation is decreasing according to

$$\frac{\mathrm{d}}{\mathrm{d}t}H(f(t)) = -\mathscr{D}(f(t)), \qquad t \geqq 0,$$

one can prove that, for H-solutions $f = f(t,v)$ to the Landau equation, it holds that

$$\int_0^T \mathscr{D}(f(t))\mathrm{d}t \leqq H(f_{\mathrm{in}}) \qquad T > 0,$$

and one deduces the following result from Desvillettes [18,19]:

Corollary A.6. *If f_{in} and $f = f(t,v)$ define a solution to Eq. (1.1) in the sense of Definition 1.15 then*

$$\int_0^T \mathrm{d}t \int_{\mathbb{R}^d} \left|\nabla\left(\langle v\rangle^{\frac{\gamma}{2}}\sqrt{f(t,v)}\right)\right|^2 \mathrm{d}v \leqq \boldsymbol{C}_T(f_{\mathrm{in}}),$$

where $\boldsymbol{C}_T(f_{\mathrm{in}})$ is an explicit positive constant depending only on T, d, γ, $\varrho_{\mathrm{in}}, E_{\mathrm{in}}$ and $H(f_{\mathrm{in}})$.

Remark A.7. Notice that, since $f \in L^\infty([0,T];\ L^1(\mathbb{R}^d))$, the above can be reformulated in the equivalent way as

$$\int_0^T \mathrm{d}t \int_{\mathbb{R}^d} \left|\nabla\sqrt{f(t,v)}\right|^2 \langle v\rangle^\gamma \mathrm{d}v \leqq \boldsymbol{C}_T(f_{\mathrm{in}})$$

for some $\boldsymbol{C}_T(f_{\mathrm{in}})$ different from the previous one. It is in this form that the result is stated in Desvillettes [18].

Finally, recall the *a priori* growth of moments of solutions to the Landau equation, referring to Carrapatoso [15]; Carrapatoso et al. [16] for a proof in dimension $d = 3$. The general case is following the same lines (see for instance Alonso et al. [7] for a proof in the case of the Landau-Fermi-Dirac equation which is easily adapted to the Landau setting).

Proposition A.8. *We consider $d \in \mathbb{N}$, $d \geqq 3$ and $\gamma \in [-d,0)$, and assume that f_{in} and $f = f(t,v)$ define a solution to Eq. (1.1) in the sense of Definition 1.15. Assume that*

$$f_{\mathrm{in}} \in L^1_s(\mathbb{R}^d), \qquad s > 2,$$

then

$$\boldsymbol{m}_s(f(t)) \leqq C_s\,(1+t) \qquad t \geqq 0$$

for some positive constant C_s depending on d, γ, s, ρ_{in}, E_{in}, $H(f_{\mathrm{in}})$. In particular,

$$\sup_{t\in[0,T]} \|f(t)\|_{L^1_s} \leqq C_s(1+T) \qquad T > 0.$$

We complement also the above with the following properties of the matrix $\mathcal{A}[h]$ and vector $\boldsymbol{b}[h]$. Notice that the result applies without any sign assumption on the mapping $h : \mathbb{R}^d \to \mathbb{R}$.

Lemma A.9. *Let $d \in \mathbb{N}$, $d \geqq 3$, $-d \leqq \gamma < -2$. For any*

$$q \in \left(\frac{d}{d+\gamma+2}, \infty\right), \qquad m > d\left(1 - \frac{1}{q}\right),$$

there is $C_{m,d,\gamma,q} > 0$ *depending only on* m, d, γ, q *such that*

$$\|\mathcal{A}[h]\|_\infty \leqq C_{m,d,\gamma,q} \left\|\langle\cdot\rangle^m h\right\|_q . \tag{A.7}$$

