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The Rayleigh–Bénard Problem for Compressible Fluid Flows

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Abstract

We consider the physically relevant fully compressible setting of the Rayleigh–Bénard problem of a fluid confined between two parallel plates, heated from the bottom, and subjected to gravitational force. Under suitable restrictions imposed on the constitutive relations we show that this open system is dissipative in the sense of Levinson, meaning there exists a bounded absorbing set for any global-in-time weak solution. In addition, global-in-time trajectories are asymptotically compact in suitable topologies and the system possesses a global compact trajectory attractor $\mathcal{A}$. The standard technique of Krylov and Bogolyubov then yields the existence of an invariant measure—a stationary statistical solution sitting on $\mathcal{A}$. In addition, the Birkhoff–Khinchin ergodic theorem provides convergence of ergodic averages of solutions belonging to $\mathcal{A}$ a.s. with respect to the invariant measure.

1. Introduction

The Rayleigh–Bénard problem concerns the motion of a fluid confined between two parallel planes, where the temperature of the bottom plane is maintained at the level $\Theta_{\rm B}$, while the top plane has the ambient temperature $\Theta_{\rm U}$, typically $\Theta_{\rm U} \leqq \Theta_{\rm B}$. The only volume force is due to gravitation acting in the downward vertical direction. The fluid mass density $\varrho = \varrho(t, x)$, the temperature $\vartheta(t, x)$, and the

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velocity $\mathbf{u} = \mathbf{u}(t, x)$ obey the standard system of equations of continuum fluid mechanics

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \varrho \nabla_x G, \tag{1.2}$$

$$\partial_t(\varrho e(\varrho, \vartheta)) + \mathrm{div}_x(\varrho e(\varrho, \vartheta)\mathbf{u}) + \nabla_x \mathbf{q} = \mathbb{S} : \mathbb{D}_x \mathbf{u} - p(\varrho, \vartheta)\mathrm{div}_x \mathbf{u}, \tag{1.3}$$

where we have denoted

the pressure .... $p = p(\varrho, \vartheta)$,
the (specific) internal energy .... $e = e(\varrho, \vartheta)$,
the viscous stress tensor .... $\mathbb{S}$,
the gravitational potential .... $G = -g x_3$,
the heat flux .... $\mathbf{q}$
the symmetric part of the velocity gradient .... $\mathbb{D}_x \mathbf{u} = \frac{1}{2}\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}\right)$.

For the sake of simplicity, we consider the periodic boundary conditions with respect to the horizontal variables. Accordingly, the fluid domain Ω can be identified with

$$\Omega = \mathbb{T}^2 \times (0, 1), \tag{1.4}$$

where $\mathbb{T}^2$ is the two-dimensional flat torus. If the boundary planes are impermeable and the viscous fluid sticks to them, the relevant boundary conditions read as

$$\mathbf{u}|_{x_3=0} = \mathbf{u}|_{x_3=1} = 0, \tag{1.5}$$

$$\vartheta|_{x_3=0} = \Theta_{\mathrm{B}}, \ \vartheta|_{x_3=1} = \Theta_{\mathrm{U}}. \tag{1.6}$$

We suppose that the fluid is Newtonian, with the viscous stress

$$\mathbb{S}(\vartheta, \mathbb{D}_x \mathbf{u}) = \mu(\vartheta)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta(\vartheta)\mathrm{div}_x \mathbf{u}\mathbb{I}. \tag{1.7}$$

The heat flux is given by Fourier's law

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta)\nabla_x \vartheta, \tag{1.8}$$

where κ is the conductivity. The field equations (1.1)–(1.3), endowed with the constitutive relations (1.7), (1.8), will be referred to as *Navier–Stokes–Fourier system*.

The behaviour of the fluid under the boundary conditions (1.5), (1.6) with a sufficiently large difference between the top and bottom temperatures is a prominent example of turbulence, see for example Davidson [13]. There is a large amount of mathematical literature devoted to the asymptotic behaviour of solutions to the Rayleigh–Bénard problem in the simplified incompressible framework, where the system (1.1)–(1.3) is replaced by the Oberbeck–Boussinesq approximation, see Constantin et al. [11], Foias et al. [22], Cao et al. [8] and the references therein. Recently, the problem motivated a series of studies by Otto et al. [10, 30, 34] concerning the associated scaling laws.

Much less seems to be known in the original and physically relevant framework of compressible and heat conducting fluids. Here, the rigorous analysis is hampered by the following notoriously known difficulties:

- Navier–Stokes–Fourier system endowed with the boundary conditions (1.5), (1.6) is an open system in the regime far from equilibrium. Unfortunately, the existence of *global-in-time* smooth solutions is known only in the case of closed systems approaching the equilibrium solution in the long run, see Matsumura and Nishida [28, 29], or Valli and Zajączkowski [36].
- The available theory of weak solutions developed in [16] (see also the alternative approach by Bresch and Desjardins [6] and Bresch and Jabin [7]) applies to conservative or periodic boundary conditions pertinent to the closed systems. Note that the dynamics of the Navier–Stokes–Fourier system with *conservative* boundary conditions is nowadays well understood, see [20]. Indeed, in accordance with the celebrated statement of Clausius:

"Die Energie der Welt ist konstant. Die Entropie der Welt strebt einem Maximum zu"

Rudolf Clausius, Poggendorff's Annals of Physics 1865 (125), 400;

any global-in-time weak solution of the Navier–Stokes–Fourier system with conservative boundary conditions and driven by a conservative volume force tends to an equilibrium, see for example [18, 20] or Novotný and Pokorný [31], Novotný and Straškraba [32, 33]. Here, conservative means that the system is thermally insulated, the boundary conditions (1.6) being replaced by

$$\mathbf{q} \cdot \mathbf{n} = q_3 = 0 \text{ on } \partial\Omega.$$

- The weak solutions are not (known to be) uniquely determined by the initial/boundary data.

Recently, the theory of weak solutions has been extended to non-zero in/out flux boundary conditions in [17], and, finally, to general Dirichlet boundary conditions in [9]. In particular, the theory of weak solutions proposed in [9] yields a suitable platform to attack the Rayleigh–Bénard problem (1.5), (1.6). As pointed out above, the weak solutions are not known to be uniquely determined by the initial/boundary data. Accordingly, we follow the approach advocated by Sell [35] and Málek and Nečas [27] replacing the standard phase space by the space of trajectories.

The principal objective of the paper is to establish the following two basic results:

- *Levinson dissipativity or bounded absorbing set.* Any global-in-time weak solution to the Navier–Stokes–Fourier system endowed with the boundary conditions (1.5), (1.6) enters eventually a bounded absorbing set. In comparison with [15], we relax the hypothesis of the hard sphere pressure and consider the physically relevant equation of state of a general monoatomic gas with the effect of radiation proposed in [16].
- *Asymptotic compactness.* Similarly to [21, Chapter 4, Theorem 4.2], we show that any bounded family of global solutions is precompact in a suitable topology of the trajectory space, whereas any of its accumulation points represents a weak solution of the same problem.

Using the above results, we establish the existence of a compact trajectory attractor, an invariant measure and the existence of stationary statistical solutions generated by bounded trajectories. Finally, we also discuss the existence of the ergodic averages in the spirit of [14].

As pointed out, the key point of the analysis is the Levinson dissipativity or the existence of a universal bounded absorbing set for the "monoatomic" equation of state introduced in [16, Chapters 1,2]. This is rather surprising as this constitutive equation can be seen as a temperature dependent counterpart of the isentropic pressure law $p(\varrho) \approx \varrho^\gamma$, with $\gamma = \frac{5}{3}$. Note that for the isentropic model, the existence of a bounded absorbing set is known only if $\gamma > \frac{5}{3}$, see [19], whereas the limit case $\gamma = \frac{5}{3}$ requires smallness of the total mass of the fluid, see Wang and Wang [37]. Moreover, uniform boundedness of global trajectories for the Navier–Stokes–Fourier system is a delicate issue. As is known, see [20,21], the energy of *all* global-in-time solutions tends to infinity with growing time as soon as the system is energetically closed and driven by a non-conservative volume force.

Similarly to the incompressible Navier–Stokes system with conservative boundary conditions studied by Málek and Nečas [27] and Sell [35], the large time asymptotic behaviour of solutions to the Rayleigh–Bénard problem is captured by the set of *entire* trajectories $\mathcal{A}$ defined for all $t \in R$ and with uniformly bounded total energy and mass. We show that the set $\mathcal{A}$ is **(i)** non-empty, **(ii)** time shift invariant, and **(iii)** compact if endowed by a suitable metrics. The standard Krylov–Bogolyubov technique then yields the existence of an invariant measure supported in $\mathcal{A}$—a stationary statistical solution of the Rayleigh–Bénard problem. Moreover, the standard Birkhoff–Khinchin ergodic theorems yields the convergence of the ergodic averages a.s. with respect to the invariant measure. Uniqueness of the invariant measure for solutions with the same total mass remains an outstanding open problem.

The results obtained could be also used to establish the existence of the *standard* attractor in the spirit of the work of Sell [35] or Málek and Nečas [27] for the incompressible fluid flows.

The paper is organized as follows. In Section 2, we recall the principal constitutive hypotheses and introduce the concept of weak solution. The main results—the existence of a bounded absorbing set and asymptotic compactness of global-in-time solutions—are stated in Section 3. In Section 4, we show the existence of a bounded absorbing set in terms of the total energy. The implications of the main results on the long-time behavior of the system are discussed in Section 5.

2. Principal hypotheses, weak solutions

Following [9] we introduce the concept of weak solution to the Navier–Stokes–Fourier system based on the combination of the balance equations for the entropy and the *ballistic energy*

$$E_{\tilde{\vartheta}}(\varrho, \vartheta, \mathbf{u}) = E(\varrho, \vartheta, \mathbf{u}) - \tilde{\vartheta}\varrho s(\varrho, \vartheta), \ E(\varrho, \vartheta, \mathbf{u}) = \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho, \vartheta),$$

where s is the entropy related to the other thermodynamic functions through Gibbs' equation

$$\vartheta Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + pD\left(\frac{1}{\varrho}\right), \tag{2.1}$$

and $\tilde{\vartheta}$ is an arbitrary continuously differentiable function of (t, x) satisfying

$$\tilde{\vartheta} > 0, \ \tilde{\vartheta}|_{x_3=0} = \Theta_{\rm B}, \ \tilde{\vartheta}|_{x_3=1} = \Theta_{\rm U}. \tag{2.2}$$

For the sake of simplicity, we suppose that $\Theta_{\rm B}$, $\Theta_{\rm U}$ are positive *constants*. A generalization of the results of the present paper to the space or even time dependent boundary temperatures is possible with obvious modifications in the proofs.

2.1. Weak solution

As we are interested in the long time behaviour of solutions, the specific value of the initial data is irrelevant. We therefore consider solutions of the Navier–Stokes–Fourier system defined on the open time interval (T, ∞), $T \in R$.

