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Micro-Slip-Induced Multiplicative Plasticity: Existence of Energy Minimizers

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Abstract

To account for material slips at microscopic scale, we take deformation mappings as SBV functions φ, which are orientation-preserving outside a jump set taken to be two-dimensional and rectifiable. For their distributional derivative $F = D\varphi$ we examine the common multiplicative decomposition $F = F^e F^p$ into so-called elastic and plastic factors, the latter a measure. Then, we consider a polyconvex energy with respect to F^e, augmented by the measure $|\mathrm{curl}\, F^p|$. For this type of energy we prove the existence of minimizers in the space of SBV maps. We avoid self-penetration of matter. Our analysis rests on a representation of the slip system in terms of currents (in the sense of geometric measure theory) with both $\mathbb{Z}^3$ and $\mathbb{R}^3$ valued multiplicity. The two choices make sense at different spatial scales; they describe separate but not alternative modeling options. The first one is particularly significant for periodic crystalline materials at a lattice level. The latter covers a more general setting and requires to account for an energy extra term involving the slip boundary size. We include a generalized (and weak) tangency condition; the resulting setting embraces gliding and cross-slip mechanisms. The possible highly articulate structure of the jump set allows one to consider also the resulting setting even as an approximation of climbing mechanisms.

1. Introduction

We consider energies of bodies undergoing irrecoverable strain that emerges from the cumulative effects of internal slips. We investigate the existence of pertinent minimizers in the space of special bounded variation functions. The scheme is particularly appropriate for (periodic) crystalline bodies, where slips are associated with dislocations, but we can consider it as a rather appreciable approximated picture even for those amorphous bodies in which slips among grains of various nature are a dominant mechanism among the sources of irrecoverable strain.

Consider the traditional picture of body morphology: the choice of a region in space (a minimalist choice, indeed). In such a setting, to represent the presence of irrecoverable strain, we commonly accept the multiplicative decomposition of the deformation gradient F, into so-called *elastic* (F^e) and *plastic* (F^p) factors, namely

$$F = F^e F^p \tag{1.1}$$

(the first authors introducing the decomposition are those of references [4–6,17,19]; for historical reasons we tend to call it the Kröner-Lee decomposition).

Let Ω be a fit region in the three-dimensional point space, endowed with piecewise Lipschitz boundary, a region that we take as a reference. At every $x \in \Omega$, F^p maps tangent vectors to Ω at x onto a linear space where, at least pictorially, we think of representing the local rearrangement of matter in a small neighborhood of x. Then, F^e maps that linear space onto the tangent space of a configuration considered deformed with respect to Ω. This last mapping does not involve any structural irreversible change in the matter. So, F is compatible with a deformation, that is, $F = D\varphi$, with $\varphi : \Omega \to \mathbb{R}^3$, which we consider to be orientation preserving, while in general F^e and F^p are not compatible. Unless we are in very special conditions, such as those that we can assimilate to the sliding of a deck of cards, we cannot write φ as a composition of two maps, one of elastic nature (to be definite in some way), the other with plastic character (see the detailed analyses on crystal lattices developed in references [9,28–30]). By varying x in Ω, the union of all linear spaces reached by $F^p(x)$ is not (or better, *not necessarily*) the tangent bundle of some intermediate configuration. Instead, we prefer to speak of *intermediate spaces*, which visualize the ideal decomposition of recoverable strain from irreversible rearrangements of matter depicted by the product $F^e F^p$.

This view is also compatible with a scheme in which material rearrangements leading to irrecoverable strain can be described through a multiplicity of reference shapes, a parameterized family of configurations with infinitesimal generator a volume-preserving vector field, as proposed in reference [24]. In this setting, a mechanical dissipation inequality written relatively to such changes allows us to derive from a unique invariance requirement all pertinent rules [24]. At variance, here, we just consider equilibrium along a deformation allowing slips over two-dimensional rectifiable sets.

For this reason, we consider φ to be a special function of bounded variation, namely $\varphi \in SBV(\Omega, \mathbb{R}^3)$, and assume that it preserves orientation outside its jump set $S(\varphi)$ and avoids self-penetration of matter. As such the distributional derivative $F = D\varphi$ of φ is a measure that is compatible with the multiplicative decomposition (1.1), as shown for single crystal slips in reference [31] and further analyzed in terms of lattice-to-continuum limit [32]. So, we take the plastic factor F^p to be an $\mathbb{R}^{3\times 3}$-valued bounded measure in Ω, which decomposes as

$$F^p = a(x) I \mathscr{L}^3 + \hat{F}(\bar{S}, \overline{\Gamma}),$$

where I is the 3×3 identity matrix, $\mathscr{L}^3$ is the Lebesgue measure, and $a(x)$ is a measurable function in Ω satisfying

$$C^{-1} \leq a(x) \leq C \qquad \forall x \in \Omega$$

for some given real constant $C > 1$. The presence of a accounts for possible plastic volume changes. In reference [31] the case $a = 1$ is considered and the last addendum in the structure of F^p is the Schmid tensor associated with a crystal slip system. Here, instead, we substitute that tensor with $\hat{F}(\bar{S}, \overline{\Gamma})$. It is a tensor-valued rectifiable measure supported by a 2-rectifiable set in such a way that the measure curl $F^p = \text{curl}\,\hat{F}(\bar{S}, \overline{\Gamma})$ is supported on a 1-rectifiable set. This last one coincides with a dislocation when we refer to (periodic or even quasi-periodic, although to an extent in the latter case) crystals. More in general such a set describes a generic line defect. This type of defect is rather ubiquitous, from bubble rafts to liquid crystals etc.

The symbols $\bar{S}$ and $\bar{\Gamma}$ indicate a 2-current and its boundary (in the sense of geometric measure theory), the supports of which are, respectively, the jump set S and its boundary.

We analyze two cases:

- First $\bar{S}$ is taken to be a $\mathbb{Z}^3$-valued rectifiable current with boundary $\overline{\Gamma}$ that describes the dislocation measure curl F^p. This is a sort of toy model, which has however a meaning when we refer to periodic crystals and look at the lattice scale. In that case, in fact, it works (or can be considered as an appropriate approximation) when we normalize the Burgers vector with respect to the lattice spacing (see [18] for an example on edge dislocations).
- Then, $\bar{S}$ is taken to be a size bounded rectifiable current with $\mathbb{R}^3$-valued multiplicity Θ. This choice covers, for example, standard issues in the analysis of crystal dislocations at continuum scale.

We consider the elastic factor F^e as a tensor-valued summable field

$$F^e \in L^1(\Omega, \mathbb{R}^{3\times 3}; |F^p|),$$

where $|F^p|$ is the total variation of F^p. Being F^e as such, its minors are required to satisfy some integrability assumptions with respect to the Lebesgue measure. It means that, once F^e is assigned, we can evaluate averaged strain over lines, surfaces, and volumes.

The multiplicative decomposition (1.1) implies that the jump set $S(\varphi)$ identifies the 2-rectifiable set corresponding to the current $\bar{S}$.

In any case, we avoid to be abundant in analytical assumptions that we cannot interpret in physical terms at least to the extent of our knowledge, although with this we pay something in terms of convenience and results.

In this setting we first consider an energy given by

$$\mathscr{F}_{p,s}(\varphi) := \int_\Omega \left(|M(F^e(x))|^p + |\det F^e(x)|^{-s}\right) \mathrm{d}x + |\text{curl}\, F^p|(\Omega), \qquad (1.2)$$

where $M(F^e)$ is the vector with entries all minors of the elastic factor F^e, and $p > 1$, $s > 0$ are real exponents. We prove for $\mathscr{F}_{p,s}$ existence of minimizers in the SBV space, under Dirichlet-type boundary conditions, in the $\mathbb{Z}^3$-valued case.

Then, we analyze the energy

$$\widetilde{\mathscr{F}}_{p,s}(\varphi) := \int_\Omega \left(|M(F^e(x))|^p + |\det F^e(x)|^{-s}\right) \mathrm{d}x + |\mathrm{curl}\, F^p|(\Omega) + \mathbf{S}(\overline{\Gamma}), \tag{1.3}$$

where $\mathbf{S}(\overline{\Gamma})$ is a constant-density line energy, that is, the size of a current associated with a line-defect. For $\widetilde{\mathscr{F}}_{p,s}(\varphi)$ we prove existence of minimizers satisfying Dirichlet-type boundary conditions when we deal with $\mathbb{R}^3$-valued current multiplicity. This last choice imposes the boundary current $\overline{\Gamma} = \partial \bar{S}$ to be with bounded size.

Unitary dimensional constants that grant correct physical dimensions for the energy expressions are left implicit.

Our results apply also in the more general case in which the energy dependence on F^e is through a density which is a convex function of the F^e minors. Also, beyond plasticity, the SBV setting seems to be natural for the mechanics of elastic microcracked bodies [23].

2. Background material

2.1. Special functions of bounded variation

A real valued summable function $v \in L^1(\Omega)$ is said to be *of bounded variation* when its distributional derivative Dv is a finite $\mathbb{R}^3$-valued measure in Ω. In this case, v is approximately differentiable $\mathscr{L}^3$-a.e. in Ω, and the approximate gradient ∇v agrees with the density of the Radon-Nikodym derivative of Dv with respect to the Lebesgue measure $\mathscr{L}^3$. Therefore, the decomposition $Dv = \nabla v\, \mathscr{L}^3 + D^s v$ holds, where the component $D^s v$ is singular with respect to $\mathscr{L}^3$.

The *jump set* $S(v)$ of v is a countably 2-rectifiable subset of Ω that agrees $\mathscr{H}^2$-essentially with the complement of the Lebesgue set of v. $\mathscr{H}^2$ is the two-dimensional Hausdorff measure. If, in addition, the singular component $D^s v$ is concentrated on the jump set $S(v)$, we say that v is a *special function of bounded variation*, and write in short $v \in SBV(\Omega)$. In this case, we find $D^s v = D^J v$, with $D^J v = (v^+ - v^-)\nu\, \mathscr{H}^2 \llcorner S(v)$, where $v^\pm$ are the one-sided limits at points in the jump set $S(v)$ with respect to the given unit normal ν to $S(v)$.

A vector field $u : \Omega \to \mathbb{R}^3$ belongs to the class $SBV(\Omega, \mathbb{R}^3)$ if all its components u^j are in $SBV(\Omega)$. Therefore, the distributional derivative Du belongs to the class $\mathscr{M}_b(\Omega, \mathbb{R}^{3\times 3})$ of *matrix-valued bounded Radon measures*; it decomposes as

$$Du = \nabla u\, \mathscr{L}^3 + D^J u\,,$$

where the approximate gradient ∇u belongs to $L^1(\Omega, \mathbb{R}^{3\times 3})$. The pertinent jump component reads as $D^J u = (u^+ - u^-) \otimes \nu\, \mathscr{H}^2 \llcorner S(u)$, where the jump set $S(u) := \cup_{j=1}^3 S(u^j)$ is oriented by the unit normal ν and the one-sided limits $u^\pm$ are defined componentwise. Therefore, the total variation $|Du|(B)$ of Du is

$$|Du|(B) = \int_B |\nabla u|\, \mathrm{d}x + \int_{B\cap S(u)} |u^+ - u^-|\, \mathrm{d}\mathscr{H}^2$$

for each Borel set $B \subset \Omega$ (the treatise [2] presents an accurate analysis of SBV functions).

2.2. Compatibility condition

We take two isomorphic copies of the three-dimensional real space, say $\tilde{\mathbb{R}}^3$ and $\mathbb{R}^3$, with the isomorphism being just the identification. We select Ω in $\mathbb{R}^3$ and consider it as a reference configuration. For every $x \in \Omega$, we refer to a pertinent local basis $\{\mathbf{e}_A\}$. Capital indices indicate from now on coordinates in the reference configuration.

Orientation preserving differentiable maps φ select deformed shapes, with respect to Ω, in $\tilde{\mathbb{R}}^3$, endowed with basis $\{\tilde{\mathbf{e}}_i\}$. Lower-case indices indicate coordinates in the deformed shape.

Here and below F is a linear operator mapping at each $x \in \Omega$ the tangent space $T_x\Omega$ into $\tilde{\mathbb{R}}^3$, so that we write $F(x) \in \mathrm{Hom}(T_x\Omega, \tilde{\mathbb{R}}^3)$, intending F of the form $F = F^i_A \tilde{\mathbf{e}}_i \otimes \mathbf{e}^A$. By exploiting the identification mentioned above, for short-hand notation we write just $\mathbb{R}^3$ in what follows. The setting will distinguish whether we are in the reference space or the current one.

Take $\psi \in C^1_c(\Omega, \mathbb{R}^{3\times3})$ as a tensor valued field with components ψ^i_A. We set $\mathrm{curl}\,\psi \in C^0_c(\Omega, \mathbb{R}^{3\times3})$ as the tensor valued field $\nabla \times \psi$ with components $(\mathrm{curl}\,\psi)^i_A := (\epsilon_A{}^B{}_C(\partial\psi^i_B)^C)^\mathsf{T}$, where ϵ is the Levi-Civita alternating symbol. The superscript T means transposition.

We intend $\mathrm{curl}\,F$ as a measure defined in a distributional sense for any $F \in \mathscr{M}_b(\Omega, \mathbb{R}^{3\times3})$ of the form above, so that

$$\langle \mathrm{curl}\,F, \psi\rangle := \langle F, \mathrm{curl}\,\psi\rangle = \sum_{i,A=1}^{3} \langle F^i_A, (\mathrm{curl}\,\psi)^i_A\rangle, \quad \psi \in C^1_c(\Omega, \mathbb{R}^{3\times3}).$$

If $F = Du$ for some $u \in BV(\Omega, \mathbb{R}^3)$, since $\mathrm{div}(\mathrm{curl}\,\psi) = 0$ for every $\psi \in C^2_c(\Omega, \mathbb{R}^{3\times3})$, then F itself satisfies the *compatibility condition*

$$\mathrm{curl}\,F = 0. \tag{2.1}$$

Actually, the inverse implication holds, too. In fact, a result by M. Miranda [25] tells us that any $\mathbb{R}^3$-valued distribution T in Ω, with distributional derivative DT a finite measure in $\mathscr{M}_b(\Omega, \mathbb{R}^{3\times3})$, can be represented as $T = u\,\mathscr{L}^3$ for some $u \in L^1(\Omega, \mathbb{R}^3)$ [33]. Moreover, since the domain Ω is simply-connected, any measure $F \in \mathscr{M}_b(\Omega, \mathbb{R}^{3\times3})$ satisfying the compatibility condition (2.1) is equal to the derivative DT of a distribution T, so that $F = Du$ with $u \in BV(\Omega, \mathbb{R}^3)$. In particular, if F is absolutely continuous with density in $L^q(\Omega, \mathbb{R}^{3\times3})$ for some $q \geq 1$, we set $F = \nabla u\,\mathscr{L}^3$ for some Sobolev vector field $u \in W^{1,q}(\Omega, \mathbb{R}^3)$.

