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Alexandrov's Theorem for Anisotropic Capillary Hypersurfaces in the Half-Space

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Abstract

In this paper, we show that any embedded capillary hypersurface in the half-space with anisotropic constant mean curvature is a truncated Wulff shape. This extends Wente's result (Pac J Math 88:387–397, 1980. https://doi.org/10.2140/pjm.1980.88.387) to the anisotropic case and He–Li–Ma–Ge's result (Indiana Univ Math J 58(2):853–868, 2009. https://doi.org/10.1512/iumj.2009.58.3515) to the capillary boundary case. The main ingredients in the proof are a new Heintze-Karcher inequality and a new Minkowski formula, which have their own interest.

1. Introduction

Capillary phenomena appear in the study of the equilibrium shape of liquid drops and crystals in a given solid container. The mathematical model has been established through the work of Young, Laplace, Gauss and others, as a variational problem on minimizing a free energy functional under a volume constraint. A modern formulation of Gauss' model includes a possibly anisotropic surface tension density, which we are interested in. For more detailed description of the isotropic and anisotropic capillary phenomena, we refer to [12] and [6].

For our purposes, we consider the anisotropic capillary problem in the half-space

$$\mathbb{R}^{n+1}_+ = \{x \in \mathbb{R}^{n+1} : \langle x, E_{n+1}\rangle > 0\}.$$

Here E_{n+1} denotes the $(n+1)$-coordinate unit vector. Let Σ be a compact orientable embedded hypersurface in $\overline{\mathbb{R}^{n+1}_+}$ with boundary $\partial\Sigma$ lying on $\partial\mathbb{R}^{n+1}_+$, which, together with $\partial\mathbb{R}^{n+1}_+$, encloses a bounded domain Ω. Let ν be the unit normal of Σ pointing outward Ω. We consider the free energy functional

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$$\mathcal{E}(\Sigma) = \int_{\Sigma} F(\nu)\mathrm{d}A + \omega_0|\partial\Omega \cap \partial\mathbb{R}^{n+1}_{+}|,$$

where the term $\int_{\Sigma} F(\nu)\mathrm{d}A$ is the anisotropic surface tension and the term $\omega_0|\partial\Omega \cap \partial\mathbb{R}^{n+1}_{+}|$ is the wetting energy accounting for the adhesion between the fluid and the walls of the container. Here $F : \mathbb{S}^n \to \mathbb{R}_+$ is a C^2 positive function on $\mathbb{S}^n$, such that $(\nabla^2 F + F\sigma) > 0$, where σ is the canonical metric on $\mathbb{S}^n$ and ∇^2 is the Hessian on $\mathbb{S}^n$, and $\omega_0 \in \mathbb{R}$ is a given constant. The Cahn-Hoffman map associated with F is given by

$$\Phi : \mathbb{S}^n \to \mathbb{R}^{n+1}, \quad \Phi(x) = \nabla F(x) + F(x)x,$$

where ∇ denote the gradient on $\mathbb{S}^n$. One easily sees that $\Phi(x) = D\tilde{F}(x)$, where $\tilde{F}$ is the positive one-homogeneous extension of F to $\mathbb{R}^{n+1}$ and D denotes the Euclidean derivative. The image $\Phi(\mathbb{S}^n)$ of Φ is a strictly convex, closed hypersurface in $\mathbb{R}^{n+1}$, which is the unit Wulff shape with respect to F, which we denote by $\mathcal{W}_F$.

In the isotropic case $F \equiv 1$, the global minimizer of $\mathcal{E}$ under a volume constraint is characterized as a spherical cap by De Giorgi, which is the solution to the relative isoperimetric problem; see for example [27, Chapter 19]. In the anisotropic case, the global minimizer of $\mathcal{E}$ under volume constraint has been characterized by Winterbottom [11] as a truncated Wulff shape, which is also called a Winterbottom shape, or Winterbottom construction in applied mathematics, especially in material science; see for example [4] and references therein. The Winterbottom construction can be viewed as the capillary counterpart of Wulff construction, which characterizes the global minimizer for purely anisotropic surface tension, see [13,32,35]. For anisotropic free energy functionals involving a gravitational potential energy term, the existence, the regularity and boundary regularity of global minimizers have been studied by De Giorgi [7], Almgren, [1] and Taylor [33]; see also the recent work by De Philippis and Maggi [6,8]. For the symmetry and uniqueness of global minimizers we refer to the work of Baer [5] for a class of F with certain symmetry, following the work of Gonzalez [15] in the isotropic case, via a symmetrization technique.

In this paper, we shall study the rigidity for the stationary surfaces for the free energy functional $\mathcal{E}$ under a volume constraint. Given a variation $\{\Sigma_t\}$ of Σ, whose boundary $\partial\Sigma_t$ moves freely on $\partial\mathbb{R}^{n+1}_{+}$ and according to a variational vector field Y such that $Y\,|_{\partial\Sigma} \in T(\partial\mathbb{R}^{n+1}_{+})$, the first variation formula of $\mathcal{E}$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\,|_{t=0}\, \mathcal{E}(\Sigma_t) = \int_{\Sigma} H^F \langle Y, \nu\rangle\, \mathrm{d}A + \int_{\partial\Sigma} \langle Y, R(p(\Phi(\nu)))\rangle\, \mathrm{d}s + \omega_0 \int_{\partial\Sigma} \langle Y, \mu\rangle\, \mathrm{d}s,$$

where H^F is the anisotropic mean curvature of Σ, p is the projection onto the $\{\nu, E_{n+1}\}$-plane, R is the $\pi/2$-rotation in the $\{\nu, E_{n+1}\}$-plane, μ is the conormal of $\partial\Sigma \subset \Sigma$, see [22,23,30]. For its proof we refer to the one of [23, Proposition 2]. It follows that the stationary points of $\mathcal{E}$ among C^2 hypersurfaces under a volume constraint are anisotropic ω_0-capillary hypersurfaces with constant anisotropic mean curvature. In this paper we say a hypersurface in $\overline{\mathbb{R}^{n+1}_+}$ with boundary $\partial\Sigma \subset \partial\mathbb{R}^{n+1}_+$ *anisotropic ω_0-capillary* if

$$\langle \Phi(\nu), -E_{n+1} \rangle = \omega_0, \quad \text{on } \partial\Sigma. \tag{1.1}$$

We emphasize that it is not necessarily a constant anisotropic mean curvature hypersurface. Moreover we are interested in hypersurfaces which intersect with $\partial\mathbb{R}^{n+1}_+$ transversely.

