Oscillation and Concentration in Sequences of PDE Constrained Measures

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Abstract

We show that for constant rank partial differential operators $\mathscr{A}$ whose wave cones are spanning, generalized Young measures generated by bounded sequences of $\mathscr{A}$-free measures can be characterized by duality with $\mathscr{A}$-quasiconvex integrands of linear growth. This includes a characterization of the concentration effects in such sequences that allows us to conclude that, in sharp contrast to the oscillation effects, the concentration always has $\mathscr{A}$-free structure.

1. Introduction

We investigate the oscillation and concentration effects in weakly* converging sequences of vector measures that satisfy a system of pde constraints on a bounded open subset of $\mathbb{R}^n$. The pde constraints are given in terms of linear homogeneous differential operators with constant coefficients on finite dimensional inner product spaces $\mathbb{V}$, $\mathbb{W}$

$$\mathscr{A} = \mathscr{A}(\partial) \equiv \sum_{|\alpha|=k} A_\alpha \partial^\alpha, \quad A_\alpha \in \mathscr{L}(\mathbb{V}, \mathbb{W}) \tag{1.1}$$

that we in the main parts of the paper will assume have *constant rank*,

$$\operatorname{rank}\big(\mathscr{A}(\xi)\big) = r_{\mathscr{A}} \quad \text{for} \ \ \xi \in \mathbb{R}^n \backslash \{0\}, \tag{1.2}$$

and *spanning wave cone*,

$$\operatorname{span}\big(\Lambda_{\mathscr{A}}\big) = \mathbb{V}, \quad \text{where} \ \ \Lambda_{\mathscr{A}} = \bigcup_{\xi \in \mathbb{S}^{n-1}} \ker \mathscr{A}(\xi). \tag{1.3}$$

These conditions are standard in the theory of compensated compactness [18,21, 38,51], as far as nonlinear integral functionals acting on spaces of maps satisfying linear pde constraints are concerned.

The purpose of the present paper is threefold. We provide an abstract characterization in the spirit of Kinderlehrer and Pedregal [27,28] of $\mathscr{A}$-free generalized Young measures by duality with $\mathscr{A}$-quasiconvex integrands of linear growth via Jensen-type inequalities as in [4,12,29,34,45] under the assumptions that the operator $\mathscr{A}$ has constant rank and that its wave cone is spanning. For the terminology used in the statements below we refer to Section 2. On the other hand, we hope that our relatively simple approach might have a revitalizing effect on the literature: also when restricting our proof to the basic gradient case, the argument is significantly shorter and more streamlined than the existing ones. This is even so when comparing with proofs in the literature confined to the case without concentration effects (including [18,28,30,31,48] without concentration effects and [4,6,12,16,34,45] with concentration effects). We illustrate the utility of our approach with a result about generation of Young measures by pde constrained maps in cases that include general differential operators $\mathscr{A}$ that do not have constant rank nor spanning wave cone. The third purpose of this paper concerns the concentration effects in an $\mathscr{A}$-free sequence. It is known that the oscillation effects in the considered case can fail completely to have $\mathscr{A}$-free structure [1], [31, Ex. 7.6]. We were therefore surprised to find that the situation for concentration effects is entirely different since, as we show, they always have $\mathscr{A}$-free structure. We recently established this result in the simpler case of gradient Young measures corresponding to $\mathscr{A} = \mathrm{curl}$ [32] and our results here can be seen as a far reaching generalization. Recall that the situation is very different for sequences converging weakly in L^p, where it is well-known that for exponents $1 < p < \infty$ one may separate concentration and oscillation effects and show that each have $\mathscr{A}$-free structure (see [19,28,30,31] for the case $\mathscr{A} = \mathrm{curl}$ and [16,20] for the general case).

In order to make the discussion more precise let us state our main results. As mentioned already the reader should consult Section 2 for undefined notation.

Proposition 1.1. *Let $\mathscr{A}$ be a linear homogeneous differential operator of order k on $\mathbb{R}^n$ from $\mathbb{V}$ to $\mathbb{W}$ as in (1.1). Assume it has constant rank (1.2) and satisfies the spanning cone condition (1.3). Let Ω be a bounded open subset of $\mathbb{R}^n$, $v \in \mathscr{M}(\Omega, \mathbb{V})$ and (v_j) a sequence in $\mathscr{M}(\Omega, \mathbb{V})$ satisfying $v_j \overset{*}{\rightharpoonup} v$ in $\mathscr{M}(\Omega, \mathbb{V})$, $\mathscr{A} v_j \to \mathscr{A} v$ in $\mathrm{W}^{-k,p}_{\mathrm{loc}}(\Omega, \mathbb{W})$ for some $p > 1$ and (v_j) generates $\nu = (\nu_x, \lambda, \nu_x^\infty)$. Let $\lambda = \lambda^a \mathscr{L}^n \llcorner \Omega + \lambda^s$ be the Lebesgue–Radon–Nikodým decomposition with respect to $\mathscr{L}^n$. Then there exists an $\mathscr{L}^n$ negligible subset N^a of Ω such that*

$$\int_{\mathbb{V}} f \, d\nu_x + \lambda^a(x) \int_{\mathbb{S}_{\mathbb{V}}} f^\infty \, d\nu_x^\infty \geqq f\left(\overline{v}_x + \lambda^a(x)\overline{v}_x^\infty\right) \tag{1.4}$$

holds for all $x \in \Omega \backslash N^a$ and all $f : \mathbb{V} \to \mathbb{R}$ that are $\mathscr{A}$-quasiconvex and of linear growth. Furthermore, if $\lambda^s = \lambda^s / |v|^s |v|^s + \lambda^*$ is the Lebesgue–Radon–Nikodým decomposition with respect to $|v|^s$, then

$$\begin{cases} \overline{v}_x^\infty = \dfrac{v^s}{|v|^s}(x) \in \Lambda_{\mathscr{A}} \cap \mathbb{S}_{\mathbb{V}} & \text{for} \quad |v|^s \text{ almost all } x, \\ \overline{v}_x^\infty = 0 & \text{for} \quad \lambda^* \llcorner \Omega \text{ almost all } x. \end{cases} \tag{1.5}$$

The Jensen inequality (1.4) was established by ARROYO-RABASA et al. [5, Thm. 1.6] under additional assumptions. More precisely they confined attention to $\mathscr{A}$-quasiconvex integrands of linear growth that are bounded from below and they also assumed that $\mathscr{A}v = 0$. While the lower bound on the considered integrands might seem natural when discussing lower semicontinuity, and admittedly it has been a standard assumption in the literature, it is essential that we allow truly signed integrands here when we seek conditions that allow us to characterize the possible concentration effects of a sequence. We prove (1.4) for signed integrands by use of an approximation result similar to that found in [29, Sect. 6] that concerned the classical case $\mathscr{A} = \mathrm{curl}$. The constraint (1.5) on the centres of mass for the concentration angle measures is a remarkable result proved in [11]. That our result also applies when the limit v is not $\mathscr{A}$-free follows from general facts about constant rank differential operators and more precisely that for such operators and exponents $p \in (1, \infty)$ one can prove the closed-range inequality:

$$\inf_{\psi \in C_c^\infty(\mathbb{R}^n, \mathbb{V}), \, \mathscr{A}\psi = 0} \left\| \phi - \psi \right\|_p \leqq c_p \left\| \mathscr{A}\phi \right\|_{W^{-k,p}} \tag{1.6}$$

holds for all $\phi \in L^p(\mathbb{R}^n, \mathbb{V})$. Note in particular that we take infimum over $\mathscr{A}$-free $C_c^\infty(\mathbb{R}^n, \mathbb{V})$ fields on the left hand side. This inequality follows if we combine the local potentials constructed in [42,43] with an inequality due to MURAT [38]. Namely, if $P_{\mathscr{A}} \colon L^2(\mathbb{R}^n, \mathbb{V}) \to L^2(\mathbb{R}^n, \mathbb{V})$ denotes orthogonal projection onto the kernel $\ker \mathscr{A} \equiv \{\phi \in L^2(\mathbb{R}^n, \mathbb{V}) \colon \mathscr{A}\phi = 0\}$, then $P_{\mathscr{A}}$ extends by continuity to $L^p(\mathbb{R}^n, \mathbb{V})$ for each $p \in (1, \infty)$ and

$$\left\| \phi - P_{\mathscr{A}}\phi \right\|_p \leqq c_p \left\| \mathscr{A}\phi \right\|_{W^{-k,p}} \tag{1.7}$$

holds for all $\phi \in L^p(\mathbb{R}^n, \mathbb{V})$. Murat proved this inequality using Fourier multipliers and the Hörmander–Mihlin theorem when the operator $\mathscr{A}$ has order $k = 1$, but the same argument applies to the case of general order $k \in \mathbb{N}$. The inequality (1.7) is also central in the work by FONSECA and MÜLLER [18], where it is adapted and stated for periodic fields. We remark that it was recently observed in [22] that validity of (1.7) for a $p \in (1, \infty)$ in fact is equivalent to the constant rank condition (1.2) for $\mathscr{A}$. While (1.6) is proved by use of (1.7) it is possible that the former could hold even if $\mathscr{A}$ did not have constant rank. However, as observed in [22], (1.6) is for $p = 2$ equivalent to the constant rank condition (1.2), and we believe this remains true for all exponents in the range $p \in (1, \infty)$. We present an almost self-contained, short proof of Proposition 1.1 in Section 4. A brief discussion of the above inequalities (1.6) and (1.7) can be found in the second half of Section 3.

Observe that (1.5) together with the automatic convexity result of [29] yield the Jensen inequality

$$\int_{\mathbb{S}_{\mathbb{V}}} h \, d\nu_x^\infty \geqq h\left(\overline{\nu}_x^\infty\right) \tag{1.8}$$

for all positively 1-homogeneous and $\Lambda_{\mathscr{A}}$-convex $h \colon \mathbb{V} \to \mathbb{R}$ and all $x \in \Omega \setminus N^s$, where $\lambda^s(N^s) = 0$. Here we may in particular take $h = f^\infty$, the upper recession integrand of any $\mathscr{A}$-quasiconvex integrand $f \colon \mathbb{V} \to \mathbb{R}$ of linear growth. General

results about Young measures [3,33,34,46] and the Jensen inequalities (1.4), (1.8) then imply lower semicontinuity results in a routine manner.

The next result goes in the opposite direction in that it starts with a Jensen type inequality for a Young measure and then asserts the existence of a generating sequence of certain $\mathscr{A}$-free fields. We emphasize that we allow exactly the same differential operators $\mathscr{A}$ to which Proposition 1.1 applies. The result extends [29,34] from the classical curl-free setting to the setting of differential operators (1.1) and amounts to a characterization of $\mathscr{A}$-freeness at the level of Young measures in the spirit of KINDERLEHRER and PEDREGAL [27,28].

Theorem 1.2. *Let $\mathscr{A}$ be a linear homogeneous differential operator of order k on $\mathbb{R}^n$ from $\mathbb{V}$ to $\mathbb{W}$ as in (1.1). Assume it has constant rank (1.2) and satisfies the spanning cone condition (1.3). Then there exists a linear homogeneous differential operator $\mathscr{B}$ of order $l \leqq 2kr_{\mathscr{A}}$ on $\mathbb{R}^n$ from $\mathbb{V}$ to $\mathbb{V}$ that is annihilated by $\mathscr{A}$ and that has the following property. Let Ω be an open bounded subset of $\mathbb{R}^n$ and assume that $\nu = (\nu_x, \lambda, \nu_x^\infty)$ is a Young measure on Ω such that $\lambda(\partial\Omega) = 0$. Let $v \in \mathscr{M}(\Omega, \mathbb{V})$ be its barycentre and let $\lambda = \lambda^a \mathscr{L}^n \llcorner \Omega + \lambda^s$ be the Lebesgue–Radon–Nikodým decomposition of λ with respect to $\mathscr{L}^n$.*

Suppose $\mathscr{A} v = 0$ and that

$$\int_{\mathbb{V}} f \mathrm{d}\nu_x + \lambda^a(x) \int_{\mathbb{S}_{\mathbb{V}}} f^\infty \mathrm{d}\nu_x^\infty \geqq f(\overline{\nu}_x + \lambda^a(x)\overline{\nu}_x^\infty) \ \ for$$
$$\mathscr{L}^n \llcorner \Omega \, almost \, all \, x \qquad (1.9)$$

holds for all $\mathscr{A}$-quasiconvex $f : \mathbb{V} \to \mathbb{R}$ of linear growth for which one can find $r = r(f) > 0$ such that $f(z) = f^\infty(z)$ when $|z| > r$.

Then there exists a sequence (u_j) of maps in $\mathrm{C}_c^\infty(\Omega, \mathbb{V})$ such that for any sequence of mollifiers $\phi_j \overset{*}{\rightharpoonup} \delta_0$ in $\mathscr{M}(\mathbb{R}^n)$ we have

$$\begin{cases} (\phi_j * (v \llcorner \Omega) + \mathscr{B}u_j) \ generates \ \nu \\ \|u_j\|_{\mathrm{W}^{l-1,1}} \to 0 \end{cases}$$

We consider our main new contribution here to be the proof, that is, the construction of the generating sequence, rather than the actual result itself. This construction has two steps. We first deal with the case of homogeneous Young measures, and as was the case in the KINDERLEHRER and PEDREGAL papers [27,28], this step relies on a Dacorogna type formula for the $\mathscr{A}$-quasiconvex envelope and an abstract argument based on the Hahn–Banach separation theorem. While this strategy for the homogeneous measures is well-established we believe our flexible and streamlined implementation might be of independent interest. In the second step the homogeneous Young measures are merged, or inhomogenized, by an approximation and gluing procedure. Because we treat both oscillation and concentration effects that in the considered situation necessarily must interact and that cannot be separated, this procedure is quite delicate. The inhomogenization is the main technical novelty of this work and is presented in Section 5.3. We emphasize that it is completely measure theoretic and that it in particular does not rely on any fine structure properties of $\mathscr{A}$-free measures. The use of these is confined to the homogeneous step

that is presented in Section 5.2. Thus no understanding of the blow up limits at singular points of $\mathscr{A}$-free measures is necessary to perform the approximation and gluing procedure. This sets our approach apart from those in the literature, including [4,12,34,45], where the arguments are arguably also more involved. We obtained the result of Theorem 1.2 corresponding to the case when the barycentre v has no singular part in [41, Ch. 3]. As we show here, the strategy used there is flexible enough to also deal with the interaction between the singular parts of the barycentre and concentration measures, as we implement in Lemmas 5.6 and 5.7. We record that a similar result is discussed in the recent paper [4]; there, the Helmholtz-type decomposition from [18] plays a key role, whereas our approach here is more elementary and has a larger scope, cf. Propositions 1.4 and 7.3. Our use of local potentials as in [42,43] ensures that we may work with compactly supported $\mathscr{A}$-free test fields.

In Section 6 we will focus on the concentration angle measures in the diffuse part of the concentration, a subject which has been largely unexplored. We recently discussed the problem in the basic case of gradient Young measures [32] and our intention here is to carry the analysis through in the wider framework of constant rank differential operators with spanning wave cone. Recall that as a consequence of the main results in [11,29], the measures v_x^∞ are unconstrained for λ^s almost every x. Here, we will present a new necessary Jensen-type inequality that v_x^∞ satisfies for $\lambda^a \mathscr{L}^n$ almost every x if v is $\mathscr{A}$-free. In fact, we will show that this inequality, together with a natural condition on the barycenter, constitute a characterization of the concentration part of an $\mathscr{A}$-free Young measure. The latter, stated in Proposition 6.4, is new even for the basic gradient case $\mathscr{A} = \mathrm{curl}$, where it also adds precision to the result of [32]. As a consequence of this new Jensen-type inequality we establish the surprising facts that for a λ null set $N \subset \Omega$,

$$\int_{\mathbb{S}^\mathbb{V}} f^\infty \mathrm{d}v_x^\infty \geqq f^\infty(\bar{v}_x^\infty) \quad \text{for} \quad x \in \Omega \backslash N$$

holds for $\mathscr{A}$-quasiconvex f of linear growth, and that the concentration part of an $\mathscr{A}$-free measure is $\mathscr{A}$-free, a feature that so far was only believed to be true in the reflexive case $p \in (1, \infty)$. We highlight the main result in this direction:

Theorem 1.3. *Let $\mathscr{A}$, $\mathscr{B}$ be differential operators as in Theorem 1.2, let Ω be an open bounded subset of $\mathbb{R}^n$ and $p \in (1, \frac{n}{n-1})$ an exponent. Assume $v_j \overset{*}{\rightharpoonup} v$ in $\mathscr{M}(\Omega, \mathbb{V})$, $\mathscr{A}v_j \to \mathscr{A}v$ in $\mathrm{W}_{\mathrm{loc}}^{-k,p}(\Omega, \mathbb{W})$ and that (v_j) generates the Young measure $v = (v_x, \lambda, v_x^\infty)$. Then there exists a sequence (u_j) of maps in $\mathrm{C}_c^\infty(\Omega, \mathbb{V})$ such that for any sequence of mollifiers $\phi_j \overset{*}{\rightharpoonup} \delta_0$ in $\mathscr{M}(\mathbb{R}^n)$ we have that*

$$\begin{cases} \left(\phi_j * (v \llcorner \Omega) + \mathscr{B}u_j\right) \text{ generates } \left(\delta_{\bar{v}_x}, \lambda, v_x^\infty\right) \\ \|u_j\|_{\mathrm{W}^{l-1,1}} \to 0. \end{cases}$$

We finish the paper with an abstract result that applies to general differential operators (1.1). The main motivation is to illustrate the flexibility of our method.

Proposition 1.4. *Let $\mathscr{A}$ be a linear homogeneous differential operator of order k on $\mathbb{R}^n$ from $\mathbb{V}$ to $\mathbb{W}$ as in (1.1) (possibly having non-constant rank and a non-spanning wave cone). Let Ω be an open bounded subset of $\mathbb{R}^n$ and assume that $v = (v_x, \lambda, v_x^\infty)$ is a Young measure on Ω. Let $v \in \mathscr{M}(\Omega, \mathbb{V})$ be its barycentre, let $v = v^a + v^s$ and $\lambda = \lambda^a \mathscr{L}^n \llcorner \Omega + \lambda^s$ be the Lebesgue–Radon–Nikodým decompositions with respect to $\mathscr{L}^n$.*

Suppose $\mathscr{A} v = 0$ and that

$$\int_{\mathbb{V}} f \mathrm{d}v_x + \lambda^a(x) \int_{\mathbb{S}_{\mathbb{V}}} f^\infty \mathrm{d}v_x^\infty \geqq Qf(\overline{v}_x + \lambda^a(x)\overline{v}_x^\infty)$$

$$\text{for} \quad \mathscr{L}^n \llcorner \Omega \text{ almost all } x \qquad (1.10)$$

holds for all Lipschitz integrands $f : \mathbb{V} \to \mathbb{R}$, where

$$Qf(z) \equiv \inf \left\{ \int_{\mathbb{X}} f(z + \phi(x)) \, \mathrm{d}x : \phi \in \mathrm{C}_c^\infty(\mathbb{X}, \mathbb{V}) \text{ and } \mathscr{A}\phi = 0 \right\}$$

$$\text{with} \quad \mathbb{X} \equiv \left(-\tfrac{1}{2}, \tfrac{1}{2}\right)^n. \qquad (1.11)$$

Then there exists a sequence (v_j) of maps in $\mathrm{C}_c^\infty(\Omega, \mathbb{V})$ with $\mathscr{A} v_j = 0$ in Ω such that for any sequence of mollifiers $\phi_j \overset{*}{\rightharpoonup} \delta_0$ in $\mathscr{M}(\mathbb{R}^n)$ we have

$$\left(\phi_j * \left(v^a \llcorner \Omega\right) + v_j\right) \text{ generates } \left((v_x)_{x\in\Omega}, \lambda^a \mathscr{L}^n \llcorner \Omega, (v_x^\infty)_{x\in\Omega^a}\right),$$

where $\Omega^a \equiv \{x \in \Omega : \lambda^a(x) > 0\}$. In fact, the $\mathscr{A}$-free fields v_j can be realized as $\mathscr{B}\psi_j$, where ψ_j are smooth and compactly supported and $\mathscr{B}$ is a certain differential operator constructed from $\mathscr{A}$.

The paper is organized as follows: In Section 2 we recall some properties of the Kantorovich norm, generalized Young measures, linear partial differential operators, and quasiconvex and directionally convex integrands. In Section 3 we use results of HÖRMANDER [26] to construct vector potentials for $\mathscr{A}$-free fields in C_c^∞ for general differential operators $\mathscr{A}$ that do not necessarily satisfy the constant rank condition. These results generalize and add precision to results of Malgrange. The second part of Section 3 is devoted to a proof of a Dacorogna type formula for the $\mathscr{A}$-quasiconvex envelope. In Section 4 we prove the Jensen inequality of Proposition 1.1 and discuss the spanning cone condition. In Section 5.2 we prove Theorem 1.2 in the case of homogeneous Young measures, whereas in Section 5.3 we perform the crucial approximation argument. In Section 6 we prove Theorem 1.3 concerning $\mathscr{A}$-freeness of concentration effects and state and prove the related characterization of the concentration part of an $\mathscr{A}$-free Young measure. In the final Section 7 we discuss possibilities and challenges beyond the constant rank condition and briefly sketch the proof of Proposition 1.4 concerning general differential constraints.

2. Preliminaries

We use standard notation for measures, distributions and function spaces as can be found in, for instance, [7,10,17,25,46]. We also follow the convention that (unimportant) constants can change values within a string of estimates without this being reflected in our notation. Below we highlight particularly important notation and adapt background results used in our proofs.

2.1. Basic Notation

Throughout the paper $\mathbb{V}$, $\mathbb{W}$ denote finite dimensional vector spaces over $\mathbb{R}$ equipped with an inner product and the associated norm denoted $v_1 \cdot v_2$ and $|v| = \sqrt{v \cdot v}$, respectively, where the ambient space will be clear from context. The distance between two subsets S, T of $\mathbb{V}$ is $\mathrm{dist}(S, T) \equiv \inf\{|s - t| : s \in S, t \in T\}$ (understood as ∞ if one of the sets is empty) and when $S = \{s\}$ we write $\mathrm{dist}(s, T) \equiv \mathrm{dist}(\{s\}, T)$. The open ball in $\mathbb{V}$ of center $x \in \mathbb{V}$ and radius $r > 0$ is $B_r(x)$, the unit sphere in $\mathbb{V}$ is $\mathbb{S}_{\mathbb{V}}$ and we write $B_r(S)$ for the r-metric neighbourhood of a subset S of $\mathbb{V}$, thus $B_r(S) \equiv \{v \in \mathbb{V} : \mathrm{dist}(x, S) < r\}$. The closure of the set S is denoted by $\overline{S}$. The space of linear maps $T : \mathbb{V} \to \mathbb{W}$ is denoted by $\mathscr{L}(\mathbb{V}, \mathbb{W})$ and equipped with the usual operator norm (again denoted by $|T|$).

For an open subset Ω of $\mathbb{R}^n$ we denote by $\mathrm{C}_c^\infty(\Omega, \mathbb{V})$ the space of $\mathbb{V}$-valued test maps on Ω, namely the C^∞ maps $\phi : \Omega \to \mathbb{V}$ whose support $\mathrm{supp}(\phi)$ is compact and contained in Ω. The space of $\mathbb{V}$-valued distributions on Ω is denoted by $\mathscr{D}'(\Omega, \mathbb{V})$. The Lebesgue space relative to Lebesgue measure $\mathscr{L}^n$ of (equivalence classes of) p-integrable $\mathbb{V}$-valued maps on Ω (a bounded open subset of $\mathbb{R}^n$ or the whole of $\mathbb{R}^n$) is denoted by $\mathrm{L}^p(\Omega, \mathbb{V})$ and its norm by $\|\cdot\|_{p,\Omega}$ or $\|\cdot\|_p$ when the set Ω is clear from context. The Sobolev space of $\mathbb{V}$-valued maps on Ω whose distributional derivatives up to order $k \in \mathbb{N}$ are in L^p is denoted $\mathrm{W}^{k,p}(\Omega, \mathbb{V})$ and endowed with the norm $\|u\|_{\mathrm{W}^{k,p}} \equiv \left(\sum_{|\alpha| \leq k} \|\partial^\alpha u\|_p^p\right)^{1/p}$ (where standard multi-index notation is used and with the usual modification for $p = \infty$). The subspace $\mathrm{W}_0^{k,p}(\Omega, \mathbb{V})$ is defined as the closure of the space of test maps, $\mathrm{C}_c^\infty(\Omega, \mathbb{V})$, in $\mathrm{W}^{k,p}(\Omega, \mathbb{V})$ when $p < \infty$ (and the weak* closure when $p = \infty$). We usually think of L^p and $\mathrm{W}^{k,p}$ maps in terms of their precise representatives. Negative order Sobolev spaces are defined by duality: for $k \in \mathbb{N}$ and $p \in (1, \infty)$ we let $\mathrm{W}^{-k,p}(\Omega, \mathbb{V}) \equiv \mathrm{W}_0^{k,p'}(\Omega, \mathbb{V})^*$, where $p' = p/(p-1)$. Accordingly the $\mathrm{W}^{-k,p}$ norm is the dual norm of the $\mathrm{W}^{k,p'}$ norm:

$$\|u\|_{\mathrm{W}^{-k,p}} \equiv \sup_{\varphi \in \mathrm{C}_c^\infty(\Omega, \mathbb{V}), \|\varphi\|_{\mathrm{W}^{k,p'}} \leq 1} \langle u, \varphi \rangle.$$

It is well-known that its elements are those $\mathbb{V}$-valued distributions on Ω that can be represented as a sum of distributional derivatives of $\mathbb{V}$-valued L^p maps, and more precisely as $\sum_{|\alpha| \leq k} \partial^\alpha f_\alpha$, where each $f_\alpha \in \mathrm{L}^p(\Omega, \mathbb{V})$. The local variants of these spaces are defined in the usual way. For instance, $u \in \mathrm{W}_{\mathrm{loc}}^{-k,p}(\Omega, \mathbb{V})$ provided $\rho u \in \mathrm{W}^{-k,p}(\mathbb{R}^n, \mathbb{V})$ for each $\rho \in \mathrm{C}_c^\infty(\Omega)$, or, what is the same, if the restriction of the distribution $u|_{\Omega'} \in \mathrm{W}^{-k,p}(\Omega', \mathbb{V})$ for each $\Omega' \Subset \Omega$.