In the same way, if $q \in \left(\frac{d}{d+\gamma+1}, \infty\right)$*,*

$$\|\boldsymbol{b}[h]\|_\infty \leqq \tilde{C}_{m,d,\gamma,q} \left\|\langle\cdot\rangle^m h\right\|_q , \qquad m > d\left(1 - \frac{1}{q}\right) , \tag{A.8}$$

for $\tilde{C}_{m,d,\gamma,q} > 0$ *depending only on* m, d, γ, q*.*
Then, for

$$\frac{d}{d+\gamma+2} < q < \frac{d}{d+\gamma+1},$$

there exists $C_{d,\gamma,q} > 0$ *depending only on* d*,* γ*,* q *such that*

$$\|\boldsymbol{b}[h]\|_{p_\gamma^*} \leqq C_{d,\gamma,q}\|h\|_q \qquad p_\gamma^* = \frac{qd}{d - q(1+\gamma+d)}. \tag{A.9}$$

Finally, if $q = \frac{d}{d+\gamma+1}$*, for any* $p \in (\frac{d}{d+\gamma+2}, q)$ *and any* $m > |\gamma+1|$*, there exists* $C_{m,d,\gamma,p} > 0$ *such that*

$$\|\boldsymbol{b}[h]\|_{\bar{p}_\gamma^*} \leqq C_{m,d,\gamma,p}\|\langle\cdot\rangle^m h\|_q, \qquad \bar{p}_\gamma^* = \frac{pd}{d - p(1+\gamma+d)} = \frac{pq}{q-p}. \tag{A.10}$$

In the case when $q = \frac{d}{d+\gamma+1} = 2$*, the estimate becomes, for any* $p \in (\frac{2d}{d+2}, 2)$*,*

$$\|\boldsymbol{b}[h]\|_{\frac{2p}{2-p}} \leqq C_{m,d,\gamma,p}\|\langle\cdot\rangle^m h\|_2 \tag{A.11}$$

Proof. Let us write, for a smooth h,

$$I_s[h](v) = \int_{\mathbb{R}^d} |v-w|^s\, |h(w)|\mathrm{d}w, \qquad s \in \mathbb{R},$$

and notice that $|\mathcal{A}[h]| \leqq 2I_{\gamma+2}[h]$ and $|\boldsymbol{b}[h]| \leqq (d-1)I_{\gamma+1}[h]$. We prove the result for $I_{\gamma+2}[h]$. One has, for any $R > 0$

$$\begin{aligned} I_{\gamma+2}[h](v) &\leqq \int_{|v-w|\leqq R} |v-w|^{\gamma+2}|h(w)|\mathrm{d}w + \int_{|v-w|>R} |v-w|^{\gamma+2}|h(w)|\mathrm{d}w \\ &\leqq \left(\int_{|v-w|\leqq R} |v-w|^{q'(\gamma+2)}\mathrm{d}w\right)^{\frac{1}{q'}} \|h\|_q + R^{\gamma+2}\|h\|_1, \end{aligned}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ and $q > \frac{d}{2+d+\gamma}$. One has then easily that there is $C > 0$ (depending on γ, d and q) such that

$$I_{\gamma+2}[h](v) \leqq C\left(R^{\gamma+2+\frac{d}{q'}}\|h\|_q + R^{\gamma+2}\|h\|_1\right),$$

which, after optimizing R (recall that $\gamma+2 < 0$ while $\gamma+2+\frac{d}{q'} > 0$ since $\frac{1}{q} < 1 + \frac{\gamma+2}{d}$), reads simply

$$|I_{\gamma+2}[h](v)| \leqq C\, \|h\|_q^{-\frac{\gamma+2}{d}q'}\, \|h\|_1^{\frac{(\gamma+2)q'+d}{d}}.$$