3. Slip planes

As a special case consider Ω to be endowed only with a slip plane S (this assumption holds only in this section). Assume that the slip activates along a deformation $\varphi : \Omega \to \mathbb{R}^3$ so that φ jumps across the slip and is per se a SBV map with distributional derivative

$$D\varphi = \nabla\varphi\, \mathscr{L}^3 + b \otimes \nu\, \mathscr{H}^2 \llcorner S, \tag{3.1}$$

where $b \in \mathbb{R}^3$ is the pertinent Burgers vector. The orientation preserving condition outside the slip plane is $\det \nabla\varphi > 0$ almost everywhere in Ω. Therefore, since $\operatorname{curl} D\varphi = 0$, we get

$$\operatorname{curl}\,(\nabla\varphi\, \mathscr{L}^3) = -b \otimes \tau\, \mathscr{H}^1 \llcorner \Gamma, \tag{3.2}$$

where $\Gamma = \partial S$ is oriented by τ. The multiplicative decomposition (1.1) of $F = D\varphi$ holds true and we have (if the body is crystalline we are at crystal scale)

$$F^e = \nabla\varphi, \qquad F^p = I\, \mathscr{L}^3 + (\nabla\varphi)^{-1}(b \otimes \nu)\, \mathscr{H}^2 \llcorner S\,. \tag{3.3}$$

A measure $F^e \mathscr{L}^3 \in \mathscr{M}_b(\Omega, \mathbb{R}^{3\times 3})$ is associated with what we call an *elastic factor*. For the specific case of a single slip, if we can imagine to *fix* a slipped configuration, we could even think of F^e as the gradient of a differentiable, orientation preserving map φ^e defined over the slipped shape, so that $\operatorname{curl} F^e = (\operatorname{curl} \nabla\varphi^e)\mathscr{L}^3 = 0$. In general it is not so. In fact, as already recalled, at every $x \in \Omega$, the linear operator F^p maps the tangent space $T_x\Omega$ onto a linear space, say $\mathscr{L}_x$, which is isomorphic to $\mathbb{R}^k$, with k selected into $\{1, 2, 3\}$. In principle, varying x in Ω, $\mathscr{L}_x$ also varies. Not necessarily the union of all $\mathscr{L}_x$, as x ranges in Ω, is the tangent bundle of some *intermediate configuration* reached from Ω by means of a specific deformation. Consequently, in general, although F is compatible, in the sense that $F = D\varphi$, its elastic and plastic factors F^e and F^p are incompatible, that is, $\operatorname{curl}\,(F^e \mathscr{L}^3) \neq 0$ and $\operatorname{curl} F^p \neq 0$. However, assuming F^e to be invertible, using that $\operatorname{curl} F = 0$, and writing $F^p = (F^e)^{-1} F$, their incompatibilities are related by the formula (see [31])

$$\operatorname{curl} F^p = (\det F)\, \operatorname{curl}\,[(F^e)^{-1}]\, F^{-\mathsf{T}}, \tag{3.4}$$

where $\operatorname{curl}\,[(F^e)^{-1}]$ is computed in the deformed configuration $\varphi(\Omega)$.

In the presence of N slip planes, we can write

$$F^p = I\mathscr{L}^3 + \sum_{h=1}^{N} b_h \otimes \nu_h \mathscr{H}^2 \llcorner S_h,$$

where S_h is a smooth flat surface in Ω with boundary $\Gamma_h = \partial S_h$ a smooth, simple, and closed curve in Ω, and ν_h a smooth unit normal to S_h. Also, the Burgers vector b_h is constant. Then, the pertinent dislocation density tensor is

$$\operatorname{curl} F^p = \sum_{h=1}^{N} b_h \otimes \tau_h \mathscr{H}^1 \llcorner \Gamma_h,$$

where τ_h is a tangent unit vector orienting the closed curve Γ_h in a consistent way with respect to the orientation induced by ν_h on S_h (see also [27] and [31]).

We confront these already known issues in enlarging the view.

4. Line defects and rectifiable currents

Both measures previously considered, namely

$$\mu_S := \sum_{h=1}^{N} b_h \otimes \nu_h \mathscr{H}^2 \llcorner S_h \quad \text{and} \quad \mu_\Gamma := \sum_{h=1}^{N} b_h \otimes \tau_h \mathscr{H}^1 \llcorner \Gamma_h, \tag{4.1}$$

can be seen as triplets of integer multiplicity (in short i.m.) rectifiable currents of dimension $k = 2$ and $k = 1$, respectively, each triplet living on the same k-rectifiable set, in such a way that the equality curl $\mu_S = \mu_\Gamma$ reduces to a boundary condition in the sense of currents. To discuss the issue, first we fix some general notions.

4.1. A few elements of exterior algebra

Take a basis $\{\mathbf{e}_i\}$ in $\mathbb{R}^n$. The wedge product $\mathbf{e}_i \wedge \mathbf{e}_j$ is defined to be $\mathbf{e}_i \wedge \mathbf{e}_j :=$ skew$(\mathbf{e}_i \otimes \mathbf{e}_j)$. The same notation holds for the dual basis $\mathbf{e}^i$.

Consider a skew-symmetric tensor A, of type $(0, k)$ over $\mathbb{R}^n$: it is a k-linear skew-symmetric map with fully covariant components given by $A_{\sigma(i_1),\ldots,\sigma(i_k)} = \operatorname{sign}(\sigma) A_{i_1,\ldots,i_k}$, where $\sigma(\cdot)$ is a permutation. In particular, if $k > n$, we have $A = 0$, while if $k = n$, there is on $\mathbb{R}^n$ a unique skew-symmetric tensor $(0, n)$ to within a factor of proportionality. A basis for skew-symmetric tensors of the type $(0, k)$ on $\mathbb{R}^n$ is given by $\mathbf{e}^{i_1} \wedge \cdots \wedge \mathbf{e}^{i_k}$. However, since every vector field over $\mathbb{R}^n$ (but also over a smooth manifold) can be seen as a linear differentiation over smooth functions, if we refer to coordinates, $\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$ is also a basis.

To every skew-symmetric tensor component $A_{i_1,\ldots,i_k}$ we may associate what we call a *differential form* ω given by $\omega := A_{i_1,\ldots,i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$.

The wedge product applies also to forms, so we can write $\omega_1 \wedge \omega_2$ for a form of rank that is equal to a sum of the ranks pertaining to the two forms. Such a product is additive, bilinear in both arguments, and anti-commutative.

We will focus on smooth maps $x \mapsto \omega(x)$ compactly supported on open sets $U \subset \mathbb{R}^n$. We will denote by $\mathscr{D}^k(U)$, where the exponent k indicates the order of the forms considered.

A differentiation d, commonly called the *exterior derivative* associates with a $(0, k)$ tensor a $(0, k+1)$ one. Precisely, for ω as above, we have $\mathrm{d}\omega = \sum_{j_1 < \cdots < j_{k+1}} \mathrm{d}A_{j_1,\cdots,j_{k+1}} \mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_{k+1}}$, with

$$\mathrm{d}A_{j_1,\cdots,j_{k+1}} = \sum_{m=1}^{k+1} (-1)^{m+1} \frac{\partial A_{j_1,\cdots,\bar{j}_m \cdots j_{k+1}}}{\partial j_m},$$

where the superposed bar means that the pertinent index must be skipped. By the Schwarz theorem on symmetry of mixed derivatives, we have that $\mathrm{d} \circ \mathrm{d} = \mathrm{d}^2 = 0$.

Consider a generic metric g in $\mathbb{R}^n$. To every skew-symmetric tensor A of the type $(0, k)$, with $k < n$, we may associate a tensor $*A$ of $(0, n-k)$, through the linear operator $*$ commonly called the Hodge star. In terms of components, we have

$$(*A)_{i_{k+1},\dots,i_n} = \frac{1}{k!}\sqrt{\det g} \quad \epsilon_{i_1,\dots,i_k} A^{i_1,\dots,i_k},$$

where $A^{i_1,\dots,i_k} = g^{i_1 j_1} \cdots g^{i_k j_k} A_{i_1,\dots,i_k}$. For tensors of the type $(0,k)$, with $n \geq k > 1$, we have $*(*A) = (-1)^{k(n-k)}\mathrm{sign}(\det g)A$. Therefore, when $n = 3$ the Hodge star is involutive over 1-forms, that is, covectors.

Set $n = 3$. Let a be a covector field, that is, a tensor of the type $(0,1)$. Its exterior differential $\mathrm{d}a$ is a second-rank covariant skew-symmetric tensor. Then, we have $\mathrm{curl}\, a = *(\mathrm{d}a)$. For A a skew-symmetric tensor field of the type $(0,2)$, in the same three-dimensional setting, we also have $\mathrm{div} A = *^{-1}\mathrm{d}(*A)$.

Smooth tensor fields with values skew-symmetric tensors of the type $(k,0)$ are dual to forms. Such tensors are often called *k-vectors*. Consider the deformation gradient F in $3D$ space. A 3-vector, which can be identified with a vector in $\mathbb{R}^{19}$, indicated by $M(F)$, is associated with it. Its components are those in the list $(F, \mathrm{cof}\, F, \det F)$, where $\mathrm{cof}\, F$ is the cofactor of F. Reminding this will be crucial in understanding the physical significance of currents associated with a deformation mapping, as we will discuss below, when appropriate.

4.2. Integer rectifiable currents

If $U \subset \mathbb{R}^n$ is an open set, and $k = 0,\dots,n$, we denote by $\mathscr{D}_k(U)$ the strong dual of the space $\mathscr{D}^k(U)$ of compactly supported smooth k-forms, whence $\mathscr{D}_0(U)$ is the class of distributions in U. For any $T \in \mathscr{D}_k(U)$, we define its *mass* $\mathbf{M}(T)$ as

$$\mathbf{M}(T) := \sup\{\langle T, \omega\rangle \mid \omega \in \mathscr{D}^k(U),\ \|\omega\| \leq 1\}$$

and (for $k \geq 1$) its *boundary* as the $(k-1)$-current ∂T defined by the relation

$$\langle \partial T, \eta\rangle := \langle T, \mathrm{d}\eta\rangle, \qquad \forall \eta \in \mathscr{D}^{k-1}(U),$$

where $\mathrm{d}\eta$ is the differential of η. The *weak convergence* $T_h \rightharpoonup T$ in the sense of currents in $\mathscr{D}_k(U)$ is defined through the formula

$$\lim_{h\to\infty} \langle T_h, \omega\rangle = \langle T, \omega\rangle, \qquad \forall\, \omega \in \mathscr{D}^k(U).$$

If $T_h \rightharpoonup T$, by lower semicontinuity we also have

$$\mathbf{M}(T) \leq \liminf_{h\to\infty} \mathbf{M}(T_h).$$

For $k \geq 1$, a k-current T with finite mass is called *rectifiable* if

$$\langle T, \omega\rangle = \int_{\mathscr{M}} \theta\, \langle \omega, \xi\rangle\, \mathrm{d}\mathscr{H}^k, \qquad \forall\, \omega \in \mathscr{D}^k(U),$$

with $\mathscr{M}$ a k-rectifiable set in U, $\xi : \mathscr{M} \to \Lambda^k \mathbb{R}^n$ a $\mathscr{H}^k \llcorner \mathscr{M}$-measurable function such that $\xi(x)$ is a simple unit k-vector orienting the approximate tangent space to

$\mathscr{M}$ at $\mathscr{H}^k$-a.e. $x \in \mathscr{M}$, and $\theta : \mathscr{M} \to [0, +\infty)$ a $\mathscr{H}^k \llcorner \mathscr{M}$-summable and non-negative function. Therefore, we get $\mathbf{M}(T) = \int_{\mathscr{M}} \theta \, \mathrm{d}\mathscr{H}^k < \infty$ and the short-hand notation $T = [\![\mathscr{M}, \xi, \theta]\!]$ is commonly adopted.

In addition, if the multiplicity function θ is integer-valued, the current T is called *i.m. rectifiable* and the corresponding class is denoted by $\mathscr{R}_k(U)$.

The usefulness of currents in the calculus of variations emerges from Federer-Fleming's compactness theorem [10]. It states that if a sequence $\{T_h\} \subset \mathscr{R}_k(U)$ satisfies $\sup_h \mathbf{M}(T_h) < \infty$ and $\sup_h \mathbf{M}((\partial T_h) \llcorner U) < \infty$, there exists $T \in \mathscr{R}_k(U)$ and a (not relabeled) subsequence of $\{T_h\}$ such that $T_h \rightharpoonup T$ weakly in $\mathscr{D}_k(U)$. As a consequence, if $T \in \mathscr{R}_k(U)$ satisfies $\mathbf{M}((\partial T) \llcorner U) < \infty$, a boundary rectifiability theorem holds true; it states that $\partial T \in \mathscr{R}_{k-1}(U)$.

When we say (roughly speaking) that a current $\bar{S}$ is associated with a $2D$ set S, we are essentially referring to the integration of k-forms over S. Currents can be also associated with maps. If that is the case, as we adopt in the following sections, the physical meaning of a current as a generalized inner work along the deformation clearly appears. We will add the details below. An extended treatment of currents is in the treatise [14] (see also [15]).