A **bounded $\mathbb{V}$-valued Radon measure** on Ω is a countably additive $\mathbb{V}$-valued set function defined on the Borel subsets of Ω. A bounded $\mathbb{V}$-valued Radon measure u on Ω has a finite total variation and its total variation measure, denoted $|u|$, is a bounded positive Radon measure. The space of bounded $\mathbb{V}$-valued Radon measures on Ω is denoted $\mathscr{M}(\Omega, \mathbb{V})$ and is normed by total variation. It is well-known that hereby $\mathscr{M}(\Omega, \mathbb{V})$ is isometrically isomorphic to the dual space of $C_0(\Omega, \mathbb{V})$, the space of continuous $\mathbb{V}$-valued maps on Ω that vanish on the boundary $\partial\Omega$ (at ∞ if $\Omega = \mathbb{R}^n$) equipped with the sup-norm. In addition to weak* convergence of a sequence (u_j) of $\mathbb{V}$-valued measures we shall also use $\langle\cdot\rangle$-strict convergence on a few occasions: $u_j \to u \; \langle\cdot\rangle$ strictly on Ω means here that $u_j \overset{*}{\rightharpoonup} u$ in $C_0(\Omega, \mathbb{V})^*$ and $\langle u_j\rangle(\Omega) \to \langle u\rangle(\Omega)$, where $\langle u\rangle$ is the total variation measure of the $\mathbb{R} \oplus \mathbb{V}$-valued measure $(\mathscr{L}^n, u)$. Recall that $\langle z\rangle = \sqrt{1 + |z|^2}$ and that hereby $\langle u\rangle = \langle u^a\rangle\mathscr{L}^n\llcorner\Omega + |u|^s$, where $u = u^a\mathscr{L}^n\llcorner\Omega + (u/|u|^s)|u|^s$ is the Lebesgue–Radon–Nikodým decomposition with respect to $\mathscr{L}^n$. The Radon–Nikodým derivative $u/|u|^s$ is occasionally also denoted $\mathrm{d}u/\mathrm{d}|u|^s$.

When u is a bounded $\mathbb{V}$-valued Radon measure on Ω and B is a Borel subset of Ω, then the measure $u\llcorner B$ is defined for all Borel subsets A of $\mathbb{R}^n$ by $(u\llcorner B)(A) \equiv u(A \cap B)$. Hereby $u\llcorner B$ is again a bounded $\mathbb{V}$-valued Radon measure on $\mathbb{R}^n$.

2.2. Kantorovich Norm

Let (X, d_X) be a compact and separable metric space. We recall that $C(X)$ with the supremum norm is a separable Banach space whose dual can be isometrically identified with the space $\mathscr{M}(X)$ of signed bounded Radon measures on X normed by the total variation. The subspace $\mathrm{LIP}(X)$ consisting of all Lipschitz functions $\Phi\colon X \to \mathbb{R}$ is a (non-separable) Banach space under the norm

$$\|\Phi\|_{\mathrm{LIP}} \equiv \sup_{x\in X}|\Phi(x)| + \mathrm{lip}(\Phi), \quad \mathrm{lip}(\Phi) \equiv \sup_{x,y\in X,\, x\neq y} \frac{|\Phi(x) - \Phi(y)|}{d_X(x, y)}.$$

The Kantorovich norm of a signed bounded Radon measure μ on X is here defined as

$$\|\mu\|_{\mathrm{K}} \equiv \sup\Big\{\langle\mu, \Phi\rangle : \Phi \in \mathrm{LIP}(X),\ \|\Phi\|_{\mathrm{LIP}} \leqq 1\Big\},$$

so it is the dual norm of $\|\cdot\|_{\mathrm{LIP}}$ restricted to $\mathscr{M}(X)$. We shall be interested in its restriction to the space $\mathscr{M}^+(X)$ of positive bounded Radon measures that becomes a metric space under the Kantorovich metric $d_{\mathrm{K}}(\mu, \nu) = \|\mu - \nu\|_{\mathrm{K}}$. We start with the useful fact that for $\mu \in \mathscr{M}^+(X)$

$$\|\mu\|_{\mathrm{K}} = \mu(X). \tag{2.1}$$

Proof. Since $\mathbf{1}_X \in \mathrm{LIP}(X)$ and $\|\mathbf{1}_X\|_{\mathrm{LIP}} = 1$ we clearly have $\|\mu\|_{\mathrm{K}} \geqq \mu(X)$. Conversely, if $\Phi \in \mathrm{LIP}(X)$ with $\|\Phi\|_{\mathrm{LIP}} \leqq 1$, then in particular $\max_X |\Phi| \leqq 1$ so using that μ is a positive measure we get

$$\langle\mu, \Phi\rangle \leqq \langle\mu, \mathbf{1}_X\rangle = \mu(X),$$

hence taking supremum over such Φ we arrive at the opposite inequality. $\square$

As a subset of $C(X)^*$ the space $\mathscr{M}^+(X)$ also inherits the weak* topology:

$$\left\{ \mathcal{O} \cap \mathscr{M}^+(X) : \mathcal{O} \in \sigma\big(C(X)^*, C(X)\big) \right\}.$$

Lemma 2.1. *On $\mathscr{M}^+(X)$ the relative weak* topology is exactly the topology determined by the Kantorovich metric d_K.*

Proof. In order to show that $\mathscr{M}^+(X) \cap \mathcal{O}$ is open relative to the Kantorovich metric for each weak* open $\mathcal{O}$ it suffices to show that for each $\Phi \in C(X)$ and $t \in \mathbb{R}$ the set

$$\left\{ \mu \in \mathscr{M}^+(X) : \langle \mu, \Phi \rangle < t \right\} \tag{2.2}$$

is open relative to the Kantorovich metric on $\mathscr{M}^+(X)$. To that end we fix $\mu_0 \in \mathscr{M}^+(X)$ with $\langle \mu_0, \Phi \rangle < t$. Next, we employ a standard approximation scheme and put for each $j \in \mathbb{N}$,

$$\Phi_j(x) = \sup\left\{ \Phi(y) - j d_X(x, y) : y \in X \right\}.$$

Hereby $\mathrm{lip}(\Phi_j) \leqq j$ and $\Phi_j(x) \searrow \Phi(x)$ pointwise in $x \in X$ (hence uniformly by Dini) as $j \nearrow \infty$. Select $j \in \mathbb{N}$ so

$$\langle \mu_0, \Phi_j \rangle < \frac{t + \langle \mu_0, \Phi \rangle}{2}.$$

If we take any positive $r < (t - \langle \mu_0, \Phi \rangle)/2j$, then we have for each $\mu \in \mathscr{M}^+(X)$ with $\|\mu - \mu_0\|_K < r$ that

$$\langle \mu, \Phi \rangle \leqq \langle \mu, \Phi_j \rangle$$
$$\leqq \langle \mu - \mu_0, \Phi_j \rangle + \frac{t + \langle \mu_0, \Phi \rangle}{2}$$
$$< r.$$

It follows that the set defined at (2.2) is open relative to d_K.

For the opposite inclusion we fix $\mu_0 \in \mathscr{M}^+(X)$ and $r > 0$ and must show that the open ball $\left\{ \mu \in \mathscr{M}^+(X) : \|\mu - \mu_0\|_K < r \right\}$ is weak* open relative to $\mathscr{M}^+(X)$. To get started we note that by the Arzéla–Ascoli Theorem the set $\mathbb{S} = \left\{ \Phi : \|\Phi\|_{\mathrm{LIP}} = 1 \right\}$ is totally bounded in $C(X)$. Hence for each $\delta > 0$ we can find a finite δ-net Δ in $\mathbb{S}$. Without loss in generality we may assume that $\mathbf{1}_X \in \Delta$. Now put, for a $t \in \mathbb{R}$ to be specified,

$$H_\Delta = \left\{ \mu \in \mathscr{M}^+(X) : \max_{\Phi \in \Delta} \langle \mu - \mu_0, \Phi \rangle < t \right\}$$

and note that for $\mu \in H_\Delta$ and $\Psi \in \mathbb{S}$ we have

$$\langle \mu - \mu_0, \Psi \rangle = \min_{\Phi \in \Delta}\left(\langle \mu - \mu_0, \Phi \rangle + \langle \mu - \mu_0, \Psi - \Phi \rangle \right)$$
$$< t + \big(\mu(X) + \mu_0(X)\big)\delta.$$

In order to bound the last term we use that $\mathbf{1}_X \in \Delta$. It entails that $\mu(X) < t + \mu_0(X)$ and consequently

$$\langle \mu - \mu_0, \Psi \rangle < t + (2\mu_0(X) + t)\delta.$$

We leave it to the reader to check that we may choose

$$t = \frac{r}{2} \text{ and then any } \delta \in \left(0, \frac{r}{4\mu_0(X) + r}\right)$$

to complete the proof. $\square$

2.3. Linear Partial Differential Operators

We will work with linear homogeneous partial differential operators of order k on $\mathbb{R}^n$ from $\mathbb{V}$ to $\mathbb{W}$ (two finite-dimensional real inner product spaces):

$$\mathscr{A} = \mathscr{A}(\partial) \equiv \sum_{|\alpha|=k} A_\alpha \partial^\alpha, \tag{2.3}$$

where the coefficients are linear maps $A_\alpha \in \mathscr{L}(\mathbb{V}, \mathbb{W})$ for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = k$. Its Fourier symbol is

$$\mathscr{A}(\xi) \equiv \sum_{|\alpha|=k} \xi^\alpha A_\alpha,$$

and so for each $\xi \in \mathbb{R}^n$ we have that $\mathscr{A}(\xi) \in \mathscr{L}(\mathbb{V}, \mathbb{W})$. We say that $\mathscr{A}$ has **constant rank** when rank $\mathscr{A}(\xi) = r_\mathscr{A}$ holds for all $\xi \in \mathbb{R}^n\backslash\{0\}$ for some constant $r_\mathscr{A} \in \mathbb{N}_0$. Moreover, it turns out this assumption is equivalent to the existence of a linear homogeneous differential operator of order l on $\mathbb{R}^n$ from $\mathbb{V}$ to $\mathbb{V}$, say

$$\mathscr{B} = \mathscr{B}(\partial) \equiv \sum_{|\beta|=l} B_\beta \partial^\beta, \tag{2.4}$$

where $B_\beta \in \mathscr{L}(\mathbb{V}, \mathbb{V})$ for $\beta \in \mathbb{N}_0^n$ with $|\beta| = l$ and the order $l \leqq 2kr_\mathscr{A}$, such that we have the exactness relation at the level of Fourier symbols:

$$\ker \mathscr{A}(\xi) = \operatorname{im} \mathscr{B}(\xi) \quad \text{for} \quad \xi \in \mathbb{R}^n\backslash\{0\}, \tag{2.5}$$

see [43, Theorem 1] (and also [42] for a precursor). While the existence of such $\mathscr{B}$ is assured by this result, it is not in general unique and we emphasize that our results apply with any choice of operator $\mathscr{B}$ whose Fourier symbol satisfies (2.5).

We also recall the definition of the **wave cone of** $\mathscr{A}$,

$$\Lambda_\mathscr{A} \equiv \bigcup_{\xi \in \mathbb{S}^{n-1}} \ker \mathscr{A}(\xi).$$

This is the set of vector amplitudes where the operator $\mathscr{A}$ fails to be elliptic [38,51]. We record the following Leibniz rule for $u \in \mathscr{D}'(\Omega, \mathbb{V})$, $\eta \in C_c^\infty(\Omega)$:

$$\mathscr{A}(\eta u) = \sum_{|\alpha|\leqq k} \frac{1}{\alpha!} \partial^\alpha \eta \mathscr{A}^{(\alpha)} u, \tag{2.6}$$

where $\mathscr{A}^{(\alpha)}$ is the differential operator corresponding to the Fourier symbol $\mathscr{A}^{(\alpha)}(\xi) \equiv \partial_\xi^\alpha \mathscr{A}(\xi)$. Our core object of study will be $\mathscr{A}$-free measures, for which we record the following structure theorem of DE PHILIPPIS and RINDLER [11], a remarkable generalization of Alberti's rank-one theorem [2]:

Theorem 2.2. *Let $\mathscr{A}$ be a linear homogeneous differential operator of order k on $\mathbb{R}^n$ from $\mathbb{V}$ to $\mathbb{W}$ as in (1.1). If $v \in \mathscr{M}(\Omega, \mathbb{V})$ satisfies $\mathscr{A}v = 0$ in the sense of distributions, then*

$$\frac{\mathrm{d}v}{\mathrm{d}|v|^s} \in \Lambda_\mathscr{A} \quad |v|^s \text{ almost everywhere.}$$

2.4. Young Measures

We follow the approach of [3,14] as implemented in [34]. We start by fixing some terminology. We refer to any real-valued function defined on $\mathbb{V}$, $\Omega \times \mathbb{V}$ or $\overline{\Omega} \times \mathbb{V}$ as an **integrand**. We say the integrand $f \colon \overline{\Omega} \times \mathbb{V} \to \mathbb{R}$ has **linear growth** if there exists a constant $c \in \mathbb{R}$ such that $|f(x, z)| \leqq c(|z| + 1)$ holds for all $(x, z) \in \overline{\Omega} \times \mathbb{V}$. The space of continuous integrands $f \colon \overline{\Omega} \times \mathbb{V} \to \mathbb{R}$ of linear growth is denoted by $C_1(\overline{\Omega} \times \mathbb{V})$ and normed by

$$\sup_{(x,z) \in \overline{\Omega} \times \mathbb{V}} \frac{|f(x, z)|}{|z| + 1}.$$

An integrand $\Phi \in C_1(\overline{\Omega} \times \mathbb{V})$ is an **admissible integrand** if it admits a **regular recession integrand**, meaning that the limit

$$\Phi^\infty(x, z) \equiv \lim_{t \to \infty} \frac{\Phi(x, tz)}{t}$$

exists in $\mathbb{R}$ uniformly in $(x, z) \in \overline{\Omega} \times \mathbb{S}_\mathbb{V}$. The space of admissible integrands is denoted by $\mathbb{E}(\Omega, \mathbb{V})$. For the proof of existence and representation of Young measures given in [3,34] it was convenient to transform integrands on $\overline{\Omega} \times \mathbb{V}$ to functions defined on $\overline{\Omega} \times \mathbb{B}_\mathbb{V}$, where $\mathbb{B}_\mathbb{V}$ denotes the open unit ball in $\mathbb{V}$. Define the transformation $T \colon C_1(\overline{\Omega} \times \mathbb{V}) \to BC(\overline{\Omega} \times \mathbb{B}_\mathbb{V})$ by

$$(T\Phi)(x, \hat{z}) \equiv (1 - |\hat{z}|)\Phi\left(x, \frac{\hat{z}}{1 - |\hat{z}|}\right), \quad (x, \hat{z}) \in \overline{\Omega} \times \mathbb{B}_\mathbb{V}.$$

Here $BC(\overline{\Omega} \times \mathbb{B}_\mathbb{V})$ is the space of bounded continuous functions, and when it is equipped with the sup-norm, it follows that T becomes an isometric isomorphism. We can now rephase the definition of admissible integrand by declaring that a continuous integrand of linear growth, $\Phi \colon \overline{\Omega} \times \mathbb{V} \to \mathbb{R}$, belongs to $\mathbb{E}(\Omega, \mathbb{V})$ precisely if the transformed integrand $T\Phi \in BUC(\overline{\Omega} \times \mathbb{B}_\mathbb{V})$, the space of bounded and uniformly continuous functions. These are of course exactly the functions on $\overline{\Omega} \times \mathbb{B}_\mathbb{V}$ that admit a continuous (real-valued) extension to the closure $\overline{\Omega} \times \overline{\mathbb{B}_\mathbb{V}}$, where the extension of $T\Phi$ then is given by Φ^∞ on $\overline{\Omega} \times \mathbb{S}_\mathbb{V}$. By a slight abuse of notation we therefore have an isometric isomorphism $T \colon \mathbb{E}(\Omega, \mathbb{V}) \to C(\overline{\Omega} \times \overline{\mathbb{B}_\mathbb{V}})$.

Its dual transformation is by general results then also an isometric isomorphism $T^*\colon \mathscr{M}(\overline{\Omega \times \mathbb{B}_{\mathbb{V}}}) \to \mathbb{E}(\Omega, \mathbb{V})^*$ providing a convenient identification of the dual space $\mathbb{E}(\Omega, \mathbb{V})^*$ with $\mathscr{M}(\overline{\Omega \times \mathbb{B}_{\mathbb{V}}})$, the space of (signed) Radon measures on $\overline{\Omega \times \mathbb{B}_{\mathbb{V}}}$. Well-known compactness results then apply to norm bounded sequences in $\mathbb{E}(\Omega, \mathbb{V})^*$ that we employ as follows. A bounded $\mathbb{V}$-valued Radon measure $v \in \mathscr{M}(\Omega, \mathbb{V})$ is identified with $\boldsymbol{\varepsilon}_v \in \mathbb{E}(\Omega, \mathbb{V})^*$ according to the rule

$$
\begin{aligned}
\langle \boldsymbol{\varepsilon}_v, \Phi \rangle_{\mathbb{E}^*, \mathbb{E}} \equiv & \int_{\Omega} \Phi(x, v^a(x)) \mathrm{d}x \\
& + \int_{\Omega} \Phi^{\infty}\left(x, \frac{\mathrm{d}v^s}{\mathrm{d}|v^s|}(x)\right) \mathrm{d}|v^s|(x) \quad \text{for} \quad \Phi \in \mathbb{E}(\Omega, \mathbb{V}),
\end{aligned}
$$

where $|v^s|$ denotes the total variation measure and $v = v^a \mathscr{L}^n \llcorner \Omega + v^s$ is the Lebesgue–Radon–Nikodým decomposition of the measure v. We refer to $\boldsymbol{\varepsilon}_v$ as an **elementary Young measure** and it is clear that hereby $v \mapsto \boldsymbol{\varepsilon}_v$ is an embedding of $\mathscr{M}(\Omega, \mathbb{V})$ into $\mathbb{E}(\Omega, \mathbb{V})^*$. If (v_j) is a sequence in $\mathscr{M}(\Omega, \mathbb{V})$ such that $\boldsymbol{\varepsilon}_{v_j} \overset{*}{\rightharpoonup} \nu$ in $\mathbb{E}(\Omega, \mathbb{V})^*$, then we write $v_j \overset{Y}{\to} \nu$ and say that (v_j) **generates the Young measure** ν. The Young measure ν will in general not be elementary, but according to [3] (see also [34] whose notation we follow) its image $\left(T^*\right)^{-1}\nu$ belongs to the set

$$
\bigcap_{\varphi \in \mathrm{C}(\overline{\Omega})} \left\{ m \in \mathscr{M}^+(\overline{\Omega \times \mathbb{B}_{\mathbb{V}}}) : \int_{\overline{\Omega \times \mathbb{B}_{\mathbb{V}}}} \varphi(x)\left(1 - |\hat{z}|\right) \mathrm{d}m(x, \hat{z}) = \int_{\Omega} \varphi \, \mathrm{d}x \right\}. \tag{2.7}
$$

This is a weak* closed convex subset of $\mathscr{M}(\overline{\Omega \times \mathbb{B}_{\mathbb{V}}})$, where the relative weak* topology is determined by the Kantorovich metric discussed in Section 2.2, so that the Young measure generation $v_j \overset{Y}{\to} \nu$ amounts to

$$
\|\boldsymbol{\varepsilon}_{v_j} - \nu\|_{\mathrm{K}} \equiv \|\left(T^*\right)^{-1}\left(\boldsymbol{\varepsilon}_{v_j} - \nu\right)\|_{\mathrm{K}} \to 0.
$$

Lemma 2.3. (ALIBERT and BOUCHITTÉ [3]) *Any $\nu \in \mathbb{E}(\Omega, \mathbb{V})^*$ such that $\left(T^*\right)^{-1}\nu$ belongs to the set (2.7) can be identified uniquely with a triple $\left((\nu_x)_{x \in \Omega}, \lambda, (\nu_x^{\infty})_{x \in \overline{\Omega}}\right)$, where*

(1) *$\nu_x \in \mathscr{M}_1^+(\mathbb{V})$ for $\mathscr{L}^n$-almost every $x \in \Omega$ and $x \mapsto \nu_x$ is weakly* $\mathscr{L}^n$ measurable;*
(2) *$\lambda \in \mathscr{M}^+(\overline{\Omega})$;*
(3) *$\nu_x^{\infty} \in \mathscr{M}_1^+(\mathbb{S}_{\mathbb{V}})$ for λ-almost every $x \in \overline{\Omega}$ and $x \mapsto \nu_x$ is weakly* λ measurable;*
(4) *$\int_{\Omega} \int_{\mathbb{V}} |z| \mathrm{d}\nu_x(z) \mathrm{d}x < \infty$;*

and where

$$
\begin{aligned}
\langle \nu, \Phi \rangle_{\mathbb{E}^*, \mathbb{E}} \equiv & \int_{\Omega} \int_{\mathbb{V}} \Phi(x, z) \mathrm{d}\nu_x(z) \mathrm{d}x \\
& + \int_{\overline{\Omega}} \int_{\mathbb{S}_{\mathbb{V}}} \Phi^{\infty}(x, z) \mathrm{d}\nu_x^{\infty}(z) \mathrm{d}\lambda(x) \quad \text{for} \quad \Phi \in \mathbb{E}(\Omega, \mathbb{V}).
\end{aligned}
$$

Strictly speaking this is not the statement given in [3], but the proof given there easily adapts to give it (see [34]). In view of this result we write $\nu = \big((\nu_x)_{x\in\Omega}, \lambda, (\nu_x^\infty)_{x\in\overline{\Omega}}\big)$, or briefly $\nu = (\nu_x, \lambda, \nu_x^\infty)$ when the set Ω is clear from context. Following [34] we say that $(\nu_x)_{x\in\Omega}$ is the **oscillation measure**, λ is the **concentration measure**, and $(\nu_x^\infty)_{x\in\overline{\Omega}}$ is the **concentration-angle measure**. Note that the conditions (1)–(4) entail that the centres of mass

$$\overline{\nu}_x \equiv \int_{\mathbb{V}} z \, d\nu_x(z) \quad \text{and} \quad \overline{\nu}_x^\infty \equiv \int_{\mathbb{S}_\mathbb{V}} z \, d\nu_x^\infty(z)$$

are well-defined for $\mathscr{L}^n$ almost all $x \in \Omega$ and λ almost all $x \in \overline{\Omega}$, respectively. Furthermore, $x \mapsto \overline{\nu}_x$ is a map in $\mathrm{L}^1(\Omega, \mathbb{V})$ and $x \mapsto \overline{\nu}_x^\infty$ is a λ measurable map valued in the closed unit ball $\overline{\mathbb{B}}_\mathbb{V}$. The **barycenter** of ν is the $\mathbb{V}$-valued Radon measure $\overline{\nu} \equiv \overline{\nu}.\mathscr{L}^n \llcorner \Omega + \overline{\nu}.^\infty \lambda$. We record that when ν is generated by the sequence (ν_j) of $\mathbb{V}$-valued Radon measures on Ω, then it follows in particular that $\nu_j \overset{*}{\rightharpoonup} \overline{\nu} \llcorner \Omega$ in $\mathscr{M}(\Omega, \mathbb{V})$ since we may test the Young measure generation with the integrands $(x, z) \mapsto \eta(x) z \cdot e_i$, where $\eta \in \mathrm{C}_0(\Omega)$ and (e_i) is an orthonormal basis for $\mathbb{V}$. We also have for $\nu_j^a = \nu_j/\mathscr{L}^n$ that $\nu_j^a \to \overline{\nu}.$ in the biting sense (see [3]) and that the Young measure ν is elementary, that is, $\nu = \varepsilon_v$ for some $v \in \mathscr{M}(\Omega, \mathbb{V})$, if and only if $\nu_j \to \nu$ in the $\langle\cdot\rangle$-strict sense (see [33,34]). We refer to [3,34,46] for further discussion and examples.