Definition 2.1. We say that $(\varrho, \vartheta, \mathbf{u})$ is a *weak solution* of the Navier–Stokes–Fourier system (1.1)–(1.3), (1.7), (1.8), with the boundary conditions (1.5), (1.6) defined on the time interval (T, ∞) if the following holds:

- The solution belongs to the following **regularity class**:

$$\begin{aligned} \varrho &\in L^\infty_{\rm loc}(T,\infty; L^\gamma(\Omega)) \text{ for some } \gamma > 1, \\ \mathbf{u} &\in L^2_{\rm loc}(T,\infty; W^{1,2}_0(\Omega; R^3)), \\ \vartheta^{\beta/2},\ \log(\vartheta) &\in L^2_{\rm loc}(T,\infty; W^{1,2}(\Omega) \text{ for some } \beta \geqq 2, \\ (\vartheta - \vartheta_B) &\in L^2_{\rm loc}(T,\infty; W^{1,2}_0(\Omega)). \end{aligned} \tag{2.3}$$

- The **equation of continuity** (1.1) along with its renormalization are satisfied in the sense of distributions, specifically,

$$\int_T^\infty \int_\Omega \left[\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi\right] \ {\rm d}x\,{\rm d}t = 0, \tag{2.4}$$

$$\int_T^\infty \int_\Omega \left[b(\varrho)\partial_t \varphi + b(\varrho)\mathbf{u} \cdot \nabla_x \varphi + \Big(b(\varrho) - b'(\varrho)\varrho\Big) {\rm div}_x \mathbf{u} \varphi\right] \ {\rm d}x\,{\rm d}t = 0 \tag{2.5}$$

  for any $\varphi \in C^1_c((T,\infty) \times \overline{\Omega})$, and any $b \in C^1(R)$, $b' \in C_c(R)$.
- The **momentum equation** (1.2) is satisfied in the sense of distributions,

$$\begin{aligned} &\int_T^\infty \int_\Omega \left[\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi} + p {\rm div}_x \boldsymbol{\varphi}\right] \ {\rm d}x\,{\rm d}t \\ &\quad = \int_T^\infty \int_\Omega \left[\mathbb{S} : \nabla_x \boldsymbol{\varphi} - \varrho \nabla_x G \cdot \boldsymbol{\varphi}\right] \ {\rm d}x\,{\rm d}t, \end{aligned} \tag{2.6}$$

  for any $\boldsymbol{\varphi} \in C^1_c((T,\infty) \times \Omega; R^3)$.
- The internal energy equation (1.3) is replaced by the **entropy inequality**

$$\begin{aligned} &- \int_T^\infty \int_\Omega \left[\varrho s \partial_t \varphi + \varrho s \mathbf{u} \cdot \nabla_x \varphi + \frac{\mathbf{q}}{\vartheta} \cdot \nabla_x \varphi\right] \ {\rm d}x\,{\rm d}t \\ &\quad \geqq \int_T^\infty \int_\Omega \frac{\varphi}{\vartheta}\left[\mathbb{S} : \mathbb{D}_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right] \ {\rm d}x\,{\rm d}t \end{aligned} \tag{2.7}$$

  for any $\varphi \in C^1_c((T,\infty) \times \Omega)$, $\varphi \geqq 0$; and the **ballistic energy balance**,

$$\begin{aligned} &- \int_T^\infty \partial_t \psi \int_\Omega \left[\frac{1}{2}\varrho |\mathbf{u}|^2 + \varrho e - \tilde{\vartheta} \varrho s\right] \ {\rm d}x\,{\rm d}t \\ &\quad + \int_T^\infty \psi \int_\Omega \frac{\tilde{\vartheta}}{\vartheta}\left[\mathbb{S} : \mathbb{D}_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right] \ {\rm d}x\,{\rm d}t \\ &\quad \leqq \int_T^\infty \psi \int_\Omega \left[\varrho \mathbf{u} \cdot \nabla_x G - \varrho s \mathbf{u} \cdot \nabla_x \tilde{\vartheta} - \frac{\mathbf{q}}{\vartheta} \cdot \nabla_x \tilde{\vartheta}\right] \ {\rm d}x \end{aligned} \tag{2.8}$$

  for any $\psi \in C^1_c(T;\infty)$, $\psi \geqq 0$, and any $\tilde{\vartheta} \in C^1([T;\infty) \times \overline{\Omega})$,

$$\tilde{\vartheta} > 0,\ \tilde{\vartheta}|_{x_3=0} = \Theta_{\rm B},\ \tilde{\vartheta}|_{x_3=1} = \Theta_{\rm U}.$$

The existence of global-in-time weak solutions under the constitutive restrictions specified in the forthcoming section was proved in [9, Theorem 4.2]. In addition, the weak solutions comply with the weak–strong uniqueness principle and coincide with strong solutions as soon as they are smooth.

2.2. Constitutive relations

Following [16, Chapters 1,2] we consider the equation of state

$$p(\varrho, \vartheta) = p_{\rm m}(\varrho, \vartheta) + p_{\rm rad}(\vartheta),$$

where $p_{\rm m}$ is the pressure of a general *monoatomic* gas,

$$p_{\rm m}(\varrho, \vartheta) = \frac{2}{3}\varrho e_{\rm m}(\varrho, \vartheta), \tag{2.9}$$

enhanced by the radiation pressure

$$p_{\rm rad}(\vartheta) = \frac{a}{3}\vartheta^4,\ a > 0.$$

Accordingly, the internal energy reads

$$e(\varrho, \vartheta) = e_{\rm m}(\varrho, \vartheta) + e_{\rm rad}(\varrho, \vartheta),\ e_{\rm rad}(\varrho, \vartheta) = \frac{a}{\varrho}\vartheta^4.$$

To identify the specific form of p_m we successively employ several physical principles, see [16, Chapter 1] for details.

- *Gibbs' relation* together with (2.9) yield

$$p_m(\varrho, \vartheta) = \vartheta^{\frac{5}{2}} P\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right)$$

  for a certain $P \in C^1[0, \infty)$. Consequently,

$$p(\varrho, \vartheta) = \vartheta^{\frac{5}{2}} P\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right) + \frac{a}{3}\vartheta^4, \quad e(\varrho, \vartheta) = \frac{3}{2}\frac{\vartheta^{\frac{5}{2}}}{\varrho} P\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right) + \frac{a}{\varrho}\vartheta^4,$$
$$a > 0. \tag{2.10}$$

- *Hypothesis of thermodynamics stability*, cf. Bechtel et al. [1], expressed in terms of P, reads

$$P(0) = 0, \quad P'(Z) > 0 \ \text{ for } \ Z \geqq 0, \quad 0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} \leqq c \ \text{ for } \ Z > 0. \tag{2.11}$$

In particular, the function $Z \mapsto P(Z)/Z^{\frac{5}{3}}$ is decreasing, and we suppose

$$\lim_{Z \to \infty} \frac{P(Z)}{Z^{\frac{5}{3}}} = p_\infty > 0. \tag{2.12}$$

- In accordance with Gibbs' relation (2.1), the associated entropy takes the form

$$s(\varrho, \vartheta) = \mathcal{S}\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho}, \tag{2.13}$$

where

$$\mathcal{S}'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2} < 0. \tag{2.14}$$

In addition, the **Third law of thermodynamics**, cf. Belgiorno [2,3], requires the entropy to vanish as soon as the abolute temperature approaches zero,

$$\lim_{Z \to \infty} \mathcal{S}(Z) = 0. \tag{2.15}$$

Note that (2.11)–(2.15) imply

$$0 \leqq \varrho \mathcal{S}\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right) \leqq c\left(1 + \varrho \log^+(\varrho) + \varrho \log^+(\vartheta)\right). \tag{2.16}$$

As for the transport coefficients, we suppose that they are continuously differentiable functions satisfying

$$\begin{aligned} 0 < \underline{\mu}(1 + \vartheta) \leqq \mu(\vartheta),\ |\mu'(\vartheta)| \leqq \overline{\mu}, \\ 0 \leqq \eta(\vartheta) \leqq \overline{\eta}(1 + \vartheta), \\ 0 < \underline{\kappa}(1 + \vartheta^\beta) \leqq \kappa(\vartheta) \leqq \overline{\kappa}(1 + \vartheta^\beta). \end{aligned} \tag{2.17}$$

The existence theory developed in [9] requires that

$$\beta > 6. \tag{2.18}$$

The state equation specified above, together with the fact that the transport coefficients depend on the temperature, are pertinent to models of gaseous stars discussed by Bormann [4,5].

3. Main results

Our main result states that the Navier–Stokes–Fourier system in the Rayleigh–Bénard regime (1.5), (1.6) admits a bounded absorbing set.

Theorem 3.1. (Bounded absorbing set) *Let $\Theta_{\rm B}$, $\Theta_{\rm U}$ be two strictly positive constants. Let the pressure p, the internal energy e, and the entropy s satisfy the hypotheses* (2.10)–(2.15)*. Let the transport coefficients μ, η, and κ satisfy* (2.17), (2.18).

Then there exists a constant $\mathcal{E}_\infty$ that depends only on $\Theta_{\rm B}$, $\Theta_{\rm U}$ and the total mass of the fluid

$$M = \int_\Omega \varrho \ {\rm d}x,$$

such that for any global-in-time weak solution $(\varrho, \vartheta, \mathbf{u})$ defined on a time interval (T, ∞), we have

$$\operatorname{ess}\limsup_{t\to\infty} \int_\Omega E(\varrho, \vartheta, \mathbf{u})(t, \cdot) \ {\rm d}x \leqq \mathcal{E}_\infty. \tag{3.1}$$

If, moreover,

$$\operatorname{ess}\limsup_{t\to T+} \int_\Omega E(\varrho, \vartheta, \mathbf{u})(t, \cdot) \ {\rm d}x \leqq \mathcal{E}_0 < \infty,$$

then the convergence is uniform in $\mathcal{E}_0$. Specifically, for any $\varepsilon > 0$, there exists a time $T(\varepsilon, \mathcal{E}_0)$ such that

$$\operatorname{ess} \sup_{t > T(\varepsilon, \mathcal{E}_0)} \int_\Omega E(\varrho, \vartheta, \mathbf{u})(t, \cdot) \ {\rm d}x \leqq \mathcal{E}_\infty + \varepsilon. \tag{3.2}$$

Remark 3.2. The same result can be shown for a general bounded domain with an arbitrary (nonconstant) profile of the boundary temperature and a general potential volume force $\mathbf{g} = \nabla_x G$, $G = G(x)$. In particular, the problem posed in the inclined layer studied for example by Daniels et al. [12] can be included.

Remark 3.3. Of course the quantity $\mathcal{E}_\infty$ depends also on the specific choice of the transport coefficients as well as the form of the constitutive relations $p = p(\varrho, \vartheta)$, $e = e(\varrho, \vartheta)$, $s = s(\varrho, \vartheta)$

The existence of a bounded absorbing set for the isentropic ($p = a\varrho^\gamma$) Navier–Stokes system with the no-slip boundary conditions was established in [19] under the condition $\gamma > \frac{5}{3}$, see also Wang and Wang [37]. For similar results related to the conservative boundary conditions see [18] and the monograph [21]. The existence of a bounded absorbing set for the Navier–Stokes–Fourier system with general Dirichlet boundary conditions was shown in [15] under rather restrictive assumption postulating a hard sphere equation of state for the pressure. Note that this considerably simplifies the analysis as *uniform* bounds on the fluid density are *a priori* available. It is exactly this missing piece of information that makes the analysis of the present paper much more delicate.

The second result concerns the asymptotic compactness of bounded trajectories.

Theorem 3.4. (Asymptotic compactness) *Under the hypotheses of Theorem 3.1, let $(\varrho_n, \vartheta_n, \mathbf{u}_n)_{n=1}^{\infty}$ be a sequence of weak solutions to the Navier–Stokes–Fourier system in the sense of Definition 2.1 on the time intervals*

$$(T_n, \infty), \quad T_n \geqq -\infty, \quad T_n \to -\infty \ \ as \ \ n \to \infty,$$

such that

$$\operatorname*{ess\,lim\,sup}_{t \to T_n+} \int_{\Omega} E(\varrho_n, \vartheta_n, \mathbf{u}_n)(t, \cdot) \ \mathrm{d}x \leqq \mathcal{E}_0, \ \int_{\Omega} \varrho_n \ \mathrm{d}x = M > 0,$$

uniformly for $n \to \infty$.