Since (for some C depends on d, m, q) $\|h\|_1 \leqq C\,\|h\langle\cdot\rangle^m\|_q$ for any $m > \frac{d}{q'}$, this yields (A.7).
Now, the same exact estimate holds for $|\boldsymbol{b}[h]| \leqq (d-1)I_{\gamma+1}[h]$ under the assumption $q > \frac{d}{d+\gamma+1}$ and this proves (A.8).
Let us now prove (A.9), recalling that $\frac{d}{d+\gamma+2} < q < \frac{d}{d+\gamma+1}$. According to Hardy-Littlewood-Sobolev inequality (Proposition 3.1), for any $q < \frac{d}{d+\gamma+1}$, a simple duality argument shows that

$$\| I_{\gamma+1}[h]\|_{r'} \leqq C_{\mathrm{HLS}}\|h\|_q, \qquad \frac{1}{r'} = 1 - \frac{1}{r} = \frac{1}{q} - \frac{1+\gamma}{d} - 1 = \frac{d-q(1+\gamma+d)}{qd}. \tag{A.12}$$

Notice that $r' > 1 \iff q > \frac{d}{1+\gamma+2d}$ which is satisfied since $\frac{d}{d+\gamma+2} > \frac{d}{1+\gamma+2d}$. This proves (A.9).
Strangely, the case $q = \frac{d}{1+\gamma+d}$ is not covered by the above statements and deserves a separate treatment. Indeed, Hardy-Littlewood-Sobolev inequality (A.12) for $q = \frac{d}{d+\gamma+1}$ would give $r' = \infty$, which is excluded by (3.1). We rather apply this with $p < q = \frac{d}{d+\gamma+1}$ and, as above, deduce that

$$\| I_{\gamma+1}[h]\|_{r'} \leqq C_{\mathrm{HLS}}\|h\|_p, \qquad \frac{1}{r'} = \frac{d-p(1+\gamma+d)}{pd} = \frac{1}{p} - \frac{1}{q}.$$

Since $1 < p < q$, we use a further interpolation $\|h\|_p \leqq \|h\|_1^{\theta}\,\|h\|_q^{1-\theta}$ with $\theta = \frac{q-p}{p(q-1)}$ and, with Young's inequality

$$\|h\|_p \leqq \|h\|_1 + \|h\|_q \leqq C\|\langle\cdot\rangle^m h\|_q, \qquad m > d\left(1 - \frac{1}{q}\right).$$

This concludes the proof of the Lemma. □

We finally recall a result regarding the ε-Poincaré inequality in the (critical) case $\gamma = -2$ in any dimension established in Alonso et al. [5].

Proposition A.10. *Let $d \in \mathbb{N}$, $d \geqq 3$. Assume f_{in} be as in Definition A.1. Then there exists $C_0 > 0$ depending only on $\|f_{\mathrm{in}}\|_{L^1_2}$ and $H(f_{\mathrm{in}})$ such that, for any $\varepsilon > 0$,*

$$\int_{\mathbb{R}^d} \phi^2 \boldsymbol{c}_{-2}[g]\mathrm{d}v \leqq \varepsilon \int_{\mathbb{R}^d} \left|\nabla\left(\langle v\rangle^{-1}\phi(v)\right)\right|^2 \mathrm{d}v$$
$$+C_0 \exp\left(\varepsilon^{-\frac{s}{s-2}}\,\boldsymbol{m}_s(g)^{\frac{s}{s-2}}\right)\int_{\mathbb{R}^d} \phi^2\langle v\rangle^{-2}\mathrm{d}v$$

holds true for any suitable nonnegative functions $g \in \mathcal{Y}_0(f_{\mathrm{in}}) \cap L^1_s(\mathbb{R}^d)$ ($s > 2$) and ϕ.

Proof. We refer to Alonso et al. [5] for a full proof of the result in general dimension $d \geqq 3$ which uses Lorentz spaces. We just explain the main steps of it, referring to Alonso et al. [5] for details and to Bahouri et al. [10] for definition and properties of Lorentz spaces. As in the proof of Proposition 1.7, eq. (3.14), we decompose, according to $|v - v_*| \gtreqqless \frac{\langle v\rangle}{2}$ or not,