For later reference we record the following consequence of the Stone–Weierstrass theorem:

Lemma 2.4. *Suppose that $\mu \in \mathscr{M}(\overline{\Omega} \times \overline{\mathbb{B}}_\mathbb{V})$ is such that $\langle\mu, \eta \otimes \Psi\rangle = 0$ for all $\eta \in \mathrm{LIP}(\overline{\Omega})$ and $\Psi \in \mathrm{LIP}(\overline{\mathbb{B}}_\mathbb{V})$. Then $\mu \equiv 0$. Consequently, to identify an element in $\mathbb{E}(\Omega, \mathbb{V})^*$, it suffices to test only with $\eta \otimes \Phi$, where $\|\eta\|_{\mathrm{LIP}(\Omega)} \leqq 1$ and $\|T\Phi\|_{\mathrm{LIP}(\mathbb{B}_\mathbb{V})} \leqq 1$.*

We extend the notion of (upper) **recession integrand** to general continuous integrands $f : \mathbb{V} \to \mathbb{R}$ of linear growth with the definition

$$f^\infty(z) = \limsup_{\substack{t\to\infty \\ z'\to z}} \frac{f(tz')}{t} \quad (z \in \mathbb{V}) \tag{2.8}$$

Hereby $f^\infty : \mathbb{V} \to \mathbb{R}$ is a positively 1-homogeneous integrand of linear growth. It is Lipschitz when f is, and in that case, the formula simplifies to

$$f^\infty(z) = \limsup_{t\to\infty} \frac{f(tz)}{t} \quad (z \in \mathbb{V}).$$

2.5. Directionally Convex and $\mathscr{A}$-Quasiconvex Integrands

Let Λ be a **balanced cone** in $\mathbb{V}$, meaning that $tv \in \Lambda$ when $v \in \Lambda$ and $t \in \mathbb{R}$. An integrand $f : \mathbb{V} \to \mathbb{R}$ is Λ-**convex** provided that for each $z \in \mathbb{V}$ and $v \in \Lambda$ the univariate function

$$\mathbb{R} \ni t \mapsto f(z + tv)$$

is convex. We say that the balanced cone Λ is **spanning** if $\mathrm{span}(\Lambda) = \mathbb{V}$. It is well-known that real-valued Λ-convex integrands are locally Lipschitz when Λ is spanning, see [37, Theorem 4.4.1] (or [10] and [29, Lemma 2.3] for a slightly more precise bound). We summarize these observations and include an elementary consequence of the three-slope inequality that can also be found in [29, Lemma 2.5] in the next result.

Lemma 2.5. *Let Λ be a balanced and spanning cone in $\mathbb{V}$. If $f : \mathbb{V} \to \mathbb{R}$ is Λ-convex, then f is locally Lipschitz. Furthermore, if f is also of linear growth, then f is Lipschitz and for all $z \in \mathbb{V}$ and $w \in \Lambda$ we have*

$$f(z + w) \leqq f(z) + f^{\infty}(w).$$

For an invertible linear map of $\mathbb{R}^n$, $\theta \in \mathrm{GL}(n)$, we denote by $\theta \mathbb{Z}^n$ the deformed integer lattice and say that a map $v : \mathbb{R}^n \to \mathbb{V}$ is $\theta \mathbb{Z}^n$-periodic if $v(x + \theta e_j) = v(x)$ holds for all $x \in \mathbb{R}^n$ and each direction $1 \leqq j \leqq n$. Here $(e_j)_{j=1}^n$ is the standard basis for $\mathbb{R}^n$.

Definition 2.6. Let $\mathscr{A}$ be a linear homogeneous differential operator of order k on $\mathbb{R}^n$ from $\mathbb{V}$ to $\mathbb{W}$ as in (1.1). A continuous integrand $f : \mathbb{V} \to \mathbb{R}$ is said to be $\mathscr{A}$-**quasiconvex** at $z \in \mathbb{V}$ if

$$f(z) \leqq \fint_{\theta \mathbb{X}} f(z + v(x)) \, \mathrm{d}x$$

holds for all $\theta \in \mathrm{GL}(n)$ and all $v : \mathbb{R}^n \to \mathbb{V}$ of class C^∞ that are $\theta \mathbb{Z}^n$-periodic, $\int_{\theta \mathbb{X}} v \, \mathrm{d}x = 0$ and $\mathscr{A}v = 0$. Here $\mathbb{X}$ denotes the open unit cube $(-\frac{1}{2}, \frac{1}{2})^n$. The integrand f is said to be $\mathscr{A}$-quasiconvex if it is so at every point $z \in \mathbb{V}$.

We will show that for a large class of operators $\mathscr{A}$ this notion coincides with the one defined in [18, Def. 3.1] that only requires that the above Jensen inequality holds when θ is the identity map of $\mathbb{R}^n$. The standard argument based on the Riemann–Lebesgue lemma (see [37, Proof of Theorem 4.4.2]) shows that $\mathscr{A}$-quasiconvexity of f is a necessary condition for the following lower semicontinuity property: if $\phi_j, \phi : \mathbb{X} \to \mathbb{V}$ are $\mathscr{A}$-free and $\phi_j \overset{*}{\rightharpoonup} \phi$ in $\mathrm{L}^\infty(\mathbb{X}, \mathbb{V})$, then

$$\liminf_{j \to \infty} \int_{\mathbb{X}} f(\phi_j) \, \mathrm{d}x \geqq \int_{\mathbb{X}} f(\phi) \, \mathrm{d}x.$$

The next result is therefore a reformulation of a well-known result due to Tartar:

Lemma 2.7. (TARTAR [51]) *If $q : \mathbb{V} \to \mathbb{R}$ is a quadratic form, then q is $\mathscr{A}$-quasiconvex if and only if it is $\Lambda_{\mathscr{A}}$-convex.*

Corollary 2.8. *If $f : \mathbb{V} \to \mathbb{R}$ is $\mathscr{A}$-quasiconvex, then it is also $\Lambda_{\mathscr{A}}$-convex. Hence if $\mathrm{span}(\Lambda_{\mathscr{A}}) = \mathbb{V}$, then $\mathscr{A}$-quasiconvex integrands are in particular locally Lipschitz.*

Proof. Let $\left(\rho_\varepsilon\right)_{\varepsilon>0}$ be a standard smooth mollifier. It is then routine to check that $f_\varepsilon = \rho_\varepsilon * f$ is $\mathscr{A}$-quasiconvex too, and hence for each $z \in \mathbb{V}$, by the second order necessary condition for a minimum, that

$$\int_{\mathbb{X}} f_\varepsilon''(z)\big(\phi(x), \phi(x)\big) \, \mathrm{d}x \geqq 0$$

holds for all $\mathbb{X}$-periodic $\phi \in \mathrm{C}^\infty(\mathbb{R}^n, \mathbb{V})$ with $\mathscr{A}\phi = 0$ and $\int_{\mathbb{X}} \phi \, \mathrm{d}x = 0$. The quadratic form $w \mapsto f_\varepsilon''(z)\big(w, w\big)$ is therefore by Tartar's lemma $\Lambda_{\mathscr{A}}$-convex, hence, by arbitrariness of z, f_ε is $\Lambda_{\mathscr{A}}$-convex. Finally we conclude that f is $\Lambda_{\mathscr{A}}$-convex as a pointwise limit of such integrands. □

We recall that an $\mathscr{A}$-quasiconvex integrand of linear growth, $f: \mathbb{V} \to \mathbb{R}$, might not admit a recession integrand in the usual sense of convex analysis (see [39]). In these situations f^∞ denotes the upper recession integrand that was defined at (2.8). It is routine to check that $f^\infty: \mathbb{V} \to \mathbb{R}$ hereby is an $\mathscr{A}$-quasiconvex integrand that is positively 1-homogeneous, meaning that $f^\infty(tz) = tf^\infty(z)$ holds for all $z \in \mathbb{V}$ and $t > 0$. Finally, we also record a consequence of the main result of [29]:

Lemma 2.9. *Let $f: \mathbb{V} \to \mathbb{R}$ be a positively 1-homogeneous and $\Lambda_{\mathscr{A}}$-convex integrand, where $\Lambda_{\mathscr{A}}$ is spanning. Then f is convex at each point of $\Lambda_{\mathscr{A}}$.*

3. Potential Operators and the $\mathscr{A}$-Quasiconvex Envelope

3.1. Differential Operators and Vector Potentials

Choosing orthonormal bases for $\mathbb{V}$ and $\mathbb{W}$ we may identify $\mathbb{V} = \mathbb{R}^N$ and $\mathbb{W} = \mathbb{R}^M$. Hence if $R = \mathbb{R}[\xi]$ denotes the ring of real polynomials in n indeterminates, then the Fourier symbol $\mathscr{A}(\xi) = \sum_{|\alpha|=k} A_\alpha \xi^\alpha$ becomes an $M \times N$ matrix with entries from R that naturally defines an R-linear map between free modules over R: $R^N \ni v(\xi) \mapsto \mathscr{A}(\xi)v(\xi) \in R^M$. Consider its kernel

$$\ker_R \mathscr{A}(\xi) \equiv \big\{v(\xi) \in R^N : \mathscr{A}(\xi)v(\xi) = 0\big\}.$$

This is clearly a submodule of R^N that is graded by the degree. Each $v(\xi) \in \ker_R \mathscr{A}(\xi)$ can be written $v(\xi) = \sum_{s=0}^d v_s(\xi)$, where $v_s(\xi) \in R^N$ is a column vector with s-homogeneous polynomials as entries. Now from $0 = \mathscr{A}(\xi)v(\xi)$ in R^M follows that $0 = \mathscr{A}(\xi)v(\xi)$ holds for all $\xi \in \mathbb{R}^n$, and because the entries in the matrix $\mathscr{A}(\xi)$ are k-homogeneous polynomials we get $\mathscr{A}(\xi)v_s(\xi) = 0$ for each $s \in \{0, \ldots, d\}$. In particular we record that

$$v_0 \in \bigcap_{|\alpha|=k} \ker A_\alpha. \tag{3.1}$$

The corresponding conditions for the higher degree terms in $v(\xi)$ are more involved and seem less useful at this stage. Instead we note that the module $\ker_R \mathscr{A}(\xi)$ is generated by column vectors of homogeneous polynomials. By Hilbert's basis

theorem R is a Noetherian ring so the module $\ker_R \mathscr{A}(\xi)$ is finitely generated (see for instance [15, Theorem 1.2 and Proposition 1.4]), say

$$\ker_R \mathscr{A}(\xi) = \left\{ \sum_{l=1}^{L} p_l(\xi) b_l(\xi) \colon p_1(\xi), \ldots, p_L(\xi) \in R \right\},$$

where any proper subcollection of the generators will not generate $\ker_R \mathscr{A}(\xi)$ and where each generator $b_l(\xi)$ is a column vector of real and homogeneous polynomials. Reordering the generators if necessary we can arrange that $b_l(\xi)$ is a column vector of k_l-homogeneous polynomials with $k_1 \leqq \cdots \leqq k_L$. Define $\mathscr{B}(\xi)$ to be the $N \times L$ matrix whose columns are the generators $b_l(\xi)$:

$$\mathscr{B}(\xi) \equiv \left[b_1(\xi) \ldots b_L(\xi) \right].$$

We record that the exactness relation $\ker_R \mathscr{A}(\xi) = \operatorname{im}_R \mathscr{B}(\xi)$ holds in the sense of R-modules, but that no assertion about pointwise exactness entails from this, see [43] and [24, Thm. 1.3]. In general we only have that $\operatorname{im} \mathscr{B}(\xi) \subseteq \ker \mathscr{A}(\xi)$ holds for each $\xi \neq 0$. Note that if we have another matrix $\mathscr{C}(\xi) \in R^{N \times K}$ such that $\ker_R \mathscr{A}(\xi) = \operatorname{im}_R \mathscr{C}(\xi)$ holds, then $\mathscr{C}(\xi) = \mathscr{B}(\xi) X(\xi)$ for some $X(\xi) \in R^{L \times K}$.

We now consider the corresponding differential operators. Corresponding to $\mathscr{B}(\xi)$ we have

$$\mathscr{B}\phi \equiv \mathscr{B}(\partial)\big(\phi_1, \ldots, \phi_L\big)^{\mathrm{tr}} = \sum_{l=1}^{L} b_l(\partial)\phi_l \tag{3.2}$$

We call it a potential operator for $\mathscr{A}$ since we may use Fourier transform to see that $\mathscr{A}\mathscr{B}\phi = 0$ holds for all $\mathbb{C}^L$-valued tempered distributions ϕ and hence for all $\phi \in \mathscr{D}'(\Omega, \mathbb{C}^L)$ by approximation. It is remarkable that $\mathscr{B}$ is in fact an exact potential operator for $\mathscr{A}$ on certain classes of distributions, namely flat R-modules, such as C_c^∞, $\mathscr{S}$, $\mathscr{E}'$; see [47] for a discussion and further references. Here we focus on C_c^∞ and sharpen the approach of Hörmander [26, Chapter 7]. Let $\varrho, \varphi \colon \mathbb{C}^n \to \mathbb{R}$ be two functions such that

$$\begin{cases} 0 < \varrho(\zeta) \leqq 1, \ \varrho(\xi + \zeta) \leqq 2\varrho(\zeta) \\ \quad \text{for} \ \ \xi, \zeta \in \mathbb{C}^n \ \text{satisfying} \ \big|\mathrm{Re}\xi_j\big|, \ \ \big|\mathrm{Im}\xi_j\big| \leqq 1 \ (1 \leqq j \leqq n) \\ -\log \varrho \ \text{is plurisubharmonic [26, Def. 2.6.1]} \end{cases} \tag{3.3}$$

and

$$\begin{cases} \big|\varphi(\xi + \zeta) - \varphi(\zeta)\big| \leqq c_0 \\ \quad \text{for} \ \ \xi, \zeta \in \mathbb{C}^n \ \text{satisfying} \ |\xi_j| < \varrho(\zeta) \ (1 \leqq j \leqq n) \\ \varphi \ \text{is plurisubharmonic,} \end{cases} \tag{3.4}$$

where c_0 is any positive constant. For such ϱ, φ and any $m \in \mathbb{Z}$ define

$$\varphi_m(\xi) \equiv \varphi(\xi) - m \log \varrho(\xi) + m \log\big(1 + |\xi|^2\big). \tag{3.5}$$

Theorem 3.1. [26, Corollary 7.6.12] *Given $\mathscr{A}(\xi)$, $\mathscr{B}(\xi)$ as above there exists $m \in \mathbb{Z}$ such that for ϱ, φ, φ_m as in (3.3), (3.4), (3.5) and all entire maps $\Phi \colon \mathbb{C}^n \to \mathbb{C}^N$ with $\mathscr{A}(\xi)\Phi(\xi) = 0$ on $\mathbb{C}^n$ one can find an entire map $\Psi \colon \mathbb{C}^n \to \mathbb{C}^L$ satisfying $\Phi(\xi) = \mathscr{B}(\xi)\Psi(\xi)$ on $\mathbb{C}^n$ and*

$$\int_{\mathbb{R}^{2n}} |g(\xi)|^2 e^{-\varphi_m(\xi)} \, \mathrm{d}(\xi', \xi'') \leqq c \int_{\mathbb{R}^{2n}} |f(\xi)|^2 e^{-\varphi(\xi)} \, \mathrm{d}(\xi', \xi''),$$

where $\xi = \xi' + \mathrm{i}\xi'' \in \mathbb{C}^n$ and c is a constant that only depends on $\mathscr{A}$, $\mathscr{B}$ and c_0 (the constant from (3.4)).

With this result at our disposal the next goal is to prove existence of rough vector potentials.

Lemma 3.2. *Given differential operators $\mathscr{A}$, $\mathscr{B}$ as above. Let $\phi \in \mathrm{C}_c^\infty(\mathbb{R}^n, \mathbb{V})$ with $\mathscr{A}\phi = 0$ and put $K = \mathrm{supp}(\phi)^{\mathrm{co}}$, the convex hull of the support of ϕ. Then there exists $\psi \in \mathscr{D}'(\mathbb{R}^n, \mathbb{R}^L)$ such that $\mathrm{supp}(\psi) \subseteq K$ and $\mathscr{B}\psi = \phi$.*

Proof. The proof relies on Theorem 3.1 and proceeds via a standard application of the Paley–Wiener–Schwartz theorem (in the form [25, Theorem 7.3.1]). First, the Fourier–Laplace transform defined as

$$\widehat{\phi}(\zeta) \equiv \int_{\mathbb{R}^n} \phi(x) e^{-\mathrm{i}\zeta \cdot x} \, \mathrm{d}x, \quad \zeta \in \mathbb{C}^n,$$

where $\zeta \cdot x \equiv \zeta_1 x_1 + \cdots + \zeta_n x_n$ (no complex conjugation) is an entire $\mathbb{V}$-valued map of exponential type. If

$$H_K(\xi) \equiv \sup_{\eta \in K} \eta \cdot \xi, \quad \xi \in \mathbb{R}^n,$$

is the support function for the set K, then for each $j \in \mathbb{N}$ we find according to the direct part of the Paley–Wiener–Schwartz theorem a constant $c_j > 0$ such that

$$\left| \widehat{\phi}(\zeta) \right| \leqq \frac{c_j}{\left(1 + |\zeta|\right)^j} e^{H_K(\mathrm{Im}\,\zeta)} \tag{3.6}$$

holds for all $\zeta \in \mathbb{C}^n$. We employ Theorem 3.1 with the choices $\varrho \equiv 1$, $\varphi(\zeta) \equiv 2H_K\big(\mathrm{Im}(\zeta)\big)$ and the Fourier symbols for $\mathscr{A}$, $\mathscr{B}$. It is clear that hereby (3.3) holds. Because H_K is convex and positively 1-homogeneous also (3.4) holds with $c_0 = \max_{\|\xi\|_{\ell_\infty^n} \leqq 1} H_K(\xi)$, and since clearly $\mathscr{A}(\mathrm{i}\zeta)\widehat{\phi}(\zeta) = 0$ we find accordingly an entire map $\Psi \colon \mathbb{C}^n \to \mathbb{C}^L$ such that $\widehat{\phi}(\zeta) = \mathscr{B}(\mathrm{i}\zeta)\Psi(\zeta)$ and

$$\int_{\mathbb{R}^{2n}} \left| \Psi(\zeta) \right|^2 \frac{e^{-2H_K(\mathrm{Im}(\zeta))}}{\left(1 + |\zeta|^2\right)^m} \, \mathrm{d}(\zeta', \zeta'') \leqq c \int_{\mathbb{R}^{2n}} \left| \widehat{\phi}(\zeta) \right|^2 e^{-2H_K(\mathrm{Im}(\zeta))} \, \mathrm{d}(\zeta', \zeta'')$$

$$\overset{(3.6)}{\leqq} c \int_{\mathbb{R}^{2n}} \frac{c_j^2}{\left(1 + |\zeta|\right)^{2j}} \, \mathrm{d}(\zeta', \zeta'') \equiv C_1 < \infty$$

provided we take $j > n$. Because Ψ is entire we estimate for a constant c, then Cauchy–Schwarz and the above bound

$$
\begin{aligned}
\left|\Psi(\zeta)\right| &\leqq c \int_{B_1(\zeta)} \left|\Psi(\xi)\right| \mathrm{d}(\xi', \xi'') \\
&\leqq c \left(\int_{B_1(\zeta)} \left|\Psi(\xi)\right|^2 \frac{\mathrm{e}^{-2H_K(\xi'')}}{\left(1 + |\xi|^2\right)^m} \mathrm{d}(\xi', \xi'') \right)^{\frac{1}{2}} \\
&\qquad \left(\int_{B_1(\zeta)} \left(1 + |\xi|^2\right)^m \mathrm{e}^{2H_K(\xi'')} \mathrm{d}(\xi', \xi'') \right)^{\frac{1}{2}} \\
&\leqq c C_1^{\frac{1}{2}} \left(\int_{B_1(\zeta)} \left(1 + |\xi|^2\right)^m \mathrm{e}^{2H_K(\xi'')} \mathrm{d}(\xi', \xi'') \right)^{\frac{1}{2}}
\end{aligned}
$$

Here we have for $\xi = \xi' + \mathrm{i}\xi'' \in B_1(\zeta)$, $\left(1 + |\xi|^2\right)^m \leqq 4^m \left(1 + |\zeta|^2\right)^m$ and $H_K(\xi'') \leqq H_K(\zeta'') + c_K$, where $c_K = \max_{\eta \in K} |\eta|$ is the Lipschitz constant for H_K. Consequently we have for some new constant c that

$$
\left|\Psi(\zeta)\right| \leqq c\left(1 + |\zeta|\right)^m \mathrm{e}^{H_K(\mathrm{Im}(\zeta))} \tag{3.7}
$$

holds for all $\zeta \in \mathbb{C}^n$. But then the Paley–Wiener–Schwartz theorem yields $\psi \in \mathscr{D}'(\mathbb{R}^n, \mathbb{C}^L)$ supported in K of order at most m such that $\widehat{\psi} = \Psi$, and consequently that $\phi = \mathscr{B}\psi$ by Fourier inversion. Finally, since ϕ and $\mathscr{B}$ are real we have that $\phi = \mathscr{B}\psi = \mathscr{B}(\mathrm{Re}\,\psi)$, so if necessary we replace ψ by its real part to achieve a real vector potential $\psi \in \mathscr{D}'(\mathbb{R}^n, \mathbb{R}^L)$.   □

In order to improve the regularity of the vector potential found in Lemma 3.2 we let

$$
\mathbb{V}_0 \equiv \bigcap_{|\xi|=1} \ker_{\mathbb{R}} \mathscr{A}(\xi), \tag{3.8}
$$

where we emphasize that the kernel here is of the map $\mathscr{A}(\xi) \colon \mathbb{V} \to \mathbb{W}$ so that $\mathbb{V}_0$ is a subspace of $\mathbb{V}$. Let $\pi \colon \mathbb{V} \to \mathbb{V}_0$ be the orthogonal projection. Then π^* is the orthogonal projection onto $\mathbb{V}_0^\perp$, and we can apply the above construction to the differential operator $\mathscr{A}\pi^*$ above instead of $\mathscr{A}$. We consider $\mathscr{A}\pi^*$ as a differential operator on $\mathbb{R}^n$ from $\mathbb{V}_0^\perp$ to $\mathbb{W}$ and we find hereby a corresponding potential operator $\mathscr{B}^{\mathbb{V}_0}$ on $\mathbb{R}^n$ from $\mathbb{R}^L$ to $\mathbb{V}_0^\perp$ similar to the one in (3.2). This time there is no zero order part (compare (3.1)) so the orders of the differential operators in the columns of $\mathscr{B}^{\mathbb{V}_0}$ are $1 \leqq k_1 \leqq \cdots \leqq k_L$. Put $\ell_0 \equiv \dim \mathbb{V}_0$ and choose an orthonormal basis (e_l) for $\mathbb{V}_0$. Define the differential operator $\mathscr{B}$ on $\mathbb{R}^n$ from $\mathbb{R}^{\ell_0+L}$ to $\mathbb{V}$ by

$$
\mathscr{B}\psi \equiv \sum_{l=1}^{L} \mathscr{B}_l \psi_l + \sum_{l=1}^{\ell_0} \psi_{L+l} e_l, \tag{3.9}
$$

where $\psi = \left(\psi_1, \ldots, \psi_L, \psi_{L+1}, \ldots, \psi_{L+\ell_0}\right)^{\mathrm{tr}}$. We record that

$$
\pi^* \mathscr{B}\psi = \mathscr{B}^{\mathbb{V}_0}\left(\psi_1, \ldots, \psi_L\right)^{\mathrm{tr}} \quad \text{and} \quad \pi \mathscr{B}\psi = \sum_{l=1}^{\ell_0} \psi_{L+l} e_l \tag{3.10}
$$

holds for all ψ. Next, we consider the submodule $\mathcal{M}$ of $R^{1 \times N}$ consisting of all row vectors $a(\xi) \in R^{1 \times N}$ satisfying $a(\xi)\mathcal{B}(\xi) = 0$ in $R^{1 \times L}$. As before it follows that $\mathcal{M}$ is finitely generated:

$$\mathcal{M} = \left\{ \sum_{j=1}^{J} p_j(\xi) a_j(\xi) : p_1(\xi), \ldots, p_J(\xi) \in R \right\},$$

where $a_j(\xi) \in R^{1 \times N}$ and the collection of generators is irreducible. Put

$$\widetilde{\mathscr{A}}(\xi) \equiv \begin{bmatrix} a_1(\xi) \\ \cdots \cdots \\ \cdots \cdots \\ a_J(\xi) \end{bmatrix} \in R^{J \times N}$$

and note that since $\mathscr{A}(\xi)$ annihilates $\mathcal{B}(\xi)$ there exists $Y(\xi) \in R^{M \times J}$ such that $\mathscr{A}(\xi) = Y(\xi)\widetilde{\mathscr{A}}(\xi)$. It is also not difficult to check that $\ker_R \mathscr{A}(\xi) = \ker_R \widetilde{\mathscr{A}}(\xi)$. We now let $\widetilde{\mathscr{A}} = \widetilde{\mathscr{A}}(\partial)$ be the corresponding differential operator. It is remarkable that $\widetilde{\mathscr{A}}$ is an exact annihilator for $\mathcal{B}$ on C^∞:

Theorem 3.3. [26, Theorem 7.6.13] *Let $\mathcal{B}$ and $\widetilde{\mathscr{A}}$ be as above and let O be an open convex subset of $\mathbb{R}^n$. Then for $\phi \in C^\infty(O, \mathbb{R}^N)$ the equation $\mathcal{B}\psi = \phi$ on O admits a solution $\psi \in C^\infty(O, \mathbb{R}^{L+\ell_0})$ if and only if $\widetilde{\mathscr{A}}\phi = 0$ on O.*

We are now ready to state and prove the main result of this subsection.