Then there is a subsequence (not relabelled) such that

$$\begin{aligned} &\varrho_n \to \varrho \ in \ C_{\mathrm{weak}}([-N, N]; L^{\frac{5}{3}}(\Omega)) \cap C([-N, N]; L^1(\Omega)), \\ &\vartheta_n \to \vartheta \ in \ L^q((-N, N); L^4(\Omega)) \ \ for\ any \ \ 1 \leqq q < \infty, \\ &\mathbf{u}_n \to \mathbf{u} \ weakly\ in \ L^2((-N, N); W^{1,2}(\Omega; R^3)) \end{aligned} \tag{3.3}$$

for any $N > 0$, where the limit $(\varrho, \vartheta, \mathbf{u})$ is an entire weak solution of the Navier–Stokes–Fourier system defined for $t \in R$ and satisfying

$$\int_{\Omega} E(\varrho, \vartheta, \mathbf{u})(t, \cdot) \ \mathrm{d}x \leqq \mathcal{E}_{\infty} \ \ for\ a.a. \ \ t \in R. \tag{3.4}$$

The heart of the paper is the proof of Theorem 3.1. Once the uniform bounds on the energy are established, the proof of Theorem 3.4 reduces to showing compactness of a sequence of bounded solutions. To certain extent, this is similar to the existence proof, where the only essential issue is the strong (almost everywhere pointwise) convergence of the densities in (3.3). Unlike in the existence proof, compactness of the densities at an appropriate "initial" time is not available here. Fortunately, this problem is nowadays well understood and we refer the reader to [14, Section 3, Theorem 3.1] for a detailed proof.

The next section is devoted to the proof of Theorem 3.1. In view of the hypotheses (2.10), (2.12),

$$p(\varrho, \vartheta) \approx \varrho^{\frac{5}{3}} + \vartheta^4.$$

As already pointed out, the exponent $\gamma = \frac{5}{3}$ is critical in the simplified isentropic case. To handle this problem we use the fact that **(i)** the gravitational force acting on the fluid is of potential type, and **(ii)** the entropy satisfies the Third law of thermodynamics, notably (2.15).

4. Dissipativity

Our goal is to prove Theorem 3.1. Suppose that we are given a global-in-time solution $(\varrho, \vartheta, \mathbf{u})$ defined on a time interval (T, ∞). The proof of asymptotic boundedness leans on several estimates that follow from the basic physical conservation laws. Here and hereafter, we fix $\tilde{\vartheta}$ to be the unique solution of the Dirichlet problem

$$\Delta_x \tilde{\vartheta} = 0 \text{ in } \Omega,\ \tilde{\vartheta}|_{x_3=0} = \Theta_{\rm B},\ \tilde{\vartheta}|_{x_3=1} = \Theta_{\rm U}. \tag{4.1}$$

As $\Theta_{\rm B}$, $\Theta_{\rm U}$ are constant, we easily compute

$$\tilde{\vartheta} = \tilde{\vartheta}(x_3) = \Theta_{\rm B} + x_3\left(\Theta_{\rm U} - \Theta_{\rm B}\right).$$

Obviously, the same ansatz can be used in the case of general $x-$dependent boundary data.

4.1. Mass conservation

It follows from the equation of continuity (2.4) that the total mass of the fluid is a constant of motion,

$$M = \int_\Omega \varrho(t,\cdot)\ {\rm d}x \quad \text{for any} \quad t > T. \tag{4.2}$$

In addition, as the volume force is potential, we can write

$$\int_\Omega \varrho \mathbf{u} \cdot \nabla_x G\ {\rm d}x = \frac{\rm d}{{\rm d}t}\int_\Omega \varrho G\ {\rm d}x, \quad G = -x_3.$$

Consequently, the ballistic energy balance (2.8) takes the form

$$\frac{\rm d}{{\rm d}t}\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e - \tilde{\vartheta}\varrho s - \varrho G\right]\ {\rm d}x + \int_\Omega \frac{\tilde{\vartheta}}{\vartheta}\left[\mathbb{S} : \mathbb{D}_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right]\ {\rm d}x$$
$$\leqq -\int_\Omega \left[\varrho s \mathbf{u}\cdot\nabla_x\tilde{\vartheta} + \frac{\mathbf{q}}{\vartheta}\cdot\nabla_x\tilde{\vartheta}\right]\ {\rm d}x \text{ in } \mathcal{D}'(T,\infty). \tag{4.3}$$

It is worth-noting that the same argument applies for a general Lipschitz potential $G = G(x)$.

4.2. Coercivity of the dissipative term

It follows from the hypotheses (2.17) and Korn–Poincaré inequality that

$$\int_\Omega \frac{\tilde{\vartheta}}{\vartheta}\left(\mathbb{S}(\vartheta,\mathbb{D}_x\mathbf{u}) : \mathbb{D}_x\mathbf{u} - \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right)\ {\rm d}x$$
$$\geqq c\inf\{\Theta_{\rm U},\Theta_{\rm B}\}\left(\|\mathbf{u}\|^2_{W^{1,2}(\Omega;R^3)} + \|\nabla_x\vartheta^{\frac{\beta}{2}}\|^2_{L^2(\Omega;R^3)} + \|\nabla_x\log(\vartheta)\|^2_{L^2(\Omega;R^3)}\right).$$

Consequently, adding the boundary integrals to the left-hand side and using Poincaré inequality, we get

$$\left(\|\mathbf{u}\|^2_{W^{1,2}(\Omega;R^3)} + \|\vartheta^{\frac{\beta}{2}}\|^2_{W^{1,2}(\Omega)} + \|\log(\vartheta)\|^2_{W^{1,2}(\Omega)}\right)$$
$$\leqq c(\Theta_{\rm U},\Theta_{\rm B})\left[1 + \int_\Omega \frac{\tilde{\vartheta}}{\vartheta}\left(\mathbb{S}(\vartheta,\mathbb{D}_x\mathbf{u}) : \mathbb{D}_x\mathbf{u} - \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right)\ {\rm d}x\right] \tag{4.4}$$

4.3. Energy estimates

To simplify the ballistic energy inequality (4.3), we first realize, by virtue of (4.1),

$$\int_{\Omega} \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \tilde{\vartheta} \ \mathrm{d}x = - \int_{\Omega} \frac{\kappa(\vartheta)}{\vartheta} \nabla_x \vartheta \cdot \nabla_x \tilde{\vartheta} \ \mathrm{d}x = \int_{\partial\Omega} \mathcal{K}(\tilde{\vartheta}) \nabla_x \tilde{\vartheta} \ \mathrm{d}\sigma_x,$$

where

$$\mathcal{K}'(\vartheta) = \kappa(\vartheta).$$

Consequently, the ballistic energy inequality (4.3) reduces to

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e - \tilde{\vartheta} \varrho s - \varrho G \right) \ \mathrm{d}x \\ &\quad + \int_{\Omega} \frac{\tilde{\vartheta}}{\vartheta} \left(\mathbb{S}(\vartheta, \mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \ \mathrm{d}x \\ &\quad \leqq - \int_{\Omega} \left[\varrho s \mathbf{u} \cdot \nabla_x \tilde{\vartheta} \right] \ \mathrm{d}x + c(\Theta_{\mathrm{U}}, \Theta_{\mathrm{B}}), \end{aligned} \tag{4.5}$$

holding in the sense of distributions.

4.4. Entropy estimates

In accordance with hypothesis (2.13),

$$\int_{\Omega} \varrho s \mathbf{u} \cdot \nabla \tilde{\vartheta} \ \mathrm{d}x = \int_{\Omega} \varrho \mathcal{S} \left(\frac{\varrho}{\vartheta^{\frac{3}{2}}} \right) \mathbf{u} \cdot \nabla_x \tilde{\vartheta} \ \mathrm{d}x + \frac{4a}{3} \int_{\Omega} \vartheta^3 \mathbf{u} \cdot \nabla \tilde{\vartheta} \ \mathrm{d}x. \tag{4.6}$$

To estimate further these terms we observe that by virtue of monotonicity of $\mathcal{S}$, that is (2.14), for $\frac{\varrho}{\vartheta^{\frac{3}{2}}} \geqq r$

$$\varrho \mathcal{S} \left(\frac{\varrho}{\vartheta^{\frac{3}{2}}} \right) \leqq \varrho \mathcal{S}(r) \tag{4.7}$$

and (2.15) provides that

$$\mathcal{S}(r) \to 0 \quad \text{as} \quad r \to \infty. \tag{4.8}$$

If

$$\frac{\varrho}{\vartheta^{\frac{3}{2}}} < r, \quad \text{which implies that} \quad \varrho < r \vartheta^{\frac{3}{2}},$$

we get, by virtue of (2.16),

$$0 \leqq \varrho \mathcal{S} \left(\frac{\varrho}{\vartheta^{\frac{3}{2}}} \right) \leqq c \left(1 + r \vartheta^{\frac{3}{2}} \left[\log^+(r \vartheta^{\frac{3}{2}}) + \log^+(\vartheta) \right] \right). \tag{4.9}$$

Thus we may estimate the first term of the right-hand side of (4.6) as follows

$$
\begin{aligned}
&\int_\Omega \varrho \mathcal{S}\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right)\mathbf{u}\cdot\nabla_x\tilde{\vartheta}\ \mathrm{d}x \\
&\quad = \int_\Omega \mathbb{1}_{\left\{\frac{\varrho}{\vartheta^{\frac{3}{2}}}\geqq r\right\}}\varrho \mathcal{S}\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right)\mathbf{u}\cdot\nabla_x\tilde{\vartheta}\ \mathrm{d}x + \int_\Omega \mathbb{1}_{\left\{\frac{\varrho}{\vartheta^{\frac{3}{2}}}< r\right\}}\varrho \mathcal{S}\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right)\mathbf{u}\cdot\nabla_x\tilde{\vartheta}\ \mathrm{d}x \\
&\quad \leqq c_2(\Theta_{\mathrm{U}},\Theta_{\mathrm{B}})\mathcal{S}(r)\int_\Omega \varrho|\mathbf{u}|\ \mathrm{d}x + c_3(\Theta_{\mathrm{U}},\Theta_{\mathrm{B}}) \\
&\quad \int_\Omega \left(1 + r\vartheta^{\frac{3}{2}}\left[\log^+(r\vartheta^{\frac{3}{2}}) + \log^+(\vartheta)\right]\right)|\mathbf{u}|\ \mathrm{d}x. \qquad (4.10)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\int_\Omega r\vartheta^{\frac{3}{2}}\left[\log^+(r\vartheta^{\frac{3}{2}}) + \log^+(\vartheta)\right]|\mathbf{u}|\ \mathrm{d}x \\
&\quad \leqq c\left(\int_\Omega r\log^+(r)\vartheta^{\frac{3}{2}}|\mathbf{u}|\ \mathrm{d}x + \int_\Omega [r\vartheta^{\frac{3}{2}}\log^+(\vartheta)|\mathbf{u}|\ \mathrm{d}x\right) \\
&\quad \leqq cr\log^+(r)\left(\int_\Omega \vartheta^{\frac{3}{2}\cdot 4}\ \mathrm{d}x\right)^{1/4}\left(\int_\Omega |\mathbf{u}|^2\ \mathrm{d}x\right)^{1/2} \\
&\qquad + cr\left(\int_\Omega \vartheta^{2\cdot 4}\ \mathrm{d}x\right)^{1/4}\left(\int_\Omega |\mathbf{u}|^2\ \mathrm{d}x\right)^{1/2} \\
&\quad \leqq cr^4\log^4(r) + c_3(\|\vartheta^3\|^2_{L^2(\Omega)} + \|\vartheta^4\|^2_{L^2(\Omega)} + \|\mathbf{u}\|^2_{L^2(\Omega;R^3)}), \qquad (4.11)
\end{aligned}
$$

where the constant c_3 may be chosen small enough, dependent on Θ_{U} and Θ_{B}, such that the corresponding terms will be absorbed by the left-hand side of the entropy estimate. We treat the second term of the right-hand side of (4.6) accordingly. As $\beta > 6$ in hypothesis (2.17), we may combine (4.4) with (4.5)–(4.11) to obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e - \tilde{\vartheta}\varrho s - \varrho G\right)\ \mathrm{d}x \\
&\qquad + c_1(\Theta_{\mathrm{U}},\Theta_{\mathrm{B}})\left(\|\mathbf{u}\|^2_{W^{1,2}(\Omega;R^3)} + \|\vartheta^{\frac{\beta}{2}}\|^2_{W^{1,2}(\Omega)} + \|\log(\vartheta)\|^2_{W^{1,2}(\Omega)}\right) \\
&\quad \leqq c_2(\Theta_{\mathrm{U}},\Theta_{\mathrm{B}})\mathcal{S}(r)\int_\Omega \varrho|\mathbf{u}|\ \mathrm{d}x + \Lambda(\Theta_{\mathrm{U}},\Theta_{\mathrm{B}},r), \qquad (4.12)
\end{aligned}
$$

holding in the sense of distributions for any value of the parameter $r > 0$ where $c_1 > 0$ and $c_2 > 0$. The exact form of the function $\Lambda(r)$ may be concluded from (4.12). Notice that $\Lambda(r) \to \infty$ if $r \to \infty$, however for a fixed r the value of $\Lambda(r)$ is finite.