4.3. $\mathbb{Z}^m$-Valued rectifiable currents

Let $m \in \mathbb{N}^+$ and $k = 1, \dots, n$. We define $\mathbb{Z}^m$-*valued rectifiable k-currents* $\overline{T}$ in U by triplets $(\mathscr{M}, \xi, \Theta)$, where $\mathscr{M}$ and ξ are as above, but $\Theta : \mathscr{M} \to \mathbb{Z}^m$ is a $\mathbb{Z}^m$-valued $\mathscr{H}^k \llcorner \mathscr{M}$-summable multiplicity function. More precisely, setting $\Theta = (\theta^1, \dots, \theta^m)$, we look at $\overline{T} = (T^1, \dots, T^m)$ as an *ordered m-tuple of i.m. rectifiable currents* $T^j \in \mathscr{R}_k(U)$, with $T^j = [\![\mathscr{M}, \sigma^j \xi, \sigma^j \theta^j]\!]$ for $j = 1, \dots, m$, where $\sigma^j = 0$ if $\theta^j = 0$ and $\sigma^j = \theta^j / |\theta^j|$ otherwise. Denote by $\overline{\omega} = (\omega_1, \dots, \omega_m)$ an ordered m-tuple of k-forms $\omega_j \in \mathscr{D}^k(U)$, in short $\overline{\omega} \in [\mathscr{D}^k(U)]^m$, and by $[\mathscr{R}_k(U)]^m$ the class of $\mathbb{Z}^m$-valued rectifiable k-currents $\overline{T}$ as above. The action of $\overline{T}$ on $\overline{\omega}$ is defined through its components by

$$\langle \overline{T}, \overline{\omega} \rangle := \sum_{j=1}^{m} \langle T^j, \omega_j \rangle.$$

Differently from, for example, reference [8], *we are not dealing with* rectifiable k-currents $\hat{T}$ with coefficients in $\mathbb{Z}^m$, in short $\hat{T} \in \mathscr{R}_k(U, \mathbb{Z}^m)$, with action on a form $\omega \in \mathscr{D}^k(U)$ defined by $\langle \hat{T}, \omega \rangle := \int_{\mathscr{M}} \Theta \, \langle \omega, \xi \rangle \mathrm{d}\mathscr{H}^k$ for some triplet $(\mathscr{M}, \xi, \Theta)$ as above. In order to recover $\hat{T}$ from $\overline{T} = (T^1, \dots, T^m)$, it suffices to observe that $\langle \hat{T}, \omega \rangle = \sum_{j=1}^{m} \langle T^j, \omega \rangle e_j$, where $(e_1, \dots, e_m)$ is the canonical basis in $\mathbb{R}^m$.

Remark 4.1. If $T^j \in \mathscr{R}_k(U)$ for $j = 1, \dots, m$ we find a current $\overline{T} \in [\mathscr{R}_k(U)]^m$ with components $\overline{T} = (T^1, \dots, T^m)$. In fact, letting $T^j = [\![\mathscr{M}_j, \xi_j, \theta_j]\!]$, we choose $\mathscr{M}$ as the set of points x in $\hat{\mathscr{M}} := \bigcup_{j=1}^{m} \mathscr{M}_j$ with unitary k-dimensional density Θ^k, namely $\Theta^k(\hat{\mathscr{M}}, x) = 1$. Then, we equip $\mathscr{M}$ with an orientation ξ. Eventually, it suffices to define the multiplicity $\Theta = (\theta^1, \dots, \theta^m)$ as follows: for $x \in \mathscr{M}$ and $j = 1, \dots, m$, if $\Theta^k(\mathscr{M}^j, x) = 0$ we let $\theta^j(x) = 0$, whereas if

$\Theta^k(\mathscr{M}^j, x) = 1$ we let $\theta^j(x) = \pm\theta_j(x)$, according to the sign in the equality $\xi_j(x) = \pm\xi(x)$.

The *weak convergence* $\overline{T}_h \rightharpoonup \overline{T}$ in the class $[\mathscr{R}_k(U)]^m$ is defined by components through the formula $\langle \overline{T}_h, \overline{\omega}\rangle \to \langle \overline{T}, \overline{\omega}\rangle$ for each $\overline{\omega} \in [\mathscr{D}^k(U)]^m$. In a similar way, the *boundary* of a current $\overline{T} \in [\mathscr{R}_k(U)]^m$ is defined by the formula $\langle \partial\overline{T}, \overline{\omega}\rangle := \langle \overline{T}, \mathrm{d}\overline{\omega}\rangle$ for any $\overline{\omega} \in [\mathscr{D}^{k-1}(U)]^m$, where $\mathrm{d}\overline{\omega} := (\mathrm{d}\omega_1, \ldots, \mathrm{d}\omega_m)$ is in $[\mathscr{D}^k(U)]^m$. We also define the *mass* $\mathbf{M}(\overline{T}) := \sum_{j=1}^m \mathbf{M}(T^j) < \infty$ and the *boundary mass* $\mathbf{M}((\partial\overline{T}) \llcorner U) := \sum_{j=1}^m \mathbf{M}((\partial T^j) \llcorner U)$ if $\overline{T} = (T^1, \ldots, T^m)$ as above.

If $\overline{T} \in [\mathscr{R}_k(\Omega)]^m$ satisfies $\mathbf{M}((\partial\overline{T}) \llcorner \Omega) < \infty$, on account of the previous remark, the boundary rectifiability theorem yields $\partial\overline{T} \in [\mathscr{R}_{k-1}(\Omega)]^m$.

In a similar way, if a sequence $\{\overline{T}_h\} \subset [\mathscr{R}_k(\Omega)]^m$ satisfies $\sup_h \mathbf{M}(\overline{T}_h) < \infty$ and $\sup_h \mathbf{M}((\partial\overline{T}_h) \llcorner \Omega) < \infty$, by using Federer-Fleming's compactness theorem and a diagonal argument, we can find a current $\overline{T} \in [\mathscr{R}_k(\Omega)]^m$ and a (not relabeled) subsequence of $\{\overline{T}_h\}$ such that $\overline{T}_h \rightharpoonup \overline{T}$.

4.4. A physically significant choice

Consider $n = m = 3$, and $U = \Omega$. Vector fields ϕ in $C_c^\infty(\Omega, \mathbb{R}^3)$ agree with 0-forms $\overline{\omega}$ in $[\mathscr{D}^1(\Omega)]^3$, say $\overline{\omega} = \omega_\phi^{(0)}$. A 1-form $\overline{\omega} \in [\mathscr{D}^1(\Omega)]^3$ is identified by a tensor valued field $\psi \in C_c^\infty(\Omega, \mathbb{R}^{3\times 3})$ with components ψ_{jA} by letting $\omega_j = \sum_{A=1}^3 \psi_{jA}\mathrm{d}x^A$ for $j = 1, 2, 3$. In this case, we write $\overline{\omega} = \omega_\psi^{(1)}$. In a similar way, a 2-form $\overline{\omega} \in [\mathscr{D}^2(\Omega)]^3$ is identified by a tensor valued field $\zeta \in C_c^\infty(\Omega, \mathbb{R}^{3\times 3})$ with components ζ_{jA} by letting $\omega_j = \sum_{A=1}^3 (-1)^{A-1}\zeta_{jA}\widehat{\mathrm{d}x^A}$, where $\mathrm{d}x^A \wedge \widehat{\mathrm{d}x^A} = (-1)^{A-1}\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$. In this case, we write $\overline{\omega} = \omega_\zeta^{(2)}$. Finally, a 3-form $\overline{\omega} \in [\mathscr{D}^3(\Omega)]^3$ is identified by a covector field $\eta \in C_c^\infty(\Omega, \mathbb{R}^3)$ with $\eta = (\eta_1, \eta_2, \eta_3)$, by letting $\omega_j = \eta_j \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$, and we write $\overline{\omega} = \omega_\eta^{(3)}$. With this notation, we have

$$\mathrm{d}\omega_\phi^{(0)} = \omega_{\nabla\phi}^{(1)}, \qquad \mathrm{d}\omega_\psi^{(1)} = \omega_{\operatorname{curl}\psi}^{(2)}, \qquad \mathrm{d}\omega_\zeta^{(2)} = \omega_{\operatorname{div}\zeta}^{(3)}, \tag{4.2}$$

and hence the identities $\operatorname{curl}\nabla\phi = 0$ and $\operatorname{div}(\operatorname{curl}\psi) = 0$ are equivalent to the closure relations $\mathrm{d} \circ \mathrm{d} = \mathrm{d}^2 = 0$ for 0-forms and 1-forms, respectively.

For $k = 2$, a current $\overline{T} = \overline{T}_S \in [\mathscr{R}_2(\Omega)]^3$ is naturally associated with the measure μ_S defined in (4.1). Assuming for the sake of simplicity that $\mathscr{H}^2(S_{h_1} \cap S_{h_2}) = 0$ for $1 \le h_1 < h_2 \le N$, it suffices to take $\mathscr{M} = \cup_{h=1}^N S_h$ and define $\xi \equiv *\nu_h$ and $\Theta \equiv b_h$ on each S_h, where $*$ is the Hodge operator in $\mathbb{R}^3$. Similarly, for $k = 1$, a current $\overline{T} = \overline{T}_\Gamma \in [\mathscr{R}_1(\Omega)]^3$ is naturally associated with the measure μ_Γ in (4.1). By assuming again $\mathscr{H}^1(\Gamma_{h_1} \cap \Gamma_{h_2}) = 0$ for $1 \le h_1 < h_2 \le N$, it suffices to take $\mathscr{M} = \cup_{h=1}^N \Gamma_h$, setting $\xi \equiv \tau_h$ and $\Theta \equiv b_h$ on each Γ_h. We also notice that

$$\operatorname{curl}\mu_S = \mu_\Gamma \iff \partial\overline{T}_S = \overline{T}_\Gamma.$$

Since $\langle \mu_\Gamma, \psi\rangle = \langle \overline{T}_\Gamma, \omega_\psi^{(1)}\rangle$ and $\langle \mu_S, \zeta\rangle = \langle \overline{T}_S, \omega_\zeta^{(2)}\rangle$, it suffices, in fact, to recall that $\langle \partial\overline{T}_S, \omega_\psi^{(1)}\rangle = \langle \overline{T}_S, \mathrm{d}\omega_\psi^{(1)}\rangle$ and to use the second formula in (4.2). Therefore, the implication $\Rightarrow$ readily follows, whereas the reverse $\Leftarrow$ holds true by a standard density argument based on the dominated convergence theorem. Finally, the closure relation $\mathrm{d} \circ \mathrm{d} = 0$ yields that $\partial \overline{T}_\Gamma = 0$ if $\mu_\Gamma = \operatorname{curl} \mu_S$. On account of identities (4.2), the null-boundary property $\partial \overline{T}_\Gamma = 0$ is equivalent to the requirement that μ_Γ is a divergence-free line defect measure (see [8]). For a given current $\bar{S} \in [\mathscr{R}_2(\Omega)]^3$, with a slight abuse of notation we let $\tilde{F} = \tilde{F}(\bar{S})$ denote the tensor valued distribution in Ω acting on test functions $\zeta \in C_c^\infty(\Omega, \mathbb{R}^{3\times 3})$ as

$$\langle \tilde{F}, \zeta \rangle := \langle \bar{S}, \omega_\zeta^{(2)} \rangle.$$

Since $\mathbf{M}(\bar{S}) < \infty$, the distribution $\tilde{F}$ is extended to a measure $\tilde{F} \in \mathscr{M}_b(\Omega, \mathbb{R}^{3\times 3})$ with total variation bounded by the mass of $\bar{S}$, namely $|\tilde{F}|(\Omega) \leq \mathbf{M}(\bar{S})$. Moreover, by using the notation (S, ξ_S, Θ_S) for $\bar{S} \in [\mathscr{R}_2(\Omega)]^3$, and choosing the unit normal ν_S to S as a covector $\nu_S = (\nu_{S1}, \nu_{S2}, \nu_{S3})$ in such a way that $\xi_S = *\nu_S^\sharp$, with $\nu_S^\sharp$ the vector associated with the covector ν_S by the metric in Ω, we have $(-1)^{A-1} \langle \phi \, \widehat{\mathrm{d}x^A}, \xi_S \rangle = \phi \, \nu_{SA}$ for each $A = 1, 2, 3$ and $\phi \in C_c^\infty(\Omega)$, where, as above, the extended hat indicates the form complementing the one covered by the hat itself, with respect to the volume form.

The component F_A^j acts on bounded and continuous functions $\phi \in C_b(\Omega)$ as

$$\langle \tilde{F}_A^j, \phi \rangle = \int_\Omega \phi \, \mathrm{d}\tilde{F}_A^j = \int_S \Theta_S^j \nu_{SA} \, \phi \, \mathrm{d}\mathscr{H}^2$$

and hence we can write

$$\tilde{F} = \tilde{F}(\bar{S}) = \Theta_S \otimes \nu_S \, \mathscr{H}^2 \llcorner S.$$

4.5. *Confinement condition*

A *confinement condition* for line defects in crystals has been discussed in reference [21]. We can translate it within the setting discussed here by requiring that the current $\bar{S}$ has compact support contained in Ω. By looking at the corresponding measure, this property becomes

$$\operatorname{spt} \tilde{F}(\bar{S}) \subset \Omega,$$

where $\operatorname{spt}(\cdot)$ indicates the *support* of $(\cdot)$. In addition, if the boundary current $\partial \bar{S}$ has finite mass, there exists a current $\overline{\Gamma} \in [\mathscr{R}_1(\Omega)]^3$ with support contained in Ω such that $\partial \bar{S} = \overline{\Gamma}$. By adopting the notation $(\Gamma, \tau_\Gamma, \Theta_\Gamma)$ as above, it turns out that the tensor valued distribution $\tilde{F} = \tilde{F}(\overline{\Gamma})$ in Ω acting on test functions $\psi \in C_c^\infty(\Omega, \mathbb{R}^{3\times 3})$ as

$$\langle \tilde{F}, \psi \rangle := \langle \overline{\Gamma}, \omega_\psi^{(1)} \rangle$$

can be extended to a measure $F \in \mathscr{M}_b(\Omega, \mathbb{R}^{3\times 3})$, with total variation bounded by the mass of $\overline{\Gamma}$, $|\tilde{F}|(\Omega) \leq \mathbf{M}(\overline{\Gamma})$, and

$$\tilde{F}(\overline{\Gamma}) = \Theta_\Gamma \otimes \tau_\Gamma \mathscr{H}^1 \llcorner \Gamma.$$

Moreover, the boundary condition $\partial \bar{S} = \overline{\Gamma}$ is equivalent to

$$\langle \text{curl } \tilde{F}(\bar{S}), \psi \rangle = \langle \bar{S}, \mathrm{d}\omega_\psi^{(1)} \rangle = \langle \overline{\Gamma}, \omega_\psi^{(1)} \rangle \qquad \forall \psi \in C_c^\infty(\Omega, \mathbb{R}^{3\times 3}).$$

We thus have, for all $\phi \in C_b(\Omega)$ and $A, j = 1, 2, 3$,

$$\langle (\text{curl } \tilde{F}(\bar{S}))_A^j, \phi \rangle = \int_\Omega \phi \, \mathrm{dcurl} \tilde{F}(\bar{S})_A^j = \int_\Gamma \Theta_\Gamma^j \, \phi \, \tau_{\Gamma A} \, \mathrm{d}\mathscr{H}^1,$$

which is

$$\text{curl } \tilde{F}(\bar{S}) = \Theta_\Gamma \otimes \tau_\Gamma \mathscr{H}^1 \llcorner \Gamma.$$

Finally, the support condition on $\overline{\Gamma}$ is equivalent to the confinement condition

$$\text{spt } (\text{curl } F(\bar{S})) \subset \Omega.$$

4.6. Tangency condition

As a guiding picture, consider dislocations in a periodic crystal. When they glide, the Burgers vector b is parallel to the slip plane. When they climb the geometry involved is not so simple. When material grains relatively move, for example in polycrystalline bodies, we might also accept at least in approximate sense a tangency condition of the (averaged) Burgers vector of the line defect forest at the intergranular interstices.