Theorem 3.4. *Let the differential operators $\mathscr{A}$, $\mathcal{B}$ be as above. Then for $\varepsilon > 0$ and $\phi \in C_c^\infty(\mathbb{R}^n, \mathbb{V})$ with $\mathscr{A}\phi = 0$ we can find $\psi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^{L+\ell_0})$ such that $\mathrm{supp}(\psi) \subseteq B_\varepsilon(\mathrm{supp}(\phi)^{co})$ and $\mathcal{B}\psi = \phi$.*

Proof. First we note that by virtue of Lemma 3.2 we find a compactly supported $\Psi \in \mathscr{D}'(\mathbb{R}^n, \mathbb{R}^{L+\ell_0})$ such that $\mathcal{B}\Psi = \phi$. But then we have for a standard smooth mollifier on $\mathbb{R}^n$, $(\rho_t)_{t>0}$, that $\mathcal{B}(\rho_t * \Psi) = \rho_t * \phi$, so by the easy part of Theorem 3.3, $\rho_t * \widetilde{\mathscr{A}}\phi = \widetilde{\mathscr{A}}(\mathcal{B}(\rho_t * \Psi)) = 0$. Consequently, $\widetilde{\mathscr{A}}\phi = 0$ and we find by Theorem 3.3 a $\psi \in C^\infty(\mathbb{R}^n, \mathbb{R}^{L+\ell_0})$ such that $\phi = \mathcal{B}\psi$. According to (3.10),

$$\pi^*\phi = \mathcal{B}^{\mathbb{V}_0}(\psi_1, \ldots, \psi_L)^{\mathrm{tr}} \text{ and } \pi\phi = \sum_{l=1}^{\ell_0} \psi_{l+L} e_l.$$

It follows that $\psi_{L+1}, \ldots, \psi_{L+\ell_0} \in C_c^\infty(\mathbb{R}^n)$ with supports contained in the support of ϕ. Put $K = \mathrm{supp}(\phi)^{co}$, let $t > 0$ be a small number that will be specified below, take $\eta = \rho_t * \mathbf{1}_{B_{2t}(K)}$ and $\psi_0 \equiv (\psi_1, \ldots, \psi_L)^{\mathrm{tr}}$. Observe that $\eta = 1$ on $B_t(K)$ so that by the Leibniz rule (2.6)

$$\mathcal{B}^{\mathbb{V}_0}(\eta\phi_0) = \pi^*\phi + \sum_{l=1}^{L} \sum_{0 < |\alpha| \leq k_l} \frac{1}{\alpha!} \pi^* B_l^{(\alpha)} \psi_l \partial^\alpha \eta = \pi^*\phi + e,$$

say. Here $e = 0$ on $B_t(K)$ and on $\mathbb{R}^n \setminus B_{3t}(K)$, so taking $\kappa = \rho_s * \mathbf{1}_{B_{3t+s}(K) \setminus B_{t-s}(K)}$ for $s > 0$ small, we have $\kappa = 1$ on $B_{3t}(K) \setminus B_t(K)$ and $\kappa = 0$ near K, whereby another use of Leibniz yields

$$\mathscr{B}^{\mathbb{V}_0}(\kappa \eta \psi_0) = \kappa \mathscr{B}^{\mathbb{V}_0}(\eta \psi) + \sum_{l=1}^{L} \sum_{0 < |\alpha| \leq k_l} \frac{1}{\alpha!} \pi^* B_l^{(\alpha)}(\eta \psi_0) \partial^\alpha \kappa = \kappa(\pi^* \phi + e) = e.$$

Now $\psi' \equiv \eta \psi_0 - \kappa \eta \psi_0 \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^L)$ with $\operatorname{supp}(\psi') \subseteq B_{3t+s}(K)$ and

$$\mathscr{B}^{\mathbb{V}_0} \psi' = \mathscr{B}^{\mathbb{V}_0}(\eta \psi_0) - \mathscr{B}^{\mathbb{V}_0}(\kappa \eta \psi_0) = \phi + e - e = \phi.$$

Consequently, taking $3t + s = \varepsilon$ we can use $\psi' + \sum_{l=1}^{\ell_0} \psi_{l+L} e_l$ as vector potential with the asserted properties. $\qquad \square$

3.2. A Dacorogna Type Formula for the Envelope

We recall that for a continuous integrand $f : \mathbb{V} \to \mathbb{R}$ of linear growth its $\mathscr{A}$-quasiconvex envelope is the extended real-valued integrand

$$f^{qc}(z) \equiv \sup\{g(z) : g : \mathbb{V} \to \mathbb{R} \ \mathscr{A}\text{-quasiconvex and } g \leqq f\}.$$

It follows immediately from the definition that either $f^{qc} \equiv -\infty$ or f^{qc} is real-valued. In the latter case we check that, since $\mathscr{A}$-quasiconvexity implies $\Lambda_{\mathscr{A}}$-convexity by Corollary 2.8, f^{qc} is $\Lambda_{\mathscr{A}}$-convex. Since it is bounded from above by f and f has linear growth a standard argument (see for instance [31]) implies that f^{qc} has linear growth too. But then it is Lipschitz by Corollary 2.8. It is then easy to check that f^{qc} is $\mathscr{A}$-quasiconvex. The following result is important for our approach:

Theorem 3.5. *Let $\mathscr{A}$ be a differential operator of the form* (1.1) *that satisfies the constant rank condition* (1.2) *and the spanning cone condition* (1.3). *Let $\mathscr{B}$ be a potential operator as in* (2.4) *satisfying* (2.5). *Then for any continuous integrand $f : \mathbb{V} \to \mathbb{R}$ of linear growth we have for each non-empty, bounded open subset O of $\mathbb{R}^n$,*

$$f^{qc}(z) = \inf\left\{ \fint_O f(z + \mathscr{B}\psi(x)) \, dx : \psi \in C_c^\infty(O, \mathbb{V}) \right\} \tag{3.11}$$

for each $z \in \mathbb{V}$. It is not excluded that the common value is $-\infty$ here. Furthermore, for each $\varepsilon > 0$ it suffices to take infimum over potentials ψ satisfying additionally

$$\max_{x \in O} |\nabla^j \psi(x)| < \varepsilon \tag{3.12}$$

for derivative orders $0 \leqq j < l$, where we recall that l is the order of $\mathscr{B}$.

We merely sketch the proof as it is similar to those presented in [10, Theorem 6.9], [27, Appendix] and [42]. We start with the following lemma.

Lemma 3.6. *Assume that $\mathscr{B}$ is an operator as in Theorem 3.5. Then for $f : \mathbb{V} \to \mathbb{R}$ continuous and $z \in \mathbb{V}$ the quantity*

$$R(z) \equiv \inf\left\{ \fint_O f(z + \mathscr{B}\psi(x))\,dx \colon \psi \in C_c^\infty(O, \mathbb{V}) \right\}$$

is independent of the set O within the class of all non-empty, bounded and open subsets of $\mathbb{R}^n$. Furthermore, the value is unchanged if we only take infimum over potentials ψ satisfying additionally (3.12). Finally, either $R(z) = -\infty$ for all $z \in \mathbb{V}$ or R is real-valued.

We leave it to the reader to check that [10, Step 1 in proof of Theorem 6.9] and [27, Appendix] still apply in the present context. Next, it is clear that

$$f^{qc}(z) \le R(z) \le f(z) \tag{3.13}$$

holds for all $z \in \mathbb{V}$. We may therefore assume that $R(z) > -\infty$ for all $z \in \mathbb{V}$ as otherwise there is nothing to prove. Our next step is

Lemma 3.7. *Assume that $\mathscr{A}, \mathscr{B}$ are operators as in Theorem 3.5 and let $p \in (1, \infty)$. Let $\phi \in L^p(\mathbb{R}^n, \mathbb{V})$ with $\mathscr{A}\phi = 0$. Then for any $\chi \in C_c^\infty(\mathbb{R}^n)$ with $\chi = 1$ near 0 we have $\phi = (\mathcal{F}^{-1}\chi) * \phi + \mathscr{B}\psi$, where*

$$\psi = \mathrm{Re}\left(\mathcal{F}^{-1}\left(\mathscr{B}(i\xi)^\dagger (1 - \chi(\xi))\widehat{\phi}(\xi) \right) \right) \in W^{l,p}(\mathbb{R}^n, \mathbb{V}).$$

Here $\mathrm{Re}(\cdot)$ denotes the real part taken component-wise, $\mathcal{F}$ is the Fourier transform and $\mathscr{B}(i\xi)^\dagger = i^{-l}\mathscr{B}(\xi)^\dagger$ is the Moore–Penrose inverse of $\mathscr{B}(i\xi) = i^l\mathscr{B}(\xi)$.

Proof. We clearly have

$$\phi = (\mathcal{F}^{-1}\chi) * \phi + \left(\mathcal{F}^{-1}(1 - \chi) \right) * \phi$$

and that both terms on the right-hand side are $\mathscr{A}$-free. For the second term this means that $\mathscr{A}(\xi)(1 - \chi(\xi))\widehat{\phi}(\xi) = 0$ for all $\xi \in \mathbb{R}^n$. To see that it implies that we have a vector potential we invoke the exactness relation (2.5) and properties of the Moore–Penrose inverse. First, the exactness relation implies that also the operator $\mathscr{B}$ has constant rank and so the Moore–Penrose inverse $\mathscr{B}(\xi)^\dagger$ is C^∞ and $-l$-homogeneous on $\mathbb{R}^n \setminus \{0\}$ (see [42,43] for an easy proof of this). Now with ψ as defined above we calculate (in sense of $\mathbb{V}$-valued tempered distributions):

$$\mathscr{B}\psi = \mathrm{Re}\left(\mathcal{F}^{-1}\left(\mathscr{B}(i\xi)\mathscr{B}(i\xi)^\dagger (1 - \chi(\xi))\widehat{\phi}(\xi) \right) \right)$$

$$= \mathrm{Re}\left(\mathcal{F}^{-1}\left(\mathscr{B}(\xi)\mathscr{B}(\xi)^\dagger (1 - \chi(\xi))\widehat{\phi}(\xi) \right) \right)$$

$$= \mathrm{Re}\left(\mathcal{F}^{-1}\left(\mathrm{proj}_{\mathrm{im}\,\mathscr{B}(\xi)}\left((1 - \chi(\xi))\widehat{\phi}(\xi) \right) \right) \right)$$

$$\overset{(2.5)}{=} \mathrm{Re}\left(\mathcal{F}^{-1}(1 - \chi)\widehat{\phi} \right) = \left(\mathcal{F}^{-1}(1 - \chi) \right) * \phi,$$

as required. Now using that the $W^{l,p}$ norm is equivalent to the corresponding one given in terms of Bessel potentials we conclude from the Hörmander–Mihlin theorem that $\psi \in W^{l,p}(\mathbb{R}^n, \mathbb{V})$. Indeed, the function

$$\xi \mapsto \langle \xi \rangle^l \mathscr{B}(\xi)^\dagger (1 - \chi(\xi))$$

is a bounded map of class $C^\infty(\mathbb{R}^n, \mathscr{L}(\mathbb{V}, \mathbb{V}))$ and so in particular a multiplier of the required type. □

The next result is a slight strengthening of the closed range inequality (1.6) mentioned in the Introduction. It is obtained by combination of Lemma 3.7 and Murat's inequality (1.7):

Lemma 3.8. *Assume that $\mathscr{A}$, $\mathscr{B}$ are operators as in Theorem* 3.5. *Then for each* $p \in (1, \infty)$ *there exists a constant $C_p > 0$ such that*

$$\inf_{\psi \in C_c^\infty(\mathbb{R}^n, \mathbb{V})} \|\phi - \mathscr{B}\psi\|_p \leqq C_p \|\mathscr{A}\phi\|_{W^{-k,p}}$$

holds for all $\phi \in L^p(\mathbb{R}^n, \mathbb{V})$.

Proof. Let $\phi \in L^p(\mathbb{R}^n, \mathbb{V})$. By Murat's inequality (1.7) we have $\|\phi - P_{\mathscr{A}}\phi\|_p \leqq c_p \|\mathscr{A}\phi\|_{W^{-k,p}}$, where $P_{\mathscr{A}}\phi \in L^p(\mathbb{R}^n, \mathbb{V})$ is $\mathscr{A}$-free. Let $\varepsilon > 0$. By use of Lemma 3.7 we find $u \in W^{l,p}(\mathbb{R}^n, \mathbb{V})$ such that $\|P_{\mathscr{A}}\phi - \mathscr{B}u\|_p < \varepsilon$. Next for a standard smooth mollifier $\rho_j(x) = j^n \rho(jx)$ we take $j \in \mathbb{N}$ so large that $\|P_{\mathscr{A}}\phi - \mathscr{B}(\rho_j * u)\|_p < \varepsilon$. Fix such j and put $\chi_r = \rho_j * \mathbf{1}_{B_r(0)}$. Now by the Leibniz rule (2.6) we have $\|\mathscr{B}(\rho_j * u) - \mathscr{B}(\chi_r \rho_j * u)\|_p \to 0$ as $r \to \infty$. Hence for large $r > 0$ we have with $\psi = \chi_r \rho_j * u \in C_c^\infty(\mathbb{R}^n, \mathbb{V})$ that $\|\phi - \mathscr{B}\psi\|_p < c_p \|\mathscr{A}\phi\|_p + \varepsilon$ concluding the proof. □

The proof of Theorem 3.5 can now be completed as in [42].

4. Proof of Proposition 1.1

Recall that we are given a linear homogeneous differential operator of order k on $\mathbb{R}^n$ from $\mathbb{V}$ to $\mathbb{W}$, two finite-dimensional inner product spaces over $\mathbb{R}$:

$$\mathscr{A} = \sum_{|\alpha|=k} A_\alpha \partial^\alpha, \quad A_\alpha \in \mathscr{L}(\mathbb{V}, \mathbb{W}).$$

We do not impose any further restrictions on $\mathscr{A}$ at this stage. Instead we prefer to introduce the hypotheses on $\mathscr{A}$ in the course of the proof as it will then be clear why they are natural.

For a bounded open set Ω in $\mathbb{R}^n$ and an exponent $p \in (1, \frac{n}{n-1})$ we consider $\mathbb{V}$-valued measures $v_j, v \in \mathscr{M}(\Omega, \mathbb{V})$ satisfying

$$\begin{cases} v_j \overset{*}{\rightharpoonup} v & \text{in } \mathscr{M}(\Omega, \mathbb{V}) \\ \mathscr{A}v_j \to \mathscr{A}v & \text{in } W^{-k,p}_{\text{loc}}(\Omega, \mathbb{W}). \end{cases} \quad (4.1)$$

Under the assumption $(4.1)_1$ we can extract a subsequence (not relabelled) such that

$$v_j \xrightarrow{\text{Y}} v, \quad v = \big((v_x), \lambda, (v_x^\infty)\big) \tag{4.2}$$

The task is to find conditions at the level of the Young measure v in the form of Jensen type inequalities that reflect the convergence $(4.1)_2$.

4.1. A First Reduction: the Spanning Cone Condition

Recall that the wave cone for $\mathscr{A}$ is defined as

$$\Lambda_{\mathscr{A}} \equiv \bigcup_{\xi \in \mathbb{R}^n \setminus \{0\}} \ker \mathscr{A}(\xi), \quad \text{where} \quad \mathscr{A}(\xi) \equiv \sum_{|\alpha|=k} A_\alpha \xi^\alpha \in \mathscr{L}(\mathbb{V}, \mathbb{W}).$$

Put

$$\begin{cases} \mathbb{V}_1 \equiv \text{span} \Lambda_{\mathscr{A}}, \quad \mathbb{V}_2 \equiv \mathbb{V}_1^\perp, \\ \pi_i : \mathbb{V} \to \mathbb{V}_i \ \text{the corresponding orthogonal projections.} \end{cases} \tag{4.3}$$

Observe that the differential operator $\mathscr{A}\pi_2 = \sum_{|\alpha|=k} A_\alpha \pi_2 \partial^\alpha$ is partially elliptic in the sense that its wave cone $\Lambda_{\mathscr{A}\pi_2} = \mathbb{V}_1$. A standard use of the Hörmander–Mihlin multiplier theorem yields a constant $c_p > 0$ such that

$$\big\| \pi_2 v \big\|_p \leqq c_p \big\| \mathscr{A} \pi_2 v \big\|_{\text{W}^{-k,p}} \tag{4.4}$$

holds for all $v \in \text{L}^p(\mathbb{R}^n, \mathbb{V})$. The next result relates the actions of $\mathscr{A}$ on a field and its projection on $\mathbb{V}_2$.

Lemma 4.1. *Suppose that $\mathscr{A}$ is a constant rank operator. Then*

$$\big\| \pi_2 v \big\|_p \leqq c_p \big\| \mathscr{A} v \big\|_{\text{W}^{-k,p}}$$

for all $v \in \text{C}_c^\infty(\mathbb{R}^n, \mathbb{V})$.

Proof. We note that for $z \in \mathbb{V}$, and $\xi \in \mathbb{S}^{n-1}$

$$\pi_2 z = \Pi_{\Lambda_{\mathscr{A}}^\perp} z = \Pi_{\bigcap_{|\zeta|=1} \ker \mathscr{A}(\zeta)^\perp} z = \Pi_{\bigcap_{|\zeta|=1} \ker \mathscr{A}(\zeta)^\perp} \Pi_{\ker \mathscr{A}(\xi)^\perp} z,$$

where Π_P denotes orthogonal projection on the linear space P. Therefore we can write in Fourier space, for $v \in \text{C}_c^\infty(\mathbb{R}^n, \mathbb{V})$

$$\pi_2 \hat{v}(\xi) = \pi_2 \Pi_{\ker \mathscr{A}(\xi)^\perp} \hat{v}(\xi) = \pi_2 \mathscr{A}^\dagger(\xi) \mathscr{A}(\xi) \hat{v}(\xi) = \pi_2 \mathscr{A}^\dagger \left(\frac{\xi}{|\xi|} \right) \frac{\widehat{\mathscr{A} v}(\xi)}{|\xi|^k}$$

and the multiplier theorem gives the result. Here $M^\dagger$ denotes the Moore–Penrose inverse of a matrix M. The constant rank condition of $\mathscr{A}$ is known to be equivalent with smoothness of $\mathscr{A}^\dagger(\cdot)$ on $\mathbb{S}^{n-1}$ [42]. $\qquad \square$

We return to the Young measure ν and fix $\Omega' \Subset \Omega$ with $\lambda(\partial\Omega') = 0$. Write

$$\nu \llcorner \Omega' = \left((\nu_x)_{x \in \Omega'}, \lambda \llcorner \Omega', (\nu_x^\infty)_{x \in \Omega'}\right).$$

Choose $\eta \in C_c^\infty(\Omega)$ satisfying $\mathbf{1}_{\Omega'} \leqq \eta \leqq \mathbf{1}_\Omega$ and record the convergences $\eta v_j \overset{*}{\rightharpoonup} \eta v$ in $\mathscr{M}(\mathbb{R}^n, \mathbb{V})$, $\mathscr{A}(\eta v_j) \to \mathscr{A}(\eta v)$ in $W^{-k,p}(\mathbb{R}^n, \mathbb{W})$ and $(\eta v_j) \llcorner \Omega' = v_j \llcorner \Omega' \overset{Y}{\to} \nu \llcorner \Omega'$. If $(\rho_\varepsilon)_{\varepsilon > 0}$ is a standard smooth mollifier on $\mathbb{R}^n$, then using a result of Reshetnyak [44] and the definition of Young measure (see for instance [34] for such arguments) we find a null sequence $\varepsilon_j \searrow 0$ such that $(\rho_{\varepsilon_j} * (\eta v_j))|_{\Omega'} \overset{Y}{\to} \nu \llcorner \Omega'$. Put $\tilde{v} \equiv \eta v$ and $\tilde{v}_j \equiv \rho_{\varepsilon_j} * (\eta v_j)$. By inspection $\tilde{v}_j \overset{*}{\rightharpoonup} \tilde{v}$ in $\mathscr{M}(\mathbb{R}^n, \mathbb{V})$, $\mathscr{A}\tilde{v}_j \to \mathscr{A}\tilde{v}$ in $W^{-k,p}(\mathbb{R}^n, \mathbb{W})$ and in view of Lemma 4.1, $\mathscr{A}\pi_2\tilde{v}_j \to \mathscr{A}\pi_2\tilde{v}$ in $W^{-k,p}(\mathbb{R}^n, \mathbb{W})$. In particular, $\mathscr{A}\pi_2\tilde{v} \in W^{-k,p}(\mathbb{R}^n, \mathbb{W})$ and so from (4.4) infer $\pi_2\tilde{v} \in L^p(\mathbb{R}^n, \mathbb{V})$. But then another application of (4.4) yields $\pi_2\tilde{v}_j \to \pi_2\tilde{v}$ in $L^p(\mathbb{R}^n, \mathbb{V})$, and $(\pi_1\tilde{v}_j + \pi_2\tilde{v})|_{\Omega'} \overset{Y}{\to} \nu \llcorner \Omega'$ follows. Let us summarize the above discussion:

Lemma 4.2. *Assume* (4.1), (4.2), (4.3). *Then* $\pi_2 v \in L^p_{\text{loc}}(\Omega, \mathbb{V})$, $\pi_1 v_j + \pi_2 v \overset{Y}{\to} \nu$, $\pi_1 v_j \overset{*}{\rightharpoonup} \pi_1 v$ *in* $\mathscr{M}(\Omega, \mathbb{V})$ *and* $\mathscr{A}\pi_1 v_j \to \mathscr{A}\pi_1 v$ *in* $W^{-k,p}_{\text{loc}}(\Omega, \mathbb{W})$. *In particular,*

$$\pi_1 v_j \overset{Y_{\text{loc}}}{\to} \left((\delta_{-(\pi_2 v)(x)} * \nu_x), \lambda, (\nu_x^\infty)\right)$$

and

$$\begin{cases} \text{supp}(\delta_{-(\pi_2 v)(x)} * \nu_x) \subseteq \mathbb{V}_1 & \text{for} \ \ \mathscr{L}^n \text{ almost everywhere } x \in \Omega \\ \text{supp}(\nu_x^\infty) \subseteq \mathbb{S}_{\mathbb{V}_1} & \text{for} \ \ \lambda \text{ almost everywhere } x \in \Omega. \end{cases}$$

Here the suffix *loc* on the Young measure generation should be understood as $(\pi_1 v_j \llcorner \Omega')$ generates the listed Young measure restricted to Ω' for each $\Omega' \Subset \Omega$ with $\lambda(\partial\Omega') = 0$. In view of this result we can focus on the Young measure generated by the projected sequence $(\pi_1 v_j)$. To avoid heavy notation we assume in the sequel that

$$\text{span}(\Lambda_{\mathscr{A}}) = \mathbb{V}. \tag{4.5}$$

4.2. The Closed-Range Inequality and Conclusion of Proof

Because $\mathscr{A}$ satisfies the constant rank condition (1.2) we infer in particular from Lemma 3.8 that there is a constant $C_p > 0$ such that for all $\phi \in L^p(\mathbb{R}^n, \mathbb{V})$ We have

$$\inf_{\psi \in C_c^\infty(B_1\text{supp}(\phi), \mathbb{V}), \mathscr{A}\psi = 0} \|\phi - \psi\|_p \leqq C_p \|\mathscr{A}\phi\|_{W^{-k,p}} \tag{4.6}$$

In view of our definition of $\mathscr{A}$-quasiconvexity the hypothesis is natural because, as we will see below, it allows us to use the convergence $(4.2)_2$ to relate the fields v_j, v in a straight forward way to $\mathscr{A}$-free fields and hence use $\mathscr{A}$-quasiconvexity to establish the Jensen type inequalities for the Young measure ν. Without (4.6) the

relation to $\mathscr{A}$-free fields will be much weaker and, apart from some special cases [35,40], any known proof breaks down.