The main problem to conclude is the fact that forcing term $\int_\Omega \varrho|\mathbf{u}|\ \mathrm{d}x$ on the right-hand side is not directly controlled by the dissipation on the left-hand side. To this end, we need the so-called pressure estimates which we recall in the next section.

4.5. Pressure estimates

To continue, we recall the inverse of the divergence known as Bogovskii operator:

$$
\begin{aligned}
\mathcal{B}: L^q_0(\Omega) \equiv \left\{ f \in L^q(\Omega) \ \Big| \ \int_\Omega f \ \mathrm{d}x = 0 \right\} &\to W^{1,q}_0(\Omega, R^d), \ 1 < q < \infty, \\
\mathrm{div}_x \mathcal{B}[f] &= f, \\
\|\mathcal{B}[f]\|_{W^{1,q}_0(\Omega, R^d)} &\leqq c \|f\|_{L^q_0(\Omega)}, \\
\|\mathcal{B}[\mathrm{div}_x \mathbf{g}]\|_{L^r(\Omega)} &\leqq c \|\mathbf{g}\|_{L^r(\Omega)}, \ 1 < r < \infty \text{ whenever } \mathbf{g} \cdot \mathbf{n}|_{\partial\Omega} = 0.
\end{aligned}
\tag{4.13}
$$

see for example Galdi [25, Chapter 3] or Geissert et al. [26].

Now, the test function

$$
\varphi(t,x) = \mathcal{B}\left[b(\varrho) - \frac{1}{|\Omega|} \int_\Omega b(\varrho) \ \mathrm{d}x \right]
$$

in the momentum equation yields

$$
\begin{aligned}
\int_\tau^{\tau+1} \int_\Omega p(\varrho, \vartheta) b(\varrho) \ \mathrm{d}x \, \mathrm{d}t &= \int_\tau^{\tau+1} \frac{1}{|\Omega|} \left(\int_\Omega b(\varrho) \ \mathrm{d}x \right) \left(\int_\Omega p(\varrho, \vartheta) \ \mathrm{d}x \right) \mathrm{d}t \\
&- \int_\tau^{\tau+1} \int_\Omega \varrho (\mathbf{u} \otimes \mathbf{u}) : \nabla_x \mathcal{B}\left[b(\varrho) - \frac{1}{|\Omega|} \int_\Omega b(\varrho) \ \mathrm{d}x \right] \mathrm{d}x \, \mathrm{d}t \\
&+ \int_\tau^{\tau+1} \int_\Omega \mathbb{S}(\vartheta, \mathbb{D}_x \mathbf{u}) : \nabla_x \mathcal{B}\left[b(\varrho) - \frac{1}{|\Omega|} \int_\Omega b(\varrho) \ \mathrm{d}x \right] \mathrm{d}x \, \mathrm{d}t \\
&- \int_\tau^{\tau+1} \int_\Omega \varrho \nabla_x G \cdot \mathcal{B}\left[b(\varrho) - \frac{1}{|\Omega|} \int_\Omega b(\varrho) \ \mathrm{d}x \right] \mathrm{d}x \, \mathrm{d}t \\
&+ \left[\int_\Omega \varrho \mathbf{u} \cdot \mathcal{B}\left[b(\varrho) - \frac{1}{|\Omega|} \int_\Omega b(\varrho) \ \mathrm{d}x \right] \mathrm{d}x \right]_{t=\tau}^{t=\tau+1} \\
&- \int_\tau^{\tau+1} \int_\Omega \varrho \mathbf{u} \cdot \partial_t \mathcal{B}\left[b(\varrho) - \frac{1}{|\Omega|} \int_\Omega b(\varrho) \ \mathrm{d}x \right] \mathrm{d}x \, \mathrm{d}t.
\end{aligned}
\tag{4.14}
$$

In addition, as ϱ satisfies the renormalized equation of continuity, we obtain

$$
\begin{aligned}
&\int_\tau^{\tau+1} \int_\Omega \varrho \mathbf{u} \cdot \partial_t \mathcal{B}\left[b(\varrho) - \frac{1}{|\Omega|} \int_\Omega b(\varrho) \ \mathrm{d}x \right] \mathrm{d}x \, \mathrm{d}t \\
&= - \int_\tau^{\tau+1} \int_\Omega \varrho \mathbf{u} \cdot \mathcal{B}[\mathrm{div}_x (b(\varrho) \mathbf{u}] \ \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_\tau^{\tau+1} \int_\Omega \varrho \mathbf{u} \cdot \mathcal{B}\Big[(b(\varrho) - b'(\varrho)\varrho) \mathrm{div}_x \mathbf{u} \\
&\quad - \frac{1}{|\Omega|} \int_\Omega (b(\varrho) - b'(\varrho)\varrho) \mathrm{div}_x \mathbf{u} \ \mathrm{d}x \Big] \, \mathrm{d}t.
\end{aligned}
\tag{4.15}
$$

The unit length of the time interval has been chosen just for simplicity.

4.6. Uniform bounds

In view of the structural restrictions imposed through hypotheses (2.11), (2.13) and (2.16), for any $\lambda > 1$ there exist two constants $c_1(\lambda, \text{data})$, $c_2(\lambda, \text{data})$ such that

$$c_1(\lambda, \text{data}) + \frac{1}{\lambda} E(\varrho, \vartheta, \mathbf{u}) \leqq E_{\tilde{\vartheta}}(\varrho, \vartheta, \mathbf{u}) - \varrho G \leqq \lambda E(\varrho, \vartheta, \mathbf{u}) + c_2(\lambda, \text{data}). \tag{4.16}$$

Here and hereafter the term *data* refers only to the boundary data Θ_U, Θ_B, but not to the initial data. The following result is crucial for showing the existence of a bounded absorbing set.

Lemma 4.1. *Suppose that*

$$\int_\Omega \left[E_{\tilde{\vartheta}}(\tau, \cdot) - \varrho(\tau, \cdot) G \right] \ \mathrm{d}x - \int_\Omega \left[E_{\tilde{\vartheta}}(\tau + 1, \cdot) - \varrho(\tau + 1, \cdot) G \right] \ \mathrm{d}x \leqq K. \tag{4.17}$$

Then there exists $L = L(K, M, \text{data})$ *such that*

$$\operatorname*{ess\ sup}_{\tau \leqq t \leqq \tau+1} \int_\Omega E(t, \cdot) \ \mathrm{d}x \leqq L. \tag{4.18}$$

Remark 4.2. Strictly speaking, the pointwise values of the ballistic energy appearing in (4.17) are defined only for a.a. $\tau \in (T; \infty)$. However, thanks to the inequality (4.12), we may identify $E_{\tilde{\vartheta}}$ with its, say, càglàd representative defined for any $\tau > T$.

The rest of this subsection is devoted to the proof of Lemma 4.1. If (4.17) holds, it follows from (4.12) that

$$\begin{aligned} &\int_\tau^{\tau+1} \left(\|\mathbf{u}\|^2_{W^{1,2}(\Omega; R^3)} + \|\vartheta^{\frac{\beta}{2}}\|_{W^{1,2}(\Omega)} + \|\log(\vartheta)\|^2_{W^{1,2}(\Omega)} \right) \mathrm{d}t \\ &\quad \leqq c(\text{data}, K) \left(1 + \mathcal{S}(r) \int_\tau^{\tau+1} \int_\Omega \varrho |\mathbf{u}| \ \mathrm{d}x \, \mathrm{d}t \right) + \Lambda(\text{data}, K, r). \end{aligned} \tag{4.19}$$

4.6.1. Pressure estimates revisited At this stage, we use the pressure estimates (4.14), with

$$b(\varrho) = \varrho^\alpha, \quad \alpha > 0.$$

In view of the hypotheses (2.10), (2.12) imposed on the equation of state, we have

$$c_1 \left(\varrho^{\frac{5}{3}} + \vartheta^4 \right) \leqq p(\varrho, \vartheta) \leqq c_2 \left(\varrho^{\frac{5}{3}} + \vartheta^4 + 1 \right), \quad c_1, c_2 > 0. \tag{4.20}$$

Moreover, as the total mass is constant via (4.2), the smoothing properties of $\mathcal{B}$ stated in (4.13) imply

$$\left| \mathcal{B} \left[\varrho^\alpha - \frac{1}{|\Omega|} \int_\Omega \varrho^\alpha \ \mathrm{d}x \right] \right| \leqq c(M) \quad \text{as soon as} \quad \alpha < \frac{1}{3}$$

to provide that $W^{1,\frac{1}{\alpha}}(\Omega) \subset L^\infty(\Omega)$. Thus inequality (4.14) gives rise to

$$\begin{aligned}\int_\tau^{\tau+1}\int_\Omega \varrho^{\frac{5}{3}+\alpha}\ \mathrm{d}x\,\mathrm{d}t \leqq c(M)\Bigg(1+\int_\tau^{\tau+1}\int_\Omega \vartheta^4\ \mathrm{d}x\,\mathrm{d}t\\ -\int_\tau^{\tau+1}\int_\Omega \varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla_x\mathcal{B}\left[\varrho^\alpha - \frac{1}{|\Omega|}\int_\Omega \varrho^\alpha\ \mathrm{d}x\right]\ \mathrm{d}x\,\mathrm{d}t\\ +\int_\tau^{\tau+1}\int_\Omega \mathbb{S}(\vartheta,\mathbb{D}_x\mathbf{u}) : \nabla_x\mathcal{B}\left[\varrho^\alpha - \frac{1}{|\Omega|}\int_\Omega \varrho^\alpha\ \mathrm{d}x\right]\ \mathrm{d}x\,\mathrm{d}t\\ +\left[\int_\Omega \varrho\mathbf{u}\cdot\mathcal{B}\left[\varrho^\alpha - \frac{1}{|\Omega|}\int_\Omega \varrho^\alpha\ \mathrm{d}x\right]\ \mathrm{d}x\right]_{t=\tau}^{t=\tau+1}\\ -\int_\tau^{\tau+1}\int_\Omega \varrho\mathbf{u}\cdot\partial_t\mathcal{B}\left[\varrho^\alpha - \frac{1}{|\Omega|}\int_\Omega \varrho^\alpha\ \mathrm{d}x\right]\ \mathrm{d}x\,\mathrm{d}t\Bigg).\end{aligned} \tag{4.21}$$