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |v-v_*|^{-2} g(v_*)\phi^2(v)\mathrm{d}v = I_1 + I_2,$$

with $I_1 \leqq \|g\|_1 \|\psi^2\|_1$, and $I_2 \leqq C_{d,\gamma} \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - v_*|^{-2} \psi(v)^2 F(v_*) \mathrm{d}v_* \mathrm{d}v$, still using the notations $F(v) = \langle v \rangle^2 g(v)$, $\psi(v) = \langle v \rangle^{-1} \phi(v)$. Then $I_2 \leqq J_1 + J_2$, with

$$J_1 := \int_{|v - v_*| > 1} |v - v_*|^{-2} F(v_*) \psi^2(v) \mathrm{d}v \mathrm{d}v_* \leqq \|F\|_{L^1} \|\psi\|_2^2,$$

and

$$J_2 := \int_{|v - v_*| \leqq 1} |v - v_*|^{-2} F(v_*) \psi^2(v) \mathrm{d}v \mathrm{d}v_* .$$

The term J_2 is then further split, for $R > 0$, according to

$$F = F_R^+ + F_R^-, \qquad F_R^+ = F \mathbf{1}_{F > R}, \qquad F_R^- = F \mathbf{1}_{F \leqq R},$$

and

$$J_{2,R}^{\pm} := \int_{|v - v_*| \leqq 1} |v - v_*|^{-2} \, F_R^{\pm}(v_*) \, \psi^2(v) \mathrm{d}v \mathrm{d}v_*.$$

Then, $J_{2,R}^- \leqq C_d \, R \, \|\psi\|_2^2$, while, thanks to the use of Lorentz spaces,

$$J_{2,R}^+ \leqq C_d \|F_R^+\|_1 \, \|\psi\|^2_{L^{\frac{2d}{d-2},2}} \leqq C_d \|F_R^+\|_1 \, \|\nabla \psi\|_2^2.$$

Then, we end up with

$$\int_{\mathbb{R}^d} \phi^2 \boldsymbol{c}_{-2}[g] \mathrm{d}v \leqq C_{d,\gamma} \left(\|F\|_1 \|\psi\|_2^2 + R \|\psi\|_2^2 + \|F_R^+\|_1 \, \|\nabla \psi\|_2^2 \right), \qquad \forall R > 0 \, .$$

In order to conclude the proposition, one needs to prove that $\|F_R^+\|_1$ is small as R is sufficiently large. This comes from the observation that for, $s > 2$ and $\theta_s = 1 - \frac{2}{s}$ and any suitable function $g \in L_s^1 \cap L \log L(\mathbb{R}^d)$,

$$\|\langle \cdot \rangle^2 g \mathbf{1}_{f > R}\|_1 \leqq \|\langle \cdot \rangle^s g \mathbf{1}_{g > R}\|_1^{1-\theta_s} \|g \mathbf{1}_{g > R}\|_1^{\theta_s} \leqq \|\langle \cdot \rangle^s g\|_1^{1-\theta_s} \left(\frac{\|g \log g\|_1}{\log(R)} \right)^{\theta_s}.$$

We refer the reader to Alonso et al. [5] for more details. □

Appendix B: Evolution of Weighted L^p-Norms

We briefly explain here how the analysis of Sections 2 and 3 can be modified to allow the appearance of *weighted* L^p-norms. We recall the notations

$$\mathbb{M}_{k,p}(t) := \int_{\mathbb{R}^d} f(t,v)^p \langle v \rangle^k \mathrm{d}v, \qquad \mathbb{D}_{k,p}(t) := \int_{\mathbb{R}^d} \left| \nabla \left(\langle v \rangle^{\frac{k}{2}} f^{\frac{p}{2}}(t,v) \right) \right|^2 \mathrm{d}v$$

for $k \in \mathbb{R}$ and $p \in (1, \infty)$. Let us dig directly into the technical lemma as follows:

Lemma B.1. *Consider $d \in \mathbb{N}$, $d \geqq 3$ and $\gamma \in [-d, 0)$, and assume that f_{in} and $f = f(t,v)$ define a solution to Eq.* (1.1) *in the sense of Definition* 1.15*. Then, for all $k \in \mathbb{R}_+$ and $p \in (1, \infty)$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{M}_{k,p}(t) + 2K(p) \mathbb{D}_{k+\gamma,p}(t) \leqq K(p)(k+\gamma)^2 \mathbb{M}_{k+\gamma,p}(t)$$

$$+C_{k,\gamma,p}\sum_{i=0}^{2}\int_{\mathbb{R}^d}\langle v\rangle^{k-i}\boldsymbol{c}_{\gamma+i}[f(t)](v)\,f^p(t,v)\mathrm{d}v, \tag{B.1}$$

with (K_0 being the constant appearing in Remark 1.13)

$$K(p) := \frac{p-1}{p}K_0, \qquad C_{k,\gamma,p} := \max\left(p-1,\, \frac{2k}{\gamma+1+d},\, \frac{d^2k^2}{(d-1)(d+\gamma+2)}\right).$$

Proof. We easily check that, for any $k \geqq 0$,

$$\begin{aligned}
\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} f^p(t,v)\langle v\rangle^k \mathrm{d}v &= \int_{\mathbb{R}^d}\langle v\rangle^k f^{p-1}(t,v)\nabla\cdot(\mathcal{A}[f]\nabla f - \boldsymbol{b}[f]f)\,\mathrm{d}v\\
&= -(p-1)\int_{\mathbb{R}^d}\langle v\rangle^k f^{p-2}\mathcal{A}[f]\nabla f\cdot\nabla f\mathrm{d}v + (p-1)\int_{\mathbb{R}^d}\langle v\rangle^k f^{p-1}\boldsymbol{b}[f]\cdot\nabla f\mathrm{d}v\\
&\quad -k\int_{\mathbb{R}^d}\langle v\rangle^{k-2} f^{p-1}\,(\mathcal{A}[f]\nabla f)\cdot v\mathrm{d}v + k\int_{\mathbb{R}^d}\langle v\rangle^{k-2} f^p\boldsymbol{b}[f]\cdot v\,\mathrm{d}v.
\end{aligned}$$

Arguing as in the proof of Proposition 3.3, we obtain that

$$(p-1)\int_{\mathbb{R}^d}\langle v\rangle^k f^{p-2}\mathcal{A}[f]\nabla f\cdot\nabla f\mathrm{d}v \geqq \frac{4K_0(p-1)}{p^2}\int_{\mathbb{R}^d}\langle v\rangle^{k+\gamma}\left|\nabla(f^{\frac{p}{2}})\right|^2\mathrm{d}v.$$

Moreover, writing

$$\nabla\left(\langle v\rangle^{\frac{k+\gamma}{2}} f^{\frac{p}{2}}\right) = \langle v\rangle^{\frac{k+\gamma}{2}}\nabla(f^{\frac{p}{2}}) + \frac{k+\gamma}{2}v\,\langle v\rangle^{\frac{k+\gamma}{2}-2} f^{\frac{p}{2}},$$

from which

$$\langle v\rangle^{k+\gamma}\left|\nabla(f^{\frac{p}{2}})\right|^2 \geqq \frac{1}{2}\left|\nabla\left(\langle v\rangle^{\frac{k+\gamma}{2}} f^{\frac{p}{2}}\right)\right|^2 - \frac{1}{4}(k+\gamma)^2\langle v\rangle^{k+\gamma-2} f^p, \tag{B.2}$$

we observe that

$$\begin{aligned}
(p-1)\int_{\mathbb{R}^d}\langle v\rangle^k f^{p-2}\mathcal{A}[f]\nabla f\cdot\nabla f\mathrm{d}v &\geqq \frac{2K_0(p-1)}{p^2}\int_{\mathbb{R}^d}\left|\nabla\left(\langle v\rangle^{\frac{k+\gamma}{2}} f^{\frac{p}{2}}\right)\right|^2\mathrm{d}v\\
&\quad -\frac{K_0(p-1)(k+\gamma)^2}{p^2}\int_{\mathbb{R}^d}\langle v\rangle^{k+\gamma-2} f^p\mathrm{d}v.
\end{aligned}$$