With the hypotheses (4.5), (4.6) in place we return to the arbitrary set $\Omega' \Subset \Omega$ with $\lambda(\partial\Omega') = 0$ and a cut-off function $\eta \in C^\infty(\Omega)$ satisfying $\mathbf{1}_{\Omega'} \leqq \eta \leqq \mathbf{1}_\Omega$. As above we find for a standard smooth mollifier $(\rho_\varepsilon)_{\varepsilon > 0}$ on $\mathbb{R}^n$ a null sequence $\varepsilon_j \searrow 0$ such that for $\tilde{v}_j \equiv \rho_{\varepsilon_j} * (\eta v_j)$ we have $\tilde{v}_j|_{\Omega'} \overset{Y}{\to} v \llcorner \Omega'$. Put $\tilde{v} \equiv \eta v$, $w_j \equiv \rho_{\varepsilon_j} * (\tilde{v}_j)$ and note $w_j \to \tilde{v}$ $\langle \cdot \rangle$-strictly on $\mathbb{R}^n$, $\tilde{v}_j - w_j \overset{*}{\rightharpoonup} 0$ in $\mathscr{M}(\mathbb{R}^n, \mathbb{V})$ and $\mathscr{A}(\tilde{v}_j - w_j) \to 0$ in $W^{-k,p}(\mathbb{R}^n, \mathbb{W})$. We first establish (1.5). Find by (4.6) $\psi_j \in C_c^\infty(\mathbb{R}^n, \mathbb{V})$ such that $\mathscr{A}\psi_j = 0$ and $\|w_j - \psi_j\|_p \leqq c_p\|\mathscr{A}w_j\|_{W^{-k,p}}$. Since especially $\mathscr{A}\tilde{v} \in W^{-k,p}(\mathbb{R}^n, \mathbb{W})$ we have $\mathscr{A}w_j = \rho_{\varepsilon_j} * \mathscr{A}\tilde{v} \to \mathscr{A}\tilde{v}$ in $W^{-k,p}(\mathbb{R}^n, \mathbb{W})$, hence it follows easily that $(w_j - \psi_j)$ is Cauchy in $L^p(\mathbb{R}^n, \mathbb{V})$, say $w_j - \psi_j \to a$ in $L^p(\mathbb{R}^n, \mathbb{V})$. In particular, $w_j \overset{*}{\rightharpoonup} \tilde{v}$ in $\mathscr{M}(\mathbb{R}^n, \mathbb{V})$, hence also $\psi_j \overset{*}{\rightharpoonup} \tilde{v} + a$ in $\mathscr{M}(\mathbb{R}^n, \mathbb{V})$. But then $\mathscr{A}(\tilde{v}+a) = 0$ and because $\tilde{v}^s = (\tilde{v}+a)^s$ we infer from Lemma 2.2 that the polar $\tilde{v}^s/|\tilde{v}|^s \in \Lambda_{\mathscr{A}}$ for $|\tilde{v}|^s$ almost everywhere. If we restrict the measures to Ω', then we find $v \llcorner \Omega' = \tilde{v} \llcorner \Omega' = \overline{v}_x \mathscr{L}^n \llcorner \Omega' + \overline{v}_x^\infty \lambda \llcorner \Omega'$, and thus $\overline{v}_x^\infty \in \Lambda_{\mathscr{A}}$ for λ^s almost every $x \in \Omega'$. After a further Lebesgue decomposition of $\lambda^s \llcorner \Omega'$ with respect to $|v|^s \llcorner \Omega'$ we deduce (1.5) by identification of terms in the above expression for $v \llcorner \Omega'$.

For the proof of (1.4) we require an auxiliary result that we adapt from [29]. An integrand $f : \mathbb{V} \to \mathbb{R}$ is called a **special $\mathscr{A}$-quasiconvex integrand** if it is $\mathscr{A}$-quasiconvex, of linear growth and there exists $r = r_f > 0$ such that $f(z) = f^\infty(z)$ holds for all $|z| \geqq r$. The class of such integrands is denoted by SQ. It is perhaps not even clear if there are any such integrands at all. But in fact there are plenty of them as the following approximation result shows.

Lemma 4.3. *For each $\mathscr{A}$-quasiconvex integrand $f : \mathbb{V} \to \mathbb{R}$ of linear growth there exists a sequence (f_j) from SQ such that*

$$f_j(z) \geqq f_{j+1}(z) \searrow f(z) \quad and \quad f_j^\infty(z) \geqq f_{j+1}^\infty(z) \searrow f^\infty(z) \quad as \quad j \nearrow \infty$$

for each $z \in \mathbb{V}$.

A proof in the case $\mathscr{A} = \mathrm{curl}$ acting row-wise on $\mathbb{V} = \mathbb{R}^{N \times n}$ valued fields can be found in [29, Lemma 6.3] and we leave it to the interested reader to check that it in view of Theorem 3.5 also applies to the present more general situation.

Recall the space $\mathbb{E}(\Omega, \mathbb{V})$ of admissible integrands defined in Section 2.4 and that it is a separable Banach space. We define an autonomous version of it by letting

$$\mathbb{H} \equiv \{f : \mathbf{1}_{\overline{\Omega}} \otimes f \in \mathbb{E}(\Omega, \mathbb{V})\} \quad \text{and} \quad \|f\| \equiv \sup_{z \in \mathbb{V}} \frac{|f(z)|}{1 + |z|}.$$

It is easy to check that $\mathbb{H}$ hereby is a separable Banach space (compare also Section 5.1), hence we can find a countable family Q of $\mathscr{A}$-quasiconvex integrands whose closure $\overline{\mathrm{Q}}$ in $\mathbb{H}$ is the cone of all $\mathscr{A}$-quasiconvex integrands in $\mathbb{H}$. Let $f \in \mathrm{Q}$. Fix concentric balls $B_r = B_r(x_0) \Subset B_s = B_s(x_0) \Subset \Omega$ such that $\lambda(\partial B_r \cup \partial B_s) = 0$ and let $\eta \in C_c^\infty(\mathbb{R}^n)$ be chosen so $\mathbf{1}_{B_r} \leqq \eta \leqq \mathbf{1}_{B_s}$. Next select as before a null sequence $\varepsilon_j \searrow 0$ such that $\rho_{\varepsilon_j} * (\eta v_j)|_{B_r} \overset{Y}{\to} v \llcorner B_r$.

Put $\tilde{v}_j \equiv \rho_{\varepsilon_j} * (\eta v_j)$, $w_j \equiv \rho_{\varepsilon_j} * (\eta v)$ and record $\tilde{v}_j - w_j \overset{*}{\rightharpoonup} 0$ in $\mathscr{M}(\mathbb{R}^n, \mathbb{V})$, $\mathscr{A}(\tilde{v}_j - w_j) \to 0$ in $W^{-k,p}(\mathbb{R}^n, \mathbb{W})$ and $w_j \to \eta v \langle\cdot\rangle$ strictly on $\mathbb{R}^n$. From (4.6) we find $\psi_j \in C_c^\infty(B_{s+1}, \mathbb{V})$ such that $\mathscr{A}\psi_j = 0$ and $\tilde{v}_j - w_j - \psi_j \to 0$ in $L^p(\mathbb{R}^n, \mathbb{V})$. Put $e_j \equiv \tilde{v}_j - w_j - \psi_j$ and note that

$$\psi_j = -e_j \quad \text{on} \quad \mathbb{R}^n \backslash B_s. \tag{4.7}$$

Considering $f - f(0)$ instead of f we can assume $f(0) = 0$. By virtue of Lemma 2.8 the integrand f is Lipschitz, say with Lipschitz constant ℓ. From our hypotheses

$$\limsup_{j \to \infty} \int_{B_s} f(\tilde{v}_j)\, dx \leqq \int_{B_r} \langle \nu_x, f \rangle\, dx + \int_{B_r} \langle \nu_x^\infty, f^\infty \rangle\, d\lambda$$
$$+ \ell \left(\int_{B_s \backslash B_r} \langle \nu_x, |\cdot| \rangle\, dx + \lambda(B_s \backslash B_r) \right). \tag{4.8}$$

Let $\nu = \nu^a \mathscr{L}^n \llcorner \Omega + \nu^s$ be the Lebesgue–Radon–Nikodým decomposition with respect to $\mathscr{L}^n$. Then using the Lipschitz continuity of f and $\tilde{v}_j = w_j + \psi_j + e_j$ we estimate:

$$\limsup_{j \to \infty} \int_{B_s} f(\tilde{v}_j)\, dx \geqq \limsup_{j \to \infty} \int_{B_s} f(w_j + \psi_j)\, dx + 0$$
$$\geqq \limsup_{j \to \infty} \int_{\mathbb{R}^n} \left(f(\nu^a(x_0) + \psi_j) - f(\nu^a(x_0)) \right) dx$$
$$+ \mathscr{L}^n(B_s) f(\nu^a(x_0))$$
$$- \ell \limsup_{j \to \infty} \left(\int_{B_s} |w_j - \nu^a(x_0)|\, dx + \int_{\mathbb{R}^n \backslash B_s} |\psi_j|\, dx \right)$$
$$\overset{(4.7)}{\geqq} \mathscr{L}^n(B_s) f(\nu^a(x_0)) - \ell \int_{B_s} |\eta \nu^a - \nu^a(x_0)|\, dx - \ell |\nu|^s(B_s).$$

In concert with (4.8) this yields

$$\int_{B_r} \langle \nu_x, f \rangle\, dx + \int_{B_r} \langle \nu_x^\infty, f^\infty \rangle\, d\lambda \geqq f(\nu^a(x_0)) \mathscr{L}^n(B_s)$$
$$- \ell \int_{B_s} |\eta \nu^a - \nu^a(x_0)|\, dx - \ell |\nu|^s(B_s)$$
$$- \ell \left(\int_{B_s \backslash B_r} \langle \nu_x, |\cdot| \rangle\, dx + \lambda(B_s \backslash B_r) \right).$$

Now take $\eta \searrow \mathbf{1}_{B_r}$ and then $s \searrow r$ to get

$$\int_{B_r} \langle \nu_x, f \rangle\, dx + \int_{B_r} \langle \nu_x^\infty, f^\infty \rangle\, d\lambda \geqq f(\nu^a(x_0)) \mathscr{L}^n(B_r)$$
$$- \ell \int_{B_r} |\nu^a - \nu^a(x_0)|\, dx - \ell |\nu|^s(B_r),$$

divide by $\mathscr{L}^n(B_r)$, take $r \searrow 0$ and refer to standard differentiation results for measures (see for instance [17]) to arrive at (1.4) for $x_0 \in \Omega \setminus N_f$, where N_f is $\mathscr{L}^n$ negligible. Put $N^a = \bigcup_{f \in Q} N_f$ and observe that (1.4) holds for all $\mathscr{A}$-quasiconvex $f \in \mathbb{H}$ and all $x \in \Omega \setminus N^a$. In view of the approximation Lemma 4.3 we may extend (1.4) for each $x \in \Omega \setminus N^a$ to all $\mathscr{A}$-quasiconvex integrands of linear growth. The proof is therefore complete.

5. Proof of Theorem 1.2

Fix a differential operator $\mathscr{B}$ as in (2.4) satisfying (2.5). By virtue of the approximation lemma 4.3 and separability considerations the assumed Jensen-type inequality (1.9) imply that for some $\mathscr{L}^n$ negligible set $N \subset \Omega$ the Jensen-type inequality (1.9) holds for all $x \in \Omega \setminus N$ and all $\mathscr{A}$-quasiconvex integrands f of linear growth.

As in [27,28] we start by characterizing the homogeneous Young measures. Because we consider Young measures for oscillation and concentration this requires some additional care.

5.1. The Functional Set-Up

It is an autonomous version of that used in Section 2.4 for Young measures. Recall the space $\mathbb{H}$ from Section 4.2 consisting of continuous autonomous integrands $\Phi \colon \mathbb{V} \to \mathbb{R}$ of linear growth that admit a regular recession integrand,

$$\Phi^\infty(z) = \lim_{t \to \infty} \frac{\Phi(tz)}{t} \quad \text{exists locally uniformly in } z \in \mathbb{V}$$

and that $\mathbb{H}$ is normed by

$$\|\Phi\| = \sup_{z \in \mathbb{V}} \frac{|\Phi(z)|}{1 + |z|}.$$

We recall that $\Phi \in \mathbb{H}$ precisely when the transformed integrand

$$\hat{z} \mapsto \left(1 - |\hat{z}|\right) \Phi\left(\frac{\hat{z}}{1 - |\hat{z}|}\right)$$

is bounded and uniformly continuous on the open unit ball $\mathbb{B}_\mathbb{V}(0, 1)$. The mapping $T \colon \mathbb{H} \to C(\overline{\mathbb{B}_\mathbb{V}})$ defined by

$$T\Phi(\hat{z}) = \begin{cases} \left(1 - |\hat{z}|\right) \Phi\left(\frac{\hat{z}}{1-|\hat{z}|}\right) & \text{if } |\hat{z}| < 1 \\ \Phi^\infty(\hat{z}) & \text{if } |\hat{z}| = 1, \end{cases}$$

is then easily seen to be an isometric isomorphism. Hence by general results so is the dual mapping $T^* \colon C(\overline{\mathbb{B}_\mathbb{V}})^* \to \mathbb{H}^*$. By virtue of the Riesz representation theorem we may identify the dual space $C(\overline{\mathbb{B}_\mathbb{V}})^*$ with the space $\mathscr{M}(\overline{\mathbb{B}_\mathbb{V}})$ of bounded signed

Radon measures on the closed unit ball $\overline{\mathbb{B}_\mathbb{V}}$. Hence given $\ell \in \mathbb{H}^*$ there is a unique $\mu \in \mathcal{M}(\overline{\mathbb{B}_\mathbb{V}})$ such that $T^*\mu = \ell$. Consequently we have for $\Phi \in \mathbb{H}$,

$$
\begin{aligned}
\ell(\Phi) &= \langle T^*\mu, \Phi \rangle = \int_{\mathbb{B}_\mathbb{V}} T\Phi \, d\mu \\
&= \int_{\mathbb{B}_\mathbb{V}(0,1)} (1 - |\hat{z}|)\Phi\left(\frac{\hat{z}}{1 - |\hat{z}|}\right) d\mu(\hat{z}) + \int_{\mathbb{S}_\mathbb{V}} \Phi^\infty \, d\mu.
\end{aligned}
$$

If we put $S(\hat{z}) = \frac{\hat{z}}{1-|\hat{z}|}$ for $\hat{z} \in \mathbb{B}_\mathbb{V}(0, 1)$ and

$$
\begin{aligned}
\mu^0 &= S_\#\left[\left(1 - |\hat{z}|\right)\mu \llcorner \mathbb{B}_\mathbb{V}(0, 1)\right], \\
\mu^\infty &= \mu \llcorner \mathbb{S}_\mathbb{V},
\end{aligned}
$$

then $\mu^0 \in \mathcal{M}(\mathbb{V})$ has a total variation measure with finite first moment, that is, $\langle |\mu^0|, | \cdot | \rangle < \infty$, $\mu^\infty \in \mathcal{M}(\mathbb{S}_\mathbb{V})$, and

$$
\ell(\Phi) = \int_\mathbb{V} \Phi \, d\mu^0 + \int_{\mathbb{S}_\mathbb{V}} \Phi^\infty \, d\mu^\infty. \tag{5.1}
$$

In view of this representation we identify in the following each $\ell \in \mathbb{H}^*$ with the unique pair of measures (μ^0, μ^∞) as above. We also record that

$$
\|\ell\| = \int_\mathbb{V} (1 + | \cdot |) \, d|\mu^0| + |\mu^\infty|(\mathbb{S}_\mathbb{V})
$$

and that $\ell \geqq 0$ when $\mu^0 \geqq 0$, $\mu^\infty \geqq 0$, so that T^* is a positive operator. This means that T^* is a homeomorphism of the positive cones, $\mathcal{M}^+(\overline{\mathbb{B}_\mathbb{V}})$ onto $(\mathbb{H}^*)^+$ with their respective relative weak* topologies, and consequently, by virtue of Lemma 2.1, that the relative weak* topology on $(\mathbb{H}^*)^+$ is determined by the (push-forward) Kantorovich metric defined as

$$
d_K\left((\mu^0, \mu^\infty), (\nu^0, \nu^\infty)\right) = \|(\mu^0, \mu^\infty) - (\nu^0, \nu^\infty)\|_K
$$

for $(\mu^0, \mu^\infty), (\nu^0, \nu^\infty) \in \mathbb{H}^*$, where the (push-forward) Kantorovich norm is

$$
\|(\mu^0, \mu^\infty)\|_K = \sup_{\Phi \in \mathbb{H}, \|T\Phi\|_{\mathrm{LIP}} \leqq 1} \left| \int_\mathbb{V} \Phi \, d\mu^0 + \int_{\mathbb{S}_\mathbb{V}} \Phi^\infty \, d\mu^\infty \right|. \tag{5.2}
$$

If v is a bounded $\mathbb{V}$-valued Radon measure on $\mathbb{X} = \left(-\frac{1}{2}, \frac{1}{2}\right)^n$ with Lebesgue–Radon–Nikodým decomposition $v = v^a dx + v^s$, then we define $\boldsymbol{\varepsilon}_v \in \mathbb{H}^*$ by

$$
\boldsymbol{\varepsilon}_v(\Phi) = \int_\mathbb{X} \Phi(v^a(x)) \, dx + \int_\mathbb{X} \Phi^\infty(v^s) \quad (\Phi \in \mathbb{H}). \tag{5.3}
$$

5.2. The Hahn–Banach Argument

Fix $z \in \mathbb{V}$ and denote by $Y = Y_z$ the set of all pairs $(\nu^0, \nu^\infty) \in \mathscr{M}_1^+(\mathbb{V}) \times \mathscr{M}^+(\mathbb{S}_\mathbb{V})$ for which we can find a sequence (v_j) in $C_c^\infty(\mathbb{X}, \mathbb{V})$ such that, as $j \to \infty$, $\|v_j\|_{W^{l-1,1}} \to 0$ and

$$\int_{\mathbb{X}} \Phi(z + \mathscr{B}v_j(x))\, dx \to \int_{\mathbb{V}} \Phi\, d\nu^0 + \int_{\mathbb{S}_\mathbb{V}} \Phi^\infty\, d\nu^\infty \quad \forall\, \Phi \in \mathbb{H}. \qquad (5.4)$$

Obviously, we can consider Y as a subset of $(\mathbb{H}^*)^+$ and we clarify that $\mathscr{M}^+(\mathbb{S}_\mathbb{V})$ denotes the set of positive Radon measures on $\mathbb{S}_\mathbb{V}$, whereas $\mathscr{M}_1^+(\mathbb{V})$ denotes the set of probability measures on $\mathbb{V}$. The above convergence condition (5.4) for membership of Y can therefore also be stated in terms of the Kantorovich metric (5.2), where it takes the form that $\|\boldsymbol{\varepsilon}_{z+\mathscr{B}v_j} - (\nu^0, \nu^\infty)\|_K \to 0$ as $j \to \infty$.

Remark 5.1. Using a standard exhaustion argument it is not difficult to show that each $\nu \in Y$ can be generated by sequences with apparently much better convergence properties: Let $\nu = (\nu^0, \nu^\infty) \in Y$ and let $\omega: [0, \infty) \to [0, \infty)$ be a sublinear modulus of continuity (so $\omega(0) = 0$, ω is continuous, increasing, concave and $\omega(t)/t \to \infty$ as $t \searrow 0$). Then there exists a sequence (φ_j) in $C_c^\infty(\mathbb{X}, \mathbb{V})$ so $\|\varphi_j\|_{W^{l-1,\infty}} \to 0$, $\|\nu - \boldsymbol{\varepsilon}_{z+\mathscr{B}\varphi_j}\|_K \to 0$ and

$$\sup_{\substack{x,y \in \mathbb{X} \\ x \neq y}} \frac{\left|\nabla^{l-1}\varphi_j(x) - \nabla^{l-1}\varphi_j(y)\right|}{\omega(|x - y|)} \to 0.$$

Lemma 5.2. *The family $\left\{\boldsymbol{\varepsilon}_{z+\mathscr{B}u} : u \in C_c^\infty(\mathbb{X}, \mathbb{V})\right\}$ is a weakly-* dense subset of Y.*

Proof. In view of the definition of Y it suffices to show that $\boldsymbol{\varepsilon}_{z+\mathscr{B}u} \in Y$ for each $u \in C_c^\infty(\mathbb{X}, \mathbb{V})$. But this is a consequence of a standard exhaustion argument that runs as follows: We extend u to $\mathbb{R}^n$ by zero without changing notation and subdivide each side of $\mathbb{X}$ into $j \in \mathbb{N}$ disjoint congruent open intervals, and consider the resulting mesh of j^n disjoint congruent cubes. We fix an arrangement of these cubes, say

$$\mathbb{X}_i = x_i + j^{-1}\mathbb{X}, \quad i \in \{1, \ldots, j^n\},$$

and define

$$\phi_j(x) = \sum_{i=1}^{j^n} j^{-l}u(j(x - x_i)).$$

It is then simple to compute for $\Phi \in \mathbb{H}$ that

$$\int_{\mathbb{X}} \Phi(z + \mathscr{B}u)\, dx = \int_{\mathbb{X}} \Phi(z + \mathscr{B}\phi_j)\, dx$$

$$\|\nabla^{l-1}\phi_j\|_{L^1(\mathbb{X})} = j^{-1}\|\nabla^{l-1}u\|_{L^1(\mathbb{X})} \to 0 \quad \text{as } j \to \infty.$$

That also $\|\phi_j\|_{W^{l-1,1}} \to 0$ follows from Poincaré's inequality. $\qquad \square$

Lemma 5.3. *The set Y is weakly-* closed in $\mathbb{H}^*$.*

Proof. Let $\mu = (\mu^0, \mu^\infty) \in \overline{Y}^{w^*}$. Let $\{\Phi_j\}_{j\in\mathbb{N}}$ be a countable and dense subset of $\mathbb{H}$ and put $\Phi_0 = |\cdot|$. Now for each $i \in \mathbb{N}$ we may select $v_i \in C_c^\infty(\mathbb{X}, \mathbb{V})$ such that

$$\|v_i\|_{W^{l-1,1}} < \frac{1}{i}, \quad \max_{0 \le j \le i} \left| \int_{\mathbb{X}} \Phi_j(z + \mathscr{B}v_i) \, dx - \mu(\Phi_j) \right| < \frac{1}{i}. \quad (5.5)$$

Since $\Phi_0 = |\cdot|$ is included above we infer that the sequence $(z + \mathscr{B}v_i)$ is bounded in $L^1(\mathbb{X}, \mathbb{V})$ and so, by Banach–Alaoglu's compactness theorem and separability of $\mathbb{H}$, each of its subsequences admit a further subsequence (not relabelled) so

$$\int_{\mathbb{X}} \Phi(z + \mathscr{B}v_i) \, dx \to \ell(\Phi) \quad \forall \Phi \in \mathbb{H} \quad (5.6)$$

for some $\ell \in \mathbb{H}^*$ that at this stage of course can depend on the particular subsequence. But the density of $\{\Phi_j\}_{j\in\mathbb{N}_0}$ and the identification of $\mathbb{H}^*$ with pairs of measures allow us, by virtue of (5.5), to conclude that $\ell = \mu$ and so that (5.6) in fact holds true for the full sequence. Taken together with (5.5) we have shown that $\mu \in Y$, and the proof is complete. $\qquad\qquad\square$

Lemma 5.4. *The set Y is convex.*

Proof. Let $\nu_0, \nu_1 \in Y$, $t \in (0, 1)$ and $\nu_t = (1-t)\nu_0 + t\nu_1$. In view of Lemmas 5.2 and 5.3 it suffices to show that $\nu_t \in Y$ when

$$\nu_0 = \varepsilon_{z+\mathscr{B}u_0}, \; \nu_1 = \varepsilon_{z+\mathscr{B}u_1} \quad \text{and} \quad t = \left(\frac{p}{q}\right)^n,$$

where $u_0, u_1 \in C_c^\infty(\mathbb{X}, \mathbb{V})$ and $p, q \in \mathbb{N}$ are coprime. In fact, we shall show that $\nu_t = \varepsilon_{z+\mathscr{B}\phi}$ for some $\phi \in C_c^\infty(\mathbb{X}, \mathbb{V})$. We became aware of this possibility when reading [49].

In order to see this we subdivide each side of $\mathbb{X}$ into q disjoint congruent half-open intervals, and consider the resulting mesh of q^n disjoint congruent cubes. We fix an arrangement of the cubes, say

$$\mathbb{X}_i = x_i + q^{-1}\mathbb{X}, \quad i \in \{1, \ldots, q^n\},$$

and define

$$\phi(x) = \sum_{i=1}^{p^n} q^{-l} u_1\big(q(x - x_i)\big) + \sum_{i=p^n+1}^{q^n} q^{-l} u_0\big(q(x - x_i)\big) \quad (x \in \mathbb{X})$$

where we extend u_0, u_1 to $\mathbb{R}^n \backslash \mathbb{X}$ by 0. Clearly, $\phi \in C_c^\infty(\mathbb{X}, \mathbb{V})$ and for $\Phi \in \mathbb{H}$ we check that $\varepsilon_{z+\mathscr{B}\phi}(\Phi) = \nu_t(\Phi)$, and therefore $\nu_t \in Y$ by Lemma 5.2. $\qquad\square$

Lemma 5.5. *Let $\nu = (\nu^0, \nu^\infty) \in \mathscr{M}_1^+(\mathbb{V}) \times \mathscr{M}^+(\mathbb{S}_{\mathbb{V}})$ and $z \in \mathbb{V}$. Then $\nu \in Y$ if, and only if, $\overline{\nu}^0 + \overline{\nu}^\infty = z$ and*

$$\int_{\mathbb{V}} f \, d\nu^0 + \int_{\mathbb{S}_{\mathbb{V}}} f^\infty \, d\nu^\infty \geqq f(z)$$

holds for all $\mathscr{A}$-quasiconvex integrands of linear growth.