The rigidity of embedded closed constant mean curvature hypersurfaces was obtained by Alexandrov [2] in the celebrated Alexandrov's theorem, that any embedded closed hypersurface of constant mean curvature in $\mathbb{R}^{n+1}$ must be a sphere. In the proof Alexandrov introduced the famous moving plane method. Wente [34] showed that any embedded compact hypersurface of constant mean curvature with capillary boundary in $\overline{\mathbb{R}^{n+1}_+}$ is a spherical cap. Taking into account of the anisotropy, He–Li–Ma–Ge [17] proved that any embedded closed hypersurface in $\mathbb{R}^{n+1}$ with constant anisotropic mean curvature must be a Wulff shape. See also a related result by Morgan [28] in $\mathbb{R}^2$ for a more general anisotropic function F. We also mention that the Alexandrov-type theorem for general finite perimeter sets has been proved in the isotropic setting by Delgadino-Maggi [9], and in the anisotropic setting by De Rosa-Kolasiński-Santilli [10]. For the closely related work on the stability problem of constant anisotropic mean curvature hypersurface without boundary or with capillary boundary, we refer to [16,22–26,30] and references therein.

Our main result in this paper is the following Alexandrov-type theorem for embedded anisotropic capillary hypersurfaces of constant anisotropic mean curvature in $\overline{\mathbb{R}^{n+1}_+}$.

Theorem 1.1. *Let $\omega_0 \in (-F(E_{n+1}), F(-E_{n+1}))$. Let $\Sigma \subset \overline{\mathbb{R}^{n+1}_+}$ be a C^2 embedded compact anisotropic ω_0-capillary hypersurface with constant anisotropic mean curvature. Then Σ is an ω_0-capillary Wulff shape.*

An ω_0-capillary Wulff shape is part of a Wulff shape in $\overline{\mathbb{R}^{n+1}_+}$ such that the anisotropic capillary boundary condition (1.1) holds. We remark that the assumption $\omega_0 \in (-F(E_{n+1}), F(-E_{n+1}))$ is a necessary condition so that Wulff shapes intersect with $\partial\mathbb{R}^{n+1}_+$ transversely, see Remark 2.1.

As mentioned above, Theorem 1.1 for the isotropic case was proved by Wente in [34], where he used Alexandrov's moving plane method. However, the moving plane method fails in general for the anisotropic case, at least if F has less symmetry. A new proof of Wente's result has been done by the authors [21] through the establishment of a Heintze-Kacher-type inequality in the capillary problem, which is inspired by the original idea of Heintze-Karcher [18] (see also Montiel-Ros [29]). This method is flexible to the anisotropic case and this is the way we achieve Theorem 1.1.

Following this way we first need to establish a Heintze-Harcher type inequality for anisotropic capillary hypersurfaces. In order to state the inequality, we need a constant vector $E^F_{n+1} \in \mathbb{R}^{n+1}$ defined as

$$E^F_{n+1} = \begin{cases} \frac{\Phi(E_{n+1})}{F(E_{n+1})}, & \text{if } \omega_0 < 0, \\ -\frac{\Phi(-E_{n+1})}{F(-E_{n+1})}, & \text{if } \omega_0 > 0. \end{cases} \tag{1.2}$$

Note that E^F_{n+1} is the unique vector in the direction $\Phi(E_{n+1})$, whose scalar product with E_{n+1} is 1. When $\omega_0 = 0$, one can define it by any unit vector. This constant vector plays a crucial role in the paper. A hypersurface is said to be strictly anisotropic-mean convex if $H^F > 0$. Now we state our anisotropic Heintze-Karcher inequality.

Theorem 1.2. *Let $\omega_0 \in (-F(E_{n+1}), F(-E_{n+1}))$ and $\Sigma \subset \overline{\mathbb{R}^{n+1}_+}$ be a C^2 compact embedded strictly anisotropic-mean convex hypersurface with boundary $\partial\Sigma \subset \partial\mathbb{R}^{n+1}_+$ such that*

$$\langle \Phi(\nu(x)), -E_{n+1} \rangle = \omega(x) \leq \omega_0, \quad \textit{for any } x \in \partial\Sigma. \tag{1.3}$$

Then it holds that

$$\int_\Sigma \frac{F(\nu) + \omega_0 \left\langle \nu, E^F_{n+1} \right\rangle}{H^F} dA \geq \frac{n+1}{n} |\Omega|. \tag{1.4}$$

Equality in (1.4) holds if and only if Σ is an ω_0-capillary Wulff shape.

We will follow the argument in [21] to prove Theorem 1.2. The main idea is to define suitable parallel hypersurfaces $\zeta_F(\cdot, t)$, in order to sweepout the enclosed domain Ω and use the area formula to compute the volume. A crucial ingredient is an anisotropic angle comparison principle in Proposition 3.1 which enables us to prove the surjectivity of ζ_F.

Then we need the following anisotropic Minkowski-type formula:

Theorem 1.3. *Let $\omega_0 \in (-F(E_{n+1}), F(-E_{n+1}))$ and $\Sigma \subset \overline{\mathbb{R}^{n+1}_+}$ be a C^2 compact anisotropic ω_0-capillary hypersurface. Let H^F_r be the (normalized) anisotropic r-th mean curvature for some $r \in \{1, \ldots, n\}$ and $H^F_0 \equiv 1$ by convention. Then it holds*

$$\int_\Sigma H^F_{r-1} \left(F(\nu) + \omega_0 \left\langle \nu, E^F_{n+1} \right\rangle \right) - H^F_r \langle x, \nu \rangle \, dA = 0. \tag{1.5}$$

In particular,

$$\int_\Sigma \left(F(\nu) + \omega_0 \left\langle \nu, E^F_{n+1} \right\rangle \right) - H^F_1 \langle x, \nu \rangle \, dA = 0. \tag{1.6}$$

Remark 1.1. We remark the importance of using the constant vector E^F_{n+1}. In fact, (1.4) and (1.5) hold true, if we replace E^F_{n+1} by E_{n+1}. However, if one used E_{n+1} instead of E^F_{n+1}, we could only prove our main Theorem 1.1, for a smaller range $\omega_0 \in (-1/F^o(E_{n+1}), 1/F^o(-E_{n+1}))$. For one of reasons see Proposition 3.2. The main reason lies in the proof of the Heintze-Karcher inequality. For details, see Remarks 3.1 and 3.2. This is one of the crucial differences between the isotropic case and the anisotropic case.

For the isotropic case $F \equiv 1$, (1.4) and (1.5) were proved by the authors [21]. We refer to [21] and [36] for a historical description of the Heintze-Karcher inequality and the Minkowski formula respectively, and references therein.

The anisotropic Heintze-Karcher inequality and the anisotropic Minkowski formula for closed hypersurfaces have been proved by He–Li–Ma–Ge [17, Theorem 4.4] and He–Li [19]. In this case, our argument provides a slight improvement for the anisotropic Heintze-Karcher inequality, at least when F is even, see Remark 3.3.

Corollary 1.1. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a C^2 closed embedded strictly anisotropic-mean convex hypersurface. Then it holds that*

$$\int_\Sigma \frac{F(\nu)}{H^F} dA \geq \frac{n+1}{n}|\Omega| + \max\left\{0, \max_{e \in \mathbb{S}^n} \int_\Sigma \frac{\langle \nu, \Phi(e) \rangle}{H^F} dA\right\}. \tag{1.7}$$

Equality holds if and only if Σ is a Wulff shape.