Also, note that

$$\begin{aligned}
\int_{\mathbb{R}^d}\langle v\rangle^k f^{p-1}\boldsymbol{b}[f]\cdot\nabla f\mathrm{d}v &= -\frac{1}{p}\int_{\mathbb{R}^d} f^p\nabla\cdot\left(\boldsymbol{b}[f]\langle v\rangle^k\right)\mathrm{d}v\\
&= -\frac{k}{p}\int_{\mathbb{R}^d}\langle v\rangle^{k-2} f^p\boldsymbol{b}[f]\cdot v\,\mathrm{d}v + \frac{1}{p}\int_{\mathbb{R}^d}\langle v\rangle^k f^p\boldsymbol{c}_\gamma[f]\,\mathrm{d}v.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}_{k,p}(t) + \frac{2K_0(p-1)}{p}\mathbb{D}_{k+\gamma,p}(t)\\
&\leqq (p-1)\int_{\mathbb{R}^d}\langle v\rangle^k f^p\,\boldsymbol{c}_\gamma[f]\mathrm{d}v + k\int_{\mathbb{R}^d}\langle v\rangle^{k-2} f^p\;(\boldsymbol{b}[f]\cdot v)\,\mathrm{d}v\\
&\quad + \frac{K_0(p-1)(k+\gamma)^2}{p}\int_{\mathbb{R}^d}\langle v\rangle^{k+\gamma-2} f^p\mathrm{d}v - kp\int_{\mathbb{R}^d}\langle v\rangle^{k-2} f^{p-1}\,(\mathcal{A}[f]\nabla f\cdot v)\,\mathrm{d}v.
\end{aligned}$$

Let us investigate in more detail the latter term. Integration by parts shows that

$$-kp\int_{\mathbb{R}^d}\langle v\rangle^{k-2}f^{p-1}\left(\mathcal{A}[f]\nabla f\right)\cdot v\mathrm{d}v = -k\int_{\mathbb{R}^d}\nabla(f^p)\cdot\left(\mathcal{A}[f]\,\langle v\rangle^{k-2}v\right)\mathrm{d}v$$
$$= k\int_{\mathbb{R}^d}f^p\,\nabla\cdot\left(\mathcal{A}[f]\,\langle v\rangle^{k-2}v\right)\mathrm{d}v\,.$$

Using the product rule

$$\nabla\cdot\left(\mathcal{A}[f]\,\langle v\rangle^{k-2}v\right) = \langle v\rangle^{k-2}\,\boldsymbol{b}[f]\cdot v\,+\mathrm{Trace}\left(\mathcal{A}[f]\cdot\,D_v\left(\langle v\rangle^{k-2}v\right)\right),$$

where $D_v\big(\langle v\rangle^{k-2}v\big)$ is the matrix with entries $\partial_{v_i}\big(\langle v\rangle^{k-2}v_j\big)$, $i,j=1,\ldots,d$, or more compactly,

$$D_v\big(\langle v\rangle^{k-2}v\big) = \langle v\rangle^{k-4}\boldsymbol{A}(v),$$

where $\boldsymbol{A}(v)=\langle v\rangle^2\mathbf{Id}+(k-2)\,v\otimes v$, $v\in\mathbb{R}^d$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}_{k,p}(t)+\frac{2K_0(p-1)}{p}\mathbb{D}_{k+\gamma,p}(t) &\leqq (p-1)\int_{\mathbb{R}^d}\langle v\rangle^k\boldsymbol{c}_\gamma[f](v)\,f^p\mathrm{d}v\\ &+2k\int_{\mathbb{R}^d}\langle v\rangle^{k-2}\,f^p\,(\boldsymbol{b}[f]\cdot v)\,\mathrm{d}v+\frac{K_0(p-1)(k+\gamma)^2}{p}\int_{\mathbb{R}^d}\langle v\rangle^{k+\gamma-2}f^p\mathrm{d}v\\ &+k\int_{\mathbb{R}^d}\langle v\rangle^{k-4}f^p\;\mathrm{Trace}\left(\mathcal{A}[f]\cdot\boldsymbol{A}(v)\right)\mathrm{d}v.\end{aligned}\tag{B.3}$$

