



Invariance Principles and Log-Distance of F-KPP Fronts in a Random Medium

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Communicated by J. BEDROSSIAN

Abstract

We study the front of the solution to the F-KPP equation with randomized non-linearity. Under suitable assumptions on the randomness including spatial mixing behavior and boundedness, we show that the front of the solution lags at most logarithmically in time behind the front of the solution of the corresponding linearized equation, i.e. the parabolic Anderson model. This can be interpreted as a partial generalization of Bramson’s findings (Bramson in *Commun Pure Appl Math* 31(5):531–581, 1978) for the homogeneous setting. Partially building on this result and its derivation, we establish functional central limit theorems for the fronts of the solutions to both equations.

1. Introduction and Main Results

1.1. The classical F-KPP equation

The F-KPP equation is the initial value problem given by

$$\begin{aligned} w_t(t, x) &= \frac{1}{2} w_{xx}(t, x) + w(t, x)(1 - w(t, x)), \quad t > 0, \quad x \in \mathbb{R}, \\ w(0, x) &= w_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{1.1}$$

with initial condition $w_0 : \mathbb{R} \rightarrow [0, 1]$. Its investigation has a long history, with seminal results dating back to FISHER [13] and KOLMOGOROV–PETROVSKII–PISKUNOV [24]. Their research had been motivated by pioneering works in genetics, where the equation has been used to model a randomly mating diploid population living in a one-dimensional habitat. Further applications can be found in chemical combustion theory or flame propagation, see [1, 12], as well as [37] and references therein.

In [24] it has been shown that for reasonably general non-linearities (see (SC) at the beginning of Sect. 1.4 for further details) and initial conditions of Heaviside type

$w_0 = \mathbb{1}_{(-\infty, 0]}$, the solution to (1.1) approaches a *traveling wave* $g : \mathbb{R} \rightarrow [0, 1]$. I.e., there exists a function $m : [0, \infty) \rightarrow \mathbb{R}$ —generally referred to as the (*position of the*) *front* or *breakpoint*—such that

$$w(t, \cdot + m(t)) \xrightarrow[t \rightarrow \infty]{} g \quad \text{uniformly.}$$

The limit g is known (see [24, Theorems 14 and 17]) to solve the differential equation

$$\begin{aligned} \frac{1}{2}g''(x) + \sqrt{2}g'(x) + g(x)(1 - g(x)) &= 0, \quad x \in \mathbb{R}, \\ 0 < g(x) < 1 \quad \text{for all } x \in \mathbb{R} \quad \text{and} \quad g(x) &\xrightarrow[x \rightarrow \infty]{} 0, \quad g(x) \xrightarrow[x \rightarrow -\infty]{} 1, \end{aligned}$$

and it is unique up to spatial translations. In particular, in this case the first order asymptotics $m(t)/t \xrightarrow[t \rightarrow \infty]{} \sqrt{2}$ has been derived. A couple of decades later, Bramson in his seminal work [8] improved this result by computing the second order correction up to additive constants. More precisely, he showed that for each $\varepsilon \in (0, 1)$, the choice $m(t) = m^\varepsilon(t) := \sup\{x \in \mathbb{R} : w(t, x) = \varepsilon\}$ fulfills

$$m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \ln t + \mathcal{O}(1) \quad \text{as } t \rightarrow \infty. \tag{1.2}$$

Later on, BRAMSON [7, Theorem 3 (p. 141)] extended this result to more general initial conditions; roughly speaking, he required the initial condition $w_0(x)$ to have a sufficiently fast exponential decay for $x \rightarrow \infty$ and to be non-vanishing for $x \rightarrow -\infty$. One of the main tools employed in the proof was the McKean representation of the solution to (1.1) in terms of expectations of branching Brownian motion, see [20] and [27]. Another important ingredient was the comparison of the solution of (1.1) to the solution of its linearized version

$$\begin{aligned} u_t(t, x) &= \frac{1}{2}u_{xx}(t, x) + u(t, x), \quad t > 0, \quad x \in \mathbb{R}, \\ u(0, x) &= w_0(x). \end{aligned} \tag{1.3}$$

Indeed, since $m(t)$ describes the front of the solution where u is small and hence $1 - u \approx 1$, the heuristics is that $m(t)$ is in some sense well-approximated by the front of the solution to (1.3). More precisely, for $\bar{m}(t) := \sup\{x \in \mathbb{R} : u(t, x) = \varepsilon\}$ and Heaviside-type initial condition $w_0 = \mathbb{1}_{(-\infty, 0]}$, standard Gaussian computations entail

$$\bar{m}(t) = \sqrt{2}t - \frac{1}{2\sqrt{2}} \ln t + \mathcal{O}(1). \tag{1.4}$$

In combination with (1.2), this results in a respective logarithmic backlog of the two fronts in the sense that

$$\bar{m}(t) - m(t) = \frac{1}{\sqrt{2}} \ln t + \mathcal{O}(1). \tag{1.5}$$

The main goal of this article is to investigate the effect of introducing a random potential in the non-linearity of (1.1) as well as in its linearization (1.3), on the

logarithmic backlog derived in (1.5) (cf. Theorem 1.5). Taking advantage of this result we will also derive functional central limit theorems for the fronts of the respective solutions to the randomized equations, see Theorem 1.4 and Corollary 1.6 below, which can be interpreted as analogues to homogeneous-case results (1.4) and (1.2).

1.2. *The randomized F-KPP equation and the parabolic Anderson model*

Already in Fisher’s seminal paper [13], where he investigated the setting (1.1) of homogeneous branching rates, it has been observed that a more realistic model would be obtained by considering spatially heterogeneous rates of the transformation of recessive to advantageous alleles. This, alongside a mathematical interest, is our guiding motivation to consider the setting of random ξ . Replacing the term $w(1 - w)$ of (1.1) by a more general non-linearity $F(w)$ fulfilling suitable *standard conditions* (see (SC) below), and overriding the notation u and w from the homogeneous setting of the previous section, we then arrive at the equation

$$\begin{aligned} w_t &= \frac{1}{2}w_{xx} + \xi(x, \omega) \cdot F(w), \quad t > 0, x \in \mathbb{R}, \\ w(0, x) &= w_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{F-KPP}$$

as well as its linearized version, the *parabolic Anderson model*

$$\begin{aligned} u_t &= \frac{1}{2}u_{xx} + \xi(x, \omega) \cdot u, \quad t > 0, x \in \mathbb{R}, \\ u(0, x) &= u_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{PAM}$$

Here, the stochastic process $\xi : \mathbb{R} \times \Omega \rightarrow (0, \infty)$ models the random medium; most of the time we will keep the dependence on Ω implicit, and write ξ instead of $\xi(\cdot, \omega)$ as is common in probability theory. For the case $F(w) = w(1 - w)$ and degenerate $\xi \equiv 1$, these two equations yield the special cases (1.1) and (1.3), respectively. It has long been known, see e.g. FREIDLIN [14, Theorem 7.6.1], that under suitable assumptions there exists $v_0 > 0$, such that the solution $w(t, x)$ to (F-KPP) converges to 0 (resp. 1), uniformly for all $x \geq \bar{v}t$ with $\bar{v} > v_0$ (resp. for all $x \leq \underline{v}t$ with $\underline{v} < v_0$), as t tends to infinity. In a similar way, this result can be shown to hold for the solution to (PAM) with the same v_0 as well, showing that the *speeds* or *velocities* of both fronts, the one of the solution to (F-KPP) as well as the one of the solution to (PAM), coincide. Consequently, as in the homogeneous case, the question of second order corrections arises naturally.

1.3. *Summary of results*

In order to address this question and to be able to summarize our results, we introduce some notation. Let $\varepsilon \in (0, 1)$ and $a > 0$. Furthermore, write $u = u^{\varepsilon, u_0}$ for the solution to (PAM) with initial condition u_0 , and $w = w^{\varepsilon, F, w_0}$ for the solution

to (F-KPP) with initial condition w_0 . As in the previous section, the fronts of the respective solutions, denoted by

$$\begin{aligned} \bar{m}^{\xi, u_0, a}(t) &:= \sup \{x \in \mathbb{R} : u(t, x) \geq a\}, \\ m^{\xi, F, w_0, \varepsilon}(t) &:= \sup \{x \in \mathbb{R} : w(t, x) \geq \varepsilon\}, \end{aligned} \tag{1.6}$$

are of special interest. Since we will from now on focus on the heterogeneous setting, we override the notation from Sect. 1.1, where it was used to denote the fronts in the homogeneous case, and define

$$\begin{aligned} \bar{m}(t) &:= \bar{m}^a(t) := \bar{m}^{\xi, \mathbb{1}_{(-\infty, 0]}, a}(t), \\ m(t) &:= m^\varepsilon(t) := m^{\xi, F, \mathbb{1}_{(-\infty, 0]}, \varepsilon}(t). \end{aligned} \tag{1.7}$$

Our findings are motivated by the respective results of (1.4), (1.2), and (1.5) for the homogeneous case, which provide information about the position of the fronts of the solutions to the respective equations, and thus their respective backlog as well. Under suitable assumptions, our results can be summarized verbally in the following two statements:

- (a) There exist a constant $C \in (0, \infty)$ and a \mathbb{P} -a.s. finite random time $\mathcal{T}(\omega)$ such that for all $t \geq \mathcal{T}(\omega)$,

$$\bar{m}(t) - m(t) \leq C \ln t; \tag{1.8}$$

see Theorem 1.5 below.

- (b) After centering and diffusive rescaling, the stochastic processes $[0, \infty) \ni t \rightarrow \bar{m}(t)$ and $[0, \infty) \ni t \rightarrow m(t)$ satisfy invariance principles; see Theorem 1.4 and Corollary 1.6 below.

As is shown in the companion article [10, Theorems 2.3 and 2.4], in a certain sense there is a logarithmic lower bound for $\bar{m}(t) - m(t)$ corresponding to (1.8) as well; cf. also Sect. 1.7 below for further discussion.

1.4. Further notation

In order to be able to precisely formulate the previously summarized results, we have to introduce some further notation. We start with introducing the *standard conditions* for the non-linearity, i.e., F in (F-KPP) has to fulfill the following:

$$\begin{aligned} F &\in C^1([0, 1]), \quad F(0) = F(1) = 0, \quad F(w) > 0 \quad \forall w \in (0, 1), \\ F'(0) &= 1 = \sup_{w>0} F(w)w^{-1}, \quad F'(1) < 0, \quad \limsup_{w \downarrow 0} \frac{1 - F'(w)}{w} < \infty; \end{aligned} \tag{SC}$$

see Fig. 1 for an illustration. Note that the last condition in (SC) is essentially a $C^{1,1}$ -condition on F at 0. Among others, the conditions in (SC) are used to employ a sandwiching argument for being able to deduce the desired result (1.8) not only for non-linearities F which have the form of a probability generating function, see (PROB) below, but also for all F fulfilling (SC). The sandwiching

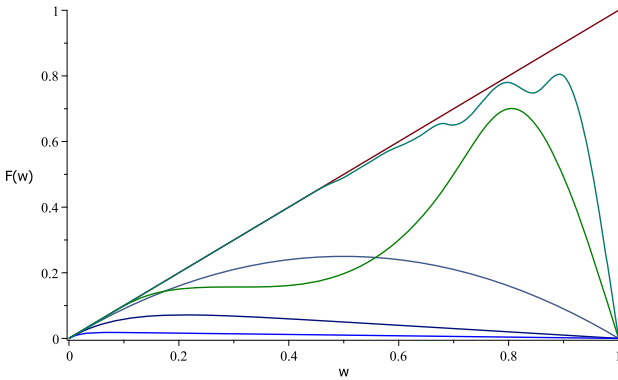


Fig. 1. Sketches of functions fulfilling (SC), all of which are dominated by the identity function

argument is inspired by the proof of [8, Theorem 2], where the author investigates the homogeneous case $\xi \equiv \text{const}$.

We will now specify the classes of initial conditions under consideration for both, (F-KPP) and (PAM). For this purpose, we fix $\delta' \in (0, 1)$ and $C' > 1$, and require an initial condition u_0 of (PAM) to fulfill

$$\delta' \mathbb{1}_{[-\delta', 0]} \leq u_0 \leq C' \mathbb{1}_{(-\infty, 0]}. \tag{PAM-INI}$$

Our results also hold for initial conditions that decay sufficiently fast at infinity and grow towards minus infinity with sufficiently small exponential rate; i.e., the condition (PAM-INI) can be relaxed to

$$\delta' \mathbb{1}_{[-\delta', 0]}(x) \leq u_0(x) \leq C' \mathbb{1}_{(-\infty, 0]} \vee C'(e^{-Cx} \wedge e^{-x/C}) \quad \forall x \in \mathbb{R},$$

where $C \in (0, \infty)$ is a large enough constant. However, in order to avoid further technical complications we stick to the above set of initial conditions.

In addition, let us introduce a tail condition for the initial condition of (F-KPP), which is the same as the one for the case $\xi \equiv 1$ stated in [7, (1.17)]. For this purpose, we fix $N, N' > 0$, and require w_0 as in (F-KPP) to fulfill

$$0 \leq w_0 \leq \mathbb{1}_{(-\infty, 0]} \quad \text{and} \quad \int_{[x-N, x]} w_0(y) dy \geq \delta' \quad \forall x \leq -N'. \tag{KPP-INI}$$

Denote by \mathcal{S}^1 the class of functions $f : \mathbb{R} \rightarrow [0, \infty)$ which are pointwise limits of increasing sequences of continuous functions, and let

$$\mathcal{I}_{\text{PAM}} := \{u_0 \in \mathcal{S}^1 : u_0 \text{ fulfills (PAM-INI)}\}, \quad \mathcal{I}_{\text{F-KPP}} := \{w_0 \in \mathcal{S}^1 : w_0 \text{ fulfills (KPP-INI)}\},$$

which will be the classes of initial conditions under consideration. An emblematic example which is contained in both, $\mathcal{I}_{\text{F-KPP}}$ and \mathcal{I}_{PAM} , is the function $\mathbb{1}_{(-\infty, 0]}$ of Heaviside type.

We will assume $\xi = (\xi(x))_{x \in \mathbb{R}}$ to be a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ having locally Hölder continuous paths, i.e., there exists $\alpha = \alpha(\xi) > 0$ such that for every interval $I \subset \mathbb{R}$ there exists $C = C(\xi, I) > 0$, such that

$$|\xi(x) - \xi(y)| \leq C |x - y|^\alpha \quad \forall x, y \in I, \tag{HÖL}$$

and such that the following conditions are fulfilled:

- ξ is *uniformly bounded away from 0 and ∞* , i.e., there exist constants $0 < e_i \leq e_s < \infty$ such that \mathbb{P} -a.s.,

$$e_i \leq \xi(x) \leq e_s \quad \text{for all } x \in \mathbb{R}; \tag{BDD}$$

- ξ is *stationary*, i.e. for every $h \in \mathbb{R}$ we have

$$(\xi(x))_{x \in \mathbb{R}} \stackrel{d}{=} (\xi(x+h))_{x \in \mathbb{R}}; \tag{STAT}$$

- ξ fulfills a *ψ -mixing condition*: Let $\mathcal{F}_x := \sigma(\xi(z) : z \leq x)$ and $\mathcal{F}^y := \sigma(\xi(z) : z \geq y)$, $x, y \in \mathbb{R}$ and assume that there is a continuous, non-increasing function $\psi : [0, \infty) \rightarrow [0, \infty)$, such that for all $j, k \in \mathbb{Z}$ with $j \leq k$ as well as $X \in \mathcal{L}^1(\Omega, \mathcal{F}_j, \mathbb{P})$ and $Y \in \mathcal{L}^1(\Omega, \mathcal{F}^k, \mathbb{P})$ we have

$$\begin{aligned} |\mathbb{E}[X - \mathbb{E}[X] | \mathcal{F}^k]| &\leq \mathbb{E}[|X|] \cdot \psi(k-j), \\ |\mathbb{E}[Y - \mathbb{E}[Y] | \mathcal{F}_j]| &\leq \mathbb{E}[|Y|] \cdot \psi(k-j), \\ \sum_{k=1}^{\infty} \psi(k) &< \infty. \end{aligned} \tag{MIX}$$

Note that (MIX) implies the ergodicity of ξ with respect to the shift operator θ_y acting via $\xi(\cdot) \circ \theta_y = \xi(\cdot + y)$, $y \in \mathbb{R}$.

Summarizing, we arrive at the following standing assumptions:

We will assume conditions (HÖL), (BDD), (STAT) and (MIX)
to be fulfilled from now on, if not explicitly mentioned otherwise.
(Standing assumptions)

We will provide here two prototypical examples for a suitable potential ξ satisfying the conditions of (Standing assumptions).

Example 1.1. (a) The Ornstein–Uhlenbeck process (Y_x) , $x \in \mathbb{R}$, is an important stochastic process, which has all the nice properties of being stationary, Markovian, and Gaussian. It can be written as

$$Y_x = e^{-x} B_{e^{2x}}, \text{ for } x \in \mathbb{R}, \quad \text{where } (B_t), t \in [0, \infty), \text{ is a Brownian motion; } \tag{1.9}$$

see Fig. 2 for a realization. For $0 < \varepsilon < M < \infty$ we now consider the potential $\xi(x) = (\varepsilon \vee Y_x) \wedge M$, $x \in \mathbb{R}$, and note that it satisfies (BDD) by definition. It furthermore fulfills (STAT), since the process (Y_x) , $x \in \mathbb{R}$, is stationary. We also note that for any $\gamma \in (0, 1/2)$ the local γ -Hölder continuity of (Y_x) follows

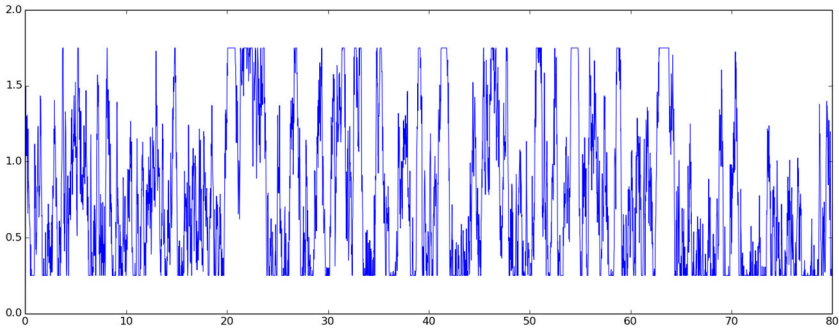


Fig. 2. Realization of the potential $\xi(x) = (\varepsilon \vee Y_x) \wedge M$ with (Y_x) , $x \in \mathbb{R}$, an Ornstein–Uhlenbeck process and the choices $\varepsilon = 0.25$ and $M = 1.75$

from (1.9) in combination with the respective Hölder continuity of Brownian motion (see [28, Corollary 1.20]); hence, (HÖL) holds true. Property (MIX) is somewhat harder to establish. One can for instance do so by taking advantage of the fact that the Ornstein–Uhlenbeck process can also be characterized as the solution to the stochastic differential equation

$$dY_x = -Y_x dx + \sqrt{2} dB_x, \tag{1.10}$$

and then use the fact that the Ornstein–Uhlenbeck process started from a single point converges to its stationary distribution sufficiently fast (cf. [21, Example 6.8]) as well as the Markov property of the process. The details of the proof are slightly technical and omitted at this point. See [21,28] for further details on the Ornstein–Uhlenbeck process.

- (b) This example is instructive in the sense that not only does it fulfill these conditions, but at the same time it serves as an example in [10] to demonstrate that the transition front of the solution to (F-KPP) can grow logarithmically in time along subsequences in case $\frac{es}{ei} > 2$; see [10] for further details. We choose $0 < ei < es < \infty$ and let $\chi : [0, \infty) \rightarrow [0, 1]$ be a continuous non-increasing function with $\chi(x) = 1$ for $x \leq 1$ and $\chi(x) = 0$ for $x \geq 2$. Furthermore, let $\omega = (\omega^i)_{i \in \mathbb{Z}}$ be a Poisson point process on \mathbb{R} with homogeneous intensity one. We then define

$$\xi(x) := ei + (es - ei) \cdot \sup\{\chi(|x - \omega^i|) : i \in \mathbb{Z}\}.$$

Observe that by construction the potential ξ satisfies (HÖL). Furthermore, $\xi(x) \in [ei, es]$ for all $x \in \mathbb{R}$, $\xi(x) = ei$ if $|x - \omega^i| > 2$ for all i , and $\xi(x) = es$ if there exists ω^i such that $|x - \omega^i| \leq 1$. Also, using the properties of the Poisson point process, it is not hard to show that ξ fulfills (BDD), (STAT) and (MIX). See Fig. 3 for an illustration of this potential, which highlights the fact that the potential has logarithmically in time long stretches where it equals ei that are adjacent to comparably long stretches where it equals es ; cf. [10, Lemma 5.3] for precise statements.

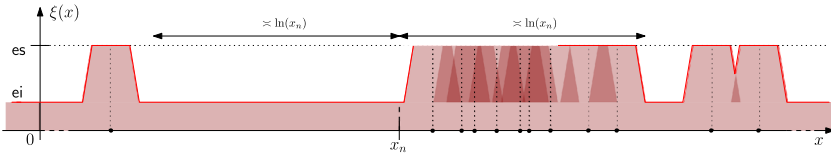


Fig. 3. Realization of a potential ξ (top red line) emanating as described from a Poisson point process. This realization exhibits the appearance of long stretches of connected components in space where it takes the values e_i and e_s , respectively

Since we allow for non-smooth initial conditions in both equations, (F-KPP) and (PAM), we shortly comment on the notion of solution in this setting. We call a function w to (F-KPP) a *generalized solution to (F-KPP)*, if it satisfies

$$w(t, x) = E_x \left[\exp \left\{ \int_0^t \xi(B_s) F(w(t-s, B_s)) / w(t-s, B_s) ds \right\} w_0(B_t) \right], \quad \forall (t, x) \in [0, \infty) \times \mathbb{R}, \tag{1.11}$$

where E_x is the corresponding expectation of the probability measure P_x , under which $(B_t)_{t \geq 0}$ is a standard Brownian motion starting in $x \in \mathbb{R}$. This equation can be interpreted as a mild formulation of (F-KPP) and one can show (see e.g. [14, (1.4), p. 354, and (a), p. 355]) that every classical solution to (F-KPP) also is a generalized one. Generalized solutions can be shown to exist under weak assumptions, and, vice versa, in many instances they turn out to be classical solutions indeed, see Proposition A.9 in Appendix A.

Remark 1.2. A useful observation is that for initial conditions in $\mathcal{I}_{\text{F-KPP}}$, generalized solutions can be approximated by classical solutions. That is, if $(w_0^{(n)})_{n \in \mathbb{N}} \subset \mathcal{I}_{\text{F-KPP}}$ is a sequence of continuous functions which increase pointwise to w_0 , then by Corollary A.11 the corresponding sequence of (by Proposition A.9 classical) solutions $(w^{(n)})_{n \in \mathbb{N}} = (w_0^{(n)})_{n \in \mathbb{N}}$ to (F-KPP) is also monotone and thus the limit $w(t, x) := \lim_{n \rightarrow \infty} w^{(n)}(t, x)$ exists for all $(t, x) \in [0, \infty) \times \mathbb{R}$. Dominated convergence and the fact that $w^{(n)}$ also is a generalized solution then imply

$$\begin{aligned} w(t, x) &= \lim_{n \rightarrow \infty} w^{(n)}(t, x) = \lim_{n \rightarrow \infty} E_x \left[\exp \left\{ \int_0^t \xi(B_s) \frac{F(w^{(n)}(t-s, B_s))}{w^{(n)}(t-s, B_s)} ds \right\} w_0^{(n)}(B_t) \right] \\ &= E_x \left[\exp \left\{ \int_0^t \xi(B_s) \frac{F(w(t-s, B_s))}{w(t-s, B_s)} ds \right\} w_0(B_t) \right]. \end{aligned}$$

I.e., w is the generalized solution to (F-KPP) with initial condition w_0 .

A similar concept can be introduced for the solution to (PAM) as well. That is, for $u_0 \in \mathcal{I}_{\text{PAM}}$, we call a function

$$u(t, x) = E_x \left[\exp \left\{ \int_0^t \xi(B_s) ds \right\} u_0(B_t) \right], \tag{1.12}$$

a *generalized solution to (PAM)*. Note that this is an explicit expression of the solution in terms of a Brownian path, while in (1.11) the solution is only given

implicitly. Furthermore, if u_0 is continuous, then by [21, Remark 4.4.4], there exists a unique solution u to (PAM) which fulfills $u \in C^{1,2}((0, \infty) \times \mathbb{R})$ and $u(0, \cdot) = u_0$. As a consequence of these observations, in the following, we will always consider generalized solutions.

1.5. The linearized equation

As already mentioned in the first section, we expect that investigating the solution to (PAM) might also provide some insight into the solution to (F-KPP). Therefore, starting with the first order of the front as a function of time, it turns out useful to consider the so-called *Lyapunov exponent*

$$\Lambda(v) := \lim_{t \rightarrow \infty} \frac{1}{t} \ln u(t, vt). \tag{1.13}$$

Due to Proposition A.3, the Lyapunov exponent exists \mathbb{P} -a.s. for all $v \in \mathbb{R}$, is non-random, and—as a consequence of Corollary 3.10—does not depend on the initial condition in \mathcal{I}_{PAM} under consideration. Furthermore, the function $[0, \infty) \ni v \mapsto \Lambda(v)$ is concave, tends to $-\infty$ as $v \rightarrow \infty$ and $\Lambda(0) = \text{es}$, where es is defined in (BDD). $\Lambda(v)$ describes the asymptotic exponential growth of the solution in the linear regime with speed v . By Proposition A.3, there exists a unique $v_0 > 0$ such that

$$\Lambda(v_0) = 0,$$

which we will call *velocity* or *speed* of the solution to (PAM). Using the properties of the Lyapunov exponent, we immediately infer the first order asymptotics for \bar{m} to \mathbb{P} -a.s. satisfy

$$\frac{\bar{m}(t)}{t} \xrightarrow{t \rightarrow \infty} v_0. \tag{1.14}$$

It will turn out that our methods work if we require v_0 to be strictly larger than some “critical” value v_c , defined in Lemma 2.4 (d). Roughly speaking, the condition $v > v_c$ allows for the application of change-of-measure techniques; more precisely, it allows us to find a suitable additive tilting parameter in the exponent of the Feynman–Kac representation (1.12), which depends on v and makes the solution $u(t, x)$ to (PAM) amenable to the investigation by standard tools for values $x \approx vt$ and large t . Hence, we will work under the assumption

$$v_0 > v_c \tag{VEL}$$

from now on. As will be shown in Sect. 4.4, this assumption is fulfilled for a rich class of potentials ξ .

We start with investigating the fluctuations of the function $t \mapsto \ln u(t, vt)$ around $t\Lambda(v)$ for values v in a neighborhood of v_0 , which are interesting in their own right. To this end, on the space $C([0, \infty))$ of continuous functions from $[0, \infty)$ to \mathbb{R} , we define the metric

$$\rho(f, g) := \sum_{j=1}^{\infty} 2^{-j} \frac{\|f - g\|_j}{1 + \|f - g\|_j}, \quad f, g \in C([0, \infty)), \tag{1.15}$$

where we write $\|f - g\|_j := \sup_{x \in [0, j]} |f(x) - g(x)|$. This makes $(C([0, \infty)), \rho)$ a complete separable metric space.

Theorem 1.3. *Let (VEL) be fulfilled, $u_0 \in \mathcal{I}_{\text{PAM}}$, and $u = u^{\xi, u_0}$ be the corresponding solution to (PAM). Furthermore, let $V \subset (v_c, \infty)$ be a compact interval such that $v_0 \in \text{int}(V)$. Then for each $v \in V$, as $n \rightarrow \infty$ the sequence of random variables $(nv)^{-1/2}(\ln u(n, vn) - n\Lambda(v))$, $n \in \mathbb{N}$, converges in \mathbb{P} -distribution to a centered Gaussian random variable with variance $\sigma_v^2 \in [0, \infty)$, where σ_v^2 is defined in (3.1). If $\sigma_v^2 > 0$, the sequence of processes*

$$[0, \infty) \ni t \mapsto \frac{1}{\sqrt{nv\sigma_v^2}} (\ln u(nt, vnt) - nt\Lambda(v)), \quad n \in \mathbb{N},$$

converges as $n \rightarrow \infty$ in \mathbb{P} -distribution to a standard Brownian motion in the sense of weak convergence of measures on $C([0, \infty))$ endowed with the metric ρ from (1.15).

Note that in the above one should replace $[0, \infty)$ by $(0, \infty)$ for u_0 such that $[0, \infty) \ni t \mapsto u(t, vt)$ is not continuous in 0 (the latter might e.g. be the case for u_0 of Heaviside type). In combination with perturbation estimates for u , we will use this result in order to infer an invariance principle for the front of the solution to (PAM). Note that since the function $t \mapsto \bar{m}_t$ may be discontinuous, we consider convergence in the Skorohod space $D([0, \infty))$ in the following result; details on the latter can for example be found in [11].

Theorem 1.4. *Let (VEL) be fulfilled, $u_0 \in \mathcal{I}_{\text{PAM}}$ and $a > 0$. Then for $n \rightarrow \infty$, the sequence $n^{-1/2}(\bar{m}^{\xi, u_0, a}(n) - v_0n)$, $n \in \mathbb{N}$, converges in \mathbb{P} -distribution to a centered Gaussian random variable with variance $\tilde{\sigma}_{v_0}^2 \in [0, \infty)$, where $\tilde{\sigma}_{v_0}^2$ is defined in (3.58). If $\tilde{\sigma}_{v_0}^2 > 0$, the sequence of processes*

$$[0, \infty) \ni t \mapsto \frac{\bar{m}^{\xi, u_0, a}(nt) - v_0nt}{\sqrt{n\tilde{\sigma}_{v_0}^2}}, \quad n \in \mathbb{N},$$

converges as $n \rightarrow \infty$ in \mathbb{P} -distribution to a standard Brownian motion in the Skorohod space $D([0, \infty))$.

We underline that in the above theorems the case $\sigma_v^2 = 0$ (resp. $\tilde{\sigma}_{v_0}^2 = 0$) is allowed and leads to a degenerate limit of the corresponding sequences. This can be excluded, e.g. if the finite-dimensional projections of the stochastic process $\xi = (\xi(x))_{x \in \mathbb{R}}$ are associated (see e.g. [32] or [31] for definitions and results). In this case the covariances in (3.1) are nonnegative and $\sigma_v^2 > 0$ follows. NOLEN [29, Proposition 2.1] provides an example of a potential, which is generated by an i.i.d. sequence of random variables and thus associated.

1.6. The non-linear equation

Coming back to the original equation of interest, it is natural to ask whether we can obtain results for (F-KPP), which are in some sense counterparts to those derived in latter section for (PAM). For the solution to (F-KPP) it is known (see e.g. [14, §7.6]) that the first order of the position of the front is linear as well and moves with the same velocity as the front of (PAM). Indeed, by FREIDLIN [14, Theorem 7.6.1] we have that \mathbb{P} -a.s.,

$$\frac{m(t)}{t} \xrightarrow{t \rightarrow \infty} v_0.$$

As mentioned in Sect. 1.2, the next result, which is one of the main results of the paper, states that there is an at most logarithmic distance of the fronts of (F-KPP) and (PAM).

Theorem 1.5. *Let (VEL) be fulfilled. Then for each F fulfilling (SC) there exists a constant $C_1 > 0$ such that the following holds: For all $a > 0$, $\varepsilon \in (0, 1)$, $u_0 \in \mathcal{I}_{\text{PAM}}$ and $w_0 \in \mathcal{I}_{\text{F-KPP}}$, there exists a non-random $C = C(\varepsilon, a, u_0, w_0) > 0$ and a \mathbb{P} -a.s. finite random time $\mathcal{T} = \mathcal{T}(\xi, \varepsilon, a, u_0, w_0) \geq 0$, such that*

$$-C \leq \bar{m}^{\xi, u_0, a}(t) - m^{\xi, F, w_0, \varepsilon}(t) \leq C_1 \ln t + C \quad \forall t \geq \mathcal{T}. \quad (1.16)$$

Moreover for $w_0 = u_0$ and $a \leq \varepsilon$, the left inequality in (1.16) can be replaced by $0 \leq \bar{m}^{\xi, u_0, a}(t) - m^{\xi, F, w_0, \varepsilon}(t)$ for all $t \geq 0$.

Furthermore, combining Theorems 1.4 and 1.5, we can deduce an invariance principle for the front of (F-KPP) as well.

Corollary 1.6. *Let (VEL) be fulfilled as well as F fulfill (SC), $w_0 \in \mathcal{I}_{\text{F-KPP}}$ and $\varepsilon \in (0, 1)$. Then as $n \rightarrow \infty$, the sequence $n^{-1/2}(m^{\xi, F, w_0, \varepsilon}(n) - v_0 n)$, $n \in \mathbb{N}$, converges in \mathbb{P} -distribution to a centered Gaussian random variable with variance $\tilde{\sigma}_{v_0}^2 \in [0, \infty)$, where $\tilde{\sigma}_{v_0}^2$ is defined in (3.58). If $\tilde{\sigma}_{v_0}^2 > 0$, the sequence of processes*

$$[0, \infty) \ni t \mapsto \frac{m^{\xi, F, w_0, \varepsilon}(nt) - v_0 nt}{\sqrt{n \tilde{\sigma}_{v_0}^2}}, \quad n \in \mathbb{N},$$

converges as $n \rightarrow \infty$ in \mathbb{P} -distribution to a standard Brownian motion in the Skorohod space $D([0, \infty))$.

1.7. Discussion and previous results

As already alluded to above, in the homogeneous case (1.1) of constant potential the front has been well-understood by now. This is indeed the case to a much wider extent than illustrated in the Introduction, see e.g. [6] and references therein for further details. Also the heterogeneous setting (F-KPP) of random potential and the properties of its solution have been investigated. Specifically, under fairly general assumptions, the existence and characterization of the propagation speed (i.e., the linear order $\lim_{t \rightarrow \infty} m(t)/t$ of the position of the front) have been derived by Freidlin and Gärtner, see e.g. [15] as well as [14, Chapter VII], using large

deviation principles. Incidentally, the Feynman–Kac formula (see Proposition 2.3 below), which characterizes the solution to the linearization (PAM), has also played an important role in the derivation.

Second order corrections for the position have been investigated by NOLEN [29]. Also making the detour along (PAM), he examines (F-KPP) for a potential ξ under similar assumptions, but requires the (random) initial conditions to satisfy

$$C_1(\xi)g^{\xi,\gamma}(x) \leq w_0(x, \xi) \leq C_2(\xi)g^{\xi,\gamma}(x) \quad \forall x > 0. \tag{1.17}$$

Here, $C_1(\xi)$ and $C_2(\xi)$ are positive random variables, and $g = g^{\xi,\gamma}$ is a solution to

$$g''(x) + (\xi(x) - \gamma)g(x) = 0, \quad x > 0,$$

for $\gamma > \bar{\gamma}$, for a certain $\bar{\gamma} > 0$. For technical reasons, he additionally requires $\gamma < \gamma^*$, i.e., the initial condition $w_0(x)$ must not decay too fast as x tends to infinity. The technical assumption (1.17) thus entails being in the *supercritical regime* which corresponds to waves that move faster than the minimal speed. As his main result, Nolen obtains a (functional) central limit theorem for $m(t)$ in this case also, see [29, Theorem 1.4]. In this lingo, our set of initial conditions corresponds to the *critical regime*, and Corollary 1.6 (for the critical regime) also suggests that the randomness in Nolen’s central limit theorem is already coming from the environment, and does not necessarily require the randomness of the initial condition.

Furthermore, in [30] a corresponding invariance principle for the front has been derived in the case where the non-linearity in (F-KPP) is either ignition type or bistable.

In [18], on the other hand, for the case of periodic instead of random ξ , the authors have investigated the respective logarithmic correction, corresponding to (1.2) in the homogeneous setting; here, the authors have been able to characterize the constant in front of the logarithmic correction as a certain minimizer.

In our main Theorem 1.5, we establish a corresponding logarithmic upper bound for the difference (1.5), also in the setting of a random environment. However, our methods are currently too coarse for identifying a sharp prefactor. Nevertheless, it is a natural question whether the logarithmic upper bound we derive captures the correct order at least. And indeed, in some sense, a partial positive answer to this questions is provided in the companion article [10]. There, the authors show that there exist an increasing sequence (t_n) of times with $t_n \in (0, \infty)$ such that $\lim_{n \rightarrow \infty} t_n = \infty$ and a sequence (x_n) of reals such that $\bar{m}^{\frac{1}{2}}(t_n) - x_n \geq c \ln t_n$ such that for all $n \in \mathbb{N}$ one has

$$w(t_n, x_n) < \frac{1}{2} \quad \text{and (by definition)} \quad u(t_n, \bar{m}^{\frac{1}{2}}(t_n)) = \frac{1}{2}; \tag{1.18}$$

see the discussion around [10, (2.11)] for further details.

As explained above, there is a profound connection between the PDEs we consider and branching Brownian motion, which is more involved than in the setting of constant ξ . Indeed, related results for branching random walk in random environment (BRWRE) have been recently derived in [9]. In this source the authors analyze the distribution of the maximal particle of a branching random walk in

random environment, which itself is closely related to a discrete-space version of (F-KPP). In particular, a corresponding logarithmic upper bound on the distance of the expected position of the maximal particle and the median of the distribution of the maximal particle is given. Furthermore, an invariance principle for the median of the position of the maximal particle of BRWRE is derived. It is therefore no surprise that, on the one hand, principal techniques we employ in this paper are generalizations and adaptations of respective discrete space analogues from [9]. On the other hand, we work under weaker independence assumptions and softer requirements on the non-linearity which we only require to be contained in (SC).

Recall that directly before Theorem 1.3 we have introduced our assumption (VEL), which reads $v_0 > v_c$; from a technical point of view, this will be necessary for our change of measure argument to work. It is not hard to show that for a rich class of potentials, this condition is satisfied indeed, cf. Sect. 4.4. What is more, however, in Proposition 4.10 below we also obtain a more profound understanding of the scope of the methods employed. I.e., there exist potentials ξ fulfilling (Standing assumptions), but such that $v_0 < v_c$ holds true. Therefore, it remains an open and interesting question to understand the behavior of the solutions to the above partial differential equations in case condition (VEL) is not fulfilled.