Proof. To prove the *only if* part we fix $v \in Y$ and an $\mathscr{A}$-quasiconvex integrand $f : \mathbb{V} \to \mathbb{R}$ of linear growth. Select a sequence (u_j) in $C_c^\infty(\mathbb{X}, \mathbb{V})$ such that $\|u_j\|_{W^{l-1,1}} \to 0$ and

$$\int_{\mathbb{X}} \Phi(z + \mathscr{B}u_j)\, dx \to v(\Phi)$$

for each $\Phi \in \mathbb{H}$. Thus for $\Phi \in \mathbb{H}$ with $\Phi \geq f$ we get by $\mathscr{A}$-quasiconvexity of f at z that

$$v(\Phi) \geq \limsup_{j\to\infty} \int_{\mathbb{X}} f(z + \mathscr{B}u_j)\, dx \geq f(z).$$

To obtain the required bound we use the approximation Lemma 4.3 to find $\mathscr{A}$-quasiconvex integrands $\Phi_j \in \mathbb{H}$ with $\Phi_j \searrow f$ and $\Phi_j^\infty \searrow f^\infty$ pointwise. Note that

$$|\Phi_j - f| = \Phi_j - f \leq \Phi_1 - f \leq |\Phi_1| + |f|,$$

and similarly $|\Phi_j^\infty - f^\infty| \leq |\Phi_1^\infty| + |f^\infty|$. By Lebesgue's dominated convergence theorem, we have

$$\int_{\mathbb{V}} f\, dv^0 + \int_{S_{\mathbb{V}}} f^\infty\, dv^\infty = \lim_{j\to\infty} v(\Phi_j) \geq f(z),$$

which concludes the proof of this implication.

We turn to the *if* part of the statement. According to Lemmas 5.3 and 5.4, Y is a weakly* closed and convex subset of $\mathbb{H}^*$, hence by the Hahn–Banach separation theorem we can write $Y = \bigcap H$, where we take intersection over all weakly* closed half-spaces H in $\mathbb{H}^*$ that contain Y. Fix such a half-space H; it is well-known that we can find $\Phi \in \mathbb{H}$, $t \in \mathbb{R}$ such that

$$H = \big\{ \ell \in \mathbb{H}^* : \ell(\Phi) \geq t \big\}.$$

Because $Y \subset H$ it follows in particular from Lemma 5.2 that $\varepsilon_{z+\mathscr{B}u} \in H$ for all $u \in C_c^\infty(\mathbb{X}, \mathbb{V})$. Consequently, the inequality

$$t \leq \varepsilon_{z+\mathscr{B}u}(\Phi) = \int_{\mathbb{X}} \Phi(z + \mathscr{B}u)\, dx$$

holds for all $u \in C_c^\infty(\mathbb{X}, \mathbb{V})$. But then the relaxation formula Theorem 3.5 yields $t \leq \Phi^{qc}(z)$ and it follows that the envelope Φ^{qc} is real-valued, $\mathscr{A}$-quasiconvex and of linear growth. Returning to our assumptions on v with this information we deduce that

$$v(\Phi) \geq \int_{\mathbb{V}} \Phi^{qc}\, dv^0 + \int_{S_{\mathbb{V}}} (\Phi^{qc})^\infty\, dv^\infty \geq \Phi^{qc}(z) \geq t,$$

that is, $v \in H$. $\square$

5.3. Inhomogenization

Let $\left(\phi_t\right)_{t>0}$ be a standard smooth mollifier on $\mathbb{R}^n$, so obtained by L^1 dilation by $t > 0$ of a nonnegative C^∞ smooth and compactly supported function $\phi \colon \mathbb{R}^n \to \mathbb{R}$ that integrates to 1: $\phi_t(x) = t^{-n}\phi(x/t)$. It is convenient to assume that the support of ϕ is the closed unit cube $\overline{\mathbb{X}} = [-\frac{1}{2}, \frac{1}{2}]^n$ and that ϕ is strictly positive on its interior $\mathbb{X}$. For later reference we put

$$M \equiv \max |\nabla \phi|. \tag{5.7}$$

We shall also in this subsection use the ℓ_∞^n-metric on $\mathbb{R}^n$ and for convenience of notation we denote it simply by $\| \cdot \|$, thus

$$\|x\| \equiv \|x\|_{\ell_\infty^n} = \max\{|x_1|, \ldots, |x_n|\} \quad \text{for} \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

The following result is the key step in our construction. We emphasize that it is elementary and purely measure theoretic.

Lemma 5.6. *Under the assumptions of Theorem* 1.2*, given* $\varepsilon > 0$ *we can find* $t_\varepsilon > 0$ *such that for each* $t \in (0, t_\varepsilon]$ *there exists* $\varphi = \varphi_t \in \mathrm{C}_c^\infty(\Omega, \mathbb{V})$ *with* $\|\varphi\|_{\mathrm{W}^{l-1,1}(\Omega, \mathbb{V})} < \varepsilon$ *so*

$$\left| \int_\Omega \eta(x)\Phi(0)\mathrm{d}x + \int_\Omega \eta(x)\langle \Phi^\infty, \nu_x^\infty \rangle \, \mathrm{d}\lambda^s \right.$$
$$\left. - \int_\Omega \eta\Phi\left(\phi_t * (\overline{\nu_\cdot^\infty}\lambda^s \llcorner \Omega) + \mathscr{B}\varphi\right)\mathrm{d}x \right| < \varepsilon \tag{5.8}$$

holds uniformly in $\eta \colon \overline{\Omega} \to \mathbb{R}$ *and* $\Phi \colon \mathbb{V} \to \mathbb{R}$ *of class* $\mathbb{H}$ *with*

$$\|\eta\|_{\mathrm{LIP}} \leqq 1 \text{ and } \|T\Phi\|_{\mathrm{LIP}} \leqq 1. \tag{5.9}$$

In line with what we stated at the start of this subsection we use the ℓ_∞^n-metric in the first bound of (5.9):

$$\|\eta\|_{\mathrm{LIP}} \equiv \sup_\Omega |\eta| + \sup_{x,y \in \Omega\, x \neq y} \frac{|\eta(x) - \eta(y)|}{\|x - y\|}.$$

It is elementary to check that $\mathrm{lip}(\Phi) \leqq 2\mathrm{lip}(T\Phi) + \max |T\Phi| \leqq 2\|T\Phi\|_{\mathrm{LIP}}$.

Proof. Let $\varepsilon \in (0, 1)$. Apply Luzin's theorem (see [7, Theorem 3.1.13] for a statement that applies in the considered generality) to the λ^s measurable map

$$\overline{\Omega} \ni x \mapsto \left(\delta_0, \nu_x^\infty\right) \in \mathscr{M}_1^+(\mathbb{V}) \times \mathscr{M}^+(\mathbb{S}_\mathbb{V}) \hookrightarrow \left(\mathbb{H}^*\right)^+$$

to find a compact subset $C^s = C^s(\varepsilon) \subset \overline{\Omega}$ such that $\lambda^s(\overline{\Omega}\backslash C^s) < \varepsilon\lambda^s(\overline{\Omega})$ and the map

$$C^s \ni x \mapsto \left(\delta_0, \nu_x^\infty\right) \in \mathscr{M}_1^+(\mathbb{V}) \times \mathscr{M}^+(\mathbb{S}_\mathbb{V})$$

is (uniformly) continuous. The latter can of course be expressed quantitatively in the sense that we can find a modulus of continuity $\omega^s = \omega^s_\varepsilon \colon [0, \infty) \to [0, \infty)$ such that for all $x, y \in C^s$ the inequality

$$\left\| \left(\delta_0, \nu^\infty_x \right) - \left(\delta_0, \nu^\infty_y \right) \right\|_{\mathrm{K}} \leqq \omega^s \left(\|x - y\| \right) \tag{5.10}$$

holds. Because λ^s and $\mathscr{L}^n$ are mutually singular we can assume that $\mathscr{L}^n(C^s) = 0$ and because $\lambda^s(\partial\Omega) = 0$ we can also assume that

$$\Delta = \Delta_\varepsilon \equiv \inf \left\{ \|x - y\| : x \in C^s,\ y \in \partial\Omega \right\} \in (0, 1). \tag{5.11}$$

Let $d, m \in \mathbb{N}$ be two natural numbers whose precise values will be specified in the course of the proof. Put $t = 2^{-d}$ and for convenience of notation write $\phi \equiv \phi_t$ so that we in particular have

$$\phi(x) \begin{cases} > 0 & \text{if } \|x\| < t \\ = 0 & \text{if } \|x\| \geq t. \end{cases} \tag{5.12}$$

Our first condition on $d \in \mathbb{N}$ is that it is so large that

$$2t \leqq \Delta \text{ that is } d \geqq \log_2 \left(\tfrac{2}{\Delta} \right). \tag{5.13}$$

The collection of $(d + m)$-th generation dyadic cubes $Q \in \mathscr{D}_{d+m}$ in $\mathbb{R}^n$ with $\mathrm{dist}(Q, \partial\Omega) \geqq t$ is denoted $\mathcal{F}$. For each $Q \in \mathcal{F}$ we define

$$r_Q \equiv \fint_Q \phi * (\lambda^s \llcorner C^s) \, dx. \tag{5.14}$$

In view of the choice of mollifier ϕ we have for cubes Q with $r_Q > 0$ that

$$\mathrm{dist}(Q, C^s) \equiv \inf \left\{ \|x - y\| : x \in Q,\ y \in C^s \right\} < t,$$

and may therefore select $x_Q \in C^s$ so

$$\mathrm{dist}(Q, x_Q) \equiv \inf \left\{ \|x - x_Q\| : x \in Q \right\} < t.$$

Summarizing, we denote $\mathcal{F}^s \equiv \{ Q \in \mathcal{F} : r_Q > 0 \}$ and have that

$$\forall\, Q \in \mathcal{F}^s\, \exists\, x_Q \in C^s \text{ such that } \sup_{x \in Q} \|x - x_Q\| < 2t. \tag{5.15}$$

We continue the selection process for these cubes and fix $Q \in \mathcal{F}^s$. Then for integrands $f \in \mathbb{H}$ that are $\mathscr{A}$-quasiconvex we get by Lemmas 2.5, 2.2 and 2.9,

$$f\left(r_Q \overline{\nu}^\infty_{x_Q} \right) \leqq f(0) + r_Q f^\infty \left(\overline{\nu}^\infty_{x_Q} \right)$$
$$\leqq f(0) + r_Q \int_{\mathbb{S}_{\mathbb{V}}} f^\infty \, d\nu^\infty_{x_Q}.$$

In connection with the Hahn–Banach Lemma 5.5 this bound allows us to select $\varphi^Q \in \mathrm{C}^\infty_c(Q, \mathbb{V})$ satisfying $\|\varphi^Q\|_{\mathrm{W}^{-1,1}(Q,\mathbb{V})} < \varepsilon \lambda^s(Q)$ and

$$\left\| \left(\delta_0, \nu^\infty_{x_Q} r_Q \right) - \boldsymbol{\varepsilon}_{r_Q \overline{\nu}^\infty_{x_Q} + \mathscr{B}\varphi^Q} \right\|_{\mathrm{K}} < \varepsilon. \tag{5.16}$$

We recall that this amounts to (where we realize the elementary Young measure on Q instead of on $\mathbb{X}$),

$$\sup_{\|T\Phi\|_{\mathrm{LIP}} \leq 1} \left| \Phi(0) + \int_{\mathbb{S}_\mathbb{V}} \Phi^\infty \, \mathrm{d}\nu^\infty_{x_Q} r_Q - \fint_Q \Phi\big(r_Q \overline{\nu}^\infty_{x_Q} + \mathscr{B}\varphi^Q\big) \, \mathrm{d}x \right| < \varepsilon.$$

If therefore we define

$$\varphi \equiv \sum_{Q \in \mathcal{F}^s} \varphi^Q,$$

then clearly

$$\varphi \in C^\infty_c(\Omega, \mathbb{V}), \quad \|\varphi\|_{\mathrm{W}^{l-1,1}(\Omega,\mathbb{V})} < \varepsilon \lambda^s(\Omega) \tag{5.17}$$

and we have (5.16) for each $Q \in \mathcal{F}^s$. The sought-after map is now

$$\xi^s \equiv \phi * \big(\overline{\nu}^\infty_\cdot \lambda^s \mathop{\llcorner} \Omega\big) + \mathscr{B}\varphi, \tag{5.18}$$

where we define $\varphi \equiv 0$ off Ω so that in particular $\xi^s \in C^\infty_c(\mathbb{R}^n, \mathbb{V})$. In order to check that ξ^s has the required properties, in addition to (5.17), we fix $\eta \colon \overline{\Omega} \to \mathbb{R}$ and $\Phi \colon \mathbb{V} \to \mathbb{R}$ of class $\mathbb{H}$ satisfying (5.9). We start by writing

$$\int_\Omega \eta \langle \Phi^\infty, \phi * \big(\nu^\infty_\cdot \lambda^s \mathop{\llcorner} \Omega\big) \rangle \, \mathrm{d}x = \int_\Omega \eta \langle \Phi^\infty, \phi * \big(\nu^\infty_\cdot \lambda^s \mathop{\llcorner} C^s\big) \rangle \, \mathrm{d}x \\ + \mathcal{E}_1, \tag{5.19}$$

where

$$|\mathcal{E}_1| \leq \int_\Omega \phi * \big(\lambda^s \mathop{\llcorner} \Omega \backslash C^s\big) \, \mathrm{d}x \leq \varepsilon \lambda^s(\Omega).$$

We emphasize that the integrals in (5.19) should be understood as

$$\int_\Omega \eta \langle \Phi^\infty, \phi * \big(\nu^\infty_\cdot \lambda^s \mathop{\llcorner} \Omega\big) \rangle \, \mathrm{d}x \equiv \int_\Omega \eta(x) \int_\Omega \left(\phi(x - y) \int_{\mathbb{S}_\mathbb{V}} \Phi^\infty \, \mathrm{d}\nu^\infty_y \right) \mathrm{d}\lambda^s(y) \, \mathrm{d}x,$$

and likewise for the integral on the right-hand side, except that the y-integration is over the set C^s instead of Ω. Since for each $Q \in \mathcal{F}$ with $r_Q = 0$ we have

$$\int_Q \eta \langle \Phi^\infty, \phi * \big(\nu^\infty_\cdot \lambda^s \mathop{\llcorner} C^s\big) \rangle \, \mathrm{d}x = 0,$$

and $\Omega \backslash \bigcup \mathcal{F} \subset B_{2t}(\partial \Omega)$ we get

$$\int_\Omega \eta \langle \Phi^\infty, \phi * \big(\nu^\infty_\cdot \lambda^s \mathop{\llcorner} C^s\big) \rangle \, \mathrm{d}x = \sum_{Q \in \mathcal{F}^s} \int_Q \eta \langle \Phi^\infty, \phi * \big(\nu^\infty_\cdot \lambda^s \mathop{\llcorner} C^s\big) \rangle \, \mathrm{d}x + \mathcal{E}_2$$

$$= \sum_{Q \in \mathcal{F}^s} \left(\int_Q \eta \, \mathrm{d}x \langle \Phi^\infty, \nu^\infty_{x_Q} \rangle r_Q + \mathcal{E}^Q_3 \right) + \mathcal{E}_2,$$

where

$$\left|\mathcal{E}_2\right| \leqq \lambda^s\big(C^s \cap B_{2t}(\partial\Omega)\big) = 0,$$

since $C_s \cap B_{2t}(\partial\Omega) = \emptyset$ according to (5.11) and (5.13). The local error terms $\mathcal{E}_3^Q$ are estimated as follows. First,

$$\left|\mathcal{E}_3^Q\right| \leqq \left|\int_Q \Big(\eta - \fint_Q \eta\Big)\langle\Phi^\infty, \phi * (v_\cdot^\infty\lambda^s \lfloor C^s)\rangle\,\mathrm{d}x\right|$$

$$+ \left|\int_Q \eta\left(\fint_Q\langle\Phi^\infty, \phi * (v_\cdot^\infty\lambda^s\lfloor C^s)\rangle\,\mathrm{d}x - \langle\Phi^\infty, v_{x_Q}^\infty\rangle r_Q\right)\right|$$

$$\leqq \|\eta\|_{\mathrm{LIP}}\mathscr{L}^n(Q)^{\frac{1}{n}} \max_{\mathbb{S}_\mathbb{V}} |\Phi^\infty| \int_Q \phi * \lambda^s\,\mathrm{d}x$$

$$+\|\eta\|_{\mathrm{LIP}}\int_Q\int_{C^s} \phi(x - y)\langle\Phi^\infty, v_y^\infty - v_{x_Q}^\infty\rangle\,\mathrm{d}\lambda^s(y)\,\mathrm{d}x$$

and invoking (5.9), (5.10) we continue with

$$\left|\mathcal{E}_3^Q\right| \leqq t \int_Q \phi * \lambda^s\,\mathrm{d}x$$

$$+ \int_Q\int_{C^s} \phi(x - y)\omega^s\big(\|y - x_Q\|\big)\,\mathrm{d}\lambda^s(y)\,\mathrm{d}x$$

$$\leqq \big(t + \omega^s(3t)\big)\int_Q \phi * \lambda^s\,\mathrm{d}x,$$

where the last inequality follows from the triangle inequality and (5.15). Next, for each $Q \in \mathcal{F}^s$ we get from (5.16) that

$$\Phi(0) + \langle\Phi^\infty, v_{x_Q}^\infty\rangle r_Q = \fint_Q \Phi(r_Q\bar{v}_{x_Q}^\infty + \mathscr{B}\varphi^Q)\,\mathrm{d}x + \mathcal{E}_4^Q,$$

where $|\mathcal{E}_4^Q| \leqq \varepsilon$ uniformly in $\Phi \in \mathbb{H}$ satisfying (5.9). Using the triangle inequality, (5.9) and (5.16) again we estimate

$$\left|\fint_Q\big(\Phi(r_Q\bar{v}_{x_Q}^\infty + \mathscr{B}\varphi^Q) - \Phi(0)\big)\,\mathrm{d}x\right| \leqq \|\boldsymbol{\varepsilon}_{r_Q\bar{v}_{x_Q}^\infty + \mathscr{B}\varphi^Q}\|_\mathrm{K}$$

$$< \|(\delta_0, v_{x_Q}^\infty r_Q)\|_\mathrm{K} + \varepsilon$$

$$\leqq 1 + r_Q + \varepsilon$$

and consequently,

$$\int_Q \eta\,\mathrm{d}x\fint_Q \Phi(r_Q\bar{v}_{x_Q}^\infty + \mathscr{B}\varphi^Q)\,\mathrm{d}x = \int_Q \eta\Phi(r_Q\bar{v}_{x_Q}^\infty + \mathscr{B}\varphi^Q)\,\mathrm{d}x + \mathcal{E}_5^Q,$$

where (recall $\varepsilon < 1$ and (5.9))

$$|\mathcal{E}_5^Q| \leqq \sup_Q \left|\eta - \fint_Q \eta\right|(1 + r_Q + \varepsilon)\mathscr{L}^n(Q)$$

$$\leqq \mathscr{L}^n(Q)^{\frac{1}{n}}\int_Q\big(2 + \phi * \lambda^s\big)\,\mathrm{d}x.$$

Finally we turn to

$$\int_Q \eta \Phi(r_Q \overline{v}_{x_Q}^\infty + \mathscr{B}\varphi^Q) \, dx = \int_Q \eta \Phi\big(\phi * (\overline{v}_\cdot^\infty \lambda^s) + \mathscr{B}\varphi^Q\big) \, dx + \mathcal{E}_6^Q.$$

To estimate the local error term $\mathcal{E}_6^Q$, $Q \in \mathcal{F}^s$, we start by using (5.9) to get the bound

$$\left| \int_Q \eta \bigg(\Phi\big(\phi * (\overline{v}_\cdot^\infty \lambda^s) + \mathscr{B}\varphi^Q\big) - \Phi\big(\phi * (\overline{v}_\cdot^\infty \lambda^s \llcorner C^s) + \mathscr{B}\varphi^Q\big) \bigg) \, dx \right|$$

$$\leqq \int_Q \phi * \big(\lambda^s \llcorner \Omega \backslash C^s\big) \, dx.$$

Another application of (5.9) yields

$$\left| \int_Q \eta \bigg(\Phi\big(r_Q \overline{v}_{x_Q}^\infty + \mathscr{B}\varphi^Q\big) - \Phi\big(\phi * (\overline{v}_\cdot^\infty \lambda^s \llcorner C^s) + \mathscr{B}\varphi^Q\big) \bigg) \, dx \right|$$

$$\leqq \int_Q \big| r_Q \overline{v}_{x_Q}^\infty - \phi * (\overline{v}_\cdot^\infty \lambda^s \llcorner C^s) \big| \, dx.$$

We estimate the last term as follows:

$$\int_Q \big| r_Q \overline{v}_{x_Q}^\infty - \phi * (\overline{v}_\cdot^\infty \lambda^s \llcorner C^s) \big| \, dx$$

$$\leqq \left| \int_Q \phi * \big((\overline{v}_{x_Q}^\infty - \overline{v}_\cdot^\infty)\lambda^s \llcorner C^s\big) \, dx \right|$$

$$+ \int_Q \left| \fint_Q \phi * (\overline{v}_\cdot^\infty \lambda^s \llcorner C^s) \, dx' - \phi * (\overline{v}_\cdot^\infty \lambda^s \llcorner C^s) \right| \, dx$$

$$\leqq \int_Q \int_{C^s} \phi(x - y) \big| \overline{v}_{x_Q}^\infty - \overline{v}_y^\infty \big| \, d\lambda^s(y) \, dx$$

$$+ \int_Q \left| \fint_Q \phi * (\overline{v}_\cdot^\infty \lambda^s \llcorner C^s) \, dx' - \phi * (\overline{v}_\cdot^\infty \lambda^s \llcorner C^s) \right| \, dx$$

$$\overset{(5.16)}{\leqq} \omega^s(3t) \int_Q \phi * \lambda^s \, dx + \mathcal{E}_7^Q,$$

where the local error terms $\mathcal{E}_7^Q$ are the mean oscillations on $Q \in \mathcal{F}^s$ times $\mathscr{L}^n(Q)$ appearing in the penultimate line. Writing out the convolutions we estimate

$$\mathcal{E}_7^Q \leqq \int_Q \fint_Q \int_{C^s} \big| \phi(x' - y) - \phi(x - y) \big| |\overline{v}_y^\infty| \, d\lambda^s(y) \, dx' \, dx$$

$$\leqq \sqrt{n} \int_Q \fint_Q \int_{Q+t\mathbb{X}} \|x' - x\| \int_0^1 |\nabla\phi(x - y + \tau(x' - x))| \, d\tau \, d\lambda^s(y) \, dx' \, dx$$

We now recall that $\phi = \phi_t$ and (5.7), so that we have the pointwise bounds $|\nabla\phi| \leqq t^{-1} M(\mathbf{1}_\mathbb{X})_t$, which implies for $x, x' \in Q$, $y \in Q + t\mathbb{X}$, and $\tau \in [0, 1]$ that

$$|\nabla\phi(x - y + \tau(x' - x))| \leqq t^{-1} M(\mathbf{1}_{2\mathbb{X}})_t(x - y),$$

so that

$$\mathcal{E}_7^Q \leqq \sqrt{n} M \frac{\mathscr{L}^n(Q)^{\frac{1}{n}}}{t} \int_Q (\mathbf{1}_{2\mathbb{X}})_t * \lambda^s \, \mathrm{d}x$$

We now have all the necessary bounds and can start to backtrack through the estimates to conclude the proof of Lemma 5.6. First we recall that we have $t = 2^{-d}$ with $d \in \mathbb{N}$ satisfying (5.13) and that the considered dyadic cubes are of higher generation $Q \in \mathscr{D}_{d+m}$ (so $\mathscr{L}^n(Q) = 2^{-n(d+m)}$). With ξ^s defined in (5.18) we get by combination of the above

$$\int_\Omega \eta \langle \Phi^\infty, \phi * (v_\cdot^\infty \lambda^s \llcorner \Omega) \rangle \, \mathrm{d}x = \int_{\bigcup \mathcal{F}^s} \eta \Phi(\xi^s) \, \mathrm{d}x + \mathcal{E}$$

where

$$|\mathcal{E}| \leqq \varepsilon \lambda^s(\Omega) + \left(t + \omega^s(3t) \right) \int_{\bigcup \mathcal{F}^s} \phi * \lambda^s \, \mathrm{d}x$$

$$+ 2^{-d-m} \int_{\bigcup \mathcal{F}^s} \left(2 + \phi * \lambda^s \right) \mathrm{d}x + \int_{\bigcup \mathcal{F}^s} \phi * \left(\lambda^s \llcorner \Omega \backslash C^s \right) \mathrm{d}x$$

$$+ \omega^s(3t) \int_{\bigcup \mathcal{F}^s} \phi * \lambda^s \, \mathrm{d}x + \sqrt{n} M 2^{-m} \int_{\bigcup \mathcal{F}^s} (\mathbf{1}_{2\mathbb{X}})_t * \lambda^s \, \mathrm{d}x$$

$$\leqq \left(2\varepsilon + 2^{-d} + 2\omega^s(3 \cdot 2^{-d}) + 2^{-d-m} + c_n M 2^{-m} \right) \lambda^s(\Omega) + 2^{1-d-m} \mathscr{L}^n(\Omega).$$

To conclude we add

$$\int_{\Omega \backslash \bigcup \mathcal{F}^s} \eta \Phi(\xi^s) \, \mathrm{d}x = \int_{\Omega \backslash \bigcup \mathcal{F}^s} \eta \, \mathrm{d}x \, \Phi(0)$$

to both sides, whereby we get

$$\int_\Omega \eta \left(\Phi(0) + \langle \Phi^\infty, \phi * (v_\cdot^\infty \lambda^s \llcorner \Omega) \rangle \right) \mathrm{d}x$$

$$= \int_\Omega \eta \Phi(\xi^s) \, \mathrm{d}x + \mathcal{E} + \int_{\bigcup \mathcal{F}^s} \eta \, \mathrm{d}x \, \Phi(0). \qquad (5.20)$$

Here we have that $\bigcup \mathcal{F}^s \subset \overline{B_{2t}(C^s)}$ and since C_s is compact with $\mathscr{L}^n(C^s) = 0$ we may find $d_\varepsilon, m_\varepsilon \in \mathbb{N}$ that in view of (5.9) only depend on $\varepsilon > 0$ so that for $d \geq d_\varepsilon$, $m \geq m_\varepsilon$ we have that

$$|\mathcal{E}| + \left| \int_{\bigcup \mathcal{F}^s} \eta \, \mathrm{d}x \, \Phi(0) \right| \leqq 3\varepsilon (\mathscr{L}^n + \lambda^s)(\Omega).$$

This completes the proof since the left-hand side of (5.20) clearly tends to

$$\int_\Omega \eta \, \mathrm{d}x \, \Phi(0) + \int_\Omega \eta \langle \Phi^\infty, v_x^\infty \rangle \, \mathrm{d}\lambda^s(x)$$

uniformly in η, Φ satisfying (5.9) as $d \to \infty$. $\qquad \qquad \square$

We turn to the absolutely continuous part, where the proof is similar though a little less technical.