We denote by I_1, I_2, I_3, I_4 the various terms on the right-hand-side of (B.3) and control each of them separately. We get

$$\begin{aligned}|I_2| &\leqq 2k\int_{\mathbb{R}^d}\langle v\rangle^{k-1}f^p(t,v)\,|\boldsymbol{b}[f(t)](v)|\mathrm{d}v\\ &\leqq 2k(d-1)\int_{\mathbb{R}^{2d}}\langle v\rangle^{k-1}f^p(t,v)|v-v_*|^{\gamma+1}f(t,v_*)\mathrm{d}v_*\mathrm{d}v\,,\end{aligned}\tag{B.4}$$

that is,

$$|I_2|\leqq\frac{2k}{\gamma+1+d}\int_{\mathbb{R}^d}\langle v\rangle^{k-1}f^p(t,v)\boldsymbol{c}_{\gamma+1}[f(t)](v)\mathrm{d}v.$$

Clearly, we also get

$$|I_3|\leqq\frac{K_0(p-1)(k+\gamma)^2}{p}\mathbb{M}_{k+\gamma,p}(t).$$

For the term I_4, one checks easily that, for any $i,j\in\{1,\ldots,d\}$,

$$\big|\mathcal{A}_{i,j}[f]\big|\leqq|\cdot|^{\gamma+2}*f,\qquad\big|\boldsymbol{A}_{i,j}(v)\big|\leqq k\langle v\rangle^2,$$

and

$$|I_4|\leqq d^2k^2\int_{\mathbb{R}^{2d}}\langle v\rangle^{k-2}f^p(t,v)|v-v_*|^{\gamma+2}f(t,v_*)\mathrm{d}v\mathrm{d}v_*.$$

Therefore,

$$|I_4|\leqq\frac{d^2k^2}{(d-1)(d+\gamma+2)}\int_{\mathbb{R}^d}\langle v\rangle^{k-2}f^p(t,v)\boldsymbol{c}_{\gamma+2}[f](v)\mathrm{d}v,$$

which proves the result thanks to (B.3). □

We now estimate the various terms

$$\int_{\mathbb{R}^d} \langle v\rangle^{k-i} \boldsymbol{c}_{\gamma+i}[f(t)](v) f^p(t,v)\mathrm{d}v, \qquad i = 0, 1, 2,$$

and observe that the less favourable estimate corresponds to the most singular case $i = 0$.

Lemma B.2. *Consider $d \in \mathbb{N}$. For any $-d < \gamma < 0$, $p \in (1,\infty)$, $k \in \mathbb{R}_+$, and suitable functions $f \geqq 0$ and h, the following estimate holds:*

$$\sum_{i=0}^{2} \int_{\mathbb{R}^d} \langle v\rangle^{k-i} \boldsymbol{c}_{\gamma+i}[f](v)\,|h(v)|^p \mathrm{d}v$$
$$\leqq C_{d,\gamma} \int_{\mathbb{R}^d} \langle v\rangle^k \overline{\boldsymbol{c}}_\gamma[f](v)|h(v)|^p\mathrm{d}v + c_{d,\gamma}\int_{\mathbb{R}^d} \langle v\rangle^k |h(v)|^p\mathrm{d}v.$$

Here $C_{d,\gamma} := 3\frac{\gamma+d+2}{\gamma+d}$, $c_{d,\gamma} := 3(d-1)\,(\gamma+d+2)\,\|f\|_{L^1_2}$, and where

$$\overline{\boldsymbol{c}}_\gamma[f] := (d-1)(d+\gamma)\int_{|v-v_*|\leqq 1} |v-v_*|^\gamma f(v_*)\mathrm{d}v_*.$$