1.8. Strategy of the proof

Our proofs will be essentially based on techniques from probability theory. One of the principal tools for our investigations is given by the probabilistic representation of solutions to the partial differential equations (PAM) and (F-KPP) through branching Brownian motion in the random (branching) environment ξ ; the details will be discussed in Sect. 2.1. Further key roles in the proofs of Theorems 1.3 and 1.4 are played by the tilting of probability measures, concentration estimates, as well as perturbation estimates for the solution to (PAM). In the proof of Theorem 1.5 in particular, a modified second moment method will be fundamental. We now provide some further details on the proofs of the main results.

Proof of Theorem 1.3 Considering the initial condition $u_0 = \mathbb{1}_{(-\infty, 0]}$ for the sake of exposition, the Feynman–Kac formula (1.12) supplies us with

$$u(t, vt) = E_{vt} \left[\exp \left\{ \int_0^t \xi(B_s) ds \right\} \mathbb{1}_{(-\infty, 0]}(B_t) \right], \tag{1.19}$$

where $v > 0$ stands for velocity. While in probabilistic terms, this expression can be interpreted as the expected number of particles of a branching Brownian motion—starting in vt and with branching rate ξ —to the left of the origin at time t , we will, after normalization, rather interpret it as a Brownian motion in the random potential ξ . In this vein, if $v > v_c$, then the expectation in (1.19) can be investigated by a random change of measure which depends on the potential ξ , cf. Sect. 2.2, in particular Lemma 2.4. Under this perturbed probability measure, the *typical behavior* of the Brownian motion in the random potential ξ (which then has a drift towards the left) started in vt is to arrive in 0 approximately at time t . Since furthermore the first hitting time of the origin by this Brownian motion is in fact

concentrated around t (the respective results can be found in Sect. 3.2), one can take advantage of a resulting renewal structure via independent—but not identically distributed—random variables, which is formulated in terms of empirical Legendre transforms. For a suitably centered and rescaled version of these, one can then show a functional central limit theorem by the use of martingale methods, see Proposition 3.1. Again using Proposition 3.5, one can then infer the desired functional central limit theorem stated in Theorem 1.3; the details are implemented in Sect. 3.3.

Proof of Theorem 1.4 The proof of this result boils down to understanding the behavior of the position of the front $\bar{m}(t)$ of the solution to (PAM) in dependence on the time t . As already observed in (1.14), to first order it is not hard to observe that the position of the front of the wave is given by $v_0 t$. Hence, the challenge is to establish the normally distributed second order correction (in functional form). For the purpose of the proof, we can take advantage of the understanding obtained along the proof of Theorem 1.3. More precisely, using perturbation estimates for the solution to (PAM) in time and space which are derived in Sects. 3.4 and 3.5, the functional central limit theorem for the suitably centered and rescaled version of the sequence $[0, \infty) \ni t \mapsto \ln u(nt, vnt)$, $n \in \mathbb{N}$, obtained in Theorem 1.3 can be transferred to the functional central limit theorem for the suitably centered and rescaled version of $[0, \infty) \ni t \mapsto \bar{m}(nt)$, $n \in \mathbb{N}$, which is the content of Theorem 1.4; see Sect. 3.7 for further details.

Proof of Theorem 1.5 A key difficulty in the proofs stems from the fact that the accuracy up to which we want to understand the behavior of the solutions is logarithmic in time, and hence much smaller than the typical fluctuations of the fronts of the solutions to (PAM) and (F-KPP).

In probabilistic terms, the solution $w(t, x)$ of (F-KPP) with initial condition $w(0, \cdot) = \mathbb{1}_{(-\infty, 0]}$ corresponds to the probability that the leftmost particle of the branching Brownian motion in the random environment ξ is to the left of the origin at time t . As a consequence, proving that the implications of Theorem 1.5 are fulfilled therefore amounts to showing that if one starts at most logarithmically to the left of $\bar{m}(t)$, the probability of actually seeing a particle to the left of or equal to the origin at time t , is bounded from below by ε . In order to do so, we take advantage of a second moment method, see (4.3), to obtain a lower bound on the probability to observe a particle in the branching Brownian motion in the random environment ξ to the left of the origin at time t . As in the homogeneous setting, such an approach is set to fail when considering all particles in the process; indeed, in this case the second moments would be prohibitively large in comparison to the square of the first moment. Hence, one has to restrict to so-called “leading particles” known from branching Brownian motion in homogeneous branching potential. This notion has to be adapted here in a careful manner so as to cater for the necessities of our random branching potential ξ . As in the homogeneous case, the remaining steps then boil down to finding good upper bounds for the second moment of leading particles (see Lemma 4.4 in Sect. 4.2) as well as a good lower bound on the first moment of leading particles, cf. Lemma 4.1 in Sect. 4.1; the latter is the harder part

in our context. These findings then constitute the main steps in proving Theorem 1.5, the implementation of which is found in Sect. 4.3.

1.9. Notational conventions

We will frequently use sums of real-indexed quantities $A_x, x \in \mathbb{R}$. In this case, we write

$$\sum_{i=1}^x A_i := \sum_{i=1}^{\lfloor x \rfloor} A_i + A_x, \quad x \in [0, \infty) \setminus \mathbb{N}_0,$$

where $\sum_{i=1}^0 := 0$. This notion remains consistent if we also allow for additive constants $b \in \mathbb{R}$, i.e.

$$\sum_{i=1}^x (A_i + b) = \sum_{i=1}^{\lfloor x \rfloor} A_i + A_x + bx, \quad x \in [0, \infty).$$

Finally, we set

$$\sum_{i=x+1}^y A_i := \begin{cases} \sum_{i=1}^y A_i - \sum_{i=1}^x A_i, & x \leq y, \\ \sum_{i=1}^x A_i - \sum_{i=1}^y A_i, & x > y, \end{cases} \quad x, y \in [0, \infty).$$

A prime example is the quantity $A_x = \ln E_x[e^{\int_0^{H_y} (\xi(B_s) - \varepsilon s) ds}]$, where $H_y := \inf\{t \geq 0 : B_t = y\}$. Indeed, by the strong Markov property we have $\ln E_x[e^{\int_0^{H_0} (\xi(B_s) - \varepsilon s) ds}] = \sum_{i=1}^{\lfloor x \rfloor} A_i + A_x$ for all $x \in [0, \infty) \setminus \mathbb{N}_0$.

Furthermore, we will often use positive finite constants c_1, c_2, \dots in the proofs. This numbering is consistent within any of the proofs, and it is reset after each proof. On the other hand, C_1, C_2, \dots will be used to denote positive finite constants that are fixed throughout the article, and they will oftentimes depend on each other. Other constants like $c, C, \varepsilon, \delta$ etc. in the proofs are used to compare certain quantities and are also reset after each proof.

The structure of this article is as follows. In Sect. 2 we will introduce various tools which play a seminal role in our investigations. In particular, these comprise the connections between the partial differential equations under consideration and branching processes, change of measure techniques, as well as concentration inequalities. Taking advantage of these results and further exact large deviation estimates, Sect. 3 first leads to a proof of Theorem 1.3. After developing perturbation results in space and time for the solution to (PAM), Theorem 1.3 will play a crucial role for the proof of Theorem 1.4 as well. The main goal of Sect. 4 then is to show Theorem 1.5, i.e., that the front of the solution to the non-linear equation (F-KPP) lags at most logarithmically in time behind the front of the solution to the linear equation (PAM).

2. Some Technical Tools

In this section we will introduce some further tools that will be helpful in the proof of the main results.

2.1. Connection to branching processes

We define a *branching Brownian motion in random environment* (BBMRE) as follows: Conditionally on the realization of ξ and for fixed $x \in \mathbb{R}$, consider an initial particle starting at a point x and moving as a standard Brownian motion $(B_t)_{t \geq 0}$ on \mathbb{R} . While at site y , the particle dies at rate $\xi(y)$. More precisely, for an exponentially distributed random variable S with parameter one, independent of everything else, the first particle dies at time $\inf \{t \geq 0 : S < \int_0^t \xi(B_s) ds\}$. When the initial particle dies, it gives birth to k new particles with probability $p_k, k \in \mathbb{N}$ [see (PROB) below for the precise assumptions on the p_k]. The new particle(s) start their evolution at the site where their parent particle had died, and they evolve independently of everything else and according to the same stochastic behavior as their parent. Note that on the one hand we assume $p_0 = 0$, so genealogies do not die out at any finite time. On the other hand, we allow $p_1 > 0$, i.e., it is possible to die and give birth to one descendant. This implies that a particle at site y branches into more than one particle with rate $\xi(y)(1 - p_1)$. We denote the corresponding probability measure by \mathbb{P}_x^ξ and write \mathbb{E}_x^ξ for the respective expectation. By $N(t)$ we denote the set of particles alive at time t . For $\nu \in N(t)$ we let $(X_s^\nu)_{s \in [0, t]}$ be the spatial trajectory of the genealogy of ancestral particles of ν up to time t . For a Borel set $A \subset \mathbb{R}$ and $y \in \mathbb{R}$ we define

$$N(t, A) := \{\nu \in N(t) : X_t^\nu \in A\} \quad \text{and} \quad N^{\leq}(t, y) := |N(t, (-\infty, y])|, \quad (2.1)$$

i.e., the set of particles which are in A at time t , and the number of particles to the left of y at time t .

Let us now introduce a set of functions, formulated in terms of the probability generating function of the sequence $(p_k)_{k \in \mathbb{N}}$ as above. More precisely, assume $(p_k)_{k \in \mathbb{N}}$ and $F = F^{(p_k)_{k \in \mathbb{N}}}$ to fulfill

$$p_k \in [0, 1] \forall k \in \mathbb{N}, \quad \sum_{k=1}^\infty p_k = 1, \quad \sum_{k=1}^\infty k p_k = 2, \quad m_2 := \sum_{k=1}^\infty k^2 p_k < \infty; \tag{PROB}$$

$$F(u) = 1 - u - \sum_{k=1}^\infty p_k (1 - u)^k, \quad u \in [0, 1].$$

Note that such F always fulfills (SC), is smooth and strictly concave on $(0, 1)$. Note that the assumption $\sum_{k=1}^\infty k p_k = 2$. A fundamental result is the connection of the solution to (F-KPP) and the expected number of particles of the branching process. This is sometimes referred to as the McKean representation. The respective result in the homogeneous setting can be found in [20] as well as [27].

Proposition 2.1. *Let $w_0 \in \mathcal{I}_{F\text{-KPP}}$, ξ satisfy (HÖL), and F fulfill (PROB). Then \mathbb{P} -a.s. the solution to (F-KPP) is given by*

$$w(t, x) = 1 - \mathbb{E}_x^\xi \left[\prod_{v \in N(t)} (1 - w_0(X_t^v)) \right], \quad (t, x) \in [0, \infty) \times \mathbb{R}. \quad (\text{McKean})$$

For the proof see Sect. A.5.

Remark 2.2. We will frequently use the application of the latter result to functions $w_0 = \mathbb{1}_{(-\infty, 0]}$, resulting in

$$w(t, x) = 1 - \mathbb{E}_x^\xi \left[\prod_{v \in N(t)} \mathbb{1}_{(0, \infty)}(X_t^v) \right] = 1 - \mathbb{P}_x^\xi(X_t^v > 0 \ \forall v \in N(t)) = \mathbb{P}_x^\xi(N^{\leq}(t, 0) \geq 1).$$

Furthermore, we will need the so-called *many-to-few* lemma, which breaks down the moments of the branching process to a functional of the Brownian paths. For our purposes, it suffices to state it up to second moments (*many-to-one* and *many-to-two formula*).

Proposition 2.3. *Let $(p_k)_{k \in \mathbb{N}}$ fulfill (PROB) and let $\varphi_1, \varphi_2 : [0, \infty) \rightarrow [-\infty, \infty]$ be càdlàg functions satisfying $\varphi_1 \leq \varphi_2$. Then the first and second moments of the number of particles in $N(t)$ whose genealogy stays between φ_1 and φ_2 in the time interval $[0, t]$ are given by*

$$\begin{aligned} & \mathbb{E}_x^\xi \left[\left| \nu \in N(t) : \varphi_1(s) \leq X_s^v \leq \varphi_2(s) \ \forall s \in [0, t] \right| \right] && (\text{FK-1}) \\ & = E_x \left[\exp \left\{ \int_0^t \xi(B_r) dr \right\}; \varphi_1(s) \leq B_s \leq \varphi_2(s) \ \forall s \in [0, t] \right] \end{aligned}$$

and

$$\begin{aligned} & \mathbb{E}_x^\xi \left[\left| \nu \in N(t) : \varphi_1(s) \leq X_s^v \leq \varphi_2(s) \ \forall s \in [0, t] \right|^2 \right] && (\text{FK-2}) \\ & = E_x \left[\exp \left\{ \int_0^t \xi(B_r) dr \right\}; \varphi_1(s) \leq B_s \leq \varphi_2(s) \ \forall s \in [0, t] \right] \\ & \quad + (m_2 - 2) \int_0^t E_x \left[\exp \left\{ \int_0^s \xi(B_r) dr \right\} \xi(B_s) \mathbb{1}_{\{\varphi_1(r) \leq B_r \leq \varphi_2(r) \ \forall 0 \leq r \leq s\}} \right. \\ & \quad \left. \times \left(E_y \left[\exp \left\{ \int_0^{t-s} \xi(B_r) dr \right\} \mathbb{1}_{\{\varphi_1(r+s) \leq B_r \leq \varphi_2(r+s) \ \forall 0 \leq r \leq t-s\}} \right] \right)_{|y=B_s}^2 \right] ds, \end{aligned}$$

respectively.

The proof of the first identity follows from [19, Section 4.1]. The second identity can be shown by using [19, Lemma 1], conditioning on the first splitting of the so-called “spines”, similarly to the proof of [16, (2.6)] for $n = 2$ there. Indeed, one has to consider branching Brownian motion instead of branching random walk and replace binary branching by general branching. Then the expectation of the quantity in display [16, (2.15)] turns into the second summand in (FK-2), because the first splitting rate of the two “spine particles” at site y is $(m_2 - 2)\xi(y)$.

As a consequence of Proposition 2.3, the solution to (PAM) can be expressed by a functional of the branching process, i.e. we have that the solution to (PAM) is given by

$$u(t, x) = E_x \left[\exp \left\{ \int_0^t \xi(B_s) ds \right\} u_0(B_t) \right] = E_x^\xi \left[\sum_{v \in N(t)} u_0(X_t^v) \right].$$

As a special case for $u_0 = \mathbb{1}_{(-\infty, 0]}$, this turns into

$$u(t, x) = E_x^\xi [N^{\leq}(t, 0)] = E_x \left[\exp \left\{ \int_0^t \xi(B_s) ds \right\}; B_t \leq 0 \right]. \tag{2.2}$$

2.2. Change of measure

As a consequence of, among others, the previous display, it will be important to obtain a profound understanding of Brownian motion in the potential ξ . For this purpose, we introduce a change of measure which makes it typical for the Brownian motion in the Feynman–Kac formula started at tv , some $v \neq 0$, to be close to the origin at time t . This perturbed Brownian motion in the random potential ξ will then be amenable to an investigation through a certain regeneration structure, as we will show in Sect. 3. More precisely, let $(\xi(x))_{x \in \mathbb{R}}$ be as in (BDD) and define the shifted potential

$$\zeta := \xi - es.$$

Then \mathbb{P} -a.s.,

$$\zeta(x) \in [-(es - ei), 0] \quad \forall x \in \mathbb{R}. \tag{2.3}$$

We will oftentimes write

$$H_y := \inf \{ t \geq 0 : B_t = y \}, \quad y \in \mathbb{R}, \quad \text{and} \quad \tau_i := H_{i-1} - H_i, \quad i \in \mathbb{Z}, \tag{2.4}$$

for the first hitting times and their pairwise differences. The above shift of ξ gives rise to a change of measure which will play a crucial role in the following. For $x, y \in \mathbb{R}$ as well as $\eta \leq 0$ define the probability measures $P_{x,y}^{\zeta,\eta}$ via

$$P_{x,y}^{\zeta,\eta}(A) := \frac{1}{Z_{x,y}^{\zeta,\eta}} E_x \left[\exp \left\{ \int_0^{H_{x-y}} (\zeta(B_s) + \eta) ds \right\}; A \right], \quad A \in \sigma(B_{t \wedge H_{x-y}} : t \geq 0), \tag{2.5}$$

with normalizing constant

$$Z_{x,y}^{\zeta,\eta} := E_x \left[\exp \left\{ \int_0^{H_{x-y}} (\zeta(B_s) + \eta) ds \right\} \right] \in (0, \infty),$$

where P_x and E_x , $x \in \mathbb{R}$, are defined below (1.12). For $A \in \sigma(B_{t \wedge H_{x-y}} : t \geq 0)$, using the strong Markov property at time H_{x-y} , we infer that $P_{x,y}^{\zeta,\eta}(A) = P_{x,y'}^{\zeta,\eta}(A)$

for all $y' \geq y$. Thus, by the classical Kolmogorov’s extension theorem (see e.g. [36, Theorem 2.4.3]),

$$(P_{x,y}^{\zeta,\eta})_{y \geq 0} \text{ can be extended to a unique probability measure } P_x^{\zeta,\eta} \text{ on } \sigma(B_t : t \geq 0). \tag{2.6}$$

We write $E_x^{\zeta,\eta}$ for the corresponding expectation and introduce the logarithmic moment generating functions

$$L_x^\zeta(\eta) := \ln E_x \left[\exp \left\{ \int_0^{H_{[x]}-1} (\zeta(B_s) + \eta) ds \right\} \right], \quad x > 0, \tag{2.7}$$

$$\bar{L}_x^\zeta(\eta) := \frac{1}{x} \sum_{i=1}^x L_i^\zeta(\eta) = \frac{1}{x} \ln E_x \left[\exp \left\{ \int_0^{H_0} (\zeta(B_s) + \eta) ds \right\} \right], \quad x > 0,$$

where we recall the notation introduced in Sect. 1.9, and where the last equality is due to the strong Markov property. In addition, set

$$L(\eta) := \mathbb{E}[L_1^\zeta(\eta)]. \tag{2.8}$$

Due to (2.3), for any $\eta \leq 0$ the quantities above are well-defined, and it is easy to check that in this case and under (BDD), the expressions defined in (2.7)–(2.8) are finite. We have the following useful properties.

Lemma 2.4. (a) *The function $(-\infty, 0) \ni \eta \mapsto L(\eta)$ is infinitely differentiable and its derivative $L'(\eta)$ is positive and monotonically strictly increasing.*

(b) *We have \mathbb{P} -a.s. that*

$$\lim_{x \rightarrow \infty} \bar{L}_x^\zeta(\eta) = L(\eta) \text{ for all } \eta \leq 0. \tag{2.9}$$

(c) *$L'(\eta) \downarrow 0$ as $\eta \downarrow -\infty$*

(d) *For every $v > v_c := \frac{1}{L'(0-)}$, where $\frac{1}{+\infty} := 0$ and where we call v_c critical velocity, there exists a*

$$\text{unique solution } \bar{\eta}(v) < 0 \text{ to the equation } L'(\bar{\eta}(v)) = \frac{1}{v}. \tag{2.10}$$

$\bar{\eta}(v)$ can be characterized as the unique maximizer to $(-\infty, 0] \ni \eta \mapsto \frac{\eta}{v} - L(\eta)$, i.e.

$$\sup_{\eta \leq 0} \left(\frac{\eta}{v} - L(\eta) \right) = \frac{\bar{\eta}(v)}{v} - L(\bar{\eta}(v)). \tag{2.11}$$

The function $(v_c, \infty) \ni v \mapsto \bar{\eta}(v)$ is continuously differentiable and strictly decreasing.

Proof of Lemma 2.4. (a) This follows from Lemma A.1.

(b) By [14, Theorem 7.5.1], for every $\eta \leq 0$ we get \mathbb{P} -a.s. that $\lim_{x \rightarrow \infty} \bar{L}_x^\zeta(\eta) = \mathbb{E}[L_1^\zeta(\eta) | \mathcal{F}_{\text{inv}}^\zeta]$, where $\mathcal{F}_{\text{inv}}^\zeta$ is the σ -algebra of all \mathbb{P} -invariant sets. Due to our standing assumptions, the family $\zeta(x)$, $x \in \mathbb{R}$, is mixing and thus ergodic. Thus, $\mathcal{F}_{\text{inv}}^\zeta$ is \mathbb{P} -trivial, i.e., $\mathbb{E}[L_1^\zeta(\eta) | \mathcal{F}_{\text{inv}}^\zeta] = L(\eta)$. By continuity of the functions \bar{L}_x^ζ and L , the statement follows.

- (c) We note that L is strictly increasing and strictly convex on $(-\infty, 0)$ by (a), and that

$$L(\eta) \geq \mathbb{E} \left[\ln E_1 \left[e^{-(es - ei - \eta)H_0} \right] \right] = -\sqrt{2(es - ei - \eta)} \quad \text{for all } \eta \leq 0,$$

where the equality is due to [4, (2.0.1), p. 204]. Thus, we infer that its derivative $L'(\eta)$ must tend to 0 as $\eta \rightarrow -\infty$.

- (d) Using that $1/v_c > 1/v$ (where $\frac{1}{0} := +\infty$) and the fact that L' is strictly increasing and continuous with $L'(\eta) \downarrow 0$ for $\eta \downarrow -\infty$, we can find a unique $\bar{\eta}(v) < 0$ such that $L'(\bar{\eta}(v)) = 1/v$, giving (2.10). Display (2.11) is a direct consequence of (2.10) and standard properties of the Legendre transformations of strictly convex functions. In order to show the remaining part, we observe that since L' is strictly increasing and smooth on $(-\infty, 0)$, it has a strictly increasing inverse function $(L')^{-1}$, which is differentiable on $(0, 1/v_c)$. By (2.10), for $v > v_c$ we have $\bar{\eta}(v) = (L')^{-1}(1/v)$. Hence, using the formula for the derivative of the inverse function we get that

$$\bar{\eta}'(v) = -\frac{1}{v^2} \cdot \frac{1}{L''(\bar{\eta}(v))}.$$

Since the right-hand side of the latter display is continuous in v and negative, we can conclude.

We use the standard notation $L^* : \mathbb{R} \rightarrow (-\infty, \infty]$ to denote the Legendre transformation

$$v \mapsto \sup_{\eta \leq 0} (\eta v - L(\eta))$$

of L . Lemma 2.4 entails that

$$L^*(1/v) = \frac{\bar{\eta}(v)}{v} - L(\bar{\eta}(v)). \tag{2.12}$$

In the next part of this section, we are interested in the existence and the properties of a suitable *tilting parameter* $\eta_x^\zeta(v)$ such that

$$E_x^{\zeta, \eta_x^\zeta(v)}[H_0] = \frac{x}{v}, \quad x > 0, \quad v > 0, \tag{2.13}$$

holds true (setting $\eta_x^\zeta(v) := 0$ if no such parameter exists). For $\eta_x^\zeta(v)$ fulfilling (2.13) we observe that under $P_x^{\zeta, \eta_x^\zeta(v)}$, the Brownian motion is tilted to have time-averaged velocity v until it reaches the origin. More precisely, in Lemma 2.5 we will show that for suitable v and x large enough, a tilting parameter as postulated in (2.13) actually exists. Furthermore, we will show that the random parameter $\eta_x^\zeta(v)$ concentrates around the deterministic quantity $\bar{\eta}(v)$ defined in (2.10). The last result is a perturbation estimate for $\eta_x^\zeta(v)$ in x , cf. Lemma 2.7.

2.3. Concentration inequalities

We have the following result regarding the existence, negativity, and concentration properties of the postulated parameter $\eta_x^\zeta(v)$.

- Lemma 2.5.** (a) *For every $v > v_c$ there exists a finite random variable $\mathcal{N} = \mathcal{N}(v)$ such that for all $x \geq \mathcal{N}$ the solution $\eta_x^\zeta(v) < 0$ to (2.13) exists.*
 (b) *For each $q \in \mathbb{N}$ and each compact interval $V \subset (v_c, \infty)$, there exists $C_2 := C_2(V, q) \in (0, \infty)$ such that*

$$\mathbb{P} \left(\sup_{v \in V} \sup_{x \in [n, n+1]} |\eta_x^\zeta(v) - \bar{\eta}(v)| \geq C_2 \sqrt{\frac{\ln n}{n}} \right) \leq C_2 n^{-q} \quad \text{for all } n \in \mathbb{N}. \tag{2.14}$$

Proof. We recall that due to Lemma A.1, the tilting parameter $\eta_x^\zeta(v)$ can alternatively be characterized as the unique solution $\eta_x^\zeta(v) \in (-\infty, 0)$ to

$$(\bar{L}_x^\zeta)'(\eta_x^\zeta(v)) = \frac{1}{v}, \tag{2.15}$$

if the solution exists, and $\eta_x^\zeta(v) = 0$ otherwise. We start with noting that Part (a) directly follows from Part (b). Indeed, let $A_n, n \in \mathbb{N}$, be the event in the probability on the left-hand side of (2.14). Then $\sum_n \mathbb{P}(A_n) < \infty$ for $q \geq 2$. By the first Borel–Cantelli lemma, \mathbb{P} -a.s. only finitely many of the A_n occur. In combination with the fact that $\bar{\eta}(v) < 0$, cf. (2.10), this implies that \mathbb{P} -a.s., the value of $\eta_x^\zeta(v)$ can only vanish for $x > 0$ small enough. In particular, we deduce the existence of a \mathbb{P} -a.s. finite random variable \mathcal{N} as postulated.

Hence, it remains to show (2.14). For this purpose, in the following lemma we investigate the fluctuations of the functions through which the parameters $\eta_x^\zeta(v)$ and $\bar{\eta}(v)$ are implicitly defined; we will then infer the desired bounds on the fluctuations of the parameters themselves through perturbation estimates for these functions.

- Lemma 2.6.** *For every compact interval $\Delta \subset (-\infty, 0)$ and each $q \in \mathbb{N}$, there exists a constant $C_3 = C_3(\Delta, q) \in (0, \infty)$ such that*

$$\mathbb{P} \left(\sup_{\eta \in \Delta} \sup_{x \in [n, n+1]} \left| (\bar{L}_x^\zeta)'(\eta) - L'(\eta) \right| \geq C_3 \sqrt{\frac{\ln n}{n}} \right) \leq C_3 n^{-q} \quad \text{for all } n \in \mathbb{N}. \tag{2.16}$$

In order not to hinder the flow of reading, we postpone the proof of this auxiliary result to the end of the proof of Lemma 2.5 and finish the proof of Lemma 2.5 (b) first. Let $q \in \mathbb{N}$ and $V \subset (v_c, \infty)$ be a compact interval. By Lemma A.1, for each compact $\Delta \subset (-\infty, 0)$ we have \mathbb{P} -a.s.,

$$\begin{aligned} C_{18}^{-1} &\leq \inf_{\eta \in \Delta} L''(\eta) \leq \sup_{\eta \in \Delta} L''(\eta) \leq C_{18}, \\ C_{18}^{-1} &\leq \inf_{x \geq 1} \inf_{\eta \in \Delta} (\bar{L}_x^\zeta)''(\eta) \leq \sup_{x \geq 1} \sup_{\eta \in \Delta} (\bar{L}_x^\zeta)''(\eta) \leq C_{18}. \end{aligned} \tag{2.17}$$

Therefore, and because the function $V \ni v \mapsto \bar{\eta}(v)$ is strictly decreasing by Lemma 2.4, it is possible to find $N = N(V) \in \mathbb{N}$ and a compact interval $\Delta = \Delta(N, V) \subset (-\infty, 0)$, where for notational convenience we write

$$V = [v_*, v^*] \quad \text{and} \quad \Delta = [\eta_*, \eta^*], \tag{2.18}$$

such that, using standard calculus for sets,

$$(\bar{\eta}(V) - C_3 C_{18} \sqrt{\ln n/n}) \cup (\bar{\eta}(V) + C_3 C_{18} \sqrt{\ln n/n}) \subset \Delta \quad \text{for all } n \geq N.$$

Let $n \geq N$ and assume that the complement of the event on the left-hand side of (2.16),

$$\sup_{\eta \in \Delta} \sup_{x \in [n, n+1)} |(\bar{L}_x^\zeta)'(\eta) - L'(\eta)| < C_3 \sqrt{\frac{\ln n}{n}}, \tag{2.19}$$

occurs. On this event, for all $v \in V$ and all $x \in [n, n + 1)$,

$$(\bar{L}_x^\zeta)'(\bar{\eta}(v) - C_3 C_{18} \sqrt{\ln n/n}) \leq \frac{1}{v} \leq (\bar{L}_x^\zeta)'(\bar{\eta}(v) + C_3 C_{18} \sqrt{\ln n/n}) \tag{2.20}$$

and thus, due to the strict monotonicity of $(\bar{L}_x^\zeta)'$ as well as its continuity implied by (2.17), there exists a unique $\eta_x^\zeta(v) \in \Delta$ such that $(\bar{L}_x^\zeta)'(\eta_x^\zeta(v)) = 1/v$. Due to (2.20), still assuming (2.19), we have

$$\sup_{v \in V} \sup_{x \in [n, n+1)} |\bar{\eta}(v) - \eta_x^\zeta(v)| \leq C_3 C_{18} \sqrt{\frac{\ln n}{n}}.$$

Thus, for $n \geq N$, choosing $C_2 > C_3 C_{18}$, the probability in (2.14) is upper bounded by the right-hand side of (2.16), which finishes the proof. \square

It remains to prove Lemma 2.6.

Proof of Lemma 2.6. Applying the strong Markov property at time $H_{\lfloor x \rfloor}$, we get

$$x(\bar{L}_x^\zeta)'(\eta) = E_x^{\zeta, \eta}(H_0) = E_x^{\zeta, \eta}[H_{\lfloor x \rfloor}] + E_{\lfloor x \rfloor}^{\zeta, \eta}[H_0] = E_x^{\zeta, \eta}[H_{\lfloor x \rfloor}] + \lfloor x \rfloor (\bar{L}_{\lfloor x \rfloor}^\zeta)'(\eta).$$

Furthermore, $0 \leq E_x^{\zeta, \eta}[H_{\lfloor x \rfloor}] = (L_x^\zeta)'(\eta) \leq C_{18}$ by (A.1) and Lemma A.1(b), and thus also $0 \leq (\bar{L}_x^\zeta)'(\eta) = \frac{1}{x} \sum_{i=1}^x (L_i^\zeta)'(\eta) \leq C_{18}$ for all $x \geq 1$ and all $\eta \in \Delta$, \mathbb{P} -a.s. As a consequence, we get that for all $x \geq 1$,

$$|(\bar{L}_x^\zeta)'(\eta) - L'(\eta)| \leq |(\bar{L}_{\lfloor x \rfloor}^\zeta)'(\eta) - L'(\eta)| + \frac{2C_{18}}{\lfloor x \rfloor}.$$

It is therefore enough to prove

$$\mathbb{P} \left(\sup_{\eta \in \Delta} \left| (\bar{L}_n^\zeta)'(\eta) - L'(\eta) \right| \geq C_3 \sqrt{\frac{\ln n}{n}} \right) \leq C_3 n^{-q} \quad \text{for all } n \in \mathbb{N}. \tag{2.21}$$

For each $\eta \in \Delta$, the sequence $((L_i^\zeta)'(\eta) - L'(\eta))_{i \in \mathbb{Z}}$ is a family of stationary, centered and bounded random variables. Furthermore, they fulfill the exponential

mixing condition (A.11) due to Lemma A.2. Due to $\sigma((L_i^\zeta)'(\eta) : i \geq k) \subset \sigma(\xi(x) : x \geq k - 1)$ and (MIX), setting $Y_i := (L_i^\zeta)'(\eta) - L'(\eta)$, the left-hand side in (A.15) is bounded from above by some constant $c_2 > 0$, uniformly for every i . Then setting $m_i := c_1$, condition (A.15) is fulfilled and we can apply the Hoeffding-type inequality from Corollary A.5 to infer the existence of $c_2 > 0$ such that

$$\mathbb{P}\left(\left|(\overline{L}_n^\zeta)'(\eta) - L'(\eta)\right| \geq c_2\sqrt{\frac{\ln n}{n}}\right) \leq c_2n^{-q-1} \quad \text{for all } \eta \in \Delta \text{ and all } n \in \mathbb{N}.$$

Let $\Delta_n := (\Delta \cap \frac{1}{n}\mathbb{Z}) \cup \{\eta_*, \eta^*\}$, recalling the notation of (2.18). Because $|\Delta \cap \frac{1}{n}\mathbb{Z}| \leq n \cdot \text{diam}(\Delta) + 1$, taking advantage of the previous display we infer

$$\begin{aligned} \mathbb{P}\left(\sup_{\eta \in \Delta_n} \left|(\overline{L}_n^\zeta)'(\eta) - L'(\eta)\right| \geq c_2\sqrt{\frac{\ln n}{n}}\right) \\ \leq |\Delta_n| \cdot \sup_{\eta \in \Delta_n} \mathbb{P}\left(\left|(\overline{L}_n^\zeta)'(\eta) - L'(\eta)\right| \geq c_2\sqrt{\frac{\ln n}{n}}\right) \leq c_3(\Delta)n^{-q}, \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

By Lemma A.1(b), we have \mathbb{P} -a.s. that $\sup_n \sup_{\eta \in \Delta} |(\overline{L}_n^\zeta(\eta))''| \vee |L''(\eta)| \leq C_{18}$. Thus, the mean value theorem entails that

$$\sup_{\eta \in \Delta} \left|(\overline{L}_n^\zeta)'(\eta) - L'(\eta)\right| \leq \sup_{\eta \in \Delta_n} \left|(\overline{L}_n^\zeta)'(\eta) - L'(\eta)\right| + \frac{2C_{18}}{n},$$

and thus we find $C_3 > 0$ such that (2.21) and hence (2.16) hold true.

In what comes below, many results will implicitly depend on the choice of the compact intervals V and Δ , which have already occurred before. Thus, in order to avoid ambiguity and due to assumption (VEL), we will now

$$\text{fix arbitrary compact intervals } V \subset (v_c, \infty) \text{ and } \Delta = \Delta(V) \subset (-\infty, 0) \text{ such that} \quad (2.22) \\ v_0 \in \text{int}(V) \text{ and } \bar{\eta}(V) \subset \text{int}(\Delta).$$

Furthermore, due to Lemma 2.5, there exists a \mathbb{P} -a.s. finite random variable $\mathcal{N}_1 = \mathcal{N}_1(\xi, C_2(V, 2))$ such that

$$\mathcal{H}_x := \mathcal{H}_x(V) := \{\eta_x^\zeta(v) \in \Delta \text{ for all } v \in V\} \quad \text{occurs for all } x \geq \mathcal{N}_1. \quad (2.23)$$

We write

$$(\overline{L}_x^\zeta)^* \left(\frac{1}{v}\right) = \sup_{\eta < 0} \left(\frac{\eta}{v} - \overline{L}_x^\zeta(\eta)\right) = \frac{\eta_x^\zeta(v)}{v} - \overline{L}_x^\zeta(\eta_x^\zeta(v)), \quad x \geq 1,$$

for the Legendre transformation of the weighted averages. We also recall that $\eta_x^\zeta(v) = 0$ if there is no solution $\eta_x^\zeta(v) \in \Delta$ to (2.15); note that this can only happen on \mathcal{H}_x^c .

In order to show an invariance principle for the Legendre transformation $(\overline{L}_x^\zeta)^*$ in the following section, we now derive a perturbation result on the tilting parameter $\eta_x^\zeta(v)$ in x .

Lemma 2.7. *There exists a constant $C_4 > 0$ such that \mathbb{P} -a.s., for all $x \in (0, \infty)$ large enough, uniformly in $v \in V$ and $0 \leq h \leq x$,*

$$|\eta_x^\zeta(v) - \eta_{x+h}^\zeta(v)| \leq C_4 \frac{h}{x}. \tag{2.24}$$

Proof. By Lemma 2.5 we can choose x large enough such that $\eta_y^\zeta(v) \in \Delta$ for all $y \geq x$ and all $v \in V$. For $h = 0$, the statement is obvious. For $0 < h \leq x$, it suffices to show that there exists $c_1 > 0$ such that

$$\sup_{\eta \in \Delta} |(\overline{L}_{x+h}^\zeta)'(\eta) - (\overline{L}_x^\zeta)'(\eta)| \leq c_1 \frac{h}{x}. \tag{2.25}$$

Indeed, using (2.15) we can write

$$\begin{aligned} (\overline{L}_{x+h}^\zeta)'(\eta_{x+h}^\zeta(v)) - (\overline{L}_x^\zeta)'(\eta_{x+h}^\zeta(v)) &= (\overline{L}_x^\zeta)'(\eta_x^\zeta(v)) - (\overline{L}_x^\zeta)'(\eta_{x+h}^\zeta(v)) \\ &= (\overline{L}_x^\zeta)''(\tilde{\eta})(\eta_x^\zeta(v) - \eta_{x+h}^\zeta(v)) \end{aligned}$$

for some $\tilde{\eta} \in \Delta$ between $\eta_x^\zeta(v)$ and $\eta_{x+h}^\zeta(v)$. By the second display in (2.17) we know that \mathbb{P} -a.s. $\inf_{\eta \in \Delta, x \geq 1} (\overline{L}_x^\zeta)''(\eta) \geq C_{18}^{-1}$. Using this, inequality (2.24) is a direct consequence of (2.25) with $C_4 := c_1 C_{18}$. To prove (2.25), recall that for all $\eta \in \Delta, x \geq 1$, and $0 < h \leq x$, by the strong Markov property applied at time H_x ,

$$\begin{aligned} (\overline{L}_{x+h}^\zeta)'(\eta) - (\overline{L}_x^\zeta)'(\eta) &= \frac{1}{x+h} (E_{x+h}^{\zeta, \eta}[H_x] + E_x^{\zeta, \eta}[H_0]) - \frac{1}{x} E_x^{\zeta, \eta}[H_0] \\ &= -\frac{h}{x+h} (\overline{L}_x^\zeta)'(\eta) + \frac{h}{x+h} \frac{1}{h} E_{x+h}^{\zeta, \eta}[H_x]. \end{aligned}$$

Finally, recall that by Lemma A.1 there exists $C_{18} = C_{18}(\Delta) > 0$ such that \mathbb{P} -a.s. we have $\sup_{\eta \in \Delta, x > 0} |(\overline{L}_x^\zeta)'(\eta)| \leq C_{18}$. By exactly the same argument used for the proof of the latter inequality [see proof of (A.1)], one can show that also $\sup_{\eta \in \Delta, x, h > 0} |\frac{1}{h} E_{x+h}^{\zeta, \eta}[H_x]| \leq C_{18}$ holds \mathbb{P} -a.s. with the same constant C_{18} . (2.25) now follows choosing $c_1 := 2C_{18}$.