In the generalized sense adopted here, we consider a tangency condition by assuming that the multiplicity Θ_S is orthogonal to the unit normal ν_S, that is, $\Theta_S(x) \bullet \nu_S(x) = 0$ at $\mathscr{H}^2$-a.e. $x \in S$, where $\bullet$ denotes the scalar product in $\mathbb{R}^3$. By taking $\zeta_{jA} = \delta_{jA} \, \phi(x)$, with δ_{jA} the Kronecker delta, for some $\phi \in C_c^\infty(\Omega)$, we find that

$$\langle \bar{S}, \omega_\zeta^{(2)} \rangle = \int_S (\Theta_S \bullet \nu_S) \, \phi \, \mathrm{d}\mathscr{H}^2,$$

and hence, in terms of currents, the *tangency condition* reads as

$$\langle \bar{S}, \overline{\omega}_\phi \rangle = 0 \qquad \forall \phi \in C_c^\infty(\Omega), \tag{4.3}$$

where $\overline{\omega}_\phi = (\omega_{1\phi}, \omega_{2\phi}, \omega_{3\phi})$, with $\omega_{j\phi} := (-1)^{j-1} \sum_{A=1}^3 \delta_{jA} \phi(x) \widehat{\mathrm{d}x^A}$ for $j = 1, 2, 3$.

If the condition (4.3) holds, $\Theta_S(x) \bullet \nu_S(x) = 0$ at $\mathscr{H}^2$-a.e. $x \in S$.

In the smooth case, if we assume S to be a flat surface contained in the slip plane with a constant Burgers vector $\Theta_S \equiv b$, the multiplicity Θ_Γ of the boundary current $\overline{\Gamma} = \partial \overline{S}$ is tangential to the osculating plane to Γ. However, for currents $\overline{S} \in [\mathscr{R}_2(\Omega)]^3$ associated with a smooth surface S with multiplicity Θ_S, the tangency condition (4.3) does not imply in general a geometric property concerning the multiplicity Θ_Γ of Γ.

4.7. Irrecoverable strains and rectifiable line defects

Definition 4.1. We call *generalized slip surface* any $\mathbb{Z}^3$-valued current $\bar{S}$ in $[\mathscr{R}_2(\Omega)]^3$ satisfying confinement condition spt $\bar{S} \subset \Omega$, tangency one (4.3), and being such that the boundary current $\overline{\Gamma} := \partial \bar{S}$ has finite mass. The $\mathbb{Z}^3$-valued current $\overline{\Gamma} = [\mathscr{R}_1(\Omega)]^3$ is called a *rectifiable line defect* in Ω, and we write $\overline{\Gamma} \in \mathrm{r} - \mathrm{ld}\,(\Omega)$.

As we have seen, $\partial\overline{\Gamma} = 0$ and spt $\overline{\Gamma} \subset \Omega$ for every $\overline{\Gamma} \in \mathrm{r} - \mathrm{ld}\,(\Omega)$.

In addition, since in general no energy contribution is associated to the slip surface, as a constitutive condition we require that

$$\mathbf{M}(\bar{S}) \leq c\,\mathbf{M}(\overline{\Gamma})^2 \tag{4.4}$$

for some fixed real constant $c > 0$. This bound holds true when $\overline{\Gamma}$ is associated with loop Γ strictly contained in Ω and lying in a slip plane, and $\Theta_\Gamma \equiv b$ for some vector b tangential to the slip plane.

More precisely, choose $\overline{S}$ as the 2-current generated by the triplet (S, ξ, b), where S is the flat surface in Ω with boundary Γ. The 2-vector ξ is chosen in accordance to the orientation τ_Γ of $\overline{\Gamma}$. In this case, the inequality (4.4) holds true with c equal to square of the isoperimetric constant in $\mathbb{R}^2$.

Definition 4.2. A tensor-valued bounded measure $F \in \mathscr{M}_b(\Omega, \mathbb{R}^{3\times 3})$ is said to be *associated with a generalized slip surface* $\bar{S}$ *and line defect* $\overline{\Gamma}$, writing $\tilde{F} = \hat{F}(\bar{S}, \overline{\Gamma})$, if

$$\langle \tilde{F}, \zeta \rangle = \langle \bar{S}, \omega_\zeta^{(2)} \rangle \qquad \forall\, \zeta \in C_c^\infty(\Omega, \mathbb{R}^{3\times 3})$$

for some generalized slip surface $\bar{S}$ with $\overline{\Gamma} = \partial \bar{S}$ in $\mathrm{r} - \mathrm{ld}\,(\Omega)$.

Definition 4.3. A tensor-valued measure $F^p \in \mathscr{M}(\Omega, \mathbb{R}^{3\times 3})$ is called a *plastic deformation factor with generalized slip surface* $\bar{S}$ *and line defect* $\overline{\Gamma}$ if

$$F^p = a(x) I\, \mathscr{L}^3 + \hat{F}(\bar{S}, \overline{\Gamma}),$$

where $\hat{F}(\bar{S}, \overline{\Gamma})$ is defined as above and $a(x)$ is a Borel function in Ω satisfying

$$C^{-1} \leq a(x) \leq C \qquad \forall\, x \in \Omega \tag{4.5}$$

for some given real constant $C > 1$.

With these assumptions, curl F^p agrees with curl $\hat{F}(\bar{S}, \overline{\Gamma})$ and hence we can identify it through the formula

$$\langle \mathrm{curl}\, F^p, \psi \rangle = \langle \overline{\Gamma}, \omega_\psi^{(1)} \rangle \qquad \forall\, \psi \in C_c^\infty(\Omega, \mathbb{R}^{3\times 3}). \tag{4.6}$$

Eventually, for any given plastic deformation as above, due to the bound (4.4), we get

$$|\hat{F}(\bar{S}, \overline{\Gamma})|(\Omega) \leq c\,\mathbf{M}(\overline{\Gamma})^2, \qquad 2^{-1}\mathbf{M}(\overline{\Gamma}) \leq |\mathrm{curl}\, F^p|(\Omega) \leq \mathbf{M}(\overline{\Gamma}).$$

4.8. Stability of the tangency condition

Weakly converging sequences of generalized slip surfaces with bounded masses preserve the tangency condition.

Proposition 4.1. *Let* $\{\bar{S}_h\}_h \subset [\mathscr{R}_2(\Omega)]^3$ *be a sequence of generalized slip surfaces satisfying* $\cup_h \mathrm{spt}\, \bar{S}_h \subset \mathscr{K}$ *for some compact set* $\mathscr{K} \subset \Omega$ *and*

$$\sup_h \big(\mathbf{M}(\bar{S}_h) + \mathbf{M}(\partial \bar{S}_h)\big) < \infty.$$

Then, there exists a (not relabeled) subsequence and a generalized slip surface $\bar{S} \in [\mathscr{R}_2(\Omega)]^3$ *such that* $\bar{S}_h \rightharpoonup \bar{S}$ *weakly in* $[\mathscr{D}_2(\Omega)]^3$ *and* $\mathrm{spt}\, \bar{S} \subset \mathscr{K}$.

Proof. Due to the validity of Federer-Fleming's compactness theorem, we only have to check that the limit current $\bar{S}$ satisfies the tangency condition. We have seen that such a geometric condition is equivalent to the identity (4.3), whereas the weak convergence $\bar{S}_h \rightharpoonup \bar{S}$ implies that $\langle \bar{S}_h, \overline{\omega}_\phi \rangle \to \langle \bar{S}, \overline{\omega}_\phi \rangle$ for every $\phi \in C_c^\infty(\Omega)$, whence property (4.3) is preserved, as required. □

Another question to be investigated is a stability of the corresponding tangency condition concerning the deformation map $\varphi \in SBV(\Omega, \mathbb{R}^3)$, namely, that the jump of φ is tangential to the approximate tangent space to $S(\varphi)$ at $\mathscr{H}^2 \llcorner S(\varphi)$-a.e. point. This tangency condition is equivalent to

$$\int_{S(\varphi)} \phi \,\mathrm{tr}((\varphi^+ - \varphi^-) \otimes \nu)\, \mathrm{d}\mathscr{H}^2 = 0 \qquad \forall \phi \in C_c^\infty(\Omega).$$

Proposition 4.2. *Let* $p > 1$ *and* $\{\varphi_h\} \subset SBV(\Omega, \mathbb{R}^3)$ *satisfy*

$$\sup_h \Big(\|\varphi_h\|_\infty + \int_\Omega |\nabla \varphi_h|^p \,\mathrm{d}x + \mathscr{H}^2(S(\varphi_h)) \Big) < \infty$$

where the tangency condition holds for each φ_h. *Then, there exists a (not relabeled) subsequence and a vector field* $\varphi \in SBV(\Omega, \mathbb{R}^3)$ *that satisfies the tangency condition and is such that* $\varphi_h \to \varphi$ *in* $L^1(\Omega, \mathbb{R}^3)$, $\nabla \varphi_h \rightharpoonup \nabla \varphi$ *weakly in* $L^p(\Omega, \mathbb{R}^{3\times 3})$, *and* $(\varphi_h^+ - \varphi_h^-) \otimes \nu_h \mathscr{H}^2 \llcorner S(\varphi_h)$ *weakly converges in the sense of measures to* $(\varphi^+ - \varphi^-) \otimes \nu \mathscr{H}^2 \llcorner S(\varphi) \otimes \nu$. *More generally, the tangency condition holds true for any weak limit point* φ.

Proof. As before, due to the validity of the compactness theorem in SBV, we only have to check that if a (not relabeled) subsequence of $\{\varphi_h\}$ weakly converges to φ in the BV-sense, then the tangency condition is preserved. For this purpose, we observe that the current $[\![\, \partial SG\varphi_h^j \,]\!]$ carried by the subgraph of the j-th component of φ_h weakly converges to the current $[\![\, \partial SG\varphi^j \,]\!]$ carried by the subgraph of the j-th component of φ, for $j = 1, 2, 3$. On the other hand, for each function $\phi \in C_c^\infty(\Omega)$ we get

$$\begin{aligned}(-1)^A \langle [\![\, \partial SG\varphi^j \,]\!], \mathrm{d}\phi(x) \widehat{\mathrm{d}x^A} \rangle = &\int_\Omega (\partial_A \phi(x) \varphi^j(x) + \phi(x) \partial_A \varphi^j(x))\, \mathrm{d}x \\ &+ \int_{S(\varphi)} \phi(x)(\varphi^{j+}(x) - \varphi^{j-}(x)) \nu_A(x)\, \mathrm{d}\mathscr{H}^2,\end{aligned}$$

and a similar formula holds true for φ_h^j. As a consequence of the weak convergence $\nabla\varphi_h \rightharpoonup \nabla\varphi$ in $L^p(\Omega, \mathbb{R}^{3\times 3})$ and of the identity

$$\lim_{h\to\infty} \langle [\![\partial SG\varphi_h^j]\!], \mathrm{d}\phi(x)\widehat{\mathrm{d}x^A}\rangle = \langle [\![\partial SG\varphi^j]\!], \mathrm{d}\phi(x)\widehat{\mathrm{d}x^A}\rangle \qquad \forall A, j = 1, 2, 3,$$

we infer that

$$\lim_{h\to\infty} \int_{S(\varphi_h)} \phi \operatorname{tr}((\varphi_h^+ - \varphi_h^-) \otimes \nu_h)\, \mathrm{d}\mathscr{H}^2 = \int_{S(\varphi)} \phi \operatorname{tr}((\varphi^+ - \varphi^-) \otimes \nu)\, \mathrm{d}\mathscr{H}^2$$

for each $\phi \in C_c^\infty(\Omega)$, which yields stability of the tangency condition. □

The jump set S is a $2D$ rectifiable set: its difference with the union of images of countably many continuously differentiable maps from $\mathbb{R}^2$ into $\mathbb{R}^3$ has zero $\mathscr{H}^2$ measure. This geometry includes even complicated slip systems. The normal ν is taken to the approximate tangent plane of such a set at every point where it can be defined in the reference space.

The current multiplicity involved takes values at each point in the intermediate space determined by F^p. The trace involved above is the scalar product between the two elements constituting the dyad. To allow it making sense, we need to "project" the difference $\varphi^+ - \varphi^-$ into the reference space by a shifter. The result is not (or not necessarily) the pull-back along F^{p-1} of the vector B through Nanson's formula describing how oriented surfaces deform.

Our picture agrees with gliding and cross-slip mechanisms. However, the possible rather intricate geometry that S can assume allows us to imagine that a $2D$ rectifiable set like S can at least approximate the articulate geometry associated with climbing, taking also into account that the tangency condition is not pointwise, rather it is expressed in a weak form.

4.9. Lack of stability of the bound

The inequality (4.4) is not preserved by the weak convergence in the sense of currents. Namely, if $\{\bar{S}_h\} \subset [\mathscr{R}_2(\Omega)]^3$ satisfies

$$\mathbf{M}(\bar{S}_h) \le c\,\mathbf{M}(\overline{\Gamma}_h)^2 < \widetilde{c} < \infty \qquad \forall h$$

for some fixed real constants $c, \widetilde{c} > 0$, where $\partial\bar{S}_h = \overline{\Gamma}_h \in [\mathscr{R}_1(\Omega)]^3$ and $\operatorname{spt}\bar{S}_h \subset \mathscr{K}$ for each h and for some given compact set $\mathscr{K} \subset \Omega$, by Federer-Fleming's compactness theorem, possibly passing to a (not relabeled) subsequence, $\{\bar{S}_h\}$ weakly converges to some current $\bar{S} \in [\mathscr{R}_2(\Omega)]^3$ and $\{\overline{\Gamma}_h\}$ to some current $\overline{\Gamma} \in [\mathscr{R}_1(\Omega)]^3$ such that $\partial\bar{S} = \overline{\Gamma}$ and $\operatorname{spt}\bar{S} \subset \mathscr{K}$. By lower semicontinuity of the mass, we have $\mathbf{M}(\bar{S}) < \widetilde{c}$ and $\mathbf{M}(\overline{\Gamma})^2 < \widetilde{c}$. However, in general the weak limit currents $\bar{S}$ and $\overline{\Gamma}$ fail to satisfy the inequality (4.4).