Finally we follow an argument of Ros [31] by combining Theorems 1.2 and 1.3 to establish the Alexandrov-type theorem for capillary hypersurfaces with constant anisotropic mean curvature, Theorem 1.1, and also the Alexandrov-type theorem for capillary hypersurfaces with constant higher order anisotropic mean curvature whose definition will be given in Sect. 2.

Theorem 1.4. *Let $\omega_0 \in (-F(E_{n+1}), F(-E_{n+1}))$. Let $\Sigma \subset \overline{\mathbb{R}^{n+1}_+}$ be a C^2 embedded compact anisotropic ω_0-capillary hypersurface with constant r-th anisotropic mean curvature for some $r \in \{2, \ldots, n\}$. Then Σ is an ω_0-capillary Wulff shape.*

The rest of the paper is organized as follows. In Sect. 2, we provide more details about the anisotropic mean curvature and the higher order anisotropic mean curvature, together with the Wulff shape and the ω_0-capillary Wulff shape. In Sect. 3, we prove the Minkowski-type formula in Theorem 1.3 and the Heintze-Karcher-type inequality in Theorem 1.2. In Sect. 4, we prove the Alexandrov-type theorem, Theorems 1.1 and 1.4.

2. Preliminaries

Let $F : \mathbb{S}^n \to \mathbb{R}_+$ be a C^2 positive function on $\mathbb{S}^n$ such that $(\nabla^2 F + F\sigma) > 0$. We denote

$$A_F = \nabla^2 F + F\sigma.$$

Let $F^o : \mathbb{R}^{n+1} \to \mathbb{R}$ be defined by

$$F^o(x) = \sup\left\{\frac{\langle x, z \rangle}{F(z)} \mid z \in \mathbb{S}^n\right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product. We collect some well-known facts on F and F^o, see e.g. [17].

Proposition 2.1. *For any $z \in \mathbb{S}^n$ and $t > 0$, the following statements hold:*

(i) $F^o(tz) = tF^o(z)$.
(ii) $\langle \Phi(z), z\rangle = F(z)$.
(iii) $F^o(\Phi(z)) = 1$.
(iv) *The following Cauchy–Schwarz inequality holds:*

$$\langle x, z\rangle \le F^o(x)F(z). \tag{2.1}$$

(v) *The unit Wulff shape $\mathcal{W}_F$ can be interpreted by F^o as*

$$\mathcal{W}_F = \{x \in \mathbb{R}^{n+1} : F^o(x) = 1\}.$$

A Wulff shape of radius r centered at $x_0 \in \mathbb{R}^{n+1}$ is given by

$$\mathcal{W}_{r_0}(x_0) = \{x \in \mathbb{R}^{n+1} : F^o(x - x_0) = r_0\}.$$

Let $\Sigma \subset \overline{\mathbb{R}^{n+1}_+}$ be a C^2 hypersurface with $\partial\Sigma \subset \partial\mathbb{R}^{n+1}_+$, which encloses a bounded domain Ω. Let ν be the unit normal of Σ pointing outward Ω. The anisotropic normal of Σ is given

$$\nu_F = \Phi(\nu) = \nabla F(\nu) + F(\nu)\nu,$$

and the anisotropic principal curvatures $\{\kappa_i^F\}_{i=1}^n$ of Σ are given by the eigenvalues of the anisotropic Weingarten map

$$d\nu_F = A_F(\nu) \circ d\nu : T_p\Sigma \to T_p\Sigma.$$

The eigenvalues are real since (A_F) is positive definite and symmetric. For $r \in \{1, \cdots, n\}$, the (normalized) r-th anisotropic mean curvature is defined by

$$H_r^F = \frac{1}{\binom{n}{r}}\sigma_r^F,$$

where σ_r^F be the r-th elementary symmetric function on the anisotropic principal curvatures $\{\kappa_i^F\}_{i=1}^n$, namely,

$$\sigma_r^F = \sum_{1\le i_1<\cdots<i_r\le n} \kappa_{i_1}^F \cdots \kappa_{i_r}^F,$$

In particular, $H^F = \sigma_1^F$ is the anisotropic mean curvature and H_1^F the normalized anisotropic mean curvature. Alternatively, the r-th anisotropic mean curvature H_r^F of Σ can be defined through the identity

$$\mathcal{P}_n(t) = \prod_{i=1}^n (1 + t\kappa_i^F) = \sum_{i=0}^n \binom{n}{i} H_i^F t^i \tag{2.2}$$

for all real numbers t.

It is easy to check that the anisotropic principal curvatures of $\mathcal{W}_r(x_0)$ are $\frac{1}{r}$, since

$$\nu_F(x) = \frac{x - x_0}{r}, \quad \text{on } \mathcal{W}_r(x_0). \tag{2.3}$$

For the convenience of the reader, we provide a proof of (2.3). We first consider the unit Wulff shape, $\mathcal{W}_F$. Since the unit normal vector ν of $\mathcal{W}_F$ at $x \in \mathcal{W}_F$ is given by $\Phi^{-1}(x)$, then the anisotripic normal is just x. For the general case, one use a translation and a scaling.

A truncated Wulff shape is a part of a Wulff shape cut by a hyperplane, say $\{x_{n+1} = 0\}$. Namely, it is an intersection of a Wulff shape and $\mathbb{R}^{n+1}_+$. As mentioned above, it was used by Winterbottem [11]. At a first glimpse, it is not very easy to image why the hyperplane intersects a Wulff shape at a "constant angle" as in the isotropic case, namely (1.1) holds. It follows from

$$\langle \nu_F, -E_{n+1} \rangle = \langle \frac{x - x_0}{r}, -E_{n+1} \rangle = \langle \frac{x_0}{r}, -E_{n+1} \rangle,$$

which is a constant.

Remark 2.1. The boundary condition $\langle \Phi(\nu), -E_{n+1} \rangle = \omega_0$ implies $\omega_0 \in (-F(E_{n+1}), F(-E_{n+1}))$. Indeed, by the Cauchy-Schwarz inequality (2.1),

$$\begin{aligned} -F(E_{n+1}) = -F(E_{n+1})F^o(\Phi(\nu)) \leq \langle \Phi(\nu), \\ - E_{n+1} \rangle \leq F^o(\Phi(\nu))F(-E_{n+1}) = F(-E_{n+1}). \end{aligned} \tag{2.4}$$

Since Σ is embedded, Σ intersects $\partial \mathbb{R}^{n+1}_+$ transversely. It follows that equality in (2.4) cannot hold. Therefore, $\omega_0 \in (-F(E_{n+1}), F(-E_{n+1}))$ is a necessary condition for anisotropic ω_0-capillary hypersurfaces.

From our work in this paper, one can in fact introduce a notion of "anisotropic contact angle" as follows, which is a natural generalization of the contact angle in the isotropic case. We define $\theta : \partial \Sigma \to (0, \pi)$ by

$$-\cos\theta = \begin{cases} F(E_{n+1})^{-1} \langle \nu_F, -E_{n+1} \rangle, & \text{if } \langle \nu_F, -E_{n+1} \rangle < 0, \\ 0, & \text{if } \langle \nu_F, -E_{n+1} \rangle = 0, \\ F(-E_{n+1})^{-1} \langle \nu_F, -E_{n+1} \rangle, & \text{if } \langle \nu_F, -E_{n+1} \rangle > 0. \end{cases}$$

If $\theta = \pi/2$, or equivalently $\langle \nu_F, -E_{n+1} \rangle = 0$, we call that the anisotropic hypersurface intersects $\partial \mathbb{R}^{n+1}_+$ perpendicularly, or it is a free boundary anisotropic hypersurface.