3. Functional Central Limit Theorems, Large Deviations and Perturbation Results for the PAM

The principal objective of this section is to establish our first main results, i.e. the functional central limit theorems stated in Theorems 1.3 and 1.4. In order to prepare for this, we start with proving a functional central limit theorem for a suitably centered and rescaled version of the empirical Legendre transforms in Proposition 3.1 of Sect. 3.1. In Sect. 3.2 we then show how this limit theorem can be transferred to an auxiliary quantity Y_v^\approx , see Proposition 3.5. In combination with concentration results (referred to as exact large deviations in probability theory) also obtained in Proposition 3.5, we can then deduce that the auxiliary quantity Y_v^\approx provides a good description of the (Feynman–Kac representation of the) solution to

(PAM), see Corollary 3.8 also. In Sect. 3.3 we can then put these findings together to prove Theorem 1.3.

We next obtain perturbation results for the solution to (PAM) in Sects. 3.4 and 3.5. In Sect. 3.7, using the approximation results established in Sect. 3.6 in combination with Theorem 1.3, we can then transfer the latter functional central limit theorem to a functional central limit theorem for the position of the front of the solution to (PAM), hence completing the proof of Theorem 1.4.

3.1. A first functional central limit theorem

We start with a functional central limit theorem for a centered and rescaled version of the empirical Legendre transforms.

For this purpose let

$$\begin{aligned}
 V_x^{\zeta, v}(\eta) &:= \frac{\eta}{v} - L_x^\zeta(\eta), \quad x \in \mathbb{R}, \\
 \sigma_v^2 &:= \text{Var}_{\mathbb{P}}(V_1^{\zeta, v}(\bar{\eta}(v))) + 2 \sum_{i=2}^{\infty} \text{Cov}_{\mathbb{P}}(V_1^{\zeta, v}(\bar{\eta}(v)), V_i^{\zeta, v}(\bar{\eta}(v))), \quad \sigma_v := \sqrt{\sigma_v^2}, \quad v \in V.
 \end{aligned}
 \tag{3.1}$$

We start with observing that $\sigma_v^2 \in [0, \infty)$ for all $v \in V$. Indeed, $(\tilde{L}_i)_{i \in \mathbb{N}}$, where $\tilde{L}_i := L_i^\zeta(\bar{\eta}(v)) - \mathbb{E}[L_i^\zeta(\bar{\eta}(v))]$, is a sequence of bounded (see Lemma A.1), centered and mixing (see Lemma A.2) random variables, giving

$$\sum_{i=1}^{\infty} |\text{Cov}_{\mathbb{P}}(V_1^{\zeta, v}(\bar{\eta}(v)), V_i^{\zeta, v}(\bar{\eta}(v)))| = \sum_{i=1}^{\infty} |\mathbb{E}[\tilde{L}_1 \tilde{L}_i]| = \sum_{i=1}^{\infty} |\mathbb{E}[\tilde{L}_i \mathbb{E}[\tilde{L}_1 | \mathcal{F}^{i-1}]]| < \infty,
 \tag{3.2}$$

where the last inequality is due to uniform boundedness of \tilde{L}_i in i , (A.9) and the summability criterion in (MIX). Thus, σ_v^2 is well-defined and finite. Furthermore, the non-negativity $\sigma_v^2 \geq 0$ is due to (3.2) and [34, Lemma 1.1].

We now introduce the process $W_x^v(t)$ of empirical Legendre transformations

$$W_x^v(t) := t\sqrt{x} \left((\bar{L}_{xt}^\zeta)^*(1/v) - L^*(1/v) \right), \quad t, x > 0, \quad v \in V, \tag{3.3}$$

set $W_0^v(t) = W_x^v(0) = 0$ for $t, x > 0, v \in V$, and obtain a first functional central limit theorem for it.

Proposition 3.1. *For every $v \in V$, $W_n^v(1)$ converges in \mathbb{P} -distribution to a centered Gaussian random variable with variance $\sigma_v^2 \geq 0$. If $\sigma_v^2 > 0$, the sequence of processes*

$$[0, \infty) \ni t \mapsto \frac{1}{\sigma_v} W_n^v(t), \quad n \in \mathbb{N},$$

converges in \mathbb{P} -distribution to a standard Brownian motion in the sense of weak convergence of measures on $C([0, \infty))$, endowed with the topology induced by the metric ρ from (1.15).

Proof. It is sufficient to show the claim if $(W_n^v(t))_{t \in [0, \infty)}$ is replaced by $(W_n^v(t) \cdot \mathbb{1}_{\mathcal{H}_{nt}})_{t \in [0, \infty)}$, $n \in \mathbb{N}$, with \mathcal{H}_{nt} as defined in (2.23), since the \mathbb{P} -probability of \mathcal{H}_{nt} tends to 1 for $n \rightarrow \infty$ by Lemma 2.5. In the notation of (3.1), setting

$$S_x^{\zeta, v}(\eta) := \sum_{i=1}^x V_i^{\zeta, v}(\eta), \quad x \in \mathbb{R}, \tag{3.4}$$

on \mathcal{H}_{nt} we have

$$(\overline{L}_{nt}^\zeta)^* \left(\frac{1}{v} \right) = \frac{\eta_{nt}^\zeta(v)}{v} - \overline{L}_{nt}^\zeta(\eta_{nt}^\zeta(v)) = \frac{1}{nt} \sum_{i=1}^{nt} V_i^{\zeta, v}(\eta_{nt}^\zeta(v)) = \frac{1}{nt} S_{nt}^{\zeta, v}(\eta_{nt}^\zeta(v)).$$

Thus, we can rewrite the relevant term as a sum of three differences

$$\begin{aligned} nt \left((\overline{L}_{nt}^\zeta)^* (1/v) - L^*(1/v) \right) &= \left(S_{nt}^{\zeta, v}(\eta_{nt}^\zeta(v)) - S_{nt}^{\zeta, v}(\overline{\eta}(v)) \right) \\ &\quad + \left(S_{nt}^{\zeta, v}(\overline{\eta}(v)) - \mathbb{E}[S_{nt}^{\zeta, v}(\overline{\eta}(v))] \right) \\ &\quad + \left(\mathbb{E}[S_{nt}^{\zeta, v}(\overline{\eta}(v))] - nt L^*(1/v) \right), \end{aligned} \tag{3.5}$$

where we note that the third summand vanishes. Indeed, we have

$$\mathbb{E}[S_{nt}^{\zeta, v}(\overline{\eta}(v))] = nt \left(\frac{\overline{\eta}(v)}{v} - \mathbb{E}[L_1^\zeta(\overline{\eta}(v))] \right) = nt L^*(1/v),$$

where the last equality is due to (2.11) and the definition of the Legendre transform. The proof is completed by the use of Lemmas 3.3 and 3.2 below, which show that the second summand of (3.5) exhibits the postulated diffusive behavior whereas the first summand is negligible in that scaling.

Lemma 3.2. *For every $v \in V$ and $t > 0$, the sequence of random variables $\frac{1}{\sqrt{n}} \left(S_{nt}^{\zeta, v}(\overline{\eta}(v)) - \mathbb{E}[S_{nt}^{\zeta, v}(\overline{\eta}(v))] \right)$, $n \in \mathbb{N}$, converges in \mathbb{P} -distribution to a centered Gaussian random variable with variance $\sigma_v^2 \geq 0$. If $\sigma_v^2 > 0$, the sequence of processes*

$$[0, \infty) \ni t \mapsto \frac{1}{\sigma_v \sqrt{n}} \left(S_{nt}^{\zeta, v}(\overline{\eta}(v)) - \mathbb{E}[S_{nt}^{\zeta, v}(\overline{\eta}(v))] \right), \quad n \in \mathbb{N},$$

converges in \mathbb{P} -distribution to a standard Brownian motion in the sense of weak convergence of measures on $C([0, \infty))$, endowed with the topology induced by the metric ρ from (1.15).

Proof. Let $\tilde{L}_i := L_i^\zeta(\overline{\eta}(v)) - \mathbb{E}[L_i^\zeta(\overline{\eta}(v))]$, $\tilde{V}_i := V_i^\zeta(\overline{\eta}(v)) - \mathbb{E}[V_i^\zeta(\overline{\eta}(v))]$ and $M \in \mathbb{N}$. Further set $\tilde{L}_i^{(M)} := \sum_{j=1+(i-1)M}^{iM} \tilde{L}_j$. Then $(\tilde{L}_i^{(M)})_{i \in \mathbb{Z}}$ is a sequence of centered, stationary and (by Lemma A.1) bounded random variables. To show the central limit theorem on $C([0, M])$, we will use the method of martingale approximation from [17], which is summarized as a theorem by NOLEN in [29, Section 2.3] and turns out to be applicable in our situation. That is, we have to

make sure that condition [29, (2.36)] is fulfilled. Indeed, replacing \mathcal{F}^j in (A.13) by \mathcal{F}_k and noting that quantity A in (A.13) is \mathcal{F}_k -measurable we get

$$\sum_{k=1}^{\infty} |\tilde{L}_0^{(M)} - \mathbb{E}[\tilde{L}_0^{(M)} | \mathcal{F}_k]| \leq c_1 \sum_{k=1}^{\infty} e^{-k/c_1} < \infty,$$

giving the convergence of the first series in [29, (2.36)]. Furthermore, using that $\tilde{L}_k^{(M)}$ is $\mathcal{F}^{(k-1)M}$ -measurable and bounded, also recalling (MIX), we get

$$\sum_{k=1}^{\infty} |\mathbb{E}[\tilde{L}_k^{(M)} | \mathcal{F}_0]| \leq \sum_{k=1}^{\infty} \psi(k-1) \mathbb{E}[|\tilde{L}_0^{(M)}|] < \infty.$$

Because the series in (3.1) is absolutely convergent, by [34, Lemma 1.1] and [29, (2.37)] we have $\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[(\sum_{k=1}^n \tilde{L}_k^{(M)})^2] = M \cdot \lim_{n \rightarrow \infty} \frac{1}{Mn} \mathbb{E}[(\sum_{k=1}^{Mn} \tilde{L}_k)^2] = M \cdot \sigma_v^2 \in [0, \infty)$. Furthermore, if $\sigma_v^2 > 0$, [29, Theorem 2.1] entails that the sequence of processes

$$[0, 1] \ni t \mapsto X_n^{(M)}(t) := \frac{1}{\sigma_v \sqrt{nM}} \left(\sum_{k=1}^{\lfloor nt \rfloor} \tilde{L}_k^{(M)} + (nt - \lfloor nt \rfloor) \tilde{L}_{\lfloor nt \rfloor + 1}^{(M)} \right), \quad n \in \mathbb{N},$$

converges in \mathbb{P} -distribution to a standard Brownian motion $(B_t)_{t \in [0,1]}$ in the sense of weak convergence of measures on $C([0, 1])$ with the topology induced by the uniform metric. Then by definition, the above convergence also holds true for $(\tilde{V}_i)_{i \geq 1}$ instead of $(\tilde{L}_i)_{i \geq 1}$. Furthermore, we have the uniform bound

$$\sup_{t \in [0, M], n \in \mathbb{N}} \left| S_{nt}^{\zeta, v}(\bar{\eta}(v)) - \left(\sum_{i=1}^{\lfloor nt/M \rfloor} V_i^{\zeta, v} + (nt - \lfloor nt/M \rfloor) \sum_{i=1 + \lfloor nt/M \rfloor}^{M + \lfloor nt/M \rfloor} V_i^{\zeta, v} \right) \right| \leq c_2 \quad \mathbb{P}\text{-a.s.}$$

Consequently, the sequence $[0, M] \ni t \mapsto \frac{1}{\sigma_v \sqrt{n}} \left(S_{nt}^{\zeta, v}(\bar{\eta}(v)) - \mathbb{E}[S_{nt}^{\zeta, v}(\bar{\eta}(v))] \right)$ has the same weak limit as $(\sqrt{M} \cdot X_n^{(M)}(t/M))_{t \in [0, M]}$, $n \in \mathbb{N}$, which converges to $(\sqrt{M} \cdot B(t/M))_{t \in [0, M]}$ and the latter process is a standard Brownian motion on $[0, M]$. Because $M \in \mathbb{N}$ was arbitrary, WHITT [38, Theorem 5] gives weak convergence on $C([0, \infty))$.

To show that the first summand in (3.5) is asymptotically negligible, we use the following result.

Lemma 3.3. *There exists a constant $C_5 \in (0, \infty)$ such that for every $v \in V$ and $M > 0$,*

$$\limsup_{n \rightarrow \infty} \frac{1}{\ln n} \sup_{0 \leq t \leq M} \left| S_{nt}^{\zeta, v}(\eta_{nt}^{\zeta}(v)) - S_{nt}^{\zeta, v}(\bar{\eta}(v)) \right| \leq C_5 \quad \mathbb{P}\text{-a.s.}$$

Proof. There exists a \mathbb{P} -a.s. finite time $\mathcal{N}_1(\omega)$, defined before (2.23), such that for all $x \geq \mathcal{N}_1$ and all $v \in V$ we have $\eta_x^\zeta(v) \in \Delta$. Furthermore, by Lemma A.1, $S_x^{\zeta,v}$ is infinitely differentiable on $(-\infty, 0)$, so for all $x \geq \mathcal{N}_1$ there exists $\tilde{\eta}_x^\zeta(v) \in [\bar{\eta}(v) \wedge \eta_x^\zeta(v), \bar{\eta}(v) \vee \eta_x^\zeta(v)]$ such that

$$S_x^{\zeta,v}(\bar{\eta}(v)) = S_x^{\zeta,v}(\eta_x^\zeta(v)) + (S_x^{\zeta,v})'(\eta_x^\zeta(v))(\bar{\eta}(v) - \eta_x^\zeta(v)) + \frac{(S_x^{\zeta,v})''(\tilde{\eta}_x^\zeta(v))}{2}(\bar{\eta}(v) - \eta_x^\zeta(v))^2.$$

Due to (2.15), $(S_x^{\zeta,v})'(\eta_x^\zeta(v)) = 0$ and by Lemma A.1 we have $\sup_{\eta \in \Delta} \sup_{x \geq 1} |(S_x^{\zeta,v})''(\eta)|/x \leq c_1$. By (2.14) and the first Borel–Cantelli lemma, there exists a finite random variable $\mathcal{N}_2 \geq \mathcal{N}_1$ such that for $x \geq \mathcal{N}_2$ the complementary event on the left-hand side of (2.14) occurs, hence

$$\sup_{x \geq \mathcal{N}_1} (\eta_x^\zeta(v) - \bar{\eta}(v))^2 \frac{x}{\ln x} \leq C_2^2(V, 2)$$

and thus

$$\sup_{x \geq \mathcal{N}_1} \frac{1}{\ln x} |S_x^{\zeta,v}(\eta_x^\zeta(v)) - S_x^{\zeta,v}(\bar{\eta}(v))| \leq C_5 \tag{3.6}$$

with $C_5 := \frac{c_1 C_2^2(V, 2)}{2}$. Finally, we have

$$\begin{aligned} \sup_{0 \leq t \leq M} \left| S_{nt}^{\zeta,v}(\eta_{nt}^\zeta(v)) - S_{nt}^{\zeta,v}(\bar{\eta}(v)) \right| &\leq \sup_{0 \leq x \leq \mathcal{N}_1} \left| S_x^{\zeta,v}(\eta_x^\zeta(v)) - S_x^{\zeta,v}(\bar{\eta}(v)) \right| \\ &\quad + \sup_{\mathcal{N}_1/n \leq t \leq M} \left| S_{nt}^{\zeta,v}(\eta_{nt}^\zeta(v)) - S_{nt}^{\zeta,v}(\bar{\eta}(v)) \right| \\ &\leq 2\mathcal{N}_1 c_2 + C_5 \ln M + C_5 \ln n, \end{aligned}$$

where in the last inequality we used that \mathbb{P} -a.s., every summand in the definition of $S_n^{\zeta,\eta}$ is uniformly bounded by c_2 . The \mathbb{P} -a.s. finiteness of \mathcal{N}_1 gives the claim.

As a by-product of the proof above we get an approximation result of W_x^v be a centered stationary sequence.

Corollary 3.4. *For every $v \in V$ and all t such that $vt \geq \mathcal{N}_1$, we have*

$$\left| \sqrt{vt} W_{vt}^v(1) - \sum_{i=1}^{vt} (L(\bar{\eta}(v)) - L_i^\zeta(\bar{\eta}(v))) \right| \leq C_5 \ln v + C_5 \ln t.$$

Proof. By the definition of $W_x^v(t)$ and $S_x^{\zeta,v}(\eta)$ from (3.3) and (3.4), as well as the definition in (2.12) for the corresponding Legendre transformations, we have

$$\sqrt{vt} W_{vt}^v(1) = vt \left((\bar{L}_{vt}^\zeta)^*(1/v) - L^*(1/v) \right)$$

$$\begin{aligned}
 &= vt \left(\frac{\eta_{vt}^\zeta(v)}{v} - \bar{L}_{vt}(\eta_{vt}^\zeta(v)) - \frac{\bar{\eta}(v)}{v} + L(\bar{\eta}(v)) \right) \\
 &= S_{vt}^{\zeta,v}(\eta_{vt}^\zeta(v)) - \sum_{i=1}^{vt} \left(\frac{\bar{\eta}(v)}{v} - L_i^\zeta(\bar{\eta}(v)) \right) + \sum_{i=1}^{vt} (L(\bar{\eta}(v)) - L_i^\zeta(\bar{\eta}(v))) \\
 &= S_{vt}^{\zeta,v}(\eta_{vt}^\zeta(v)) - S_{vt}^{\zeta,v}(\bar{\eta}(v)) + \sum_{i=1}^{vt} (L(\bar{\eta}(v)) - L_i^\zeta(\bar{\eta}(v))).
 \end{aligned}$$

Then we can conclude using (3.6).

3.2. An exact large deviation result for auxiliary processes

The main result of this subsection is Proposition 3.5, where we show—cf. (3.10)—that the probability for the perturbed Brownian motion in shifted potential to hit the origin at time x/v for the first time exhibits certain concentration properties. These concentration properties then allow us to investigate Y_v^\approx instead of Y_v , see (3.7) and (3.9) below. The virtue of Y_v^\approx is that we will be able to employ Proposition 3.1 to deduce a functional central limit theorem for the process induced by it, see also (3.22) below. This again will be beneficial since Y_v^\approx provides a good description of the (Feynman–Kac representation of the) solution to (PAM), see Corollary 3.8 below.

For $x \geq 0$ and $v > 0$ we introduce

$$\begin{aligned}
 Y_v^\approx(x) &:= E_x \left[e^{\int_0^{H_0} \zeta(B_s) ds}; H_0 \in \left[\frac{x}{v} - K, \frac{x}{v} \right] \right], \\
 Y_v^>(x) &:= E_x \left[e^{\int_0^{H_0} \zeta(B_s) ds}; H_0 < \frac{x}{v} - K \right], \quad \text{and} \\
 Y_v(x) &:= E_x \left[e^{\int_0^{H_0} \zeta(B_s) ds}; H_0 \leq \frac{x}{v} \right] = Y_v^\approx(x) + Y_v^>(x),
 \end{aligned} \tag{3.7}$$

where $K > 0$ is a constant, defined in (3.17) below. For $v \in V$ and $x \geq 1$ we define the random quantity

$$\sigma_x^\zeta(v) := \begin{cases} |\eta_x^\zeta(v)| \sqrt{\text{Var}_x^{\zeta, \eta_x^\zeta(v)}(H_0)}, & \text{on } \mathcal{H}_x, \\ \sup_{\eta \in \Delta} |\eta| \sqrt{\text{Var}_x^{\zeta, \sup_{\eta \in \Delta}}(H_0)}, & \text{on } \mathcal{H}_x^c. \end{cases}$$

Furthermore, by Lemma A.1, there exists some $C_{18} > 1$ such that $\text{Var}_x^{\zeta, \eta}(H_0) = x(\bar{L}_x^\zeta)''(\eta) \in [xC_{18}^{-1}, xC_{18}]$ for all $x \geq 1$. Thus, there is some constant $C_6 = C_6(\Delta) > 1$ such that

$$C_6^{-1} \sqrt{x} \leq \sigma_x^\zeta(v) \leq C_6 \sqrt{x} \quad \text{for all } v \in V, x \geq 1, \text{ on } \mathcal{H}_x. \tag{3.8}$$

We now prove the following result.

Proposition 3.5. *Let V be as in (2.22), σ_v defined by (3.1) and $W_x^v(t)$ as in (3.3), $v \in V$, and $K > 0$ be such that (3.17) holds. Then there exists a constant $C_7 \in (1, \infty)$, such that for all $v \in V$ and all $x \geq 1$, on \mathcal{H}_x we have*

$$\sigma_x^\zeta(v) Y_v^>(x) \exp \left\{ x L^*(1/v) + \sqrt{x} W_x^v(1) \right\} \in \left[C_7^{-1}, C_7 \right]. \tag{3.9}$$

Furthermore, for all $v \in V$ and all $x \geq 1$, on \mathcal{H}_x we have

$$\frac{Y_v^\approx(x)}{Y_v^>(x)} \in \left[C_7^{-1}, C_7 \right], \tag{3.10}$$

and the sequence $n^{-1/2} (\ln Y_v^\approx(n) - n L^*(1/v))$, $n \in \mathbb{N}$, converges to a centered Gaussian random variable with variance $\sigma_v^2 \in [0, \infty)$, where σ_v^2 is defined in (3.1). If furthermore $\sigma_v^2 > 0$, then the sequence of processes

$$[0, \infty) \ni t \mapsto \frac{1}{\sigma_v \sqrt{n}} (\ln Y_v^\approx(nt) + nt L^*(1/v)), \quad n \in \mathbb{N}, \tag{3.11}$$

converges in \mathbb{P} -distribution to standard Brownian motion in $C([0, \infty))$.

Proof. We start with proving (3.9) and for this purpose let $x \geq 1$ such that $\mathcal{H}_{\lfloor x \rfloor}$ occurs. Write $\eta := \eta_x^\zeta(v)$ and $\sigma := \sigma_x^\zeta(v)$, and recall the notation introduced in (2.4), i.e. $\tau_i = H_i - H_{i-1}$, $i = 1, \dots, \lfloor x \rfloor - 1$, and set $\tau_x := H_{\lfloor x \rfloor - 1} - H_x$, $x \in \mathbb{R} \setminus \mathbb{Z}$, (which is consistent with the definition in (2.4) for x integer) to define $\widehat{\tau}_i := \widehat{\tau}_i^{(x)} := \tau_i - E_x^{\zeta, \eta} [\tau_i]$. Then $\sum_{i=1}^x E_x^{\zeta, \eta} [\tau_i] = E_x^{\zeta, \eta} [H_0] = \frac{x}{v}$. We now rewrite

$$\begin{aligned} Y_v^\approx(x) &= E_x \left[e^{\int_0^{H_0} (\zeta(B_s) + \eta) ds} \exp \left\{ -\eta \sum_{i=1}^x \widehat{\tau}_i \right\}; \sum_{i=1}^x \widehat{\tau}_i \in [-K, 0] \right] \exp \left\{ -x \frac{\eta}{v} \right\} \\ &= E_x^{\zeta, \eta} \left[\exp \left\{ -\sigma \frac{\eta}{\sigma} \sum_{i=1}^x \widehat{\tau}_i \right\}; \frac{\eta}{\sigma} \sum_{i=1}^x \widehat{\tau}_i \in \left[0, -\frac{K\eta}{\sigma} \right] \right] \exp \left\{ -x \left(\frac{\eta}{v} - \bar{L}_x^\zeta(\eta) \right) \right\}. \end{aligned} \tag{3.12}$$

Analogously, we get

$$Y_v^>(x) = E_x^{\zeta, \eta} \left[\exp \left\{ -\sigma \frac{\eta}{\sigma} \sum_{i=1}^x \widehat{\tau}_i \right\}; \frac{\eta}{\sigma} \sum_{i=1}^x \widehat{\tau}_i > -\frac{K\eta}{\sigma} \right] \exp \left\{ -x \left(\frac{\eta}{v} - \bar{L}_x^\zeta(\eta) \right) \right\}.$$

We define $\mu_x^{\zeta, v}$ as the distribution of $\frac{\eta}{\sigma} \sum_{i=1}^x \widehat{\tau}_i$ under $P_x^{\zeta, \eta}$. Then

$$Y_v^\approx(x) = e^{-x(\frac{\eta}{v} - \bar{L}_x^\zeta(\eta))} \int_0^{-\frac{K\eta}{\sigma}} e^{-\sigma y} d\mu_x^{\zeta, v}(y) \tag{3.13}$$

and

$$Y_v^>(x) = e^{-x(\frac{\eta}{v} - \bar{L}_x^\zeta(\eta))} \int_{-\frac{K\eta}{\sigma}}^\infty e^{-\sigma y} d\mu_x^{\zeta, v}(y). \tag{3.14}$$

Using Lemma 3.6 below, the integrals on the right-hand side of (3.13) and (3.14), multiplied by σ , are bounded from below and above by positive constants. Display (3.9) now follows by the definition of W_x^v , and (3.10) is a direct consequence of (3.13)–(3.16). The last two statements, are a consequence of (3.9), (3.10), $W_{nt}^v(1) = \frac{1}{\sqrt{t}} W_n^v(t)$ and Proposition 3.1.

To complete the previous proof, it remains to prove the following.

Lemma 3.6. *Under the conditions of Proposition 3.5, there exists a constant $C_8 > 1$ such that for all $v \in V$ and $x \geq 1$, on \mathcal{H}_x ,*

$$\sigma_x^\zeta(v) \int_0^{-K\eta_x^\zeta(v)/\sigma_x^\zeta(v)} e^{-\sigma_x^\zeta(v)y} d\mu_x^{\zeta,v}(y) \in [C_8^{-1}, C_8] \tag{3.15}$$

and

$$\sigma_x^\zeta(v) \int_{-K\eta_x^\zeta(v)/\sigma_x^\zeta(v)}^\infty e^{-\sigma_x^\zeta(v)y} d\mu_x^{\zeta,v}(y) \in [C_8^{-1}, C_8], \tag{3.16}$$

with $\mu_x^{\zeta,v}$ as in the proof of Proposition 3.5.

Proof. We write $n := \lfloor x \rfloor$ and recall that under $P_x^{\zeta,\eta}$, the sequence $\left(\sqrt{n} \frac{\eta_x^\zeta(v)}{\sigma_x^\zeta(v)} \widehat{\tau}_i\right)_{i=1,\dots,\lfloor x \rfloor,x}$ is a sequence of independent, centered random variables. Thus, on \mathcal{H}_x we obtain

$$\frac{1}{n} \sum_{i=1}^x \text{Var}_x^{\zeta,\eta} \left(\sqrt{n} \frac{\eta_x^\zeta(v)}{\sigma_x^\zeta(v)} \widehat{\tau}_i \right) = \left(\frac{\eta_x^\zeta(v)}{\sigma_x^\zeta(v)} \right)^2 \text{Var}_x^{\zeta,\eta} \left(\sum_{i=1}^x \widehat{\tau}_i \right) = \left(\frac{\eta_x^\zeta(v)}{\sigma_x^\zeta(v)} \right)^2 \text{Var}_x^{\zeta,\eta}(H_0) = 1.$$

Additionally, the $\widehat{\tau}_i, i \in \mathbb{N}$, have uniform exponential moments. Thus, the conditions of [3, Theorem 13.3] are fulfilled and an application of [3, (13.43)] yields

$$\sup_{\mathcal{C}} |\mu_x^{\zeta,v}(\mathcal{C}) - \Phi(\mathcal{C})| \leq c_1 n^{-1/2},$$

where the supremum is taken over all Borel-measurable convex subsets of \mathbb{R} , Φ denotes the standard Gaussian measure on \mathbb{R} and c_1 only depends on the uniform bound of the exponential moments of the $\widehat{\tau}_i, i \in \mathbb{N}$. Without loss of generality, we will assume $c_1 > 4$. Then, due to (3.8), by denoting $\mathcal{C} := [0, -K\eta_x^\zeta(v)/\sigma_x^\zeta(v)]$ we can choose $K > 0$ large enough, so that

$$\Phi(\mathcal{C}) \geq 2c_1^{-1} n^{-1/2} \quad \text{for all } n \in \mathbb{N} \text{ and } v \in V. \tag{3.17}$$

We thus get

$$\begin{aligned} c_1^{-1} n^{-1/2} &\leq \Phi(\mathcal{C}) - |\mu_x^{\zeta,v}(\mathcal{C}) - \Phi(\mathcal{C})| \leq \mu_x^{\zeta,v}(\mathcal{C}) \\ &\leq |\mu_x^{\zeta,v}(\mathcal{C}) - \Phi(\mathcal{C})| + \Phi(\mathcal{C}) \leq c_2(K) \cdot n^{-1/2}. \end{aligned} \tag{3.18}$$

Because the integrand in (3.15) is bounded away from 0 and infinity on the respective interval of integration (uniformly in $n \in \mathbb{N}$), (3.15) is a direct consequence of (3.8) and (3.18). For (3.16), we split the integral into a sum:

$$\begin{aligned} \int_{-K\eta_x^\zeta(v)/\sigma_x^\zeta(v)}^\infty e^{-\sigma_x^\zeta(v)y} d\mu_x^{\zeta,v}(y) &= \sum_{k=1}^\infty \int_{-kK\eta_x^\zeta(v)/\sigma_x^\zeta(v)}^{-(k+1)K\eta_x^\zeta(v)/\sigma_x^\zeta(v)} e^{-\sigma_x^\zeta(v)y} d\mu_x^{\zeta,v}(y) \\ &\leq \sum_{k=1}^\infty \mu_x^{\zeta,v} \left(-\frac{K\eta_x^\zeta(v)}{\sigma_x^\zeta(v)} [k, k+1] \right) e^{-k \cdot K|\eta_x^\zeta(v)|} \leq c_2 n^{-1/2} \sum_{k=1}^\infty e^{-k \cdot K|\eta^{*}|} \leq C_8 n^{-1/2}, \end{aligned}$$

where we recall the notation from (2.18). The lower bound in (3.16) can be obtained by noting that

$$\begin{aligned} \int_{-K\eta_x^\zeta(v)/\sigma_x^\zeta(v)}^\infty e^{-\sigma_x^\zeta(v)y} d\mu_x^{\zeta,v}(y) &\geq \int_{-K\eta_x^\zeta(v)/\sigma_x^\zeta(v)}^{-2K\eta_x^\zeta(v)/\sigma_x^\zeta(v)} e^{-\sigma_x^\zeta(v)y} d\mu_x^{\zeta,v}(y) \\ &\geq e^{2K\eta_x^\zeta(v)} \mu_x^{\zeta,v}([-K\eta_x^\zeta(v)/\sigma_x^\zeta(v), -2K\eta_x^\zeta(v)/\sigma_x^\zeta(v)]). \end{aligned}$$

Analogously to (3.18), choosing C_8 large enough, the last expression is bounded from below by $C_8^{-1}n^{-1/2}$. Combining this with (3.8), we finally arrive at (3.16).

We are now in the position to prove the following result.

Lemma 3.7. *Under the conditions of Proposition 3.5, for each $\delta > 0$ there exists a constant $C_9 = C_9(\delta) \in (1, \infty)$ such that for all $v \in V$, $t > 0$, on \mathcal{H}_{vt} we have*

$$C_9^{-1}Y_v^\approx(vt) \leq E_{vt} \left[e^{\int_0^t \zeta(B_s)ds}; B_t \in [-\delta, 0] \right] \leq E_{vt} \left[e^{\int_0^t \zeta(B_s)ds}; B_t \leq 0 \right] \leq C_9 Y_v^\approx(vt). \tag{3.19}$$

Proof. The second inequality is obvious. Since $\{B_t \leq 0\} \subset \{H_0 \leq t\}$ and $\zeta \leq 0$, we get $E_{vt} \left[e^{\int_0^t \zeta(B_s)ds}; B_t \leq 0 \right] \leq Y_v(vt) \leq (1 + C_7)Y_v^\approx(vt)$ by Proposition 3.5 and thus the last inequality in (3.19) is obtained. Therefore, it remains to show the first inequality. For this purpose, define the random function $p(s) := E_0 \left[e^{\int_0^s \zeta(B_r)dr}; B_s \in [-\delta, 0] \right]$ which almost surely is bounded from below by a deterministic constant $c_1(K, \delta) > 0$ for all $s \in [0, K]$. Using the strong Markov property at H_0 , we finally get

$$\begin{aligned} Y_v^\approx(vt) &= E_{vt} \left[e^{\int_0^{H_0} \zeta(B_r)dr}; H_0 \in [t - K, t] \right] \\ &\leq c_1(K, \delta)^{-1} E_{vt} \left[e^{\int_0^{H_0} \zeta(B_r)dr} p(t - s)|_{s=H_0}; H_0 \in [t - K, t] \right] \\ &\leq c_1(K, \delta)^{-1} E_{vt} \left[e^{\int_0^{H_0} \zeta(B_r)dr} p(t - s)|_{s=H_0} \right] \\ &= c_1(K, \delta)^{-1} E_{vt} \left[e^{\int_0^t \zeta(B_r)dr}; B_t \in [-\delta, 0] \right]. \end{aligned}$$

and the claim follows by choosing $C_9 := c_1(K, \delta) \vee (1 + C_7)$.

Plugging the relation $\xi(x) = \zeta(x) + \text{es}$, $x \in \mathbb{R}$, into Lemma 3.7 immediately supplies us with the following corollary.

Corollary 3.8. *Let C_9 be as in Lemma 3.7. Then for all $v \in V$, $t > 0$, on \mathcal{H}_{vt} we have*

$$C_9^{-1} e^{\text{es} \cdot t} Y_v^\approx(vt) \leq E_{vt} \left[e^{\int_0^t \xi(B_s) ds}; B_t \leq 0 \right] \leq C_9 e^{\text{es} \cdot t} Y_v^\approx(vt).$$

Using the Feynman–Kac formula (1.12), Lemma 3.7 also directly entails the following result; recall that u^{u_0} denotes the solution to (PAM) with initial condition $u_0 \in \mathcal{I}_{\text{PAM}}$.

Corollary 3.9. *Let $C_9 = C_9(\delta)$ be as in Lemma 3.7. Then for all $v \in V$, $t > 0$, on \mathcal{H}_{vt} we have*

$$u^{\mathbb{1}_{[-\delta, 0]}}(t, vt) \leq u^{\mathbb{1}_{(-\infty, 0]}}(t, vt) \leq C_9^2 \cdot u^{\mathbb{1}_{[-\delta, 0]}}(t, vt).$$

The previous results are fundamental for proving perturbation statements in the next section, which themselves will allow us to analyze path probabilities of the branching process.

3.3. Proof of Theorem 1.3

The previous findings already enable us to prove our first main result.

Proof of Theorem 1.3. We first assume $\sigma_v^2 > 0$ and consider the case $u_0 = \mathbb{1}_{(-\infty, 0]}$ to show the second part of the claim, i.e. that the sequence of processes

$$[0, \infty) \ni t \mapsto \frac{1}{\sqrt{nv\sigma_v^2}} (\ln u(nt, vnt) - nt\Lambda(v)), \quad n \in \mathbb{N}, \tag{3.20}$$

converges in \mathbb{P} -distribution to standard Brownian motion. Because $[0, \infty) \ni t \mapsto \ln u(t, vt)$ might be discontinuous *only* 0 (in which case we replace $[0, \infty)$ by $(0, \infty)$ in (3.20)), we cover both cases (continuous and discontinuous in 0) in a unified setting by showing the invariance principle for a sequence of auxiliary processes $(X_n^v(t))_{t \geq 0}$, $n \in \mathbb{N}$, where for every $n \in \mathbb{N}$ and $t \geq \frac{1}{n}$, the term $X_n^v(t)$ is the same as in (3.20), whereas for $t \in [0, 1/n]$ the term $\ln u(nt, vnt)$ in (3.20) is replaced by $(1 - nt) \ln u(0, 0) + nt \ln u(1, v)$, making $(X_n^v(t))_{t \geq 0}$ continuous. Because the difference of the processes in (3.20) and $(X_n^v(t))_{t \geq 0}$ converges uniformly to zero as $n \rightarrow \infty$, convergence of the processes in (3.20) to a standard Brownian motion is equivalent to the convergence of the processes $(X_n^v(t))_{t \geq 0}$, $n \in \mathbb{N}$, to a standard Brownian motion in $C([0, \infty))$ (or $C((0, \infty))$ in case of a discontinuity in 0) with topology induced by the metric ρ from (1.15).

By Proposition 3.5 and Corollary 3.8, on \mathcal{H}_{nvt} (recall the notation from (2.23)) we have

$$\begin{aligned} -\ln C_7 &\leq \ln \sigma_{nvt}^\zeta(v) + \ln Y_v^\approx(nvt) + nvt L^*(1/v) + \sqrt{nvt} W_{nvt}^v(1) \leq \ln C_7, \quad \text{and} \\ -\ln C_9 &\leq \ln u(nt, vnt) - \text{es} \cdot nt - \ln Y_v^\approx(vnt) \leq \ln C_9. \end{aligned} \tag{3.21}$$

Recall that a sequence of processes $t \mapsto A_n(t)$, $n \in \mathbb{N}$, converges in \mathbb{P} -distribution to standard Brownian motion if and only if for each $c > 0$ the sequence $t \mapsto c^{-1}A_n(c^2t)$, $n \in \mathbb{N}$, converges in \mathbb{P} -distribution to a standard Brownian motion. Applying this to (3.11), the sequence of processes

$$[0, \infty) \ni t \mapsto \frac{1}{\sqrt{nv\sigma_v^2}} \left(\ln Y_v^\approx(vnt) + vntL^*(1/v) \right), \quad n \in \mathbb{N}, \tag{3.22}$$

converges in \mathbb{P} -distribution to a standard Brownian motion. Further, by the second line in (3.21),

$$-\ln C_9 \leq (\ln u(nt, vnt) - nt(\text{es} - vL^*(1/v))) - (\ln Y_v^\approx(vnt) + vntL^*(1/v)) \leq \ln C_9$$

holds. Consequently, if we can prove that

$$\Lambda(v) = \text{es} - vL^*(1/v) \quad \forall v \in V, \tag{3.23}$$

the claim follows from (3.22). To prove (3.23), we set $n = 1$ in (3.21) and note that $\frac{W_v^v(1)}{\sqrt{t}} \xrightarrow[t \rightarrow \infty]{} 0$ \mathbb{P} -a.s. for all $v \in V$, because $W_n(1)$ converges in \mathbb{P} -distribution to a centered normally distributed random variable by Proposition 3.1. Using (3.25), (3.21) and (3.8), we get (3.23).