Notice that the bound (4.4) is preserved if for example we assume that the compact set $\mathscr{K}$ above is a convex (or star-shaped) subset of a 2-dimensional plane of $\mathbb{R}^3$. In this case, in fact, by a cone construction it turns out that, for each $\overline{\Gamma} \in [\mathscr{R}_1(\Omega)]^3$ with $\operatorname{spt}\overline{\Gamma} \subset \mathscr{K}$, there is a unique current $\bar{S} \in [\mathscr{R}_2(\Omega)]^3$ satisfying $\partial\bar{S} = \overline{\Gamma}$ and $\operatorname{spt}\bar{S} \subset \mathscr{K}$. Therefore, the bound (4.4) holds true with c equal to the square of the isoperimetric constant in $\mathbb{R}^2$.

5. The elastic factor F^e

By summarizing, $F = D\varphi$ for some function $\varphi \in SBV(\Omega, \mathbb{R}^3)$ satisfying

$$D\varphi = \nabla\varphi\, \mathscr{L}^3 + (\varphi^+ - \varphi^-) \otimes \nu \mathscr{H}^2 \llcorner S(\varphi),$$

whence $F \in \mathscr{M}_b(\Omega, \mathbb{R}^{3\times 3})$ and the compatibility condition curl $F = 0$ holds.

We see that F also satisfies a multiplicative decomposition as in (1.1), that is, we can write $F = F^e F^p$, where $F^p \in \mathscr{M}_b(\Omega, \mathbb{R}^{3\times 3})$ is the plastic factor previously defined, and the elastic factor F^e is summable as a function of x, namely $F^e \in L^1(\Omega, \mathbb{R}^{3\times 3}; |F^p|)$, with det $F^e > 0$ almost everywhere in Ω.

Since the total variation of F^p decomposes as

$$|F^p| = \sqrt{3}\, a(x)\mathscr{L}^3 + |\Theta_S \otimes \nu_S|\, \mathscr{H}^2 \llcorner S,$$

the absolutely continuous and singular components are strictly related to the elastic and plastic factors, respectively.

As to the absolutely continuous component, by assumption (4.5), that is, the boundedness of plastic volume changes, we set

$$F^e(x) = a(x)^{-1} \nabla\varphi(x)\, \mathscr{L}^3 \tag{5.1}$$

for almost everywhere $x \in \Omega$, whence the L^1-norms of $\nabla\varphi$ and F^e are comparable, namely

$$C^{-1}\, \|F^e\|_{L^1(\Omega)} \leq \|\nabla\varphi\|_{L^1(\Omega)} \leq C\, \|F^e\|_{L^1(\Omega)}.$$

When $a(x)$ is identically 1, we recover the traditional choice of considering just volume-preserving plastic strain. Otherwise, as already anticipated, we allow the possibility of plastic changes in volume (we call such a circumstance *the Bell effect*, after James Bell).

Concerning the singular component of F^e, we make use of the local bound (5.8) of the mass of the boundary current ∂G_φ in terms of the total variation of the plastic component F^p, a condition that we shall assume in the sequel.

Since the size of $S(\varphi)$, that is, the jump set of φ, is controlled by the mass of ∂G_φ (compare (5.5) below), the local bound (5.8) implies that $S(\varphi)$ is $\mathscr{H}^2$-essentially contained in the set S corresponding to the generalized slip surface $\bar{S}$, under the assumption that $|\Theta_S| > 0$ on S, namely

$$(5.8) \quad \Longrightarrow \quad \mathscr{H}^2(S(\varphi) \setminus S) = 0. \tag{5.2}$$

$\mathscr{H}^2(S \setminus S(\varphi)) > 0$ corresponds to existence of pieces of the generalized slip surface where the deformation φ does not jump. If $\mathscr{H}^2(S \setminus S(\varphi) = 0$, instead, the jump set of φ agrees $\mathscr{H}^2$-essentially with the generalized slip surface.

By the identity (5.2) we may and do choose the unit normal ν to $S(\varphi)$ equal to the unit normal ν_S at $\mathscr{H}^2$-a.e. point in the jump set. Requiring that $\varphi^+ - \varphi^- \in \mathbb{R}^3$ is tangent to the jump set $S(\varphi)$ at $\mathscr{H}^2$-a.e. point implies that for $\mathscr{H}^2$-a.e. $x \in S(\varphi)$, both unit vectors

$$v(x) = \frac{\Theta_S(x)}{|\Theta_S(x)|}, \qquad w(x) = \frac{\varphi^+(x) - \varphi^-(x)}{|\varphi^+(x) - \varphi^-(x)|}$$

lie in the approximate tangent plane to S at x. Therefore, there exists a unique rotation matrix $R(x) \in SO(3)$, with rotation axis oriented by the unit normal $\nu_S(x)$, such that $w(x) = R(x)v(x)$. We thus define F^e on $\mathscr{H}^2$-a.e. point x in $S(\varphi)$ as

$$F^e(x) = \frac{|\varphi^+(x) - \varphi^-(x)|}{|\Theta_S(x)|} R(x)$$

so that condition $\det F^e > 0$ extends to points in the jump set $S(\varphi)$. Of course, we may speak of tangency of $S(\varphi)$ only after exploiting of the reference space with the actual one through the appropriate shifter. Furthermore, in case $\mathscr{H}^2(S \backslash S(\varphi)) > 0$ we let $F^e(x) = 0$ at $\mathscr{H}^2$-a.e. point $x \in S \backslash S(\varphi)$.

Using that $\varphi^+ = \varphi^-$ at $\mathscr{H}^2$-a.e. point in $S \backslash S(\varphi)$, we infer that $\mathscr{H}^2$-a.e. on S

$$(\varphi^+ - \varphi^-) \otimes \nu = F^e (\Theta_S \otimes \nu_S) \quad \Longrightarrow \quad \varphi^+ - \varphi^- = F^e \Theta_S. \tag{5.3}$$

In conclusion, the multiplicative decomposition (1.1) is satisfied in the previous generalized sense.

Remark 5.1. For future use, we point out that the multiplicative decomposition (1.1) is stable along minimizing sequences as for example in the proof of the existence theorem 6.1. This is due to the fact that the bound (5.8) is preserved. Therefore, the minimum point φ satisfies inequality (5.2), and hence we argue as above to check the validity of the multiplicative decomposition of the derivative $F = D\varphi$. This aspect may be compared with results in reference [32], where lack of stability of the multiplicative decomposition is also discussed.

5.1. A closure theorem in SBV

As recalled above, with $G \in \mathbb{R}^{3\times3}$, we write $M(G)$ for the vector given by the list $(G, \operatorname{cof} G, \det G) \in \mathbb{R}^{19}$. We require a summability condition on the function $M(F^e)$, actually on $M(\nabla\varphi)$. In the SBV setting, the weak L^1 convergence of the minors holds true as a consequence of a closure theorem proven in reference [11]; its version used here reads as follows:

Theorem 5.1. *Let $\{u_h\}$ a sequence in $SBV(\Omega, \mathbb{R}^3)$ converging in $L^1(\Omega, \mathbb{R}^3)$ to a summable function $u : \Omega \to \mathbb{R}^3$. Assume that for some real exponents $p \geq 2$, $q \geq p/(p-1)$, and $r > 1$,*

$$\sup_h \Big\{ \|u_h\|_\infty + \int_\Omega \Big(|\nabla u_h|^p + |\operatorname{cof} \nabla u_h|^q + |\det \nabla u_h|^r \Big)\, \mathrm{d}x + \mathscr{H}^2(S(u_h)) \Big\} < \infty.$$

Then, $u \in SBV(\Omega, \mathbb{R}^3)$, the sequence $\mathscr{H}^2 \llcorner S(u_h)$ weakly converges in Ω to a measure μ greater than $\mathscr{H}^2 \llcorner S(u)$, and $\{M(\nabla u_h)\}$ converges to $M(\nabla u)$ weakly in $L^1(\Omega, \mathbb{R}^{19})$.

In reference [27] certain weak regularity properties on φ are assumed outside the fixed loop Γ, in order to obtain the closure property in the minimization process. However, the identity (3.2) implies that the gradient $\nabla\varphi$ fails to be in L^2 around the loop Γ. An analogous problem would emerge in our treatment if we would rely on Theorem 5.1.

To avoid the circumstance, we follow a different approach, based on Federer-Fleming's closure theorem, for the *currents G_φ carried by the graph* of maps $\varphi : \Omega \to \mathbb{R}^3$. We point out that a similar approach is followed in the second main result presented in reference [27], where the authors assume that the elastic deformation is a *Cartesian map* from V into $\mathbb{R}^3$ for each open set V contained in $\Omega \setminus \Gamma$, where Γ is fixed.

5.2. Currents carried by approximately differentiable maps

Let $u \in L^1(\Omega, \mathbb{R}^3)$ be an $\mathscr{L}^3$-a.e. approximately differentiable map (as for example an SBV vector field). The map u has a Lusin representative on the subset $\widetilde{\Omega}$ of Lebesgue points pertaining to both u and ∇u, where $\mathscr{L}^3(\Omega \setminus \widetilde{\Omega}) = 0$. (Recall that by Lusin's theorem, measurable functions f into topological spaces with a countable basis can be approximated by continuous functions on arbitrarily large portions of their domain. They are the Lusin representatives.) We say that $u \in \mathscr{A}^1(\Omega, \mathbb{R}^3)$ if $M(\nabla u) \in L^1(\Omega, \mathbb{R}^{19})$.

The *graph* of a map $u \in \mathscr{A}^1(\Omega, \mathbb{R}^3)$ is defined by

$$\mathscr{G}_u := \Big\{(x, y) \in \Omega \times \mathbb{R}^3 \mid x \in \widetilde{\Omega},\ y = \widetilde{u}(x)\Big\},$$

where $\widetilde{u}(x)$ is the Lebesgue value of u.[1] We see that $\mathscr{G}_u$ is a 3-rectifiable set of $U = \Omega \times \mathbb{R}^3$, with $\mathscr{H}^3(\mathscr{G}_u) < \infty$. The approximate tangent 3-plane at $(x, \widetilde{u}(x))$ is generated by the vectors $\mathbf{t}_A(x) = (e_A, \partial_A u(x)) \in \mathbb{R}^{3+3}$, for $A = 1, 2, 3$, where $\partial_A u$ is the A-th column vector of the gradient matrix ∇u, and $\nabla u(x)$ is the Lebesgue value of ∇u at $x \in \widetilde{\Omega}$. Therefore, the unit 3-vector

$$\xi(x) := \frac{\mathbf{t}_1(x) \wedge \mathbf{t}_2(x) \wedge \mathbf{t}_3(x)}{|\mathbf{t}_1(x) \wedge \mathbf{t}_2(x) \wedge \mathbf{t}_3(x)|}$$

provides an orientation to the graph $\mathscr{G}_u$, and the current $G_u = [\![\, \mathscr{G}_u, \xi, 1 \,]\!]$ carried by the graph of u is i.m. rectifiable in $\mathscr{R}_3(\Omega \times \mathbb{R}^3)$, with mass

$$\mathbf{M}(G_u) = \mathscr{H}^3(\mathscr{G}_u) = \int_\Omega \sqrt{1 + |M(\nabla u)|^2}\, \mathrm{d}\mathscr{L}^3 < \infty.$$

---

[1] If $f : \Omega \to \mathbb{R}^N$ is locally summable in Lebesgue's sense, by the Lebesgue differentiation theorem, almost every x in Ω is a Lebesgue point of f, that is, a point such that for some $\lambda \in \mathbb{R}^N$

$$\lim_{r \to 0^+} \frac{1}{|B(x, r)|} \int_{B(x,r)} |f(z) - \lambda|\, \mathrm{d}x = 0$$

with $B(x, r)$ a ball of radius r, centered at x, which Lebesgue measure is $|B(x, r)|$. The number $\lambda = f(x)$ is called the Lebesgue value of f at x.

The current G_u carried by the graph of u is such that on 3-forms ω on $\mathscr{G}_u$ we have

$$\langle G_u, \omega \rangle := \int_{\mathscr{G}_u} \langle \omega, \xi \rangle \, \mathrm{d}\mathscr{H}^n, \qquad \omega \in \mathscr{D}^3(\Omega \times \mathbb{R}^3),$$

where $\langle , \rangle$ indicates as above the duality pairing. Consequently, since G_u is a linear functional over $\mathscr{D}^3(\Omega \times \mathbb{R}^3)$, it is an element of the (strong) dual of the space $\mathscr{D}^3(\Omega \times \mathbb{R}^3)$.

By the area formula

$$\langle G_u, \omega \rangle = \int_{\Omega} \langle \omega(x, u(x)), (1, M(\nabla u(x))) \rangle \, \mathrm{d}x$$

for any $\omega \in \mathscr{D}^3(\Omega \times \mathbb{R}^3)$.

Roughly, the coefficient 1 corresponds to the duality with the component $\phi(x, y)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ of ω, whereas the terms ∇u, cof ∇u, and det ∇u of the vector $M(\nabla u)$ are associated by duality with the components $\phi(x, y)\, \widehat{\mathrm{d}x^A} \wedge \mathrm{d}y^j$, $\phi(x, y)\, \mathrm{d}x^A \wedge \widehat{\mathrm{d}y^j}$, and $\phi(x, y)\, \mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}y^3$, respectively. The circumstance allows us to interpret the value $\langle G_u, \omega \rangle$ as an inner work that accounts separately for the work expenditures in varying lines, surfaces, and volumes because of the presence of the gradient ∇u (so, the ingredient mapping tangent vectors to Ω onto the tangent space of the current body shape), its cofactor (that is, surface variations), and its determinant (volume variations).

For further details on the physical significance of currents associated with approximately differentiable and orientation preserving maps see reference [13].

If u is of class C^2, by the Stokes theorem we have

$$\langle \partial G_u, \eta \rangle = \langle G_u, \mathrm{d}\eta \rangle = \int_{\mathscr{G}_u} \mathrm{d}\eta = \int_{\partial \mathscr{G}_u} \eta = 0$$

for every 2-form $\eta \in \mathscr{D}^2(\Omega \times \mathbb{R}^3)$, which is tantamount to write the null-boundary condition

$$(\partial G_u) \llcorner \Omega \times \mathbb{R}^3 = 0. \tag{5.4}$$

This property holds also, by approximation, for Sobolev maps in $W^{1,3}(\Omega, \mathbb{R}^3)$. It defines the class of Cartesian maps $u \in \mathrm{cart}^1(\Omega, \mathbb{R}^3)$. However, in general, the boundary ∂G_u does not vanish and may not have finite mass.

Since G_u can be interpreted as above, we can think of ∂G_u as a generalized surface inner work performed over the body boundary.