Next, using again the smoothing properties (4.13) of $\mathcal{B}$ we get

$$\begin{aligned}&\left|\int_\tau^{\tau+1}\int_\Omega \varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla_x\mathcal{B}\left[\varrho^\alpha - \frac{1}{|\Omega|}\int_\Omega \varrho^\alpha\ \mathrm{d}x\right]\ \mathrm{d}x\,\mathrm{d}t\right|\\ &\leqq \int_\tau^{\tau+1} \|\varrho\|_{L^\gamma(\Omega)}\|\mathbf{u}\|^2_{L^6(\Omega;R^3)}\|\varrho^\alpha\|_{L^q(\Omega)}\,\mathrm{d}t\\ &\leqq \sup_{t\in(\tau,\tau+1)}\|\varrho\|_{L^\gamma(\Omega)}\int_\tau^{\tau+T}\|\mathbf{u}\|^2_{W^{1,2}(\Omega;R^3)}\sup_{t\in(\tau,\tau+1)}\|\varrho^\alpha\|_{L^q(\Omega)}\,\mathrm{d}t,\end{aligned} \tag{4.22}$$

where

$$q = \frac{3\gamma}{2\gamma-3} > 1 \quad \text{provided} \quad \gamma > \frac{3}{2}.$$

Thus setting

$$\gamma = \frac{5}{3},\quad \alpha = \frac{2\gamma-3}{3\gamma} = \frac{1}{15} < \frac{1}{3}, \tag{4.23}$$

we may use the total mass conservation (4.2) to conclude

$$\begin{aligned}&\left|\int_\tau^{\tau+1}\int_\Omega \varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla_x\mathcal{B}\left[\varrho^\alpha - \frac{1}{|\Omega|}\int_\Omega \varrho^\alpha\ \mathrm{d}x\right]\ \mathrm{d}x\,\mathrm{d}t\right|\\ &\leqq c(M)\sup_{t\in(\tau,\tau+1)}\|\varrho\|_{L^{\frac{5}{3}}(\Omega)}\int_\tau^{\tau+1}\|\mathbf{u}\|^2_{W^{1,2}(\Omega;R^3)}\,\mathrm{d}t,\end{aligned} \tag{4.24}$$

Similarly, going back to (4.15) we have

$$\begin{aligned}&\left|\int_\tau^{\tau+1}\int_\Omega \varrho\mathbf{u}\cdot\mathcal{B}[\mathrm{div}_x(\varrho^\alpha\mathbf{u})]\ \mathrm{d}x\right|\\ &\leqq \int_0^{\tau+1}\|\varrho\|_{L^\gamma(\Omega)}\|\mathbf{u}\|_{L^6(\Omega;R^3)}\|\varrho^\alpha\mathbf{u}\|_{L^q(\Omega;R^3)}\,\mathrm{d}t,\end{aligned}$$

where

$$\frac{1}{\gamma} + \frac{1}{6} + \frac{1}{q} = 1.$$

Moreover,

$$\|\varrho^\alpha \mathbf{u}\|_{L^q(\Omega;R^3)} \leqq \|\mathbf{u}\|_{L^6(\Omega;R^3)} \|\varrho^\alpha\|_{L^p(\Omega)}, \text{ where } \frac{1}{q} = \frac{1}{6} + \frac{1}{p};$$

whence

$$\left| \int_\tau^{\tau+1} \int_\Omega \varrho \mathbf{u} \cdot \mathcal{B}[\mathrm{div}_x(\varrho^\alpha \mathbf{u})] \ \mathrm{d}x \right| \\ \leqq c(M) \sup_{t\in(\tau,\tau+1)} \|\varrho\|_{L^{\frac{5}{3}}(\Omega)} \int_\tau^{\tau+1} \|\mathbf{u}\|^2_{W^{1,2}(\Omega;R^3)} \,\mathrm{d}t \tag{4.25}$$

as soon as (4.23) holds.

Finally,

$$\left| \int_\tau^{\tau+1} \int_\Omega \varrho \mathbf{u} \cdot \mathcal{B} \left[\varrho^\alpha \mathrm{div}_x \mathbf{u} - \frac{1}{|\Omega|} \int_\Omega \varrho^\alpha \mathrm{div}_x \mathbf{u} \ \mathrm{d}x \right] \ \mathrm{d}x \, \mathrm{d}t \right| \\ \leqq \int_\tau^{\tau+1} \|\varrho\|_{L^\gamma(\Omega)} \|\mathbf{u}\|_{L^6(\Omega;R^3)} \left\| \mathcal{B} \left[\varrho^\alpha \mathrm{div}_x \mathbf{u} - \frac{1}{|\Omega|} \int_\Omega \varrho^\alpha \mathrm{div}_x \mathbf{u} \ \mathrm{d}x \right] \right\|_{L^q(\Omega;R^3)} \mathrm{d}t,$$

where

$$\frac{1}{\gamma} + \frac{1}{6} + \frac{1}{q} = 1.$$

Furthermore,

$$\left\| \mathcal{B} \left[\varrho^\alpha \mathrm{div}_x \mathbf{u} - \frac{1}{|\Omega|} \int_\Omega \varrho^\alpha \mathrm{div}_x \mathbf{u} \ \mathrm{d}x \right] \right\|_{L^q(\Omega;R^3)} \\ \lesssim \|\varrho^\alpha \mathrm{div}_x \mathbf{u}\|_{L^r(\Omega;R^3)}, \quad q = \frac{3r}{3-r},$$

and

$$\|\varrho^\alpha \mathrm{div}_x \mathbf{u}\|_{L^r(\Omega;R^3)} \leqq \|\mathbf{u}\|_{W^{1,2}(\Omega;R^3)} \|\varrho^\alpha\|_{L^p(\Omega)}, \ \text{with } \frac{1}{2} + \frac{1}{p} = \frac{1}{r}.$$

Consequently, condition (4.23) yields

$$\left| \int_\tau^{\tau+1} \int_\Omega \varrho \mathbf{u} \cdot \mathcal{B} \left[\varrho^\alpha \mathrm{div}_x \mathbf{u} - \frac{1}{|\Omega|} \int_\Omega \varrho^\alpha \mathrm{div}_x \mathbf{u} \ \mathrm{d}x \right] \ \mathrm{d}x \, \mathrm{d}t \right| \\ \leqq c(M) \sup_{t\in(\tau,\tau+1)} \|\varrho\|_{L^{\frac{5}{3}}(\Omega)} \int_\tau^{\tau+1} \|\mathbf{u}\|^2_{W^{1,2}(\Omega;R^3)} \,\mathrm{d}t. \tag{4.26}$$

Summing up the previous inequalities and going back to (4.21), we get

$$\int_\tau^{\tau+1}\int_\Omega \varrho^{\frac{5}{3}+\alpha}\ \mathrm{d}x\,\mathrm{d}t \leqq c(M)\left(1+\int_\tau^{\tau+1}\int_\Omega \vartheta^4\ \mathrm{d}x\,\mathrm{d}t\right.$$
$$+\sup_{t\in(\tau,\tau+1)}\|\varrho\|_{L^{\frac{5}{3}}(\Omega)}\int_\tau^{\tau+1}\|\mathbf{u}\|^2_{W^{1,2}(\Omega;R^3)}\,\mathrm{d}t$$
$$+\int_\tau^{\tau+T}\int_\Omega \mathbb{S}(\vartheta,\mathbb{D}_x\mathbf{u}):\nabla_x\mathcal{B}\left[\varrho^\alpha-\frac{1}{|\Omega|}\int_\Omega\varrho^\alpha\ \mathrm{d}x\right]\ \mathrm{d}x\,\mathrm{d}t$$
$$\left.+\left[\int_\Omega\varrho\mathbf{u}\cdot\mathcal{B}\left[\varrho^\alpha-\frac{1}{|\Omega|}\int_\Omega\varrho^\alpha\ \mathrm{d}x\right]\ \mathrm{d}x\right]_{t=\tau}^{t=\tau+1}\right),\ \alpha=\frac{1}{15}. \tag{4.27}$$

Now,

$$\int_\Omega \mathbb{S}(\vartheta,\mathbb{D}_x\mathbf{u}):\nabla_x\mathcal{B}\left[\varrho^\alpha-\frac{1}{|\Omega|}\int_\Omega\varrho^\alpha\ \mathrm{d}x\right]\ \mathrm{d}x$$
$$\leqq(1+\|\vartheta\|_{L^4(\Omega)})\|\mathbf{u}\|_{W^{1,2}(\Omega;R^3)}\left\|\nabla_x\mathcal{B}\left[\varrho^\alpha-\frac{1}{|\Omega|}\int_\Omega\varrho^\alpha\ \mathrm{d}x\right]\right\|_{L^4(\Omega;R^3)}$$
$$\leqq c(M)(1+\|\vartheta\|_{L^4(\Omega)})\|\mathbf{u}\|_{W^{1,2}(\Omega;R^3)}.$$

We therefore conclude

$$\int_\tau^{\tau+1}\int_\Omega \varrho^{\frac{5}{3}+\alpha}\ \mathrm{d}x\,\mathrm{d}t \leqq c(M)\left[1+\int_\tau^{\tau+1}\int_\Omega\vartheta^4\ \mathrm{d}x\,\mathrm{d}t\right.$$
$$+\left(1+\sup_{t\in(\tau,\tau+1)}\|\varrho\|_{L^{\frac{5}{3}}(\Omega)}\right)\int_\tau^{\tau+1}\|\mathbf{u}\|^2_{W^{1,2}(\Omega;R^3)}\,\mathrm{d}t$$
$$\left.+\sup_{t\in(\tau,\tau+1)}\int_\Omega\varrho|\mathbf{u}|\ \mathrm{d}x\right],\quad \alpha=\frac{1}{15}. \tag{4.28}$$

4.6.2. Proof of Lemma 4.1 Now, in accordance with (4.19),

$$\int_\tau^{\tau+1}\int_\Omega\vartheta^4\ \mathrm{d}x\leqq c(\mathrm{data})\left(1+\int_\tau^{\tau+1}\|\vartheta^{\frac{\beta}{2}}\|^2_{W^{1,2}(\Omega)}\,\mathrm{d}t\right)$$
$$\leqq c(\mathrm{data},K)\left(1+\mathcal{S}(r)\int_\tau^{\tau+1}\int_\Omega\varrho|\mathbf{u}|\ \mathrm{d}x\,\mathrm{d}t\right)+\Lambda(\mathrm{data},K,r),$$

where we may fix $r=1$. Consequently, inequality (4.28) reduces to

$$\int_\tau^{\tau+1}\int_\Omega\varrho^{\frac{5}{3}+\alpha}\ \mathrm{d}x\,\mathrm{d}t\leqq c(K,M,\mathrm{data})\left[\left(1+\sup_{t\in(\tau,\tau+1)}\|\varrho\|_{L^{\frac{5}{3}}(\Omega)}\right)\right.$$
$$\int_\tau^{\tau+1}\|\mathbf{u}\|^2_{W^{1,2}(\Omega;R^3)}\,\mathrm{d}t$$
$$\left.+\sup_{t\in(\tau,\tau+1)}\int_\Omega\varrho|\mathbf{u}|\ \mathrm{d}x+1\right],\ \alpha=\frac{1}{15}. \tag{4.29}$$

Next, it follows from the hypotheses (4.17), (4.19), and (4.12) that

$$\sup_{t\in(\tau,\tau+1)} \int_\Omega E(t,\cdot)\ \mathrm{d}x \leqq c(\mathrm{data})\left(1+\int_\tau^{\tau+1} E(s,\cdot)\mathrm{d}s\right). \tag{4.30}$$