It remains to show the claim for arbitrary $u_0 \in \mathcal{I}_{\text{PAM}}$. Recall that there exist $\delta' \in (0, 1)$ and $C' > 1$, such that $\delta' \mathbb{1}_{[-\delta', 0]}(x) \leq u_0(x) \leq C' \mathbb{1}_{(-\infty, 0]}(x)$ for all $x \in \mathbb{R}$. Therefore, using Corollary 3.9 we have

$$\delta' C_9^{-2} u^{\mathbb{1}_{(-\infty, 0]}}(t, vt) \leq u^{u_0}(t, vt) \leq C' u^{\mathbb{1}_{(-\infty, 0]}}(t, vt), \tag{3.24}$$

where we used that the solution to (PAM) is linear in its initial condition. Thus, the convergence of (3.20) for arbitrary initial condition $u_0 \in \mathcal{I}_{\text{PAM}}$ follows from the convergence with initial condition $\mathbb{1}_{(-\infty, 0]}$. This gives the second part of Theorem 1.3.

It remains to show that $(nv)^{-1/2}(\ln u(n, vn) - n\Lambda(v))$ converges in \mathbb{P} -distribution to a Gaussian random variable. For $u_0 = \mathbb{1}_{(-\infty, 0]}$, this is a direct consequence of (3.21) for $t = 1$, (3.23) and the second part of Proposition 3.5. For general u_0 , the claim follows from (3.24).

In view of Corollary 3.9, Proposition A.3 and (3.23), the Lyapunov exponent Λ , defined in (1.13), determines the exponential decay (growth, resp.) for solutions to (PAM) for arbitrary initial conditions $u_0 \in \mathcal{I}_{\text{PAM}}$ (and not only for those with compact support).

Corollary 3.10. *For all $v \geq 0$ and all $u_0 \in \mathcal{I}_{\text{PAM}}$ we have that \mathbb{P} -a.s.,*

$$\Lambda(v) = \lim_{t \rightarrow \infty} \frac{1}{t} \ln u^{u_0}(t, vt). \tag{3.25}$$

Furthermore, Λ is linear on $[0, v_c]$ and strictly concave on (v_c, ∞) , and the convergence in (3.25) holds uniformly on any compact interval $K \subset [0, \infty)$.

Proof. Note that by Proposition A.3, $\delta' \mathbb{1}_{[-\delta', 0]} \leq u_0 \leq C' \mathbb{1}_{(-\infty, 0]}$ and Corollary 3.9 we have that (3.25) holds for all $v \in V$ and all compact $V \subset (v_c, \infty)$, so (3.25) is true for all $v > v_c$. The strict concavity of Λ on (v_c, ∞) follows from the strict convexity of $L^*(1/v)$, which in turn follows from the strict convexity of L and standard properties of the Legendre transformation. If $v_c = 0$, the proof is complete due to $\lim_{t \rightarrow \infty} \frac{1}{t} \ln u^{\delta' \mathbb{1}_{[-\delta', 0]}} = \Lambda(0) = \varepsilon s$ by Proposition A.3 and $u^{u_0} \leq e^{\varepsilon s t}$ for all $u_0 \in \mathcal{I}_{\text{PAM}}$. Thus, let us assume $v_c > 0$ from now on. First observe that $L^*(1/v)$ tends to $L^*(1/v_c) = -L(0)$ as $v \downarrow v_c$. Indeed, due to Lemma 2.4 (d), for each $v > v_c$ there exists a unique $\bar{\eta}(v) \in (-\infty, 0)$, characterized via $L'(\bar{\eta}(v)) = \frac{1}{v}$, such that $L^*(1/v) = \frac{\bar{\eta}(v)}{v} - L(\bar{\eta}(v))$. Furthermore, $(v_c, \infty) \ni v \mapsto \bar{\eta}(v)$ is continuously differentiable and strictly decreasing, bounded from above by 0. In addition, $(-\infty, 0) \ni \eta \mapsto L'(\eta)$ is smooth and strictly monotone and tends to $L'(0-)$ as $\eta \uparrow 0$. As a consequence, we get that $\eta(v) \uparrow 0$ as $v \downarrow v_c$ and thus $L^*(1/v) \rightarrow L^*(1/v_c)$ as $v \downarrow v_c$. Therefore, we deduce that $\Lambda(v) = \varepsilon s - vL^*(1/v)$ for all $v \in [v_c, \infty)$. Furthermore, due to (BDD), for all $u_0 \in \mathcal{I}_{\text{PAM}}$ we have

$$\begin{aligned} u^{u_0}(t, vt) &\leq C' e^{\varepsilon s t} E_{v_t} [e^{\int_0^t \zeta(B_s) ds}, B_t \leq 0] \leq C' e^{\varepsilon s t} E_{v_t} [e^{\int_0^{H_0} \zeta(B_s) ds}, H_0 \leq t] \\ &\leq C' \exp \{t(\varepsilon s + v \bar{L}_{v_t}(0))\}. \end{aligned} \tag{3.26}$$

Using Lemma 2.4 (b), taking logarithms, and dividing by t , we get that the (due to Proposition A.3) concave function Λ is bounded from above by the linear function $[0, \infty) \ni v \mapsto \varepsilon s - vL(0) = \varepsilon s + vL^*(1/v_c)$ and coincides with this function at $v = 0$ as well as $v = v_c$, and hence on the whole interval $[0, v_c]$. Using (3.26) again, we infer (3.25) for all $v \geq 0$ and $u_0 \in \mathcal{I}_{\text{PAM}}$.

To show that the convergence is uniform on every compact interval $K \subset [0, \infty)$, for $\varepsilon > 0$ arbitrary we consider $\varepsilon \mathbb{Z} := \{k\varepsilon : k \in \mathbb{Z}\}$, and for $y \in \mathbb{R}$ set $\lfloor y \rfloor_\varepsilon := \sup\{x \in \varepsilon \mathbb{Z} : x \leq y\}$. Then the convergence is uniform on $K \cap \varepsilon \mathbb{Z}$. A fortiori, for t large enough, uniformly in $y \in K$,

$$u(t, t \lfloor y \rfloor_\varepsilon) \geq e^{t(\Lambda(\lfloor y \rfloor_\varepsilon) - \varepsilon)}.$$

Lemma A.8 then entails that

$$\inf_{z \in [t \lfloor y \rfloor_\varepsilon - 1, t \lfloor y \rfloor_\varepsilon + 1]} u(t + 1, z) \geq \frac{1}{C_{19}} u(t, t \lfloor y \rfloor_\varepsilon) \geq \frac{1}{C_{19}} e^{t(\Lambda(\lfloor y \rfloor_\varepsilon) - \varepsilon)}. \tag{3.27}$$

Furthermore, using $0 \leq y - \lfloor y \rfloor_\varepsilon \leq \varepsilon$, we have

$$\begin{aligned} P_{y_t}(B_{\varepsilon t} \in [t \lfloor y \rfloor_\varepsilon - 1, t \lfloor y \rfloor_\varepsilon + 1]) &\geq \sqrt{\frac{2}{\pi \varepsilon t}} \inf_{x \in [t \lfloor y \rfloor_\varepsilon - 1, t \lfloor y \rfloor_\varepsilon + 1]} e^{-\frac{(x-y)^2}{2\varepsilon t}} \\ &\geq \sqrt{\frac{2}{\pi}} \cdot \exp \left\{ -\frac{(\varepsilon t + 1)^2}{2\varepsilon t} - \frac{\ln(\varepsilon t)}{2} \right\}. \end{aligned} \tag{3.28}$$

Using the Feynman–Kac formula in the equality, the Markov property at time εt , and (BDD) in the first inequality, we infer that

$$\begin{aligned} u(t + 1 + \varepsilon t, y) &= E_{y_t} \left[e^{\int_0^{t+1+\varepsilon t} \xi(B_s) ds} u_0(B_{t+1+\varepsilon t}) \right] \\ &\geq e^{\varepsilon i \varepsilon t} \cdot P_{y_t}(B_{\varepsilon t} \in [t \lfloor y \rfloor_\varepsilon - 1, t \lfloor y \rfloor_\varepsilon + 1]) \cdot \inf_{z \in [t \lfloor y \rfloor_\varepsilon - 1, t \lfloor y \rfloor_\varepsilon + 1]} u(t + 1, z) \\ &\geq c_1 \exp \left\{ t \left(\Lambda(\lfloor y \rfloor_\varepsilon) + (\varepsilon i - 3/2)\varepsilon - \frac{1}{t} - \frac{1}{2\varepsilon t^2} - \frac{\ln(\varepsilon t)}{2t} \right) \right\}, \end{aligned} \tag{3.29}$$

where in the last inequality we used (3.27) and (3.28). Setting $t' := t + 1 + \varepsilon t$ and $y' := \frac{t'}{t}y$, we get

$$\frac{1}{t'} \ln u(t', t'y) - \Lambda(y) = \frac{t}{t'} \left(\frac{1}{t} \ln u(t', t'y') - \Lambda(\lfloor y' \rfloor_\varepsilon) \right) + \frac{t}{t'} \Lambda(\lfloor y' \rfloor_\varepsilon) - \Lambda(y).$$

Since (3.29) holds uniformly for all $y \in K$, we infer that

$$\begin{aligned} \inf_{y \in K} \left(\frac{1}{t'} \ln u(t', t'y) - \Lambda(y) \right) &\geq \frac{t}{t'} \left(\frac{\ln c_1}{t} + (\varepsilon i - 3/2)\varepsilon - \frac{1}{t} - \frac{1}{2\varepsilon t^2} - \frac{\ln(\varepsilon t)}{2t} \right) \\ &\quad - \sup_{y \in K} \left| \frac{t}{t'} \Lambda(\lfloor y' \rfloor_\varepsilon) - \Lambda(y) \right|. \end{aligned}$$

Since Λ is concave and finite, it is uniformly continuous on compact intervals. As a consequence, since $\varepsilon > 0$ was chosen arbitrarily, we deduce the lower bound

$$\liminf_{t \rightarrow \infty} \inf_{y \in K} \left(\frac{1}{t} \ln u(t, ty) - \Lambda(y) \right) \geq 0. \tag{3.30}$$

To derive the matching upper bound, we assume that the convergence does not hold uniformly on K . Then, due to (3.30), there exist $\alpha > 0$ and sequences $(t_n)_{n \in \mathbb{N}} \subset [0, \infty)$ and $(y_n)_{n \in \mathbb{N}} \subset K$ such that $t_n \rightarrow \infty$ and

$$\frac{1}{t_n} \ln u(t_n, t_n y_n) - \Lambda(y_n) \geq \alpha \quad \forall n \in \mathbb{N}. \tag{3.31}$$

Retreating to a suitable subsequence, we can assume $y_n \xrightarrow{n \rightarrow \infty} y \in K$. For n such that $|y_n - y| \leq \varepsilon$, we have, similarly to (3.28),

$$\begin{aligned} P_{y(t_n+1+\varepsilon t_n)}(B_{\varepsilon t_n} \in [t_n y_n - 1, t_n y_n + 1]) &\geq \sqrt{\frac{2}{\pi \varepsilon t_n}} \inf_{x \in [t_n y_n - 1, t_n y_n + 1]} e^{-\frac{(x-y(t_n+1+\varepsilon t_n))^2}{2\varepsilon t_n}} \\ &\geq \sqrt{\frac{2}{\pi \varepsilon t_n}} \exp \left\{ -\frac{(1+y)^2}{2\varepsilon t_n} (t_n \varepsilon + 1)^2 \right\}. \end{aligned}$$

Therefore, taking advantage of (3.27) again and using an argument as in the derivation of (3.29), we infer that

$$u(t_n + 1 + \varepsilon t_n, (t_n + 1 + \varepsilon t_n)y) \geq c_1 \cdot u(t_n, t_n y_n) \cdot \sqrt{\frac{1}{\varepsilon t_n}}$$

$$\exp \left\{ \varepsilon e^{\pm t_n} - \frac{(1+y)^2}{2\varepsilon t_n} (t_n \varepsilon + 1)^2 \right\} \tag{3.32}$$

for all n such that $|y_n - y| \leq \varepsilon$. Now recall that $y_n \rightarrow_{n \rightarrow \infty} y$, that Λ is continuous, as well as $\frac{1}{t} \ln u(t, ty) \rightarrow_{t \rightarrow \infty} \Lambda(y)$. Therefore, first taking logarithms, dividing by t_n , and then taking $n \rightarrow \infty$, the left-hand side in (3.32) converges to $(1 + \varepsilon)\Lambda(y)$. Contrarily, by (3.31), the limit of the right-hand side is bounded from below by $\Lambda(y) + \alpha + \varepsilon e^{\pm 1} - \frac{(1+y)^2}{2}\varepsilon$. Choosing $\varepsilon > 0$ small enough, this leads to a contradiction. As a consequence, we deduce the uniform convergence on K .

3.4. Time perturbation

In the next step we prove perturbation results, i.e., the time and space perturbation Lemmas 3.11 and 3.12. These statements will be useful when comparing the expected number of particles of the BBMRE which are slightly slower or faster than the ones with given velocity. As usual, $u = u^{u_0}$ denotes the solution to (PAM) with initial condition $u_0 \in \mathcal{I}_{\text{PAM}}$.

Lemma 3.11. (a) *Let $u_0 \in \mathcal{I}_{\text{PAM}}$ and let $\varepsilon : (0, \infty) \rightarrow (0, \infty)$ be a function such that $\varepsilon(t) \rightarrow 0$ and $t\varepsilon(t) \rightarrow \infty$ as $t \rightarrow \infty$. Then there exists $C_{10} = C_{10}((\varepsilon(t))_{t \geq 0}, u_0)$ such that \mathbb{P} -a.s., for all t large enough,*

$$\sup_{(v,h) \in \mathcal{E}_t} \left| \ln \left(\frac{u^{u_0}(t+h, vt)}{u^{u_0}(t, vt)} \right) - h(e^{\varepsilon} - \bar{\eta}(v)) \right| \leq C_{10} + C_{10}|h| \left(\sqrt{\frac{\ln t}{t}} + \frac{|h|}{t} \right), \tag{3.33}$$

where $\mathcal{E}_t := \left\{ (v, h) : v \in V, |h| \leq t\varepsilon(t), \frac{vt}{t+h} \in V \right\}$.

(b) *For all $\varepsilon > 0$ and $u_0 \in \mathcal{I}_{\text{PAM}}$ there exists a constant $C_{11} > 1$ and a \mathbb{P} -a.s. finite random variable T_1 such that for all $t \geq T_1$, uniformly in $0 \leq h \leq t^{1-\varepsilon}$, $v \in V$ and $\frac{vt}{t+h} \in V$,*

$$C_{11}^{-1} e^{h/C_{11}} u^{u_0}(t, vt) \leq u^{u_0}(t+h, vt) \leq C_{11} e^{C_{11}h} u^{u_0}(t, vt). \tag{3.34}$$

Proof. (a) Note that it suffices to show the claim for $u_0 = \mathbb{1}_{(-\infty, 0]}$. Indeed, for all $u_0 \in \mathcal{I}_{\text{PAM}}$ we have $\delta' \mathbb{1}_{[-\delta', 0]} \leq u_0 \leq C' \mathbb{1}_{(-\infty, 0]}$. Using Corollary 3.9, we infer that for all $u_0 \in \mathcal{I}_{\text{PAM}}$, all $v \in V$, and all t large enough

$$\delta' C_9^{-2} u^{\mathbb{1}_{(-\infty, 0]}}(t, vt) \leq u^{u_0}(t, vt) \leq C' u^{\mathbb{1}_{(-\infty, 0]}}(t, vt).$$

where u^{u_0} denotes the solution to (PAM) with initial condition u_0 .

by the same argument as at the end of the proof of Theorem 1.3, the solutions to (PAM) for different initial conditions $u_0 \in \mathcal{I}_{\text{PAM}}$ differ at most by a multiplicative constant. Let t be large enough such that \mathcal{H}_{vt} occurs for all $v \in V$, which is possible by (2.23). Letting $(v, h) \in \mathcal{E}_t$ and writing $v' := \frac{vt}{t+h} \in V$, we infer that

$$\frac{u^{\mathbb{1}_{(-\infty, 0]}}(t+h, vt)}{u^{\mathbb{1}_{(-\infty, 0]}}(t, vt)} = \frac{E_{vt} \left[e^{\int_0^{t+h} \xi(B_s) ds}; B_{t+h} \leq 0 \right]}{E_{vt} \left[e^{\int_0^t \xi(B_s) ds}; B_t \leq 0 \right]}$$

$$= e^{\varepsilon s \cdot h} \frac{E_{vt} \left[e^{\int_0^{t+h} \zeta(B_s) ds}; B_{t+h} \leq 0 \right]}{E_{vt} \left[e^{\int_0^t \zeta(B_s) ds}; B_t \leq 0 \right]}. \tag{3.35}$$

Using Lemma 3.7, on \mathcal{H}_{vt} , the last fraction divided by $\frac{Y_{v'}^{\approx}(vt)}{Y_v^{\approx}(vt)}$ is bounded away from 0 and infinity for all t large enough. As in the derivation of (3.12), the term $\frac{Y_{v'}^{\approx}(vt)}{Y_v^{\approx}(vt)}$ can be written as

$$\begin{aligned} & \frac{E_{vt}^{\zeta, \eta_{vt}^{\zeta}(v')} \left[\exp \left\{ -\eta_{vt}^{\zeta}(v') \sum_{i=1}^{vt} \widehat{\tau}_i \right\}; \sum_{i=1}^{vt} \widehat{\tau}_i \in \left[-K, 0 \right] \right]}{E_{vt}^{\zeta, \eta_{vt}^{\zeta}(v)} \left[\exp \left\{ -\eta_{vt}^{\zeta}(v) \sum_{i=1}^{vt} \widetilde{\tau}_i \right\}; \sum_{i=1}^{vt} \widetilde{\tau}_i \in \left[-K, 0 \right] \right]} \\ & \times \frac{\exp \left\{ -vt \left(\frac{\eta_{vt}^{\zeta}(v')}{v'} - \overline{L}_{vt}^{\zeta}(\eta_{vt}^{\zeta}(v')) \right) \right\}}{\exp \left\{ -vt \left(\frac{\eta_{vt}^{\zeta}(v)}{v} - \overline{L}_{vt}^{\zeta}(\eta_{vt}^{\zeta}(v)) \right) \right\}}, \end{aligned} \tag{3.36}$$

where $\widehat{\tau}_i = \tau_i - E_{vt}^{\zeta, \eta_{vt}^{\zeta}(v')} [\tau_i]$ and $\widetilde{\tau}_i := \tau_i - E_{vt}^{\zeta, \eta_{vt}^{\zeta}(v)} [\tau_i]$. But now, since $v' \in V$, as in the proof of Proposition 3.5, the first fraction of the previous display is bounded from below and above by positive constants, for all t large enough. Indeed, setting $x = vt$ in (3.13), the denominator in (3.36) equals the integral in (3.13) and, replacing v by v' in (3.13), the numerator equals the integral in (3.13). The claim then follows due to Lemma 3.6 and using (3.8). Therefore, taking logarithms in (3.35) and recalling the definition of $S_{vt}^{\zeta, v}(\eta)$ in (3.4), according to the previous considerations it suffices to show that the logarithm of the second fraction in (3.36) plus $\overline{\eta}(v) \cdot h$, i.e.

$$\left(S_{vt}^{\zeta, v}(\eta_{vt}^{\zeta}(v)) - S_{vt}^{\zeta, v}(\eta_{vt}^{\zeta}(v')) \right) + \left(S_{vt}^{\zeta, v}(\eta_{vt}^{\zeta}(v')) - S_{vt}^{\zeta, v'}(\eta_{vt}^{\zeta}(v')) \right) + \overline{\eta}(v) \cdot h, \tag{3.37}$$

satisfies bound on the right-hand side of (3.33), uniformly in $(v, h) \in \mathcal{E}_t$. Recall that $\frac{1}{v'} = \frac{1}{v} \left(1 + \frac{h}{t} \right)$, thus the second summand in (3.37) is $-h \cdot \eta_{vt}^{\zeta}(v')$. The triangle inequality entails

$$|\eta_{vt}^{\zeta}(v') - \overline{\eta}(v)| \leq |\eta_{vt}^{\zeta}(v') - \overline{\eta}(v')| + |\overline{\eta}(v') - \overline{\eta}(v)|, \tag{3.38}$$

and so by Lemma 2.5, uniformly for $v' \in V$ and t large enough, the first term on the right-hand side of (3.38) can be upper bounded by $C_2 \sqrt{\frac{\ln vt}{vt}}$, \mathbb{P} -a.s. Furthermore, by Lemma 2.4(d) we know that $\overline{\eta}$ is continuously differentiable and strictly decreasing, having uniform positive bounds of the derivative on every bounded subinterval of (v_c, ∞) . Hence, $\overline{\eta}(\cdot)$ is Lipschitz continuous on V and we therefore get that the second summand in (3.38) can be upper bounded by $c_1 |v - v'| = c_1 v \frac{|h|}{|t+h|} = c_1 v \frac{|h|}{t} \cdot \frac{t}{|t+h|} \leq c_2 \frac{|h|}{t}$, uniformly for all $v, v' \in V$ and all t large enough, where the last inequality is due to $|h|/t \leq \varepsilon(t) \rightarrow 0$. Therefore, the absolute value of the sum of the second and third summand in (3.37) is upper bounded by $C_{10} |h| \left(\sqrt{\frac{\ln t}{t}} + \frac{|h|}{t} \right)$ with $C_{10} := c_2 \vee C_2$.

It remains to show that the first summand in (3.37) tends to 0 as t tends to ∞ . We write

$$S_{vt}^{\zeta,v}(\eta_{vt}^{\zeta}(v')) = S_{vt}^{\zeta,v}(\eta_{vt}^{\zeta}(v)) + (\eta_{vt}^{\zeta}(v') - \eta_{vt}^{\zeta}(v))(S_{vt}^{\zeta,v})'(\eta_{vt}^{\zeta}(v)) + \frac{1}{2}(\eta_{vt}^{\zeta}(v') - \eta_{vt}^{\zeta}(v))^2(S_{vt}^{\zeta,v})''(\tilde{\eta})$$

for some $\tilde{\eta} \in [\eta_{vt}^{\zeta}(v') \wedge \eta_{vt}^{\zeta}(v), \eta_{vt}^{\zeta}(v') \vee \eta_{vt}^{\zeta}(v)]$. Recall that $S_{vt}^{\zeta,v}(\eta) = vt(\frac{\eta}{v} - \overline{L}_{vt}^{\zeta}(\eta))$ and by definition $(\overline{L}_{vt}^{\zeta})'(\eta_{vt}^{\zeta}(v)) = \frac{1}{v}$. Thus, $(S_{vt}^{\zeta,v})'(\eta_{vt}^{\zeta}(v)) = 0$. Furthermore, $(S_{vt}^{\zeta,v})''(\eta) = -vt(\overline{L}_{vt}^{\zeta})''(\eta)$ and the function $(\overline{L}_{vt}^{\zeta})''$ is uniformly bounded away from 0 and infinity by Lemma A.1 on V . As a consequence, by the characterizing equation (2.15) and the implicit function theorem, on \mathcal{H}_{vt} , the function $\eta_{vt}^{\zeta}(\cdot)$ is differentiable with uniformly bounded first derivative, i.e.

$$|\eta_{vt}^{\zeta}(v') - \eta_{vt}^{\zeta}(v)| \leq c_3|v' - v| \leq c_3v \frac{|h|}{t(1 - \varepsilon(t))}. \tag{3.39}$$

Thus, on \mathcal{H}_{vt} , the first summand in (3.37) can be bounded by $c_4 \cdot t \cdot \frac{h^2}{t^2} = c_4 \cdot \frac{h^2}{t}$, uniformly in $(v, h) \in \mathcal{E}_t$, for all t large enough. This implies (a).

(b) The first part of the proof is similar to that of (a); indeed, dividing by $u^{u_0}(t, vt)$ and taking logarithms in (3.34), one arrives at (3.37) again. The second part then consists of showing that (3.37) is lower and upper bounded by two strictly increasing linear functions for all $0 < h \leq t^{1-\varepsilon}$. Following the same computations as in the proof of (a), for $v \in V$, and $h > 0$ such that $v' \in V$, we have up to some additive constant, which is independent of h , that

$$\ln \frac{u^{u_0}(t + h, vt)}{u^{u_0}(t, vt)} \asymp c_t \frac{vt}{2}(v - v')^2 - \eta_{vt}^{\zeta}(v') \cdot h + \varepsilon s \cdot h,$$

where c_t is a function which for t large enough is positive and bounded away from 0 and infinity. Because $\eta_{vt}^{\zeta}(v') < 0$ is negative as well as bounded away from 0 and minus infinity, the second expression in the previous display is bounded from below by h/C_{11} and bounded from above by $C_{11} \cdot h$, some constant C_{11} large enough, for our choice of parameters, and hence we can conclude.

3.5. Space perturbation

While in the previous subsection we have been investigating the effects of time perturbations of u and related quantities, here we will consider space perturbations. As before, let u^{u_0} denotes the solution to (PAM) with initial condition $u_0 \in \mathcal{I}_{\text{PAM}}$.

Lemma 3.12. *Let $\varepsilon(t)$ be a positive function such that $\varepsilon(t) \rightarrow 0$ and $\frac{t\varepsilon(t)}{\ln t} \rightarrow \infty$ as $t \rightarrow \infty$. Then for all $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that \mathbb{P} -a.s., for all $u_0 \in \mathcal{I}_{\text{PAM}}$ we have*

(a)

$$\limsup_{t \rightarrow \infty} \sup \left\{ \left| \frac{1}{h} \ln \left(\frac{u(t, vt + h)}{u(t, vt)} \right) - L(\bar{\eta}(v)) \right| : (v, h) \in \mathcal{E}_t \right\} \leq \varepsilon, \tag{3.40}$$

where $\mathcal{E}_t := \{(v, h) : v, v + \frac{h}{t} \in V, C(\varepsilon) \ln t \leq |h| \leq t\varepsilon(t)\}$.

(b) Let $\varepsilon(t)$ be a positive function such that $\varepsilon(t) \rightarrow 0$. Then there exists a constant $C_{12} < \infty$ and a \mathbb{P} -a.s. finite random variable \mathcal{T}_2 such that for all $t \geq \mathcal{T}_2$, uniformly in $0 \leq h \leq t\varepsilon(t)$, $v \in V$, $v + \frac{h}{t} \in V$ and $u_0 \in \mathcal{I}_{PAM}$ we have

$$C_{12}^{-1} e^{-C_{12}h} \cdot u(t, vt) \leq u(t, vt + h) \leq C_{12} e^{-h/C_{12}} \cdot u(t, vt). \tag{3.41}$$

The proof of Lemma 3.12 can be found in the companion article [10]; indeed, it is central to the results of [10] and we hence prefer not to duplicate it here due to space constraints.

3.6. Approximation results

In this section we mainly show how moment generating functions can be used in order to approximate quantities related to the solution to (PAM) and expectations of BBMRE. As usual, for $u_0 \in \mathcal{I}_{PAM}$ let u^{u_0} be the solution to (PAM) with initial condition u_0 .

Lemma 3.13. *There exists a constant $C_{13} > 0$ and a \mathbb{P} -a.s. finite random variable \mathcal{T}_3 such that for all $u_0 \in \mathcal{I}_{PAM}$ and $t \geq \mathcal{T}_3$,*

$$\left| \ln u^{u_0}(t, v_0 t) - \sum_{i=1}^{v_0 t} (L_i^\zeta(\bar{\eta}(v_0)) - L(\bar{\eta}(v_0))) \right| \leq C_{13} \ln t. \tag{3.42}$$

Proof. By (3.23) and $\Lambda(v_0) = 0$, we have $L^*(1/v_0) = \frac{e\bar{s}}{v_0}$. Also, by (PAM-INI) and the monotonicity of the solution to (PAM) in its initial condition, we have $u^{\delta' \mathbb{1}_{[-\delta', 0]}} \leq u^{u_0} \leq u^{C' \mathbb{1}_{(-\infty, 0]}}$. Thus, on the one hand, by the many-to-few lemma (Proposition 2.3) and Lemma 3.7, for all $u_0 \in \mathcal{I}_{PAM}$ and t such that $v_0 t \geq \mathcal{N}_1$, where \mathcal{N}_1 was defined in (2.23), we have

$$\left| \ln u^{u_0}(t, v_0 t) - (\ln Y_{v_0}^\sim(v_0 t) + v_0 t L^*(1/v_0)) \right| \leq \ln(C_9/\delta').$$

On the other hand, due to Proposition 3.5 and Corollary 3.4, there exists a finite random variable \mathcal{N} , such that for all $t \geq \mathcal{N}$ we have

$$\begin{aligned} & \left| \ln Y_{v_0}^\sim(v_0 t) + v_0 t L^*(1/v_0) + \ln \sigma_{v_0 t}^\zeta(v_0) - \sum_{i=1}^{v_0 t} (L_i^\zeta(\bar{\eta}(v_0)) - L(\bar{\eta}(v_0))) \right| \\ & \leq \ln C_7 + C_5 \ln v_0 + C_5 \ln t. \end{aligned}$$

Finally, by (3.8), $|\ln \sigma_{v_0 t}^\zeta(v_0) - \frac{1}{2} \ln t| \leq \ln C_6 + \frac{1}{2} |\ln v_0|$ for all t such that $v_0 t \geq \mathcal{N}_1$. Combining this with the previous two displays, inequality (3.42) follows with $\mathcal{T}_3 := (\mathcal{N} \vee \mathcal{N}_1)/v_0$ and C_{13} suitable.

We introduce the so-called *breakpoint inverse*

$$T_x^{u_0, a} := \inf \{t \geq 0 : u^{u_0}(t, x) \geq a\}, \quad x \in \mathbb{R}, a \in [0, \infty), u_0 \in \mathcal{I}_{PAM}, \quad (3.43)$$

and abbreviate

$$T_x^{(a)} := T_x^{\mathbb{1}_{(-\infty, 0]}, a}. \quad (3.44)$$

Next, we state an important approximation result for $T_x^{u_0, a}$, $x \geq 0$, in terms of the centered logarithmic moment generating functions.

Lemma 3.14. *There exists a constant $C_{14} < \infty$ and a \mathbb{P} -a.s. finite random variable $\mathcal{C}_1 = \mathcal{C}_1(a, u_0)$, $a > 0$, $u_0 \in \mathcal{I}_{PAM}$, such that for all $x \geq 1$,*

$$\left| T_x^{u_0, a} - \frac{1}{v_0 L(\bar{\eta}(v_0))} \sum_{i=1}^x L_i^\zeta(\bar{\eta}(v_0)) \right| \leq \mathcal{C}_1 + C_{14} \ln x. \quad (3.45)$$

Additionally, for each $u_0 \in \mathcal{I}_{PAM}$ and $a > 0$,

$$\lim_{x \rightarrow \infty} \frac{T_x^{u_0, a}}{x} = \frac{1}{v_0} \quad \mathbb{P}\text{-a.s.} \quad (3.46)$$

Proof. We set $t = x/v_0$ and let

$$h_t := \frac{1}{v_0 L(\bar{\eta}(v_0))} \sum_{i=1}^{v_0 t} (L(\bar{\eta}(v_0)) - L_i^\zeta(\bar{\eta}(v_0))).$$

We first note that due to Lemma A.2, the family $(L(\bar{\eta}(v_0)) - L_i^\zeta(\bar{\eta}(v_0)))_{i \in \mathbb{Z}}$ satisfies the assumptions of Corollary A.5 with all m_i equal to some large enough finite constant and thus $\sum_n \mathbb{P}(|h_n| \geq C\sqrt{n} \ln n) < \infty$ for some $C > 1$ large enough. The first Borel-Cantelli lemma then readily supplies us with $|h_n| < C\sqrt{n} \ln n$ \mathbb{P} -a.s. for all n large enough. To control non-integer t , we recall

$$h_t - h_{\lfloor t \rfloor} = \frac{1}{v_0 L(\bar{\eta}(v_0))} (v_0(t - \lfloor t \rfloor)L(\bar{\eta}(v_0)) - \ln E_{v_0 t} [e^{\int_0^{H_{v_0 \lfloor t \rfloor}} (\zeta(B_s) + \bar{\eta}(v_0)) ds}])$$

and hence \mathbb{P} -a.s. that $|h_t - h_{\lfloor t \rfloor}| \leq 1 + \frac{\sqrt{2(\text{es} - \text{ei} - \bar{\eta}(v_0))}}{|L(\bar{\eta}(v_0))|}$ by (BDD) and [4, (2.0.1), p. 204], thus giving

$$|h_t| < c_1 \sqrt{t \ln t} \quad \mathbb{P}\text{-a.s. for all } t \text{ large enough.} \quad (3.47)$$

To show the desired inequality, we note that (3.45) is equivalent to

$$|T_{v_0 t}^{u_0, a} - (t - h_t)| \leq \mathcal{C}_1 + C_{14} \ln(v_0 t). \quad (3.48)$$

For proving the latter, observe that it is sufficient to show that we can choose $C_{14} > 0$ as well as a \mathbb{P} -a.s. finite random variable \mathcal{T} , such that

$$\begin{aligned} u^{\delta' \mathbb{1}_{[-\delta', 0]}}(t - h_t + C_{14} \ln t, v_0 t) &\geq a \quad \text{and} \\ u^{\mathbb{1}_{(-\infty, 0]}}(t - h_t - C_{14} \ln t, v_0 t) &< \frac{a}{2C'}, \quad \forall t \geq \mathcal{T}. \end{aligned} \quad (3.49)$$

with δ', C' from (PAM-INI). Indeed, due to (PAM-INI), the first inequality in (3.49) implies $T_{v_0 t}^{u_0, a} \leq T_{v_0 t}^{\delta' \mathbb{1}_{[-\delta', 0]}, a} \leq t - h_t + C_{14} \ln t$ for $t \geq \mathcal{T}$. To use the second inequality, first note that $T_{v_0 t}^{u_0, a} \geq T_{v_0 t}^{C' \mathbb{1}_{(-\infty, 0]}, a} = T_{v_0 t}^{\mathbb{1}_{(-\infty, 0]}, a/C'}$. Then, using $u^{\mathbb{1}_{(-\infty, 0]}}(s, x) = \mathbb{E}_x^\xi [N^{\leq}(s, 0)]$ and Lemma A.6, (3.49) implies $T_{v_0 t}^{\mathbb{1}_{(-\infty, 0]}, a/C'} \geq t - h_t - C_{14} \ln t$ for all $t \geq \mathcal{T}$. For $t < \mathcal{T}$ on the other hand, we can use that \mathbb{P} -a.s., the family $T_{v_0 t}^{u_0, a}$, $t < \mathcal{T}$, as well as the $L_i^\xi(\bar{\eta}(v_0))$ are uniformly bounded, allowing us to upper bound the remaining cases of (3.48) by some finite random variable C_1 .

Thus, in order to show (3.49), note that for $\alpha \in \mathbb{R}$ and uniformly in $u_0 \in \mathcal{I}_{\text{PAM}}$ we have that

$$\begin{aligned} & \ln u^{u_0}(t - h_t + \alpha \ln t, v_0 t) \\ &= \ln \left(\frac{u^{u_0}(t - h_t + \alpha \ln t, v_0 t)}{u^{u_0}(t, v_0 t)} \right) + \sum_{i=1}^{v_0 t} (L_i^\xi(\bar{\eta}(v_0)) - L(\bar{\eta}(v_0))) + a_t \\ &= (-h_t + \alpha \ln t)(\text{es} - \bar{\eta}(v_0)) + v_0 L(\bar{\eta}(v_0)) h_t + b(\alpha, t) = \alpha(\text{es} - \bar{\eta}(v_0)) \ln t + b(\alpha, t), \end{aligned}$$

for some error terms a_t and $b(\alpha, t)$ fulfilling $|a_t| \leq C_{13} \ln t$ and

$$|b(\alpha, t)| \leq C_{13} \ln t + C_{10} + C_{10} \cdot |\alpha \ln t - h_t| \cdot \left(\sqrt{\frac{\ln t}{t}} + \frac{|\alpha \ln t - h_t|}{t} \right)$$

for all t large enough. Indeed, the first equality is due to Lemma 3.13, the second due to the time perturbation Lemma 3.11, the last one due to the identity $\text{es} - \bar{\eta}(v_0) = v_0 L(\bar{\eta}(v_0))$. Then due to $|h_t| \leq c_1 \sqrt{t} \ln t$ for large t [cf. (3.47)], choosing $C_{14} = \alpha > \frac{2C_{10} \cdot C^2}{\text{es} - \bar{\eta}(v_0)}$ the latter term tends to infinity, supplying us with (3.49).

To complete the proof, equation (3.46) is a direct consequence of (3.45) and (2.9).

Recall the definition $\bar{m}^{u_0, a} = \bar{m}^{\xi, u_0, a}$ from (1.6).

Corollary 3.15. *For all $u_0 \in \mathcal{I}_{\text{PAM}}$ and $a > 0$ we have*

$$\frac{\bar{m}^{u_0, a}(t)}{t} \xrightarrow[t \rightarrow \infty]{} v_0 \quad \mathbb{P}\text{-a.s.} \tag{3.50}$$

Proof. For an upper bound, we have $\limsup_{t \rightarrow \infty} \frac{\bar{m}^{u_0, a}(t)}{t} = \limsup_{t \rightarrow \infty} \frac{\bar{m}^{u_0, a}(t)}{T_{\bar{m}^{u_0, a}(t)}^{u_0, a}(t)}$
 $\frac{T_{\bar{m}^{u_0, a}(t)}^{u_0, a}(t)}{t} \leq v_0$, where the last inequality is due to $T_{\bar{m}^{u_0, a}(t)}^{u_0, a}(t) \leq t$ and (3.46). To get a lower bound, we can use the properties of the Lyapunov exponent from Proposition A.3, giving $\liminf_{t \rightarrow \infty} \frac{\bar{m}^{u_0, a}(t)}{t} \geq v$ for all $v \in (0, v_0)$, and we can conclude.