If $u \in \mathscr{A}^1(\Omega, \mathbb{R}^3)$ is such that ∂G_u has finite mass in $\Omega \times \mathbb{R}^3$, the boundary rectifiability theorem yields that $\partial G_u \in \mathscr{R}_2(\Omega \times \mathbb{R}^3)$, that is, the boundary current is supported by a 2-rectifiable set in $\Omega \times \mathbb{R}^3$, and actually $u \in SBV(\Omega, \mathbb{R}^3)$, with

$$\mathscr{H}^2(S(u) \cap V) \leq \mathbf{M}((\partial G_u) \llcorner V \times \mathbb{R}^3) < \infty \tag{5.5}$$

for each open set $V \subset \Omega$.

5.3. Weak convergence of minors

Federer-Fleming's compactness theorem grants the *weak convergence of minors* of Du [14, Vol. I, Sec. 3.3.2].

Theorem 5.2. *Let $\{u_h\}$ be a sequence in $\mathscr{A}^1(\Omega, \mathbb{R}^3)$, $u \in L^1(\Omega, \mathbb{R}^3)$ an almost everywhere approximately differentiable map, and $v \in L^1(\Omega, \mathbb{R}^{19})$. Assume that $u_h \to u$ strongly in $L^1(\Omega, \mathbb{R}^3)$ and that $M(\nabla u_h) \rightharpoonup v$ weakly in $L^1(\Omega, \mathbb{R}^{19})$. If in addition*

$$\sup_h \mathbf{M}((\partial G_{u_h}) \llcorner \Omega \times \mathbb{R}^3) < \infty, \tag{5.6}$$

we get $u \in \mathscr{A}^1(\Omega, \mathbb{R}^N)$ and $v(x) = M(\nabla u(x))$ for $\mathscr{L}^3$-a.e $x \in \Omega$. Moreover, $G_{u_h} \rightharpoonup G_u$ weakly in $\mathscr{D}_3(\Omega \times \mathbb{R}^3)$, whence by lower semicontinuity

$$\mathbf{M}(G_u) \leq \liminf_{h\to\infty} \mathbf{M}(G_{u_h}) < \infty$$
$$\mathbf{M}((\partial G_u) \llcorner \Omega \times \mathbb{R}^3) \leq \liminf_{h\to\infty} \mathbf{M}((\partial G_{u_h}) \llcorner \Omega \times \mathbb{R}^3) < \infty.$$

The boundary mass equi-boundedness (that is, the estimate (5.6)) is automatically satisfied by sequences of Cartesian maps $\{u_h\} \subset \mathrm{cart}^1(\Omega, \mathbb{R}^3)$, that is, those for which the condition (5.4) holds true, so it prevents the formation of discontinuities that describe holes and/or cracks in the material. The limit u is a Cartesian map too, since the weak convergence in terms of currents preserves condition (5.4).

5.4. The graph boundary of the deformation

In order to apply the Closure Theorem 5.2, we need to ensure that the SBV deformation $\varphi : \Omega \to \mathbb{R}^3$ belongs to the class $\mathscr{A}^1(\Omega, \mathbb{R}^3)$. Equation (5.1) gives

$$\begin{aligned} \nabla\varphi(x) &= a(x)\, F^e(x) \\ \mathrm{cof}\, \nabla\varphi(x) &= a(x)^2 \mathrm{cof}\, F^e(x) \\ \det \nabla\varphi(x) &= a(x)^3 \det F^e(x) \end{aligned} \tag{5.7}$$

almost everywhere in Ω. Therefore, on account of the bounds (4.5), it suffices to require that

$$M(F^e) \in L^1(\Omega, \mathbb{R}^{19}),$$

where $\mathscr{L}^3$ is the undertaking measure. ∂G_φ describes 'vertical' parts in the graph of φ. They may represent shear bands in this setting.

Here, we find it physically reasonable to require that the projection of ∂G_φ onto Ω falls within the 2-rectifiable set S that correspond to the singular component $\hat{F}(\bar{S}, \overline{\Gamma})$ of the plastic factor F^p in the multiplicative decomposition $D\varphi = F^e F^p$. This condition generalizes the requirement in reference [27] on the summability of distributional determinant and adjoint of $\nabla\varphi$ outside a given loop Γ. This is enclosed into the bound

$$\mathbf{M}((\partial G_\varphi) \llcorner V \times \mathbb{R}^3) \leq c_1 |\hat{F}(\bar{S}, \overline{\Gamma})|(V) \tag{5.8}$$

for each open set $V \subset \Omega$ and for some absolute constant $c_1 > 0$. We have already seen that the latter bound implies inequality (5.2) and hence the validity of the multiplicative decomposition (1.1).

Also, on account of the ansatz (4.4), and recalling that $\mathbf{M}(\overline{\Gamma}) \leq 2|\mathrm{curl}\, F^p|(\Omega)$, the bound (5.8) implies

$$\mathbf{M}((\partial G_\varphi) \llcorner \Omega \times \mathbb{R}^3) \leq 4c_1 c|\mathrm{curl}\, F^p|(\Omega)^2. \tag{5.9}$$

Proposition 5.1. *With the previous assumptions, the inequality (5.9) only depends on the minors $M(F^e)$ of the elastic factor and on the total variation of the dislocation measure* curl F^p.

Proof. If $u \in \mathscr{A}^1(\Omega, \mathbb{R}^3)$, condition $\mathbf{M}((\partial G_u) \llcorner \Omega \times \mathbb{R}^3) < \infty$ is equivalent to a bound for all A, $j = 1, 2, 3$ of the entities

$$\sup\langle \partial G_u, \phi(x, y)\, \widehat{\mathrm{d}x^A} \rangle, \quad \sup\langle \partial G_u, \phi(x, y)\, \mathrm{d}x^A \wedge \mathrm{d}y^j \rangle, \quad \sup\langle \partial G_u, \phi(x, y)\, \widehat{\mathrm{d}y^j} \rangle,$$

where each supremum is taken among all test functions $\phi \in C_c^\infty(\Omega \times \mathbb{R}^3)$ such that $\|\phi\|_\infty \leq 1$. For 2-forms of the type $\phi(x, y)\, \widehat{\mathrm{d}x^A}$, that is, those based on the reference configuration, we have

$$\langle \partial G_u, \phi(x, y)\, \widehat{\mathrm{d}x^A} \rangle = (-1)^{A-1} \int_\Omega \partial_A[\phi(x, u(x))]\, \mathrm{d}x,$$

while for those 2-forms $\phi(x, y)\, \widehat{\mathrm{d}y^j}$ defined on the current configuration,

$$\langle \partial G_u, \phi(x, y)\, \widehat{\mathrm{d}y^j} \rangle = (-1)^{j-1} \sum_{A=1}^{3} \int_\Omega \partial_A[\phi(x, u(x))]\, (\mathrm{adj}\, \nabla u(x))^j_A\, \mathrm{d}x.$$

The Laplace formulas imply

$$(-1)^{j-1} \sum_{A=1}^{3} \partial_A[\phi(x, u)](\mathrm{adj}\, \nabla u)^j_A = \sum_{A=1}^{3} (-1)^{A-1} \frac{\partial \phi}{\partial x_A}(x, u)\, M^{\overline{j}}_{\overline{A}}(\nabla u) + (-1)^{j-1} \frac{\partial \phi}{\partial y_j}(x, u)\, \det \nabla u,$$

where $M^{\overline{j}}_{\overline{A}}(G)$ indicates the 2×2-minor of G obtained by deleting the j-th raw and A-th column (compare [14, Vol. I, Sec. 3.3.2]). Since a similar formula holds for the terms $\phi(x, y)\, \mathrm{d}x^A \wedge \mathrm{d}y^j$, too, by condition (4.5) and formulas (5.7), the left-hand side of inequality (5.9) only depends on $M(F^e)$, as required. $\square$

5.5. By avoiding self-penetration

To avoid self-penetration of matter along deformations while allowing self-contact between distant portions of the boundary, in 1987 P. Ciarlet and J. Nečas [7] proposed the introduction of the constraint

$$\int_{\Omega'} \det \nabla\varphi(x)\,\mathrm{d}x \leq \mathscr{L}^3(\widetilde{\varphi}(\widetilde{\Omega}'))$$

for any sub-domain Ω' of Ω, where $\widetilde{\Omega}'$ is intersection of Ω' with the domain $\widetilde{\Omega}$ of the Lebesgue's representative $\widetilde{\varphi}$ of φ.

In 1989, M. Giaquinta, G. Modica, and J. Souček proposed a version of such a constraint [14, Vol. II, Sec. 2.3.2], that is,

$$\int_{\Omega} f(x,\varphi(x))\,\det \nabla\varphi(x)\,\mathrm{d}x \leq \int_{\mathbb{R}^3} \sup_{x\in\Omega} f(x,y)\,\mathrm{d}y \quad \forall f \in C_c^\infty(\Omega\times\mathbb{R}^3), \quad f \geq 0. \tag{5.10}$$

We adopt this here. Again by the identities (5.7), it turns out that (5.10) is essentially a property of the elastic factor F^e. We also point out that this condition is preserved by the weak convergence of currents, namely $G_{\varphi_h} \rightharpoonup G_\varphi$.

6. Existence of minimizers

What we have discussed so far deals with kinematics and allows us to define a class of physically admissible competitors minimizing the energy (1.2) and related variants that can be analyzed in the same way.

6.1. The admissible class

The class $\mathscr{A} = \mathscr{A}_{M,C,c_1,\mathscr{K}}(\Omega)$ of admissible competitors depends on

- the reference body shape $\Omega \subset \mathbb{R}^3$, taken to be open, simply connected, and endowed with a surface like Lipschitz boundary $\partial\Omega$,
- positive constants M, C, c_1, with $C > 1$, and
- a compact set $\mathscr{K}$ contained in Ω; it is the key ingredient of the confinement condition.

A map $\varphi : \Omega \to \mathbb{R}^3$ belongs to the admissible class $\mathscr{A}$ provided that the properties listed below hold true.

(1) $\varphi \in SBV(\Omega, \mathbb{R}^3)$ satisfies $\|\varphi\|_\infty \leq M$ for some fixed constant $M > 0$ and the tangency condition: the jump of φ is tangential to the approximate tangent plane at $\mathscr{H}^2$-a.e. point in the slip set $S(\varphi)$.

(2) The distributional derivative of φ, namely $D\varphi$, satisfies the multiplicative decomposition $D\varphi = F^e F^p$. The plastic factor F^p belongs to $\mathscr{M}_b(\Omega, \mathbb{R}^{3\times 3})$. It indicates the effects of slips over a generalized jump set S represented by the current $\bar{S}$ with associated rectifiable line defect indicated by $\overline{\Gamma}$ (see Definition 4.3).

(3) $\bar{S}$ is supported over some given compact set $\mathscr{K}$, namely spt $\bar{S} \subset \mathscr{K}$, and the bounding constant $C > 1$ for the term $a(x)$ in (4.5) is fixed.
(4) The so-called elastic factor F^e in the multiplicative decomposition belongs to $L^1(\Omega, \mathbb{R}^{3\times 3}; |F^p|)$ and is such that $M(F^e) \in L^1(\Omega, \mathbb{R}^{19})$ and $\det F^e > 0$ almost everywhere in Ω.
(5) The boundary of the graph current G_φ, associated with φ, satisfies the mass bound (5.8) for each open set $V \subset \Omega$ and for some absolute constant $c_1 > 0$.
(6) Condition (5.10) holds, that is, we exclude self-penetration of matter.

Due to the lack of stability of the bound (4.4), as explained in Sec. 4.9, we are led to introduce the following:

Definition 6.1. Given some real constant $c > 0$, we denote by $\widetilde{\mathscr{A}}_{M,C,c_1,\mathscr{K},c}(\Omega)$ the subclass of maps φ in $\mathscr{A}_{M,C,c_1,\mathscr{K}}(\Omega)$ such that the bound (4.4) on the mass of $\bar{S}$ in terms of the mass of $\overline{\Gamma}$ holds.

6.2. The energy functional

As already recalled in the Introduction, we consider a homogeneous material admitting an energy that is polyconvex with respect to the elastic factor F^e and includes weakly non-local effects encoded by curl F^p. Its simplest form reads as

$$\mathscr{F}_{p,s}(\varphi) := \int_\Omega \left(|M(F^e(x))|^p + |\det F^e(x)|^{-s}\right) \mathrm{d}x + |\mathrm{curl}\, F^p|(\Omega),$$

where $D\varphi = F^e F^p$ as above, while $p > 1$ and $s > 0$ are real exponents.

Essentially, the analysis we propose does not change if we replace the integrand depending on F^e with, for example, a non-negative convex function f on $\mathbb{R}^{19}$ such that

$$f(M(G)) \geq c_2(|M(G)|^p + |\det G|^{-s})$$

for all $G \in \mathbb{R}^{3\times 3}$, where $c_2 > 0$ is a real constant.

6.3. Dirichlet-Type boundary conditions

If $\varphi \in \mathscr{A}_{M,C,c_1,\mathscr{K}}(\Omega)$, the slip set $S(\varphi)$ is $\mathscr{H}^2$-essentially contained in the given compact subset $\mathscr{K}$ of Ω, whence the restriction of φ to the open set $\Omega \setminus \mathscr{K}$ is a Sobolev map. Therefore, if φ has bounded energy, namely $\mathscr{F}_{p,s}(\varphi) < \infty$, the restriction $\varphi_{|\Omega\setminus\mathscr{K}} \in W^{1,p}(\Omega \setminus \mathscr{K}, \mathbb{R}^3)$. We thus may impose a Dirichlet-type condition by choosing a function γ in the trace space $W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3)$ and requiring that the equality $\mathrm{Tr}(\varphi) = \gamma$ holds $\mathscr{H}^2$-a.e. in $\partial\Omega$, where $\mathrm{Tr}(\varphi)$ is the trace of φ on the boundary of Ω. We thus define

$$\begin{aligned}\mathscr{A}_{\underset{\sim}{\gamma}} &:= \{\varphi \in \mathscr{A}_{M,C,c_1,\mathscr{K}}(\Omega) \mid \mathscr{F}_{p,s}(\varphi) < \infty, \quad \mathrm{Tr}(\varphi) = \gamma\} \\ \mathscr{A}_\gamma &:= \{\varphi \in \widetilde{\mathscr{A}}_{M,C,c_1,\mathscr{K},c}(\Omega) \mid \mathscr{F}_{p,s}(\varphi) < \infty, \quad \mathrm{Tr}(\varphi) = \gamma\}.\end{aligned}$$

The absence of line defects implies $|\mathrm{curl}\, F^p|(\Omega) = 0$. In this case, the bound (4.4) reduces F^p to $a(x)\, I\, \mathscr{L}^3$ and the deformation φ is a Sobolev map in $W^{1,p}(\Omega, \mathbb{R}^3)$.