3. Minkowski-type formula and Heintze-Karcher-type inequality

3.1. Minkowski-type formula

To prove the Minkowski-type formula, we need the following structural lemma for compact hypersurfaces in $\mathbb{R}^{n+1}$ with boundary, which is well-known and widely use; see, for example [3,20].

Lemma 3.1. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a compact hypersurface with boundary. Then it holds that*

$$n \int_\Sigma \nu dA = \int_{\partial \Sigma} \{ \langle x, \mu \rangle \, \nu - \langle x, \nu \rangle \, \mu \} ds. \tag{3.1}$$

In the paper we denote μ the unit outward co-normal of $\partial\Sigma$ in Σ. Recall E^F_{n+1} defined in (1.2). It is easy to check that

$$\langle E^F_{n+1}, E_{n+1}\rangle = 1. \tag{3.2}$$

Proof of Theorem 1.3. We first prove (1.6). We begin by introducing the following C^1 vector field along Σ:

$$X_F(x) = F(\nu(x))x - \langle x, \nu(x)\rangle\, \nu_F(x).$$

Observe that X_F is indeed a tangential vector field along Σ, since

$$\langle X_F, \nu\rangle = F(\nu)\,\langle x, \nu\rangle - \langle x, \nu\rangle\,\langle \nu_F, \nu\rangle = 0.$$

Notice also that along Σ we have

$$\begin{aligned}\mathrm{div}_\Sigma(X_F) &= nF(\nu) + \langle d\nu(\nabla F), x\rangle - \left\langle x, d\nu(\nu_F^T)\right\rangle \\ &\quad - \langle x, \nu\rangle\, H^F = nF(\nu) - H^F\,\langle x, \nu\rangle\,,\end{aligned} \tag{3.3}$$

where div_Σ is the divergence on Σ. In the second equality, we have used the self-adjointness of $d\nu$. Here ν_F^T and x^T denote the tangential projection on Σ of ν_F and x respectively. In particular, $\nu_F^T = \nu_F - \langle \nu_F, \nu\rangle\nu = \nabla F(\nu)$. On one hand, integrating (3.3) along Σ and using the divergence theorem, we find

$$\int_\Sigma nF(\nu) - H^F(x)\,\langle x, \nu\rangle\,\mathrm{d}A = \int_{\partial\Sigma} F(\nu)\,\langle x, \mu\rangle - \langle x, \nu\rangle\,\langle \nu_F, \mu\rangle\,\mathrm{d}s. \tag{3.4}$$

On the other hand, by (3.1) we have

$$-n\omega_0 \int_\Sigma \left\langle \nu, E^F_{n+1}\right\rangle \mathrm{d}A = \int_{\partial\Sigma} (-\,\langle x, \mu\rangle \left\langle \nu, E^F_{n+1}\right\rangle \omega_0 + \langle x, \nu\rangle \left\langle \mu, E^F_{n+1}\right\rangle \omega_0)\mathrm{d}s. \tag{3.5}$$

It is easy to see that at any $x \in \partial\Sigma \subset \partial\mathbb{R}^{n+1}_+$

$$E_{n+1} = \langle \nu, E_{n+1}\rangle\,\nu + \langle \mu, E_{n+1}\rangle\,\mu,$$

and hence we have

$$-\omega_0 = \langle E_{n+1}, \nu_F\rangle = \langle \nu, E_{n+1}\rangle\, F(\nu) + \langle \mu, E_{n+1}\rangle\,\langle \nu_F, \mu\rangle\,, \tag{3.6}$$

$$0 = \langle E_{n+1}, x\rangle = \langle \nu, E_{n+1}\rangle\,\langle x, \nu\rangle + \langle \mu, E_{n+1}\rangle\,\langle x, \mu\rangle\,. \tag{3.7}$$

Moreover by (3.2) we have

$$1 = \langle E^F_{n+1}, E_{n+1}\rangle = \langle \nu, E_{n+1}\rangle\langle E^F_{n+1}, \nu\rangle + \langle \mu, E_{n+1}\rangle\langle E^F_{n+1}, \mu\rangle. \tag{3.8}$$

This yields that

$$-\,\langle x, \mu\rangle \left\langle \nu, E^F_{n+1}\right\rangle \omega_0 + \langle x, \nu\rangle \left\langle \mu, E^F_{n+1}\right\rangle \omega_0$$

$$
\begin{aligned}
&= \langle x,\mu\rangle \left\langle \nu, E^F_{n+1}\right\rangle \langle \nu, E_{n+1}\rangle\, F(\nu) - \langle x,\nu\rangle \left\langle \mu, E^F_{n+1}\right\rangle \langle \nu, E_{n+1}\rangle\, F(\nu)\\
&+ \langle x,\mu\rangle \left\langle \nu, E^F_{n+1}\right\rangle \langle \mu, E_{n+1}\rangle \langle \nu_F,\mu\rangle - \langle x,\nu\rangle \left\langle \mu, E^F_{n+1}\right\rangle \langle \mu, E_{n+1}\rangle \langle \nu_F,\mu\rangle\\
&= \langle x,\mu\rangle \left\langle \nu, E^F_{n+1}\right\rangle \langle \nu, E_{n+1}\rangle\, F(\nu) + \langle x,\mu\rangle \langle \mu, E_{n+1}\rangle \left\langle \mu, E^F_{n+1}\right\rangle F(\nu)\\
&- \langle x,\nu\rangle \langle \nu, E_{n+1}\rangle \left\langle \nu, E^F_{n+1}\right\rangle \langle \nu_F,\mu\rangle + \langle x,\nu\rangle \left\langle \mu, E^F_{n+1}\right\rangle \langle \mu, E_{n+1}\rangle \langle \nu_F,\mu\rangle\\
&= F(\nu)\langle x,\mu\rangle - \langle x,\nu\rangle \langle \nu_F,\mu\rangle,
\end{aligned}
$$

where we have used (3.6) in the first equality, (3.7) in the second equality and (3.8) in the last one. In particular, this identity, together with (3.4) and (3.5), implies

$$
\int_\Sigma nF(\nu) - H^F(x)\langle x,\nu\rangle \,\mathrm{d}A = -n\int_\Sigma \left\langle \nu, \omega_0 E^F_{n+1}\right\rangle \mathrm{d}A,
$$

which is (1.6).