Lemma 3.16. *For every $a > 0$ there exists a constant $C_{15} = C_{15}(a) > 0$ and a \mathbb{P} -a.s. finite random variable $\mathcal{T}_4 = \mathcal{T}_4(a)$ such that for all $u_0 \in \mathcal{I}_{\text{PAM}}$ and $t \geq \mathcal{T}_4$,*

$$t - C_{15} \leq T_{\bar{m}^{u_0, a}(t)}^{u_0, a} \leq t. \tag{3.51}$$

Proof. By definition, the inequality $T_{\bar{m}^{u_0,a}(t)}^{u_0,a} \leq t$ follows directly. To show $t - C_{15} \leq T_{\bar{m}^{u_0,a}(t)}^{u_0,a}$, recall that due to (3.50) we can use time perturbation. Indeed, by defining $C_{15} := C_{11} \ln(C_{11} \cdot 3C'/a)$ with C' from (PAM-INI) and C_{11} from Lemma 3.11(b), for all t large enough

$$u^{\mathbb{1}_{(-\infty,0]}}(t - C_{15}, \bar{m}^{u_0,a}(t)) \leq C_{11} e^{-C_{15}/C_{11}} u^{u_0}(t, \bar{m}^{u_0,a}(t)) < \frac{a}{2C'}$$

and thus, recalling $u^{\mathbb{1}_{(-\infty,0]}}(s, x) = \mathbb{E}_x^\xi[N^{\leq}(s, 0)]$ and Lemma A.6, we get the lower bound $T_{\bar{m}^{u_0,a}(t)}^{u_0,a} \geq T_{\bar{m}^{u_0,a}(t)}^{C' \mathbb{1}_{(-\infty,0]}, a} = T_{\bar{m}^{u_0,a}(t)}^{\mathbb{1}_{(-\infty,0]}, a/C'} \geq t - C_{15}$ for all t large enough and we can conclude.

Recall definition (3.44) for $T_x^{(a)}$.

Corollary 3.17. *There exists $\bar{K} \in (1, \infty)$ such that \mathbb{P} -a.s., for all $a > 0$ and for all x large enough*

$$\sup_{|y| \leq 1} T_{x+y}^{(a)} - \bar{K} \leq T_x^{(a)} \leq \inf_{|y| \leq 1} T_{x+y}^{(a)} + \bar{K}. \tag{3.52}$$

Proof. We set $\bar{K} := 1 + C_{11}(\ln(2C_{11}C_{12}) + C_{12})$. Then due to (3.46), \mathbb{P} -a.s. for all x large enough we have

$$\frac{x + y'}{T_{x+y''}^{(a)} \pm \bar{K}} \in V \quad \forall y', y'' \in [-1, 1].$$

This allows us to apply the inequalities (3.34) and (3.41) for $u_0 = \mathbb{1}_{(-\infty,0]}$. Indeed, for all $|y| \leq 1$,

$$\begin{aligned} u(T_{x+y}^{(a)} - \bar{K}, x) &\leq C_{12} e^{C_{12}} u(T_{x+y}^{(a)} - \bar{K}, x + y) \\ &\leq C_{12} e^{C_{12}} C_{11} e^{-(\bar{K}-1)/C_{11}} u(T_{x+y}^{(a)} - 1, x + y) < \frac{a}{2}, \end{aligned}$$

where the first inequality is due to (3.41), the second one due to (3.34) and the last one uses $u(T_{x+y}^{(a)} - 1, x + y) < a$. By Lemma A.6 we get $u(t, x) < a$ for all $t \leq T_{x+y}^{(a)} - \bar{K}$ and thus the left-hand side in (3.52). Analogously, first applying (3.41) and then (3.34), we have

$$u(T_{x+y}^{(a)} + \bar{K}, x) \geq C_{12}^{-1} e^{-C_{12}} u(T_{x+y}^{(a)} + \bar{K}, x + y) \geq C_{12}^{-1} e^{-C_{12}} C_{11}^{-1} e^{\bar{K}/C_{11}} u(T_{x+y}^{(a)}, x + y) \geq a$$

for all $|y| \leq 1$, giving the right-hand side of (3.52).

Corollary 3.18. *Let $\bar{m}^\varepsilon(t) = \bar{m}^{\xi, \mathbb{1}_{(-\infty,0]}, a}(t)$, $a > 0$, be defined in (1.6). Then for all $0 < \varepsilon \leq M$ there exists $C = C(\varepsilon, M)$ such that \mathbb{P} -a.s. for all t large enough*

$$0 \leq \bar{m}^\varepsilon(t) - \bar{m}^M(t) \leq C.$$

Proof. The first inequality is clear. By Corollary 3.15, we can use the second inequality from Lemma 3.12(b) and get the claim by defining $C := C_{12} \ln(C_{12} \cdot M/\varepsilon)$ with C_{12} from Lemma 3.12(b) to get $u^{\mathbb{1}_{(-\infty,0]}}(t, \bar{m}^\varepsilon(t) - C) \geq M$ and thus $\bar{m}^M(t) \geq \bar{m}^\varepsilon(t) - C$ for all t large enough.

3.7. Proof of Theorem 1.4

Using the preparatory results from the previous sections, it is now possible to obtain an invariance principle for the front of the solution to (PAM). Roughly speaking, up to some error which can be controlled by the results from the previous sections, we have $\bar{m}(t) \approx \ln u(t, v_0 t)$ and can then use the invariance principle from Theorem 1.3 to conclude.

Proof of Theorem 1.4. Let $u_0 \in \mathcal{I}_{\text{PAM}}$, $a > 0$ and abbreviate $u = u^{u_0}$ and $\bar{m} := \bar{m}^{u_0, a}$. We first assume $\sigma_{v_0}^2 > 0$. Then we have to show that the sequence of processes

$$[0, \infty) \ni t \mapsto \frac{\bar{m}(nt) - v_0 nt}{\sqrt{n\tilde{\sigma}_{v_0}^2}}, \quad n \in \mathbb{N}, \tag{3.53}$$

where $\tilde{\sigma}_{v_0}^2 > 0$ is given in (3.58) below, converges in \mathbb{P} -distribution to standard Brownian motion in the Skorohod space $D([0, \infty))$. Notice that $[0, \infty) \ni t \mapsto \bar{m}(t)$ might not be càdlàg *only* in 0. To avoid this issue, the above convergence is defined as convergence of the sequence of processes in (3.53) where in a slight abuse we redefine $\bar{m}(t) \equiv 0$ for t such that $\bar{m}(t) \leq 0$, making it càdlàg.

Due to the limiting behavior and the continuity of $x \mapsto u(t, x)$ for $t > 0$, the value $r(t) := \bar{m}(t) - v_0 t$ is the largest solution to

$$\ln u(t, v_0 t + r(t)) = -\ln 2.$$

We define

$$\begin{aligned} \mathcal{L}(t, h) &:= \ln \frac{u(t, v_0 t + h)}{u(t, v_0 t)}, \quad t > 0, h \in \mathbb{R}, \\ U(t) &:= -\ln u(t, v_0 t) - \ln 2 = \mathcal{L}(t, r(t)), \quad t \geq 0, \end{aligned}$$

let

$$\delta \in (0, |L(\bar{\eta}(v_0))|)$$

and $\varepsilon(t)$ be a positive function such that $\varepsilon(t) \rightarrow 0$ and $\varepsilon(t)t^{1/2} \rightarrow \infty$. Then by Lemma 3.12, there is $C(\varepsilon) > 0$ such that for t large enough and all $h \in \mathbb{R}$ fulfilling $C(\varepsilon) \ln t \leq |h| \leq \varepsilon(t)t$ and $v_0 + \frac{h}{t} \in V$ we have

$$-(|L(\bar{\eta}(v_0))| + \delta)h \leq \mathcal{L}(t, h) \leq -(|L(\bar{\eta}(v_0))| - \delta)h. \tag{3.54}$$

Now, multiplying (3.45) by v_0 , replacing x by $\bar{m}(t)$ in (3.45) and recalling that $t - C_{15} \leq T_{\bar{m}(t)} \leq t$ by Lemma 3.16, we get

$$\left| (\bar{m}(t) - v_0 t) - \frac{1}{L(\bar{\eta}(v_0))} \sum_{i=1}^{\bar{m}(t)} (L_i^\zeta(\bar{\eta}(v_0)) - L(\bar{\eta}(v_0))) \right| \leq v_0 C_1 + C_{15} + v_0 C_{14} \ln(\bar{m}(t)) \tag{3.55}$$

for all t large enough. Next, recall that $\frac{\bar{m}(t)}{t} \rightarrow v_0$ by Corollary 3.15 and that the standardized sum $\frac{1}{\sqrt{n}} \sum_{i=1}^{nt} (L_i^\zeta(\bar{\eta}(v_0)) - L(\bar{\eta}(v_0)))$ converges in distribution to a non-degenerate Gaussian random variable by Lemma 3.2. As a consequence, in

combination with (3.55), we infer that $|r(t)| = |\bar{m}(t) - v_0 t| \in [C(\varepsilon) \ln t, \varepsilon(t)t]$ with probability tending to 1 as t tends to infinity. This and (3.54) implies

$$r(t) \in \left[\frac{U(t)}{|L(\bar{\eta}(v_0))| \mp \delta}, \frac{U(t)}{|L(\bar{\eta}(v_0))| \pm \delta} \right], \tag{3.56}$$

with probability tending to 1 as t tends to ∞ , where the upper sign is chosen if $U(t) > 0$ and the lower sign if $U(t) < 0$. If $\sigma_{v_0}^2 > 0$, due to $\Lambda(v_0) = 0$ and Theorem 1.3, the sequence of processes

$$[0, \infty) \ni t \mapsto \frac{1}{\sqrt{nv_0\sigma_{v_0}^2}} U(nt), \quad n \in \mathbb{N}, \tag{3.57}$$

converges in \mathbb{P} -distribution to standard Brownian motion. Because (3.56) holds for all $\delta > 0$ small enough, Theorem 1.4 is a direct consequence of the convergence in distribution of (3.57) by choosing

$$\tilde{\sigma}_{v_0} := \frac{\sqrt{\sigma_{v_0}^2 v_0}}{|L(\bar{\eta}(v_0))|}, \tag{3.58}$$

where $\sigma_{v_0}^2$ is defined in (3.1). This gives the second part of Theorem 1.4. If $\sigma_{v_0}^2 = 0$, we can proceed analogously and the first part of Theorem 1.4 follows from the first part of Theorem 1.3 and (3.56).

4. Log-Distance of the Fronts of the Solutions to PAM and F-KPP

We finally prove our last main result, Theorem 1.5. In Sects. 4.1 and 4.2, we will assume that $u_0 = w_0 = \mathbb{1}_{(-\infty, 0]}$. Indeed, using a comparison argument in the proof of Theorem 1.5, it will turn out that this is actually sufficient for our purposes. Let us emphasize again that the tools we employ are inherently probabilistic. As a consequence, and for notational convenience, we will mostly formulate the respective results in terms of the BBMRE in what follows below; the correspondence to the results in PDE terms is immediate from (2.2) and Remark 2.2.

In the case $u_0 = w_0 = \mathbb{1}_{(-\infty, 0]}$, using Markov’s inequality we infer

$$\mathbb{P}_x^\xi \left(N^{\leq}(t, 0) \geq 1 \right) \leq \mathbb{E}_x^\xi \left[N^{\leq}(t, 0) \right]$$

and thus $\bar{m}(t) \geq m(t)$ for all $t \geq 0$, which establishes the first inequality in (1.16). The rest of this section will be dedicated to deriving the second inequality in (1.16), i.e., that the front of the randomized F-KPP equation lags behind the front of the solution of the parabolic Anderson model at most logarithmically. We introduce some notation and, recalling the notation X^v introduced before (2.1), start with considering certain “well-behaved” particles

$$N_{s,u,t}^{\mathcal{L},a} := \left\{ \left\{ v \in N(s) : X_s^v \leq 0, H_k^v \geq u - T_k^{(a)} - 5\chi_1(\bar{m}^a(t)) \forall k \in \{1, \dots, \lfloor \bar{m}^a(t) \rfloor \} \right\} \right\}, \tag{4.1}$$

$a > 0, s, t, u \geq 0;$

here, $H_k^v := \inf\{t \geq 0 : X_t^v = k\}$, the random variables $T_k^{(a)}$ have been defined in (3.44), and we set

$$\chi_b(x) := \mathcal{C}_1 + b(1 + \overline{K} + C_{15}) + C_{14}(\ln x \vee 1), \quad x \in (0, \infty), \quad b \in \mathbb{R}, \quad (4.2)$$

where \mathcal{C}_1 and C_{14} have been defined in Lemma 3.14, \overline{K} is taken from Corollary 3.17, the constant C_{15} from Lemma 3.16. We abbreviate $N_t^{\mathcal{L}} := N_{t,t,t}^{\mathcal{L}}$ and call the particles contributing to $N_t^{\mathcal{L}}$ *leading particles at time t*. Cauchy-Schwarz immediately gives

$$\mathbb{P}_x^\xi \left(N^{\leq}(t, 0) \geq 1 \right) \geq \mathbb{P}_x^\xi \left(N_t^{\mathcal{L}} \geq 1 \right) \geq \frac{\mathbb{E}_x^\xi \left[N_t^{\mathcal{L}} \right]^2}{\mathbb{E}_x^\xi \left[(N_t^{\mathcal{L}})^2 \right]}. \quad (4.3)$$

The next two sections are dedicated to deriving an upper bound for the denominator and a lower bound for the numerator of the right-hand side, both for x in a neighborhood of $\overline{m}(t)$. These bounds will then form a key part in the proof of Theorem 1.5 in Sect. 4.3, together with a so-called amplification argument.

4.1. First moment of leading particles

The biggest chunk of this section will consist of proving the following first moment bound on the number of leading particles. Recall the notation $\overline{m}^a(t)$ from (1.7).

Lemma 4.1. *For all $a > 0$ there exists $\gamma_1 = \gamma_1(a) \in (0, \infty)$ such that \mathbb{P} -a.s., for all t large enough*

$$\inf_{x \in [\overline{m}^a(t)-1, \overline{m}^a(t)+1]} \mathbb{E}_x^\xi \left[N_t^{\mathcal{L},a} \right] \geq t^{-\gamma_1}.$$

Proof. Let $a > 0$. To simplify notation, we will omit the index $a > 0$ in the quantities involved and write $N_{s,u,t}^{\mathcal{L}} := N_{s,u,t}^{\mathcal{L},a}$, $T_x^{(a)} := T_x$, $\overline{m}^a(t) := \overline{m}(t)$ from now on.

Let $A_{u,t} := \{H_k \geq u - T_k - 5\chi_1(\overline{m}(t)) \forall k \in \{1, \dots, \lfloor m(t) \rfloor\}\}$, let K be such that (3.17) holds and set $\bar{t} := T_{\lfloor \overline{m}(t) \rfloor}$. We obtain for all t large enough that

$$\begin{aligned} \inf_{x \in [\overline{m}(t)-1, \overline{m}(t)+1]} \mathbb{E}_x^\xi \left[N_t^{\mathcal{L}} \right] &\geq \frac{\inf_{x \in [\overline{m}(t)-1, \overline{m}(t)+1]} \mathbb{E}_x^\xi \left[N_t^{\mathcal{L}} \right]}{2\mathbb{E}_{\lfloor \overline{m}(t) \rfloor}^\xi \left[N^{\leq}(\bar{t}, 0) \right]} \geq \frac{c}{2} \frac{\mathbb{E}_{\lfloor m(t) \rfloor}^\xi \left[N_{t-1,t,t+1}^{\mathcal{L}} \right]}{\mathbb{E}_{\lfloor m(t) \rfloor}^\xi \left[N^{\leq}(\bar{t}, 0) \right]} \\ &= \frac{c}{2} \frac{E_{\lfloor m(t) \rfloor} \left[e^{\int_0^{t-1} \xi(B_s) ds}; B_{t-1} \leq 0, A_{t+1,t} \right]}{E_{\lfloor m(t) \rfloor} \left[e^{\int_0^{\bar{t}} \xi(B_s) ds}; B_{\bar{t}} \leq 0 \right]} \\ &\geq c_1 \frac{E_{\lfloor m(t) \rfloor} \left[e^{\int_0^{t-1} \zeta(B_s) ds}; B_{t-1} \leq 0, A_{t+1,t} \right]}{E_{\lfloor m(t) \rfloor} \left[e^{\int_0^{\bar{t}} \zeta(B_s) ds}; B_{\bar{t}} \leq 0 \right]}, \end{aligned} \quad (4.4)$$

here, the first inequality follows from the definition of $T_{\lfloor \overline{m}(t) \rfloor}$, the second inequality is due to Lemma A.7, the equality follows using Proposition 2.3 and the last

inequality is due to $\xi = \zeta + \varepsilon s$, as well as (3.52) which gives $\bar{t} = T_{\lfloor \bar{m}(t) \rfloor} \leq T_{\bar{m}(t)} + \bar{K} \leq t + \bar{K}$. The numerator can be bounded from below by

$$\begin{aligned} & E_{\lfloor \bar{m}(t) \rfloor} \left[e^{\int_0^{t-1} \zeta(B_s) ds}; H_0 \in [t - 3\bar{K} - C_{15}, t - 1], B_{t-1} \leq 0, A_{t+1,t} \right] \\ & \geq E_{\lfloor \bar{m}(t) \rfloor} \left[e^{\int_0^{H_0} \zeta(B_s) ds} E_0 \left[e^{\int_0^r \zeta(B_s) ds}; B_{t-1-r} \leq 0 \right] \Big|_{r=t-1-H_0}; H_0 \in [t - 3\bar{K} - C_{15}, t - 1], A_{t+1,t} \right] \\ & \geq c_2 E_{\lfloor \bar{m}(t) \rfloor} \left[e^{\int_0^{H_0} \zeta(B_s) ds}; H_0 \in [t - 3\bar{K} - C_{15}, t - 1], A_{t+1,t} \right], \end{aligned}$$

where the second inequality is due to $\zeta \geq -(\varepsilon s - \varepsilon i)$ and $P_0(B_s \leq 0) \geq 1/2$ for all $s \geq 0$. Now using the inclusion $\{B_{\bar{t}} \leq 0\} \subset \{H_0 \leq \bar{t}\}$ in combination with $\zeta \leq 0$, we infer $E_{\lfloor \bar{m}(t) \rfloor} \left[e^{\int_0^{\bar{t}} \zeta(B_s) ds}; B_{\bar{t}} \leq 0 \right] \leq E_{\lfloor \bar{m}(t) \rfloor} \left[e^{\int_0^{H_0} \zeta(B_s) ds}; H_0 \leq \bar{t} \right]$. Thus, recalling $\bar{\eta}(v_0) < 0$ and (2.5), we can continue to lower bound (4.4) via

$$\begin{aligned} & \inf_{x \in [\bar{m}(t)-1, \bar{m}(t)+1]} \mathbb{E}_x^\xi \left[N_t^\xi \right] \geq c_3 \frac{E_{\lfloor \bar{m}(t) \rfloor}^{\zeta, \bar{\eta}(v_0)} \left[e^{-\bar{\eta}(v_0)H_0}; H_0 \in [t - 3\bar{K} - C_{15}, t - 1], A_{t+1,t} \right]}{E_{\lfloor \bar{m}(t) \rfloor}^{\zeta, \bar{\eta}(v_0)} \left[e^{-\bar{\eta}(v_0)H_0}; H_0 \leq \bar{t} \right]} \\ & \geq c_4 \frac{P_{\lfloor \bar{m}(t) \rfloor}^{\zeta, \bar{\eta}(v_0)} \left(H_0 \in [t - 3\bar{K} - C_{15}, t - 1], H_k \geq t + 1 - T_k - 5\chi_1(\bar{m}(t)), \forall 1 \leq k \leq \lfloor \bar{m}(t) \rfloor \right)}{P_{\lfloor \bar{m}(t) \rfloor}^{\zeta, \bar{\eta}(v_0)} \left(H_0 \leq \bar{t} \right)} \\ & \geq c_4 P_{\lfloor \bar{m}(t) \rfloor}^{\zeta, \bar{\eta}(v_0)} \left(H_0 \in [\bar{t} - 2\bar{K}, \bar{t} - \bar{K} - 1], H_k \geq \bar{t} - T_k - 5\chi_0(\bar{m}(t)), \forall 1 \leq k \leq \lfloor \bar{m}(t) \rfloor \right), \end{aligned}$$

where the last inequality is due to $t \geq T_{\bar{m}(t)} \geq \bar{t} - \bar{K}$ and $\bar{t} \geq T_{\bar{m}(t)} - C_{15} - \bar{K}$ [by (3.52) and (3.51)]. Now as we recall that $\frac{\lfloor \bar{m}(t) \rfloor}{t} \rightarrow v_0$, abbreviating $\eta = \bar{\eta}(v_0)$, $n := \lfloor \bar{m}(t) \rfloor$ and thus $\bar{t} = T_n$, we see that in order to finish the proof, it suffices to show that there exists $\gamma \in (0, \infty)$ such that \mathbb{P} -a.s., for all $n \in \mathbb{N}$ large enough,

$$P_n^{\zeta, \eta} \left(H_0 \in [T_n - 2\bar{K}, T_n - \bar{K} - 1], H_k \geq T_n - T_k - 5\chi_0(n) \forall k \in \{1, \dots, n\} \right) \geq n^{-\gamma}. \tag{4.5}$$

Using the notation

$$\widehat{H}_k^{(n)} := H_k - E_n^{\zeta, \eta} [H_k] \quad \text{as well as} \quad R_k^{(n)} := T_n - T_k - E_n^{\zeta, \eta} [H_k], \tag{4.6}$$

the probability in (4.5) can be rewritten as

$$P_n^{\zeta, \eta} \left(\widehat{H}_0^{(n)} \in [R_0^{(n)} - 2\bar{K}, R_0^{(n)} - \bar{K} - 1], \widehat{H}_k^{(n)} \geq R_k^{(n)} - 5\chi_0(n) \forall k \in \{1, \dots, n\} \right). \tag{4.7}$$

In order to facilitate computations, we approximate the sequence $(R_k^{(n)})$ by a stationary one, setting

$$\rho_i := \frac{L_i^\zeta(\eta)}{v_0 L(\eta)} - (L_i^\zeta)'(\eta) = \frac{1}{v_0 L(\eta)} (L_i^\zeta(\eta) - L(\eta)) - (E_n^{\zeta, \eta}[\tau_{i-1}] - \mathbb{E}[E_n^{\zeta, \eta}[\tau_{i-1}]]) \quad \text{and} \tag{4.8}$$

$$\widehat{R}_k^{(n)} := \sum_{i=k+1}^n \rho_i, \quad k < n, \tag{4.9}$$

where $\tau_{i-1} = H_{i-1} - H_i$, and in the equality we used $E_n^{\zeta,\eta}[\tau_{i-1}] = (L_i^\zeta)'(\eta)$ and $\mathbb{E}[E_n^{\zeta,\eta}[H_k]] = \frac{n-k}{v_0}$. Applying inequality (3.45) from Lemma 3.14 and using the identity $\mathbb{E}[E_n^{\zeta,\eta}[H_k]] = \frac{n-k}{v_0}$, we get that \mathbb{P} -a.s.,

$$|R_k^{(n)} - \widehat{R}_k^{(n)}| \leq 2(C_1 + C_{14} \ln n) \quad \text{for all } n \in \mathbb{N} \text{ and each } k \in \{0, \dots, n\}. \tag{4.10}$$

From now on we will write $\chi := \chi_0$. Then by (4.10), the probability in (4.7) can be lower bounded by

$$P_n^{\zeta,\eta}(\widehat{H}_0^{(n)} \in [R_0^{(n)} - 2\overline{K}, R_0^{(n)} - \overline{K} - 1]; \widehat{H}_k^{(n)} \geq \widehat{R}_k^{(n)} - 3\chi(n) \forall k \in \{1, \dots, n\}). \tag{4.11}$$

Now, for every n , enlarging the underlying probability space if necessary, we introduce two processes $(B_t^{(i,n)})_{t \geq 0}$, $i = 1, 2$, which are independent from everything else and Brownian motions under P_n , starting in n , and, without further formal definition, we tacitly assume in the following that the tilting of the probability measure $P_n^{\zeta,\eta}$ of our original Brownian motion *also applies to* $(B_t^{(i,n)})_{t \geq 0}$, $i = 1, 2$, in the obvious way. For $i = 1, 2$, let $H_k^{(i,n)} := \inf\{t \geq 0 : B_t^{(i,n)} = k\}$, $k \in \mathbb{Z}$, be the corresponding hitting times, $\widehat{H}_k^{(i,n)} := H_k^{(i,n)} - E_n^{\zeta,\eta}[H_k^{(i,n)}]$ and let Σ_n be a random variable which, under P_n , is uniformly distributed on $\{1, \dots, n - 1\}$ and independent of everything else. We define

$$\begin{aligned} \beta_k^{(i,n)} &:= \widehat{H}_k^{(i,n)} - \widehat{R}_k^{(n)}, \quad k = n - 1, n - 2, \dots, \quad i = 1, 2, \\ \beta_k^{(n)} &:= \begin{cases} \beta_k^{(1,n)}, & \Sigma_n \leq k < n, \\ \beta_{\Sigma_n}^{(1,n)} + (\beta_k^{(2,n)} - \beta_{\Sigma_n}^{(2,n)}), & k < \Sigma_n. \end{cases} \end{aligned}$$

The ξ -adaptedness of the process $(\widehat{R}_k^{(n)})_{k < n}$ implies that the processes $(\beta_k^{(i,n)})_{k < n}$, $i = 1, 2$, are $P_n^{\zeta,\eta}$ -independent and have the same distribution as $(\beta_k^{(n)})_{k < n}$. We can therefore rewrite (4.11) as

$$P_n^{\zeta,\eta}(\beta_k^{(n)} \geq -3\chi(n) \forall k \in \{1, \dots, n\}, \beta_0^{(n)} \in I_n), \tag{4.12}$$

where $I_n := [R_0^{(n)} - \widehat{R}_0^{(n)} - 2\overline{K}, R_0^{(n)} - \widehat{R}_0^{(n)} - \overline{K} - 1]$. Due to (4.10) we have that \mathbb{P} -a.s. for all n large enough, $R_0^{(n)} - \widehat{R}_0^{(n)} - 2\overline{K} \geq -3\chi(n)$, i.e.

$$I_n \subset [-3\chi(n), \infty). \tag{4.13}$$

For each $k \in \{0, \dots, n\}$ we introduce

$$\overline{\beta}_k^{(1,n)} := \beta_{n-1-k}^{(1,n)} - \beta_{n-1}^{(1,n)}, \quad \overline{\beta}_k^{(2,n)} := \beta_k^{(2,n)} - \beta_0^{(2,n)},$$

and note that

$$\beta_0^{(n)} = \overline{\beta}_{n-1-\Sigma_n}^{(1,n)} - \overline{\beta}_{\Sigma_n}^{(2,n)} + \beta_{n-1}^{(1,n)}. \tag{4.14}$$

An illustration of the various processes introduced above is given in Fig. 4 below. Now the key to bound the probability in (4.12) is the following lemma.

Lemma 4.2. (a) *There exists $\gamma' < \infty$ such that \mathbb{P} -a.s. for all n large enough,*

$$P_n^{\zeta, \eta}(\bar{\beta}_k^{(1, n)} \geq 0 \forall 0 \leq k \leq n, \bar{\beta}_n^{(1, n)} \geq n^{1/4}) \geq n^{-\gamma'}, \quad \text{and}$$

$$P_n^{\zeta, \eta}(\bar{\beta}_k^{(2, n)} \geq 0 \forall 0 \leq k \leq n, \bar{\beta}_n^{(2, n)} \geq n^{1/4}) \geq n^{-\gamma'}.$$

(b) *There exists $C(\gamma') > 0$ such that \mathbb{P} -a.s. for all n large enough,*

$$P_n^{\zeta, \eta}\left(\max_{1 \leq k \leq n, i \in \{1, 2\}} |\beta_k^{(i, n)} - \beta_{k-1}^{(i, n)}| \leq C(\gamma') \ln n\right) \geq 1 - n^{-3\gamma'}. \quad (4.15)$$

(c) *Let $\delta \in (0, 1)$. There exists $c > 0$ such that for all $x \geq 1$ and all $n \in \mathbb{Z}$,*

$$P_n^{\zeta, \eta}(\beta_{n-1}^{(1, n)} \in [x, x + \delta]) \geq c\delta e^{-x/c}.$$

Before proving Lemma 4.2 we will finish the current proof in order not to interrupt the Reader. To this end let

$$J_n := \sup \{k \in \{1, \dots, n - 1\} : I_n - \bar{\beta}_{n-k+1}^{(1, n)} + \bar{\beta}_k^{(2, n)} \subset [0, 2C(\gamma') \ln n]\},$$

where as always $\sup \emptyset := -\infty$. We have

$$\begin{aligned} & \left\{ \beta_k^{(n)} \geq -3\chi(n) \forall 0 \leq k \leq n - 1, \beta_0^{(n)} \in I_n \right\} \\ & \supset \left(\left\{ \bar{\beta}_k^{(1, n)} \geq 0 \forall 0 \leq k \leq n, \bar{\beta}_n^{(1, n)} \geq n^{1/4} \right\} \cap \left\{ \max_{1 \leq k \leq n} |\bar{\beta}_k^{(1, n)} - \bar{\beta}_{k-1}^{(1, n)}| \leq C(\gamma') \ln n \right\} \right. \\ & \quad \cap \left\{ \bar{\beta}_k^{(2, n)} \geq 0 \forall 0 \leq k \leq n, \bar{\beta}_n^{(2, n)} \geq n^{1/4} \right\} \cap \left\{ \max_{1 \leq k \leq n} |\bar{\beta}_k^{(2, n)} - \bar{\beta}_{k-1}^{(2, n)}| \leq C(\gamma') \ln n \right\} \\ & \quad \left. \cap \left\{ \beta_{n-1}^{(1, n)} \in I_n - \bar{\beta}_{n-1-\Sigma_n}^{(1, n)} + \bar{\beta}_{\Sigma_n}^{(2, n)} \right\} \cap \{\Sigma_n = J_n\} \right). \end{aligned} \quad (4.16)$$

Indeed, due to (4.14), the fifth event on the right-hand side of (4.16) entails that $\beta_0^{(n)} \in I_n$ must hold. On the last two events on the right-hand side of (4.16) we have $\bar{\beta}_{n-1}^{(1, n)} \geq 0$ and thus the first event on the right-hand side of (4.16) implies that $\beta_k^{(n)}$ is non-negative for $k \geq \Sigma_n$. The third event then implies monotonicity at times $k < \Sigma_n$. Since $I_n \subset [-3\chi(n), \infty)$ due to (4.13), this gives the first condition on the left-hand side of (4.16). Now the first and third event on the right hand-side of (4.16) are independent under $P_n^{\zeta, \eta}$ and their probabilities are bounded from below by $n^{-\gamma'}$ due to Lemma 4.2(a). Thus, as a consequence of Lemma 4.2(b), for n large enough the probability of the first four events is bounded from below by $n^{-2\gamma'} - n^{-3\gamma'}$. Furthermore, the first four events imply that $J_n \in \{1, \dots, n - 1\}$. Thus, due to Lemma 4.2, conditionally on the occurrence of the first four events, the probability of the last two events on the right-hand side in (4.16) can be bounded from below by $cn^{-1}e^{-C(\gamma') \ln n/c} \geq n^{-\gamma''}$ for n large enough. The proof of (4.5) and thus of Lemma 4.1 is completed by the choice $\gamma_1 > 2\gamma' + \gamma''$.

Proof of Lemma 4.2. (b) Because $H_k^{(1, n)} - H_{k-1}^{(1, n)} \stackrel{d}{=} \tau_{k-1}$ under $P_n^{\zeta, \eta}$, by recalling $\mathbb{E}[E_n^{\zeta, \eta}[\tau_k]] = \frac{1}{n}$ and the definition of $\widehat{R}_k^{(n)}$ from (4.8), we have $\beta_k^{(1, n)} - \beta_{k-1}^{(1, n)} =$

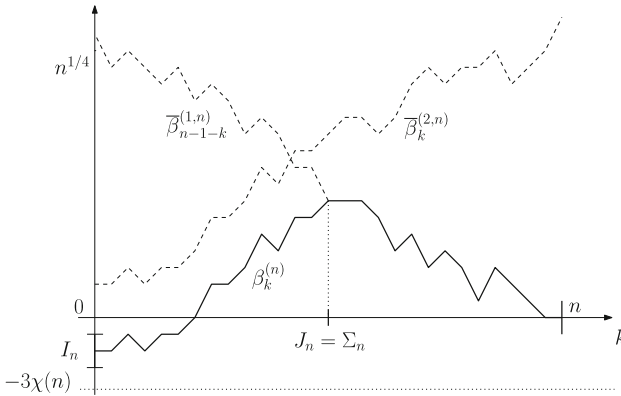


Fig. 4. Illustration of (4.16)

$\tau_{k-1} - \frac{L_k^\zeta(\eta)}{v_0 L(\eta)}$. Now $L_k^\zeta(\eta)$ is \mathbb{P} -a.s. bounded by Lemma A.1. Furthermore, for all θ such that $|\theta| \leq |\eta^*|$ (where $\Delta = [\eta_*, \eta^*]$), we have

$$0 \leq E_n^{\zeta,\eta}[e^{\theta\tau_{k-1}}] = E_k^{\zeta,\eta}[e^{\theta\tau_{k-1}}] \leq \left(E_k[e^{(-es+ei+\eta_*)H_{k-1}}]\right)^{-1} = e^{\sqrt{2(es-ei+|\eta_*|)}} < \infty,$$

where the last equation is due to [4, (2.0.1), p. 204]. I.e., τ_k has uniform exponential moments under $P_n^{\zeta,\eta}$ and thus (4.15) follows by a union bound in combination with the exponential Chebyshev inequality.

(c) We have $\beta_{n-1}^{(1,n)} = H_{n-1}^{(1,n)} - E_n^{\zeta,\eta}[H_{n-1}^{(1,n)}] - \widehat{R}_{n-1}^{(n)} = H_{n-1}^{(1,n)} - E_n^{\zeta,\eta}[\tau_{n-1}] - \rho_n$, thus recalling definition (4.8), the event in (c) is equivalent to $\{H_{n-1}^{(1,n)} \in [x, x + \delta] + \frac{L_n^\zeta(\eta)}{v_0 L(\eta)}\}$. Because $\frac{L_n^\zeta(\eta)}{v_0 L(\eta)}$ is uniformly bounded and non-negative, it suffices to check that for every $C > 0$, there exists $c > 0$ such that $\inf_{y \in [0, C]} P_n^{\zeta,\eta}(H_{n-1}^{(1,n)} \in [x + y, x + y + \delta]) \geq c\delta e^{-x/c}$ for all $x \geq 1$. Indeed, recalling (2.5), we can lower bound

$$\begin{aligned} P_n^{\zeta,\eta}(H_{n-1}^{(1,n)} \in [x + y, x + y + \delta]) &\geq E_n[e^{-(es-ei-\eta_*)H_{n-1}}; H_{n-1}^{(1,n)} \in [x + y, x + y + \delta]] \\ &\geq e^{-(es-ei-\eta_*)(x+y+\delta)} P_n(H_{n-1}^{(1,n)} \in [x + y, x + y + \delta]) \geq \frac{\delta e^{-(es-ei-\eta_*)(x+y+\delta)}}{\sqrt{2\pi(x+y+\delta)^3}} e^{-\frac{1}{2(x+y)}}, \end{aligned}$$

where the last inequality is due to [4, (2.0.2), p. 204]. Now since the latter term can be lower bounded by $c\delta e^{-x/c}$, uniformly in $y \in [0, C]$, the claim follows.

(a) We will prove the second inequality, and explain the modifications that are necessary to show the first one at the end of the proof. For later reference it will serve our purposes to exclude some potentially bad behavior of the process $(\widehat{R}_k^{(n)} - \widehat{R}_{k-1}^{(n)})_k$. To do so, we take advantage of the next claim, the proof of which we provide after concluding the proof of Lemma 4.2.

Claim 4.3. For each $n \in \mathbb{Z}$, the sequence $(\rho_i)_{i \in \mathbb{Z}}$ consists of \mathbb{P} -centered and \mathbb{P} -stationary random variables, and the family $(\rho_i)_{i \in \mathbb{Z}}$ is bounded \mathbb{P} -a.s. In addition,

ρ_i is \mathcal{F}^{i-1} -adapted and there exists $C_{16} > 0$ such that \mathbb{P} -a.s., for all $k, n \in \mathbb{Z}$, $k < n$, we have

$$\mathbb{E}[\rho_{n-k} | \mathcal{F}^n] \leq C_{16} \cdot (\psi(k/2) + e^{-k/C_{16}}). \tag{4.17}$$

Furthermore, there exists $\bar{\sigma} \in [0, \infty)$ such that $n^{-1/2} \sum_{l=1}^n \rho_l$ and $n^{-1/2} \sum_{l=1}^n \rho_{-l}$ converge in \mathbb{P} -distribution to $\bar{\sigma}X$ as $n \rightarrow \infty$, where $X \sim \mathcal{N}(0, 1)$ is a standard Normal random variable.