Suitable choices of the boundary term γ force the occurrence of defects in this setting when we impose constraints on the energy derivative with respect to $M(F^e)$; pertinent specific examples, expressed in the formal language adopted here, are in reference [13].

When we refer to crystals, the presence of curl F^p in the energy is a way to account for geometrically necessary dislocations, which can be detected by orientation imaging microscopy [22]. Numerical simulations accounting for the energetic weight of curl F^p corroborate the interpretation (see, for example, references [12,16,20]).

6.4. Existence theorem

Theorem 6.1. *Take $M, C, c_1, c > 0$, with $C > 1$, $\mathscr{K} \subset \Omega$ a compact set, and $p > 1$, $s > 0$. If for some $\gamma \in W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3)$ the class $\widetilde{\mathscr{A}_\gamma}$ is non-empty, the functional $\varphi \mapsto \mathscr{F}_{p,s}(\varphi)$ attains a minimum in $\mathscr{A}_\gamma$, that is, there exists $\varphi_0 \in \mathscr{A}_\gamma$ such that*

$$\mathscr{F}_{p,s}(\varphi_0) = \inf\{\mathscr{F}_{p,s}(\varphi) \mid \varphi \in \widetilde{\mathscr{A}_\gamma}\}.$$

Proof. We shall repeatedly extract not relabeled subsequences. Let $\{\varphi_h\} \subset \widetilde{\mathscr{A}_\gamma}$ be a minimizing sequence. Write $D\varphi_h = F_h^e F_h^p$, where

$$F_h^p = a_h(x) I\, \mathscr{L}^3 + \hat{F}(\bar{S}_h, \overline{\Gamma}_h).$$

Then, $\{\overline{\Gamma}_h\} \subset [\mathscr{R}_1(\Omega)]^3$ with spt $\overline{\Gamma}_h \subset \mathscr{K}$, $\partial\overline{\Gamma}_h = 0$, and $\mathbf{M}(\overline{\Gamma}_h) \leq 2\,|\text{curl } F_h^p|(\Omega)$ for each h. Moreover, $\{\bar{S}_h\} \subset [\mathscr{R}_2(\Omega)]^3$ with spt $\bar{S}_h \subset \mathscr{K}$, $\partial\bar{S}_h = \overline{\Gamma}_h$, and by the bound (4.4) we get the estimate

$$\mathbf{M}(\bar{S}_h) = |\hat{F}(\bar{S}_h, \overline{\Gamma}_h)|(\Omega) \leq 4c|\text{curl } F_h^p|(\Omega)^2 \qquad \forall h.$$

Therefore, by Federer-Fleming's compactness theorem we find $\bar{S} \in [\mathscr{R}_2(\Omega)]^3$ and $\overline{\Gamma} \in [\mathscr{R}_1(\Omega)]^3$ such that spt $\bar{S}$, spt $\overline{\Gamma} \subset \mathscr{K}$, $\partial\bar{S} = \overline{\Gamma}$, $\partial\overline{\Gamma} = 0$, and a subsequence such that $\bar{S}_h \rightharpoonup \bar{S}$ weakly in $[\mathscr{D}_2(\Omega)]^3$ and $\overline{\Gamma}_h \rightharpoonup \overline{\Gamma}$ weakly in $[\mathscr{D}_1(\Omega)]^3$. Proposition 4.1 implies that the weak limit current $\bar{S}$ satisfies the tangency condition, whence it is a generalized slip surface in our sense.

By (4.5) we may and do assume the existence of a function $a \in L^\infty(\Omega)$ such that $a_h \rightharpoonup a$ weakly in $L^\infty(\Omega)$ and almost everywhere, with $a(x)$ satisfying (4.5). Setting then

$$F^p = a(x)\, I\, \mathscr{L}^3 + \hat{F}(\bar{S}, \overline{\Gamma}) \tag{6.1}$$

so that curl $\hat{F}(\bar{S}, \overline{\Gamma}) =$ curl F^p, we have $\hat{F}(\bar{S}_h, \overline{\Gamma}_h) \rightharpoonup \hat{F}(\bar{S}, \overline{\Gamma})$ and curl $F_h^p \rightharpoonup$ curl F^p weakly as measures in $\mathscr{M}_b(\Omega, \mathbb{R}^{3\times3})$, where the dislocation measure curl F^p is associated to the current $\overline{\Gamma} \in \mathrm{r}-\mathrm{Id}(\Omega)$, and by lower semicontinuity of the total variation

$$|\text{curl } F^p|(\Omega) \leq \liminf_{h\to\infty} |\text{curl } F_h^p|(\Omega).$$

Moreover, since (5.7) holds true for each h by the multiplicative decomposition, condition (4.5) and the lower bound $\mathscr{F}_{p,s}(\varphi_h) \geq \|M(\nabla\varphi_h)\|^p_{L^p(\Omega,\mathbb{R}^{19})}$ furnish

$$\sup_h \int_\Omega |M(\nabla\varphi_h)|^p \,\mathrm{d}x < \infty.$$

Since the bounds (4.4) and (5.8) imply the inequality (5.9), we obtain

$$\sup_h \big(\mathbf{M}(G_{\varphi_h}) + \mathbf{M}((\partial G_{\varphi_h}) \llcorner \Omega \times \mathbb{R}^3)\big) < \infty.$$

Therefore, we can apply Theorem 5.2, which states that, possibly passing to a subsequence, $\varphi_h \to \varphi$ strongly in $L^1(\Omega, \mathbb{R}^3)$ and $M(\nabla\varphi_h) \rightharpoonup M(\nabla\varphi)$ weakly in $L^1(\Omega, \mathbb{R}^{19})$ for some function $\varphi \in \mathscr{A}^1(\Omega, \mathbb{R}^3)$ satisfying $\|\varphi\|_\infty \leq M$. Also, by lower semicontinuity of the mass with respect to the weak convergence in terms of currents, we have $\mathbf{M}(G_\varphi) + \mathbf{M}((\partial G_\varphi) \llcorner \Omega \times \mathbb{R}^3) < \infty$.

We thus infer that $\varphi \in SBV(\Omega, \mathbb{R}^3)$. Since moreover $\mathscr{H}^2(S_h \backslash S(\varphi_h)) = 0$ for each h, we get $\sup_h \mathscr{H}^2(S(\varphi_h)) < \infty$ and we can apply Proposition 4.2 to infer that the limit vector field φ satisfies the tangency condition, too.

By passing to a subsequence, we get $a_h(x) \to a(x)$ almost everywhere in Ω, $M(\nabla\varphi_h) \rightharpoonup M(\nabla\varphi)$ weakly in $L^p(\Omega, \mathbb{R}^{3\times 3})$ and almost everywhere in Ω, and $\hat{F}(\bar{S}_h, \overline{\Gamma}_h) \rightharpoonup \hat{F}(\bar{S}, \overline{\Gamma})$ as measures, whereas $\varphi_h \to \varphi$ in $L^1(\Omega, \mathbb{R}^3)$. Then, by using the multiplicative decomposition $D\varphi_h = F^e_h F^p_h$ for each h we deduce that the limit deformation φ satisfies itself the multiplicative decomposition $D\varphi = F^e F^p$, where the plastic factor F^p is given by (6.1) and the elastic factor F^e by (5.1), for almost everywhere $x \in \Omega$.

In fact, inequality (5.8) is stable with respect to the weak convergences $G_{\varphi_h} \rightharpoonup G_\varphi$ and $\hat{F}(\bar{S}_h, \overline{\Gamma}_h) \rightharpoonup \hat{F}(\bar{S}, \overline{\Gamma})$, whence it is satisfied by the weak limit current G_φ and measure $\hat{F}(\bar{S}, \overline{\Gamma})$. We follow Remark 5.1 to recover the multiplicative decomposition.

Also, $M(F^e_h) \rightharpoonup M(F^e)$ weakly in $L^p(\Omega, \mathbb{R}^{19})$ and $F^e_h \to F^e$ almost everywhere in Ω, so that $F^e \in L^1(\Omega, \mathbb{R}^{3\times 3})$, and condition $\sup_h \int_\Omega |\det F^e_h|^{-s} \,\mathrm{d}x < \infty$ implies, by lower semicontinuity, that $\int_\Omega |\det F^e|^{-s} \,\mathrm{d}x < \infty$ and hence that $\det F^e > 0$ almost everywhere in Ω. In particular, by using the notation (S, ξ_S, Θ_S) for $\bar{S} \in [\mathscr{R}_2(\Omega)]^3$ and assuming that $\Theta_S \in \mathbb{Z}^3 \setminus \{0_{\mathbb{R}^3}\}$ in S, we infer that $\mathscr{H}^2$-essentially $S(\varphi) \subset S$, see (5.2). We thus have $F^e \in L^1(\Omega, \mathbb{R}^{3\times 3}; |F^p|)$, and by the tangency conditions of both $\bar{S}$ and φ we infer that the relation (5.3) holds true.

The weak convergence of $\varphi_{h|\Omega\backslash\mathscr{K}}$ to $\varphi_{|\Omega\backslash\mathscr{K}}$ in $W^{1,p}(\Omega\backslash\mathscr{K}, \mathbb{R}^3)$ implies the $\mathscr{H}^2$-a.e. convergence $\mathrm{Tr}(\varphi_h) \to \mathrm{Tr}(\varphi)$ of the traces in $\partial\Omega$, whence the limit deformation φ satisfies the prescribed Dirichlet-type condition $\mathrm{Tr}(\varphi) = \gamma$.

Since $G_{\varphi_h} \rightharpoonup G_\varphi$ weakly in $\mathscr{D}_3(\Omega \times \mathbb{R}^3)$, we also infer that the deformation φ satisfies condition (5.10); thus it avoids self-penetration of matter.

Therefore, conditions (1)–(6) are satisfied and actually $\varphi \in \mathscr{A}_\gamma$. As already remarked in Sec. 4.9, due to the lack of stability of the bound (4.4) we cannot in general conclude that $\varphi \in \widetilde{\mathscr{A}_\gamma}$.

Since $M(F_h^e) \rightharpoonup M(F^e)$ weakly in $L^1(\Omega, \mathbb{R}^{3\times3})$ and $\operatorname{curl} F_h^p \rightharpoonup \operatorname{curl} F^p$ weakly as measures, by lower semicontinuity we get

$$\mathscr{F}_{p,s}(\varphi) \leq \liminf_{h\to\infty} \mathscr{F}_{p,s}(\varphi_h)$$

which concludes the proof. □

7. A more general class of dislocation-type defects

We move forward by considering in what follows currents with $\mathbb{R}^3$-valued multiplicity. When applied to (periodic) crystals, such a setting refers in principle the common picture of dislocations described at continuum scale. In this case, the tangency condition agrees with gliding and cross-slip behavior. However, being the deformation jump set taken to be a $2D$-rectifiable set, its possibly articulated structure may be considered as a possible reasonable approximation of geometries like those involving climbing.

The present extended setting (that is, the transition from currents with $\mathbb{Z}^3$-valued multiplicity to those with $\mathbb{R}^3$-valued ones) requires to look at their size. Thus we need to select only those currents with bounded size.

For example, in the case of a finite number N of pairwise disjoint dislocation loops Γ_h, see (4.1), the size $\mathbf{S}(\overline{\Gamma})$ of the dislocation current $\overline{\Gamma}$ is the total length

$$\mathbf{S}(\overline{\Gamma}) = \sum_{h=1}^{N} \mathscr{H}^1(\Gamma_h).$$

7.1. Size bounded currents

Let $U \subset \mathbb{R}^n$ be an open set. Take a rectifiable current T in $\mathscr{D}_k(U)$, endowed with finite mass, and write $T = [\![\mathscr{M}, \xi, \theta]\!]$. We denote $\operatorname{set}(T)$ the set of points in $\mathscr{M}$ where the k-dimensional density of the measure $\|T\| := \theta\, \mathscr{H}^k \llcorner \mathscr{M}$ is positive, and *size* of T the number $\mathbf{S}(T) := \mathscr{H}^k(\operatorname{set}(T))$.

We say that a rectifiable current T is a *size bounded* one if $\mathbf{S}(T) < \infty$. We indicate by $\mathscr{S}_k(U)$ the corresponding class of size bounded currents.

$T \in \mathscr{S}_k(U)$ implies that $\operatorname{set}(T)$ agrees $\mathscr{H}^k$-essentially with the set of points in $\mathscr{M}$ with positive multiplicity θ. Therefore, an integer multiplicity rectifiable current $T \in \mathscr{R}_k(U)$ is automatically size bounded because $\mathbf{S}(T) \leq \mathbf{M}(T)$, a property that fails to hold in general for currents with real multiplicity.

We also denote by $\mathscr{N}_k(U)$ the class of *normal currents*, those $T \in \mathscr{D}_k(U)$ such that $\mathbf{N}(T) := \mathbf{M}(T) + \mathbf{M}((\partial T) \llcorner U) < \infty$.

For $T_1, T_2 \in \mathscr{N}_k(U)$ we define a *flat distance* $d(T_1, T_2)$ by

$$d(T_1, T_2) := \inf\{\mathbf{M}(Q) + \mathbf{M}(R) \mid Q \in \mathscr{N}_k(U), \quad R \in \mathscr{N}_{k+1}(U) \\ T_1 - T_2 = Q + \partial R\},$$

where the term R does not appear in the case of top dimension $k = n$.

In general, the flat convergence $d(T_h, T) \to 0$ of normal currents in $\mathscr{N}_k(U)$ implies the weak convergence $T_h \rightharpoonup T$ in $\mathscr{D}_k(U)$. The reverse implication holds true provided that U is a smooth and bounded domain (see [34]).

By adapting a result due to Almgren [1, Prop. 2.10], and slightly weakening some assumptions adopted there, a lower semicontinuity result holds true (see the proof in reference [26]).

Theorem 7.1. *Let* $\{T_h\},\ T \subset \mathscr{S}_k(U) \cap \mathscr{N}_k(U)$ *be such that* $\sup_h \mathbf{N}(T_h) < \infty$ *and* $d(T_h, T) \to 0$. *Then,* $\mathbf{S}(T) \le \liminf_{h\to\infty} \mathbf{S}(T_h)$.

Also, a closure theorem is valid, as proven in reference [3, Thm. 8.5]. It refers to the more general setting of currents in metric spaces.

Theorem 7.2. *Let* $\{T_h\} \subset \mathscr{S}_k(U) \cap \mathscr{N}_k(U)$ *be such that*

$$\sup_h (\mathbf{S}(T_h) + \mathbf{N}(T_h)) < \infty.$$

Then, there exists a current $T \in \mathscr{S}_k(U) \cap \mathscr{N}_k(U)$ *and a (not relabeled) subsequence such that* $d(T_h, T) \to 0$.