Lemma 5.7. *Under the assumptions of Theorem* 1.2, *given* $\varepsilon > 0$ *we can find* $t_\varepsilon > 0$ *such that for each* $t \in (0, t_\varepsilon]$ *there exists* $\psi = \psi_t \in C_c^\infty(\Omega, \mathbb{V})$ *with* $\|\psi\|_{W^{l-1,1}(\Omega, \mathbb{V})} < \varepsilon$ *so*

$$\left| \int_\Omega \eta(x)\big(\langle \Phi, \nu_x \rangle + \lambda^a(x)\langle \Phi^\infty, \nu_x^\infty \rangle\big)\mathrm{d}x \right.$$
$$\left. - \int_\Omega \eta\Phi\big(\phi_t * \big((\overline{\nu}. + \lambda^a\overline{\nu}_.^\infty)\mathscr{L}^n \llcorner \Omega\big) + \mathscr{B}\psi\big)\,\mathrm{d}x \right| < \varepsilon \qquad (5.21)$$

holds uniformly in $\eta\colon \overline{\Omega} \to \mathbb{R}$ *and* $\Phi\colon \mathbb{V} \to \mathbb{R}$ *of class* $\mathbb{H}$ *with*

$$\|\eta\|_{\mathrm{LIP}} \leqq 1 \quad and \quad \|T\Phi\|_{\mathrm{LIP}} \leqq 1. \qquad (5.22)$$

Proof. Let $\varepsilon \in (0, 1)$ and put $M(x) = \langle \nu_x, 1 + |\cdot| \rangle + \lambda^a(x)$. Apply Luzin's theorem to the $\mathscr{L}^n$ measurable map

$$\Omega \ni x \mapsto \big(\nu_x, \lambda^a(x)\nu_x^\infty\big) \in \mathscr{M}_1^+(\mathbb{V}) \times \mathscr{M}^+(\mathbb{S}_\mathbb{V}) \equiv \big(\mathbb{H}^*\big)^+ \qquad (5.23)$$

to find a compact subset $C^a = C^a(\varepsilon) \subset \Omega$ such that

$$\int_{\Omega \setminus C^a} M(x)\,\mathrm{d}x < \varepsilon. \qquad (5.24)$$

Select a modulus of continuity $\omega^a = \omega_\varepsilon^a\colon [0, \infty) \to [0, \infty)$ such that for all x, $y \in C^a$ the inequality

$$\big\|\big(\nu_x, \lambda^a(x)\nu_x^\infty\big) - \big(\nu_y, \lambda^a(y)\nu_y^\infty\big)\big\|_{\mathrm{K}} \leqq \omega^a\big(\|x - y\|\big) \qquad (5.25)$$

holds. We consider for a $d \in \mathbb{N}$ and $s \in (0, 1)$ the following family of dyadic cubes in $\mathbb{R}^n$ of side-length $t = 2^{-d}$,

$$\mathcal{F}^a = \left\{ Q \in \mathscr{D}_d\colon \mathrm{dist}(Q, \partial\Omega) > t,\ \mathscr{L}^n(Q \cap C^a) > s\mathscr{L}^n(Q) \right\},$$

where we recall that we work with the ℓ^∞ norm on $\mathbb{R}^n$. We will specify conditions on $d \in \mathbb{N}$ and $s \in (0, 1)$ in the course of the proof. Select $x_Q \in Q \cap C^a$ for each $Q \in \mathcal{F}^a$. By Lemma 5.5, we find $\psi^Q \in C_c^\infty(Q, \mathbb{V})$ with $\|\psi^Q\|_{W^{l-1,1}(Q, \mathbb{V})} < \varepsilon\mathscr{L}^n(Q)/\mathscr{L}^n(\Omega)$ and

$$\big\|\big(\nu_{x_Q}, \lambda^a(x_Q)\nu_{x_Q}^\infty\big) - \varepsilon_{\overline{\nu}_{x_Q} + \lambda^a(x_Q)\overline{\nu}_{x_Q}^\infty + \mathscr{B}\psi^Q}\big\|_{\mathrm{K}} < \varepsilon. \qquad (5.26)$$

We extend ψ^Q by $0 \in \mathbb{V}$ to $\mathbb{R}^n \setminus Q$ and define $\psi \equiv \sum_{Q \in \mathcal{F}^a} \psi^Q$. Hereby $\psi \in C_c^\infty(\Omega, \mathbb{V})$ and $\|\psi\|_{W^{l-1,1}(\Omega, \mathbb{V})} < \varepsilon$. We then estimate for η, Φ as in (5.22):

$$\int_\Omega \eta\big(\langle \Phi, \nu_x \rangle + \lambda^a(x)\langle \Phi^\infty, \nu_x^\infty \rangle\big)\,\mathrm{d}x = \sum_{Q \in \mathcal{F}^a} \int_Q \eta\big(\langle \Phi, \nu_x \rangle + \lambda^a(x)\langle \Phi^\infty, \nu_x^\infty \rangle\big)\,\mathrm{d}x$$
$$+ \mathcal{E}_1$$

where, by virtue of (5.24) and the Lebesgue differentiation theorem,

$$|\mathcal{E}_1| \leqq \int_{\Omega \setminus (\bigcup \mathcal{F}^a)} M(x) \, dx < \varepsilon$$

for d sufficiently large, say $d \geqq d(\varepsilon, s)$. Next,

$$\sum_{Q \in \mathcal{F}^a} \int_Q \eta \big(\langle \Phi, \nu_x \rangle + \lambda^a(x) \langle \Phi^\infty, \nu_x^\infty \rangle \big) \, dx$$

$$= \sum_{Q \in \mathcal{F}^a} \fint_Q \eta \int_Q \big(\langle \Phi, \nu_x \rangle + \lambda^a(x) \langle \Phi^\infty, \nu_x^\infty \rangle \big) dx + \mathcal{E}_2,$$

where by (5.22),

$$|\mathcal{E}_2| \leqq t \sum_{Q \in \mathcal{F}^a} \int_Q \big| \langle \Phi, \nu_x \rangle + \lambda^a(x) \langle \Phi^\infty, \nu_x^\infty \rangle \big| \, dx \leqq t \int_\Omega M(x) \, dx.$$

Put $f(x) = \langle \Phi, \nu_x \rangle + \lambda^a(x) \langle \Phi^\infty, \nu_x^\infty \rangle$ and observe that for each $Q \in \mathcal{F}^a$,

$$\left| \fint_Q f \, dx - \mathcal{L}^n(Q) f(x_Q) \right| \leqq \frac{1}{s} \int_{Q \cap C^a} \big(|f - f(x_Q)| + (1-s)|f| \big) \, dx + \int_{Q \setminus C^a} |f| \, dx$$

and by virtue of (5.22), (5.24) we have $|f - f(x_Q)| \leqq \omega^a(t)$ on $Q \cap C^a$, hence

$$\left| \fint_Q f \, dx - \mathcal{L}^n(Q) f(x_Q) \right| \leqq \frac{\omega^a(t)}{s} \mathcal{L}^n(Q) + \frac{1-s}{s} \int_Q |f| \, dx + \int_{Q \setminus C^a} |f| \, dx.$$

Since by (5.22) we have $|f(x)| \leqq M(x)$ we infer from (5.24) that

$$\sum_{Q \in \mathcal{F}^a} \fint_Q \eta \int_Q f \, dx = \sum_{Q \in \mathcal{F}^a} f(x_Q) \int_Q \eta + \mathcal{E}_3,$$

where

$$|\mathcal{E}_3| \leqq \frac{\omega^a(t)}{s} \mathcal{L}^n(\Omega) + \frac{1-s}{s} \int_\Omega M \, dx + \varepsilon.$$

We now invoke (5.26) whereby we for each $Q \in \mathcal{F}^a$ have that

$$f(x_Q) = \fint_Q \Phi \left(\overline{\nu}_{x_Q} + \lambda^a(x_Q) \overline{\nu}_{x_Q}^\infty + \mathcal{B} \psi^Q \right) \, dx + \mathcal{E}_4^Q,$$

where $|\mathcal{E}_4^Q| \leqq \varepsilon$. Put $v^a(x) = \overline{\nu}_x + \lambda^a(x) \overline{\nu}_x^\infty$. Because $\Phi(z) = z \cdot e_i$ satisfies (5.22) for each i, where (e_i) is an orthonormal basis for $\mathbb{V}$ we get from (5.25) that $|v^a - v^a(x_Q)| \leqq \omega^a(t)$ on $Q \cap C^a$ for each $Q \in \mathcal{F}^a$. Consequently,

$$\int_Q |v^a - v^a(x_Q)| \, dx \leqq \frac{\omega^a(t)}{s} \mathcal{L}^n(Q) + \int_{Q \setminus C^a} |v^a| \, dx + \frac{1-s}{s} \int_Q |v^a| \, dx$$

for each $Q \in \mathcal{F}^a$. Using that $|v^a| \leqq M$ and $\mathrm{lip}(\Phi) \leqq 2\|T\Phi\|_{\mathrm{LIP}} \leqq 2$ we find

$$\sum_{Q \in \mathcal{F}^a} \fint_Q \eta \, \mathrm{d}x \int_Q \Phi(v^a(x^Q) + \mathscr{B}\psi^Q) \, \mathrm{d}x = \sum_{Q \in \mathcal{F}^a} \fint_Q \eta \, \mathrm{d}x \int_Q \Phi(v^a + \mathscr{B}\psi^Q) \, \mathrm{d}x + \mathcal{E}_5$$

and

$$|\mathcal{E}_5| \leqq 2 \frac{\omega^a(t)}{s} \mathscr{L}^n(\Omega) + 2\varepsilon + 2\frac{1-s}{s} \int_\Omega M \, \mathrm{d}x.$$

Using (5.22) and some of the previous estimates again we get

$$\sum_{Q \in \mathcal{F}^a} \fint_Q \eta \, \mathrm{d}x \int_Q \Phi(v^a + \mathscr{B}\psi^Q) \, \mathrm{d}x = \sum_{Q \in \mathcal{F}^a} \int_Q \eta \Phi(v^a + \mathscr{B}\psi^Q) \, \mathrm{d}x + \mathcal{E}_6,$$

and

$$|\mathcal{E}_6| \leqq t \sum_{Q \in \mathcal{F}^a} \int_Q |\Phi(v^a + \mathscr{B}\psi^Q)| \, \mathrm{d}x$$

$$\leqq t|\mathcal{E}_5| + t \sum_{Q \in \mathcal{F}^a} \int_Q |\Phi(v^a(x_Q) + \mathscr{B}\psi^Q)| \, \mathrm{d}x$$

$$\overset{(5.26)}{\leqq} t|\mathcal{E}_5| + t\varepsilon \mathscr{L}^n(\Omega) + t \sum_{Q \in \mathcal{F}^a} \mathscr{L}^n(Q)\left(\langle |\Phi|, \nu_{x_Q}\rangle + \lambda^a(x_Q)\langle |\Phi|^\infty, \nu_{x_Q}^\infty\rangle\right)$$

$$\overset{(5.25)}{\leqq} t|\mathcal{E}_5| + t\varepsilon \mathscr{L}^n(\Omega) + t\frac{\omega^a(t)}{s} \mathscr{L}^n(\Omega) + \int_{\Omega \setminus C^a} M \, \mathrm{d}x + \frac{1-s}{s} \int_\Omega M \, \mathrm{d}x.$$

Finally we have for a standard smooth mollifier $(\phi_t)_{t>0}$ that

$$\int_\Omega |\phi_t * (v^a \llcorner \Omega) - v^a| \, \mathrm{d}x \to 0 \text{ as } t \searrow 0$$

and so by (5.22),

$$\int_\Omega \eta \Phi(v^a + \mathscr{B}\psi) \, \mathrm{d}x = \int_\Omega \eta \Phi(\phi_t * (v^a \llcorner \Omega) + \mathscr{B}\psi) \, \mathrm{d}x + \mathcal{E}_7,$$

where

$$|\mathcal{E}_7| \leqq \mathrm{lip}\Phi \int_\Omega |\phi_t * (v^a \llcorner \Omega) - v^a| \, \mathrm{d}x \leqq 2 \int_\Omega |\phi_t * (v^a \llcorner \Omega) - v^a| \, \mathrm{d}x.$$

Collecting the above estimates we end the proof. □

We can now conclude the proof of Theorem 1.2 if we combine the Lemmas 5.6, 5.7, 2.4 with the following general result.

Lemma 5.8. *Suppose that the sequences (v_j), (w_j) in $L^1(\Omega, \mathbb{V})$ generate the Young measures μ, ν, respectively. Assume that the oscillation measure for μ is trivial, say $\mu_x = \delta_{v(x)}$ for some $v \in L^1(\Omega, \mathbb{V})$, and that the concentration measures for μ, ν are mutually singular, say $\Omega = \Omega_\mu \cup \Omega_\nu$ is a Borel partition with $\lambda_\mu(\Omega_\nu) = \lambda_\nu(\Omega_\mu) = 0$. Then the sum sequence $(v_j + w_j)$ generates the Young measure κ, where the oscillation measure is $\kappa_x = \delta_{v(x)} * \nu_x$ for $\mathscr{L}^n$ almost everywhere $x \in \Omega$, the concentration measure is $\lambda_\kappa = \lambda_\mu + \lambda_\nu$ and the concentration-angle measure is*

$$\kappa_x^\infty = \begin{cases} \mu_x^\infty & \text{for } \lambda_\mu \text{ almost everywhere } x \in \Omega_\mu \\ \nu_x^\infty & \text{for } \lambda_\nu \text{ almost everywhere } x \in \Omega_\nu. \end{cases}$$

The proof relies on the following decomposition result that is closely related to [17, Lemma 2.31].

Lemma 5.9. *Assume (w_j) is a sequence in $L^1(\Omega, \mathbb{V})$ that generates the Young measure $\nu = ((\nu_x), \lambda, (\nu_x^\infty))$. Then we can decompose $w_j = g_j + b_j$, where (g_j), (b_j) are sequences in $L^1(\Omega, \mathbb{V})$ such that (g_j) is equi-integrable, $g_j \xrightarrow{Y} ((\nu_x), 0, \text{n/a})$ and $b_j \xrightarrow{Y} ((\delta_0), \lambda, (\nu_x^\infty))$. The converse is also true: if (g_j) is equi-integrable, $g_j \xrightarrow{Y} ((\nu_x), 0, \text{n/a})$ and $b_j \xrightarrow{Y} ((\delta_0), \lambda, (\nu_x^\infty))$, then $g_j + b_j \xrightarrow{Y} \nu$.*

Proof. It is not difficult to see that we may choose $k_j \nearrow \infty$ such that the truncated sequence $g_j \equiv w_j \mathbf{1}_{\{|w_j| \leq k_j\}}$ is equi-integrable and generates the Young measure $((\nu_x), 0, \text{n/a})$. Put $b_j \equiv w_j - g_j$. In order to show that (b_j) generates the Young measure as asserted we use the integrands $\eta \otimes \Phi$, where $\eta \in C(\overline{\Omega})$, $\Phi \in \mathbb{H}$ with $\|T\Phi\|_{\text{LIP}} \leq 1$ (see Lemma 2.4). Now

$$\int_\Omega \eta\big(\Phi(b_j) - \Phi(w_j)\big)\, dx = \int_{|w_j| \leq k_j} \eta\big(\Phi(0) - \Phi(g_j)\big)\, dx$$
$$+ \int_{|w_j| > k_j} \eta\big(\Phi(w_j) - \Phi(w_j)\big)\, dx$$
$$= \int_\Omega \eta\big(\Phi(0) - \Phi(g_j)\big)\, dx$$
$$\to \int_\Omega \eta\big(\Phi(0) - \langle \nu_x, \Phi \rangle\big)\, dx \text{ as } j \to \infty,$$

hence we get that, as $j \to \infty$,

$$\int_\Omega \eta \Phi(b_j)\, dx \to \int_\Omega \eta \langle \nu_x, \Phi \rangle\, dx$$
$$+ \int_{\overline{\Omega}} \eta \langle \nu_x^\infty, \Phi^\infty \rangle\, d\lambda$$
$$+ \int_\Omega \eta\big(\Phi(0) - \langle \nu_x, \Phi \rangle\big)\, dx$$
$$= \int_\Omega \eta\, dx\, \Phi(0) + \int_{\overline{\Omega}} \eta \langle \nu_x^\infty, \Phi^\infty \rangle\, d\lambda.$$

This is exactly what we wanted to prove.

The opposite direction is a straightforward subsequence argument: any subsequence of $(g_j + b_j)$ admits a further subsequence that generates a Young measure. Using the properties of (g_j), (b_j) it is easily shown that the Young measure in each instance is ν. We leave the details to the interested reader.   □

Proof of Lemma 5.8. We start by writing $w_j = g_j + b_j$ as in the decomposition lemma 5.9 and put $c_j = v_j - v$. We will show that

$$ b_j + c_j \overset{Y}{\to} \big((\delta_0)_{x\in\Omega}, \lambda_\mu + \lambda_\nu, (\kappa_x^\infty)_{x\in\overline{\Omega}}\big), \tag{5.27} $$

where the concentration-angle measure is defined in Lemma 5.8. Once this is accomplished the proof is easily completed using that $(v + g_j)$ is equi-integrable and the last part of Lemma 5.9. Note that we have, $b_j + c_j \to 0$ in $\mathscr{L}^n$ measure, so the oscillation measure will be trivial. To identify the other parts of the Young measure (and to show that one is indeed generated) we employ again Lemma 2.4. Fix $\eta \in C(\overline{\Omega})$ with $\|\eta\|_\infty \leq 1$, $\Phi \in \mathbb{H}$ with $\|T\Phi\|_{\mathrm{LIP}} \leq 1$ and $\Phi(0) = 0$. Let $\varepsilon > 0$ and find disjoint compact sets $C_\mu, C_\nu \subset \overline{\Omega}$ and open sets in $\mathbb{R}^n$ with $C_\mu \subset O_\mu$, $C_\nu \subset O_\nu$ such that $\lambda_\mu(\overline{\Omega}\backslash C_\mu) + \lambda_\nu(\overline{\Omega}\cap O_\mu) < \varepsilon$, $\lambda_\nu(\overline{\Omega}\backslash C_\nu) + \lambda_\mu(\overline{\Omega}\cap O_\nu) < \varepsilon$. Next, let $\rho \in C(\mathbb{R}^n)$ be a function satisfying $\mathbf{1}_{C_\mu} \leq \rho \leq \mathbf{1}_{O_\mu}$. Write

$$ I = \int_\Omega \eta\big(\Phi(b_j + c_j) - \Phi(b_j) - \Phi(c_j)\big)\,\mathrm{d}x = II + III, $$

where

$$ II = \int_\Omega \eta\rho\big(\Phi(b_j + c_j) - \Phi(b_j) - \Phi(c_j)\big)\,\mathrm{d}x \quad\text{and}\quad III = I - II. $$

Note that

$$ \limsup_{j\to\infty} |II| \leq \limsup_{j\to\infty} 2\int_\Omega \rho|b_j|\,\mathrm{d}x = 2\int_{\overline{\Omega}} \rho\,\mathrm{d}\lambda_\nu \leq 2\lambda_\nu(\overline{\Omega}\cap O_\mu) < 2\varepsilon, $$

and similarly,

$$ \limsup_{j\to\infty} |III| \leq \limsup_{j\to\infty} 2\int_\Omega (1 - \rho)|c_j|\,\mathrm{d}x \leq 2\lambda_\mu(\overline{\Omega}\backslash C_\mu) < 2\varepsilon. $$

It follows that $I \to 0$ as $j \to \infty$, and consequently we have proved (5.27).   □

Remark 5.10. By Lemmas 2.1 and 5.7 we find a sequence (u_j) in $C_c^\infty(\Omega, \mathbb{V})$ such that $\|u_j\|_{W^{l-1,1}} \to 0$ and for any mollifier ϕ_j,

$$ \phi_j * \big((\overline{v}. + \lambda^a\overline{v}.^\infty)\mathscr{L}^n \llcorner \Omega\big) + \mathscr{B}u_j \overset{Y}{\to} \big((v_x)_{x\in\Omega}, \lambda^a\mathscr{L}^n\llcorner\Omega, (v_x^\infty)_{x\in\Omega^a}\big), $$

where $\Omega^a \equiv \{x \in \Omega : \lambda^a(x) > 0\}$. By properties of Young measures it follows that $\mathscr{B}u_j \overset{*}{\rightharpoonup} 0$ in $\mathscr{M}(\Omega, \mathbb{V})$ and

$$ \phi_j * \big(\lambda^a\overline{v}.^\infty\mathscr{L}^n\llcorner\Omega\big) + \mathscr{B}u_j \to 0 \quad\text{in the biting sense on } \Omega. $$

6. Concentration Angle Measures and Proof of Theorem 1.3

The proof relies on the notion of convex deficiency integrand that we introduced and used in [32] and that we will generalize and develop further here. Let $f: \mathbb{V} \to \mathbb{R}$ be a Lipschitz integrand. We then define its **convex deficiency integrand** $\mathrm{CD}_f: \mathbb{V} \to \mathbb{R}$ as

$$\mathrm{CD}_f(z) \equiv \sup \left\{ \frac{f(w + tz) - f(w)}{t} : w \in \mathbb{V}, \ t > 0 \right\}. \tag{6.1}$$

Note that hereby CD_f is Lipschitz with $\mathrm{lip}(\mathrm{CD}_f) = \mathrm{lip}(f)$ and that it is the smallest positively 1-homogeneous integrand such that

$$f(w + z) \leqq f(w) + \mathrm{CD}_f(z) \tag{6.2}$$

holds for all $z, w \in \mathbb{V}$. It might not be immediately clear, but CD_f is a convex integrand:

Lemma 6.1. *Let $f: \mathbb{V} \to \mathbb{R}$ be a Lipschitz integrand. Then*

$$\mathrm{CD}_f(z) = \sup \left\{ f'(w) \cdot z : f \text{ is differentiable at } w \in \mathbb{V} \right\}$$

holds for all $z \in \mathbb{V}$. Thus CD_f is the support function for the essential range of the derivative f'.

Remark 6.2. While we shall not make use of this observation here we record that the compact convex set that CD_f is support function for is the set of all Clarke subgradients of f:

$$\left\{ z' \in \mathbb{V} : z' \text{ is a Clarke subgradient for } f \right\} = \bigcap_{w \in \mathbb{V}} \left\{ z' \in \mathbb{V} : z' \cdot w \leqq \mathrm{CD}_f(w) \right\}.$$

Proof. Let $R(z)$ denote the integrand on the right-hand side and put $d = \dim \mathbb{V}$. Because f is assumed Lipschitz it follows by Rademacher's theorem that f is differentiable $\mathscr{H}^d$ almost everywhere on $\mathbb{V}$ and that the essential range of the derivative f' is compact.

Let $w \in \mathbb{V}$ be a point of differentiability for f. Then we have for all $z \in \mathbb{V}$ that

$$f'(w) \cdot z = \lim_{s \searrow 0} \frac{f(w + sz) - f(w)}{s} \leqq \mathrm{CD}_f(z),$$

and so taking the supremum over all such w we arrive at $R(z) \leqq \mathrm{CD}_f(z)$.