Now due to (4.17), $(\rho_i)_{i \in \mathbb{Z}}$ fulfills the conditions of Corollary A.5. As a consequence we deduce that for $k \in \mathbb{N}$ and $x \geq 0$ we have $\mathbb{P}(\sum_{l=1}^k \rho_l \geq x) \leq c_1 e^{-\frac{x^2}{c_1 k}}$, which, using stationarity, can be extended to the maximal inequality (e.g. by [22, Theorem 1])

$$\mathbb{P}\left(\max_{0 \leq k \leq y} (\widehat{R}_{r+k}^{(n)} - \widehat{R}_r^{(n)}) \geq x\right) = \mathbb{P}\left(\max_{0 \leq k \leq y} \sum_{l=0}^k \rho_l \geq x\right) \leq c_2 e^{-\frac{x^2}{c_2 y}} \quad \forall r, y \in \mathbb{Z}, x \geq 0. \tag{4.18}$$

Furthermore, recalling (4.6), (4.8) and (4.9), the increments of the process $(\bar{\beta}_k^{(2,n)})_k$ can be written as

$$\begin{aligned} \bar{\beta}_k^{(2,n)} - \bar{\beta}_{k-1}^{(2,n)} &= (H_k - H_{k-1} - E_n^{\xi, \eta}[H_k - H_{k-1}]) - \left(\sum_{i=k+1}^n \rho_i - \sum_{i=k}^n \rho_i\right) \\ &= (-\tau_{k-1} + (L_k^\xi)'(\eta)) - \left(-\frac{L_k^\xi(\eta)}{v_0 L(\eta)} + (L_k^\xi)'(\eta)\right) = \frac{L_k^\xi(\eta)}{v_0 L(\eta)} - \tau_{k-1}. \end{aligned}$$

Now \mathbb{P} -a.s., by Lemma A.1(b), the last fraction in the previous display is positive and uniformly bounded away from zero and infinity, whereas under $P_n^{\xi, \eta}$, τ_{k-1} is an absolutely continuous random variable with positive density on $(0, \infty)$. Therefore, for the constant

$$a := \frac{1}{4} \sup_{k \in \mathbb{Z}} \text{ess inf}_{\xi} (\bar{\beta}_k^{(2,n)} - \bar{\beta}_{k-1}^{(2,n)}),$$

we have $\text{ess inf}_{k, n \in \mathbb{Z}; k \leq n, \xi} P_n^{\xi, \eta}(\bar{\beta}_k^{(n)} - \bar{\beta}_{k-1}^{(n)} \geq 2a) \geq \delta$ for some universal constant $\delta \in (0, 1)$. We now split the environment into $\bar{\xi}(j) := (\xi(l))_{l \geq j}$ and $\underline{\xi}(j) := (\xi(l))_{l < j}$ and set $t_0 = t_{-1} := 0$ as well as $t_i := 2^i$ for $i \geq 1$. Furthermore, we introduce two constants: $\bar{c} > 0$, which is defined in (4.31) below, and $\bar{C} > 0$, which is independent of \bar{c} and will be chosen large enough such that the sums in (4.21) and (4.35) below are finite. For $i \geq 1$, we define the random variables

$$\begin{aligned} Z_i^{(n)} &:= \text{ess inf}_{\bar{\xi}(t_{i+1})} \inf_{x \geq at_{i-1}^{1/2}} P_n^{\xi, \eta}(\bar{\beta}_{t_i}^{(2,n)} \geq at_i^{1/2}, \bar{\beta}_k^{(2,n)} \geq t_i^{1/4} \forall k \in \{t_{i-1}, \dots, t_i\} | \bar{\beta}_{t_{i-1}}^{(2,n)} = x) \\ &= \text{ess inf}_{\bar{\xi}(t_{i+1})} P_n^{\xi, \eta}(\bar{\beta}_{t_i}^{(2,n)} \geq at_i^{1/2}, \bar{\beta}_k^{(2,n)} \geq t_i^{1/4} \forall k \in \{t_{i-1}, \dots, t_i\} | \bar{\beta}_{t_{i-1}}^{(2,n)} = at_{i-1}^{1/2}), \end{aligned} \tag{4.19}$$

where $\text{ess inf}_{\bar{\xi}(x)}$ means taking the essential infimum with respect to $\bar{\xi}(x)$, and where the second equality is due to the monotonicity of the first probability in (4.19) as a

function in x . Thus, as a random variable, $Z_i^{(n)}$ is measurable with respect to $\mathcal{F}_{t_{i+1}}$. Now since $\bar{\beta}_k^{(2,n)}$ is \mathcal{F}^k -measurable, we have that $Z_i^{(n)}$ is $(\mathcal{F}^{t_{i-1}} \cap \mathcal{F}_{t_{i+1}})$ -measurable. Setting $i(n) := \lfloor \ln_2(\lfloor \bar{C} \ln n \rfloor) \rfloor$, we further define

$$Y^{(n)} := P_n^{\zeta, \eta} \left(\bar{\beta}_k^{(2,n)} \geq 0 \forall \lfloor \bar{C} \ln n \rfloor \leq k \leq t_{i(n)}, \bar{\beta}_{t_{i(n)}}^{(2,n)} \geq a \lfloor \bar{C} \ln n \rfloor \mid \bar{\beta}_{\lfloor \bar{C} \ln n \rfloor}^{(2,n)} = 2a \lfloor \bar{C} \ln n \rfloor \right).$$

Writing $j(n) := \lceil \log_2(n) \rceil$, due to the Markov property of the process $\bar{\beta}^{(2,n)}$ under $P_n^{\zeta, \eta}$, we have \mathbb{P} -almost surely that for all n large enough,

$$\begin{aligned} & P_n^{\zeta, \eta} \left(\bar{\beta}_n^{(2,n)} \geq n^{1/4}, \bar{\beta}_k^{(2,n)} \geq 0 \forall k \leq n \right) \\ & \geq \prod_{k=1}^{\lfloor \bar{C} \ln n \rfloor} P_n^{\zeta, \eta} \left(\bar{\beta}_k^{(2,n)} - \bar{\beta}_{k-1}^{(2,n)} \geq 2a \right) \cdot Y^{(n)} \cdot \prod_{i=i(n)+1}^{j(n)} Z_i^{(n)} \\ & \geq \delta^{\lfloor \bar{C} \ln n \rfloor} \cdot Y^{(n)} \cdot \exp \left\{ \sum_{i=i(n)+1}^{j(n)} \ln Z_i^{(n)} \mathbb{1}_{B_i^{(n)}} \right\}, \end{aligned} \tag{4.20}$$

where the event

$$B_i^{(n)} := \left\{ \max_{r \in [t_{i-1}, t_i], 0 \leq k \leq \frac{5}{2} a t_i^{1/2} / \bar{c}} (\widehat{R}_{r+k} - \widehat{R}_r) < a t_{i-1}^{1/2} / 16 \right\}$$

occurs \mathbb{P} -almost surely for all $i \in [i(n), j(n)]$ and all n large enough. Indeed, by (4.18) we have

$$\begin{aligned} \sum_n \sum_{i=i(n)}^{j(n)} \mathbb{P}((B_i^{(n)})^c) & \leq \sum_n \sum_{i=i(n)}^{j(n)} t_{i-1} \mathbb{P} \left(\max_{0 \leq k \leq \frac{5}{2} a t_i^{1/2} / \bar{c}} (\widehat{R}_k - \widehat{R}_0) \geq a t_{i-1}^{1/2} / 16 \right) \\ & \leq c_3 \sum_n \sum_{i=i(n)}^{j(n)} e^{-a^2 \bar{c} t_{i-1}^{1/2} / c_3} \leq c_4 \sum_n n \log_2(n) e^{-a^2 \bar{C} \ln n / c_4} < \infty, \end{aligned} \tag{4.21}$$

where the last inequality holds true for \bar{C} large enough. Thus, the Borel-Cantelli lemma implies that \mathbb{P} -a.s., for all n large enough the events $B_i^{(n)}$ occur for all $i \in [i(n), j(n)]$. Furthermore, it is possible to show that \mathbb{P} -almost surely, for all n large enough we have $Y^{(n)} \geq n^{-\gamma''}$. We postpone a proof of this fact, because it uses the same arguments as the following paragraph and we will describe necessary adaptations afterwards, cf. below (4.35). Thus, for the time being it remains to show that there exists $\tilde{c} > 0$ such that \mathbb{P} -almost surely, for all n large enough,

$$\sum_{i=i(n)}^{j(n)} \ln(Z_i^{(n)}) \mathbb{1}_{B_i^{(n)}} \geq -\tilde{c} \cdot j(n). \tag{4.22}$$

The second inequality in Lemma 4.2(a) then follows from (4.20) with $\gamma' > \bar{C} \ln(1/\delta) + \gamma'' + \tilde{c} / \ln(2)$.

In order to establish (4.22), it is enough to show that there exist $c'', \theta > 0$, independent of \tilde{c} , such that for all i large enough,

$$\sup_n \mathbb{E} \left[e^{-\theta \ln(Z_i^{(n)}) \mathbb{1}_{B_i^{(n)}}} \right] \leq c''. \tag{4.23}$$

Indeed, if (4.23) holds, setting $\tilde{Z}_i^{(n)} := \ln(Z_i^{(n)}) \mathbb{1}_{B_i^{(n)}}$, by Markov's inequality we have

$$\begin{aligned} \mathbb{P} \left(\sum_{i=i(n)}^{j(n)} \tilde{Z}_i^{(n)} < -\tilde{c} \cdot j(n) \right) &\leq \mathbb{P} \left(\sum_{k=0}^3 \sum_{i=\lceil i(n)/4 \rceil}^{\lfloor \frac{j(n)}{4} \rfloor - 1} \tilde{Z}_{4i+k}^{(n)} < -\tilde{c} \cdot j(n) \right) \\ &\leq \sum_{k=0}^3 \mathbb{P} \left(\sum_{i=\lceil i(n)/4 \rceil}^{\lfloor \frac{j(n)}{4} \rfloor - 1} \tilde{Z}_{4i+k}^{(n)} < -\tilde{c} \cdot j(n)/4 \right) \leq 4e^{-\theta \tilde{c} \cdot j(n)/4} \max_{k=1, \dots, 4} \mathbb{E} \left[e^{-\theta \sum_{i=\lceil i(n)/4 \rceil}^{\lfloor \frac{j(n)}{4} \rfloor - 1} \tilde{Z}_{4i+k}^{(n)}} \right]. \end{aligned}$$

We will only estimate the above expectation for the case $k = 0$; the cases $k \in \{1, 2, 3\}$ can be estimated similarly. Now $\tilde{Z}_{4i}^{(n)}$ is $\mathcal{F}^{t_{4i-1}}$ -measurable, hence, also recalling $t_{4i-1} - t_{4i-2} = 2^{4i-2}$, by (MIX) we have

$$\mathbb{E} \left[e^{-\theta \tilde{Z}_{4i}^{(n)}} \mid \mathcal{F}_{t_{4i-2}} \right] \leq (1 + \psi(2^{4i-2})) \mathbb{E} \left[e^{-\theta \tilde{Z}_{4i}^{(n)}} \right].$$

Since furthermore $\tilde{Z}_{4(i-1)}^{(n)}$ is $\mathcal{F}^{t_{4i-2}}$ -measurable, we obtain via iterated conditioning that

$$\begin{aligned} \mathbb{E} \left[e^{-\theta \sum_{i=\lceil i(n)/4 \rceil}^{\lfloor \frac{j(n)}{4} \rfloor} \tilde{Z}_{4i}^{(n)}} \right] &= \mathbb{E} \left[\mathbb{E} \left[\dots \mathbb{E} \left[\mathbb{E} \left[e^{-\theta \sum_{i=\lceil i(n)/4 \rceil}^{\lfloor \frac{j(n)}{4} \rfloor} \tilde{Z}_{4i}^{(n)} \mid \mathcal{F}_{t_{4j(n)-2}} \right] \mid \mathcal{F}_{t_{4j(n)-6}} \right] \dots \mid \mathcal{F}_{t_2} \right] \right] \\ &\leq \prod_{i=\lceil i(n)/4 \rceil}^{\lfloor \frac{j(n)}{4} \rfloor} (1 + \psi(2^{4i-2})) \mathbb{E} \left[e^{-\theta \tilde{Z}_{4i}^{(n)}} \right] \leq (c_6 c'')^{j(n)}, \end{aligned}$$

for some $c_6 > 0$ and n large enough. Choosing \tilde{c} large enough, by a Borel-Cantelli argument similar to the proof of Lemma 2.5, inequality (4.22) would follow.

Thus, in order to show (4.23), note that because

$$Z_i^{(n)} = Z_i^{(n)}(\xi(\cdot)) = Z_i^{(n-k)}(\xi(\cdot + k)) \stackrel{d}{=} Z_i^{(n-k)}(\xi(\cdot)) = Z_i^{(n-k)}, \tag{4.24}$$

we can drop the supremum in (4.23). In the following, we first choose i large enough (and from then on fixed) such that several estimates in the remaining part of the proof hold, and afterwards we adapt $n = n(i)$ to ensure $0 \leq i \leq \lceil \log_2(n) \rceil$. For simplicity, we write $Z_i := Z_i^{(n)}$, $\beta_k := \bar{\beta}_k^{(2,n)}$, $\hat{H}_k := H_k - E_n^{\zeta, \eta}[H_k]$, $\hat{R}_k := \hat{R}_k^{(n)}$ and define

$$\bar{\rho}_k^{(j)} := \operatorname{ess\,sup}_{\tilde{\xi}^{(j)}} \rho_k, \quad \bar{R}_k^{(j)} := \sum_{l=0}^k \bar{\rho}_l^{(j)}, \quad 0 \leq k \leq j.$$

Furthermore, Thus, $\bar{\rho}_k^{(j)}$ is \mathcal{F}_j -measurable and $\bar{R}_{k+l}^{(j)} - \bar{R}_k^{(j)}$ is $(\mathcal{F}^k \cap \mathcal{F}_j)$ -measurable for all $l \geq 0$. Let $M_R := \text{ess sup } \rho_0$ and $L := at_i^{1/2}$, and note that the latter choice corresponds to diffusive scaling. Then we define

$$r_0 := t_{i-1}, \quad m := \frac{L}{16M_R}, \quad s_0 := (\inf \{k \geq r_0 : \bar{R}_k^{(k+m)} - \bar{R}_{r_0}^{(k+m)} \geq L/8\} - 1) \wedge t_i, \tag{4.25}$$

and for $j \geq 1$ let

$$r_j := s_{j-1} + \left\lceil \frac{L}{8M_R} \right\rceil, \\ s_j := (\inf \{k \geq r_j : \bar{R}_k^{(k+m)} - \bar{R}_{r_j}^{(k+m)} \geq L/8\} - 1) \wedge (r_j + (t_i - t_{i-1})). \tag{4.26}$$

Heuristically, s_j is the first time after which the process \bar{R} (and thus \widehat{R}) increases at least by the amount $L/8$ after time r_j . Such large increments of \widehat{R} are potentially troublesome, since as a consequence, the process β might decrease too much and cause the event in the definition of Z_i to have too small probability. In order to cater for this inconvenience, we start noting that by definition, $s_j - r_j$ is bounded by $t_i - t_{i-1}$ and \mathcal{F}_{s_j+m} -measurable, and $r_{j+1} - (s_j + m) \geq m$. Thus, by condition (MIX), for every non-negative measurable function f we notice for later reference that

$$\mathbb{E}[f(s_j - r_j) \mid \mathcal{F}^{r_{j+1}}] \leq (1 + \psi(m))\mathbb{E}[f(s_j - r_j)]. \tag{4.27}$$

Next, we define

$$\mathcal{G}_j := \left\{ \inf_{r_j \leq k \leq s_j} (\widehat{H}_k - \widehat{H}_{r_j}) \geq -L/8, \beta_{s_j} \geq 2L \right\}, \quad \mathcal{G}'_j := \left\{ \inf_{s_j \leq k \leq r_{j+1}} (\widehat{H}_k - \widehat{H}_{s_j}) \geq -L/8 \right\}, \\ J := \inf \{j : s_j - r_j = t_i - t_{i-1}\} \wedge \inf \{j : s_j \geq t_i\}, \quad \text{as well as } \mathcal{G} := \bigcap_{j=0}^J \mathcal{G}_j \cap \bigcap_{j=0}^{J-1} \mathcal{G}'_j,$$

and claim that

$$Z_i \geq P_n^{\zeta, \eta}(\mathcal{G} \mid \beta_{t_{i-1}} = at_{i-1}^{1/2}). \tag{4.28}$$

Indeed, on $[r_0, s_0]$, the process \bar{R} (and thus also \widehat{R}) increases by at most $L/8$, and the process \widehat{H} decreases by at most $L/8$ on \mathcal{G}_0 . Moreover, for $j \geq 1$, on $[s_{j-1}, r_j]$, the process \bar{R} increases by at most $L/8$, and \widehat{H} decreases by at most $L/8$ on \mathcal{G}'_j . Finally, on $[r_j, s_j]$, the process \bar{R} (and thus \widehat{R}) increases by at most $L/8$, and on \mathcal{G}_j , \widehat{H} decreases by at most $L/8$ and $\beta_{s_j} \geq 2L$. All in all, conditioning on $\beta_{t_{i-1}} = at_{i-1}^{1/2} = L/\sqrt{2} \geq L/2$, we have $\beta_k \geq L/4 \geq t_i^{1/4}$ for $k \in [r_0, s_0]$ and $\beta_k \geq L$ for all $k \in [s_0, s_J]$. Since by definition, $s_J \geq t_i$, we get $\beta_{t_i} \geq L = at_i^{1/2}$, implying (4.28).

Furthermore, we can continue to lower bound

$$P_n^{\zeta, \eta}(\mathcal{G} \mid \beta_{t_{i-1}} = at_{i-1}^{1/2}) \geq P_n^{\zeta, \eta}(\mathcal{G}_0 \mid \beta_{r_0} = L/\sqrt{2}) \prod_{j=0}^{J-1} P_n^{\zeta, \eta}(\mathcal{G}'_j) \prod_{j=1}^J P_n^{\zeta, \eta}(\mathcal{G}_j \mid \beta_{r_j} = 2L). \tag{4.29}$$

To see this, successively condition on $\beta_{r_j} \geq 2L$, $j = 1, \dots, J$, and use the Markov property of the process \widehat{H} as well as the fact that $x \mapsto P_n^{\zeta, \eta}(\mathcal{G}_j \mid \beta_{r_j} = x)$ is increasing. Then use the fact that under $P_n^{\zeta, \eta}$, the event \mathcal{G}'_j is independent of β_{r_j} by the independence of the increments of \widehat{H} , $j = 0, \dots, J - 1$.

In order to lower bound (4.29), observe that under $P_n^{\zeta, \eta}$, the sequence $(\widehat{H}_{k+1} - \widehat{H}_k)_{k \geq r_j}$ consists of independent and centered random variables, whose $P_n^{\zeta, \eta}$ -moment generating function is finite in a neighborhood of zero. Thus, the central limit theorem entails that for i large enough we have $P_n^{\zeta, \eta}(\mathcal{G}'_i) \geq 1/2$ for all relevant choices of j . Moreover,

$$P_n^{\zeta, \eta}(\mathcal{G}_j \mid \beta_{r_j} = 2L) \geq P_n^{\zeta, \eta}\left(\widehat{H}_{s_j} - \widehat{H}_{r_j} \geq 5L/2, \inf_{r_j \leq k \leq s_j} (\widehat{H}_k - \widehat{H}_{r_j}) \geq -L/8\right).$$

We see that both events are nondecreasing in the (independent) increments of \widehat{H} . By Harris' inequality ([5, Theorem 2.15]) we get

$$\begin{aligned} &P_n^{\zeta, \eta}\left(\widehat{H}_{s_j} - \widehat{H}_{r_j} \geq 5L/2, \inf_{r_j \leq k \leq s_j} (\widehat{H}_k - \widehat{H}_{r_j}) \geq -L/8\right) \\ &\geq P_n^{\zeta, \eta}(\widehat{H}_{s_j} - \widehat{H}_{r_j} \geq 5L/2) \cdot P_n^{\zeta, \eta}\left(\inf_{r_j \leq k \leq s_j} (\widehat{H}_k - \widehat{H}_{r_j}) \geq -L/8\right). \end{aligned}$$

Recalling $s_j - r_j \leq t_i - t_{i-1} = \frac{L^2}{2a^2}$, a Gaussian scaling yields $P_n^{\zeta, \eta}\left(\inf_{r_j \leq k \leq s_j} (\widehat{H}_k - \widehat{H}_{r_j}) \geq -L/8\right) \geq c_7 > 0$. To bound the first factor, we recall that by (A.10) and (A.12), we have \mathbb{P} -a.s.

$$0 \leq \frac{\overline{\rho}_{r_j+l}^{(r_j+k+m)}}{\rho_{r_j+l}} - \rho_{r_j+l} \leq c_8(\psi(m/2) + e^{-m/c_8}) \quad \text{for all } l \leq k \leq t_i/2.$$

Because $m = \frac{L}{16M_R}$ and $t_i/2 = \frac{L^2}{2a^2}$, due to (MIX) we finally get for all i (and thus L) large enough (due to $\psi(x) \cdot x \rightarrow 0$ ($x \rightarrow \infty$), which itself is due to summability of $\psi(k)$), that

$$\begin{aligned} 0 &\leq \left(\overline{R}_{r_j+k}^{(r_j+k+m)} - \overline{R}_{r_j}^{(r_j+k+m)}\right) - (\widehat{R}_{r_j+k} - \widehat{R}_{r_j}) = \sum_{l=1}^k (\overline{\rho}_{r_j+l}^{(r_j+k+m)} - \rho_{r_j+l}) \\ &\leq c_8 L^2 (\psi(L/16M_R) + e^{-L/c_8}) \leq L/16 \quad \text{for all } k \in \left\{0, \dots, \frac{L^2}{2a^2}\right\}. \end{aligned} \tag{4.30}$$

By $s_j - r_j \leq t_i - t_{i-1} = \frac{L^2}{2a^2}$ and (4.26), we see that $s_j - r_j \geq L/16$ for all i large enough. Recall that under $P_n^{\zeta, \eta}$, the sequence $\widehat{H}_{s_j} - \widehat{H}_{r_j}$ is a sum of independent centered random variables, whose moment generating function is uniformly bounded in a neighborhood of the origin. Then by [39, (2)], we can apply [39, Theorem 4] in the following manner: Let $c' > 0$ be as in [39, Theorem 4] and in the notation of the

latter theorem we choose $k = s_j - r_j$, $\alpha = \text{ess inf}_{\zeta, k < n} E_n^{\zeta, \eta} [(\widehat{H}_k - \widehat{H}_{k-1})^2] > 0$, $M = |\eta|/2$, $u_1 = \dots = u_k = 1$ and

$$\bar{c} := c' \cdot \frac{|\eta|}{2} \cdot \alpha \tag{4.31}$$

Then a lower Bernstein-type inequality from [39, Theorem 4] gives that on $B_i^{(n)}$ we have

$$P_n^{\zeta, \eta}(\widehat{H}_{s_j} - \widehat{H}_{r_j} \geq 5L/2) \geq c_9 e^{-\frac{L^2}{c_9(s_j - r_j)}}. \tag{4.32}$$

Note that the condition in $B_i^{(n)}$ makes [39, Theorem 4] applicable by ensuring 'enough' summands $s_j - r_j$ and is the main reason we have to introduce the sets $B_i^{(n)}$. We will write $c = c_7 \wedge c_9$ from now on. Using (4.28) in combination with the lower bounds for the factors of (4.29) just derived, the term in (4.23) can be bounded from above by

$$\begin{aligned} \mathbb{E}[e^{-\theta \ln(Z_i)}] &\leq c_{10} \mathbb{E}\left[\exp\left\{\theta J \ln(2/c) + \theta \sum_{j=0}^J \frac{L^2}{c(s_j - r_j)}\right\}\right] \\ &\leq c_{10} \sum_{k=0}^{\infty} \mathbb{E}\left[\left(\frac{2}{c}\right)^{\theta k} \exp\left\{\theta \frac{L^2}{c(s_k - r_k)} \mathbb{1}_{s_k - r_k = t_i - t_{i-1}} + \theta \sum_{j=0}^{k-1} \frac{L^2}{c(s_j - r_j)} \mathbb{1}_{s_j - r_j < t_i - t_{i-1}}\right\}\right] \\ &\leq c_{10} \sum_{k=0}^{\infty} \left(\frac{2}{c}\right)^{\theta k} (1 + \psi(m))^k e^{\frac{2\theta L^2}{c t_{i-1}}} \cdot \prod_{j=0}^{k-1} \mathbb{E}\left[\exp\left\{\frac{\theta L^2}{c(s_j - r_j)}\right\} \mathbb{1}_{s_j - r_j < t_i/2}\right], \end{aligned} \tag{4.33}$$

where we recall m from (4.26), and the last inequality is due to (4.27) in combination with $t_i - t_{i-1} = t_i/2$. For the latter expectation we have

$$\begin{aligned} \mathbb{E}\left[\exp\left\{\frac{\theta L^2}{c(s_j - r_j)}\right\} \mathbb{1}_{s_j - r_j < t_i/2}\right] &= \int_0^{\infty} \mathbb{P}\left(e^{\frac{\theta L^2}{c(s_j - r_j)}} \mathbb{1}_{s_j - r_j < t_i/2} \geq x\right) dx \\ &\leq e^{\frac{2\theta L^2}{c t_i}} \mathbb{P}(s_j - r_j < t_i/2) + \int_e^{\infty} \mathbb{P}\left(e^{\frac{\theta L^2}{c(s_j - r_j)}} \geq x\right) dx. \end{aligned} \tag{4.34}$$

Substituting $x = e^{\frac{\theta L^2}{c y}}$, the second summand can be written as

$$\int_0^{t_i/2} \frac{\theta L^2}{c y^2} e^{\frac{\theta L^2}{c y}} \mathbb{P}(s_j - r_j \leq y) dy.$$

In order to obtain an upper bound, we start with the probability inside the integral and get

$$\begin{aligned} \mathbb{P}(s_j - r_j \leq y) &= \mathbb{P}\left(\max_{1 \leq k \leq y} \sum_{l=1}^k \bar{\rho}_{r_j+l}^{(r_j+k+m)} \geq L/8\right) \leq \mathbb{P}\left(\max_{1 \leq k \leq y} \sum_{l=1}^k \rho_{r_j+l} \geq L/16\right) \\ &\leq c_{11} e^{-\frac{L^2}{c_{11} y}}, \quad \forall y \in [0, t_i/2]; \end{aligned}$$

here, the first inequality is due to (4.30) and the last inequality due to (4.18).

Putting these bounds together, the second summand in (4.34) can be bounded from above by

$$\begin{aligned} \int_0^{t_i/2} \frac{\theta L^2}{cy^2} e^{\frac{\theta L^2}{cy}} c_{10} e^{-\frac{L^2}{yc_{10}}} dy &\leq c_{11} \int_0^{t_i/2} \frac{\theta L^2}{cy^2} e^{\frac{L^2}{cy}(\theta - c_{12})} dy \leq c_{13} \int_0^{1/2a^2} \frac{\theta}{z^2} e^{\frac{1}{z^2}(\theta - c_{12})} dz \\ &\leq c_{14} \theta \int_0^\infty e^{\frac{x(\theta - c_{11})}{c}} dx. \end{aligned}$$

Now the latter term can be made arbitrarily close to zero by choosing $\theta > 0$ small enough. Furthermore, again choosing $\theta > 0$ small enough, the first term in (4.34) is strictly smaller than one by the central limit theorem from Claim 4.3. Thus, $\theta > 0$ can be chosen small enough such that for all i large enough, the sum on the right-hand side in (4.33) converges, with a finite upper bound independent of i .

To show $Y^{(n)} \geq n^{-\gamma''}$, we adapt the strategy in the proof of (4.22), i.e. set $L := 2a\bar{C} \ln n$, $r_0 = \lfloor \bar{C} \ln n \rfloor$ and $J := \inf\{j : s_j \geq \lfloor (\bar{C} \ln n)^2 \rfloor\}$ and keep the other definitions as in (4.25) and (4.26). Then by the same argument below display (4.23), $Y^{(n)} \geq n^{-\gamma''}$ for some suitable $\gamma'' > 0$ follows if $\mathbb{E}[e^{-\theta Y^{(n)}}] \leq c$ for some constant $c > 0$, some small $\theta > 0$ and all n large enough. But this follows (as in the argument leading to the definition of $B_i^{(n)}$), if the process $(\widehat{R}_k - \widehat{R}_{k-1})_k$ does not decrease too fast, see the Borel–Cantelli argument below display (4.32), which itself is a consequence of

$$\sum_n (\bar{C} \ln n)^2 \cdot \mathbb{P} \left(\sup_{0 \leq k \leq L/8\bar{c}} (\widehat{R}_k - \widehat{R}_0) \geq \frac{L}{8} \right) \leq c_{15} \sum_n (\bar{C} \ln n)^2 e^{-a\bar{C} \ln n / c_{15}} < \infty. \tag{4.35}$$

This completes the proof of the second inequality in Lemma 4.2 (a).

We will now explain how to adapt the latter arguments for the proof of the first inequality in (a). We define $\beta_k = \bar{\beta}_k^{(1,n)} - \bar{\beta}_{n-1}^{(1,n)}$. In the definition of $Z_i^{(n)}$, we have to take the essential infimum over $\bar{\xi}(n - t_{i-2})$ and have to replace the subscripts k of β_k by $n - k$, i.e. “running backwards” from n . Thus, $Z_i^{(n)}$ is $(\mathcal{F}^{n-t_i} \cap \mathcal{F}_{n-t_{i-2}})$ -measurable. It is then enough to consider the case $n = 0$ due to the argument in (4.24). Writing $Z_i := Z_i^{(0)}$, $\beta_k := \beta_k^{(0)}$, $\widehat{H}_k := H_k - E_0^{\zeta, \eta}[H_k]$, $\widehat{R}_k := \widehat{R}_k^{(0)}$ and defining

$$\bar{\rho}_k^{(j)} := \operatorname{ess\,sup}_{\bar{\xi}^{(j)}} \rho_k, \quad \bar{R}_k^{(j)} := \sum_{l=k+1}^0 \bar{\rho}_l^{(j)}, \quad k < 0, \quad k \in \mathbb{Z},$$

we have to adapt the definitions of r_j and s_j by the expressions

$$\begin{aligned} r_0 &:= -t_{i-1}, \quad s_0 := \left(\sup \{k \leq r_0 : \widehat{R}_k - \widehat{R}_{r_0} \geq L/8\} + 1 \right) \vee (-t_i), \\ r_j &:= s_{j-1} - \left\lceil \frac{L}{8M_R} \right\rceil, \quad s'_j := s_{j-1} - \frac{L}{16M_R}, \quad j \geq 1, \\ s_j &:= \left(\sup \{k \leq r_j : \bar{R}_k^{(s'_j)} - \bar{R}_{r_j}^{(s'_j)} \geq L/8\} + 1 \right) \vee (r_j - (t_i - t_{i-1})), \quad j \geq 1, \\ J &:= \inf\{j : s_j - r_j = -t_i\} \vee \sup\{j : s_j \leq -t_i\}. \end{aligned}$$

The remaining part of the proof essentially follows the same steps as for the second inequality in (a).

Proof of Claim 4.3. Boundedness, stationarity and adaptedness are direct consequences of the corresponding properties of the sequences $(L_i^\xi(\eta))_{i \in \mathbb{Z}}$ and $((L_i^\xi)'(\eta))_{i \in \mathbb{Z}}$ and Lemma A.1. Display (4.17) is due to Lemma A.2. To show that the central limit theorem, we note that the sequence $(\rho_i)_{i \in \mathbb{Z}}$ fulfills the same conditions as the sequence $(\tilde{L}_i)_{i \in \mathbb{Z}}$ in the proof of Lemma 3.2

4.2. Second moment of leading particles

Recall the notation $N_t^{\mathcal{L}}$ from below (4.2) and that of $\bar{m}(t)$ from (1.7). For the second moment of the leading particles we now prove the following upper bound.

Lemma 4.4. *For every function F fulfilling (PROB) and for every $a > 0$, there exists $\gamma_2 = \gamma_2(F, a) < \infty$ such that \mathbb{P} -a.s., for all t large enough,*

$$\sup_{x \in [\bar{m}^a(t) - 1, \bar{m}^a(t) + 1]} \mathbb{E}_x^\xi [(N_t^{\mathcal{L}, a})^2] \leq t^{\gamma_2}. \tag{4.36}$$

Proof. We omit the superscript a in the quantities involved and use the same abbreviations as in the beginning of the proof of Lemma 4.1.

We want to show (4.36) with the help of the second-moment formula (FK-2). To this end, define the function $\varphi_t^\xi : [0, t] \rightarrow \mathbb{Z} \cup \{-\infty\}$,

$$\varphi_t(s) := \varphi_t^\xi(s) := \lfloor \bar{m}(t) \rfloor \wedge \sup \{k \in \mathbb{Z} : s \in [t - T_{k+1} - 5\chi_1(\bar{m}(t)), t - T_k - 5\chi_1(\bar{m}(t))]\},$$

where $\sup \emptyset := -\infty$ and χ_1 has been defined in (4.2). Due to $T_k = 0$ for all $k \leq 0$ [recall the notation T_k from (3.43) and (3.44)], we have $1 \leq \varphi_t(0) \leq \lfloor \bar{m}(t) \rfloor$. Furthermore, $\varphi_t(t) = -\infty$, because $T_k \geq 0$ and $\chi_1^\xi(\bar{m}(t)) \geq 0$. To apply (FK-2), the following upper bound will prove useful.

Claim 4.5. *We have*

$$N_t^{\mathcal{L}} \leq |\{v \in N(t) : X_t^v \leq 0, X_s^v > \varphi_t(s) \forall s \in [0, t]\}| \tag{4.37}$$

and \mathbb{P} -a.s. for all t large enough, the function $[0, t] \ni s \mapsto \varphi_t(s)$ is a non-increasing, càdlàg step function.

In order not to hinder the flow of reading, we postpone the proof of the latter claim to the end of the proof of Lemma 4.4 (Figs. 5 and 6).

By the Feynman–Kac formula (cf. Proposition 2.3) and (4.37), we have

$$\begin{aligned} \mathbb{E}_x^\xi [(N_t^{\mathcal{L}})^2] &\leq \mathbb{E}_x^\xi [N_t^{\mathcal{L}}] + (m_2 - 2) \int_0^t E_x \left[e^{\int_0^s \xi(B_r) dr} \xi(B_s) \mathbb{1}_{\{B_r \geq \varphi_t(r) \forall r \in [0, s]\}} \right. \\ &\quad \left. \times \left(E_y \left[e^{\int_0^{t-s} \xi(B_r) dr} \mathbb{1}_{\{B_r \geq \varphi_t(r+s) \forall r \in [0, t-s]\}, B_{t-s} \leq 0} \right] \right)^2 \right] ds. \end{aligned} \tag{4.38}$$

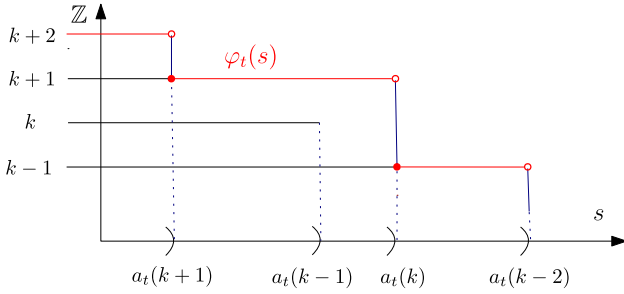


Fig. 5. Illustration of φ_t , which is the red line. We denote $a_t(k) := t - T_k - 5\chi^\xi(\bar{m}(t))$. Note that the sequence $(a_t(k))_{k \in \mathbb{Z}}$ does not have to be monotone and thus the interval $[a_t(k+1), a_t(k)]$ might be empty. In this case the graph of φ_t jumps at least two steps at time $s = a_t(k)$

For the first summand we have

$$\sup_{x \in [\bar{m}(t)-1, \bar{m}(t)+1]} E_x^\xi [N_t^\zeta] \leq \sup_{x \in [\bar{m}(t)-1, \bar{m}(t)+1]} E_x^\xi [N^{\leq}(t, 0)] \leq c_1 E_{\bar{m}(t)+1}^\xi [N^{\leq}(t, 0)] \leq \frac{c_1}{2}, \tag{4.39}$$

where the first inequality is due to the first inequality in (3.41) and the last one due to the definition of $\bar{m}(t)$. Recall that the Markov property provides us with

$$E_x \left[e^{\int_0^s \xi(B_r) dr} E_y \left[e^{\int_0^{t-s} \xi(B_r) dr} \mathbf{1}_{\{B_{t-s} \leq 0\}} \right] \Big|_{y=B_s} \right] = E_x^\xi [N^{\leq}(t, 0)].$$

Using $\xi \leq es$ and the two previous displays, the second summand in (4.38) can thus be bounded from above by

$$\begin{aligned} & es \cdot (m_2 - 2) \sup_{x \in [\bar{m}(t)-1, \bar{m}(t)+1]} E_x^\xi [N^{\leq}(t, 0)] \cdot \int_0^t \sup_{y \geq \varphi_t(s)} E_y \left[e^{\int_0^{t-s} \xi(B_r) dr}; B_{t-s} \leq 0 \right] ds \\ & \leq \frac{es(m_2 - 2)c_1}{2} \int_0^t \sup_{y \geq \varphi_t(s)} E_y^\xi [N^{\leq}(t - s, 0)] ds. \end{aligned} \tag{4.40}$$

It thus suffices to upper bound $\sup_{y \geq \varphi_t(s)} E_y^\xi [N^{\leq}(t - s, 0)]$ by a polynomial in t . We treat different areas for s and y separately and we will need an additional claim, the proof of which will be provided after this proof. It guarantees that the assumptions of the time perturbation Lemma 3.11 are satisfied in our setting.

Claim 4.6. *There exists $\tilde{C}_1 \in (0, \infty)$ such that \mathbb{P} -a.s. for all t large enough and all $y \geq \tilde{C}_1 \chi_1(\bar{m}(t))$ we have $\frac{y}{T_y - 1}, \frac{y}{T_y + K + 5\chi_1(\bar{m}(t))} \in V$, where V is defined in (2.22). Furthermore, there exists $\tilde{C}_2 = \tilde{C}_2(\tilde{C}_1) \in (0, \infty)$ such that \mathbb{P} -a.s. for all t large enough and all $s \in [0, t - \tilde{C}_2 \chi_1(\bar{m}(t))]$ we have $\varphi_t(s) \geq \tilde{C}_1 \chi_1(\bar{m}(t))$.*

We choose $\gamma' > 5C_{11}C_{14}$. Then, recalling the definition of χ_1 from (4.2) and that \mathbb{P} -a.s. $\frac{\bar{m}(t)}{t} \rightarrow v_0$ by Corollary 3.15, for t large enough, the statements from Claim 4.6

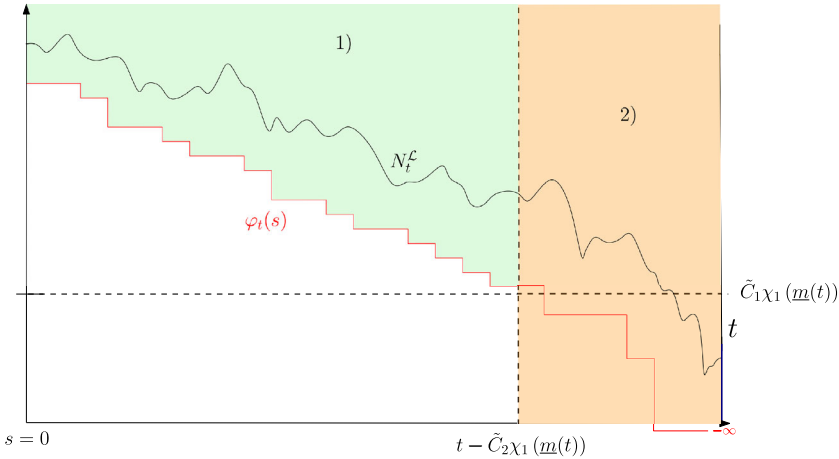


Fig. 6. Leading particles and the different areas in the proof of Lemma 4.4

hold true, we have $C_{11}e^{C_{11}(\bar{K}+1+5\chi_1(\bar{m}(t)))} \leq t^{\gamma'}$ and also $T_{\lfloor y \rfloor + 1} \leq T_y + \bar{K}$ for all $y \geq \tilde{C}_1 \chi_1(\bar{m}(t))$ by Corollary 3.17.