In the Coulomb case $\gamma = -d$, the estimate becomes

$$\sum_{i=0}^{2} \int_{\mathbb{R}^d} \langle v\rangle^{k-i} \boldsymbol{c}_{\gamma+i}[f](v)\,|h(v)|^p \mathrm{d}v \leqq c_d \int_{\mathbb{R}^d} \langle v\rangle^k\, f(v)\,|h(v)|^p\mathrm{d}v$$
$$+\tilde{C}_d \int_{\mathbb{R}^d} \langle v\rangle^{k-1} \overline{\boldsymbol{c}}_{1-d}[f](v)\,|h(v)|^p\mathrm{d}v + \tilde{c}_d \int_{\mathbb{R}^d}\langle v\rangle^{k-1}|h(v)|^p\mathrm{d}v, \tag{B.5}$$

for $\tilde{C}_d := 3(d-1)$, $\tilde{c}_d := 3(d-1)\,\|f_{\mathrm{in}}\|_1$.

Proof. For $i = 0, 1, 2$, one simply writes

$$\boldsymbol{c}_{\gamma+i}[f](v) = (d-1)\,(\gamma+d+i)\int_{\mathbb{R}^d} f(v_*)|v-v_*|^{\gamma+i}\mathrm{d}v_*,$$

and splits the above integral according to $|v-v_*| > 1$ and $|v-v_*| \leqq 1$. Clearly,

$$\int_{|v-v_*|\leqq 1} f(v_*)|v-v_*|^{\gamma+i}\mathrm{d}v_* \leqq \int_{|v-v_*|\leqq 1} f(v_*)|v-v_*|^{\gamma}\mathrm{d}v_*$$
$$\leqq \frac{1}{(d-1)(\gamma+d)}\overline{\boldsymbol{c}}_\gamma[f],$$

whereas, given that $\gamma < 0$,

$$\int_{|v-v_*|>1} f(v_*)|v-v_*|^{\gamma+i}\mathrm{d}v_* \leqq \int_{|v-v_*|>1} f(v_*)|v-v_*|^{i}\mathrm{d}v_*$$
$$\leqq \langle v\rangle^i \int_{\mathbb{R}^d} f(v_*)\langle v_*\rangle^i\mathrm{d}v_*,$$

since $|v-v_*| \leqq \langle v\rangle\langle v_*\rangle$. Recalling that $i = 0, 1, 2$,

$$\int_{\mathbb{R}^d} f(v_*)\langle v_*\rangle^i\mathrm{d}v_* \leqq \int_{\mathbb{R}^d} f(v_*)\langle v_*\rangle^2\mathrm{d}v_* = \|f\|_{L^1_2}.$$

Thus,

$$\boldsymbol{c}_{\gamma+i}[f] \leqq \frac{\gamma+d+i}{\gamma+d}\,\bar{\boldsymbol{c}}_{\gamma}[f] + (d-1)\,(\gamma+d+i)\,\langle v\rangle^{i}\,\|f\|_{L^1_2},$$

and the result follows by defining $C_{d,\gamma}$ and $c_{d,\gamma}$ as in the statement.
In the case $\gamma=-d$, the proof is almost identical and one simply needs to estimate the sum

$$\sum_{i=1}^{2}\int_{\mathbb{R}^d}\langle v\rangle^{k-i}\,\boldsymbol{c}_{i-d}[f](v)\,|h(v)|^p\mathrm{d}v.$$

Then we observe that for $i=1,2$,

$$\int_{|v-v_*|\leqq 1} f(v_*)|v-v_*|^{i-d}\mathrm{d}v_* \leqq \bar{\boldsymbol{c}}_{1-d}[f], \qquad \text{while}$$
$$\int_{|v-v_*|\geqq 1} f(v_*)|v-v_*|^{i-d}\mathrm{d}v_* \leqq \|f\|_1.$$

Therefore, $\boldsymbol{c}_{i-d}[f] \leqq (d-1)\,i\left(\bar{\boldsymbol{c}}_{1-d}[f]+\|f\|_1\right)$, which gives (B.5). □

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(*Received December 12, 2023 / Accepted April 19, 2024*)
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