Next we prove (1.5) for general r by using (1.6) as in [21]. Consider a family of hypersurfaces Σ_t with boundary for small $t > 0$, defined by

$$
\varphi_t(x) = x + t(\nu_F(x) + \omega_0 E^F_{n+1}) \quad x \in \Sigma.
$$

We claim that Σ_t is also an anisotropic ω_0-capillary hypersurface in $\overline{\mathbb{R}^{n+1}_+}$. On one hand, the ω_0-capillarity condition and (3.2) yield that for any $x \in \partial\Sigma$,

$$
\left\langle \nu_F(x) + \omega_0 E^F_{n+1}, -E_{n+1}\right\rangle = \omega_0 - \omega_0 = 0.
$$

Hence, $\varphi_t(x) \in \partial\mathbb{R}^{n+1}_+$ for $x \in \partial\Sigma$ which means $\partial\Sigma_t \subset \partial\mathbb{R}^{n+1}_+$. On the other hand, denoting by e^F_i an anisotropic principal vector at $x \in \Sigma$ corresponding to κ^F_i for $i = 1, \cdots, n$, we have

$$
(\varphi_t)_*(e^F_i) = (1 + t\kappa^F_i)e^F_i, \quad i = 1, \ldots, n. \tag{3.9}
$$

We see from (3.9) that $\nu^{\Sigma_t}(\varphi_t(x)) = \nu(x)$, and in turn $\nu^{\Sigma_t}_F(\varphi_t(x)) = \nu_F(x)$. Here ν^{Σ_t} and $\nu^{\Sigma_t}_F$ denote the outward normal and anisotropic normal to Σ_t respectively. In view of this, we have

$$
\left\langle \nu^{\Sigma_t}_F(\varphi_t(x)), -E_{n+1}\right\rangle = \langle \nu_F(x), -E_{n+1}\rangle = \omega_0.
$$

Therefore, Σ_t is also an anisotropic ω_0-capillary hypersurface in $\overline{\mathbb{R}^{n+1}_+}$ and hence (1.6) holds for Σ_t for any small t. Exploiting (1.6) for every such Σ_t, we find that

$$
\int_{\Sigma_t=\varphi_t(\Sigma)} \left(F(\nu_t) + \omega_0\left\langle \nu_t, E^F_{n+1}\right\rangle\right) - H^F_1(t)\,|_y\, \langle y,\nu_t\rangle \,\mathrm{d}A_t(y) = 0. \tag{3.10}
$$

By (3.9), the tangential Jacobian of φ_t along Σ at x is just

$$
\mathrm{J}^\Sigma\varphi_t(x) = \prod_{i=1}^n (1 + t\kappa^F_i(x)) = \mathcal{P}_n(t), \tag{3.11}
$$

where $\mathcal{P}_n(t)$ is the polynomial defined in (2.2). Moreover, using (3.9) again, we see that the corresponding anisotropic principal curvatures are given by

$$\kappa_i^F(\varphi_t(x)) = \frac{\kappa_i^F(x)}{1 + t\kappa_i^F(x)}. \tag{3.12}$$

Hence fix $x \in \Sigma$, the anisotropic mean curvature of Σ_t at $\varphi_t(x)$, say $H^F(t)$, is given by

$$H^F(t) = \frac{\mathcal{P}_n'(t)}{\mathcal{P}_n(t)} = \frac{\sum_{i=0}^n i\binom{n}{i} H_i^F t^{i-1}}{\mathcal{P}_n(t)}, \tag{3.13}$$

where $H_i^F = H_i^F(x)$ is the i-th mean curvature of Σ at x.

Using the area formula, (3.11) and (3.13), we find from (3.10) that

$$\int_\Sigma n\left(F(\nu) + \omega_0\left\langle \nu, E_{n+1}^F\right\rangle\right)\mathcal{P}_n(t) - t\left(\langle x, \nu_F(x)\rangle\right.$$
$$\left. + \omega_0\left\langle x, E_{n+1}^F\right\rangle\right)\mathcal{P}_n'(t) - \mathcal{P}_n'(t)\langle x, \nu\rangle \, \mathrm{d}A_x = 0.$$

As the left hand side in this equality is a polynomial in the time variable t, this shows that all its coefficients vanish, and hence a direct computation yields (1.5). □

We remark that the definition of the family of capillary hypersurfaces Σ_t was inspired by [21]. These are the parallel hypersurfaces in the case of capillary boundary.

Remark 3.1. If we replace E_{n+1}^F by E_{n+1} in the proof, every step above is valid and we achieve that

$$\int_\Sigma H_{r-1}^F\left(F(\nu) + \omega_0\langle \nu, E_{n+1}\rangle\right) - H_r^F\langle x, \nu\rangle \, \mathrm{d}A = 0.$$

Alternatively, we can prove directly that

$$\int_\Sigma H_{r-1}^F\left\langle \nu, E_{n+1}^F\right\rangle \mathrm{d}A = \int_\Sigma H_{r-1}^F\langle \nu, E_{n+1}\rangle \, \mathrm{d}A.$$

Since we do not need it in this paper, we omit the proof.

3.2. Heintze-Karcher-type inequality

To prove the Heintze-Karcher-type inequality, we need the following key proposition, which amounts to be an anisotropic angle comparison principle. It is clear that in the isotropic case it is trivial. However in the anisotropic case it is non-trivial.

Proposition 3.1. *Let $x, z \in \mathbb{S}^n$ be two distinct points and $y \in \mathbb{S}^n$ lie in a length-minimizing geodesic joining x and z in $\mathbb{S}^n$, then we have*

$$\langle \Phi(x), z\rangle \leq \langle \Phi(y), z\rangle.$$

Equality holds if and only if $x = y$.

Proof. We denote $d_0 = d_{\mathbb{S}^n}(x, z)$ and $d_1 = d_{\mathbb{S}^n}(x, y)$, where $d_{\mathbb{S}^n}$ denotes the intrinsic distance on $\mathbb{S}^n$. If $y \neq x$, clearly $0 < d_1 \leq d_0$. Let $\gamma : [0, d_0] \to \mathbb{S}^n$ be the arc-length parameterized geodesic with $\gamma(0) = x$, $\gamma(d_0) = z$. Considering the function

$$f = \langle \Phi(\gamma(t)), z \rangle, \quad t \in [0, d_0],$$

we have that

$$\begin{aligned}\langle \Phi(y), z \rangle - \langle \Phi(x), z \rangle &= f(d_1) - f(0)\\ &= \int_0^{d_1} \left\langle \frac{\mathrm{d}}{\mathrm{d}t} \Phi(\gamma(t)), z \right\rangle \mathrm{d}t = \int_0^{d_1} \langle D_{\dot\gamma(t)} \Phi(\gamma(t)), z \rangle \mathrm{d}t,\end{aligned} \tag{3.14}$$

where D is the Euclidean covariant derivative. Since γ is length-minimizing, it is easy to see that

$$\langle \dot\gamma(t), z \rangle \geq 0, \quad \forall t \in (0, d_0).$$

Thus z can be expressed as $z = \sin s \dot\gamma(t) + \cos s \gamma(t)$ with some $s \in (0, \pi)$. It follows that

$$\langle D_{\dot\gamma(t)} \Phi(\gamma(t)), z \rangle = \sin s (\nabla^2 F + F I)(\dot\gamma(t), \dot\gamma(t)),$$