(1) Let $s \in [0, t - \tilde{C}_2 \chi_1(\bar{m}(t))]$ and $y \geq \varphi_t(s)$. Then by Claim 4.6, $y \geq \tilde{C}_1 \chi_1(\bar{m}(t))$ and thus $\frac{y}{T_y - 1}, \frac{y}{T_y + \bar{K} + 5\chi_1(\bar{m}(t))} \in V$. By definition of φ_t we have $s \geq t - T_{\lfloor y \rfloor + 1} - 5\chi_1(\bar{m}(t))$ and thus $T_y + \bar{K} + 5\chi_1(\bar{m}(t)) \geq T_{\lfloor y \rfloor + 1} + 5\chi_1(\bar{m}(t)) \geq t - s$. Thus, by Lemma A.6, we infer that $\mathbb{E}_y^\xi [N \leq (t - s, 0)] \leq 2\mathbb{E}_y^\xi [N \leq (T_y + \bar{K} + 5\chi_1(\bar{m}(t)), 0)]$, and then the second inequality in (3.34) entails that for all t large enough,

$$\sup_{\substack{0 \leq s < t - \tilde{C}_2 \chi_1(\bar{m}(t)) \\ y \geq \varphi_t(s)}} \mathbb{E}_y^\xi [N \leq (t - s, 0)] \leq 2C_{11}e^{C_{11}(\bar{K}+1+5\chi_1(\bar{m}(t)))} \sup_{y \in \mathbb{R}} \mathbb{E}_y^\xi [N \leq ((T_y - 1) \vee 0, 0)] \leq t^{\gamma'}. \tag{4.41}$$

(2) The remaining part of the domain above the graph of φ_t not controlled by (1) is a subset of

$$\{(s, y) \in [0, t] \times \mathbb{R} : t - \tilde{C}_2 \chi_1(\bar{m}(t)) \leq s \leq t\}.$$

Recalling the definition of χ_1 from (4.2) and that \mathbb{P} -a.s., $\frac{\bar{m}(t)}{t} \rightarrow v_0$ by Corollary 3.15, choosing $\gamma'' > \varepsilon s \tilde{C}_2 C_{14}$, on the the above domain we get that \mathbb{P} -a.s., for all t large enough,

$$\mathbb{E}_y^\xi [N \leq (t - s, 0)] \leq 2\mathbb{E}_y^\xi [N \leq (\tilde{C}_2 \chi_1(\bar{m}(t)), 0)] \leq 2e^{\varepsilon s \tilde{C}_2 \chi_1(\bar{m}(t))} \leq t^{\gamma''}. \tag{4.42}$$

To conclude the proof, defining $\gamma_2 := 1 \vee \gamma' \vee \gamma'' + 1$, inequalities (4.39), (4.40) and the estimates (4.41) and (4.42) for the term $\mathbb{E}_y^\xi [N \leq (t - s, 0)]$ entail the statement of Lemma 4.4. \square

Proof of Claim 4.5. Let $a_t(k) := t - T_k - 5\chi_1^{\xi}(\bar{m}(t))$ and recall the definition of $N_t^{\mathcal{L}}$:

$$N_t^{\mathcal{L}} = \left\{ v \in N(t) : X_t^v \leq 0, H_k^v \geq a_t(k) \forall k \in \{1, \dots, \lfloor \bar{m}(t) \rfloor\} \right\}.$$

To prove (4.37), note that $H_k^v \geq a_t(k)$ if and only if $X_s^v > k$ for all $s < a_t(k)$. But the property $X_s^v > k$ for all $s \in [0, a_t(k))$ and all $k \in \{1, \dots, \lfloor \bar{m}(t) \rfloor\}$ implies that $X_s^v > \sup\{k \in \mathbb{Z} : s \in [a_t(k+1), a_t(k))\} \wedge \lfloor \bar{m}(t) \rfloor = \varphi_t(s)$ for all $s \in [0, t]$ and thus (4.37) is shown. The property of φ_t being a càdlàg step-function is a direct consequence of the use of left-closed, right-open intervals in the definition of φ_t . It remains to show that $s \mapsto \varphi_t(s)$ is non-increasing. For this purpose, let us first prove by induction in $k = \lfloor \bar{m}(t) \rfloor, \lfloor \bar{m}(t) \rfloor - 1, \dots$ that for all t large enough and all $k \leq \bar{m}(t)$,

$$[0, a_t(k-1)) \subset \bigcup_{l=k}^{\lfloor \bar{m}(t) \rfloor} [a_t(l), a_t(l-1)) \tag{4.43}$$

holds. By Corollary 3.17 and Lemma 3.16, there exist $\bar{K}, C_{15} > 0$ such that $t - T_{\lfloor \bar{m}(t) \rfloor - 1} \leq t - T_{\bar{m}(t)} + \bar{K} \leq C_{15} + \bar{K}$ and thus $a_t(\lfloor \bar{m}(t) \rfloor) \leq 0$ for all t large enough. Assume now that (4.43) holds for some $k \leq \bar{m}(t)$. Then

$$[0, a_t(k-2)) \subset [0, a_t(k-1)) \cup [a_t(k-1), a_t(k-2)) \subset \bigcup_{l=k-1}^{\lfloor \bar{m}(t) \rfloor} [a_t(l), a_t(l-1)),$$

where the last inclusion is due to induction hypothesis. Thus, we have shown (4.43). Now let $0 \leq s_1 \leq s_2$. Assume there exists k_2 such that $s_2 \in [a_t(k_2), a_t(k_2-1))$. Then by (4.43), there exists $k_1 \in \mathbb{Z}$ with $k_2 \leq k_1 \leq \lfloor \bar{m}(t) \rfloor$, such that $s_1 \in [a_t(k_1), a_t(k_1-1))$. By definition we get $\varphi_t(s_1) \geq \varphi_t(s_2)$. If no such k_2 exists, then $\varphi_t(s_2) = -\infty \leq \varphi_t(s_1)$.

Proof of Claim 4.6. We write $V = [v_*, v^*]$. Since \mathbb{P} -a.s. we have $\frac{y}{T_y} \rightarrow v_0 \in \text{int}(V)$ by (3.46), it follows that $\frac{y}{T_y - 1}, \frac{y}{T_y} \in V$ for all y large enough. Among others, there exists $\varepsilon = \varepsilon(v_*, v^*, v_0) > 0$ and $\mathcal{N}'(\xi)$ such that $v_*(1 + \varepsilon) \leq \frac{y}{T_y} \leq (1 - \varepsilon)v^*$ for all $y \geq \mathcal{N}'$. Choosing $\tilde{C}_1 > \frac{5v^*}{\varepsilon}$, this implies $1 \leq \frac{T_y + \bar{K} + 5\chi_1(\bar{m}(t))}{T_y} \leq 1 + \varepsilon$ for all $y \geq \tilde{C}_1 \cdot \chi_1(\bar{m}(t))$ and all t large enough. Thus, we get

$$v_* \leq \frac{y}{T_y} \cdot \frac{T_y}{T_y + \bar{K} + 5\chi_1(\bar{m}(t))} \leq v^* \quad \text{for all } y \geq \tilde{C}_1 \chi_1(\bar{m}(t)) \text{ and all } t \text{ large enough.}$$

This gives the first part of the Claim 4.6. For the second part, recall that $T_y \leq \frac{1}{v}y$ for all $y \geq \mathcal{N}'$. Furthermore, by the definition of φ_t we have $\varphi_t(s) \geq \lfloor y \rfloor + 1$ for all $s \in [0, t - T_{\lfloor y \rfloor} - 5\chi_1(\bar{m}(t))]$. Choosing $y := \lfloor \tilde{C}_1 5\chi_1(\bar{m}(t)) \rfloor$ and $\tilde{C}_2 > \frac{\tilde{C}_1}{v} + 1$, this implies that for t large enough we get $\varphi_t(s) \geq \tilde{C}_1 5\chi_1(\bar{m}(t))$ for all $s \in [0, t - \tilde{C}_2 5\chi_1(\bar{m}(t))]$.

4.3. Proof of Theorem 1.5

Recall $\bar{m}(t) = \sup\{x \in \mathbb{R} : u(t, x) \geq 1/2\}$ and $m(t) = \sup\{x \in \mathbb{R} : w(t, x) \geq 1/2\}$, where u and v are the solutions to (PAM) and w to (F-KPP), respectively. We start with an amplification result.

Lemma 4.7. *For every $(p_k)_{k \in \mathbb{N}}$ fulfilling (PROB) there exists $C_{17} = C_{17}((p_k)) > 1$ and $t_0 > 0$ such that \mathbb{P} -a.s., for all $t \geq t_0$,*

$$\sup_{x \in \mathbb{R}} \mathbb{P}_x^{\xi} (N(t, [x - 1, x + 1]) \leq C_{17}^t) \leq C_{17}^{-t}.$$

Proof. For the proof it is enough to show the claim for binary branching with rate $\xi(x) \equiv e^{i'}$:= $e^{i'}(1 - p_1)$ (which is the rate of first branching into more than one particle) by a straightforward coupling argument. Due to the spatial homogeneity of $e^{i'}$ it is enough to show

$$\mathbb{P}_0^{e^{i'}} (N(t, [-1, 1]) \leq C_{17}^t) \leq C_{17}^{-t}$$

for all $t \geq t_0$, where $\mathbb{P}_0^{e^{i'}}$ is the probability measure under which the branching Brownian motion starts with one particle in 0 and has constant branching rate $e^{i'} > 0$. Then for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\mathbb{P}_0^{e^{i'}} (|N(t/3, [-\varepsilon, \varepsilon])| \geq \delta t) \geq 1 - e^{-\delta t} \tag{4.44}$$

for all t large enough. Indeed, the probability that the initial particle does not leave the interval $[-\varepsilon t/2, \varepsilon t/2]$ before time $t/3$ is at least $1 - e^{-c_1 \varepsilon^2 t}$. If this happens, the particle produces more than $t e^{i'}/4$ offsprings with probability $1 - e^{-c_2 t}$ before time $t/3$, while each of these offsprings does not leave the interval $[-\varepsilon t, \varepsilon t]$ before time $t/3$ with probability at least $1 - e^{-c_1 \varepsilon^2 t/2}$. Combining these observations and choosing $\delta(\varepsilon) > 0$ small enough provides us with (4.44). For a particle $v \in N(t/3)$ let $D^v(t/3 + s)$ be the set of offsprings of v in the interval $[X_{t/3}^v - 1, X_{t/3}^v + 1]$ at time $t/3 + s$, $s \geq 0$. We will show the existence of some $p > 0$ and $c > 1$ such that

$$\mathbb{P}_0^{e^{i'}} (|D^v(t/3 + s)| \geq c^s) \geq p \tag{4.45}$$

for all s large enough. To obtain (4.45), let $r > 0$ be such that

$$\inf_{y \in [-1, +1]} \mathbb{E}_y^{e^{i'}} [N(r, [-1, +1])] =: \mu > 1,$$

(the feasibility of such a choice of r is a direct consequence of the Feynman–Kac formula). For $v \in D_\varepsilon(t/3)$ consider the following process under $\mathbb{P}_0^{e^{i'}}$, conditionally on $X_{t/3}^v$:

- the process starts with one particle at position $X_{t/3}^v$;
- between times $r(n - 1)$ and rn , $n \in \mathbb{N}$, the process evolves as a branching Brownian motion with branching rate $e^{i'}$;
- at times rn , $n \in \mathbb{N}$, particles outside of the interval $[X_{t/3}^v - 1, X_{t/3}^v + 1]$ are killed.

Using the Markov property, one readily observe that the number of particles of the latter process stochastically dominates the number of particles of a Galton-Watson process $(L_n)_{n \in \mathbb{N}}$ which starts with one particle and whose offspring distribution has expectation μ . Then by [2, Theorem 1, section I.5], the Galton-Watson process has positive probability to survive, i.e. $P(L_n > 0 \forall n \in \mathbb{N}) =: p_1 > 0$. Conditioned on survival, there exists $c > 1$ such that

$$P(L_k \geq c^k \mid L_n > 0 \forall n \in \mathbb{N}) \geq \frac{1}{2} \text{ for all } k \in \mathbb{N}.$$

One can see that for every $v \in N(t/3)$, inequality (4.45) holds true with the choice $p := p_1/2$ for all $s \in r \cdot \mathbb{N}$. By a straightforward comparison argument, this extends to all $s \geq 0$. Therefore, we can now apply (4.44) and (4.45) in order to deduce

$$P_0^{e_1'}(N(2t/3, [-\varepsilon t, \varepsilon t]) \geq c^t) \geq 1 - e^{c'(\varepsilon)t}. \tag{4.46}$$

Furthermore, we have

$$P_{\varepsilon t}(X_{t/3} \in [-1, 1]) \geq c_3 t^{-1/2} e^{-3\varepsilon^2 t/2} \geq \left(\frac{1+c}{2}\right)^{-t/3},$$

for all t large enough $\varepsilon > 0$ small enough and $c > 1$ suitable, where for the last inequality we used that ε does not depend on c . The latter inequality and a large deviation statement then gives for all $t \geq t_4 \geq t_3$

$$P_0^{e_1'}(N(t, [-1, 1]) \geq \frac{1}{2} c^t \left(\frac{1+c}{2}\right)^{-t/3} \mid N(2t/3, [-\varepsilon t, \varepsilon t]) \geq c^t) \geq 1 - e^{-c_4 t}. \tag{4.47}$$

Thus, for $t \geq t_0$, where t_0 is chosen large enough, by (4.46) and (4.47), in combination with $c > 1$, we infer the desired result.

With the help of Lemmas 4.1, 4.4, and 4.7, it is now possible to state a crucial result for the proof of Theorem 1.5.

Proposition 4.8. *For every $q > 0$, F satisfying (PROB) and $a > 0$, there exist a constant $C_1 = C_1(q, F) \in (0, \infty)$, a \mathbb{P} -a.s. finite $C = C(t) = C(t, q, F, \xi) > 0$ and a \mathbb{P} -a.s. finite random variable $\mathcal{T}_5 = \mathcal{T}_5(a, q, F, \xi)$ such that for all $t \geq \mathcal{T}_5$, we have $C(t) \leq C_1$ and*

$$P_{\bar{m}^a(t) - C \ln t}^\xi(N^{\leq}(t, 0) \neq \emptyset) \geq 1 - 2t^{-q}.$$

Proof. For simplicity, we write $\bar{m}(t) := \bar{m}^a(t)$. Without loss of generality, it is enough to show the claim for all $q > 2(\gamma_1 + \gamma_2)$, where $\gamma_1 = \gamma_1(a)$ and $\gamma_2 = \gamma_2(a)$ are defined in Lemmas 4.1 and 4.4, respectively. Let further C_{17} and t_0 be as in Lemma 4.7, and c_1 be such that for $r := c_1 \ln t$ we have $C_{17}^{-r} = t^{-q}$.

We claim that there exist $C_1, C(t)$ and \mathcal{T}_5 as above such that $\bar{m}(t - r) = \bar{m}(t) - C(t) \ln t$, $C(t) \leq C_1$ and the conclusions of Lemmas 4.1 and 4.4 hold for all $t \geq \mathcal{T}_5$. Indeed, writing $u(t, x) = \mathbb{E}_x^\xi[N^{\leq}(t, 0)]$, by the time and space perturbation Lemmas 3.11 and 3.12, defining $c_2 := C_{11} \vee C_{12}$, we deduce that

$$u(t - r, \bar{m}(t) - C_1 \ln t) \geq c_2^{-1} e^{C_1 \ln t / c_2} u(t - r, \bar{m}(t)) \geq c_2^{-2} e^{C_1 \ln t / c_2 - r / c_2} u(t, \bar{m}(t)) \geq a, \tag{4.48}$$

where for the last inequality we choose C_1 and $\bar{\mathcal{T}}_5 = \bar{\mathcal{T}}_5(\varepsilon)$ large enough such that the last inequality holds for all $t \geq \bar{\mathcal{T}}_5$. As a consequence, we infer

$$\bar{m}(t - r) \geq \bar{m}(t) - C_1 \ln t. \tag{4.49}$$

By (3.34), we also get $u(t, \bar{m}(t - r)) \geq C_{11}^{-1} e^{r/C_{11}} u(t - r, \bar{m}(t - r)) \geq a$ for all t large enough, i.e.

$$\bar{m}(t - r) \leq \bar{m}(t).$$

Thus, combining the two previous displays, we can find $C(t) \leq C_1$ such that $\bar{m}(t - r) = \bar{m}(t) - C(t) \ln t$. Now let $x := \bar{m}(t) - C(t) \ln t = \bar{m}(t - r)$. Conditioning on whether until time r there are more or less than C_{17}^r particles in $[x - 1, x + 1]$, we get for all $t \geq \bar{\mathcal{T}}_5$,

$$\begin{aligned} \mathbb{P}_x^\xi \left(N^{\leq}(t, 0) \geq 1 \right) &\geq 1 - \mathbb{P}_x^\xi \left(N(r, [x - 1, x + 1]) \leq C_{17}^r \right) \\ &\quad - \sup_{y \in [x-1, x+1]} \left(\mathbb{P}_y^\xi \left(N^{\leq}(t - r, 0) = 0 \right) \right)^{C_{17}^r} \\ &\geq 1 - C_{17}^{-r} - \sup_{y \in [x-1, x+1]} \left(\mathbb{P}_y^\xi \left(N_{t-r}^{\mathcal{L}, a} = 0 \right) \right)^{C_{17}^r}, \end{aligned}$$

using Lemma 4.7 in the last inequality. Now using Cauchy–Schwarz as in (4.3), in combination with Lemmas 4.1 and 4.4, we infer

$$\sup_{y \in [x-1, x+1]} \left(\mathbb{P}_y^\xi \left(N_{t-r}^{\mathcal{L}, a} = 0 \right) \right)^{C_{17}^r} \leq \left(1 - t^{-2\gamma_1 - \gamma_2} \right)^{r^q} \leq t^{-q},$$

where we adapt $\bar{\mathcal{T}}_5 = \bar{\mathcal{T}}_5(a, q, \xi, q)$ such that the last inequality holds for all $t \geq \bar{\mathcal{T}}_5$.

Proof of Theorem 1.5. (1) We first prove the result under the additional assumption that F fulfills (PROB). Let w^{ξ, F, w_0} be the solution to (F-KPP) with initial condition $w_0 \in \mathcal{I}_{F\text{-KPP}}$, so in particular $0 \leq w_0 \leq \mathbb{1}_{(-\infty, 0]}$. Because F fulfills (PROB), by (McKean) and the Markov property we infer

$$\begin{aligned} w^{\xi, F, w_0}(t, x) &= \mathbb{E}_x^\xi \left[1 - \prod_{v \in N(t)} (1 - w_0(X_t^v)) \right] \\ &\geq \mathbb{E}_x^\xi \left[1 - \prod_{v \in N(t)} (1 - w_0(X_t^v)); N^{\leq}(t - s, 0) \geq 1 \right] \\ &\geq \mathbb{P}_x^\xi \left(N^{\leq}(t - s, 0) \geq 1 \right) \cdot \inf_{y \leq 0} w^{\text{ei}, F, w_0}(s, y), \end{aligned} \tag{4.50}$$

where $w = w^{\text{ei}, F, w_0}$ solves the homogeneous equation $w_t = \frac{1}{2} w_{xx} + \text{ei} \cdot F(w)$ with initial condition $w(0, \cdot) = w_0$. Then we have $w^{1, F, \tilde{w}_0} = w^{\text{ei}, F, w_0} \left(\frac{t}{\text{ei}}, \frac{x}{\sqrt{\text{ei}}} \right)$ with $\tilde{w}_0(x) := w_0(x/\sqrt{\text{ei}})$. Because $w^{\text{ei}, F, w_0}(0, x) = 0$ for $x > 0$, conditions [7, (8.1) and (1.17)] are fulfilled. Together with (KPP-INI) and [7, Theorem 3, p. 141], w^{1, F, \tilde{w}_0} (and thus also w^{ei, F, w_0}) is a traveling wave solution, i.e., there

exist $m^{\text{ei}}(t) = \sqrt{2\text{ei}t} + o(t)$ and some g fulfilling $\lim_{x \rightarrow -\infty} g(x) = 1$ and $\lim_{x \rightarrow \infty} g(x) = 0$ such that

$$\sup_y |w^{\text{ei},F,w_0}(t, y + m^{\text{ei}}(t)) - g(y)| \rightarrow 0, \quad t \rightarrow \infty. \quad (4.51)$$

Now let $\varepsilon \in (0, 1)$ and choose $\delta > 0$ such that $\frac{\varepsilon}{1-\delta} \in (0, 1)$. Then by (4.51) we get

$$\begin{aligned} \inf_{y \leq 0} w^{\text{ei},F,w_0}(s, y) &= \inf_{y \leq -m^{\text{ei}}(s)} w^{\text{ei},F,w_0}(s, y + m^{\text{ei}}(s)) \\ &\geq 1 - \delta \quad \text{for all } s \geq s_0(F, w_0, \delta, \text{ei}), \end{aligned}$$

which, together with (4.50), gives

$$m^{\xi,F,w_0,\varepsilon}(t) \geq m^{\xi,F,\mathbb{1}_{(-\infty,0]},\frac{\varepsilon}{1-\delta}}(t - s_0) \quad \text{for all } t \geq s_0(F, w_0, \delta, \text{ei}). \quad (4.52)$$

The inequality

$$m^{\xi,F,\mathbb{1}_{(-\infty,0]},\frac{\varepsilon}{1-\delta}}(t - s_0) \geq \overline{m}^{\xi,\mathbb{1}_{(-\infty,0]},\frac{\varepsilon}{1-\delta}}(t - s_0) - C_1 \ln(t), \quad \text{for all } t \geq \mathcal{T}_1(\xi, F, \varepsilon, \delta),$$

follows from Proposition 4.8. By Corollary 3.18, for $C' > 1$ from (PAM-INI) we get

$$\begin{aligned} \overline{m}^{\xi,\mathbb{1}_{(-\infty,0]},\frac{\varepsilon}{1-\delta}}(t - s_0) &\geq \overline{m}^{\xi,\mathbb{1}_{(-\infty,0]},\frac{\varepsilon}{C'}}(t - s_0) - c_1(\varepsilon, \delta, C') \\ &\geq \overline{m}^{\xi,\mathbb{1}_{(-\infty,0]},\frac{\varepsilon}{C}}(t) - c_2(\varepsilon, \delta, C', s_0) \quad \text{for all } t \geq \mathcal{T}_2(\xi, \varepsilon, \delta), \end{aligned}$$

where the second inequality can be obtained similarly to the argument in (4.48) and (4.49). Combining the above inequalities, we arrive at

$$\begin{aligned} m^{\xi,F,w_0,\varepsilon}(t) &\geq \overline{m}^{\xi,\mathbb{1}_{(-\infty,0]},\frac{\varepsilon}{C'}}(t) - C_1 \ln(t) - c_3 \\ &= \overline{m}^{\xi,C'\mathbb{1}_{(-\infty,0]},\varepsilon}(t) - C_1 \ln(t) - c_3 \quad \text{for all } t \geq \mathcal{T}_3(\xi). \end{aligned}$$

Now by (PAM-INI) every $u_0 \in \mathcal{I}_{\text{PAM}}$ is upper bounded by the function $C'\mathbb{1}_{(-\infty,0]}$, and since we have $\overline{m}^{\xi,\mathbb{1}_{(-\infty,0]},\varepsilon}(t) \geq m^{\xi,F,w_0,\varepsilon}(t)$ for all $\varepsilon \in (0, 1)$ and $w_0 \in \mathcal{I}_{\text{F-KPP}}$, this finishes the proof for F fulfilling (PROB).

(2) Now let F fulfill (SC) and w_0 be continuous. By a sandwiching argument, it suffices to show that there exists some function G fulfilling (PROB), such that $F(w) \geq G(w)$ for all $w \in [0, 1]$. Indeed, by Corollary A.11 the solutions w^F and w^G to

$$w_t^F - \frac{1}{2}w_{xx}^F - \xi(x)F(w^F) = 0 = w_t^G - \frac{1}{2}w_{xx}^G - \xi(x)G(w^G)$$

(which are classical by Proposition 2.1) fulfill $w^F \geq w^G$. As a consequence, we infer that $m^{\xi,F,w_0,\varepsilon}(t) \geq m^{\xi,G,w_0,\varepsilon}(t) \geq \overline{m}^{\xi,w_0,\varepsilon}(t) - C_1 \ln(t)$, where the second inequality is due to step (1). The claim for arbitrary $w_0 \in \mathcal{I}_{\text{F-KPP}}$ is then true by an approximation argument of w_0 by continuous functions, that is if $F \geq G$, by Remark 1.2 we have $w^{w_0,F} \geq w^{w_0,G}$ and consequently $m^{\xi,w_0,F,\varepsilon}(t) \geq m^{\xi,w_0,G,\varepsilon}(t)$ for all $t \geq 0$ and we can conclude.

It remains to show that for every F fulfilling (SC) there exists G fulfilling (PROB) such that $F(x) \geq G(x)$ for all $x \in [0, 1]$. To do so, recall that (SC) implies that there exists $M \in \mathbb{N}$ such that

$$1 - F'(x) \leq \frac{M}{2}x \quad \text{and} \quad F(1-x) \geq xM^{-1} \quad \text{for all } x \in [0, M^{-1}]. \quad (4.53)$$

Define $G_n(x) := \frac{1-x}{n}(1 - (1-x)^n)$, $x \in [0, 1]$, $n \in \mathbb{N}$. Then each G_n , $n \in \mathbb{N}$, satisfies (PROB) with $p_1 = 1 - n^{-1}$ and $p_{n+1} = n^{-1}$, and our goal is to show that $G_n \leq F$ for all $x \in [0, 1]$ and all n large enough.

We start with noting that $G_{n+1} \leq G_n$ as functions on $x \in [0, 1]$, for all $n \in \mathbb{N}$, and that $G_n \downarrow 0$ uniformly as n tends to infinity. Thus, since F is continuous and $F > 0$ on $(0, 1)$ due to (SC), we only have to take care of the neighborhoods of 0 and 1. From (4.53) we immediately get $G_M(x) \leq (1-x)M^{-1} \leq F(x)$ for all $x \in [1 - M^{-1}, 1]$. To infer the desired inequality for $x \in [0, 1 - M^{-1}]$, Taylor expansion yields

$$(1-x)^M \geq 1 - Mx + \binom{M}{2}x^2 - \binom{M}{3}x^3.$$

Then for M large enough and for all $x \in [0, M^{-1}]$ we get

$$\begin{aligned} G_M(x) &\leq (1-x)\left(x - \frac{M-1}{2}x^2 + \frac{(M-1)(M-2)}{6}x^3\right) \leq x - \frac{M}{3}x^2 \\ &\leq x - \frac{M}{4}x^2 = \int_0^x (1 - Mt/2)dt \leq \int_0^x F'(t)dt = F(x), \end{aligned}$$

where the last inequality is due to (4.53) again. This finishes the proof.

4.4. Remarks on v_c and v_0

By the same argument as in [9, Lemma A.4], one can show that a rich class of potentials ξ satisfies (VEL), i.e. the inequality $v_c < v_0$. It is natural to ask if $v_c < v_0$ is always fulfilled in our setting or not; i.e., do there exist potentials which satisfy (Standing assumptions) but not the inequality $v_c < v_0$?

In order to answer this question, we will take advantage of the following result.

Claim 4.9.

$$v_0 = \inf_{\eta \leq 0} \frac{\eta - e\mathfrak{s}}{L(\eta)}. \quad (4.54)$$

Proof. As shown in [14, p. 514ff.], the function

$$I : (0, \infty) \ni y \mapsto \sup_{\eta \leq -e\mathfrak{s}} (y\eta - L(\eta + e\mathfrak{s}))$$

is strictly decreasing, finite, convex, fulfills $\lim_{y \downarrow 0} I(y) = +\infty$ and $\lim_{y \uparrow \infty} I(y) = -\infty$, there exists a unique $v^* > 0$ such that $I(1/v^*) = 0$, and one has

$$v^* = \inf_{\eta \leq 0} \frac{\eta - e\mathfrak{s}}{L(\eta)}.$$

We now show $v^* = v_0$. For this purpose, let w and u be solutions to (F-KPP) and (PAM), respectively, both with initial condition $\mathbb{1}_{[-1,0]}$. Then $w \leq u$ and thus by [14, Lemma 7.6.3] and Proposition A.3 we have for all $v > v^*$ that \mathbb{P} -a.s.,

$$\Lambda(v) = \lim_{t \rightarrow \infty} \frac{1}{t} \ln u(t, vt) \geq \liminf_{t \rightarrow \infty} \frac{1}{t} \ln w(t, vt) \geq -vI(1/v).$$

Since Λ and I are continuous, passing to the limit $v \downarrow v^*$ we deduce that $\Lambda(v^*) \geq -v^*I(1/v^*) = 0$. Furthermore, $\Lambda(v) < 0$ for all $v > v_0$ and thus we infer $v_0 \geq v^*$.

To get the converse inequality, we use that for all $v > 0$ and all $\eta \leq 0$ we have

$$\begin{aligned} u(t, vt) &= E_{vt} \left[e^{\int_0^t (\zeta(B_s) + \text{es}) ds}; B_t \in [-1, 0] \right] = E_{vt} \left[e^{\int_0^t (\zeta(B_s) + \text{es}) ds}; B_t \in [-1, 0], H_0 \leq t \right] \\ &\leq e^{(\text{es} - \eta)t} E_{vt} \left[e^{\int_0^t (\zeta(B_s) + \eta) ds} \right]. \end{aligned}$$

In combination with (2.9) this yields that for all $v > 0$,

$$\begin{aligned} \Lambda(v) &= \lim_{t \rightarrow \infty} \frac{1}{t} \ln u(t, vt) \leq \inf_{\eta \leq 0} ((\text{es} - \eta) + vL(\eta)) \\ &= -v \sup_{\eta \leq -\text{es}} \left(\frac{\eta}{v} - L(\eta + \text{es}) \right) = -vI(1/v). \end{aligned}$$

But then $\Lambda(v^*) \leq -v^*I(1/v^*) = 0$ and thus we must have $v^* \geq v_0$.

We now formulate our main result on the relation between v_0 and v_c . It implies that our results do not apply for all potentials fulfilling (BDD), (STAT) and (MIX).

Proposition 4.10. *There exist potentials ξ fulfilling (BDD), (STAT) and (MIX), and such that $v_c > v_0$; i.e., condition (VEL) is violated.*

Proof. Recalling (4.54), the definition $v_c = \frac{1}{L'(0-)}$ from Lemma 2.4(d), it is sufficient to show $L(0) + \text{es} \cdot L'(0-) < 0$, which means

$$\mathbb{E} \left[\ln E_1 \left[e^{\int_0^{H_0} \zeta(B_s) ds} \right] \right] + \text{es} \cdot \mathbb{E} \left[\frac{E_1 \left[H_0 e^{\int_0^{H_0} \zeta(B_s) ds} \right]}{E_1 \left[e^{\int_0^{H_0} \zeta(B_s) ds} \right]} \right] < 0. \tag{4.55}$$

To establish the latter, let $\tilde{\omega}$ be a one-dimensional Poisson point process with intensity one. In a slight abuse of notation, $\tilde{\omega} = (\tilde{\omega}^i)_i$ can be seen as a mapping from Ω into the set of all locally finite point configurations; i.e., $\tilde{\omega} = (\tilde{\omega}^i)_i$ can be interpreted as a random set of countably many points in \mathbb{R} , satisfying $\{|i : \tilde{\omega}^i \in B\}| < \infty$ for every bounded Borel set and $\tilde{\omega}^i \neq \tilde{\omega}^j$ for all $i \neq j$. See [33] for further details. Now denote by $\omega = (\omega^i)_i$ be the point process that is obtained from $\tilde{\omega}$ by deleting *simultaneously* all points in $\tilde{\omega}$ which have distance 1 or less to their nearest neighbor in $\tilde{\omega}$. (see [26, p. 47] for details). Let $\varphi(x)$ be a mollifier with support $[-1/2, 1/2]$, non-decreasing for $x \leq 0$, non-increasing for $x \geq 0$ with $\varphi(0) = 1$ and let $\varphi^{(\varepsilon)}(x) := \varphi(x/\varepsilon)$, $\varepsilon > 0$. Finally, for $\varepsilon \in (0, 1)$ and $a > 0$, define the potential $\zeta(x) = \zeta^{(\varepsilon, a)}(x) := -a + a \sum_i \varphi^{(\varepsilon)}(x - \omega^i)$. One can easily check that

ζ is the corresponding shifted potential as in (2.3) of some ξ fulfilling (BDD), (STAT) and (MIX), i.e. $a = \varepsilon s - \varepsilon i$. We will choose $a > \ln 2$, $\varepsilon i \in (0, \frac{a}{8})$ and $\varepsilon(a) > 0$ suitably at the end of the proof. Let us now consider both summands in (4.55) separately.

(1) We observe that \mathbb{P} -a.s., $\zeta^{(\varepsilon,a)} \downarrow -a$ as $\varepsilon \downarrow 0$ for all $x \in \mathbb{R}$, as well as by [4, (2.0.1), p. 204]

$$-\sqrt{2a} = \ln E_1[e^{-aH_0}] \leq \ln E_1[e^{\int_0^{H_0} \zeta^{(\varepsilon,a)}(B_s)ds}] \leq 0$$

for all $\varepsilon \in (0, 1)$. Thus, by dominated convergence, for all $a > 0$ there exists $\varepsilon_1 = \varepsilon_1(a) > 0$, such that

$$\mathbb{E} \left[\ln E_1[e^{\int_0^{H_0} \zeta^{(\varepsilon_1,a)}(B_s)ds}] \right] \leq -\frac{3}{4}\sqrt{2a}. \tag{4.56}$$

(2) To bound the second summand in (4.55), we lower bound its denominator by

$$E_1[e^{\int_0^{H_0} \zeta(B_s)ds}] \geq E_1[e^{-aH_0}] = e^{-\sqrt{2a}}. \tag{4.57}$$

For the numerator, define \mathbb{J} to be the set of possible point configurations of the process $(\omega^i)_i$. Let us first check that for all $a > 0$ there exists $\varepsilon = \varepsilon(a) > 0$, such that

$$\sup_{(\omega^i)_i \in \mathbb{J}} E_1[H_0 e^{\int_0^{H_0} (-a+a \sum_i \varphi^{(\varepsilon)}(B_s-\omega^i))ds}] < \infty. \tag{4.58}$$

Indeed, letting $g^{\varepsilon,(\omega^i)_i}(x) := \sum_i \varphi^{(\varepsilon)}(x - \omega^i)$, we have

$$\begin{aligned} E_1[H_0 e^{\int_0^{H_0} (-a+a \cdot g^{\varepsilon,(\omega^i)_i}(B_s))ds}] &= \sum_{n=0}^{\infty} E_1[H_0 e^{\int_0^{H_0} (-a+a \cdot g^{\varepsilon,(\omega^i)_i}(B_s))ds}; H_0 \in [n, n+1)] \\ &\leq \sum_{n=0}^{\infty} (n+1) E_1[e^{\int_0^n (-a+a \cdot g^{\varepsilon,(\omega^i)_i}(B_s))ds}]. \end{aligned} \tag{4.59}$$

Note that by the property of all point configurations in \mathbb{J} to have points with mutual distance at least one, we have

$$\sup_{(\omega^i)_i \in \mathbb{J}} \sup_{x \in \mathbb{R}} E_x \left[a \int_0^1 g^{\varepsilon,(\omega^i)_i}(B_s)ds \right] \leq a \int_0^1 E_0[\mathbb{1}_{A_\varepsilon}(B_s)]ds \leq \frac{1}{2}$$

for all $\varepsilon(a) > 0$ small enough, where $A_\varepsilon := \bigcup_{i \in \mathbb{Z}} [-\varepsilon/2 + i, \varepsilon/2 + i]$. Using Kacskinski's lemma (cf. e.g. [35, Lemma 1.2.1]) we infer $\sup_{x \in \mathbb{R}} E_x[e^{a \int_0^1 g^{\varepsilon,(\omega^i)_i}(B_s)ds}] \leq 2$. An $(n - 1)$ -fold application of the Markov property at times $1, \dots, n - 1$ supplies us with $\sup_{x \in \mathbb{R}} E_x[e^{a \int_0^n g^{\varepsilon,(\omega^i)_i}(B_s)ds}] \leq 2^n$ for all $n \in \mathbb{N}$ and all $(\omega^i)_i \in \mathbb{J}$. Plugging this into (4.59) we infer

$$\sup_{(\omega^i)_i \in \mathbb{J}} E_1 \left[H_0 e^{\int_0^{H_0} (-a+a \cdot g^{\varepsilon,(\omega^i)_i}(B_s))ds} \right] \leq \sum_{n=0}^{\infty} (n+1) e^{-na} 2^n,$$

so the right-hand side in (4.59) is finite, and (4.58) holds true for all $a > \ln 2$ and $\varepsilon(a)$ small enough as well. Since $g^{\varepsilon, (\omega^j)_i}$ decreases \mathbb{P} -a.s. to 0 monotonically as $\varepsilon \downarrow 0$, we infer

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \mathbb{E} \left[E_1 \left[H_0 e^{\int_0^{H_0} \zeta^{\varepsilon, a}(B_s) ds} \right] \right] &= E_1 \left[H_0 e^{-aH_0} \right] \\ &= -\frac{d}{da} E_1 \left[e^{-aH_0} \right] = -\frac{d}{da} \left(e^{-\sqrt{2a}} \right) = \frac{1}{\sqrt{2a}} e^{-\sqrt{2a}}, \end{aligned} \tag{4.60}$$

using [4, (2.0.1), p. 204] in the third equality. Thus, combining (4.57) and (4.60) we infer that there exists $\varepsilon_2(a) > 0$ such that the second summand on the left-hand side of (4.55) is upper bounded by $\varepsilon s \cdot \frac{4/3}{\sqrt{2a}} = (a + \varepsilon i) \cdot \frac{4/3}{\sqrt{2a}}$. Using this in combination with (4.56), we infer that for all $a > \ln 2$ [which is sufficient for (4.58)] and $\varepsilon \in (0, \varepsilon_1(a) \wedge \varepsilon_2(a))$, choosing $\varepsilon i \in (0, \frac{a}{8})$, we get that the left-hand side in (4.55) is upper bounded by $-\frac{3}{4}\sqrt{2a} + (a + \varepsilon i) \cdot \frac{4/3}{\sqrt{2a}} < 0$.