For the opposite inequality we fix $z \in \mathbb{V} \backslash \{0\}$. By Fubini's theorem it follows that for $\mathscr{H}^{d-1}$ almost all $w' \in \{z\}^\perp$, f is differentiable $\mathscr{H}^1$ almost everywhere on $w' + \mathrm{span}\{z\}$. Now fix $w \in \mathbb{V}$ such that f is differentiable $\mathscr{H}^1$ almost everywhere on $w + \mathrm{span}\{z\}$. By the fundamental theorem of calculus and the assumed differentiability we have for all $t > 0$,

$$\frac{f(w + tz) - f(w)}{t} = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} f(w + stz) \, \mathrm{d}s = \int_0^1 f'(w + stz) \cdot z \, \mathrm{d}s.$$

But the right-hand side is bounded above by $R(z)$ and since the left-hand side is continuous in w it follows that

$$\frac{f(w + tz) - f(w)}{t} \leqq R(z)$$

holds for all $w \in \mathbb{V}$ and $t > 0$, and therefore that $\mathrm{CD}_f(z) \leqq R(z)$.   □

The choice of terminology for CD_f is explained by the following result.

Lemma 6.3. *Let* $f : \mathbb{V} \to \mathbb{R}$ *be a Lipschitz integrand. Then* $f^\infty(z) \leqq \mathrm{CD}_F(z)$ *for all* $z \in \mathbb{V}$ *with equality when* f *is convex in the direction of* z *(or* $z = 0$*). Hence in particular,* $\mathrm{CD}_f = f^\infty$ *when* f *is convex.*

Proof. By Lipschitz continuity,

$$f^\infty(z) = \limsup_{t \to \infty} \frac{f(tz) - f(0)}{t} \leqq \mathrm{CD}_f(z)$$

holds for all $z \in \mathbb{V}$. It is clear that equality holds when $z = 0$. Next fix $z \neq 0$ and assume f is convex in the direction of z, that is, for each fixed $w \in \mathbb{V}$ the univariate function $t \mapsto f(tz + w)$ is convex. Hence for $s > 0$,

$$f^\infty(z) = \sup_{t > 0} \frac{f(tz + w) - f(w)}{t} \geqq \frac{f(sz + w) - f(w)}{s}$$

and so taking supremum over $s > 0$, $w \in \mathbb{V}$, we arrive at $f^\infty(z) \geqq \mathrm{CD}_f(z)$. The last argument obviously also yields the last assertion of the lemma, namely that $\mathrm{CD}_f = f^\infty$ holds when f is convex.   □

We write $\lambda = \lambda^a \mathscr{L}^n \llcorner \Omega + \lambda^s$ for the Radon–Nikodým decomposition of λ with respect to $\mathscr{L}^n$.

Proof of Theorem 1.3. From Proposition 1.1, we have for some $\mathscr{L}^n$ negligible set $N^a \subset \Omega$ that

$$F(\bar{v}_x + \lambda^a(x)\bar{v}_x^\infty) \leqq \int_{\mathbb{V}} F \mathrm{d}v_x + \lambda^a(x) \int_{\mathbb{S}_{\mathbb{V}}} F^\infty \mathrm{d}v_x^\infty \quad \text{for} \ \ x \in \Omega \backslash N^a$$

for all quasiconvex integrands F of linear growth. Suppose we are at a point $x \in \Omega$ of *diffuse concentration* meaning that $t \equiv \lambda^a(x) > 0$ (we work with the precise representative of λ^a here). Consider for $\varepsilon > 0$ and $\xi \in \mathbb{V}$ the integrand

$$f_\varepsilon(z) = \frac{F(\xi + \varepsilon z) - F(\xi)}{\varepsilon},$$

which is itself quasiconvex and $f_\varepsilon^\infty = F^\infty$. If ξ is a point of differentiability of F, then

$$\lim_{\varepsilon \searrow 0} f_\varepsilon(z) = F'(\xi) \cdot z \quad \text{locally uniformly in } z.$$

Testing the above Jensen-type inequality with f_ε and passing to the limit $\varepsilon \searrow 0$, we obtain by routine means

$$F'(\xi) \cdot \left(\bar{v}_x + t\bar{v}_x^\infty\right) \leqq \int_{\mathbb{V}} F'(\xi) \cdot z \mathrm{d}\nu_x(z) + t \int_{\mathbb{S}_{\mathbb{V}}} F^\infty \mathrm{d}\nu_x^\infty$$

$$= F'(\xi) \cdot \bar{v}_x + t \int_{\mathbb{S}_{\mathbb{V}}} F^\infty \mathrm{d}\nu_x^\infty,$$

which is rearranged as $F'(\xi) \cdot \bar{v}_x^\infty \leqq \langle F^\infty, \nu_x^\infty \rangle_{\mathcal{M},\mathrm{C}}$. Taking the supremum over such ξ on the left hand side, we arrive at the "Jensen" inequality

$$\mathrm{CD}_F(\bar{v}_x^\infty) \leqq \int_{\mathbb{S}_{\mathbb{V}}} F^\infty \mathrm{d}\nu_x^\infty \quad \text{for} \quad x \in \Omega \backslash N^a \text{ with } \lambda^a(x) > 0. \qquad (6.3)$$

For $x \in \Omega \backslash N^a$ with $\lambda^a(x) > 0$ we take $z = \bar{v}_x$, $\xi = \lambda^a(x)\bar{v}_x^\infty$ in (6.1) to get for F as above

$$F(\bar{v}_x + \lambda^a(x)\bar{v}_x^\infty) \leqq F(\bar{v}_x) + \mathrm{CD}_F(\bar{v}_x^\infty)\lambda^a(x),$$

and in combination with (6.3) we get

$$F\left(\bar{v}_x + \lambda^a(x)\bar{v}_x^\infty\right) \leqq F(\bar{v}_x) + \lambda^a(x) \int_{\mathbb{S}_{\mathbb{V}}} F^\infty \mathrm{d}\nu_x^\infty. \qquad (6.4)$$

Clearly this inequality extends to all $x \in \Omega \backslash N^a$ by declaring that 0 times undefined is 0. Hence we have obtained Jensen inequalities for all $\mathscr{A}$-quasiconvex F of linear growth against the Young measure $(\delta_{\bar{v}_x}, \lambda^a \mathscr{L}^n, \nu_x^\infty)$. We can then employ Lemma 5.7 and find a sequence (ψ_j) in $\mathrm{C}_c^\infty(\Omega, \mathbb{V})$ such that $\|\psi_j\|_{\mathrm{W}^{l-1,1}} \to 0$ and for any smooth mollifier ϕ_j on $\mathbb{R}^n$,

$$\phi_j * \left((\bar{v}_\cdot + \lambda^a \bar{v}_\cdot^\infty)\mathscr{L}^n \llcorner \Omega\right) + \mathscr{B}\psi_j \text{ generates } \left(\delta_{\bar{v}_x}, \lambda^a \mathscr{L}^n, \nu_x^\infty\right)$$

On the other hand, we can use Lemma 5.6 to find a sequence (φ_j) in $\mathrm{C}_c^\infty(\Omega, \mathbb{V})$ such that $\|\varphi_j\|_{\mathrm{W}^{l-1,1}} \to 0$ and

$$\phi_j * \left(\bar{v}_\cdot^\infty \lambda^s \llcorner \Omega\right) + \mathscr{B}\varphi_j \text{ generates } \left(\delta_0, \lambda^s, \nu_x^\infty\right).$$

Finally, Lemma 5.8 enables us to add the two generating sequences above and conclude the proof by setting $u_j = \psi_j + \varphi_j$. $\qquad \square$

The above proof constitutes one direction in the following characterization of the concentration part of an $\mathscr{A}$-free Young measure.

Proposition 6.4. *Let $\mathscr{A}$ be an operator* (1.1) *with constant rank* (1.2) *and spanning wave cone* (1.3).

Let $\lambda \in \mathscr{M}^+(\overline{\Omega})$ with $\lambda(\partial\Omega) = 0$ and let $(\mu_x)_{x \in \Omega}$ be a λ measurable parametrized family of probability measures $\mu_x \in \mathscr{M}_1^+(\mathbb{S}_{\mathbb{V}})$. Then (λ, μ_x) is the concentration part of an $\mathscr{A}$-free Young measure if, and only if,

(i) $\mathscr{A}\left(\overline{\mu}_\cdot\lambda\right) \in \mathscr{A}\left(\mathrm{L}^1(\Omega, \mathbb{V})\right)$, *and*

(ii) *for all* $f \in \mathrm{SQ}$ *(special $\mathscr{A}$-quasiconvex integrands, defined in Section* 4.2*)*

$$\mathrm{CD}_f(\overline{\mu}_x) \leqq \int_{\mathbb{S}_{\mathbb{V}}} f^{\infty} \, \mathrm{d}\mu_x \text{ for } \lambda^a \mathscr{L}^n \text{ almost all } x \in \Omega$$

holds,

where $\lambda = \lambda^a \mathscr{L}^n + \lambda^s$ *is the Lebesgue–Radon–Nikodým decomposition of λ with respect to $\mathscr{L}^n$.*

Proof. It remains to prove the *if* part of the statement. Hence assuming (i), (ii) and put $w = \overline{\mu}.\lambda$ we take $v \in \mathrm{L}^1(\Omega, \mathbb{V})$ such that $\mathscr{A}v = -\mathscr{A}w$. Then $u \equiv v\mathscr{L}^n + w \in \mathscr{M}(\Omega, \mathbb{V})$ and $\mathscr{A}u = 0$. Put $\mu = (\delta_{v(x)}, \lambda, \mu_x)$. Then μ is a Young measure with an $\mathscr{A}$-free barycenter, $\overline{\mu} = v\mathscr{L}^n \llcorner \Omega + \overline{\mu}.\lambda = u$. Next, if $f \in \mathrm{SQ}$, then by (ii) we have for $\mathscr{L}^n$ almost all $x \in \Omega$ with $\lambda^a(x) > 0$ that

$$f\big(v(x) + \overline{\mu}_x \lambda^a(x)\big) \leqq f(v(x)) + \mathrm{CD}_f(\overline{\mu}_x)\lambda^a(x)$$

$$\leqq f(v(x)) + \int_{\mathbb{S}_{\mathbb{V}}} f^{\infty} \, \mathrm{d}\mu_x \lambda^a(x).$$

Consequently Theorem 1.2 ensures that ν is an $\mathscr{A}$-free Young measure. □

7. Beyond the Constant Rank Condition

Our paper is concerned with the characterization of weak convergence effects for (asymptotically) $\mathscr{A}$-free sequences via Jensen inequalities with respect to $\mathscr{A}$-quasiconvex functions. Throughout, we assumed that the linear differential operator $\mathscr{A}$ has constant rank. In this final section, we discuss the difficulties that emerge in the absence of the rank condition, prove a generation statement for general operators that follows from our method, show how it can be used to retrieve the constant rank case, and define a large class of operators where we conjecture that our method is sufficient for the claim of Theorem 1.2 to hold (obviously, this class contains constant rank operators).

7.1. Sharp Differences in the non Constant Rank Case: Perturbations and Quasiconvexity

In this section, we will highlight three important differences between the constant rank and non constant rank cases. In particular, we will give an example of an operator $\mathscr{A}$ for which

(1) the topology given by convergence $v_j \rightharpoonup v$ and $\mathscr{A}v_j \to \mathscr{A}v$ is different from the topology given by $v_j \rightharpoonup v$ under $\mathscr{A}v = 0$;
(2) $\mathscr{A}$-quasiconvexity is different if tested with compactly supported or periodic $\mathscr{A}$-free fields;
(3) the weak convergence effects for (exactly) $\mathscr{A}$-free sequences cannot arise only from a vector potential operator $\mathscr{B}$.

As can be seen by reading our paper, none of these three phenomena can occur under the constant rank assumption.

Regarding (7.1), this has a dramatic impact on the easier implication of our characterization result, Proposition 1.1. Most known methods to prove Jensen inequalities of this flavour, rely on bounds in the spirit of (4.6), which are known to fail for non constant rank operators, cf. [22]. In fact, lower semicontinuity for variational integrals under non constant rank constraints were only established in a few examples, see [35,40].

Here we use the *separate convexity* operator from [40] to show that without the rank condition there exist asymptotically $\mathscr{A}$-free weak convergence effects that cannot be achieved by exactly $\mathscr{A}$-free sequences.

To showcase this, suppose we are now in the quadratic case and have

$$v_j \rightharpoonup v \ \ \text{in} \ \ \mathrm{L}^2(Q, \mathbb{V}), \quad \mathscr{A}v_j \to \mathscr{A}v \ \ \text{in} \ \ \mathrm{H}^{-k}(Q, \mathbb{W}), \tag{7.1}$$

where we consider only Q-periodic vector fields with zero average, $Q = (0, 1)^n$.

We can write a formula for the L^2-projections of v_j on $\ker \mathscr{A}$, independently of the rank assumption, using Fourier coefficients:

$$\widehat{\Pi v}(\xi) \equiv \Pi_{\ker \mathscr{A}(\xi)^\perp} \hat{v}(\xi) \ \ \text{for} \ \ 0 \neq \xi \in \mathbb{Z}^n.$$

In particular, since projections are uniformly bounded by 1 on Euclidean spaces, this L^2-projection is bounded as well. In particular, a continuous Helmholtz decomposition

$$v_j = \tilde{v}_j + \Pi v_j$$

with $\mathscr{A}\tilde{v}_j = 0$ survives. The issue, however, is that the compensation condition is insufficient to control the second term. It was observed in [22, Cor.] that the bound

$$\|\Pi v\|_{\mathrm{L}^2} \leqq c\|\mathscr{A}v\|_{\mathrm{H}^{-k}} \tag{7.2}$$

cannot hold, so we can conclude that the second convergence in (7.1) is insufficient to prove that $\mathrm{dist}_{\mathrm{L}^2}(v_j - v, \ker \mathscr{A}) \to 0$ in the non constant rank case. Here we give an example to show that this convergence need not hold even considering both conditions in (7.1).

Example 7.1. Let $\mathscr{A}(v^1, v^2) \equiv (\partial_2 v^1, \ \partial_1 v^2)$. There exists a sequence $v_j \in \mathrm{L}^2(Q, \mathbb{R}^2)$ such that (7.1) holds with $v = 0$ but $\mathrm{dist}_{\mathrm{L}^2}(v_j, \ker \mathscr{A}) = 1$ for all j.

To prove this, let $v_j^1(x) \equiv \exp(2\pi \mathrm{i}(j, 1) \cdot x)$ and $v_j^2 \equiv 0$. Then clearly $v_j^1 \overset{*}{\rightharpoonup} 0$ in $\mathrm{L}^\infty(Q)$ and

$$\frac{\mathscr{A}(\xi)\hat{v}_j(\xi)}{|\xi|} = \left(\frac{\xi_2}{|\xi|} \widehat{v_j^1}(\xi), 0 \right), \quad \text{so} \quad \|\mathscr{A}v_j\|_{H^{-1}} = \frac{1}{|(j, 1)|} \to 0.$$

On the other hand, $\mathrm{dist}_{\mathrm{L}^2}(v_j, \ker \mathscr{A}) = \|\Pi v_j\|_{\mathrm{L}^2} = \|v_j\|_{\mathrm{L}^2} = 1$. In particular, one needs a new argument to deal with the terms (Πv_j). In the known cases, this was performed by splitting into a strongly convergent part and a part that was controlled by directional convexity. It is very unclear how one can proceed in general or even if

it should always be the case that (7.1) implies Jensen inequalities for $\mathscr{A}$-quasiconvex f of quadratic growth.

Another sharp difference between the constant and non constant rank cases can be immediately identified by examining the very definition of $\mathscr{A}$-quasiconvexity. For the operator $\mathscr{A}$ of Example 7.1, it is easy to see that solving

$$\mathscr{A}v = 0$$

in $C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ yields only the trivial solution and that the periodic solutions are of the form $(v^1(x_1), v^2(x_2))$. In particular, $\mathscr{A}$-quasiconvexity is then equivalent with separate convexity and it does not follow by testing with compactly supported $\mathscr{A}$-free fields. The sufficiency of testing with compactly supported $\mathscr{A}$-free fields holds true for constant rank operators by [42, Cor. 1] and we used it crucially when constructing our generating sequence. We will further discuss the meaning and circumstances of testing the quasiconvexity inequality solely with compactly supported fields in the next sections; see also [24, Sect. 5] for a comparison of the notions of *compactly supported $\mathscr{A}$-quasiconvexity* and *periodic $\mathscr{A}$-quasiconvexity*. This clarifies point (7.1).

As for (7.1), we remark that an important consequence of our main results is the following: if an operator $\mathscr{A}$ has constant rank, then all weak convergence effects of $\mathscr{A}$-free sequences can be described by a vector potential operator $\mathscr{B}$. This phenomenon was first noticed in [42, Prop. 1] and refined in different contexts in [20,21,23]. Here we show that for operators of non constant rank, vector potentials cannot describe the solution space:

Proposition 7.2. *Let $\mathscr{A}$ be as in Example* 7.1. *Then for any vector space $\mathbb{U}$ there exists **no** nonzero linear differential operator with constant coefficients $\mathscr{B}$ (possibly inhomogeneous) on $\mathbb{R}^2$ from $\mathbb{U}$ to $\mathbb{R}^2$ such that*

$$\{v \in C^\infty(\mathbb{T}_2, \mathbb{R}^2) \colon \mathscr{A}v = 0, \ \hat{v}(0) = 0\} \supset \{\mathscr{B}u \colon u \in C_c^\infty((0, 1)^2, \mathbb{U})\},$$

where $(0, 1)^2$ is identified with $\mathbb{T}_2$ in an obvious way.

Proof. The functions on the left hand side are of the form $(v^1(x_1), v^2(x_2))$ for $v_i \in C^\infty(\mathbb{T}_1)$, $\hat{v}_i(0) = 0$. We notice that these maps have a nonlocal structure in $\mathbb{T}_2$, whereas the right hand side is local. Indeed, since $\mathscr{B}$ cannot be identically zero, there must exist a map $u \in C_c^\infty(Q, \mathbb{U})$ such that $\mathscr{B}u$ is not zero everywhere, say $\mathscr{B}u(X) \neq 0$. Here $Q \subset \mathbb{R}^2$ is a cube with edges parallel with the axes which we identify with the torus. If $(\mathscr{B}u(x))_i = v^i(x_i)$, then without loss of generality, we can assume that $(\mathscr{B}u(X_1, x_2))_1 = (\mathscr{B}u(X))_1 \neq 0$ for all x_2. This contradicts the fact that $\mathscr{B}u = 0$ in $Q \backslash \mathrm{supp}\,(u)$.   □

Therefore, there are examples of non constant rank operators for which our methods cannot be used. In the remainder of this section, we will focus on identifying what we believe is the largest class where our method to prove Theorem 1.2 extends.

7.2. X-Quasiconvexity and Generation: the Proof of Proposition 1.4

We will now place a very general umbrella on top of our main generation result, which includes the case of constant rank operators and other cases where our proof is or can be useful.

We will consider quasiconvex envelopes with respect to an abstract space $X = X(\mathbb{R}^n)$, which is a linear subspace of $C_c^\infty(\mathbb{R}^n, \mathbb{V})$ which is assumed to be *translation and dilation invariant*. We will also write $X(\Omega) \equiv \{v \in X(\mathbb{R}^n) : \text{supp}(v) \subset \Omega\}$ for open sets $\Omega \subset \mathbb{R}^n$.

We state the following generation result, which holds if we assume appropriate Jensen inequalities at $\mathscr{L}^n$ almost all points.

Proposition 7.3. *Let Ω be an open bounded subset of $\mathbb{R}^n$ and assume that $v = (v_x, \lambda, v_x^\infty)$ is a Young measure on Ω. Let $v \in \mathscr{M}(\Omega, \mathbb{V})$ be its barycentre and let $\lambda = \lambda^a \mathscr{L}^n \llcorner \Omega + \lambda^s$ be the Lebesgue–Radon–Nikodým decomposition of λ with respect to $\mathscr{L}^n$.*

Suppose that

$$\int_{\mathbb{V}} f \, dv_x + \lambda^a(x) \int_{\mathbb{S}\mathbb{V}} f^\infty dv_x^\infty \geqq Q_X f(\bar{v}_x + \lambda^a(x)\bar{v}_x^\infty) \quad \text{for } \mathscr{L}^n \llcorner \Omega \text{ almost all } x$$

holds for all Lipschitz integrands $f : \mathbb{V} \to \mathbb{R}$, where

$$Q_X f(z) \equiv \inf \left\{ \int_{\mathbb{X}} f(z + \phi(x)) \, dx : \phi \in X(\mathbb{X}) \right\}.$$

Then there exists a sequence (v_j) of maps in $X(\Omega)$ such that for any sequence of mollifiers $\phi_j \overset{*}{\rightharpoonup} \delta_0$ in $\mathscr{M}(\mathbb{R}^n)$ we have

$$\left(\phi_j * \left(v^a \llcorner \Omega\right) + v_j\right) \text{ generates } (v_x, \lambda^a \mathscr{L}^n \llcorner \Omega, (v_x^\infty)_{x \in \Omega^a}),$$

where $\Omega^a \equiv \{x \in \Omega : \lambda^a(x) > 0\}$.

Of course, similar results can be stated for the generation of the singular part of a Young measure, but this is not our aim; likewise for contexts of weak convergence in L^p, $p > 1$.[1] What we highlight here is the flexibility of our generation proof, which very precisely links quasiconvexification formulas to generation of weak convergence effects of sequences from very general classes.

To obtain a proof of Proposition 7.3, the reader should follow the ideas in Section 5.2 and the proof of Lemma 5.7. We remark that the Hahn–Banach argument of Proposition 5.5 only uses the invariance of the space $X(\mathbb{R}^n)$. The argument used to prove Lemma 5.7 does not use any property of this space, just the homogeneous case of Lemma 5.5 and a purely measure theoretic argument.

This is indeed the power of our main technical advancement: it easily interacts with both functional analytic and measure theoretic arguments. The only stringent constraint is the requirement of compact support, which is incompatible with the operator of Example 7.1, see also the rest of the discussion in Section 7.1.

Despite this restriction, Proposition 7.3 is the backbone of the proof of Theorem 1.2 and it is also crucial for the proof of Theorem 1.3. In particular, the limitation of compact support is overcome for constant rank operators by using [42], a point to which we will return later.

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[1] See also [20] for the implementation of our method in the case $1 < p < \infty$ of constant rank operators.

An important example of a space $X(\mathbb{R}^n)$ which is new is that of $\mathscr{A}$-free fields, with *arbitrary* operators $\mathscr{A}$ as in (1.1)

$$X(\Omega) = \{v \in C_c^\infty(\mathbb{R}^n, \mathbb{V}) \colon \mathscr{A}v = 0\}.$$

In fact, one can even take homogeneous rows in $\mathscr{A}$, of different homogeneities, which preserves the dilation invariance. In this case, Proposition 7.3 reduces to Proposition 1.4.

7.3. Sufficiency of Exactness Over Periodic Fields: the Return of Constant Rank

Of course, it may be that the class $X(\mathbb{R}^n)$ is very small and the generation result of Proposition 7.3 is trivial. This is indeed the case when looking at $X = \ker_{C_c^\infty} \mathscr{A}$ for $\mathscr{A}$ given by Example 7.1: in this case, $X = \{0\}$. We would, of course, like to know which is the class of operators $\mathscr{A}$ for which the weak convergence effects of $\mathscr{A}$-free sequences can be described by fields in $\ker_{C_c^\infty} \mathscr{A}$. This class is particularly relevant in view of Malgrange's result

$$\ker_{C_c^\infty} \mathscr{A} = \operatorname{im}_{C_c^\infty} \mathscr{B},$$

which we improve in Theorem 3.4.

It seems that if we require the exact relation above to hold for periodic fields of null average, that is,

$$\ker_{C_\#^\infty} \mathscr{A} = \operatorname{im}_{C_\#^\infty} \mathscr{B}, \quad \text{where} \quad C_\#^\infty \equiv \{v \in C^\infty(\mathbb{T}^n) \colon \hat{v}(0) = 0\}, \qquad (7.3)$$

then we have that $\mathscr{A}$-quasiconvexity can be described by fields $\mathscr{B}u$, with $u \in C_c^\infty(\mathbb{R}^n, \mathbb{V})$, similarly to [42, Cor. 1], see also [24, Sect. 5]. This explains how taking $X = \ker_{C_c^\infty} \mathscr{A}$ is sufficient to prove our main generation result, Theorem 1.2 in the case when $\mathscr{A}$ has constant rank. In that case, the exact relation (7.3) was established in [42, Lem. 1].

7.4. Speculations Concerning Weak Convergence Effects via Vector Potentials

Though sufficient, the constant rank condition need not be necessary for the exact relation (7.3). We speculate that this class contains for instance, the linear part of the formulation of the Euler equations as a differential inclusion from [13], for which it holds that

$$\ker_{C^\infty} \mathscr{A} = \operatorname{im}_{C^\infty} \mathscr{B}.$$

Related facts can be found in [50]. In fact, we will finish with a speculation that the class of operators for which the exact relation (7.3) holds, coincides with the class of operators with *(real) elliptic uncontrollable part*, as introduced in [24, Thm. 1.1] and characterized by means of algebraic geometry in [24, Cor. 4.6]. This guess is consistent with the fact that (real) constant rank operators were shown to be part of this class in [24, Thm. 1.1]. There is evidence that to stretch outside the class of constant rank operators one may need tools of nonlinear algebra, such ascommutative

algebra or, more generally, algebraic geometry. This hope is substantiated by the recent advances of Sturmfels et al in [8,9,36].

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