Since $(\nabla^2 F + F I) > 0$, we get $\langle D_{\dot\gamma(t)} \Phi(\gamma(t)), z \rangle > 0$ for any $t \in (0, d_1)$. This fact, together with (3.14), leads to the assertion. □

We note that when $y = z$, Proposition 3.1 is nothing but the Cauchy-Schwarz inequality (2.1), since one readily observes from Proposition 2.1(ii)(iii) that

$$\langle \Phi(x), z \rangle \leq F^o(\Phi(x)) F(z) = \langle \Phi(z), z \rangle .$$

The Cauchy-Schwarz inequality (2.1) also impies the following property:

Proposition 3.2. *For $\omega_0 \in (-F(E_{n+1}), F(-E_{n+1}))$, if holds that*

$$F(z) + \omega_0 \left\langle z, E^F_{n+1} \right\rangle > 0, \quad \text{for any } z \in \mathbb{S}^n.$$

Proof. It is clear that we need not to consider the case $\omega_0 = 0$. If $\omega_0 < 0$, we only need to consider the points z satisfying $\langle z, E^F_{n+1} \rangle > 0$. At any such z, since $\omega_0 > -F(E_{n+1})$, we have

$$\begin{aligned}F(z) + \omega_0 \left\langle z, E^F_{n+1} \right\rangle &> F(z) - F(E_{n+1}) \left\langle z, \frac{\Phi(E_{n+1})}{F(E_{n+1})} \right\rangle\\ &\geq F(z) - F(z) F^o(\Phi(E_{n+1})) = 0,\end{aligned}$$

where we have used the Cauchy-Schwarz inequality for the second inequality.

For the case $\omega_0 > 0$, we just need to consider the points z such that $\langle z, E^F_{n+1} \rangle < 0$. Since $\omega_0 < F(-E_{n+1})$, using the Cauchy-Schwarz inequality again, we find

$$\begin{aligned}F(z) + \omega_0 \left\langle z, E^F_{n+1} \right\rangle &> F(z) + F(-E_{n+1}) \left\langle z, -\frac{\Phi(-E_{n+1})}{F(-E_{n+1})} \right\rangle\\ &\geq F(z) - F(z) F^o(\Phi(-E_{n+1})) = 0.\end{aligned}$$

The proposition is thus proven. □

Now we can start to prove Theorem 1.2.

Proof of Theorem 1.2. Let $\Sigma \subset \overline{\mathbb{R}^{n+1}_+}$ be an anisotropic capillary hypersurface satisfying (1.3). For any $x \in \Sigma$, let $\kappa_i^F(x)$ be the anisotropic principal curvature and $e_i^F(x)$ be the corresponding anisotropic principal vector of Σ at x such that $|e_1^F \wedge e_2^F \wedge \cdots \wedge e_n^F| = 1$. Since Σ is strictly anisotropic mean convex,

$$\max_i \kappa_i^F(x) \geq \frac{1}{n} H^F(x) > 0, \text{ for } x \in \Sigma.$$

We define

$$Z = \left\{ (x,t) \in \Sigma \times \mathbb{R} : 0 < t \leq \frac{1}{\max \kappa_i^F(x)} \right\},$$

and

$$\zeta_F : Z \to \mathbb{R}^{n+1}, \tag{3.15}$$

$$\zeta_F(x,t) = x - t\left(\nu_F(x) + \omega_0 E_{n+1}^F\right). \tag{3.16}$$

Claim: $\Omega \subset \zeta_F(Z)$.

Recall $\mathcal{W}_r(x_0)$ is the Wulff shape centered at x_0 with radius r. For any $y \in \Omega$, we consider a family of Wulff shapes $\{\mathcal{W}_r(y + r\omega_0 E_{n+1}^F)\}_{r \geq 0}$. Since $y \in \Omega$ is an interior point, we definitely have $\mathcal{W}_r(y + r\omega_0 E_{n+1}^F) \subset \Omega$ for r small enough. On the other hand, by the assumption $-F(E_{n+1}) < \omega_0 < F(-E_{n+1})$, the definition (1.2) of E_{n+1}^F and Proposition 2.1(i)(iii), it is easy to see that

$$F^o(-\omega_0 E_{n+1}^F) < 1.$$

It follows that

$$F^o\big(y - (y + r\omega_0 E_{n+1}^F)\big) = rF^o(-\omega_0 E_{n+1}^F) < r,$$

which implies that for any small $r > 0$, y is always in the domain bounded by the Wulff shape $\mathcal{W}_r(y + r\omega_0 E_{n+1}^F)$. Hence $\mathcal{W}_r(y + r\omega_0 E_{n+1}^F)$ must touch Σ as we increase the radius r. Consequently, for any $y \in \Omega$, there exists $x \in \Sigma$ and $r_y > 0$, such that $\mathcal{W}_{r_y}(y + r_y\omega_0 E_{n+1}^F)$ touches Σ for the first time, at some point $x \in \Sigma$. In terms of the touching point, only the following two cases are possible:

Case 1. $x \in \mathring{\Sigma}$.

In this case, since $x \in \mathring{\Sigma}$, the Wulff shape $\mathcal{W}_{r_y}(y + r_y\omega_0 E_{n+1}^F)$ is tangent to Σ at x from the interior. Hence

$$\nu(x) = \nu^{\mathcal{W}}(x), \tag{3.17}$$

where $\nu^{\mathcal{W}}$ denotes the outward unit normal of $\mathcal{W}_{r_y}(y + r_y\omega_0 E_{n+1}^F)$. Moreover, since the touching of $\mathcal{W}_{r_y}(y + r_y\omega_0 E_{n+1}^F)$ with Σ is from interior, we see that

$$d\nu \leq d\nu^{\mathcal{W}}, \tag{3.18}$$

in the sense that the coefficient matrix of the difference of two classical Weingarten operators $d\nu - d\nu^{\mathcal{W}}$ is semi-negative definite. It follows from (3.17) and (3.18) that

$$A_F(\nu) \circ d\nu \leq A_F(\nu^{\mathcal{W}}) \circ d\nu^{\mathcal{W}}. \tag{3.19}$$

Since the anisotropic principal curvatures of $\mathcal{W}_{r_y}(y - r_y\omega_0 E^F_{n+1})$ are equal to $\frac{1}{r_y}$, we see from (3.19) that

$$\max_{1\leq i\leq n} \kappa^F_i(x) \leq \frac{1}{r_y}.$$

Invoking the definition of Z and ζ_F, we find that $y \in \zeta_F(Z)$ in this case.

Case 2. $x \in \partial\Sigma$.