7.2. $\mathbb{R}^m$*-Valued size bounded currents*

In a similar way to the class $[\mathscr{R}_k(U)]^m$, $m \in \mathbb{N}^+$, an $\mathbb{R}^m$*-valued rectifiable* k*-current* $\overline{T}$ in U is defined by a triplet $(\mathscr{M}, \xi, \Theta)$, where $\mathscr{M}$ and ξ are given as above, but $\Theta : \mathscr{M} \to \mathbb{R}^m$ is an $\mathbb{R}^m$-valued $\mathscr{H}^k \llcorner \mathscr{M}$-summable multiplicity function.

Correspondingly, we denote by $\mathrm{set}(\overline{T})$ the set of points in $\mathscr{M}$ where the k-dimensional density of the measure $\|\overline{T}\| := |\Theta|\, \mathscr{H}^k \llcorner \mathscr{M}$ is positive, and define $\mathbf{S}(\overline{T}) := \mathscr{H}^k(\mathrm{set}(\overline{T}))$.

We call $\overline{T}$ an $\mathbb{R}^m$*-valued size bounded current*, formally writing $\overline{T} \in [\mathscr{S}_k(U)]^m$, when $\mathbf{S}(\overline{T}) < \infty$.

As for the class $[\mathscr{R}_k(U)]^m$, a current $\overline{T} \in [\mathscr{S}_k(U)]^m$ can be seen as an ordered m-tuple $\overline{T} = (T^1, \ldots, T^m)$ of size bounded currents $T^j \in \mathscr{S}_k(U)$, where $\mathrm{set}(\overline{T}) = \cup_{j=1}^m \mathrm{set}(T^j)$. We also define $\mathbf{N}(\overline{T}) := \mathbf{M}(\overline{T}) + \mathbf{M}((\partial\overline{T}) \llcorner U)$ where, we recall, $\mathbf{M}(\overline{T}) := \sum_{j=1}^m \mathbf{M}(T^j) < \infty$ and $\mathbf{M}((\partial\overline{T}) \llcorner U) := \sum_{j=1}^m \mathbf{M}((\partial T^j) \llcorner U)$ if $\overline{T} = (T^1, \ldots, T^m)$.

Moreover, if $T^j \in \mathscr{S}_k(U)$ for $j = 1, \ldots, m$, we find a current $\overline{T} \in [\mathscr{S}_k(U)]^m$ with components $\overline{T} = (T^1, \ldots, T^m)$. Therefore, if U is a smooth and bounded domain, and a sequence $\{\overline{T}_h\} \subset [\mathscr{S}_k(U)]^m$ satisfies $\sup_h (\mathbf{S}(\overline{T}_h) + \mathbf{N}(\overline{T}_h)) < \infty$, by the compactness and semicontinuity results previously stated we can find a current $\overline{T} \in [\mathscr{S}_k(U)]^m$ and a (not relabeled) subsequence of $\{\overline{T}_h\}$ such that $\overline{T}_h \rightharpoonup \overline{T}$, and also

$$\mathbf{N}(\overline{T}) \le \liminf_{h\to\infty} \mathbf{N}(\overline{T}_h), \quad \mathbf{S}(\overline{T}) \le \liminf_{h\to\infty} \mathbf{S}(\overline{T}_h).$$

7.3. Plastic deformations with size bounded line defects

Set $n = m = 3$ and $U = \Omega$.

Definition 7.1. We call a *generalized slip surface* any $\mathbb{R}^3$-valued size bounded current $\bar{S} \in [\mathscr{S}_2(\Omega)]^3$ satisfying the confinement condition spt $\bar{S} \subset \Omega$, the tangency condition (4.3), and such that the boundary current $\overline{\Gamma} := \partial \bar{S}$ is an $\mathbb{R}^3$-valued size bounded current in $[\mathscr{S}_1(\Omega)]^3$. The current $\overline{\Gamma}$ is called a *size bounded line defect* in Ω, and we write $\overline{\Gamma} \in \mathrm{s-ld}\,(\Omega)$.

As before, $\partial\overline{\Gamma} = 0$ and spt $\overline{\Gamma} \subset \Omega$ for every $\overline{\Gamma} \in \mathrm{s-ld}\,(\Omega)$. In addition, as a constitutive condition we require that the bound (4.4) holds, and also that

$$\mathbf{S}(\bar{S}) \leq c\mathbf{S}(\overline{\Gamma})^2 \tag{7.1}$$

for some fixed real constant $c > 0$.

Moreover, the tangency condition (4.3) is preserved in the minimization process.

Proposition 7.1. *Let $\{\bar{S}_h\}_h \subset [\mathscr{S}_2(\Omega)]^3$ be a sequence of generalized slip surfaces satisfying*

$$\sup_h \big(\mathbf{N}(\bar{S}_h) + \mathbf{S}(\bar{S}_h)\big) < \infty.$$

Then, there exists a (not relabeled) subsequence and a generalized slip surface $\bar{S} \in [\mathscr{S}_2(\Omega)]^3$ such that $\bar{S}_h \rightharpoonup \bar{S}$ weakly in $[\mathscr{D}_2(\Omega)]^3$.

Proof. On account of the compactness theorem for size bounded currents, we argue exactly as in the proof of Proposition 4.1. □

Definition 7.2. A tensor-valued measure $F^p \in \mathscr{M}(\Omega, \mathbb{R}^{3\times 3})$ is called a *plastic deformation factor with generalized slip surface $\bar{S}$ and size bounded line defect $\overline{\Gamma}$* if

$$F^p = a(x) I \mathscr{L}^3 + \hat{F}(\bar{S}, \overline{\Gamma}),$$

where $a(x)$ is a Borel function in Ω satisfying (4.5) for some given real constant $C > 1$, and

$$\langle \hat{F}(\bar{S}, \overline{\Gamma}), \zeta \rangle = \langle \bar{S}, \omega_\zeta^{(2)} \rangle \qquad \forall\, \zeta \in C_c^\infty(\Omega, \mathbb{R}^{3\times 3})$$

for some generalized slip surface $\bar{S}$ with $\overline{\Gamma} = \partial \bar{S}$ in $\mathrm{s-ld}\,(\Omega)$.

Therefore, this time curl F^p is identified by the size bounded current $\overline{\Gamma}$ in $\mathrm{s-ld}\,(\Omega)$ through the formula (4.6).

7.4. Elastic–plastic deformations with size bounded line defects

Similarly as above, the admissible class $\mathscr{A}^{\mathfrak{s}} = \mathscr{A}^{\mathfrak{s}}_{M,C,c_1,\mathscr{K}}(\Omega)$ of *elastic–plastic deformations with size bounded line defects* is defined by the maps $\varphi : \Omega \to \mathbb{R}^3$ satisfying the properties (1), (3), (4), (5), and (6) listed in the previous section, but with property (2) replaced by

(2') $F^p \in \mathscr{M}_b(\Omega, \mathbb{R}^{3\times 3})$ is a plastic deformation factor with generalized slip surface $\bar{S}$ and size bounded line defect $\overline{\Gamma}$ (see Definition 7.2).

Also, due to the lack of stability of the bounds (4.4) and (7.1) we introduce the following:

Definition 7.3. Given some real constant $c > 0$, we denote by $\widetilde{\mathscr{A}}^{\mathfrak{s}}_{M,C,c_1,\mathscr{K},c}(\Omega)$ the subclass of maps φ in $\mathscr{A}^{\mathfrak{s}}_{M,C,c_1,\mathscr{K}}(\Omega)$ such that both the bounds (4.4) and (7.1) hold.

7.5. The energy functional

To account for the size of line defects, we modify the energy by adding the size of $\bar{\Gamma}$, that is, by including a line energy. Such a physical choice has nontrivial analytical consequences: it allows us to apply the closure theorem for size bounded currents. The resulting modified energy reads

$$\widetilde{\mathscr{F}}_{p,s}(\varphi) := \int_{\Omega} \left(|M(F^e(x))|^p + |\det F^e(x)|^{-s}\right) \mathrm{d}x + |\mathrm{curl}\, F^p|(\Omega) + \mathbf{S}(\overline{\Gamma})$$

for some real exponents $p > 1$ and $s > 0$, where in the first term we can take more general integrands as already mentioned above.

7.6. Existence result

As before, we impose a Dirichlet-type condition by choosing a function γ in $W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3)$ and defining

$$\begin{aligned} \mathscr{A}^{\mathfrak{s}}_{\gamma} &:= \{\varphi \in \mathscr{A}^{\mathfrak{s}}_{M,C,c_1,\mathscr{K}}(\Omega) \mid \widetilde{\mathscr{F}}_{p,s}(\varphi) < \infty, \quad \mathrm{Tr}(\varphi) = \gamma\} \\ \widetilde{\mathscr{A}}^{\mathfrak{s}}_{\gamma} &:= \{\varphi \in \widetilde{\mathscr{A}}^{\mathfrak{s}}_{M,C,c_1,\mathscr{K},c}(\Omega) \mid \widetilde{\mathscr{F}}_{p,s}(\varphi) < \infty, \quad \mathrm{Tr}(\varphi) = \gamma\}. \end{aligned}$$

Theorem 7.3. *Take $M, C, c_1, c > 0$, with $C > 1$, $\mathscr{K} \subset \Omega$ a compact set, and $p > 1$, $s > 0$. If for some $\gamma \in W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3)$ the class $\widetilde{\mathscr{A}}^{\mathfrak{s}}_{\gamma}$ is non-empty, the functional $\varphi \mapsto \widetilde{\mathscr{F}}_{p,s}(\varphi)$ attains a minimum in $\mathscr{A}^{\mathfrak{s}}_{\gamma}$, that is, there exists $\varphi_0 \in \mathscr{A}^{\mathfrak{s}}_{\gamma}$ such that*

$$\widetilde{\mathscr{F}}_{p,s}(\varphi_0) = \inf\{\widetilde{\mathscr{F}}_{p,s}(\varphi) \mid \varphi \in \widetilde{\mathscr{A}}^{\mathfrak{s}}_{\gamma}\}.$$

Proof. For $\{\varphi_h\} \subset \widetilde{\mathscr{A}}_{\gamma}^{\mathfrak{s}}$ a minimizing sequence, write $D\varphi_h = F_h^e F_h^p$, where $F_h^p = a_h(x) I \,\mathscr{L}^3 + \hat{F}(\bar{S}_h, \overline{\Gamma}_h)$. Then, $\overline{\Gamma}_h \in [\mathscr{S}_1(\Omega)]^3$, spt $\overline{\Gamma}_h \subset \mathscr{K}$, $\partial\overline{\Gamma}_h = 0$, and $\mathbf{M}(\overline{\Gamma}_h) \leq 2\,|\mathrm{curl}\, F_h^p|(\Omega)$ for each h. Moreover, $\bar{S}_h \in [\mathscr{S}_2(\Omega)]^3$ with spt $\bar{S}_h \subset \mathscr{K}$, $\partial\bar{S}_h = \overline{\Gamma}_h$, and by the bounds (4.4) and (7.1)

$$\mathbf{M}(\bar{S}_h) \leq 4c\,|\mathrm{curl}\, F_h^p|(\Omega)^2, \quad \mathbf{S}(\bar{S}_h) \leq c\,\mathbf{S}(\overline{\Gamma}_h)^2 \qquad \forall\, h.$$

Therefore, by the compactness theorem on size bounded currents we find $\bar{S} \in [\mathscr{S}_2(\Omega)]^3$ and $\overline{\Gamma} \in [\mathscr{S}_1(\Omega)]^3$ such that spt $\bar{S}$, spt $\overline{\Gamma} \subset \mathscr{K}$, $\partial\bar{S} = \overline{\Gamma}$, $\partial\overline{\Gamma} = 0$, and a subsequence such that $\bar{S}_h \rightharpoonup \bar{S}$ weakly in $[\mathscr{D}_2(\Omega)]^3$ and $\overline{\Gamma}_h \rightharpoonup \overline{\Gamma}$ weakly in $[\mathscr{D}_1(\Omega)]^3$. By Proposition 7.1, the weak limit current $\bar{S}$ satisfies the tangency condition, whence it is a generalized slip surface in our sense.

As a consequence of the bounds (4.5), we find again a function $a \in L^\infty(\Omega)$ such that $a_h \rightharpoonup a$ weakly in $L^\infty(\Omega)$ and almost everywhere, with $a(x)$ satisfying (4.5). Then, by setting F^p as in (6.1), so that curl $\hat{F}(\bar{S}, \overline{\Gamma}) = \mathrm{curl}\, F^p$, we have $\hat{F}(\bar{S}_h, \overline{\Gamma}_h) \rightharpoonup \hat{F}(\bar{S}, \overline{\Gamma})$ and curl $F_h^p \rightharpoonup \mathrm{curl}\, F^p$ weakly as measures in $\mathscr{M}_b(\Omega, \mathbb{R}^{3\times 3})$, where the measure curl F^p is associated with the current $\overline{\Gamma} \in \mathrm{s-ld}\,(\Omega)$, and by lower semicontinuity of the total variation and size

$$|\mathrm{curl}\, F^p|(\Omega) \leq \liminf_{h\to\infty} |\mathrm{curl}\, F_h^p|(\Omega), \quad \mathbf{S}(\overline{\Gamma}) \leq \liminf_{h\to\infty} \mathbf{S}(\overline{\Gamma}_h).$$

By following steps in the proof of Theorem 6.1, and, in particular, Remark 5.1 to recover the multiplicative decomposition, we consequently infer that $\varphi \in \mathscr{A}_{\gamma}^{\mathfrak{s}}$. Finally, since $M(F_h^e) \rightharpoonup M(F^e)$ weakly in $L^1(\Omega, \mathbb{R}^{3\times 3})$, by lower semicontinuity we get

$$\widetilde{\mathscr{F}}_{p,s}(\varphi) \leq \liminf_{h\to\infty} \widetilde{\mathscr{F}}_{p,s}(\varphi_h),$$

which concludes the proof. □

8. Additional remarks

Our proposal constitutes a framework that can be applied to circumstances in which deformation mappings appear to fall within the setting of special bounded variation functions that are orientation preserving outside a jump set, which is a $2D$ rectifiable set. Such a framework captures variegate possibilities of slip mechanisms and accounts for energy associated with the singular part of the distributional derivative of the deformation mapping.

The scheme could be extended to some classes of complex materials, by including in the energy functional suitable descriptors of the microstructural morphology and of their gradients. In general, such additional fields have to be considered as Sobolev maps taking values on a finite-dimensional, differentiable, geodesic-complete abstract manifold.

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