Moreover, relation (4.19) yields

$$\int_\tau^{\tau+1} \|\mathbf{u}\|^2_{W^{1,2}(\Omega;R^3)} \leqq c(\mathrm{data},K)\mathcal{S}(r)\int_\tau^{\tau+1}\int_\Omega \varrho|\mathbf{u}|\ \mathrm{d}x\,\mathrm{d}t + \Lambda(\mathrm{data},K,r),$$

where, by Hölder's inequality and Sobolev embedding theorem,

$$\int_\Omega \varrho|\mathbf{u}|\ \mathrm{d}x \leqq \|\sqrt{\varrho}\|_{L^2(\Omega)}\|\sqrt{\varrho}\|_{L^3(\Omega)}\|\mathbf{u}\|_{L^6(\Omega;R^3)} \leqq c\sqrt{M}\|\sqrt{\varrho}\|_{L^3(\Omega)}\|\mathbf{u}\|_{W^{1,2}(\Omega;R^3)}.$$

Thus we may infer that

$$\int_\tau^{\tau+1} \|\mathbf{u}\|^2_{W^{1,2}(\Omega;R^3)} \leqq c(\mathrm{data},K,M)\mathcal{S}(r)\int_\tau^{\tau+1} \|\varrho\|_{L^{\frac{3}{2}}(\Omega)} + \Lambda(\mathrm{data},K,r). \tag{4.31}$$

Finally, using (4.31) we may estimate the kinetic energy,

$$\begin{aligned}
\int_\tau^{\tau+1}\int_\Omega \varrho|\mathbf{u}|^2\ \mathrm{d}x\,\mathrm{d}t &\leqq \sup_{t\in(\tau,\tau+1)}\|\varrho\|_{L^{\frac{3}{2}}(\Omega)}\int_\tau^{\tau+1}\|\mathbf{u}\|^2_{L^6(\Omega;R^3)}\,\mathrm{d}t\\
&\leqq c\sup_{t\in(\tau,\tau+1)}\|\varrho\|_{L^{\frac{3}{2}}(\Omega)}\int_\tau^{\tau+1}\|\mathbf{u}\|^2_{W^{1,2}(\Omega;R^3)}\,\mathrm{d}t\\
&\leqq \Lambda(\mathrm{data},K,r)\sup_{t\in(\tau,\tau+1)}\|\varrho\|_{L^{\frac{3}{2}}(\Omega)}\\
&\quad + c(\mathrm{data},K,M)\mathcal{S}(r)\sup_{t\in(\tau,\tau+1)}\|\varrho\|_{L^{\frac{3}{2}}(\Omega)}\int_\tau^{\tau+1}\|\varrho\|_{L^{\frac{3}{2}}(\Omega)}\,\mathrm{d}t,
\end{aligned}$$

Now, by interpolation,

$$\|\varrho\|_{L^{\frac{3}{2}}(\Omega)} \leqq \|\varrho\|^{\frac{5}{6}}_{L^{\frac{5}{3}}(\Omega)}\|\varrho\|^{\frac{1}{6}}_{L^1(\Omega)};$$

whence

$$\begin{aligned}
\int_\tau^{\tau+1}\int_\Omega \varrho|\mathbf{u}|^2\ \mathrm{d}x &\leqq \Lambda(\mathrm{data},K,M,r)\sup_{t\in(\tau,\tau+1)}\|\varrho\|^{\frac{5}{6}}_{L^{\frac{5}{3}}(\Omega)}\\
&+ c(\mathrm{data},K,M)\mathcal{S}(r)\sup_{t\in(\tau,\tau+1)}\|\varrho\|^{\frac{5}{6}}_{L^{\frac{5}{3}}(\Omega)}\int_\tau^{\tau+1}\|\varrho\|^{\frac{5}{6}}_{L^{\frac{5}{3}}(\Omega)}\,\mathrm{d}t.
\end{aligned} \tag{4.32}$$

Going back to (4.29) and using (4.31) we get

$$\begin{aligned}
\int_\tau^{\tau+1}\int_\Omega \varrho^{\frac{5}{3}+\alpha}\ \mathrm{d}x\,\mathrm{d}t &\leqq \Lambda(K,M,\mathrm{data},r)\Bigg[\left(1+\sup_{t\in(\tau,\tau+1)}\|\varrho\|_{L^{\frac{5}{3}}(\Omega)}\right)\mathrm{d}t\\
&\quad + c(K,M,\mathrm{data})\mathcal{S}(r)\left(1+\sup_{t\in(\tau,\tau+1)}\|\varrho\|_{L^{\frac{5}{3}}(\Omega)}\right)\int_\tau^{\tau+1}\|\varrho\|_{L^{\frac{3}{2}}(\Omega)}\ \mathrm{d}t\\
&\quad + \sup_{t\in(\tau,\tau+1)}\int_\Omega \varrho|\mathbf{u}|\ \mathrm{d}x + 1\Bigg]\\
&\leqq \Lambda(K,M,\mathrm{data},r)\Bigg[\left(1+\sup_{t\in(\tau,\tau+1)}\|\varrho\|_{L^{\frac{5}{3}}(\Omega)}\right)\mathrm{d}t\\
&\quad + c(K,M,\mathrm{data})\mathcal{S}(r)\left(1+\sup_{t\in(\tau,\tau+1)}\|\varrho\|_{L^{\frac{5}{3}}(\Omega)}\right)\int_\tau^{\tau+1}\|\varrho\|^{\frac{5}{6}}_{L^{\frac{5}{3}}(\Omega)}\ \mathrm{d}t\\
&\quad + \sup_{t\in(\tau,\tau+1)}\int_\Omega \varrho|\mathbf{u}|\ \mathrm{d}x + 1\Bigg],\ \alpha=\frac{1}{15}.
\end{aligned}\tag{4.33}$$

Now, interpolating L^1 and $L^{\frac{5}{3}+\alpha}$, we get

$$\int_\tau^{\tau+1}\int_\Omega \varrho^{\frac{5}{3}}\ \mathrm{d}x\,\mathrm{d}t \leqq c(M)\left(\int_\tau^{\tau+1}\int_\Omega \varrho^{\frac{5}{3}+\alpha}\ \mathrm{d}x\,\mathrm{d}t\right)^{\frac{10}{11}} \text{ provided } \alpha=\frac{1}{15}.$$

Gathering the available bounds we conclude that

$$\begin{aligned}
\sup_{t\in(\tau,\tau+1)}\int_\Omega E(t,\cdot)\ \mathrm{d}x &\leqq c(\mathrm{data})\left(1+\int_\tau^{\tau+1}E(s,\cdot)\mathrm{d}s\right)\\
&\leqq c(\mathrm{data})\left(1+\int_\tau^{\tau+1}\left(\|\mathbf{u}\|^2_{W^{1,2}(\Omega;R^3)}+\|\vartheta^{\frac{\beta}{2}}\|_{W^{1,2}(\Omega)}+\|\log(\vartheta)\|^2_{W^{1,2}(\Omega)}\right)\mathrm{d}t\right)\\
&\quad + c(\mathrm{data})\left(\int_\tau^{\tau+1}\int_\Omega \varrho|\mathbf{u}|^2\ \mathrm{d}x\,\mathrm{d}t + \int_\tau^{\tau+1}\int_\Omega \varrho^{\frac{5}{3}}\ \mathrm{d}x\,\mathrm{d}t\right)\\
&\leqq \Lambda(\mathrm{data},K,r)\left[1+\left(\sup_{t\in(\tau,\tau+1)}\int_\Omega E\ \mathrm{d}x\,\mathrm{d}t\right)^\lambda\right]\\
&\quad + c(\mathrm{data},K,M)\mathcal{S}(r)\sup_{t\in(\tau,\tau+1)}\int_\Omega E\ \mathrm{d}x\,\mathrm{d}t
\end{aligned}\tag{4.34}$$

for certain $0<\lambda<1$. Consequently, choosing $r=r(\mathrm{data},K,M)$ large enough, the desired conclusion follows since $\mathcal{S}(r)\to 0$ as $r\to\infty$.

We have proved Lemma 4.1.

4.7. Bounded absorbing sets

The existence of a bounded absorbing set follows easily from Lemma 4.1. Indeed consider a global-in-time solution as in Theorem 3.1 satisfying

$$\text{ess} \limsup_{t \to T+} \int_\Omega E(\varrho, \vartheta, \mathbf{u})(t, \cdot) \ \mathrm{d}x \leqq \mathcal{E}_0.$$

Consider $K = 1$ in Lemma 4.1. In view of (4.16), there exists $\tau = \tau(\mathcal{E}_0)$ such that (4.18) holds, specifically,

$$\text{ess} \sup_{\tau \leqq t \leqq \tau + 1} \int_\Omega E(t, \cdot) \ \mathrm{d}x \leqq L(1, M, \text{data}). \tag{4.35}$$

Indeed assuming the contrary, we would obtain a sequence $T, T+1, \dots, T+n$ such that

$$\begin{aligned} \underline{E} &\leqq \int_\Omega \left[E_{\overline{\vartheta}}(T+n, \cdot) - \varrho(T+n, \cdot)G \right] \ \mathrm{d}x \\ &\leqq \int_\Omega \left[E_{\overline{\vartheta}}(T+(n-1), \cdot) - \varrho(T+(n-1), \cdot)G \right] \ \mathrm{d}x - 1 \\ &\leqq \cdots \int_\Omega \left[E_{\overline{\vartheta}}(T+(n-1), \cdot) - \varrho(T+(n-1), \cdot)G \right] \ \mathrm{d}x - n \leqq c(\mathcal{E}_0, \text{data}) - n, \end{aligned}$$

where the lower bound $\underline{E}$ depends solely on the data. We conclude that, necessarily,

$$n \leqq c(\mathcal{E}_0, \text{data}) - \underline{E}.$$

Repeating the same argument with $\mathcal{E}_0$ replaced by L given by (4.35) we deduce that there exists $H = H(L)$ such that any time interval $(s, s+H)$, $s \geqq \tau(\mathcal{E}_0)$ contains τ such that (4.35) holds. Finally, in view of inequality (4.12), the energy is growing at most exponentially and we may choose

$$\mathcal{E}_\infty = c(L, M, \text{data})(1 + \exp H(L))$$

for a sufficiently large constant $c(L, M, \text{data})$. We have proved Theorem 3.1. As pointed out in Section 3, Theorem 3.1 yields Theorem 3.4 via the existing arguments presented for example in [14].

5. Applications, long-time behavior

We finish the paper by discussing the impact of Theorems 3.1, 3.4 on the long time behavior of solutions to the Rayleigh–Bénard problem in the framework of compressible viscous and heat conducting fluids.

5.1. Trajectory space

We start by introducing a suitable *trajectory space* $\mathcal{T}$. In view of the framework of Theorems 3.1, 3.4, the "natural" trajectory space should be based on the standard phase variables $(\varrho, \vartheta, \mathbf{u})$. Unfortunately, neither ϑ nor $\mathbf{u}$ admit well defined *instantaneous values* at any time $t \in R$. It is therefore more convenient to consider the *conservative entropy* variables $(\varrho, S, \mathbf{m})$, with

$$\text{momentum } \mathbf{m} = \varrho \mathbf{u}, \quad \text{and total entropy} \quad S = \varrho s(\varrho, \vartheta).$$

On the one hand, the state variables $(\varrho, S, \mathbf{m})$ are uniquely determined by $(\varrho, \vartheta, \mathbf{u})$. On the other hand, knowing $(\varrho, S, \mathbf{m})$ we first obtain ϑ as $\vartheta \mapsto s(\varrho, \vartheta)$ is a strictly increasing function. The velocity $\mathbf{u}$ is *a priori* not well defined on the hypothetical vacuum zone, however, it can be recovered in terms of $(\varrho, \vartheta, \mathbf{m})$ from the momentum equation (2.6).