We will rule out this case by the capillarity assumption (1.3). Let $\nu^{\mathcal{W}}_F(x)$ be the outward anisotropic normal to $\mathcal{W}_{r_y}(y + r_y\omega_0 E^F_{n+1})$. It is easy to see that

$$\nu^{\mathcal{W}}_F(x) = \Phi(\nu^{\mathcal{W}}(x)) = \frac{x - (y + r_y\omega_0 E^F_{n+1})}{r_y}.$$

Recall that y lies in the interior of Ω. Thus $\langle y, E_{n+1}\rangle > 0$. On one hand, in view of (1.3) and (3.2) we have

$$\begin{aligned}\left\langle \nu^{\mathcal{W}}_F(x), -E_{n+1}\right\rangle &= 1/r_y \langle y, E_{n+1}\rangle + \omega_0\langle E^F_{n+1}, E_{n+1}\rangle \\ &> \omega_0 \geq \omega(x) = \langle \nu_F(x), -E_{n+1}\rangle .\end{aligned} \tag{3.20}$$

On the other hand, since the Wulff shape $\mathcal{W}_{r_y}(y + r_y\omega_0 E^F_{n+1})$ touches Σ from the interior, we have

$$\langle \nu(x), -E_{n+1}\rangle \geq \left\langle \nu^{\mathcal{W}}(x), -E_{n+1}\right\rangle.$$

Since ν, $\nu^{\mathcal{W}}$ and $-E_{n+1}$ lie on the two-plane orthogonal to $T_x(\partial\Sigma)$, we see that ν lies actually in the geodesic joining $\nu^{\mathcal{W}}$ and $-E_{n+1}$ in $\mathbb{S}^n$. It then follows from the angle comparison principle Proposition 3.1 that

$$\langle \Phi(\nu(x)), -E_{n+1}\rangle \geq \left\langle \Phi(\nu^{\mathcal{W}}(x)), -E_{n+1}\right\rangle.$$

This is a contradiction to (3.20). The **Claim** is thus proved.

By a simple computation, we find that

$$\begin{aligned}\partial_t\zeta_F(x,t) &= -\left(\nu_F(x) + \omega_0 E^F_{n+1}\right), \\ D_{e^F_i}\zeta_F(x,t) &= \left(1 - t\kappa^F_i(x)\right) e^F_i(x).\end{aligned}$$

Thanks to Proposition 3.2, a classical computation yields that the tangential Jacobian of ζ_F along Z at (x,t) is just

$$\mathrm{J}^Z\zeta_F(x,t) = (F(\nu) + \omega_0\langle \nu, E^F_{n+1}\rangle)\prod_{i=1}^{n}(1 - t\kappa^F_i).$$

By virtue of the fact that $\Omega \subset \zeta_F(Z)$, the area formula yields

$$|\Omega| \leq |\zeta_F(Z)| \leq \int_{\zeta_F(Z)} \mathcal{H}^0(\zeta_F^{-1}(y))dy = \int_Z \mathrm{J}^Z \zeta_F d\mathcal{H}^{n+1}$$
$$= \int_\Sigma \mathrm{d}A \int_0^{\frac{1}{\max\left\{\kappa_i^F(x)\right\}}} \left(F(\nu) + \omega_0 \left\langle \nu, E_{n+1}^F \right\rangle\right) \prod_{i=1}^n (1 - t\kappa_i^F(x))\mathrm{d}t.$$

By the AM-GM inequality, and the fact that $\max\left\{\kappa_i^F(x)\right\}_{i=1}^n \geq \frac{1}{n} H^F(x)$, we obtain

$$|\Omega| \leq \int_\Sigma \mathrm{d}A \int_0^{\frac{1}{\max\left\{\kappa_i^F(x)\right\}}} \left(F(\nu) + \omega_0 \langle \nu, E_{n+1}^F \rangle\right) \left(\frac{1}{n}\sum_{i=1}^n \left(1 - t\kappa_i^F(x)\right)\right)^n \mathrm{d}t$$
$$\leq \int_\Sigma \left(F(\nu) + \omega_0 \langle \nu, E_{n+1}^F \rangle\right) \mathrm{d}A \int_0^{\frac{n}{H^F(x)}} \left(1 - t\frac{H^F(x)}{n}\right)^n \mathrm{d}t$$
$$= \frac{n}{n+1} \int_\Sigma \frac{F(\nu) + \omega_0 \langle \nu, E_{n+1}^F \rangle}{H^F} \mathrm{d}A,$$

which gives (1.4).

If equality in (1.4) holds, then from the above argument, we see $\kappa_1^F(x) = \ldots = \kappa_n^F(x)$ for all $x \in \Sigma$. It follows from [17, Lemma 2.3] that Σ must be a part of a Wulff shape $\mathcal{W}_{r_0}(x_0)$ for some r_0 and some point x_0. Hence H^F is a constant $\frac{n}{r_0}$ and since $\nu_F(x) = \frac{x - x_0}{r_0}$, we get, for $x \in \partial\Sigma$,

$$\omega(x) = \langle \nu_F(x), -E_{n+1} \rangle = \frac{1}{r_0} \langle x_0, E_{n+1} \rangle := \tilde{\omega}_0,$$

which is a constant.

From the equality in Heintze-Karcher inequality and the Minkowski-type formula (1.5), taking into account that H_F is a constant, we deduce that

$$\omega_0 \int_\Sigma \left\langle \nu, E_{n+1}^F \right\rangle \mathrm{d}A = \tilde{\omega}_0 \int_\Sigma \left\langle \nu, E_{n+1}^F \right\rangle \mathrm{d}A.$$

By the divergence theorem and (3.2),

$$\int_\Sigma \left\langle \nu, E_{n+1}^F \right\rangle \mathrm{d}A = |\partial\Omega \cap \partial\mathbb{R}_+^{n+1}| \neq 0.$$

It follows that $\tilde{\omega}_0 = \omega_0$ which means Σ is a ω_0-capillary Wulff shape.

Conversely, for any ω_0-capillary Wulff shape, we can see easily from the Minkowski-type formula (1.5) and the fact of constant anisotropic mean curvature that equality holds in (1.4). This completes the proof. □

Remark 3.2. We may use in the proof another foliation of Wulff shapes $\{\mathcal{W}_r(y + r\omega_0 E_{n+1})\}_{r \geq 0}$. To ensure that $\mathcal{W}_r(y + r\omega_0 E_{n+1})$ intersects with Σ for large r, we need to assume

$$\omega_0 \in \left(-\frac{1}{F^o(E_{n+1})}, \frac{1}{F^o(-E_{n+1})}\right). \tag{3.21}$$

We can follow the proof to achieve that

$$\int_{\Sigma} \frac{F(\nu) + \omega_0 \langle \nu, E_{n+1} \rangle}{H^F} \mathrm{d}A \geq \frac{n+1}{n} |\Omega|. \tag{3.22}$$

under the assumption (3.21). On the other hand, by virtue of the Cauchy-Schwarz inequality, we see that (3.21) is in general more restrictive than the natural assumption $\omega_0 \in (-F(E_{n+1}), F(-E_{n+1}))$. This is the reason why we introduce E_{n+1}^F.

Proof of Corollary 1.1. It is clear that any closed hypersurface can be seen as a capillary surface in a half space with a empty boundary. For any $e \in \mathbb{S}^n$ we can see e as E_{n+1} and apply Theorem 1.2.