Acknowledgement We would like to thank Jiří Černý, Lisa Hartung and Rongfeng Sun for useful discussions and remarks on a preliminary version of this paper. We would also like to thank an anonymous referee for a very useful report.

Funding Open Access funding enabled and organized by Projekt DEAL.

Data Availability Statement Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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A. Appendix

A.1. Properties of the logarithmic moment generating functions

Lemma A.1. We recall that $P_x^{\zeta, \eta}$ has been defined in (2.6).

(a) L , L_x^ζ , and \bar{L}_x^ζ , for $x \in \mathbb{R}$, [see (2.7) to (2.8) for their definitions] are infinitely differentiable on $(-\infty, 0)$. Furthermore, for all $\eta < 0$ we have

$$(L_x^\zeta)'(\eta) = \frac{E_x \left[e^{\int_0^{H_{[x]^{-1}}(\zeta(B_r)+\eta)dr} H_{[x]^{-1}} \right]}{E_x \left[e^{\int_0^{H_{[x]^{-1}}(\zeta(B_r)+\eta)dr} \right]} = E_x^{\zeta, \eta}[\tau_{[x]^{-1}}], \quad x \in \mathbb{R}, \quad (\text{A.1})$$

$$(\bar{L}_x^\zeta)'(\eta) = \frac{1}{x} E_x^{\zeta, \eta}[H_0], \quad x > 0, \quad (\text{A.2})$$

$$L'(\eta) = \mathbb{E} \left[\frac{E_1 \left[e^{\int_0^{H_0}(\zeta(B_r)+\eta)dr} H_0 \right]}{E_1 \left[e^{\int_0^{H_0}(\zeta(B_r)+\eta)dr} \right]} \right] = \mathbb{E}[E_1^{\zeta, \eta}[H_0]], \quad (\text{A.3})$$

and

$$\begin{aligned} (L_x^\zeta)''(\eta) &= \frac{E_x \left[e^{\int_0^{H_{[x]^{-1}}(\zeta(B_r)+\eta)dr} H_{[x]^{-1}}^2 \right]}{E_x \left[e^{\int_0^{H_{[x]^{-1}}(\zeta(B_r)+\eta)dr} \right]} - \left(\frac{E_x \left[e^{\int_0^{H_{[x]^{-1}}(\zeta(B_r)+\eta)dr} H_{[x]^{-1}} \right]}{E_x \left[e^{\int_0^{H_{[x]^{-1}}(\zeta(B_r)+\eta)dr} \right]} \right)^2 \\ &= E_x^{\zeta, \eta}[\tau_{[x]^{-1}}^2] - (E_x^{\zeta, \eta}[\tau_{[x]^{-1}}])^2 = \text{Var}_x^{\zeta, \eta}(\tau_{[x]^{-1}}) > 0, \quad x \in \mathbb{R}, \end{aligned} \quad (\text{A.4})$$

$$(\bar{L}_x^\zeta)''(\eta) = \frac{1}{x} \text{Var}_x^{\zeta, \eta}(H_0), \quad x > 0, \quad (\text{A.5})$$

$$\begin{aligned} L''(\eta) &= \mathbb{E} \left[\frac{E_1 \left[e^{\int_0^{H_0}(\zeta(B_r)+\eta)dr} H_0^2 \right]}{E_1 \left[e^{\int_0^{H_0}(\zeta(B_r)+\eta)dr} \right]} - \left(\frac{E_1 \left[e^{\int_0^{H_0}(\zeta(B_r)+\eta)dr} H_0 \right]}{E_1 \left[e^{\int_0^{H_0}(\zeta(B_r)+\eta)dr} \right]} \right)^2 \right] \\ &= \mathbb{E} \left[E_1^{\zeta, \eta}[H_0^2] - (E_1^{\zeta, \eta}[H_0])^2 \right] = \mathbb{E}[\text{Var}_1^{\zeta, \eta}(H_0)] > 0. \end{aligned} \quad (\text{A.6})$$

(b) For each compact interval $\Delta \subset (-\infty, 0)$ there exists a constant $C_{18} = C_{18}(\Delta) > 0$, such that the following inequalities hold \mathbb{P} -a.s.:

$$\begin{aligned} -C_{18} &\leq \inf_{\eta \in \Delta, x \geq 1} \{L_x^\zeta(\eta), \bar{L}_x^\zeta(\eta), L(\eta)\} \leq \sup_{\eta \in \Delta, x \geq 1} \{L_x^\zeta(\eta), \bar{L}_x^\zeta(\eta), L(\eta)\} \leq -C_{18}^{-1}, \\ C_{18}^{-1} &\leq \inf_{\eta \in \Delta, x \geq 1} \{(L_x^\zeta)'(\eta), (\bar{L}_x^\zeta)'(\eta), L'(\eta)\} \leq \sup_{\eta \in \Delta, x \geq 1} \{(L_x^\zeta)'(\eta), (\bar{L}_x^\zeta)'(\eta), L'(\eta)\} \leq C_{18}, \\ C_{18}^{-1} &\leq \inf_{\eta \in \Delta, x \geq 1} \{(L_x^\zeta)''(\eta), (\bar{L}_x^\zeta)''(\eta), L''(\eta)\} \leq \sup_{\eta \in \Delta, x \geq 1} \{(L_x^\zeta)''(\eta), (\bar{L}_x^\zeta)''(\eta), L''(\eta)\} \leq C_{18}. \end{aligned}$$

Proof. (a) Due to the convexity of the exponential function we have $|\frac{e^{hx}-1}{h}| \leq x e^{hx} \vee 1$ for all $x \geq 0$ and $h \in \mathbb{R}$. If we choose $h_0 := \frac{|\eta|}{2}$, then since $\zeta \leq 0$, we have that for all $|h| \leq h_0$,

$$\left| \frac{1}{h} e^{\int_0^{H_y}(\zeta(B_r)+\eta)dr} (e^{hH_y} - 1) \right| \leq e^{\eta H_y/2} (H_y \vee 1). \quad (\text{A.7})$$

Due to $H_y e^{\eta H_y} \leq \frac{1}{|\eta|}$ for all $H_y \geq 0$ and $\eta < 0$, as well as $\lim_{h \rightarrow 0} \frac{e^{hH_y}-1}{h} = H_y e^{hH_y}$ for all $H_y \geq 0$, dominated convergence yields for all $\eta < 0$ that

$$\frac{d}{d\eta} E_x \left[e^{\int_0^{H_{[x]^{-1}}(\zeta(B_r)+\eta)dr} \right] = E_x \left[e^{\int_0^{H_{[x]^{-1}}(\zeta(B_r)+\eta)dr} H_{[x]^{-1}} \right]. \quad (\text{A.8})$$

Then (A.1) is a consequence of the chain rule and the fact that the expectation on the left-hand side in (A.8) is positive. Then (A.2) follows from linearity of the derivative. To show

(A.3), we have to apply dominated convergence once more. This time, we additionally need that the expectation on the left-hand side in (A.8) for $x = 1$ is strictly bounded from below by the constant $E_1[e^{-(es-ei-\eta)H_0}] = e^{-\sqrt{2(es-ei-\eta)}} > 0$ due to (BDD) and [4, (2.0.1), p. 204]. Using the mean value theorem and (A.7) entail the existence of $c_1 > 0$ such that

$$\sup_{|h| \leq |\eta|/2} \left| \frac{1}{h} (L_1^\zeta(\eta+h) - L_1^\zeta(\eta)) \right| \leq \sup_{|h| \leq |\eta|/2} (L_1^\zeta)'(\eta+h) \leq c_1 E_1[e^{\eta H_0/2} (H_0 \vee 1)] \leq c_1 \left(\frac{2}{|\eta|} \vee 1 \right).$$

Using (A.1) for $x = 1$ and dominated convergence, we arrive at (A.3).

By induction and similar arguments as above, it follows that for all $n \in \mathbb{N}$, the function $(-\infty, 0) \ni \eta \mapsto E_x[(H_y)^n e^{\int_0^{H_y} (\zeta(B_r) + \eta) dr}]$ is positive \mathbb{P} -a.s. and differentiable. Due to (BDD), one can interchange expectation and differentiation in η , yielding (A.4) and (A.5). The first equality in (A.6) is then again a consequence of dominated convergence. The strict inequalities in (A.4) and (A.6) are due to the fact that under P_1 , and thus also \mathbb{P} -a.s. under $P_1^{\zeta, \eta}$, the random variable H_0 is non-degenerate.

To show (b), observe that the function $\zeta \mapsto E_x[e^{\int_0^{H_{[x]^{-1}}} (\zeta(B_s) + \eta) ds}]$ is nondecreasing. Consequently, using the notation $\Delta = [\eta_*, \eta^{*}]$, we have

$$\begin{aligned} -\infty < e^{-\sqrt{2(es-ei+|\eta_*|)}} &\leq E_x[e^{(ei-es+\eta_*)H_{[x]^{-1}}}] \leq \inf_{\eta \in \Delta} \text{ess inf}_\zeta E_x[e^{\int_0^{H_{[x]^{-1}}} (\zeta(B_s) + \eta) ds}] \\ &\leq \sup_{\eta \in \Delta} \text{ess sup}_\zeta E_x[e^{\int_0^{H_{[x]^{-1}}} (\zeta(B_s) + \eta) ds}] \leq E_x[e^{\eta^* H_{[x]^{-1}}}] = e^{(x-[x]+1)\sqrt{2|\eta^*|}}, \end{aligned}$$

where we used [4, (2.0.1), p. 204] for the second inequality and last equality. Due to the inequality $e^{-xy} x \leq \frac{2}{y} e^{-xy/2}$ for all $x \geq 0$ and $y > 0$, these estimates can be used to derive similar bounds for $E_x[e^{\int_0^{H_{[x]^{-1}}} (\zeta(B_r) + \eta) dr} H_{[x]^{-1}}^k]$, $k = 1, 2$. Thus, estimating the numerator and the denominator of the corresponding expressions in (A.1) to (A.6), we can conclude.

A.2. Exponential mixing

Lemma A.2. *Let \mathcal{F}^k be as defined in (MIX), i.e. $\mathcal{F}^k = \sigma(\xi(x) : x \geq k)$, and let $\Delta \subset (-\infty, 0)$ be a compact interval. Then there exists a constant $C_\Delta > 0$ such that \mathbb{P} -a.s., for all $i, j \in \mathbb{Z}$ with $i < j$, and all $\eta \in \Delta$,*

$$|\mathbb{E}[L_i^\zeta(\eta) | \mathcal{F}^j] - L(\eta)| \leq C_\Delta \cdot \left(\psi\left(\frac{j-i}{2}\right) + e^{-(j-i)/C_\Delta} \right), \tag{A.9}$$

$$0 \leq \left(\text{ess sup}_{\xi(k); k \geq j} L_i^\zeta(\eta) \right) - L_i^\zeta(\eta) \leq C_\Delta \cdot \left(\psi\left(\frac{j-i}{2}\right) + e^{-(j-i)/C_\Delta} \right), \tag{A.10}$$

as well as

$$|\mathbb{E}[(L_i^\zeta)'(\eta) | \mathcal{F}^j] - L'(\eta)| \leq C_\Delta \cdot \left(\psi\left(\frac{j-i}{2}\right) + e^{-(j-i)/C_\Delta} \right), \tag{A.11}$$

$$0 \leq \left(\text{ess sup}_{\xi(k); k \geq j} (L_i^\zeta)'(\eta) \right) - (L_i^\zeta)'(\eta) \leq C_\Delta \cdot \left(\psi\left(\frac{j-i}{2}\right) + e^{-(j-i)/C_\Delta} \right). \tag{A.12}$$

Proof. By translation invariance, it is enough to prove (A.9) for $i = 0$ and $j \geq 2$ (the case $j = 1$ follows immediately from the uniform boundedness of L_0^ζ and L on Δ due to Lemma A.1). To show (A.9), let $\eta \in \Delta$ and write $L_0^\zeta(\eta) = \ln(A + B)$ with

$$A = A(\eta) := E_0 \left[e^{\int_0^{H-1} (\zeta(B_s) + \eta) ds}; \sup_{0 \leq s \leq H-1} B_s < \lceil j/2 \rceil \right]$$

and

$$B = B(\eta) := E_0 \left[e^{\int_0^{H-1} (\zeta(B_s) + \eta) ds}; \sup_{0 \leq s \leq H-1} B_s \geq \lceil j/2 \rceil \right].$$

Then $A \leq E_0[e^{H-1\eta}] \leq c_{\Delta,1} < 1$, and, using (2.3), at the same time we have

$$A \geq E_0 \left[e^{-(es - e\mathbf{i} + |\eta|)H-1}; \sup_{0 \leq s \leq H-1} B_s < \lceil j/2 \rceil \right] \geq c_{\Delta,2} > 0 \text{ for all } j \geq 2.$$

To bound B , we condition on $H-1$ to happen before or after time j and use the reflection principle for Brownian motion and the tail estimate from [6, Lemma 1.1] to infer for all $j \geq 2$

$$0 \leq B \leq P_0 \left(\sup_{0 \leq s \leq j} B_s \geq \lceil j/2 \rceil \right) + e^{\eta j} = 2P_0(B_j \geq \lceil j/2 \rceil) + e^{\eta j} \leq 2e^{-j(1/8 \wedge |\eta|)}.$$

As $\ln(1 + x) \leq x$, the above implies that for all $j \geq 2$

$$\ln(A) \leq L_0^\zeta(\eta) = \ln(A) + \ln \left(1 + \frac{B}{A} \right) \leq \ln(A) + c_{\Delta,3} e^{-j/c_{\Delta,3}}.$$

Since L_0^ζ is continuous on Δ , the latter display \mathbb{P} -a.s. holds uniformly for all $\eta \in \Delta$. Now $\ln(A)$ is $\mathcal{F}_{\lceil j/2 \rceil}$ -measurable and bounded, so by (MIX)

$$\begin{aligned} \sup_{\eta \in \Delta} \left| \mathbb{E}[L_0^\zeta(\eta) | \mathcal{F}^j] - L(\eta) \right| &\leq \sup_{\eta \in \Delta} \left| \mathbb{E}[\ln(A) - \mathbb{E}[\ln(A)] | \mathcal{F}^j] \right| + 2c_{\Delta,3} e^{-j/c_{\Delta,3}} \\ &\leq C_\Delta \cdot (\psi(j/2) + e^{-j/C_\Delta}). \end{aligned} \tag{A.13}$$

The proof of (A.11) is similar. Indeed, using the same notation we have $(L_0^\zeta)^\prime = \frac{A^\prime}{A+B} + \frac{B^\prime}{A+B}$. Then by $A+B \geq c_{\Delta,2}$ and $e^{\eta H-1} H_{-1} \leq \frac{2}{|\eta|} e^{\eta H-1/2}$, we can use above calculation to conclude that $\frac{B^\prime}{A+B}$ decays exponentially to 0 as $j \rightarrow \infty$. Further, $\frac{A^\prime}{A} - \frac{BA^\prime}{A(A+B)} = \frac{A^\prime}{A+B} \leq \frac{A^\prime}{A}$, $\mathcal{F}_{\lceil j/2 \rceil}$ -measurability of $\frac{A^\prime}{A}$ and above estimates give a similar bound as in (A.13) for $(L_0^\zeta)^\prime(\eta)$ and $L^\prime(\eta)$ instead of $(L_0^\zeta)(\eta)$ and $L(\eta)$. Finally, (A.10) and (A.12) follow by the same arguments as (A.9) and (A.11).

A.3. Properties of the Lyapunov exponent

Proposition A.3. *Assume (BDD) and (STAT) and let $u^{(x,\delta)}$ be a solution to (PAM) with initial condition $\mathbb{1}_{[x-\delta, x+\delta]}$. Then the limit $\Lambda(v) := \lim_{t \rightarrow \infty} \frac{1}{t} \ln u^{(x,\delta)}(t, vt)$, $v \in \mathbb{R}$, exists \mathbb{P} -a.s., is non-random and independent of $x \in \mathbb{R}$ and $\delta > 0$. The function $[0, \infty) \ni v \mapsto \Lambda(v)$ is concave, $\Lambda(0) = es$ and $\lim_{v \rightarrow \infty} \frac{\Lambda(v)}{v} = -\infty$. In particular, there exists a unique $v_0 > 0$ such that $\Lambda(v_0) = 0$.*

Proof. Let $x = 0$, $\lambda \in (0, 1)$ and $v_1, v_2 \in \mathbb{R}$, and set $v := \lambda v_1 + (1 - \lambda)v_2$ and $A := [(1 - \lambda)v_2t - \delta, (1 - \lambda)v_2t + \delta]$. By (1.12), the solution to (PAM), admits the Feynman–Kac representation $u^{(x,\delta)}(t, vt) = E_{vt} \left[e^{\int_0^t \xi(B_s) ds}; B_t \in [x - \delta, x + \delta] \right]$. Then by the Markov property, for all $y \in [vt - \delta, vt + \delta]$ we have

$$\begin{aligned} \ln E_y \left[e^{\int_0^t \xi(B_s) ds}; B_t \in [-\delta, \delta] \right] &\geq \ln E_y \left[e^{\int_0^{\lambda t} \xi(B_s) ds}; B_{\lambda t} \in A \right] \\ &\quad + \inf_{z \in A} \ln E_z \left[e^{\int_0^{(1-\lambda)t} \xi(B_s) ds}; B_{(1-\lambda)t} \in [-\delta, \delta] \right]. \end{aligned} \tag{A.14}$$

Defining $\bar{\mu}_{s,t}^\delta(v) := \inf_{y \in [vt - \delta, vt + \delta]} \ln E_y \left[e^{\int_0^{t-s} \xi(B_r) dr}; B_{t-s} \in [vs - \delta, vs + \delta] \right]$, $s < t$, by the same argument as in the last display one can see that $\bar{\mu}_{s,t}^\delta(v) \geq \bar{\mu}_{s,u}^\delta(v) + \bar{\mu}_{u,t}^\delta(v)$ for all $s < u < t$. Furthermore for the h -lateral shift θ_h on ξ (i.e. $\xi(\cdot) \circ \theta_h = \xi(\cdot + h)$) we have $\bar{\mu}_{s,t}^\delta(v) \circ \theta_h = \bar{\mu}_{s+\frac{h}{v}, t+\frac{h}{v}}^\delta(v)$ for every $h \in \mathbb{R}$, all $v \neq 0$ and $s < t$, and by (BDD) and a simple Gaussian computation there exists $K_{\delta,v} \in (0, \infty)$ such that $\bar{\mu}_{0,t}^\delta(v) \geq -K_{\delta,v} \cdot t$ for all $t \geq 1$. By (BDD) we also have $\bar{\mu}_{0,t}^\delta(v) \leq \epsilon s \cdot t$ and thus $\bar{\mu}_{0,t}^\delta(v) \in L^1$ for all $t > 0$. Thus, Kingman’s subadditive ergodic theorem [23] implies that the limit $\Lambda^\delta(v) := \lim_{t \rightarrow \infty} \frac{1}{t} \bar{\mu}_{0,t}^\delta(v)$ exists \mathbb{P} -a.s. Furthermore, by (MIX), ξ is mixing and thus ergodic, so the $\Lambda^\delta(v)$ is non-random. By standard estimates, the limit is independent of $\delta > 0$ and $x \in \mathbb{R}$. To show the concavity of $v \mapsto \Lambda(v)$, for $v_1, v_2 \geq 0$ and dividing by t , the left-hand side of (A.14) converges \mathbb{P} -a.s. to $\Lambda(\lambda v_1 + (1 - \lambda)v_2)$, while the second term on the right-hand side of (A.14) converges \mathbb{P} -a.s. to $(1 - \lambda)\Lambda(v_2)$. Dividing by t and using (STAT), the first term on the right-hand side of (A.14) converges in distribution to the constant $\lambda\Lambda(v_1)$, proving the concavity of $v \mapsto \Lambda(v)$.

By [14, Theorem 7.5.2] we have $\lim_{t \rightarrow \infty} \frac{1}{t} \ln u^{\mathbb{1}_{(-\delta,\delta)}}(t, 0) = \epsilon s$, independent of $\delta > 0$, giving $\Lambda(0) = \epsilon s$. Due to (BDD) and a standard Gaussian computation we have $\lim_{v \rightarrow \infty} \frac{\Lambda(v)}{v} = -\infty$. This, together with $\Lambda(0) = \epsilon s > 0$ and the concavity of $v \mapsto \Lambda(v)$, implies the existence of a unique $v_0 > 0$, such that $\Lambda(v_0) = 0$.

A.4. A Hoeffding-type inequality for dependent random variables

Lemma A.4. ([34, Theorem 2.4]) *Let $(X_i)_{i \in \mathbb{Z}}$ be a sequence of real-valued bounded random variables, $\tilde{\mathcal{F}}_i := \sigma(X_j : j \leq i)$, $S_n := \sum_{i=1}^n X_i$, and let (m_1, \dots, m_n) be an n -tuple of positive reals such that for all $i \in \{1, \dots, n\}$,*

$$\sup_{j \in \{i, i+1, \dots, n\}} \left(\|X_i^2\|_\infty + 2\|X_i\| \sum_{k=i+1}^j \mathbb{E}[X_k | \tilde{\mathcal{F}}_i] \|_\infty \right) \leq m_i,$$

with the convention $\sum_{k=i+1}^i \mathbb{E}[X_k | \tilde{\mathcal{F}}_i] = 0$. Then for every $x > 0$,

$$\mathbb{P}(|S_n| \geq x) \leq \sqrt{e} \exp \left\{ -x^2 / (2m_1 + \dots + 2m_n) \right\}.$$

Rearranging the quantities in the above result, we arrive at the following corollary which we primarily pronounce explicitly since its formulation tailor-made for our purposes.

Corollary A.5. *Let $(Y_i)_{i \in \mathbb{Z}}$ be a sequence of real-valued bounded random variables, $\tilde{\mathcal{F}}^k := \sigma(Y_j : j \geq k)$, and let (m_1, \dots, m_n) be an n -tuple of positive real numbers such that for all $i \in \{1, \dots, n\}$,*

$$\sup_{j \in \{1, \dots, i\}} \left(\|Y_i^2\|_\infty + 2\|Y_i\| \sum_{k=j}^{i-1} \mathbb{E}[Y_k | \tilde{\mathcal{F}}^i] \|_\infty \right) \leq m_i, \tag{A.15}$$

with the convention $\sum_{k=i}^{i-1} \mathbb{E}[Y_k | \tilde{\mathcal{F}}^i] = 0$. Then for every $x > 0$,

$$\mathbb{P}\left(\left|\sum_{i=1}^n Y_i\right| \geq x\right) \leq \sqrt{e} \exp\left\{-x^2/(2m_1 + \dots + 2m_n)\right\}.$$

Recall that u^{u_0} denotes the solution to (PAM) with initial condition u_0 .

Lemma A.6. For all $x \in \mathbb{R}$ and $0 \leq s \leq t$ we have $u^{\mathbb{1}(-\infty, 0]}(s, x) \leq 2u^{\mathbb{1}(-\infty, 0]}(t, x)$.

Proof. By the Feynman–Kac formula, $\xi \geq 0$ and the Markov property at time s we have

$$\begin{aligned} u^{\mathbb{1}(-\infty, 0]}(t, x) &= E_x\left[e^{\int_0^t \xi(B_r) dr}; B_t \leq 0\right] \geq E_x\left[e^{\int_0^s \xi(B_r) dr}; B_s \leq 0, B_t \leq 0\right] \\ &\geq E_x^\xi\left[N \leq (s, 0)\right] P_0(B_{t-s} \leq 0) = \frac{1}{2}u^{\mathbb{1}(-\infty, 0]}(s, x). \end{aligned}$$

Lemma A.7. Let $N_{s,u,t}^{\mathcal{L}}$ be as in (4.1). Then there exists $c > 0$ such that \mathbb{P} -a.s.

$$E_x^\xi[N_{t,t,t}^{\mathcal{L}}] \geq cE_{[x]}^\xi[N_{t-1,t+1,t}^{\mathcal{L}}] \quad \text{for all } t \geq 1 \text{ and } x \geq 1.$$

Proof. Write $A_{y,z,t} := \{H_k + y \geq z + t - T_k - 5\chi_1(\bar{m}(t)) \forall k \in \{1, \dots, \lfloor \bar{m}(t) \rfloor\}\}$ and $p(r) := \inf_{y \leq 0} P_y(B_r \leq 0) = \frac{1}{2}$. Then by the Feynman–Kac formula

$$\begin{aligned} E_x^\xi[N_{t,t,t}^{\mathcal{L}}] &= E_x\left[e^{\int_0^t \xi(B_s) ds}; B_t \leq 0, A_{0,0,t}\right] \\ &\geq E_x\left[e^{\int_{H_{[x]}}^{t-1+H_{[x]}} \xi(B_s) ds}; B_t \leq 0, B_{t-1+H_{[x]}} \leq 0, A_{H_{[x]},1,t}, H_{[x]} \leq 1\right] \\ &\geq E_x\left[E_{[x]}\left[e^{\int_0^{t-1} \xi(B_s) ds} \times \inf_{r \in [0,1]} p(r); B_{t-1} \leq 0, A_{0,1,t}\right]; H_{[x]} \leq 1\right] \\ &\geq \frac{1}{2}P_x(H_{[x]} \leq 1)E_{[x]}\left[e^{\int_0^{t-1} \xi(B_s) ds}; B_{t-1} \leq 0, A_{0,1,t}\right] = cE_{[x]}^\xi[N_{t-1,t+1,t}^{\mathcal{L}}], \end{aligned}$$

with $c := \frac{1}{2}P_x(H_{[x]} \leq 1)$.

The next result is an inequality of Harnack-type flavor for the solution to (PAM).

Lemma A.8. There exists a constant $C_{19} \in (0, \infty)$ such that \mathbb{P} -a.s. for all $y \in \mathbb{R}$, $t \geq 1$ and all $u_0 \in \mathcal{I}_{PAM}$ we have

$$u^{u_0}(t, y) \leq C_{19} \inf_{x \in [y-1, y+1]} u^{u_0}(t + 1, x). \tag{A.16}$$

Proof. For $x \in \mathbb{R}$ and $t > 0$ let $f_{t,x}$ be the probability density of a Brownian motion at time t , starting in x . Let us first show that for all $y, z \in \mathbb{R}$ we have

$$f_{1,y}(z) \leq \sqrt{2/e} \inf_{x \in [y-1, y+1]} f_{2,x}(z). \tag{A.17}$$

Indeed, using $f_{t,y}(z) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(z-y)^2}{2t}}$, (A.17) follows from

$$\begin{aligned} \inf_{x \in [y-1, y+1]} \{2(z-y)^2 - (z-x)^2\} &= \begin{cases} 2(z-y)^2 - (z-y+1)^2 = (z-y-1)^2 - 2, & y < z, \\ 2(z-y)^2 - (z-y-1)^2 = (z-y+1)^2 - 2, & y \geq z, \end{cases} \\ &\geq -2. \end{aligned}$$

The Feynman–Kac formula in combination with the Markov property applied at time 1 now supplies us with

$$\begin{aligned} u^{u_0}(t, y) &= E_y \left[e^{\int_0^t \xi(B_r) dr} u_0(B_t) \right] \leq e^{es} \int_{\mathbb{R}} f_{1,y}(z) E_z \left[e^{\int_0^{t-1} \xi(B_r) dr} u_0(B_{t-1}) \right] dz \\ &\leq e^{es} \sqrt{2/e} \inf_{x \in [y-1, y+1]} \int_{\mathbb{R}} f_{2,x}(z) E_z \left[e^{\int_0^{t-1} \xi(B_r) dr} u_0(B_{t-1}) \right] dz \\ &\leq e^{es} \sqrt{2/e} \inf_{x \in [y-1, y+1]} u^{u_0}(t+1, x), \end{aligned}$$

where in the second inequality we used (A.17). Then (A.16) follows choosing $C_{19} = e^{es} \sqrt{2/e}$.

A.5. Proof of the McKean representation

Proof of Proposition 2.1. We start with considering w_0 continuous and $w_0(x) \in [0, 1]$ for all $x \in \mathbb{R}$. Define $u := 1 - w$ and $u_0 := 1 - w_0$. Then by [21, Remark 4.4.4] (providing the existence of a $C^{1,2}$ -solution to (PAM)) and [21, Corollary 4.4.5] (Feynman–Kac representation), the function $u^{(1)}(t, x) = E_x \left[e^{-\int_0^t \xi(B_r) dr} u_0(B_t) \right]$ is a classical solution to $u_t^{(1)}(t, x) = \frac{1}{2} \Delta u^{(1)}(t, x) - \xi(x) u^{(1)}(t, x)$ and we have $u^{(1)}(0, x) = u_0(x)$. Furthermore, by the same argument, for all $s < t$ and $k \in \mathbb{N}$ fixed, the function

$$u^{(2)}(t, x; s, k) := E_x \left[\xi(B_{t-s}) e^{-\int_0^{t-s} \xi(B_r) dr} u(s, B_{t-s})^k \right]$$

is a classical solution to $u_t^{(2)}(t, x; s, k) = \frac{1}{2} \Delta u^{(2)}(t, x; s, k) - \xi(x) u^{(2)}(t, x; s, k)$. By dominated convergence, taking advantage of the uniform continuity of the first and second order derivatives, for every fixed $t' \leq t$, we can interchange limits to obtain the identities

$$\begin{aligned} \sum_k p_k \int_0^{t'} \frac{\partial}{\partial t} u^{(2)}(t, x; s, k) ds &= \frac{\partial}{\partial t} \sum_k p_k \int_0^{t'} u^{(2)}(t, x; s, k) ds, \\ \sum_k p_k \int_0^{t'} \left(\frac{1}{2} \Delta - \xi(x) \right) u^{(2)}(t, x; s, k) ds &= \left(\frac{1}{2} \Delta - \xi(x) \right) \sum_k p_k \int_0^{t'} u^{(2)}(t, x; s, k) ds. \end{aligned} \tag{A.18}$$

Conditioning on the first splitting time of the initial particle and on the number of children that are born, we have

$$\begin{aligned} u(t, x) &= E_x \left[e^{-\int_0^t \xi(B_r) dr} u_0(B_t) \right] + \sum_k p_k \int_0^t E_x \left[\xi(B_s) e^{-\int_0^s \xi(B_r) dr} u(t-s, B_s)^k \right] ds \\ &= u^{(1)}(t, x) + \sum_k p_k \int_0^t E_x \left[\xi(B_{t-s}) e^{-\int_0^{t-s} \xi(B_r) dr} u(s, B_{t-s})^k \right] ds, \end{aligned}$$

where we used the Markov property of the process $(B_t)_{t \geq 0}$. We deduce that for $h \neq 0$,

$$\begin{aligned} \frac{1}{h} (u(t+h, x) - u(t, x)) &= \frac{1}{h} (u^{(1)}(t+h, x) - u^{(1)}(t, x)) \\ &\quad + \sum_k p_k E_x \left[\frac{1}{h} \int_t^{t+h} \xi(B_{t+h-s}) e^{-\int_0^{t+h-s} \xi(B_r) dr} u(s, B_{t+h-s})^k ds \right] \\ &\quad + \sum_k p_k \int_0^t \frac{1}{h} (u^{(2)}(t+h, x; s, k) - u^{(2)}(t, x; s, k)) ds. \end{aligned}$$

Invoking once again the arguments from the beginning of the proof, as h tends to zero, the first summand converges to $\frac{1}{2} \Delta u^{(1)}(t, x) - \xi(x)u^{(1)}(t, x)$ and by dominated convergence, the second summand converges to $\xi(x) \sum_k p_k u(t, x)^k$. For the third term, we mention that the integrand is uniformly bounded since $u_t^{(2)}$ is continuous. Again, due to dominated convergence and (A.18), the latter term converges to

$$\left(\frac{1}{2} \Delta - \xi(x)\right) \sum_k p_k \int_0^t u^{(2)}(t, x; s, k) ds.$$

All in all, combining these observations we arrive at

$$\begin{aligned} u_t(t, x) &= u_t^{(1)}(t, x) + \xi(x) \sum_k p_k u(t, x)^k + \left(\frac{1}{2} \Delta - \xi(x)\right) \sum_k p_k \int_0^t u^{(2)}(t, x; s, k) ds \\ &= \left(\frac{1}{2} \Delta - \xi(x)\right) u(t, x) + \xi(x) \sum_k p_k u(t, x)^k, \end{aligned}$$

which is equivalent w being a solution to (F-KPP). Thus we have shown the claim for continuous w_0 . By Remark 1.2, for $w_0 \in \mathcal{I}_{\text{F-KPP}}$ the result follows by approximation.

A.6. Existence and monotonicity of solutions

We start with an existence result for generalized and also classical solutions.

Proposition A.9. ([14, Theorem 7.4.1]) *Let w_0 be measurable, non-negative and bounded, and let F fulfill (SC). Then \mathbb{P} -a.s. there exists a unique, bounded generalized solution $w \in C((0, \infty) \times \mathbb{R})$ to (F-KPP), which satisfies $0 \leq w(t, x) \leq 1 \vee \sup_x w_0(x)$ for all $(t, x) \in [0, \infty) \times \mathbb{R}$. If additionally w_0 is continuous and $x \mapsto \xi(x, \omega)$ is Hölder continuous for \mathbb{P} -a.a. $\omega \in \Omega$, then \mathbb{P} -a.s. the generalized solution w is a classical one, i.e. $w \in C^{1,2}((0, \infty) \times \mathbb{R}) \cap C([0, \infty) \times \mathbb{R})$ and w solves (F-KPP).*

For the next lemma, we introduce the differential operator

$$(\mathcal{L}_G w)(t, x) := w_t(t, x) - \frac{1}{2} w_{xx}(t, x) - G(x, w(t, x)),$$

where G is uniformly Lipschitz-continuous in w , i.e., there exists a constant $\alpha > 0$, such that

$$|G(x, u) - G(x, v)| \leq \alpha |u - v| \quad \forall x, u, v \in \mathbb{R}. \tag{A.19}$$

The next lemma is in the spirit of [1, Proposition 2.1].

Lemma A.10. *Let $T > 0$, $Q := (0, T) \times \mathbb{R}$ and G be such that (A.19) holds. Let $w^{(1)}$ and $w^{(2)}$ be non-negative and bounded functions on \overline{Q} , such that for $i \in \{1, 2\}$, $w_x^{(i)}$ and $w_{xx}^{(i)}$ are continuous on Q , and such that $w_t^{(i)}$ exists in Q . If $\mathcal{L}_G w^{(1)} \leq \mathcal{L}_G w^{(2)}$ on Q and $0 \leq w^{(1)}(0, x) \leq w^{(2)}(0, x)$ for all $x \in \mathbb{R}$, then also $w^{(1)} \leq w^{(2)}$ on Q .*

Proof. From the assumptions we infer that

$$w_t^{(2)}(t, x) - w_t^{(1)}(t, x) \geq \frac{1}{2} (w_{xx}^{(2)}(t, x) - w_{xx}^{(1)}(t, x)) + G(x, w^{(2)}(t, x)) - G(x, w^{(1)}(t, x)).$$

Then, recalling (A.19) and letting

$$v(t, x) := e^{-2\alpha t} (w^{(2)}(t, x) - w^{(1)}(t, x)),$$

we get

$$\begin{aligned} v_t(t, x) - \frac{1}{2}v_{xx}(t, x) &\geq -2\alpha v(t, x) + e^{-2\alpha t} (G(x, w^{(2)}(t, x)) - G(x, w^{(1)}(t, x))) \\ &\geq (-2 - \operatorname{sgn}(v(t, x)))\alpha \cdot v(t, x). \end{aligned}$$

Now the the first factor on the right-hand side is negative and bounded. Applying the maximum principle [25, Theorem 8.1.4] to $-v$ then implies that $v \geq 0$ on Q and we can conclude.

As an important application we get that the solution w to $\mathcal{L}_G w = 0$ is monotone in G and in the initial condition.

Corollary A.11. *Let $T > 0$, $Q := (0, T) \times \mathbb{R}$, and let G_1 and G_2 fulfill $G_1 \leq G_2$ on $\mathbb{R} \times [0, \infty)$. Furthermore, assume that G_2 satisfies (A.19). In addition, let $w^{(1)}$ and $w^{(2)}$ be non-negative and bounded functions on \overline{Q} , such that for $i \in \{1, 2\}$, $w_x^{(i)}$ and $w_{xx}^{(i)}$ are continuous on Q and $w_t^{(i)}$ exist on Q . If $\mathcal{L}_{G_1} w^{(1)} = \mathcal{L}_{G_2} w^{(2)}$ and $w^{(1)}(0, \cdot) \leq w^{(2)}(0, \cdot)$ on $x \in \mathbb{R}$, then we have $w^{(1)} \leq w^{(2)}$ on Q .*

Proof. Since the function $w^{(2)}$ is non-negative, we have $\mathcal{L}_{G_2} w^{(2)} = \mathcal{L}_{G_1} w^{(1)} \geq \mathcal{L}_{G_2} w^{(1)}$. Then by Lemma A.10, we have $w^{(1)}(t, x) \leq w^{(2)}(t, x)$ for all $(t, x) \in Q$.

Corollary A.12. *Let G fulfill (A.19) and $G(x, 0) = G(x, 1) = 0$ for all $x \in \mathbb{R}$. Let w be a solution to $\mathcal{L}_G w = 0$ with $0 \leq w(0, x) \leq 1$. Then $0 \leq w(t, x) \leq 1$ for all $(t, x) \in [0, \infty) \times \mathbb{R}$.*

Proof. The functions $w^{(1)}(t, x) = 0$ and $w^{(2)}(t, x) = 1$ are solutions to $\mathcal{L}w^{(1)} = \mathcal{L}w^{(2)} = 0$ and $w^{(1)}(0, x) \leq w(0, x) \leq w^{(2)}(0, x)$. The claim then follows from Lemma A.10.

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(Received February 5, 2021 / Accepted September 28, 2022)
 Published online November 11, 2022
 © The Author(s) (2022), corrected publication 2023