The phase variables $(\varrho, S, \mathbf{m})$ admit well defined instantaneous values understood in the weak sense. Specifically, it follows from the weak formulation (2.4), (2.6), (2.7) that the one sided limits

$$\begin{aligned}
\langle \varrho(\tau-, \cdot); \phi \rangle &\equiv \lim_{\delta \to 0+} \frac{1}{\delta} \int_{\tau-\delta}^{\tau} \int_\Omega \varrho(t, \cdot)\phi \ \mathrm{d}x \, \mathrm{d}t, \ \langle \varrho(\tau+, \cdot); \phi \rangle \\
&\equiv \lim_{\delta \to 0+} \frac{1}{\delta} \int_{\tau}^{\tau+\delta} \int_\Omega \varrho(t, \cdot)\phi \ \mathrm{d}x \, \mathrm{d}t \\
\langle \mathbf{m}(\tau-, \cdot); \boldsymbol{\varphi} \rangle &\equiv \lim_{\delta \to 0+} \frac{1}{\delta} \int_{\tau-\delta}^{\tau} \int_\Omega \mathbf{m}(t, \cdot) \cdot \boldsymbol{\varphi} \ \mathrm{d}x \, \mathrm{d}t, \ \langle \mathbf{m}(\tau+, \cdot); \boldsymbol{\varphi} \rangle \\
&\equiv \lim_{\delta \to 0+} \frac{1}{\delta} \int_{\tau}^{\tau+\delta} \int_\Omega \mathbf{m}(t, \cdot) \cdot \boldsymbol{\varphi} \ \mathrm{d}x \, \mathrm{d}t, \\
\langle S(\tau-, \cdot); \phi \rangle &\equiv \lim_{\delta \to 0+} \frac{1}{\delta} \int_{\tau-\delta}^{\tau} \int_\Omega \varrho s(t, \cdot)\phi \ \mathrm{d}x \, \mathrm{d}t, \ \langle S(\tau+, \cdot); \phi \rangle \\
&\equiv \lim_{\delta \to 0+} \frac{1}{\delta} \int_{\tau}^{\tau+\delta} \int_\Omega \varrho s(t, \cdot)\phi \ \mathrm{d}x \, \mathrm{d}t
\end{aligned}$$

exist for *any* $\tau \in R$ and any $\phi \in C^1_c(\Omega)$, $\boldsymbol{\varphi} \in C^1_c(\Omega; R^3)$. In addition,

$$\begin{aligned}
\langle \varrho(\tau-, \cdot); \phi \rangle &= \langle \varrho(\tau+, \cdot); \phi \rangle, \ \text{and } \tau \mapsto \langle \varrho(\tau, \cdot); \phi \rangle \in BC(R), \\
\langle \mathbf{m}(\tau-, \cdot); \boldsymbol{\varphi} \rangle &= \langle \mathbf{m}(\tau+, \cdot); \boldsymbol{\varphi} \rangle, \ \text{and } \tau \mapsto \langle \mathbf{m}(\tau, \cdot); \boldsymbol{\varphi} \rangle \in BC(R),
\end{aligned}$$

and

$$\begin{aligned}
&\langle S(\tau-, \cdot); \phi \rangle \leqq \langle S(\tau+, \cdot); \phi \rangle \quad \text{whenever} \quad \phi \geqq 0, \\
&\quad \text{and} \quad \tau \mapsto \langle S(\tau-, \cdot); \phi \rangle = h_\phi(\tau) + g_\phi(\tau), \ g_\phi \in C_{\mathrm{loc}}(R), \\
&\qquad h_\phi \text{ non-decreasing càglàd.}
\end{aligned} \tag{5.1}$$

The trajectory space can be therefore identified with "weakly càglàd" and bounded functions defined on R. To this end, consider the Hilbert space

$$W^{k,2}_0(\Omega), \quad k > \frac{3}{2} \quad \text{so that} \quad W^{k,2}_0 \hookrightarrow C(\overline{\Omega})$$

with an orthonormal basis of smooth functions $\{\phi_n\}_{n=1}^{\infty}$. Similarly, we consider the same space of vector valued functions $W_0^{1,2}(\Omega; R^3)$ with a basis $\{\boldsymbol{\varphi}_n\}_{n=1}^{\infty}$. Finally, we define a metrics

$$
\begin{aligned}
&d_{\mathcal{T}}\left[(\varrho^1, S^1, \mathbf{m}^1); (\varrho^2, S^2, \mathbf{m}^2)\right] \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n} \int_{-\infty}^{\infty} \exp\left(-t^2\right) G\left(\|\langle \varrho^1 - \varrho^2; \phi_n \rangle\|_{C[-t,t]}\right) \, \mathrm{d}t \\
&\quad + \sum_{n=1}^{\infty} \frac{1}{2^n} \int_{-\infty}^{\infty} \exp\left(-t^2\right) G\left(\|\langle \mathbf{m}^1 - \mathbf{m}^2; \boldsymbol{\varphi}_n \rangle\|_{C([-t,t]; R^3)}\right) \, \mathrm{d}t \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n} \int_{-\infty}^{\infty} \exp\left(-t^2\right) G\left(\left[\langle S^1; \phi_n \rangle; \langle S^2; \phi_n \rangle\right]_{D[-t,t]}\right) \, \mathrm{d}t,
\end{aligned}
\tag{5.2}
$$

where

$$
G(Z) = \frac{Z}{1+Z}
$$

and $D[-t,t]$ denotes the Skorokhod space of càglàd functions defined on $[-t,t]$ with the associated complete metrics $[\cdot; \cdot]_{D[-t,t]}$, see for example Whitt [38].

The trajectory space is defined as

$$
\mathcal{T} = \cup_{L=1}^{\infty} \mathcal{T}_L,
$$

where

$$
\begin{aligned}
\mathcal{T}_L = \Big\{ (\varrho, S, \mathbf{m}) \ \Big| \ & \varrho \in L^{\infty}(R; W^{-k,2}(\Omega)), \ \langle \varrho; \phi_n \rangle \in C(R), \ n = 1, 2, \dots, \\
& \sup_{t \in R} \|\varrho(t, \cdot)\|_{W^{-k,2}(\Omega)} \leqq L, \\
& \mathbf{m} \in L^{\infty}(R; W^{-k,2}(\Omega; R^3)), \ \langle \mathbf{m}; \boldsymbol{\varphi}_n \rangle \in C(R), \ n = 1, 2, \dots, \\
& \sup_{t \in R} \|\mathbf{m}(t, \cdot)\|_{W^{-k,2}(\Omega; R^3)} \leqq L, \\
& S \in L^{\infty}(R; W^{-k,2}(\Omega)), \ \langle S; \phi_n \rangle \text{ càglàd in } R, \ n = 1, 2, \dots, \\
& \sup_{t \in R} \|S(t, \cdot)\|_{W^{-k,2}(\Omega)} \leqq L \Big\}.
\end{aligned}
$$

Note that the trajectory space is larger then the set of entire solutions to the Navier–Stokes–Fourier system and consists of time dependent functionals ranging in the space of distributions on Ω.

Each set $\mathcal{T}_L$ endowed with the metrics $d_{\mathcal{T}}$ is a Polish space. We define inductive topology on $\mathcal{T}$:

$$
\begin{aligned}
(\varrho_n, S_n, \mathbf{m}_n) \to (\varrho, S, \mathbf{m}) \quad \text{in} \quad \mathcal{T} \\
\Leftrightarrow \textbf{(i)} \text{ there exists } L \text{ such that } (\varrho_n, S_n, \mathbf{m}_n) \in \mathcal{T}_L \text{ for all } n = 1, 2, \dots \\
\textbf{(ii)} \ d_{\mathcal{T}}\left[(\varrho_n, S_n, \mathbf{m}_n); (\varrho, S, \mathbf{m})\right] \to 0.
\end{aligned}
$$

Our choice of the topology of the trajectory space may seem a bit awkward at the first glance but accommodates the instantaneous convergence of the state variables. Alternatively, a weaker L^p topology can be used being equivalent on the attractor $\mathcal{A}$ to $d_{\mathcal{T}}$.

5.2. Attractor

As the weak solutions are not (known to be) uniquely determined by the initial/boundary data, we adopt the approach of Sell [35] and Málek and Nečas [27] replacing the standard phase space by the trajectory space $\mathcal{T}$. Here and hereafter, we always assume that the principal hypotheses of Theorem 3.1 concerning the constitutive relations are satisfied. Moreover, we fix the total mass of the fluid,

$$\int_\Omega \varrho(t,\cdot)\ \mathrm{d}x = M > 0. \tag{5.3}$$

Accordingly, the set $\mathcal{A}$,

$$\begin{aligned}\mathcal{A} = \Big\{ (\varrho, S, \mathbf{m}) \ \Big| \ & (\varrho, S, \mathbf{m}) \text{ a weak solution of the Navier–Stokes–Fourier system} \\ & \text{in the sense of Definition 2.1 on the time interval } t \in R \\ & \text{and } \sup_{t\in R} \int_\Omega E(\varrho, S, \mathbf{m})(t,\cdot)\ \mathrm{d}x < \infty \Big\},\end{aligned} \tag{5.4}$$

is a natural candidate to be *global attractor* in the trajectory space $\mathcal{T}$. As shown in Theorem 3.1,

$$\sup_{t\in R} \int_\Omega E(\varrho, S, \mathbf{m})(t,\cdot)\ \mathrm{d}x \leqq \mathcal{E}_\infty < \infty. \tag{5.5}$$

Indeed, in accordance with the definition of the set $\mathcal{A}$, there exists $\mathcal{E}_0$ such that

$$\int_\Omega E(\varrho, S, \mathbf{m})(T,\cdot)\ \mathrm{d}x \leqq \mathcal{E}_0$$

for *any* $T \in R$. Thus (5.5) follows from Theorem 3.1, specifically (3.2). In particular, $\mathcal{A} \subset \mathcal{T}_L \subset \mathcal{T}$ for a sufficiently large L.

Lemma 5.1. *Under the hypotheses of Theorem 3.1, the set $\mathcal{A}$ is*

- *non-empty;*
- *time-shift invariant,*

$$(\varrho, S, \mathbf{m}) \in \mathcal{A} \Rightarrow (\varrho, S, \mathbf{m})(\cdot + T) \in \mathcal{A} \ \ \textit{for any} \ \ T \in R;$$

- *compact in the metric topology $(\mathcal{T}_L, d_{\mathcal{T}})$ for a sufficiently large L.*

Proof. As shown in [9, Theorem 4.2], the Navier–Stokes–Fourier system with the boundary conditions (1.5), (1.6) admits a global-in-time weak solution $(\varrho, \vartheta, \mathbf{u})$ on the time interval $[0, \infty)$ for any initial data with finite energy. It follows from Theorem 3.4 that there exists a sequence of times $T_n \to \infty$ such that $((\varrho, \vartheta, \mathbf{u})(\cdot + T_n))_{n=1}^{\infty}$ converge to a weak solution $(\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}})$ of the same problem defined for all $t \in R$ with globally bounded energy. Obviously,

$$(\varrho, S, \mathbf{m}) = (\tilde{\varrho}, \tilde{\varrho} s(\tilde{\varrho}, \tilde{\vartheta}), \tilde{\varrho}\tilde{\mathbf{u}}) \in \mathcal{A}.$$

Moreover, as the underlying system is autonomous, the set $\mathcal{A}$ is time-shift invariant.

Compactness of the set $\mathcal{A}$ follows again from Theorem 3.4, where we consider $T_n = -\infty$. As pointed out, $\mathcal{A} \subset \mathcal{T}_L$, where the latter is a metric space; whence compactness is equivalent to sequential compactness. At the level of the density, convergence in the metric $d_{\mathcal{T}}$ follows from (3.3). Moreover, since the momenta $\varrho\mathbf{u}$ satisfy equation (2.6), they are precompact in the topology of the space

$$C_{\text{weak,loc}}(R; L^{\frac{5}{4}}(\Omega; R^3)),$$

which implies compactness in the momentum component of $\mathcal{T}_L$ with the $d_{\mathcal{T}}$ metrics.