First consider $\omega_0 \in (-F(E_{n+1}), 0)$. Together with the definition of E_{n+1}^F (1.4) gives us

$$\int_{\Sigma} \frac{F(\nu)}{H^F} \mathrm{d}A \geq \frac{n+1}{n} |\Omega| - \omega_0 \int_{\Sigma} \frac{\langle \nu, E_{n+1}^F \rangle}{H^F} \mathrm{d}A = \frac{n+1}{n} |\Omega| - \frac{\omega_0}{F(E_{n+1})} \int_{\Sigma} \frac{\langle \nu, \Phi(E_{n+1}) \rangle}{H^F} \mathrm{d}A.$$

It follows that

$$\int_{\Sigma} \frac{F(\nu)}{H^F} \mathrm{d}A \geq \frac{n+1}{n} |\Omega| + \max \left\{ 0, \int_{\Sigma} \frac{\langle \nu, \Phi(E_{n+1}) \rangle}{H^F} \mathrm{d}A \right\}.$$

Then we consider $\omega_0 \in (0, F(-E_{n+1}))$. Similarly, in this case (1.4) gives us

$$\int_{\Sigma} \frac{F(\nu)}{H^F} \mathrm{d}A \geq \frac{n+1}{n} |\Omega| + \max \left\{ 0, \int_{\Sigma} \frac{\langle \nu, \Phi(-E_{n+1}) \rangle}{H^F} \mathrm{d}A. \right\}.$$

This completes the proof of (1.7). □

Remark 3.3. When F is even, i.e., $F(x) = F(-x)$, we have

$$\int_{\Sigma} \frac{F(\nu)}{H^F} \mathrm{d}A \geq \frac{n+1}{n} |\Omega| + \max_{e \in \mathbb{S}^n} \int_{\Sigma} \frac{\langle \nu, \Phi(e) \rangle}{H^F} \mathrm{d}A \left(\geq \frac{n+1}{n} |\Omega| \right),$$

since in this case $\Phi(-E_{n+1}) = -\Phi(E_{n+1})$. Hence

$$\text{either } \int_{\Sigma} \frac{\langle \nu, \Phi(E_{n+1}) \rangle}{H^F} \mathrm{d}A \geq 0 \text{ or } \int_{\Sigma} \frac{\langle \nu, \Phi(-E_{n+1}) \rangle}{H^F} \mathrm{d}A \geq 0.$$

4. Alexandrov-type Theorem

We first prove a result on the existence of an elliptic point for an anisotropic capillary hypersurface.

Proposition 4.1. *Let $\omega_0 \in (-F(E_{n+1}), F(-E_{n+1}))$ and let $\Sigma \subset \overline{\mathbb{R}^{n+1}_+}$ be a C^2 compact embedded anisotropic ω_0-capillary hypersurface, then Σ has at least one elliptic point, i.e. a point where all the anisotropic principal curvatures are positive.*

Proof. We fix a point $y \in \text{int}(\partial\Omega \cap \partial\mathbb{R}^{n+1}_+)$. Consider the family of Wulff shapes $\mathcal{W}_r(y + r\omega_0 E^F_{n+1})$. Observe that for any $x \in \partial\Sigma$ and any $r > 0$, there holds

$$\left\langle \nu^{\mathcal{W}}_F(x), E_{n+1} \right\rangle = \left\langle \frac{x-y}{r} + \omega_0 E^F_{n+1}, E_{n+1} \right\rangle = \omega_0 = \langle \nu_F(x), E_{n+1} \rangle. \quad (4.1)$$

Since Σ is compact, for r large enough, Σ lies inside the domain bounded by the Wulff shape $\mathcal{W}_r(y+r\omega_0 E^F_{n+1})$. Hence we can find the smallest r, say $r_0 > 0$, such that $\mathcal{W}_{r_0}(y + r_0\omega_0 E^F_{n+1})$ touches Σ at a first time at some $x_0 \in \Sigma$ from exterior.

If $x_0 \in \mathring{\Sigma}$, then Σ and $\mathcal{W}_{r_0}(y + r_0\omega_0 E^F_{n+1})$ are tangent at x. If $x_0 \in \partial\Sigma$, from (4.1), we conclude again that Σ and $\mathcal{W}_{r_0}((y + r_0\omega_0 E^F_{n+1}))$ are tangent at x. In both cases, by a similar argument as in the proof of Theorem 1.2, we have that the anisotropic principal curvatures of Σ at x_0 are larger than or equal to $\frac{1}{r_0}$. □

Proof of Theorem 1.1 *and Theorem* 1.4 We begin by recalling that $\omega_0 \in (-F(E_{n+1}), F(-E_{n+1}))$ ensures the non-negative of $F(\nu) + \omega_0 \left\langle \nu, E^F_{n+1} \right\rangle$ pointwisely along Σ, thanks to Proposition 3.2.

On one hand, by virtue of Proposition 4.1 and Gärding's argument [14] (see also [17, Lemma 2.1]), we know that H^F_j are positive, for $j \le r$ and for any $x \in \Sigma$. Applying Theorem 1.2 and using the Maclaurin inequality $H^F_1 \ge (H^F_r)^{1/r}$ and the constancy of H^F_r, we have

$$\begin{aligned}(n+1)(H^F_r)^{1/r}|\Omega| &\le (H^F_r)^{1/r} \int_\Sigma \frac{F(\nu) + \omega_0 \left\langle \nu, E^F_{n+1} \right\rangle}{H^F_1} \mathrm{d}A \\ &\le \int_\Sigma \left(F(\nu) + \omega_0 \left\langle \nu, E^F_{n+1} \right\rangle \right) \mathrm{d}A. \end{aligned} \quad (4.2)$$

On the other hand, using the Minkowski-type formula (1.5) and the Maclaurin inequality $H^F_{r-1} \ge (H^F_r)^{\frac{r-1}{r}}$, we have

$$\begin{aligned} 0 &= \int_\Sigma H^F_{r-1} \left(F(\nu) + \omega_0 \left\langle \nu, E^F_{n+1} \right\rangle \right) - H^F_r \langle x, \nu \rangle \, \mathrm{d}A \\ &\ge \int_\Sigma (H^F_r)^{\frac{r-1}{r}} \left(F(\nu) + \omega_0 \left\langle \nu, E^F_{n+1} \right\rangle \right) - H^F_r \langle x, \nu \rangle \, \mathrm{d}A \\ &= (H^F_r)^{\frac{r-1}{r}} \int_\Sigma \left(F(\nu) + \omega_0 \left\langle \nu, E^F_{n+1} \right\rangle \right) - (H^F_r)^{\frac{1}{r}} \langle x, \nu \rangle \, \mathrm{d}A \\ &= (H^F_r)^{\frac{r-1}{r}} \left\{ \int_\Sigma \left(F(\nu) + \omega_0 \left\langle \nu, E^F_{n+1} \right\rangle \right) \mathrm{d}A - (n+1)(H^F_r)^{\frac{1}{r}} |\Omega| \right\}, \end{aligned}$$

where in the last equality we have used that

$$(n+1)|\Omega| = \int_\Omega \text{div} x \, dx = \int_\Sigma \langle x, \nu \rangle \, \mathrm{d}A.$$

Thus equality in (4.2) holds, and hence Σ is an anisotropic ω_0-capillary Wulff shape. This completes the proof. □

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Declarations

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