Thus it remains to show compactness at the level of the total entropy S. First observe that

$$\tau \mapsto \int_\Omega S(\tau, \cdot)\phi \ \mathrm{d}x - \int_0^\tau \int_\Omega \varrho s(\varrho, \vartheta)\mathbf{u} \cdot \nabla_x \phi \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega \frac{\kappa(\vartheta)}{\vartheta} \cdot \nabla_x \phi \ \mathrm{d}x \, \mathrm{d}t$$

is a non-decreasing function of τ for any test function $\phi \in C_c^1(\Omega)$, $\phi \geqq 0$. In view of boundedness of the total energy, the family

$$\tau \mapsto \int_0^\tau \int_\Omega \varrho s(\varrho, \vartheta)\mathbf{u} \cdot \nabla_x \phi \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega \frac{\kappa(\vartheta)}{\vartheta} \cdot \nabla_x \phi \ \mathrm{d}x \, \mathrm{d}t$$

is precompact in $C_{\text{loc}}(R)$; whence precompactness of S in $d_{\mathcal{T}}$ reduces to precompactness of a *non-decreasing (in time)* sequence of functions

$$\begin{aligned}\tau \mapsto \langle \widetilde{S}_n; \phi \rangle \equiv \textstyle\int_\Omega S_n(\tau, \cdot)\phi \ \mathrm{d}x - \int_0^\tau \int_\Omega \varrho_n s(\varrho_n, \vartheta_n)\mathbf{u}_n \cdot \nabla_x \phi \ \mathrm{d}x \, \mathrm{d}t \\ + \textstyle\int_0^\tau \int_\Omega \frac{\kappa(\vartheta_n)}{\vartheta_n} \cdot \nabla_x \phi \ \mathrm{d}x \, \mathrm{d}t\end{aligned}$$

with respect to the metrics d of the Skorokhod space $D[-N, N]$ of càglàd functions defined on compact time intervals $[-N, N]$. To this end, we recall the criterion due to Whitt [38, Chapter 12, Corollary 12.5.1].

Let

$$h_n : [-N, N] \to R$$

be a sequence of monotone functions.

Then

$$d[h_n; h] \to 0 \quad \text{as} \quad n \to \infty \quad \text{for some} \quad h \in D[-N, N]$$

if and only if

$$h_n(t) \to h(t) \quad \text{for all} \quad t \text{ belonging to a dense set in } [-N, N] \\ \text{including the end points} - N \text{ and } N.$$

Now, as S_n are the total entropies generated by a family of uniformly bounded weak solutions of the Navier–Stokes–Fourier system, we have

$$\begin{aligned} \langle \widetilde{S}_n(\tau, \cdot); \phi \rangle \to \langle \widetilde{S}(\tau, \cdot); \phi \rangle \\ \equiv \int_\Omega S(\tau, \cdot)\phi \ \mathrm{d}x - \int_0^\tau \int_\Omega \varrho s(\varrho, \vartheta)\mathbf{u} \cdot \nabla_x \phi \ \mathrm{d}x \, \mathrm{d}t \\ + \int_0^\tau \int_\Omega \frac{\kappa(\vartheta)}{\vartheta} \cdot \nabla_x \phi \ \mathrm{d}x \, \mathrm{d}t \end{aligned}$$

for a.a. $\tau \in R$ at least for a suitable subsequence, where $(\varrho, S = \varrho s(\varrho, \vartheta), \mathbf{m} = \varrho \mathbf{u}) \in \mathcal{A}$, is another entire solution of the same problem. In particular,

$\langle \widetilde{S}_n(\tau, \cdot); \phi \rangle \to \langle \widetilde{S}(\tau, \cdot); \phi \rangle$ for a dense set of times for any compact interval $[-N, N]$,

whence, in accordance with the above convergence criterion,

$$\langle \widetilde{S}_n(\tau, \cdot); \phi \rangle \to \langle \widetilde{S}(\tau, \cdot); \phi \rangle \quad \text{in} \quad D[-N, N] \quad \text{for a.a.} \quad N > 0,$$

yielding the desired conclusion

$$\langle S_n(\tau, \cdot); \phi \rangle \to \langle S(\tau, \cdot); \phi \rangle \quad \text{in} \quad D[-N, N] \quad \text{for a.a.} \quad N > 0.$$

□

In accordance with Theorems 3.1, 3.4, and Lemma 5.1, we may state our main result concerning the existence of a trajectory attractor for the Rayleigh–Bénard problem.

Theorem 5.2. (Trajectory attractor) *Let $M > 0$, $\mathcal{E}_0$ be given. Let $\mathcal{F}[M, \mathcal{E}_0]$ be a family of weak solutions to the Rayleigh–Bénard problem for the Navier–Stokes–Fourier system on the time interval $(0, \infty)$ satisfying*

$$\int_\Omega \varrho \ \mathrm{d}x = M, \ \operatorname{ess} \limsup_{\tau \to 0+} \int_\Omega E(\varrho, S, \mathbf{m})(\tau, \cdot) \ \mathrm{d}x \leqq \mathcal{E}_0.$$

We identify the set $\mathcal{F}[M, \mathcal{E}_0]$ with a subset of the trajectory space $\mathcal{T}$ extending

$$\varrho(\tau, \cdot) = \lim_{t \to 0+} \varrho(t, \cdot), \quad \mathbf{m}(\tau, \cdot) = \lim_{t \to 0+} \mathbf{m}(t, \cdot), \quad 0 \leqq S(\tau, \cdot) \leqq \lim_{t \to 0+} S(t, \cdot) \ \textit{ for } \ \tau < 0,$$

where the limits are understood in the weak (distributional) sense.

Then for any $\varepsilon > 0$, there exists a time $T(\varepsilon)$ such that

$$d_{\mathcal{T}}[(\varrho, S, \mathbf{m})(\cdot + T); \mathcal{A}] < \varepsilon \textit{ for any } (\varrho, S, \mathbf{m}) \in \mathcal{F}[M, \mathcal{E}_0] \textit{ and any } T > T(\varepsilon)$$

5.3. Stationary statistical solutions

Following the ideas of the preceding section we are ready to identify *statistical solutions* with shift invariant probability measures on $\mathcal{T}$, which are supported by solutions to the Navier–Stokes–Fourier system. In accordance with Theorem 5.2, these shift invariant probability measures supported by the global trajectory attractor $\mathcal{A}$.

The construction of a bounded invariant measure is the same as in [14], note that a similar approach in the incompressible setting was used by Foias et al. [23,24]. Given a trajectory $(\varrho, S, \mathbf{m}) \in \mathcal{T}$ we consider a probability measure

$$\mathcal{V}_T \equiv \frac{1}{T} \int_0^T \delta_{(\varrho, S, \mathbf{m})(\cdot + t)} \, \mathrm{d}t,$$

where δ denotes the Dirac mass. Obviously, $\mathcal{V}_T$ is a probability measure. If, in addition,

$$(\varrho, S, \mathbf{m}) \in \mathcal{A},$$

then $\mathcal{V}_T \in \mathfrak{P}(\mathcal{A})$, where $\mathfrak{P}(\mathcal{A})$ denotes the set of all probablity measures on a compact Polish space $\mathcal{A}$. In particular, the family

$$\{\mathcal{V}_T\}_{T \geqq 0} \text{ is tight.}$$

By Prokhorov theorem, there is a sequence $T_n \to \infty$ such that

$$\mathcal{V}_{T_n} \to \mathcal{V} \text{ narowly in } \mathfrak{P}(\mathcal{A}).$$

Finally, exactly as in [14, Section 5.1], we may show that the measure $\mathcal{V}$ is time-shift invariant, meaning

$$\mathcal{V}[\mathfrak{B}(\cdot + T)] = \mathcal{V}[\mathfrak{B}] \text{ for any Borel set } \mathfrak{B} \subset \mathcal{T}. \tag{5.6}$$

A Borel probability measure $\mathcal{V} \in \mathfrak{P}(\mathcal{A})$ enjoying the property (5.6) is called *statistical stationary solution* of the Rayleigh–Bénard problem for the Navier–Stokes–Fourier system.

Finally, observe that the above construction may be restricted to any shift-invariant subset $\mathcal{U} \subset \mathcal{A}$.

Theorem 5.3. *Let $\mathcal{U} \subset \mathcal{A}$ be a non-empty time-shift invariant set, meaning*

$$(\varrho, S, \mathbf{m}) \in \mathcal{U} \Rightarrow (\varrho, S, \mathbf{m})(\cdot + T) \in \mathcal{U} \text{ for any } T \in R.$$

Then there exists a stationary statistical solution $\mathcal{V}$ supported by $\overline{\mathcal{U}}$:

- *$\mathcal{V}$ is a Borel probability measure, $\mathcal{V} \in \mathfrak{P}(\overline{\mathcal{U}})$;*
- supp $\mathcal{V} \subset \overline{\mathcal{U}}$, *where the closure of a $\mathcal{U}$ is a compact invariant set;*
- *$\mathcal{V}$ is shift invariant, that is, $\mathcal{V}[\mathfrak{B}] = \mathcal{V}[\mathfrak{B}(\cdot + T)]$ for any Borel set $\mathfrak{B} \subset \mathcal{T}$ and any $T \in R$.*

5.4. Convergence of ergodic means

We conclude this section by a direct application of Birkhoff–Khinchin ergodic theorem. Similarly to [14, Section 5], we may consider the state space

$$H = W^{-k,2}(\Omega) \times W^{-k,2}(\Omega) \times W^{-k,2}(\Omega; R^3).$$

Let $\mathcal{V} \in \mathfrak{P}(\mathcal{T})$ be a statistical stationary solution, and thus a Borel probability measure. Consider a probability basis $(\mathcal{T}, \mathfrak{B}[\mathcal{T}], \mathcal{V})$ where $\mathcal{T}$ is the trajectory space and $\mathfrak{B}[\mathcal{T}]$ is the family of Borel sets. Given a trajectory $(\varrho, S, \mathbf{m}) \in \mathcal{T}$ and $\tau \in R$ we may consider the associated canonical process

$$(\varrho, S, \mathbf{m}) \times \tau \mapsto (\varrho, S, \mathbf{m})(\tau, \cdot) \in H.$$

As $\mathcal{V}$ is shift invariant, the above process is a stationary process with respect to the probability basis $(\mathcal{T}, \mathfrak{B}[\mathcal{T}], \mathcal{V})$ defined for $\tau \in R$, with càglàd paths ranging in H.

Exactly as in [14, Theorem 6.4, Section 6] we can establish the following result.

Theorem 5.4. (Convergence of ergodic averages) *Let $\mathcal{V}$ be a stationary statistical solution and $(\varrho, S, \mathbf{m})$ the associated stationary process. Let $F : H \to R$ be a Borel measurable function such that*

$$\int_{\mathcal{T}} |F(\varrho(0,\cdot), S(0,\cdot), \mathbf{m}(0,\cdot)|\, \mathrm{d}\mathcal{V} < \infty.$$

Then there exists a measurable function $\overline{F}$,

$$\overline{F} : (\mathcal{T}, \mathcal{V}) \to R$$

such that

$$\frac{1}{T}\int_0^T F(\varrho(t,\cdot), S(t,\cdot), \mathbf{m}(t,\cdot))\,\mathrm{d}t \to \overline{F} \quad as \quad T \to \infty$$

$\mathcal{V}$−a.s. and in $L^1(\mathcal{T}, \mathcal{V})$.

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